Stability for Quantitative Photoacoustic Tomography Revisited

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STABILITY FOR QUANTITATIVE PHOTOACOUSTIC TOMOGRAPHY REVISITED

ERIC BONNETIER, MOURAD CHOULLI, AND FAOUZI TRIKI

Abstract. This paper is concerned with the stability issue in determining absorption and diffusion coefficients in quantitative photoacoustic imaging. Assuming that the optical wave is generated by point sources in a region where the optical coefficients are known, we derive pointwise Hölder stability estimate of the inversion. This result shows that the reconstruction of the optical coefficients is stable in the region close to the optical illumination sources and deteriorate exponentially far away. Our stability estimate is therefore in accordance with known experimental observations.

Mathematics subject classification : 35R30.

Key words : Elliptic equations, diffusion coefficient, absorption coefficient, stability estimates, multiwave imaging.

Contents

1. Introduction

Throughout this text $n \geq 3$ is a fixed integer. If $0 < \alpha \leq 1$ we denote by $C^{0,\alpha}(\mathbb{R}^n)$ the vector space of bounded continuous functions f on $\mathbb{R}^n$ satisfying

$$[f]_\alpha = \sup\left\{ \frac{|f(x) - f(y)|}{|x-y|^\alpha};\ x, y \in \mathbb{R}^n,\ x \neq y \right\} < \infty.$$

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$C^{0,\alpha}(\mathbb{R}^n)$ is then a Banach space when it is endowed with its natural norm

$$\|f\|_{C^{0,\alpha}(\mathbb{R}^n)} = \|f\|_{L^\infty(\mathbb{R}^n)} + [f]_\alpha.$$

Define $C^{1,\alpha}(\mathbb{R}^n)$ as the vector space of functions f from $C^{0,\alpha}(\mathbb{R}^n)$ so that $\partial_j f \in C^{0,\alpha}(\mathbb{R}^n)$, $1 \leq j \leq n$. $C^{1,\alpha}(\mathbb{R}^n)$ equipped with the norm

$$\|f\|_{C^{1,\alpha}(\mathbb{R}^n)} = \|f\|_{C^{0,\alpha}(\mathbb{R}^n)} + \sum_{j=1}^{n} \|\partial_j f\|_{C^{0,\alpha}(\mathbb{R}^n)}$$

is a Banach space.

For $\lambda > 1$ and $\kappa > 1$, denote by $\mathcal{D}(\lambda, \kappa)$ the set of couples $(a, b) \in C^{1,1}(\mathbb{R}^n) \times C^{0,1}(\mathbb{R}^n)$ satisfying

$$\lambda^{-1} \leq a \quad \text{and} \quad \|a\|_{C^{1,1}(\mathbb{R}^n)} \leq \lambda, \tag{1.1}$$

$$\kappa^{-1} \leq b \quad \text{and} \quad \|b\|_{C^{0,1}(\mathbb{R}^n)} \leq \kappa, \tag{1.2}$$

Define further the elliptic operator $L_{a,b}$ acting as follows

$$L_{a,b}u(x) = -\partial_i(a(x)\partial_j u(x)) + b(x)u(x). \tag{1.3}$$

We will show in Theorem 2.2 that if $(a, b) \in \mathcal{D}(\lambda, \kappa)$ then the operator $L_{a,b}$ admits a unique fundamental solution $G_{a,b}$ satisfying

$$\begin{aligned} G_{a,b}(\cdot, \xi) &\in C(\mathbb{R}^n \setminus \{\xi\}), \quad \xi \in \mathbb{R}^n, \\ G_{a,b}(x, \cdot) &\in C(\mathbb{R}^n \setminus \{x\}), \quad x \in \mathbb{R}^n, \\ L_{a,b}G_{a,b}(\cdot, \xi) &= 0 \text{ in } \mathscr{D}'(\mathbb{R}^n \setminus \{\xi\}), \quad \xi \in \mathbb{R}^n, \end{aligned}$$

and for any $f \in C_0^\infty(\mathbb{R}^n)$,

$$u(x) = \int_{\mathbb{R}^n} G_{a,b}(x, \xi) f(\xi) d\xi,$$

belongs to $H^2(\mathbb{R}^n)$, and is the unique solution of $L_{a,b}u = f$. Moreover, for any $0 < \beta < 1$ and $\mathcal{O} \Subset \mathbb{R}^n \setminus \{\xi\}$, $G_{a,b}(\cdot, \xi)$ belongs to $C^{2,\beta}(\overline{\mathcal{O}})$ and

$$\|G_{a,b}(\cdot, \xi)\|_{C^{2,\beta}(\overline{\mathcal{O}})} \leq C,$$

the constant C only depends on n, $\mathcal{O}$, β and λ.

