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# STABILITY FOR QUANTITATIVE PHOTOACOUSTIC TOMOGRAPHY REVISITED

ERIC BONNETIER, MOURAD CHOULLI, AND FAOUZI TRIKI

**ABSTRACT.** This paper deals with the issue of stability in determining the absorption and the diffusion coefficients in quantitative photoacoustic imaging. We establish a global conditionnal Hölder stability inequality from the knowledge of two internal data obtained from optical waves, generated by two point sources in a region where the optical coefficients are known.

**Mathematics subject classification :** 35R30.

**Key words :** Elliptic equations, diffusion coefficient, absorption coefficient, stability inequality, multiwave imaging.

## Contents

### 1. INTRODUCTION

Throughout this text  $n \geq 3$  is a fixed integer. If  $0 < \beta \leq 1$  we denote by  $C^{0,\beta}(\mathbb{R}^n)$  the vector space of bounded continuous functions  $f$  on  $\mathbb{R}^n$  satisfying

$$[f]_\beta = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^\beta}; x, y \in \mathbb{R}^n, x \neq y \right\} < \infty.$$

$C^{0,\beta}(\mathbb{R}^n)$  is then a Banach space when it is endowed with its natural norm

$$\|f\|_{C^{0,\beta}(\mathbb{R}^n)} = \|f\|_{L^\infty(\mathbb{R}^n)} + [f]_\beta.$$

Define  $C^{1,\beta}(\mathbb{R}^n)$  as the vector space of functions  $f$  from  $C^{0,\beta}(\mathbb{R}^n)$  so that  $\partial_j f \in C^{0,\beta}(\mathbb{R}^n)$ ,  $1 \leq j \leq n$ . The vector space  $C^{1,\beta}(\mathbb{R}^n)$  equipped with the norm

$$\|f\|_{C^{1,\beta}(\mathbb{R}^n)} = \|f\|_{C^{0,\beta}(\mathbb{R}^n)} + \sum_{j=1}^n \|\partial_j f\|_{C^{0,\beta}(\mathbb{R}^n)}$$

is a Banach space.

The data in this paper consists in  $\xi_1, \xi_2 \in \mathbb{R}^n$ ,  $\Omega \Subset \mathbb{R}^n \setminus \{\xi_1, \xi_2\}$  a  $C^{1,1}$  bounded domain with boundary  $\Gamma$ ,  $0 < \alpha < 1$ ,  $0 < \theta < \alpha$ ,  $\lambda > 1$  and  $\kappa > 1$ . For notational convenience, the set of data will denoted by  $\mathfrak{D}$ . That is

$$\mathfrak{D} = (n, \xi_1, \xi_2, \Omega, \alpha, \theta, \lambda, \kappa).$$

Denote by  $\mathcal{D}(\lambda, \kappa)$  the set of couples  $(a, b) \in C^{1,1}(\mathbb{R}^n) \times C^{0,1}(\mathbb{R}^n)$  satisfying

$$(1.1) \quad \lambda^{-1} \leq a \quad \text{and} \quad \|a\|_{C^{1,1}(\mathbb{R}^n)} \leq \lambda,$$

$$(1.2) \quad \kappa^{-1} \leq b \quad \text{and} \quad \|b\|_{C^{0,1}(\mathbb{R}^n)} \leq \kappa,$$

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Define further the elliptic operator  $L_{a,b}$  acting as follows

$$(1.3) \quad L_{a,b}u(x) = -\operatorname{div}(a(x)\nabla u(x)) + b(x)u(x).$$

We show in Section ?? that if  $(a,b) \in \mathcal{D}(\lambda, \kappa)$  then the operator  $L_{a,b}$  admits a unique fundamental solution  $G_{a,b}$  satisfying, where  $\xi \in \mathbb{R}^n$ ,

$$G_{a,b}(\cdot, \xi) \in C_{loc}^{2,\alpha}(\mathbb{R}^n \setminus \{\xi\}), \quad L_{a,b}G_{a,b}(\cdot, \xi) = 0 \text{ in } \mathbb{R}^n \setminus \{\xi\},$$

and, for any  $f \in C_0^\infty(\mathbb{R}^n)$ ,

$$u = \int_{\mathbb{R}^n} G_{a,b}(\cdot, \xi) f(\xi) d\xi$$

belongs to  $H^2(\mathbb{R}^n)$  and it is the unique solution of  $L_{a,b}u = f$ .

We deal in the present work with the problem of reconstructing  $(a,b) \in \mathcal{D}(\lambda, \kappa)$  from energies generated by two point sources located at  $\xi_1$  and  $\xi_2$ . Precisely, if  $u_j(a,b) = G_{a,b}(\cdot, \xi_j)$ ,  $j = 1, 2$ , we want to determine  $(a,b)$  from the internal measurements

$$v_j(a,b) = bu_j(a,b) \quad \text{in } \Omega, \quad j = 1, 2.$$

This inverse problem is related to photoacoustic tomography (PAT) where optical energy absorption causes thermoelastic expansion of the tissue, which in turn generates a pressure wave [?]. This acoustic signal is measured by transducers distributed on the boundary of the sample and it is used for imaging optical properties of the sample. The internal data  $v_1(a,b)$  and  $v_2(a,b)$  are obtained by performing a first step consisting in a linear initial to boundary inverse problem for the acoustic wave equation. Therefore the inverse problem that arises from this first inversion is to determine the diffusion coefficient  $a$  and the absorption coefficient  $b$  from the internal data  $v_1(a,b)$  and  $v_2(a,b)$  that are proportional to the local absorbed optical energy inside the sample. This inverse problem is known in the literature as quantitative photoacoustic tomography [?, ?, ?, ?, ?, ?, ?, ?].

Photoacoustic imaging provides in theory images of optical contrasts and ultrasound resolution [?]. Indeed, the resolution is mainly due to the small wavelength of acoustic waves, while the contrast is somehow related to the sensitivity of optical waves to absorption and scattering properties of the medium in the diffusive regime. However, in practice, it has been observed in various experiments that the imaging depth, i.e. the maximal depth of the medium at which structures can be resolved at expected resolution, of (PAT) is still fairly limited, usually on the order of millimeters. This is mainly due to the fact that optical waves are significantly attenuated by absorption and scattering. In fact the generated optical signal decays very fast in the depth direction. This is indeed a well known faced issue in optical tomography [?]. In most physicists works dealing with quantitative (PAT), the absorption  $b > 0$  is assumed to be constant and the optical wave is simplified to  $Ce^{-bz}$ , as a function of the depth  $z$ , which is known as the Beer-Lambert-Bouguer law [?]. Recently in [?], assuming that medium is layered, the authors derived a stability estimate that shows that the reconstruction of the optical coefficients is stable in the region close to the optical illumination source and deteriorate exponentially far away.

Stability inequalities for this inverse problem were first obtained in [?, ?] under a strong non-degeneracy assumption. Later in [?], the authors improved these results by removing the non-degeneracy assumption for well-chosen boundary conditions (Definition 2.3).

Assuming that the optical waves are generated by two point sources  $\delta_{\xi_i}$ ,  $i = 1, 2$ , we aim to derive a stability estimate for the recovery of the optical coefficients from internal data. We point out that

taking the optical wave generated by a point source outside the sample seems to be more realistic than assuming a boundary condition.

In the statement of Theorem ?? below  $C = C(\mathfrak{D}) > 0$  and  $0 < \gamma = \gamma(\mathfrak{D}) < 1$  are constants.

**Theorem 1.1.** *For any  $(a, b), (\tilde{a}, \tilde{b}) \in \mathcal{D}(\lambda, \kappa)$  satisfying  $(a, b) = (\tilde{a}, \tilde{b})$  on  $\Gamma$ , we have*

$$\|a - \tilde{a}\|_{C^{1,\alpha}(\overline{\Omega})} + \|b - \tilde{b}\|_{C^{0,\alpha}(\overline{\Omega})} \leq C \left( \|v_1 - \tilde{v}_1\|_{C(\overline{\Omega})} + \|v_2 - \tilde{v}_2\|_{C(\overline{\Omega})} \right)^\gamma.$$

The rest of this text is organized as follows. In section 2 we construct a fundamental solution and give its regularity induced by that of the coefficients of the operator under consideration. We derive pointwise lower and upper bounds for the fundamental solution that are of interest themselves. These bounds show how the optical signal decays fast in the depth direction. We also establish in this section a lower bound of the local  $L^2$ -norm of the gradient of the quotient of two fundamental solutions near one of the point sources. This is the key point for establishing our stability inequality. This last result is then used in Section 3 to obtain a uniform polynomial lower bound of the local  $L^2$ -norm of the gradient in a given region. This polynomial lower bound is obtained in two steps. In the first step we derive, via a three-ball inequality for the gradient, a uniform lower bound of negative exponential type. We use then in the second step an argument based on the so-called frequency function in order to improve this lower bound. In the last section we prove our main theorem following the known method consisting in reducing the original problem to the stability of an inverse conductivity problem.

## 2. FUNDAMENTAL SOLUTIONS

**2.1. Constructing fundamental solutions.** In this subsection we construct a fundamental solution of divergence form elliptic operators. Since our construction relies on heat kernel estimates, we first recall some known results.

Consider the parabolic operator  $P_{a,b}$  acting as follows

$$P_{a,b}u(x, t) = -L_{a,b}u(x, t) - \partial_t u(x, t)$$

and set

$$Q = \{(x, t, \xi, \tau) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}; \tau < t\}.$$

Recall that a fundamental solution of the operator  $P_{a,b}$  is a function  $E_{a,b} \in C^{2,1}(Q)$  verifying  $P_{a,b}E = 0$  in  $Q$  and, for every  $f \in C_0^\infty(\mathbb{R}^n)$ ,

$$\lim_{t \downarrow \tau} \int_{\mathbb{R}^n} E_{a,b}(x, t, \xi, \tau) f(\xi) d\xi = f(x), \quad x \in \mathbb{R}^n.$$

The classical results in the monographs by A. Friedman [?], O. A. Ladyzenskaja, V. A. Solonnikov and N.N. Ural'ceva [?] show that  $P_{a,b}$  admits a non negative fundamental solution when  $(a, b) \in \mathcal{D}(\lambda, \kappa)$ .

It is worth mentioning that if  $a = c$ , for some constant  $c > 0$ , and  $b = 0$  then the fundamental solution  $E_{c,0}$  is explicitly given by

$$E_{c,0}(x, t, \xi, \tau) = \frac{1}{[4\pi c(t - \tau)]^{n/2}} e^{-\frac{|x - \xi|^2}{4c(t - \tau)}}, \quad (x, t, \xi, \tau) \in Q.$$

Examining carefully the proof of the two-sided Gaussian bounds in [?], we see that these bounds remain valid whenever  $a \in C^{1,1}(\mathbb{R}^n)$  satisfies

$$(2.1) \quad \lambda^{-1} \leq a \quad \text{and} \quad \|a\|_{C^{1,1}(\mathbb{R}^n)} \leq \lambda.$$

More precisely we have the following theorem in which

$$\mathcal{E}_c(x, t) = \frac{c}{t^{n/2}} e^{-\frac{|x|^2}{ct}}, \quad x \in \mathbb{R}^n, \quad t > 0, \quad c > 0.$$

**Theorem 2.1.** *There exists a constant  $c = c(n, \lambda) > 1$  so that, for any  $a \in C^{1,1}(\mathbb{R}^n)$  satisfying (??), we have*

$$(2.2) \quad \mathcal{E}_{c^{-1}}(x - \xi, t - \tau) \leq E_{a,0}(x, t; \xi, \tau) \leq \mathcal{E}_c(x - \xi, t - \tau),$$

for all  $(x, t, \xi, \tau) \in Q$ .

The relationship between  $\mathcal{E}_c$  and  $E_{c,0}$  is given by the formula

$$(2.3) \quad \mathcal{E}_c(x - \xi, t - \tau) = \frac{(\pi c)^{n/2+1}}{\pi} E_{c/4,0}(x, t, \xi, \tau), \quad (x, t, \xi, \tau) \in Q.$$

The following comparison principle will be useful in the sequel.

**Lemma 2.1.** *Let  $(a, b_1), (a, b_2) \in \mathcal{D}(\lambda, \kappa)$  so that  $b_1 \leq b_2$ . Then  $E_{a,b_2} \leq E_{a,b_1}$ .*

*Proof.* Pick  $0 \leq f \in C_0^\infty(\mathbb{R}^n)$ . Let  $u$  be the solution of the initial value problem

$$P_{a,b_1}u(x, t) = 0 \quad \text{in } \mathbb{R}^n \times \{t > \tau\}, \quad u(x, \tau) = f.$$

We have

$$(2.4) \quad u(x, t) = \int_{\mathbb{R}^n} E_{a,b_1}(x, t; \xi, \tau) f(\xi) d\xi \geq 0.$$

On the other hand, as  $P_{a,b_1}u(x, t) = 0$  can be rewritten as

$$P_{a,b_2}u(x, t) = [b_1(x) - b_2(x)]u(x, t),$$

we obtain

$$(2.5) \quad u(x, t) = \int_{\mathbb{R}^n} E_{a,b_2}(x, t; \xi, \tau) f(\xi) d\xi - \int_{\tau}^t \int_{\mathbb{R}^n} E_{a,b_2}(x, t; \xi, s) [b_1(\xi) - b_2(\xi)] u(\xi, s) d\xi ds.$$

Combining (??) and (??), we get

$$\int_{\mathbb{R}^n} E_{a,b_2}(x, t; \xi, \tau) f(\xi) d\xi \leq \int_{\mathbb{R}^n} E_{a,b_1}(x, t; \xi, \tau) f(\xi) d\xi,$$

which yields in a straightforward manner the expected inequality.  $\square$

Consider, for  $(a, b) \in \mathcal{D}(\lambda, \kappa)$ , the unbounded operator  $A_{a,b} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$  defined

$$A_{a,b} = -L_{a,b}, \quad D(A_{a,b}) = H^2(\mathbb{R}^n).$$

It is well known that  $A_{a,b}$  generates an analytic semigroup  $e^{tA_{a,b}}$ . Therefore in light of [?, Theorem 4 on page 30, Theorem 18 on page 44 and the proof in the beginning of Section 1.4.2 on page 35]  $k_{a,b}(t, x; \xi)$ , the Schwarz kernel of  $e^{tA_{a,b}}$ , is Hölder continuous with respect to  $x$  and  $\xi$ , satisfies

$$(2.6) \quad |k_{a,b}(t, x, \xi)| \leq e^{-\delta t} \mathcal{E}_c(x - \xi, t)$$

and, for  $|h| \leq \sqrt{t} + |x - \xi|$ ,

$$(2.7) \quad |k_{a,b}(t, x + h, \xi) - k_{a,b}(t, x, \xi)| \leq e^{-\delta t} \left( \frac{|h|}{\sqrt{t} + |x - \xi|} \right)^\eta \mathcal{E}_c(x - \xi, t),$$

$$(2.8) \quad |k_{a,b}(t, x, \xi + h) - k_{a,b}(t, x, \xi)| \leq e^{-\delta t} \left( \frac{|h|}{\sqrt{t} + |x - \xi|} \right)^\eta \mathcal{E}_c(x - \xi, t),$$

where  $c = c(n, \lambda, \kappa) > 0$  and  $\delta = \delta(n, \lambda, \kappa) > 0$  and  $\eta > 0$  are constants.