Let Ω be a bounded $C^{1,1}$ domain of $\mathbb{R}^n$ with boundary Γ. We deal with the problem of reconstructing $(a, b) \in \mathcal{D}(\lambda, \kappa)$ from energies generated by two sources located at ξ_1 and ξ_2, two distinct points in $\mathbb{R}^n \setminus \overline{\Omega}$. Precisely if $u_j(a, b) = G_{a,b}(\cdot, \xi_j)$, $j = 1, 2$, we want to determine (a, b) from the internal measurements

$$v_j(a, b) = bu_j(a, b), \quad \text{in } \Omega, \quad j = 1, 2.$$

This inverse problem is related to photoacoustic tomography (PAI) where optical energy absorption causes thermoelastic expansion of the tissue, which in turn generates a pressure wave [23]. This acoustic signal is measured by transducers distributed on the boundary of the sample and it is used for imaging optical properties of the sample. The internal data $v_1(a, b)$ and $v_2(a, b)$ are obtained by performing a first step consisting in a linear initial to boundary inverse problem for the acoustic wave equation. Therefore the inverse problem that arises from this first inversion is to determine the diffusion coefficient a and the absorption coefficient b from the

internal data $v_1(a,b)$ and $v_2(a,b)$ that are proportional to the local absorbed optical energy inside the sample. This inverse problem is known in the literature as quantitative photoacoustic tomography [1, 4, 2, 3, 9, 8, 20].

Photoacoustic imaging provides in theory images of optical contrasts and ultrasound resolution [23]. Indeed, the resolution is mainly due to the small wavelength of acoustic waves, while the contrast is somehow related to the sensitivity of optical waves to absorption and scattering properties of the medium in the diffusive regime. However in practice it has been observed in various experiments that the imaging depth, i.e. the maximal depth of the medium at which structures can be resolved at expected resolution, of PAI is still fairly limited, usually on the order of millimeters. This is mainly due to the fact that optical waves are significantly attenuated by absorption and scattering. In fact the generated optical signal decays very fast in the depth direction. This is indeed a well know faced issue in optical tomography [22]. In most physicists works dealing with quantitative PAI, the absorption $b > 0$ is assumed to be constant and the optical wave is simplified to Ce^{-bz}, as a function of the depth z, which is known as the Beer-Lambert-Bouguer law [6]. Recently in [21], assuming that medium is layered, the authors derived a stability estimate that shows that the reconstruction of the optical coefficients is stable in the region close to the optical illumination source and deteriorate exponentially far away. The main objective of this work is to provide a mathematical analysis of the issue of imaging depth in PAI in a general setting. To be more precise, assuming that the optical waves are generated by two point sources $\delta_{\xi_i}, i = 1, 2$, we aim to derive a stability estimate for the recovery of the optical coefficients from internal data. We point out that taking the optical wave generated by a point source outside the sample seems to be more realistic than assuming a boundary condition.

We show in Lemma 2.4 that there exist $\rho > 0$, $x^* \in \mathbb{R} \setminus \{\xi_1, \xi_2\}$ close to ξ_2 and $\eta > 0$ only depending on ξ_1, ξ_2, λ, κ, and n so that $B(x^*, \rho) \Subset \mathbb{R} \setminus \{\xi_1, \xi_2\}$ and

$$\eta \leq \left\| \nabla \left(\frac{u_2}{u_1} \right) \right\|_{L^2(B(x^*,\rho))}.$$

Theorem 1.1. *Let $0 < \theta < 1/2$ and $0 < \beta < 1$. Then there exists a constant $\tilde{c} > 0$, only depending on n, Ω, ξ_1, ξ_2, κ, λ, θ and β so that, for $(a,b), (\tilde{a}, \tilde{b}) \in \mathcal{D}(\lambda, \kappa)$ satisfying $(a,b) = (\tilde{a}, \tilde{b})$ on Γ, we have*

$$|a(x) - \tilde{a}(x)| + |b(x) - \tilde{b}(x)| \leq \tilde{c} e^{\tilde{c}|x-x^*|} \left(\|v_1 - \tilde{v}_1\|_{C(\overline{\Omega})} + \|v_2 - \tilde{v}_2\|_{C(\overline{\Omega})} \right)^{\mu(|x-x^*|)}, \quad x \in \Omega, \tag{1.4}$$

where

$$\mu(s) = \left(\frac{\theta}{2+\theta} \right) \frac{\beta}{ce^{cs} + \beta + n - 1}, \quad s \geq 0,$$

the constant $c > 0$ only depends on n, Ω, ξ_1, ξ_2, κ and λ.