From the uniqueness of solutions of the Cauchy problem

$$(2.9) \quad u'(t) = A_{a,b}u(t), \quad t > 0, \quad u(0) = f \in C_0^\infty(\mathbb{R}^n),$$

we deduce in a straightforward manner that  $k_{a,b}(t, x; \xi) = E_{a,b}(x, t; \xi, 0)$ .

Prior to giving the construction of the fundamental solution for the variable coefficients operators, we state a result for operators with constant coefficients. This result is proved in Appendix ??.

**Lemma 2.2.** *Let  $\mu > 0$  and  $\nu > 0$  be two constants. Then the fundamental solution for the operator  $-\mu\Delta + \nu$  is given by  $G_{\mu,\nu}(x, \xi) = \mathcal{G}_{\mu,\nu}(x - \xi)$ ,  $x, \xi \in \mathbb{R}^n$ , with*

$$\mathcal{G}_{\mu,\nu}(x) = (2\pi\mu)^{-n/2} (\sqrt{\nu\mu}/|x|)^{n/2-1} K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}), \quad x \in \mathbb{R}^n.$$

Here  $K_{n/2-1}$  is the usual modified Bessel function of second kind. Moreover the following two-sided inequality holds

$$(2.10) \quad C^{-1} \frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{n-2}} \leq \mathcal{G}_{\mu,\nu}(x) \leq C \frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}}{|x|^{n-2}}, \quad x \in \mathbb{R}^n,$$

for some constant  $C = C(n, \mu, \nu) > 1$ .

The main result of this section is the following theorem.

**Theorem 2.2.** *Let  $(a, b) \in \mathcal{D}(\lambda, \kappa)$ . Then there exists a unique function  $G_{a,b}$  satisfying  $G_{a,b}(\cdot, \xi) \in C(\mathbb{R}^n \setminus \{\xi\})$ ,  $\xi \in \mathbb{R}^n$ ,  $G_{a,b}(x, \cdot) \in C(\mathbb{R}^n \setminus \{x\})$ ,  $x \in \mathbb{R}^n$ , and*

- (i)  $L_{a,b}G_{a,b}(\cdot, \xi) = 0$  in  $\mathcal{D}'(\mathbb{R}^n \setminus \{\xi\})$ ,  $\xi \in \mathbb{R}^n$ ,
- (ii) for any  $f \in C_0^\infty(\mathbb{R}^n)$ ,

$$u(x) = \int_{\mathbb{R}^n} G_{a,b}(x, \xi) f(\xi) d\xi$$

belongs to  $H^2(\mathbb{R}^n)$  and it is the unique solution of  $L_{a,b}u = f$ ,

- (iii) there exist two constants  $c = c(n, \lambda) > 1$  and  $C = C(n, \lambda, \kappa) > 1$  so that

$$(2.11) \quad C^{-1} \frac{e^{-2\sqrt{c\kappa}|x-\xi|}}{|x-\xi|^{n-2}} \leq G_{a,b}(x, \xi) \leq C \frac{e^{-\frac{|x-\xi|}{\sqrt{c\kappa}}}}{|x-\xi|^{n-2}}.$$

*Proof.* Pick  $s \geq 1$  arbitrary and let  $f \in C_0^\infty(\mathbb{R}^n)$ . Applying Hölder's inequality, we find

$$\int_{\mathbb{R}^n} k_{a,b}(t, x, \xi) |f(\xi)| d\xi \leq \|k_{a,b}(t, x, \cdot)\|_{L^s(\mathbb{R}^n)} \|f\|_{L^{s'}(\mathbb{R}^n)},$$

where  $s'$  is the conjugate exponent of  $s$ .

But, according to (??),

$$\|k_{a,b}(t, x, \cdot)\|_{L^s(\mathbb{R}^n)}^s \leq \left( \frac{c}{t^{n/2}} \right)^s \int_{\mathbb{R}^n} e^{-\frac{s|x-\xi|^2}{ct}} d\xi.$$

Next, making the change of variable  $\xi = (\sqrt{ct/s})\eta + x$ , we get

$$\|k_{a,b}(t, x, \cdot)\|_{L^s(\mathbb{R}^n)}^s \leq \left(\frac{c}{t^{n/2}}\right)^s \left(\frac{ct}{s}\right)^{n/2} \int_{\mathbb{R}^n} e^{-|\eta|^2} d\eta.$$

Hence

$$\|k_{a,b}(t, x, \cdot)\|_{L^s(\mathbb{R}^n)} \leq t^{n(1/s-1)/2} C_s,$$

with

$$C_s = c \left(\frac{c}{s}\right)^{n/2} \left(\int_{\mathbb{R}^n} e^{-|\eta|^2} d\eta\right)^{1/s}.$$

We get, by choosing  $1 \leq s < \frac{n}{n-2} < \tilde{s}$ ,

$$\begin{aligned} & \int_0^{+\infty} \int_{\mathbb{R}^n} k_{a,b}(t, x, \xi) |f(\xi)| d\xi dt \\ &= \int_0^1 \int_{\mathbb{R}^n} k_{a,b}(t, x, \xi) |f(\xi)| d\xi dt + \int_1^{+\infty} \int_{\mathbb{R}^n} k_{a,b}(t, x, \xi) |f(\xi)| d\xi dt \\ &\leq C_s \|f\|_{L^{s'}(\mathbb{R}^n)} \int_0^1 t^{\frac{n}{2}(1/s-1)} dt + C_{\tilde{s}} \|f\|_{L^{\tilde{s}'}(\mathbb{R}^n)} \int_1^{+\infty} t^{\frac{n}{2}(1/\tilde{s}-1)} dt. \end{aligned}$$

In light of Fubini's theorem we obtain

$$(2.12) \quad \int_0^{+\infty} \int_{\mathbb{R}^n} k_{a,b}(t, x, \xi) f(\xi) d\xi dt = \int_{\mathbb{R}^n} \left( \int_0^{+\infty} k_{a,b}(t, x, \xi) dt \right) f(\xi) d\xi.$$

Define  $G_{a,b}$  as follows

$$G_{a,b}(x, \xi) = \int_0^{+\infty} k_{a,b}(t, x, \xi) dt.$$

Then (??) takes the form

$$(2.13) \quad \int_0^{+\infty} \int_{\mathbb{R}^n} k_{a,b}(t, x, \xi) f(\xi) d\xi dt = \int_{\mathbb{R}^n} G_{a,b}(x, \xi) f(\xi) d\xi.$$

Noting that  $A_{a,b}$  is invertible, we obtain

$$\begin{aligned} -A_{a,b}^{-1} f(x) &= \left( \int_0^{+\infty} e^{tA_{a,b}} f dt \right) (x) \\ &= \int_0^{+\infty} \int_{\mathbb{R}^n} k_{a,b}(t, x, \xi) f(\xi) d\xi dt, \quad x \in \mathbb{R}^n. \end{aligned}$$

This and (??) entail

$$-A_{a,b}^{-1} f(x) = \int_{\mathbb{R}^n} G_{a,b}(x, \xi) f(\xi) d\xi, \quad x \in \mathbb{R}^n.$$

In other words,  $u$  defined by

$$u(x) = \int_{\mathbb{R}^n} G_{a,b}(x, \xi) f(\xi) d\xi, \quad x \in \mathbb{R}^n,$$

belongs to  $H^2(\mathbb{R}^n)$  and satisfies  $L_{a,b} u = f$ .

Since, for  $x \neq \xi$ ,

$$\int_0^{+\infty} \frac{1}{t^{n/2}} e^{-\frac{|x-\xi|^2}{ct}} dt = \left( c^{n/2-1} \int_0^{+\infty} \tau^{n/2-2} e^{-\tau} d\tau \right) \frac{1}{|x-\xi|^{n-2}},$$

we get in light of (??)

$$|G_{a,b}(x+h, \xi) - G_{a,b}(x, \xi)| \leq \frac{C}{|x-\xi|^{n-2+\eta}} |h|^\eta, \quad x \neq \xi, |h| \leq |x-\xi|,$$

where  $C = C(n, \lambda, \kappa)$  is a constant. In particular,  $G_{a,b}(\cdot, \xi) \in C(\mathbb{R}^n \setminus \{\xi\})$ . Similarly, using (??) instead of (??), we obtain  $G_{a,b}(x, \cdot) \in C(\mathbb{R}^n \setminus \{x\})$ . More specifically we have

$$(2.14) \quad |G_{a,b}(x, \xi+h) - G_{a,b}(x, \xi)| \leq \frac{C}{|x-\xi|^{n-2+\eta}} |h|^\eta, \quad x \neq \xi, |h| \leq |x-\xi|.$$

Let  $\xi \in \mathbb{R}^n$  and  $\omega \in \mathbb{R}^n \setminus \{\xi\}$ , and pick  $g \in C_0^\infty(\omega)$ . Then set

$$w_{a,b}(y) = \int_\omega G_{a,b}(x, y) g(x) dx, \quad y \in B(\xi, \text{dist}(\xi, \bar{\omega})/2).$$

It follows from (??) that, for  $y \in B(\xi, \text{dist}(\xi, \bar{\omega}))$  and  $|h| < \text{dist}(y, \bar{\omega})$ , we have

$$|w_{a,b}(y+h) - w_{a,b}(y)| \leq \frac{C}{\text{dist}(y, \bar{\omega})^{n-2+\eta}} |h|^\eta.$$

Therefore  $w_{a,b} \in C(B(\xi, \text{dist}(\xi, \bar{\omega})/2))$ .

Let  $\mathcal{M}(\mathbb{R}^n)$  be the space of bounded measures on  $\mathbb{R}^n$ . Pick a sequence  $(f_k)$  of a positive functions of  $C_0^\infty(\mathbb{R}^n)$  converging in  $\mathcal{M}(\mathbb{R}^n)$  to  $\delta_\xi$  and let  $u_k = -A_{a,b}^{-1} f_k$ . In that case, according to Fubini's theorem, we have

$$\begin{aligned} \int_\omega u_k(x) g(x) dx &= \int_\omega \int_{\mathbb{R}^n} G_{a,b}(x, y) g(x) f_k(y) dy dx \\ &= \int_{\mathbb{R}^n} w_{a,b}(y) f_k(y) dy \longrightarrow w_{a,b}(\xi) = \int_\omega G_{a,b}(x, \xi) g(x) dx, \end{aligned}$$

where we used that  $\text{supp} f_k \subset B(\xi, \text{dist}(\xi, \bar{\omega})/2)$ , provided that  $k$  is sufficiently large. That is we proved that  $u_k$  converges to  $G_{a,b}(\cdot, \xi)$  weakly in  $L_{\text{loc}}^2(\mathbb{R}^n \setminus \{\xi\})$  (think to the fact that  $C_0^\infty(\omega)$  is dense in  $L^2(\omega)$ ).

Now, as  $L_{a,b} u_k = f_k$ , we find  $L_{a,b} G_{a,b}(\cdot, \xi) = 0$  in  $\mathbb{R}^n \setminus \{\xi\}$  in the distributional sense.

The uniqueness of  $G_{a,b}$  follows from that of  $u$  and, as  $\kappa^{-1} \leq b \leq \kappa$ , we deduce from Lemma ?? that

$$E_{a,\kappa}(x, t, \xi, 0) \leq E_{a,b}(x, t, \xi, 0) \leq E_{a,\kappa^{-1}}(x, t, \xi, 0).$$

But a simple change of variable shows that

$$(2.15) \quad E_{a,\kappa^{-1}}(x, t, \xi, 0) = e^{-\kappa^{-1}t} E_{a,0}(x, t, \xi, 0)$$

and

$$(2.16) \quad E_{a,\kappa}(x, t, \xi, 0) = e^{-\kappa t} E_{a,0}(x, t, \xi, 0).$$

Therefore, from Theorem ?? and identity (??), there exists a constant  $c = c(n, \lambda) > 1$  so that

$$\begin{aligned} e^{-\kappa t} \frac{(\pi c^{-1})^{n/2+1}}{\pi} E_{c^{-1}/4,0}(x, t, \xi, 0) &\leq E_{a,b}(x, t, \xi, 0) \\ &\leq e^{-\kappa^{-1}t} \frac{(\pi c)^{n/2+1}}{\pi} E_{c/4,0}(x, t, \xi, 0), \end{aligned}$$



which, combined with identities (??) and (??), gives

$$\begin{aligned} \frac{(\pi c^{-1})^{n/2+1}}{\pi} E_{c^{-1}/4, \kappa}(x, t, \xi, 0) &\leq E_{a,b}(x, t, \xi, 0) \\ &\leq \frac{(\pi c)^{n/2+1}}{\pi} E_{c/4, \kappa^{-1}}(x, t, \xi, 0). \end{aligned}$$

From the uniqueness of  $G_{a,b}$ , we obtain by integrating over  $(0, +\infty)$ , with respect to  $t$ , each member of the above inequalities

$$\frac{(\pi c^{-1})^{n/2+1}}{\pi} G_{c^{-1}/4, \kappa}(x, \xi) \leq G_{a,b}(x, \xi) \leq \frac{(\pi c)^{n/2+1}}{\pi} G_{c/4, \kappa^{-1}}(x, \xi).$$

These two-sided inequalities together with (??) yield in a straightforward manner (??).  $\square$

The function  $G_{a,b}$  given by the previous theorem is usually called a fundamental solution of the operator  $L_{a,b}$ .

**2.2. Regularity of fundamental solutions.** Let  $\xi \in \mathbb{R}^n$  and  $\mathcal{O} \Subset \mathcal{O}' \Subset \mathbb{R}^n \setminus \{\xi\}$  with  $\mathcal{O}'$  of class  $C^{1,1}$ . As  $G_{a,b}(\cdot, \xi) \in C(\partial\mathcal{O}')$ , we get from [?, Theorem 6.18, page 106] (interior Hölder regularity) that  $G_{a,b}(\cdot, \xi)$  belongs to  $C^{2,\alpha}(\overline{\mathcal{O}})$ .

**Proposition 2.1.** *There exist  $C = C(n, \lambda, \kappa, \alpha)$  and  $\nu = \nu(\alpha) > 2$  so that, for any  $\xi \in \mathbb{R}^n$  and  $\mathcal{O} \Subset \mathbb{R}^n \setminus \{\xi\}$ , we have*

$$(2.17) \quad \|G_{a,b}(\cdot, \xi)\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \leq C\Lambda(\mathbf{d} + \varrho)^\nu \max\left(\varrho^{-(2+\alpha)}, 1\right) \varrho^{-n+2}.$$

Here  $\varrho = \text{dist}(\xi, \overline{\mathcal{O}})$ ,  $\mathbf{d} = \text{diam}(\mathcal{O})$  and

$$\Lambda(h) = [1 + 2h + 2h^2 + h^3]\lambda, \quad h > 0.$$

The proof of this proposition is based the following lemma consisting in an adaptation of the usual interior Schauder estimates. The proof of this technical lemma will be given in Appendix A.

**Lemma 2.3.** *There exists two constants  $C = C(n, \alpha)$  and  $\nu = \nu(\alpha) > 1$  with the property that, for any bounded subset  $\mathcal{Q}$  of  $\mathbb{R}^n$ ,  $\delta > 0$  so that  $\mathcal{Q}_\delta = \{x \in \mathcal{Q}; \text{dist}(x, \partial\mathcal{Q}) > \delta\} \neq \emptyset$ ,  $w \in C^{2,\alpha}(\mathcal{Q}) \cap C(\overline{\mathcal{Q}})$  satisfying  $L_{a,b}w = 0$  in  $\mathcal{Q}$  and  $\mathcal{Q}' \subset \mathcal{Q}_\delta$ , we have*

$$(2.18) \quad \|w\|_{C^{2,\alpha}(\overline{\mathcal{Q}'})} \leq C \max\left(\delta^{-(2+\alpha)}, 1\right) \Lambda(\mathbf{d})^\nu \|w\|_{C(\overline{\mathcal{Q}})},$$

where  $\Lambda$  is as in Proposition ?? and  $\mathbf{d} = \text{diam}(\mathcal{Q})$ .

*Proof of Proposition ??.* We get, by applying Lemma ?? with  $\mathcal{Q}' = \mathcal{O}$ ,  $\delta = \varrho/2$  and  $\mathcal{Q} = \{x \in \mathbb{R}^n; \text{dist}(x, \overline{\mathcal{O}}) < \varrho/2\}$

$$\|G_{a,b}(\cdot, \xi)\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \leq C\Lambda(\mathbf{d} + \varrho)^\nu \max\left(\delta^{-(2+\alpha)}, 1\right) \|G_{a,b}(\cdot, \xi)\|_{C(\overline{\mathcal{Q}})}.$$

This and (??) yield

$$(2.19) \quad \|G_{a,b}(\cdot, \xi)\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \leq C\Lambda(\mathbf{d} + \varrho)^\nu \max\left(\delta^{-(2+\alpha)}, 1\right) \varrho^{-n+2} e^{-\varrho/\sqrt{c\kappa}},$$

with  $C = C(n, \lambda, \kappa, \alpha)$  and  $c = c(n, \lambda)$ . It is then clear that (??) implies (??).  $\square$

The preceding proposition together with Lemma ?? enable us to state the following corollary.

**Corollary 2.1.** *There exist  $C = C(n, \lambda, \kappa, \alpha, \theta)$  and  $\nu = \nu(\alpha) > 1$  so that, for any  $\xi \in \mathbb{R}^n$  and  $\mathcal{O} \in \mathbb{R}^n \setminus \{\xi\}$ , we have*

$$(2.20) \quad \begin{aligned} & \|G_{a,b}(\cdot, \xi)\|_{H^{2+\theta}(\mathcal{O})} \\ & \leq C\Lambda(\mathbf{d} + \varrho)^\nu \max\left(\mathbf{d}^{n/2}, \mathbf{d}^{n/2+\alpha-\theta}\right) \max\left(\varrho^{-(2+\alpha)}, 1\right) \varrho^{-n+2}, \end{aligned}$$

where  $\varrho = \text{dist}(\xi, \overline{\mathcal{O}})$ ,  $\mathbf{d} = \text{diam}(\mathcal{O})$ .

**Corollary 2.2.** *There exist  $C = C(n, \lambda, \kappa, \alpha)$  and  $c = c(n, \lambda, \kappa, \alpha)$  so that, for any  $\xi_1, \xi_2 \in \mathbb{R}^n$  and  $\mathcal{O} \in \mathbb{R}^n \setminus \{\xi_1, \xi_2\}$ , we have*

$$(2.21) \quad \left\| \frac{G_{a,b}(\cdot, \xi_2)}{G_{a,b}(\cdot, \xi_1)} \right\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \leq C e^{c(\mathbf{d}+\varrho_+)} \left(1 + \max\left(\varrho_-^{-(2+\alpha)}, 1\right) \varrho_-^{-n+2}\right)^4,$$

where  $\varrho_- = \min(\text{dist}(\xi_1, \mathcal{O}), \text{dist}(\xi_2, \mathcal{O}))$  and  $\varrho_+ = \max(\text{dist}(\xi_1, \mathcal{O}), \text{dist}(\xi_2, \mathcal{O}))$ .

*Proof.* In this proof  $C = C(n, \lambda, \kappa, \alpha)$ ,  $c = c(n, \lambda, \kappa, \alpha)$  and  $\nu = \nu(\alpha) > 2$  are generic constants.

From Proposition ??, we have

$$(2.22) \quad \|G_{a,b}(\cdot, \xi_j)\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \leq C\Lambda(\mathbf{d} + \varrho_+)^\nu \max\left(\varrho_-^{-(2+\alpha)}, 1\right) \varrho_-^{-n+2}, \quad j = 1, 2.$$

Let  $C_0 \geq 1$  and  $c_0 \geq 1$  be the constants in (??) and fix  $0 < \delta_0 \leq 1$ . Then the first inequality in (??) gives

$$\frac{1}{G_{a,b}(\cdot, \xi_1)} \leq C_0 (\mathbf{d} + \varrho_+)^{n-2} e^{2\sqrt{c_0\kappa}(\mathbf{d}+\varrho_+)}.$$

This inequality together with Lemma ?? in Appendix ?? yield

$$(2.23) \quad \left\| \frac{1}{G_{a,b}(\cdot, \xi_1)} \right\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \leq C e^{c(\mathbf{d}+\varrho_+)} \left(1 + \|G_{a,b}(\cdot, \xi_1)\|_{C^{2,\alpha}(\overline{\mathcal{O}})}\right)^3.$$

Then in light of (??) and (??), we get in a straightforward manner

$$\left\| \frac{G_{a,b}(\cdot, \xi_2)}{G_{a,b}(\cdot, \xi_1)} \right\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \leq C e^{c(\mathbf{d}+\varrho_+)} \left(1 + (1 + \mathbf{d})^\nu \max\left(\varrho_-^{-(2+\alpha)}, 1\right) \varrho_-^{-n+2}\right)^4,$$

and hence

$$\left\| \frac{G_{a,b}(\cdot, \xi_2)}{G_{a,b}(\cdot, \xi_1)} \right\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \leq C e^{c(\mathbf{d}+\varrho_+)} \left(1 + \max\left(\varrho_-^{-(2+\alpha)}, 1\right) \varrho_-^{-n+2}\right)^4.$$

This is the expected inequality.  $\square$

This corollary combined with Lemma ?? yields the following result.

**Corollary 2.3.** *There exist  $C = C(n, \lambda, \kappa, \alpha, \theta)$  and  $c = c(n, \lambda, \kappa, \alpha, \theta)$  so that, for any  $\xi_1, \xi_2 \in \mathbb{R}^n$  and  $\mathcal{O} \in \mathbb{R}^n \setminus \{\xi_1, \xi_2\}$ , we have*

$$(2.24) \quad \left\| \frac{G_{a,b}(\cdot, \xi_2)}{G_{a,b}(\cdot, \xi_1)} \right\|_{H^{2+\theta}(\mathcal{O})} \leq C e^{c(\mathbf{d}+\varrho_+)} \left(1 + \max\left(\varrho_-^{-(2+\alpha)}, 1\right) \varrho_-^{-n+2}\right)^4.$$

Here  $\varrho_\pm$  is the same as in Corollary ??.

**2.3. Gradient estimate of the quotient of two fundamental solutions.** The following result uses the singularity of the Green function near the location of the point source.

**Lemma 2.4.** *There exist  $x^* \in B(\xi_2, |\xi_1 - \xi_2|/2) \setminus \{\xi_2\}$ ,  $C = (n, \lambda, \kappa, |\xi_1 - \xi_2|) > 0$  and  $\rho = \rho(n, \lambda, \kappa, |\xi_1 - \xi_2|) > 0$  so that  $\bar{B}(x^*, \rho) \subset B(\xi_2, |\xi_1 - \xi_2|/2) \setminus \{\xi_2\}$  and*

$$C \leq \left\| \nabla \left( \frac{G_{a,b}(\cdot, \xi_2)}{G_{a,b}(\cdot, \xi_1)} \right) \right\|_{L^2(B(x^*, \rho))}.$$

*Proof.* We set for notational convenience  $w = G_{a,b}(\cdot, \xi_2)/G_{a,b}(\cdot, \xi_1)$ . In light of Theorem ??, we obtain by straightforward computations the following two-sided inequality

$$(2.25) \quad \frac{C^{-1}}{|x - \xi_2|^{n-2}} \leq w(x) \leq \frac{C}{|x - \xi_2|^{n-2}}, \quad x \in B(\xi_2, |\xi_1 - \xi_2|/2) \setminus \{\xi_2\}.$$

Here and until the end of this proof  $C = C(n, \lambda, \kappa, |\xi_1 - \xi_2|)$  is a generic constant.

Set  $\tilde{t} = |\xi_1 - \xi_2|/4$  and define

$$\varphi(t, \theta) = w(\xi_2 + t\theta), \quad (t, \theta) \in (0, \tilde{t}] \times \mathbb{S}^{n-1}.$$

According to Corollary ??,  $\varphi \in C_{\text{loc}}^{2,\alpha}((0, \tilde{t}] \times \mathbb{S}^{n-1})$  and hence

$$\varphi(\tilde{t}, \theta) - \varphi(t, \theta) = \int_t^{\tilde{t}} \nabla w(\xi_2 + s\theta) \cdot \theta ds,$$

which in turn gives

$$\begin{aligned} |\varphi(\tilde{t}, \theta) - \varphi(t, \theta)|^2 &\leq (\tilde{t} - t) \int_t^{\tilde{t}} |\nabla w(\xi_2 + s\theta)|^2 ds \\ &\leq \tilde{t} \int_t^{\tilde{t}} |\nabla w(\xi_2 + s\theta)|^2 ds \\ &\leq \tilde{t} \int_t^{\tilde{t}} \frac{s^{n-1}}{t^{n-1}} |\nabla w(\xi_2 + s\theta)|^2 ds, \quad (t, \theta) \in (0, \tilde{t}] \times \mathbb{S}^{n-1}. \end{aligned}$$

Whence, where  $t \in (0, \tilde{t}]$ ,

$$(2.26) \quad t^{n-1} \int_{\mathbb{S}^{n-1}} |\varphi(\tilde{t}, \theta) - \varphi(t, \theta)|^2 d\theta \leq \tilde{t} \int_{\mathcal{C}_t} |\nabla w(x)|^2 dx.$$

Here

$$\mathcal{C}_t = \{x \in \mathbb{R}^n; t < |x - \xi_2| < \tilde{t}\}.$$

On the other hand inequalities (??) imply, where  $(t, \theta) \in (0, \tilde{t}] \times \mathbb{S}^{n-1}$ ,

$$\frac{C^{-1}}{t^{n-2}} \leq \varphi(t, \theta) \leq \frac{C}{t^{n-2}}.$$

Let us then choose  $t_0 \leq \tilde{t}$  sufficiently small in such a way that

$$\frac{C^{-1}}{t^{n-2}} - \frac{C}{\tilde{t}^{n-2}} > 0, \quad t \in (0, t_0].$$

Therefore, for  $(t, \theta) \in (0, t_0] \times \mathbb{S}^{n-1}$ , we have

$$(2.27) \quad \left( \frac{C^{-1}}{t^{n-2}} - \frac{C}{\tilde{t}^{n-2}} \right)^2 \leq |\varphi(\tilde{t}, \theta) - \varphi(t, \theta)|^2.$$

We then obtain by combining inequalities (??) and (??)

$$|\mathbb{S}^{n-1}| \left( \frac{C^{-1}}{t^{n-2}} - \frac{C}{\tilde{t}^{n-2}} \right)^2 \leq \tilde{t} \int_{\mathcal{C}_t} |\nabla w(x)|^2 dx, \quad t \in (0, t_0].$$

We have in particular

$$C \leq \int_{\mathcal{C}_{t_0}} |\nabla w(x)|^2 dx.$$

Let  $\rho = t_0/4$ . Then it is straightforward to check that, for any  $x \in \overline{\mathcal{C}_{t_0}}$ ,

$$\overline{B}(x, \rho) \subset \{y \in \mathbb{R}^n; 3t_0/4 \leq |y - \xi_2| \leq 5\tilde{t}/4\} \subset B(\xi_2, |\xi_1 - \xi_2|/2) \setminus \{\xi_2\}.$$

Since  $\overline{\mathcal{C}_{t_0}}$  is compact, we find a positive integer  $N = N(\lambda, \kappa, |\xi_1 - \xi_2|)$  and  $x_j \in \overline{\mathcal{C}_{t_0}}$ ,  $j = 1, \dots, N$ , so that

$$\overline{\mathcal{C}_{t_0}} \subset \bigcup_{j=1}^N B(x_j, \rho).$$

Hence

$$C \leq \int_{\bigcup_{j=1}^N B(x_j, \rho)} |\nabla w(x)|^2 dx.$$

Pick then  $x^* \in \{x_j, 1 \leq j \leq N\}$  in such a way that

$$\int_{B(x^*, \rho)} |\nabla w(x)|^2 dx = \max_{1 \leq j \leq N} \int_{B(x_j, \rho)} |\nabla w(x)|^2 dx.$$

Therefore

$$C \leq \int_{B(x^*, \rho)} |\nabla w(x)|^2 dx.$$

This finishes the proof.  $\square$

### 3. UNIFORM LOWER BOUND FOR THE GRADIENT

Let  $\mathcal{O}$  be a Lipschitz bounded domain of  $\mathbb{R}^n$  and  $\sigma \in C^{0,1}(\overline{\mathcal{O}})$  satisfying

$$(3.1) \quad \varkappa^{-1} \leq \sigma \quad \text{and} \quad \|\sigma\|_{C^{0,1}(\overline{\mathcal{O}})} \leq \varkappa,$$

for some fixed constant  $\varkappa > 1$ .

We prove in this section a polynomial lower bound of the local  $L^2$ -norm of the gradient of solutions of

$$L_\sigma u = \operatorname{div}(\sigma \nabla u) = 0 \quad \text{in } \mathcal{O}.$$

In a first step we establish, via a three-ball inequality for the gradient, a uniform lower bound of negative exponential type. We use then in a second step an argument based on the so-called frequency function in order to improve this lower bound.

**3.1. Preliminary lower bound.** We need hereafter the following three-ball inequality for the gradient.

**Theorem 3.1.** *Let  $0 < k < \ell < m$  be real. There exist two constants  $C = C(n, \varkappa, k, \ell, m) > 0$  and  $0 < \gamma = \gamma(n, \varkappa, k, \ell, m) < 1$  so that, for any  $v \in H^1(\mathcal{O})$  satisfying  $L_\sigma v = 0$ ,  $y \in \mathcal{O}$  and  $0 < r < \text{dist}(y, \partial\mathcal{O})/m$ , we have*

$$C \|\nabla v\|_{L^2(B(y, \ell r))} \leq \|\nabla v\|_{L^2(B(y, kr))}^\gamma \|\nabla v\|_{L^2(B(y, mr))}^{1-\gamma}.$$

A proof of this theorem can be found in [?] or [?].

Define the geometric distance  $d_g^D$  on the bounded domain  $D$  of  $\mathbb{R}^n$  by

$$d_g^D(x, y) = \inf \{ \ell(\psi); \psi : [0, 1] \rightarrow D \text{ Lipschitz path joining } x \text{ to } y \},$$

where

$$\ell(\psi) = \int_0^1 |\dot{\psi}(t)| dt$$

is the length of  $\psi$ .

Note that according to Rademacher's theorem any Lipschitz continuous function  $\psi : [0, 1] \rightarrow D$  is almost everywhere differentiable with  $|\dot{\psi}(t)| \leq k$  a.e.  $t \in [0, 1]$ , where  $k$  is the Lipschitz constant of  $\psi$ .

**Lemma 3.1.** *Let  $D$  be a bounded Lipschitz domain of  $\mathbb{R}^n$ . Then  $d_g^D \in L^\infty(D \times D)$  and there exists a constant  $\mathbf{c}_D > 0$  so that*

$$(3.2) \quad |x - y| \leq d_g^D(x, y) \leq \mathbf{c}_D |x - y|, \quad x, y \in D.$$

We refer to [?, Lemma A3] for a proof.

In this subsection we use the following notations

$$\mathcal{O}^\delta = \{x \in \mathcal{O}; \text{dist}(x, \partial\mathcal{O}) > \delta\}$$

and

$$\chi(\mathcal{O}) = \sup\{\delta > 0; \mathcal{O}^\delta \neq \emptyset\}.$$

Define

$$(3.3) \quad \mathcal{S}(\mathcal{O}, x_0, M, \eta, \delta) = \{u \in H^1(\mathcal{O}); L_\sigma u = 0 \text{ in } \mathcal{O}, \\ \|\nabla u\|_{L^2(\mathcal{O})} \leq M, \|\nabla u\|_{L^2(B(x_0, \delta))} \geq \eta\},$$

with  $\delta \in (0, \chi(\mathcal{O})/3)$ ,  $x_0 \in \mathcal{O}^{3\delta}$ ,  $\eta > 0$  and  $M \geq 1$  satisfying  $\eta < M$ .

**Lemma 3.2.** *There exist two constants  $c = c(n, \varkappa) \geq 1$  and  $0 < \gamma = \gamma(n, \varkappa) < 1$  so that, for any  $u \in \mathcal{S}(\mathcal{O}, x_0, M, \eta, \delta)$  and  $x \in \mathcal{O}^{3\delta}$ , we have*

$$(3.4) \quad e^{-[\ln(cM/\eta)/\gamma]e^{[2n] \ln \gamma} |x - x_0|/\delta}} \leq \|\nabla u\|_{L^2(B(x, \delta))},$$

with  $\mathbf{c} = \mathbf{c}_\mathcal{O}$  is as in Lemma ??.

*Proof.* Pick  $u \in \mathcal{S}(\mathcal{O}, x_0, M, \eta, \delta)$ . Let  $x \in \mathcal{O}^{3\delta}$  and  $\psi : [0, 1] \rightarrow \mathcal{O}$  be a Lipschitz path joining  $x = \psi(0)$  to  $x_0 = \psi(1)$ , so that  $\ell(\psi) \leq 2d_g(x_0, x)$ . Here and henceforth, for simplicity convenience, we use  $d_g(x_0, x)$  instead of  $d_g^\mathcal{O}(x_0, x)$ .

Let  $t_0 = 0$  and  $t_{k+1} = \inf\{t \in [t_k, 1]; \psi(t) \notin B(\psi(t_k), \delta)\}$ ,  $k \geq 0$ . We claim that there exists an integer  $N \geq 1$  verifying  $\psi(1) \in B(\psi(t_N), \delta)$ . If not, we would have  $\psi(1) \notin B(\psi(t_k), \delta)$  for any  $k \geq 0$ . As the sequence  $(t_k)$  is non decreasing and bounded from above by 1, it converges to  $\hat{t} \leq 1$ . In

particular, there exists an integer  $k_0 \geq 1$  so that  $\psi(t_k) \in B(\psi(\hat{t}), \delta/2)$ ,  $k \geq k_0$ . But this contradicts the fact that  $|\psi(t_{k+1}) - \psi(t_k)| \geq \delta$ ,  $k \geq 0$ .

Let us check that  $N \leq N_0$ , where  $N_0 = N_0(n, |x - x_0|, \mathfrak{c}, \delta)$ . Pick  $1 \leq j \leq n$  so that

$$\max_{1 \leq i \leq n} |\psi_i(t_{k+1}) - \psi_i(t_k)| = |\psi_j(t_{k+1}) - \psi_j(t_k)|,$$

where  $\psi_i$  is the  $i$ th component of  $\psi$ . Then

$$\delta \leq n |\psi_j(t_{k+1}) - \psi_j(t_k)| = n \left| \int_{t_k}^{t_{k+1}} \dot{\psi}_j(t) dt \right| \leq n \int_{t_k}^{t_{k+1}} |\dot{\psi}(t)| dt.$$

Consequently, where  $t_{N+1} = 1$ ,

$$(N+1)\delta \leq n \sum_{k=0}^N \int_{t_k}^{t_{k+1}} |\dot{\psi}(t)| dt = n\ell(\psi) \leq 2nd_g(x_0, x) \leq 2n\mathfrak{c}|x - x_0|.$$

Therefore

$$N \leq N_0 = \left\lfloor \frac{2n\mathfrak{c}|x - x_0|}{\delta} \right\rfloor.$$

Let  $y_0 = x$  and  $y_k = \psi(t_k)$ ,  $1 \leq k \leq N$ . If  $|z - y_{k+1}| < \delta$ , then  $|z - y_k| \leq |z - y_{k+1}| + |y_{k+1} - y_k| < 2\delta$ . In other words  $B(y_{k+1}, \delta) \subset B(y_k, 2\delta)$ . We get from Theorem ??

$$(3.5) \quad \|\nabla u\|_{L^2(B(y_j, 2\delta))} \leq C \|\nabla u\|_{L^2(B(y_j, 3\delta))}^{1-\gamma} \|\nabla u\|_{L^2(B(y_j, \delta))}^{\gamma}, \quad 0 \leq j \leq N,$$

for some constants  $C = C(n, \varkappa) > 0$  and  $0 < \gamma = \gamma(n, \varkappa) < 1$ .

Set  $I_j = \|\nabla u\|_{L^2(B(y_j, \delta))}$ ,  $0 \leq j \leq N$  and  $I_{N+1} = \|\nabla u\|_{L^2(B(x_0, \delta))}$ . Since  $B(y_{j+1}, \delta) \subset B(y_j, 2\delta)$ ,  $1 \leq j \leq N-1$ , estimate (??) implies

$$(3.6) \quad I_{j+1} \leq CM^{1-\gamma} I_j^{\gamma}, \quad 0 \leq j \leq N.$$

Let  $C_1 = C^{1+\gamma+\dots+\gamma^{N+1}}$  and  $\beta = \gamma^{N+1}$ . Then, by a simple induction argument, estimate (??) yields

$$(3.7) \quad I_{N+1} \leq C_1 M^{1-\beta} I_0^{\beta}.$$

Without loss of generality, we assume in the sequel that  $C \geq 1$  in (??). Using that  $N \leq N_0$ , we have

$$\begin{aligned} \beta &\geq \beta_0 = \gamma^{N_0+1}, \\ C_1 &\leq C^{\frac{1}{1-\gamma}}, \\ \left(\frac{I_0}{M}\right)^{\beta} &\leq \left(\frac{I_0}{M}\right)^{\beta_0}. \end{aligned}$$

These estimates in (??) give

$$\frac{I_{N+1}}{M} \leq C^{\frac{1}{1-\gamma}} \left(\frac{I_0}{M}\right)^{\gamma^{N_0+1}},$$

from which we deduce that

$$\|\nabla u\|_{L^2(B(x_0, \delta))} \leq C^{\frac{1}{1-\gamma}} M^{1-\gamma^{N_0+1}} \|\nabla u\|_{L^2(B(x, \delta))}^{\gamma^{N_0+1}}.$$

But  $M \geq 1$ . Whence

$$\eta \leq \|\nabla u\|_{L^2(B(x_0, \delta))} \leq C^{\frac{1}{1-\gamma}} M \|\nabla u\|_{L^2(B(x, \delta))}^{\gamma^{N_0+1}}.$$

The expected inequality follows readily from this last estimate.  $\square$

**3.2. An estimate for the frequency function.** Some tools in the present section are borrowed from [?, ?, ?]. Let  $u \in H^1(\mathcal{O})$  and  $\sigma \in C^{0,1}(\overline{\mathcal{O}})$  satisfying the bounds (??). We recall that the usual frequency function, relative to the operator  $L_\sigma$ , associated to  $u$  is defined by

$$N(u)(x_0, r) = \frac{rD(u)(x_0, r)}{H(u)(x_0, r)},$$

provided that  $B(x_0, r) \Subset \mathcal{O}$ , with

$$\begin{aligned} D(u)(x_0, r) &= \int_{B(x_0, r)} \sigma(x) |\nabla u(x)|^2 dx, \\ H(u)(x_0, r) &= \int_{\partial B(x_0, r)} \sigma(x) u^2(x) dS(x). \end{aligned}$$

Define also

$$K(u)(x_0, r) = \int_{B(x_0, r)} \sigma(x) u^2(x) dx.$$

Prior to studying the properties of the frequency function, we prove some preliminary results. Fix  $u \in H^2(\mathcal{O})$  so that  $L_\sigma u = 0$  in  $\mathcal{O}$  and, for simplicity convenience, we drop in the sequel the dependence on  $u$  of  $N$ ,  $D$ ,  $H$  and  $K$ .

**Lemma 3.3.** *For  $x_0 \in \mathcal{O}^\delta$  and  $0 < r < \delta$ , we have*

$$(3.8) \quad \partial_r H(x_0, r) = \frac{n-1}{r} H(x_0, r) + \tilde{H}(x_0, r) + 2D(x_0, r),$$

$$(3.9) \quad \partial_r D(x_0, r) = \frac{n-2}{r} D(x_0, r) + \frac{1}{r} \tilde{D}(x_0, r) + 2\hat{H}(x_0, r).$$

Here

$$\begin{aligned} \tilde{H}(x_0, r) &= \int_{\partial B(x_0, r)} u^2 \nabla \sigma(x) \cdot \nu(x) dS(x), \\ \hat{H}(x_0, r) &= \int_{\partial B(x_0, r)} \sigma(x) (\partial_\nu u(x))^2 dS(x), \\ \tilde{D}(x_0, r) &= \int_{B(x_0, r)} |\nabla u(x)|^2 \nabla \sigma(x) \cdot (x - x_0) dx. \end{aligned}$$

*Proof.* Pick  $x_0 \in \mathcal{O}^\delta$  and  $0 < r < \delta$ . A simple change of variable yields

$$H(x_0, r) = \int_{\partial B(0,1)} \sigma(x_0 + ry) u^2(x_0 + ry) r^{n-1} dS(y).$$

Hence

$$\begin{aligned}
\partial_r H(x_0, r) &= \frac{n-1}{r} H(x_0, r) + \int_{\partial B(0,1)} \nabla(\sigma u^2)(x_0 + ry) \cdot yr^{n-1} dS(y) \\
&= \frac{n-1}{r} H(x_0, r) + \int_{\partial B(0,1)} u^2(x_0 + ry) \nabla \sigma(x_0 + ry) \cdot yr^{n-1} dS(y) \\
&\quad + \int_{\partial B(0,1)} \sigma(x_0 + ry) \nabla(u^2)(x_0 + ry) \cdot yr^{n-1} dS(y) \\
&= \frac{n-1}{r} H(x_0, r) + \int_{\partial B(x_0, r)} u^2(x) \nabla \sigma(x) \cdot \nu(x) dS(x) \\
&\quad + \int_{\partial B(x_0, r)} \sigma(x) \nabla(u^2)(x) \cdot \nu(x) dS(x) \\
&= \frac{n-1}{r} H(x_0, r) + \tilde{H}(x_0, r) + \int_{\partial B(x_0, r)} \sigma(x) \nabla(u^2)(x) \cdot \nu(x) dS(x).
\end{aligned}$$

Identity (??) will follow if we prove

$$(3.10) \quad 2D(x_0, r) = \int_{\partial B(x_0, r)} \sigma(x) \nabla(u^2)(x) \cdot \nu(x) dS(x).$$

To this end, we observe that  $\operatorname{div}(\sigma \nabla u) = 0$  implies

$$\operatorname{div}(\sigma \nabla(u^2)) = 2u \operatorname{div}(\sigma \nabla u) + 2\sigma |\nabla u|^2 = 2\sigma |\nabla u|^2.$$

We then get by applying the divergence theorem

$$\begin{aligned}
(3.11) \quad 2D(x_0, r) &= \int_{B(x_0, r)} \operatorname{div}(\sigma(x) \nabla(u^2)(x)) dx \\
&= \int_{\partial B(x_0, r)} \sigma(x) \nabla(u^2)(x) \cdot \nu(x) dS(x).
\end{aligned}$$

This proves (??).

By a change of variable we have

$$D(x_0, r) = \int_0^r \int_{\partial B(0,1)} \sigma(x_0 + ty) |\nabla u(x_0 + ty)|^2 t^{n-1} dS(y) dt.$$

Hence

$$\begin{aligned}
\partial_r D(x_0, r) &= \int_{\partial B(0,1)} \sigma(x_0 + ry) |\nabla u(x_0 + ry)|^2 r^{n-1} dS(y) \\
&= \int_{\partial B(x_0, r)} \sigma(x) |\nabla u(x)|^2 dS(x) \\
&= \frac{1}{r} \int_{\partial B(x_0, r)} \sigma(x) |\nabla u(x)|^2 (x - x_0) \cdot \nu(x) dS(x).
\end{aligned}$$

An application of the divergence theorem then gives

$$\partial_r D(x_0, r) = \frac{1}{r} \int_{B(x_0, r)} \operatorname{div}(\sigma(x) |\nabla u(x)|^2 (x - x_0)) dx.$$



Therefore

$$\begin{aligned}\partial_r D(x_0, r) &= \frac{1}{r} \int_{B(x_0, r)} |\nabla u(x)|^2 \operatorname{div}(\sigma(x)(x - x_0)) dx \\ &\quad + \frac{1}{r} \int_{B(x_0, r)} \sigma(x)(x - x_0) \cdot \nabla(|\nabla u(x)|^2) dx\end{aligned}$$

implying

$$\begin{aligned}(3.12) \quad \partial_r D(x_0, r) &= \frac{n}{r} D(x_0, r) + \frac{1}{r} \tilde{D}(x_0, r) \\ &\quad + \frac{1}{r} \int_{B(x_0, r)} \sigma(x)(x - x_0) \cdot \nabla(|\nabla u(x)|^2) dx.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_{B(x_0, r)} \sigma(x)(x_j - x_{0,j}) \partial_j (\partial_i u(x))^2 dx \\ &= 2 \int_{B(x_0, r)} \sigma(x)(x_j - x_{0,j}) \partial_{ij}^2 u \partial_i u(x) dx \\ &= -2 \int_{B(x_0, r)} \partial_i [\partial_i u(x) \sigma(x)(x_j - x_{0,j})] \partial_j u(x) dx \\ &\quad + 2 \int_{\partial B(x_0, r)} \sigma(x) \partial_i u(x) (x_j - x_{0,j}) \partial_j u(x) \nu_i(x) dS(x) \\ &= -2 \int_{B(x_0, r)} \partial_{ii}^2 u(x) \sigma(x)(x_j - x_{0,j}) \partial_j u(x) dx \\ &\quad - 2 \int_{B(x_0, r)} \partial_i u(x) \partial_j u(x) \partial_i [\sigma(x)(x_j - x_{0,j})] dx \\ &\quad + 2 \int_{\partial B(x_0, r)} \sigma(x) \partial_i u(x) (x_j - x_{0,j}) \partial_j u(x) \nu_i(x) dS(x).\end{aligned}$$

Thus, taking into account that  $\sigma \Delta u = -\nabla \sigma \cdot \nabla u$ ,

$$\begin{aligned}\int_{B(x_0, r)} \sigma(x)(x - x_0) \cdot \nabla(|\nabla u(x)|^2) dx &= -2 \int_{B(x_0, r)} \sigma(x) |\nabla u(x)|^2 dx \\ &\quad + 2r \int_{\partial B(x_0, r)} \sigma(x) (\partial_\nu u(x))^2 dS(x).\end{aligned}$$

This identity in (??) yields

$$\partial_r D(x_0, r) = \frac{n-2}{r} D(x_0, r) + \frac{1}{r} \tilde{D}(x_0, r) + 2\hat{H}(x_0, r).$$

That is we proved (??). □

**Lemma 3.4.** *We have*

$$K(x_0, r) \leq r e^{r^{\mathfrak{K}^2}} H(x_0, r), \quad x_0 \in \mathcal{O}^\delta, \quad 0 < r < \delta.$$

*Proof.* Taking into account that  $H(x_0, r) \geq 0$  and  $D(x_0, r) \geq 0$ , we obtain from identity (??)

$$\begin{aligned} \partial_r H(x_0, r) &\geq \int_{\partial B(x_0, r)} \partial_\nu \sigma(x) u^2(x) dS(x) \\ &\geq \int_{\partial B(x_0, r)} \frac{\partial_\nu \sigma(x)}{\sigma(x)} \sigma(x) u^2(x) dS(x) \geq -\kappa^2 H(x_0, r). \end{aligned}$$

Consequently  $r \rightarrow e^{r\kappa^2} H(x_0, r)$  is non decreasing and then

$$\begin{aligned} \int_0^r H(x_0, t) dt &\leq \int_0^r e^{t\kappa^2} H(x_0, t) dt \\ &\leq \int_0^r e^{r\kappa^2} H(x_0, r) dt \leq r e^{r\kappa^2} H(x_0, r). \end{aligned}$$

As

$$K(x_0, r) = \int_0^r H(x_0, t) dt,$$

we end up getting

$$K(x_0, r) \leq r e^{r\kappa^2} H(x_0, r).$$

This completes the proof.  $\square$

Now straightforward computations yield, for  $x_0 \in \mathcal{O}^\delta$  and  $0 < r < \delta$ ,

$$(3.13) \quad \frac{\partial_r N(x_0, r)}{N(x_0, r)} = \frac{1}{r} + \frac{\partial_r D(x_0, r)}{D(x_0, r)} - \frac{\partial_r H(x_0, r)}{H(x_0, r)}.$$

**Lemma 3.5.** *For  $x_0 \in \mathcal{O}^\delta$  and  $0 < r < \delta$ , we have*

$$N(x_0, r) \leq e^{2\kappa^2 \delta} N(x_0, \delta).$$

*Proof.* We have from formulas (??) and (??) and identity (??)

$$\begin{aligned} (3.14) \quad \frac{\partial_r N(x_0, r)}{N(x_0, r)} &= \frac{\tilde{D}(x_0, r)}{r D(x_0, r)} - \frac{\tilde{H}(x_0, r)}{H(x_0, r)} + 2 \frac{\hat{H}(x_0, r)}{D(x_0, r)} - 2 \frac{D(x_0, r)}{H(x_0, r)} \\ &= \frac{\tilde{D}(x_0, r)}{r D(x_0, r)} - \frac{\tilde{H}(x_0, r)}{H(x_0, r)} + 2 \frac{\hat{H}(x_0, r) H(x_0, r) - D(x_0, r)^2}{D(x_0, r) H(x_0, r)}. \end{aligned}$$

But from (??) we have

$$D(x_0, r) = \int_{\partial B(x_0, r)} \sigma(x) u(x) \partial_\nu u(x) dS(x).$$

Then we find by applying Cauchy-Schwarz's inequality

$$D(x_0, r)^2 \leq \left( \int_{\partial B(x_0, r)} \sigma(x) u^2(x) dS(x) \right) \left( \int_{\partial B(x_0, r)} \sigma(x) (\partial_\nu u)^2(x) dS(x) \right).$$

That is

$$(3.15) \quad D^2(x_0, r) \leq H(x_0, r) \hat{H}(x_0, r).$$

This and (??) lead

$$(3.16) \quad \frac{\partial_r N(x_0, r)}{N(x_0, r)} \geq \frac{\tilde{D}(x_0, r)}{r D(x_0, r)} - \frac{\tilde{H}(x_0, r)}{H(x_0, r)}.$$

On the other hand

$$(3.17) \quad |\tilde{H}(x_0, r)| \leq \varkappa \|\nabla a\|_\infty H(x_0, r) \leq \varkappa^2 H(x_0, r),$$

and similarly

$$(3.18) \quad |\tilde{D}(x_0, r)| \leq \varkappa^2 r D(x_0, r).$$

In light of (??), (??) and (??), we derive

$$\frac{\partial_r N(x_0, r)}{N(x_0, r)} \geq -2\varkappa^2,$$

that is to say

$$\partial_r (e^{2\varkappa^2 r} N(x_0, r)) \geq 0.$$

Consequently

$$N(x_0, r) \leq e^{2\varkappa^2(\delta-r)} N(x_0, \delta) \leq e^{2\varkappa^2 \delta} N(x_0, \delta),$$

as expected.  $\square$

### 3.3. Polynomial lower bound.

**Lemma 3.6.** *There exist a universal constant  $\varpi$  and two constants  $c = c(n, \varkappa) > 0$  and  $0 < \gamma = \gamma(n, \varkappa) < 1$  so that if*

$$\mathcal{C}_0(h) = M\varpi\varkappa^4(1 + \mathbf{d})\delta^{-1}e^{3\varkappa^2\delta + [2\ln(cM/\eta)/\gamma]e^{[6n|\ln \gamma|]\varkappa h}}, \quad h > 0,$$

then

$$\|N(u)(x, \cdot)\|_{L^\infty(0, \delta)} \leq \mathcal{C}_0(|x - x_0|/\delta),$$

for any  $u \in \mathcal{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$ , where  $\mathbf{c} = \mathbf{c}_{\mathcal{O}}$  is as in Lemma ??.

*Proof.* Pick  $x \in \mathcal{O}^\delta$ . Then from Lemma ??

$$\|\nabla u\|_{L^2(B(x, \delta/3))} \geq e^{-[\ln(cM/\eta)/\gamma]e^{[6n|\ln \gamma|]\mathbf{c}|x-x_0|/\delta}},$$

for some constant  $c = c(n, \varkappa)$  and  $0 < \gamma = \gamma(n, \varkappa) < 1$ .

On the other hand, we establish in a quite classical manner the following Caccioppoli's inequality

$$\|\nabla u\|_{L^2(B(x, \delta/3))}^2 \leq \frac{\varpi\varkappa^2(1 + \mathbf{d})}{\delta^2} \|u\|_{L^2(B(x, \delta))}^2,$$

where  $\varpi$  is a universal constant. Therefore

$$(3.19) \quad \|u\|_{L^2(B(x, \delta))}^2 \geq \tilde{\mathcal{C}}_0(|x - x_0|/\delta),$$

where

$$(3.20) \quad \tilde{\mathcal{C}}_0(h) = \frac{\delta^2}{\varpi\varkappa^2(1 + \mathbf{d})} e^{-[2\ln(cM/\eta)/\gamma]e^{[6n|\ln \gamma|]\mathbf{c}h}}, \quad h > 0.$$

Since  $K(u)(x, \delta) \geq \varkappa^{-1} \|u\|_{L^2(B(x, \delta))}^2$ , we find

$$(3.21) \quad K(u)(x, \delta) \geq \frac{\delta^2}{\varpi\varkappa^3(1 + \mathbf{d})} e^{-[2\ln(cM/\eta)/\gamma]e^{[6n|\ln \gamma|]\mathbf{c}|x-x_0|/\delta}}.$$

In light of Lemma ??, we derive from (??)

$$(3.22) \quad H(u)(x, \delta) \geq \frac{\delta e^{-\varkappa^2 \delta}}{\varpi\varkappa^3(1 + \mathbf{d})} e^{-[2\ln(cM/\eta)/\gamma]e^{[6n|\ln \gamma|]\mathbf{c}|x-x_0|/\delta}}.$$

In light of Lemma ??, we get

$$N(x, r) \leq \varkappa e^{2\varkappa^2\delta} \frac{\|\nabla u\|_{L^2(\mathcal{O})}}{H(u)(x, \delta)}, \quad 0 < r < \delta,$$

This inequality and (??) give, where  $c = c(n, \varkappa)$  is a constant,

$$N(x, r) \leq M\varpi\varkappa^4(1 + \mathbf{d})\delta^{-1}e^{3\varkappa^2\delta + [2\ln(cM/\eta)/\gamma]e^{[6n|\ln \gamma|]\varepsilon|x-x_0|/\delta}}, \quad 0 < r < \delta,$$

which is the expected inequality.  $\square$

**Proposition 3.1.** *Let  $\mathcal{C}_0$  be as in Lemma ??,  $\tilde{\mathcal{C}}_0$  as in (??) and set*

$$(3.23) \quad \mathcal{C}_1(h) = 2\mathcal{C}_0(h) + n, \quad h > 0,$$

$$(3.24) \quad \tilde{\mathcal{C}}_2(h) = \varkappa^{-2}e^{-\varkappa^2\delta}\tilde{\mathcal{C}}_0(h), \quad h > 0.$$

If  $u \in \mathcal{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$  then

$$\tilde{\mathcal{C}}_2(|x - x_0|/\delta) \left(\frac{r}{\delta}\right)^{\mathcal{C}_1(|x-x_0|/\delta)} \leq \|u\|_{L^2(B(x, r))}^2, \quad x \in \mathcal{O}^\delta, \quad 0 < r < \delta.$$

*Proof.* Observing that, where  $H = H(u)$ ,

$$\partial_r \left( \ln \frac{H(x, r)}{r^{n-1}} \right) = \frac{\partial_r H(x, r)}{H(x, r)} - \frac{n-1}{r},$$

we get from Lemma ??, (??) and the fact that  $|\tilde{H}(x, r)| \leq \varkappa^2 H(x, r)$ ,

$$\partial_r \left( \ln \frac{H(x, r)}{r^{n-1}} \right) \leq \varkappa^2 + \frac{2N(x, r)}{r} \leq \varkappa^2 + \frac{2\mathcal{C}_0(|x - x_0|/\delta)}{r}, \quad 0 < r < \delta,$$

Thus

$$\int_{sr}^{s\delta} \partial_t \left( \ln \frac{H(x, t)}{t^{n-1}} \right) dt = \ln \frac{H(x, s\delta)r^{n-1}}{H(x, sr)\delta^{n-1}} \leq \varkappa^2(\delta - r)s + 2\mathcal{C}_0(|x - x_0|/\delta) \ln \frac{\delta}{r},$$

for  $0 < s < 1$  and  $0 < r < \delta$ . Hence

$$H(x, s\delta) \leq e^{\varkappa^2\delta} \left(\frac{\delta}{r}\right)^{\mathcal{C}_1(|x-x_0|/\delta)-1} H(x, sr),$$

and then

$$\begin{aligned} \|u\|_{L^2(B(x, \delta))}^2 &\leq \varkappa\delta \int_0^1 H(x, s\delta) ds \\ &\leq \varkappa\delta e^{\varkappa^2\delta} \left(\frac{\delta}{r}\right)^{\mathcal{C}_1(|x-x_0|/\delta)-1} \int_0^1 H(x, rs) ds \\ &\leq \varkappa^2 e^{\varkappa^2\delta} \left(\frac{\delta}{r}\right)^{\mathcal{C}_1(|x-x_0|/\delta)} \|u\|_{L^2(B(x, r))}^2. \end{aligned}$$

Combined with (??) this estimate yields in a straightforward manner

$$\varkappa^{-2}e^{-\varkappa^2\delta}\tilde{\mathcal{C}}_0(|x - x_0|/\delta) \left(\frac{r}{\delta}\right)^{\mathcal{C}_1(|x-x_0|/\delta)} \leq \|u\|_{L^2(B(x, r))}^2.$$

This is the expected inequality.  $\square$

For a bounded domain  $D$ , we denote the first non zero eigenvalue of the Laplace-Neumann operator on  $D$  by  $\mu_2(D)$ . Since  $\mu_2(B(x_0, r)) = \mu_2(B(0, 1))/r^2$ , we get by applying Poincaré-Wirtinger's inequality

$$(3.25) \quad \begin{aligned} \|w - \{w\}\|_{L^2(B(x, r))}^2 &\leq \frac{1}{\mu_2(B(x, r))} \|\nabla w\|_{L^2(B(x, r))}^2 \\ &\leq \frac{r^2}{\mu_2(B(0, 1))} \|\nabla w\|_{L^2(B(x, r))}^2, \end{aligned}$$

for any  $w \in H^1(B(x, r))$ , where  $\{w\} = \frac{1}{|B(x, r)|} \int_{B(x, r)} w(x) dx$ .

Noting that  $\mathcal{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$  is invariant under the transformation  $u \rightarrow u - \{u\}$ , we can state the following consequence of Proposition ??

**Corollary 3.1.** *With the notations of Proposition ??, if  $u \in \mathcal{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$  then*

$$\mathcal{C}_2(|x - x_0|/\delta) \left(\frac{r}{\delta}\right)^{\mathcal{C}_1(|x - x_0|/\delta)} \leq \|\nabla u\|_{L^2(B(x, r))}^2, \quad x \in \mathcal{O}^\delta, \quad 0 < r < \delta,$$

with

$$(3.26) \quad \mathcal{C}_2(h) = \mu_2(B(0, 1))\delta^{-2}\tilde{\mathcal{C}}_2(h), \quad h > 0,$$

with  $\tilde{\mathcal{C}}_2$  as in Proposition ??.

It is important to remark that the argument we used to obtain Corollary ?? from Proposition ?? is no longer valid if we substitute  $L_\sigma$  by  $L_\sigma$  plus a multiplication operator by a function  $\sigma_0$ .

The following consequence of the preceding corollary will be useful in the proof of Theorem ??.

**Lemma 3.7.** *Let  $\omega \in \mathcal{O}$  and set  $\delta = \text{dist}(\omega, \partial\mathcal{O})$ . Let  $u \in \mathcal{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$  and  $f \in C^{0, \alpha}(\overline{\mathcal{O}})$ . Then we have*

$$(3.27) \quad \|f\|_{L^\infty(\omega)} \leq \hat{\mathcal{C}}_3 \|f\|_{C^{0, \alpha}(\overline{\mathcal{O}})}^{1-\hat{\mu}} \|f|\nabla u|^2\|_{L^1(\mathcal{O})}^{\hat{\mu}},$$

with

$$\begin{aligned} \hat{\mu} &= \frac{\alpha}{\max_{x \in \overline{\mathcal{O}}} \mathcal{C}_1(|x - x_0|/\delta) + \alpha}, \\ \hat{\mathcal{C}}_3 &= \max \left( 2\delta^\alpha (\max(1, (\hat{\mathcal{C}}_2\delta^\alpha)^{-1})) , \max(1, M^2) (\hat{\mathcal{C}}_2\delta^\alpha)^{-1} \right), \end{aligned}$$

where  $\hat{\mathcal{C}}_2 = \max_{x \in \overline{\mathcal{O}}} \mathcal{C}_2(|x - x_0|/\delta)$  with  $\mathcal{C}_2$  is as in Corollary ??.

*Proof.* By homogeneity it is enough to consider those functions  $f \in C^{0, \alpha}(\overline{\mathcal{O}})$  satisfying  $\|f\|_{C^{0, \alpha}(\overline{\mathcal{O}})} = 1$ . Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be respectively as in (??) and (??). Let  $u \in \mathcal{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$  and  $f \in C^{0, \alpha}(\overline{\mathcal{O}})$  satisfying  $\|f\|_{C^{0, \alpha}(\overline{\mathcal{O}})} = 1$ . Pick then  $x \in \overline{\omega}$ . From Corollary ??, we have

$$(3.28) \quad \mathcal{C}_2(|x - x_0|/\delta) \left(\frac{r}{\delta}\right)^{\mathcal{C}_1(|x - x_0|/\delta)} \leq \|\nabla u\|_{L^2(B(x, r))}^2, \quad 0 < r < \delta.$$

On the other hand, it is straightforward to check that

$$|f(x)| \leq |f(y)| + r^\alpha, \quad y \in B(x, r).$$

Whence

$$|f(x)| \int_{B(x,r)} |\nabla u(y)|^2 dy \leq \int_{B(x,r)} |f(y)| |\nabla u(y)|^2 dy + r^\alpha \int_{B(x,r)} |\nabla u(y)|^2 dy.$$

That is we have

$$|f(x)| \|\nabla u\|_{L^2(B(x,r))}^2 \leq \|f|\nabla u|^2\|_{L^1(B(x,r))} + r^\alpha \|\nabla u\|_{L^2(B(x,r))}^2.$$

Since  $u$  is non constant, by the unique continuation property, we have  $\|\nabla u\|_{L^2(B(x,r))}^2 \neq 0$ ,  $0 < r < \delta$ . Therefore

$$|f(x)| \leq \frac{\|f|\nabla u|^2\|_{L^1(B(x,r))}}{\|\nabla u\|_{L^2(B(x,r))}^2} + r^\alpha, \quad 0 < r < \delta.$$

This and (??) entail

$$|f(x)| \leq \mathcal{C}_2(|x - x_0|/\delta)^{-1} \left(\frac{\delta}{r}\right)^{\mathcal{C}_1(|x - x_0|)} \|f|\nabla u|^2\|_{L^1(B(x,r))} + r^\alpha, \quad 0 < r < \delta.$$

Hence

$$|f(x)| \leq \mathcal{C}_2(|x - x_0|/\delta)^{-1} \left(\frac{1}{s}\right)^{\mathcal{C}_1(|x - x_0|)} \|f|\nabla u|^2\|_{L^1(\mathcal{O})} + \delta^\alpha s^\alpha, \quad 0 < s < 1.$$

In consequence

$$\|f\|_{L^\infty(\omega)} \leq \hat{\mathcal{C}}_2 \left(\frac{1}{s}\right)^{\hat{\alpha}} \|f|\nabla u|^2\|_{L^1(\mathcal{O})} + \delta^\alpha s^\alpha, \quad 0 < s < 1,$$

where  $\hat{\alpha} = \max_{x \in \overline{\mathcal{O}}} \mathcal{C}_1(|x - x_0|/\delta)$ . The expected inequality follows by minimizing the right hand side of the last inequality, with respect to  $s$ .  $\square$

#### 4. PROOF THEOREM ??

Pick  $(a, b), (\tilde{a}, \tilde{b}) \in \mathcal{D}(\lambda, \kappa)$  and let  $u_j = G_{a,b}(\cdot, \xi_j)$  and  $\tilde{u}_j = G_{\tilde{a}, \tilde{b}}(\cdot, \xi_j)$ ,  $j = 1, 2$ . By simple computations we can check that  $w = u_2/u_1$  is the solution of the equation

$$\operatorname{div}(\sigma \nabla w) = 0 \quad \text{in } \mathbb{R}^n \setminus \{\xi_1, \xi_2\},$$

with

$$\sigma = au_1^2 = \frac{av_1^2}{b^2}.$$

Similarly,  $\tilde{w} = \tilde{u}_2/\tilde{u}_1$  is the solution of the equation

$$\operatorname{div}(\tilde{\sigma} \nabla \tilde{w}) = 0 \quad \text{in } \mathbb{R}^n \setminus \{\xi_1, \xi_2\},$$

with

$$\tilde{\sigma} = \tilde{a}\tilde{u}_1^2 = \frac{\tilde{a}\tilde{v}_1^2}{\tilde{b}^2}.$$

We know from Lemma ?? that there exist  $x^* \in B(\xi_2, |\xi_1 - \xi_2|/2) \setminus \{\xi_2\}$ ,  $\eta_0 = \eta_0(n, \lambda, \kappa, |\xi_1 - \xi_2|) > 0$  and  $\rho = \rho(n, \lambda, \kappa, |\xi_1 - \xi_2|) > 0$  so that  $\overline{B}(x^*, \rho) \subset B(\xi_2, |\xi_1 - \xi_2|/2) \setminus \{\xi_2\}$  and

$$(4.1) \quad \eta_0 \leq \|\nabla w\|_{L^2(B(x^*, \rho))}.$$

Fix then a bounded domain  $\mathcal{Q}$  of  $\mathbb{R}^n \setminus \{\xi_1, \xi_2\}$  is such a way that  $\Omega \cup B(x^*, \rho) \Subset \mathcal{Q}$ , and set

$$\delta = \operatorname{dist}(\Omega \cup B(x^*, \rho), \partial \mathcal{Q}).$$

In the rest of this proof  $\mathbf{d} = \text{diam}(\mathcal{Q})$ . According to Corollary ??

$$(4.2) \quad \|\nabla w\|_{L^2(\mathcal{Q})} \leq M = Ce^{c(\mathbf{d}+\varrho_+)} \left(1 + \max\left(\varrho_-^{-(2+\alpha)}, 1\right) \varrho_-^{-n+2}\right)^4,$$

with  $C = C(n, \lambda, \kappa, \alpha, \theta)$  and  $c = c(n, \lambda, \kappa, \alpha, \theta)$ ,  $\varrho_- = \min(\text{dist}(\xi_1, \mathcal{Q}), \text{dist}(\xi_2, \mathcal{Q}))$  and  $\varrho_+ = \max(\text{dist}(\xi_1, \mathcal{Q}), \text{dist}(\xi_2, \mathcal{Q}))$ .

Now, since

$$\|\sigma\|_{C^{0,1}(\overline{\mathcal{Q}})} \leq \|a\|_{C^{0,1}(\overline{\mathcal{Q}})} \|u_1\|_{C^{0,1}(\overline{\mathcal{Q}})}^2,$$

we get, similarly to the end of the proof of Corollary ??, from [?, Lemma 6.35, page 135]

$$\|\sigma\|_{C^{0,1}(\overline{\mathcal{Q}})} \leq C \|a\|_{C^{0,1}(\overline{\mathcal{Q}})} \|u_1\|_{C^{2,\alpha}(\overline{\mathcal{Q}})}^2,$$

where  $C = C(n, \lambda, \kappa, \mathbf{d}, \xi_1, \xi_2) > 0$  is a constant. This inequality together with Proposition ?? yield

$$(4.3) \quad \|\sigma\|_{C^{0,1}(\overline{\mathcal{Q}})} \leq C,$$

for some constant  $C = C(n, \lambda, \kappa, \mathbf{d}, \xi_1, \xi_2) > 0$ .

On the other hand, we have from (??)

$$(4.4) \quad C^{-1} \min_{x \in \overline{\mathcal{Q}}} \frac{e^{-2\sqrt{c\kappa}|x-\xi_1|}}{|x-\xi_1|^{n-2}} \leq u_1, \quad \text{in } \overline{\mathcal{Q}},$$

with constants  $c = c(n, \lambda) > 0$  and  $C = C(n, \lambda, \kappa) > 0$ .

We get by combining (??) and (??) that there exists  $\varkappa = \varkappa(n, \lambda, \kappa, \alpha, \Omega, \xi_1, \xi_2) > 1$  so that

$$\varkappa^{-1} \leq \sigma \quad \text{and} \quad \|\sigma\|_{C^{0,1}(\overline{\mathcal{Q}})} \leq \varkappa.$$

Next, if  $\rho \leq \delta/3$  then (??) implies obviously

$$(4.5) \quad \eta_0 \leq \|\nabla w\|_{L^2(B(x_0, \delta/3))},$$

with  $\eta_0$  as in (??). When  $\rho > \delta/3$  we can use the three-ball inequality in Theorem ?? in order to get

$$\tilde{C} \|\nabla w\|_{L^2(B(x^*, \rho))} \leq \|\nabla w\|_{L^2(B(x_0, \delta/3))}^s \|\nabla w\|_{L^2(B(x^*, \rho+\delta/3))}^{1-s},$$

where  $\tilde{C} = \tilde{C}(n, \lambda, \kappa, \Omega, \xi_1, \xi_2)$  and  $0 < s = s(n, \lambda, \kappa, \Omega, \xi_1, \xi_2) < 1$  are constants. Whence

$$(4.6) \quad (\tilde{C}\eta_0)^{1/s} M^{(s-1)/s} \leq \|\nabla w\|_{L^2(B(x_0, \delta/3))}.$$

In light of (??), (??) and (??), we can infer that, for some constant  $\eta = \eta(n, \lambda, \kappa, \Omega, \xi_1, \xi_2) > 0$ ,  $w \in \mathcal{S}(\mathcal{Q}, x^*, M, \eta, \delta/3)$ , where  $M$  is as in (??) and  $\mathcal{S}(\mathcal{Q}, x^*, M, \eta, \delta/3)$  is defined in (??).

**Lemma 4.1.** *We have*

$$(4.7) \quad C \|(\sigma - \tilde{\sigma})|\nabla w|^2\|_{L^1(\Omega)} \leq \|w - \tilde{w}\|_{L^2(\Omega)}^{\theta/(2+\theta)} + \|\sigma - \tilde{\sigma}\|_{L^\infty(\Gamma)},$$

where  $C = C(n, \lambda, \kappa, \Omega, \alpha, \theta, \xi_1, \xi_2) > 0$  is a constant.

*Proof.* Clearly, if  $\zeta = \sigma - \tilde{\sigma}$  and  $u = w - \tilde{w}$ , then

$$\text{div}(\tilde{\sigma} \nabla u) = \text{div}(\zeta \nabla w).$$

Recall that  $\text{sgn}_0$  is the sign function defined on  $\mathbb{R}$  by:  $\text{sgn}_0(t) = -1$  if  $t < 0$ ,  $\text{sgn}_0(0) = 0$  and  $\text{sgn}_0(t) = 1$  if  $t > 0$ . Since

$$\begin{aligned} \text{div}(|\zeta|\nabla w) &= \nabla|\zeta| \cdot \nabla w + |\zeta|\Delta w \\ &= \text{sgn}_0(\zeta)\nabla\zeta \cdot \nabla w + \text{sgn}_0(\zeta)\zeta\Delta w \\ &= \text{sgn}_0(\zeta)\text{div}(\zeta\nabla w) = \text{sgn}_0(\zeta)\text{div}(\tilde{\sigma}\nabla u), \end{aligned}$$

we get by integrating by parts

$$\begin{aligned} (4.8) \quad \int_{\Omega} |\zeta||\nabla w|^2 dx &= - \int_{\Omega} \text{div}(|\zeta|\nabla w)w dx + \int_{\Gamma} |\zeta|w\partial_{\nu}w dS(x) \\ &= - \int_{\Omega} \text{sgn}_0(\zeta)\text{div}(\tilde{\sigma}\nabla u)w dx + \int_{\Gamma} |\zeta|w\partial_{\nu}w dS(x). \end{aligned}$$

Thus

$$\int_{\Omega} |\zeta||\nabla w|^2 dx \leq C (\|u\|_{H^2(\Omega)} + \|\zeta\|_{L^\infty(\Gamma)}).$$

This, the following interpolation inequality

$$\|u\|_{H^2(\Omega)} \leq c_{\Omega} \|u\|_{L^2(\Omega)}^{\theta/(2+\theta)} \|u\|_{H^{2+\theta}(\Omega)}^{2/(2+\theta)}$$

and Corollary ?? give (??). □

We have from (??) in Lemma ??

$$\|\tilde{\sigma} - \sigma\|_{C(\overline{\Omega})} \leq \hat{\mathcal{C}}_3 \|\tilde{\sigma} - \sigma\|_{C^{0,\alpha}(\overline{\Omega})}^{1-\hat{\mu}} \|(\sigma - \tilde{\sigma})|\nabla w|^2\|_{L^1(\Omega)}^{\hat{\mu}},$$

from which we obtain

$$\|\tilde{\sigma} - \sigma\|_{C(\overline{\Omega})} \leq \hat{\mathcal{C}}_3 \max\left(1, \|\tilde{\sigma} - \sigma\|_{C^{0,\alpha}(\overline{\Omega})}\right) \|(\sigma - \tilde{\sigma})|\nabla w|^2\|_{L^1(\Omega)}^{\hat{\mu}}.$$

Combined with Proposition ??, this inequality gives

$$\|\tilde{\sigma} - \sigma\|_{C(\overline{\Omega})} \leq C \|(\sigma - \tilde{\sigma})|\nabla w|^2\|_{L^1(\Omega)}^{\hat{\mu}}.$$

Here and henceforward,  $C = C(n, \lambda, \kappa, \Omega, \alpha, \theta, \xi_1, \xi_2) > 0$  is a generic constant.

Therefore, we obtain in light of Lemma ??

$$\|\tilde{\sigma} - \sigma\|_{C(\overline{\Omega})} \leq C \left( \|w - \tilde{w}\|_{L^2(\Omega)}^{\theta/(2+\theta)} + \|\sigma - \tilde{\sigma}\|_{C(\Gamma)} \right)^{\hat{\mu}}.$$

Since  $\tilde{a} = a$  and  $\tilde{b} = b$  on  $\Gamma$  and regarding the regularity of  $u_i$  and  $\tilde{u}_i$ ,  $i = 1, 2$ , we finally get

$$(4.9) \quad \|\tilde{\sigma} - \sigma\|_{C(\overline{\Omega})} \leq C \left( \|v_1 - \tilde{v}_1\|_{C(\overline{\Omega})} + \|v_2 - \tilde{v}_2\|_{C(\overline{\Omega})} \right)^{\hat{\mu}_0},$$

with

$$\hat{\mu}_0 = \frac{\theta\hat{\mu}}{2+\theta}.$$

The following lemma will be used in sequel.

**Lemma 4.2.** *We have*

$$(4.10) \quad \|u_1^{-1} - \tilde{u}_1^{-1}\|_{C^{2,\alpha}(\overline{\Omega})} \leq C \left( \|v_1 - \tilde{v}_1\|_{C(\overline{\Omega})} + \|v_2 - \tilde{v}_2\|_{C(\overline{\Omega})} \right)^{\hat{\mu}_1},$$

where  $0 < \hat{\mu}_1 = \hat{\mu}_1(n, \Omega, \lambda, \kappa, \alpha, \theta, \xi_1, \xi_2) < 1$  and  $C = C(n, \Omega, \lambda, \kappa, \alpha, \theta, \xi_1, \xi_2) > 0$  are constants.



*Proof.* In this proof  $C = C(n, \Omega, \lambda, \kappa, \alpha, \theta, \xi_1, \xi_2) > 0$  is a generic constant. It is not hard to check that

$$\begin{aligned} -\operatorname{div}(\sigma \nabla u_1^{-1}) &= v_1 \quad \text{in } \Omega, \\ -\operatorname{div}(\tilde{\sigma} \nabla \tilde{u}_1^{-1}) &= \tilde{v}_1 \quad \text{in } \Omega. \end{aligned}$$

Hence

$$-\operatorname{div}(\sigma \nabla (u_1^{-1} - \tilde{u}_1^{-1})) = (v_1 - \tilde{v}_1) + \operatorname{div}((\sigma - \tilde{\sigma}) \nabla \tilde{u}_1^{-1}) \quad \text{in } \Omega.$$

By the usual Hölder a priori estimate (see [?, Theorem 6.6, page 98])

$$\begin{aligned} C \|u_1^{-1} - \tilde{u}_1^{-1}\|_{C^{2,\alpha}(\bar{\Omega})} &\leq \|v_1 - \tilde{v}_1\|_{C^{0,\alpha}(\bar{\Omega})} \\ &\quad + \|\operatorname{div}((\sigma - \tilde{\sigma}) \nabla \tilde{u}_1^{-1})\|_{C^{0,\alpha}(\bar{\Omega})} + \|u_1^{-1} - \tilde{u}_1^{-1}\|_{C^{0,\alpha}(\Gamma)}. \end{aligned}$$

Consequently

$$(4.11) \quad \|u_1^{-1} - \tilde{u}_1^{-1}\|_{C^{2,\alpha}(\bar{\Omega})} \leq C \left( \|v_1 - \tilde{v}_1\|_{C^{0,\alpha}(\bar{\Omega})} + \|\sigma - \tilde{\sigma}\|_{C^{1,\alpha}(\bar{\Omega})} \right),$$

where we used

$$\|u_1^{-1} - \tilde{u}_1^{-1}\|_{C^{0,\alpha}(\Gamma)} = \|b(v_1^{-1} - \tilde{v}_1^{-1})\|_{C^{0,\alpha}(\Gamma)}.$$

On the other hand, since

$$\|\sigma - \tilde{\sigma}\|_{C^{1,1}(\bar{\Omega})} \leq C, \quad \|v_1 - \tilde{v}_1\|_{C^{1,\alpha}(\bar{\Omega})} \leq C$$

and  $\Omega$  is  $C^{1,1}$ , we get again from the interpolation inequality in [?, Lemma 6.35, page 135]

$$(4.12) \quad \|\sigma - \tilde{\sigma}\|_{C^{1,\alpha}(\bar{\Omega})} \leq C \|\sigma - \tilde{\sigma}\|_{C(\bar{\Omega})}^\tau, \quad \|v_1 - \tilde{v}_1\|_{C^{0,\alpha}(\bar{\Omega})} \leq C \|v_1 - \tilde{v}_1\|_{C(\bar{\Omega})}^\tau,$$

where  $0 < \tau = \tau(\Omega, \alpha) < 1$  is a constant. Inequality (??) in (??) yields

$$(4.13) \quad \|u_1^{-1} - \tilde{u}_1^{-1}\|_{C^{2,\alpha}(\bar{\Omega})} \leq C \left( \|v_1 - \tilde{v}_1\|_{C(\bar{\Omega})}^\tau + \|\sigma - \tilde{\sigma}\|_{C(\bar{\Omega})}^\tau \right).$$

On the other hand, we have from (??)

$$(4.14) \quad \|\tilde{\sigma} - \sigma\|_{C(\bar{\Omega})} \leq C \left( \|v_1 - \tilde{v}_1\|_{C(\bar{\Omega})} + \|v_2 - \tilde{v}_2\|_{C(\bar{\Omega})} \right)^{\hat{\mu}_0}.$$

Whence, we get in light of inequalities (??) and (??), where  $\hat{\mu}_1 = \tau \hat{\mu}_0$ ,

$$\|u_1^{-1} - \tilde{u}_1^{-1}\|_{C^{2,\alpha}(\bar{\Omega})} \leq C \left( \|v_1 - \tilde{v}_1\|_{C(\bar{\Omega})} + \|v_2 - \tilde{v}_2\|_{C(\bar{\Omega})} \right)^{\hat{\mu}_1}.$$

This is the expected inequality. □

Also, since

$$\|\sigma - \tilde{\sigma}\|_{C^{1,1}(\bar{\Omega})} \leq C, \quad \|v_1 - \tilde{v}_1\|_{C^{2,\alpha}(\bar{\Omega})} \leq C,$$

we can proceed as in the preceding proof to get

$$(4.15) \quad \|\sigma - \tilde{\sigma}\|_{C^{1,\alpha}(\bar{\Omega})} \leq C \|\sigma - \tilde{\sigma}\|_{C(\bar{\Omega})}^\tau, \quad \|v_1 - \tilde{v}_1\|_{C^{1,\alpha}(\bar{\Omega})} \leq C \|v_1 - \tilde{v}_1\|_{C(\bar{\Omega})}^\tau,$$

the constant  $0 < \tau = \tau(\Omega, \alpha) < 1$ . But

$$\begin{aligned} a - \tilde{a} &= \sigma u_1^{-2} - \tilde{\sigma} \tilde{u}_1^{-2} = (\sigma - \tilde{\sigma}) u_1^{-2} + \tilde{\sigma} (u_1^{-2} - \tilde{u}_1^{-2}) \\ &= (\sigma - \tilde{\sigma}) u_1^{-2} + \tilde{\sigma} (u_1^{-1} + \tilde{u}_1^{-1}) (u_1^{-1} - \tilde{u}_1^{-1}). \end{aligned}$$

Hence

$$(4.16) \quad \|a - \tilde{a}\|_{C^{1,\alpha}(\bar{\Omega})} \leq C \left( \|u_1^{-1} - \tilde{u}_1^{-1}\|_{C^{1,\alpha}(\bar{\Omega})} + \|\sigma - \tilde{\sigma}\|_{C^{1,\alpha}(\bar{\Omega})} \right).$$

This inequality together with (??), (??) and (??) imply

$$(4.17) \quad \|a - \tilde{a}\|_{C^{1,\alpha}(\bar{\Omega})} \leq C \left( \|v_1 - \tilde{v}_1\|_{C(\bar{\Omega})} + \|v_2 - \tilde{v}_2\|_{C(\bar{\Omega})} \right)^{\hat{\mu}_1}.$$

We proceed similarly for  $b - \tilde{b}$ . Since

$$b - \tilde{b} = v_1 u_1^{-1} - \tilde{v}_1 \tilde{u}_1^{-1} = (v_1 - \tilde{v}_1) u_1^{-1} + \tilde{v}_1 (u_1^{-1} - \tilde{u}_1^{-1}),$$

we have

$$(4.18) \quad \|b - \tilde{b}\|_{C^{0,\alpha}(\bar{\Omega})} \leq C \left( \|v_1 - \tilde{v}_1\|_{C(\bar{\Omega})} + \|v_2 - \tilde{v}_2\|_{C(\bar{\Omega})} \right)^{\hat{\mu}_1}.$$

The expected inequality follows by putting together (??) and (??).

#### APPENDIX A. PROOF OF TECHNICAL LEMMAS

*Proof of Lemma ??.* In this proof  $C = C(n, \mu, \nu) > 1$  is a generic constant.

It is well known that  $G_{1,\nu}$ ,  $\nu > 0$ , the fundamental solution of the operator  $-\Delta + \nu$ , is given by  $G_{1,\nu}(x, \xi) = \mathcal{G}_{1,\nu}(x - \xi)$ ,  $x, \xi \in \mathbb{R}^n$ , with

$$\mathcal{G}_{1,\nu}(x) = (2\pi)^{-n/2} (\sqrt{\nu}/|x|)^{n/2-1} K_{n/2-1}(\sqrt{\nu}|x|).$$

In the particular case  $n = 3$ , we have  $K_{1/2}(z) = \sqrt{\pi/(2z)} e^{-z}$  and therefore

$$\mathcal{G}_{1,\nu}(x) = \frac{e^{-\sqrt{\nu}|x|}}{4\pi|x|}.$$

Let  $f \in C_0^\infty(\mathbb{R}^n)$ ,  $\mu > 0$  and  $\nu > 0$  be two constants, and denote by  $u$  the solution of the equation

$$(-\mu\Delta + \nu)u = f \quad \text{in } \mathbb{R}^n.$$

Then

$$(A.1) \quad u(x) = \int_{\mathbb{R}^n} G_{\mu,\nu}(x, \xi) f(\xi) d\xi, \quad x \in \mathbb{R}^n.$$

We remark that  $v(x) = u(\sqrt{\mu}x)$ ,  $x \in \mathbb{R}^n$  satisfies  $(-\Delta + \nu)v = f(\sqrt{\mu} \cdot)$ . Whence

$$\begin{aligned} u(\sqrt{\mu}x) &= v(x) = \int_{\mathbb{R}^n} \mathcal{G}_{1,\nu}(x - \xi) f(\sqrt{\mu}\xi) d\xi \\ &= \mu^{-n/2} \int_{\mathbb{R}^n} \mathcal{G}_{1,\nu}(x - \xi/\sqrt{\mu}) f(\xi) d\xi, \quad x \in \mathbb{R}^n. \end{aligned}$$

Hence

$$(A.2) \quad u(x) = \mu^{-n/2} \int_{\mathbb{R}^n} \mathcal{G}_{1,\nu}((x - \xi)/\sqrt{\mu}) f(\xi) d\xi, \quad x \in \mathbb{R}^n.$$

Comparing (??) and (??) we find

$$G_{\mu,\nu}(x, \xi) = \mu^{-n/2} \mathcal{G}_{1,\nu}((x - \xi)/\sqrt{\mu}), \quad x, \xi \in \mathbb{R}^n.$$

Consequently  $G_{\mu,\nu}(x, \xi) = \mathcal{G}_{\mu,\nu}(x - \xi)$  with

$$(A.3) \quad \mathcal{G}_{\mu,\nu}(x) = (2\pi\mu)^{-n/2} (\sqrt{\nu\mu}/|x|)^{n/2-1} K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}), \quad x \in \mathbb{R}^n.$$

By the usual asymptotic formula for modified Bessel functions of the second kind (see for instance [?, 9.7.2, page 378]) we have, when  $|x| \rightarrow \infty$ ,

$$K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}) = \left( \frac{\pi\sqrt{\mu}}{2\sqrt{\nu}|x|} \right)^{1/2} e^{-\sqrt{\nu}|x|/\sqrt{\mu}} (1 + O(1/|x|)),$$

where  $O(1/|x|)$  only depends on  $n$ ,  $\mu$  and  $\nu$ .

Consequently, there exists  $R = R(n, \mu, \nu) > 0$  so that

$$(A.4) \quad C^{-1} \frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{1/2}} \leq K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}) \leq C \frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{1/2}}, \quad |x| \geq R.$$

Substituting if necessary  $R$  by  $\max(R, 1)$ , we have

$$(A.5) \quad \frac{1}{|x|^{n/2-1}} \leq \frac{1}{|x|^{1/2}}, \quad |x| \geq R.$$

Moreover, we have

$$\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{1/2}} = \left[ |x|^{(n-3)/2} e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})} \right] \frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}}{|x|^{n/2-1}}, \quad |x| \geq R.$$

Since the function  $x \rightarrow |x|^{(n-3)/2} e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}$  is bounded in  $\mathbb{R}^n$ , we deduce

$$(A.6) \quad \frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{1/2}} \leq C \frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}}{|x|^{n/2-1}}, \quad |x| \geq R.$$

Using (??) and (??) in (??) in order to obtain

$$(A.7) \quad C^{-1} \frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{n/2-1}} \leq K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}) \leq C \frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}}{|x|^{n/2-1}}, \quad |x| \geq R.$$

We now establish a similar estimate when  $|x| \rightarrow 0$ . To this end, we recall that according to formula [?, 9.6.9, page 375] we have

$$K_{n/2-1}(\rho) \sim \frac{1}{2} \Gamma(n/2 - 1) \left( \frac{2}{\rho} \right)^{n/2-1} \quad \text{as } \rho \rightarrow 0,$$

from which we deduce in a straightforward manner that there exists  $0 < r \leq R$  so that

$$(A.8) \quad C^{-1} \frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{n/2-1}} \leq K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}) \leq C \frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}}{|x|^{n/2-1}}, \quad |x| \leq r.$$

The expected two sided inequality (??) follows by combining (??), (??) and (??).  $\square$

*Proof of Lemma ??.* Let  $\mathcal{Q}$  be an open subset of  $\mathbb{R}^n$ , set  $d = \text{diam}(\mathcal{Q})$ ,  $d_x = \text{dist}(x, \partial\mathcal{Q})$  and  $d_{x,y} = \min(d_x, d_y)$ .

We introduce the following weighted Hölder semi-norms and Hölder norms, where  $\sigma \in \mathbb{R}$ ,  $0 < \gamma \leq 1$ , and  $k$  is non-negative integer,

$$\begin{aligned} [w]_{k,0;\mathcal{Q}}^{(\sigma)} &= [w]_{k,\mathcal{Q}}^{(\sigma)} = \sup_{x \in \mathcal{Q}, |\beta|=k} d_x^{k+\sigma} |\partial^\beta w(x)|, \\ [w]_{k,\gamma;\mathcal{Q}}^{(\sigma)} &= \sup_{x,y \in \mathcal{Q}, |\beta|=k} d_{x,y}^{k+\gamma+\sigma} \frac{|\partial^\beta w(y) - \partial^\beta w(x)|}{|y-x|^\gamma}, \\ |w|_{k;\mathcal{Q}}^{(\sigma)} &= \sum_{j=0}^k [w]_{j;\mathcal{Q}}^{(\sigma)}, \\ |w|_{k,\gamma;\mathcal{Q}}^{(\sigma)} &= |w|_{k;\mathcal{Q}}^{(\sigma)} + [w]_{k,\gamma;\mathcal{Q}}^{(\sigma)}. \end{aligned}$$

In term of these notations, we have

$$\begin{aligned} |a|_{0,\alpha;\mathcal{Q}}^{(0)} &= \sup_{x \in \mathcal{Q}} |a(x)| + \sup_{x,y \in \mathcal{Q}} d_{x,y}^\alpha \frac{|a(y) - a(x)|}{|y-x|^\alpha} \leq (1 + \mathbf{d})\lambda, \\ |\partial_j a|_{0,\alpha;\mathcal{Q}}^{(1)} &= \sup_{x \in \mathcal{Q}} d_x |\partial_j a(x)| + \sup_{x,y \in \mathcal{Q}} d_{x,y}^{1+\alpha} \frac{|\partial_j a(y) - \partial_j a(x)|}{|y-x|^\alpha} \leq (\mathbf{d} + \mathbf{d}^2)\lambda, \\ |b|_{0,\alpha;\mathcal{Q}}^{(2)} &= \sup_{x \in \mathcal{Q}} d_x^2 |b(x)| + \sup_{x,y \in \mathcal{Q}} d_{x,y}^{2+\alpha} \frac{|b(y) - b(x)|}{|y-x|^\alpha} \leq (\mathbf{d}^2 + \mathbf{d}^3)\lambda. \end{aligned}$$

In consequence

$$(A.9) \quad |a|_{0,\alpha;\mathcal{Q}}^{(0)} + |\partial_j a|_{0,\alpha;\mathcal{Q}}^{(1)} + |b|_{0,\alpha;\mathcal{Q}}^{(2)} \leq \Lambda(\mathbf{d}) = [1 + 2\mathbf{d} + 2\mathbf{d}^2 + \mathbf{d}^3]\lambda.$$

Following [?] we define also

$$\begin{aligned} [w]_{k,0;\mathcal{Q}}^* &= [w]_{k,\mathcal{O}}^* = \sup_{x \in \mathcal{Q}, |\beta|=k} d_x^k |\partial^\beta w(x)|, \\ [w]_{k,\gamma;\mathcal{Q}}^* &= \sup_{x,y \in \mathcal{Q}, |\beta|=k} d_{x,y}^{k+\alpha} \frac{|\partial^\beta w(y) - \partial^\beta w(x)|}{|y-x|^\gamma}, \\ |w|_{k;\mathcal{Q}}^* &= \sum_{j=0}^k [w]_{j;\mathcal{Q}}^*, \\ |w|_{k,\gamma;\mathcal{Q}}^* &= |w|_{k;\mathcal{Q}}^* + [w]_{k,\gamma;\mathcal{Q}}^*. \end{aligned}$$

From [?, Lemma 6.32, page 130] and its proof we have the following interpolation inequalities: suppose that  $j$  and  $k$ , non negative integers, and  $0 \leq \beta, \gamma \leq 1$  are so that  $j + \beta < k + \gamma$ . Then there exist  $C = C(n, \alpha, \beta) > 0$  and  $\vartheta = \vartheta(\alpha, \beta)$  so that, for any  $w \in C^{k,\alpha}(\mathcal{Q})$  and  $\epsilon > 0$ , we have

$$(A.10) \quad [w]_{j,\beta;\mathcal{Q}}^* \leq C\epsilon^{-\vartheta} |w|_{0;\mathcal{Q}} + \epsilon [w]_{k,\gamma;\mathcal{Q}}^*,$$

$$(A.11) \quad |w|_{j,\beta;\mathcal{Q}}^* \leq C\epsilon^{-\vartheta} |w|_{0;\mathcal{Q}} + \epsilon [w]_{k,\gamma;\mathcal{Q}}^*.$$

Here  $|w|_{0;\mathcal{Q}} = \sup_{x \in \mathcal{Q}} |w(x)|$ .

Checking carefully the proof of interior Schauder estimates in [?, Theorem 6.2, page 90], we get, taking into account inequalities (??)-(??), the following result: there exist a constant  $C = C(n) > 0$  and  $\tau = \tau(\alpha)$  so that, for any  $0 < \mu \leq 1/2$  and  $w \in C^{k,\alpha}(\mathcal{Q})$  satisfying  $L_{a,b}w = 0$  in  $\mathcal{Q}$ , we have

$$(A.12) \quad [w]_{2,\alpha;\mathcal{Q}}^* \leq C\Lambda(\mathbf{d}) (\mu^{-\tau} |w|_{0;\mathcal{Q}} + \mu^\alpha [w]_{2,\alpha;\mathcal{Q}}^*).$$

Substituting in (??)  $C$  by  $\max(C, 2^{\alpha-1})$ , we may assume in (??) that  $C = C(n, \alpha) \geq 2^{\alpha-1}$ . Bearing in mind that  $\Lambda(\mathbf{d}) > 1$ , we can take in (??),  $\mu = (2C\Lambda(\mathbf{d}))^{-1/\alpha}$ . We find

$$(A.13) \quad [w]_{2,\alpha,\mathcal{Q}}^* \leq C\Lambda(\mathbf{d})^\varkappa |w|_{0;\mathcal{Q}},$$

for some constants  $C = C(n, \alpha) > 0$  and  $\varkappa = \varkappa(\alpha) > 1$ .

Using again interpolation inequalities (??) and (??), we deduce that

$$(A.14) \quad |w|_{2,\alpha,\mathcal{Q}}^* \leq C\Lambda(\mathbf{d})^\varkappa |w|_{0;\mathcal{Q}}.$$

Let  $\delta > 0$  be so that  $\mathcal{Q}_\delta = \{x \in \mathcal{Q}; \text{dist}(x, \partial\mathcal{Q}) > \delta\}$  is nonempty. If  $\mathcal{Q}'$  is an open subset of  $\mathcal{Q}_\delta$  then (??) yields in a straightforward manner

$$\|w\|_{C^{2,\alpha}(\overline{\mathcal{Q}'})} \leq C \max\left(\delta^{-(2+\alpha)}, 1\right) \Lambda(\mathbf{d})^\varkappa |w|_{0;\mathcal{Q}}.$$

This is the expected inequality.  $\square$

**Lemma A.1.** *Let  $K$  be a compact subset of  $\mathbb{R}^n$  and  $f \in C^{2,\alpha}(K)$  satisfying  $\min_K |f| \geq c_- > 0$ . Then*

$$(A.15) \quad \|1/f\|_{C^{2,\alpha}(K)} \leq Cc_+^4 (1 + \|f\|_{C^{2,\alpha}(K)})^3,$$

where  $c_+ = \max(1, c_-^{-1})$  and  $C = C(\text{diam}(K))$  is a constant.

*Proof.* Let  $x, y \in K$ . Using  $|1/f|_{0;K} \leq c_+$  and the following identities

$$\begin{aligned} \frac{1}{f^2(y)} - \frac{1}{f^2(x)} &= \left( \frac{1}{f(x)f^2(y)} + \frac{1}{f(x)^2f(y)} \right) (f(x) - f(y)), \\ \frac{1}{f^3(y)} - \frac{1}{f^3(x)} &= \left( \frac{1}{f(x)f^3(y)} + \frac{1}{f^2(x)f^2(y)} + \frac{1}{f(x)^3f(y)} \right) (f(x) - f(y)), \end{aligned}$$

we easily get

$$(A.16) \quad [1/f^j]_{\alpha;K} \leq 3c_+^4 [f]_{\alpha;K}, \quad j = 2, 3.$$

Also, we have

$$\begin{aligned} \frac{\partial_i f(y) \partial_j f(x)}{f^3(y)} - \frac{\partial_i f(y) \partial_j f(x)}{f^3(x)} &= \frac{\partial_i f(y)}{f^3(y)} (\partial_j f(y) - \partial_j f(x)) \\ &\quad + \frac{\partial_j f(x)}{f^3(y)} (\partial_i f(y) - \partial_i f(x)) + \left( \frac{1}{f^3(y)} - \frac{1}{f^3(x)} \right) (\partial_i f(y) \partial_j f(x)). \end{aligned}$$

In light of (??), this identity yields

$$(A.17) \quad \begin{aligned} [\partial_i f \partial_j f / f^3]_{\alpha;K} &\leq c_+^4 ([\partial_i f]_{\alpha;K} |\partial_j f|_{0;K} \\ &\quad + [\partial_j f]_{\alpha;K} |\partial_i f|_{0;K} + [f]_{\alpha;K} |\partial_i f|_{0;K} |\partial_j f|_{0;K}). \end{aligned}$$

On the other hand, since

$$\frac{\partial_{ij}^2 f(y)}{f^2(y)} - \frac{\partial_{ij}^2 f(x)}{f^2(x)} = \frac{1}{f^2(y)} (\partial_{ij}^2 f(y) - \partial_{ij}^2 f(x)) + \left( \frac{1}{f^2(y)} - \frac{1}{f^2(x)} \right) \partial_{ij}^2 f(x),$$

we find, by using again (??),

$$(A.18) \quad [\partial_{ij}^2 f / f^2]_{\alpha;K} \leq 3c_+^4 ([\partial_{ij}^2 f]_{\alpha;K} + [f]_{\alpha;K} |\partial_{ij}^2 f|_{0;K}).$$

Inequalities (??), (??), the identity  $\partial_{ij}^2(1/f) = 2\partial_i f \partial_j f / f^3 - \partial_{ij}^2 f / f^2$  and the interpolation inequality [?, Lemma 6.35, page 135] (by proceeding as in Corollary ??) imply

$$(A.19) \quad [\partial_{ij}^2(1/f)]_{\alpha,K} \leq Cc_+^4 (1 + \|f\|_{C^{2,\alpha}(K)})^3,$$

with  $C = C(\text{diam}(K))$  is a constant.

The other terms for  $1/f$  appearing in the norms  $\|\cdot\|_{C^{2,\alpha}(K)}$  can be estimated similarly to the semi-norm in (??). Inequality (??) then follows.  $\square$

Recall that  $0 < \theta < \alpha < 1$ .

**Lemma A.2.**  $C^{2,\alpha}(\overline{\mathcal{O}})$  is continuously embedded in  $H^{2+\theta}(\mathcal{O})$ . Furthermore, there exists  $C = C(n, \alpha - \theta)$  so that, for any  $w \in C^{2,\alpha}(\overline{\mathcal{O}})$ , we have

$$(A.20) \quad \|w\|_{H^{2+\theta}(\mathcal{O})} \leq C \max(\mathbf{d}^{n/2}, \mathbf{d}^{n/2+\alpha-\theta}) \|w\|_{C^{2,\alpha}(\overline{\mathcal{O}})},$$

where  $\mathbf{d} = \text{diam}(\mathcal{O})$ .

*Proof.* Let  $w \in C^{2,\alpha}(\overline{\mathcal{O}})$  and, for fixed  $1 \leq i, j \leq n$ , set  $g = \partial_{ij}^2 w$ . Then

$$\int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|g(x) - g(y)|^2}{|x - y|^{n+2\theta}} dx dy \leq [g]_{\alpha;\mathcal{O}}^2 \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{1}{|x - y|^{n-2(\alpha-\theta)}} dx dy.$$

In light of [?, Lemma A3, page 246], this inequality yields

$$\int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|g(x) - g(y)|^2}{|x - y|^{n+2\theta}} dx dy \leq \frac{|\mathbb{S}^{n-1}| |\mathcal{O}| \mathbf{d}^{2(\alpha-\theta)}}{2(\alpha - \theta)} [g]_{\alpha;\mathcal{O}}^2,$$

But  $|\mathcal{O}| \leq |B(0, \mathbf{d})|$ . Hence

$$(A.21) \quad \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|g(x) - g(y)|^2}{|x - y|^{n+2\theta}} dx dy \leq \frac{|\mathbb{S}^{n-1}|^2 \mathbf{d}^{n+2(\alpha-\theta)}}{2(\alpha - \theta)} [g]_{\alpha;\mathcal{O}}^2.$$

Using (??) and the inequality

$$\|h\|_{L^2(\mathcal{O})}^2 \leq |\mathbb{S}^{n-1}| \mathbf{d}^n |h|_{0,\mathcal{O}}, \quad h \in C(\overline{\mathcal{O}}),$$

we get from the definition of the norm of  $H^s$ -spaces in [?, formula (1.3.2.2), page 17]

$$\|w\|_{H^{2+\theta}(\mathcal{O})} \leq C \max(\mathbf{d}^{n/2}, \mathbf{d}^{n/2+\alpha-\theta}) \|w\|_{C^{2,\alpha}(\overline{\mathcal{O}})},$$

for some constant  $C = C(n, \alpha - \theta) > 0$ . This is the expected inequality  $\square$

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