The derived stability estimate shows clearly that the resolution of PAI may deteriorate exponentially in the depth direction far from the sources ξ_1 and ξ_2. Indeed, $\mu(|x-x^*|)$ is of order $e^{-c|x-x^*|}$ when $|x-x^*|$ is sufficiently large. Consequently the right hand side of (1.4) may be close to a constant whenever $|x-x^*|$ is sufficiently large even if the term $\|v_1 - \tilde{v}_1\|_{C(\overline{\Omega})} + \|v_2 - \tilde{v}_2\|_{C(\overline{\Omega})}$ is too small. Moreover, the term $\tilde{c}e^{\tilde{c}|x-x^*|}$ increase exponentially with respect to $|x-x^*|$.

The result of Theorem 1.1 is therefore in a good agreement with observations made in various experiments that the imaging depth is fairly limited.

The rest of this text is organized as follows. In section 2 we construct a fundamental solution and give its regularity induced by that of the coefficients of operators under consideration. We also establish in this section a lower bound of the local L^2-norm of the gradient of the quotient of two fundamental solutions near one of the point sources. This is the key point for establishing our stability estimate. This last result is then used in Section 3 to obtain a uniform polynomial lower bound of the local L^2-norm of the gradient in a given region. This polynomial lower bound is obtained in two steps. In the first step we derive, via a three-ball inequality for the gradient, a uniform lower bound of negative exponential type. We use then in the second step an argument based on the so-called frequency function in order to improve this lower bound. In the last section we prove our main theorem following the known method consisting in reducing the original problem to the stability of an inverse conductivity problem.

2. Fundamental solutions

2.1. **Constructing fundamental solutions.** In this subsection we construct a fundamental solution of divergence form elliptic operators. Since our construction relies on heat kernel estimates, we first recall some known results.

Consider the parabolic operator $P_{a,b}$ acting as follows

$$P_{a,b}u(x,t) = -L_{a,b}u(x,t) - \partial_t u(x,t)$$

and set

$$Q = \{(x,t,\xi,\tau) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R};\ \tau < t\}.$$

Recall that a fundamental solution of the operator $P_{a,b}$ is a function $E_{a,b} \in C^{2,1}(Q)$ verifying $P_{a,b}E = 0$ in Q and, for every $f \in C_0^\infty(\mathbb{R}^n)$,

$$\lim_{t\downarrow\tau} \int_{\mathbb{R}^n} E_{a,b}(x,t,\xi,\tau) f(\xi) d\xi = f(x), \quad x \in \mathbb{R}^n.$$

The classical results in the monographs by A. Friedman [13], O. A. Ladyzenskaja, V. A. Solonnikov and N.N Ural'ceva [19] show that $P_{a,b}$ admits a on negative fundamental solution when $(a,b) \in \mathcal{D}(\lambda,\kappa)$.

It is worth mentioning that if $a = c$ for some constant $c > 0$, and $b = 0$, the fundamental solution $E_{c,0}$ is explicitly given by

$$E_{c,0}(x,t,\xi,\tau) = \frac{1}{(4\pi c(t-\tau))^{n/2}} e^{-\frac{|x-\xi|^2}{4c(t-\tau)}}, \quad (x,t,\xi,\tau) \in Q.$$

Examining carefully the proof of the two-sided Gaussian bounds in [12], we see that these bounds remain valid whenever $a \in C^{1,1}(\mathbb{R}^n)$ satisfies

$$\lambda^{-1} \le a \quad \text{and} \quad \|a\|_{C^{1,1}(\mathbb{R}^n)} \le \lambda, \tag{2.1}$$

$\lambda > 1$ is a constant.

More precisely we have the following theorem in which

$$\mathcal{E}_c(x,t) = \frac{c}{t^{n/2}} e^{-\frac{|x|^2}{ct}}, \quad x \in \mathbb{R}^n,\ t > 0,\ c > 0.$$

Theorem 2.1. *There exists a constant $c > 1$ only depending on n and λ so that, for any $a \in C^{1,1}(\mathbb{R}^n)$ satisfying (2.1), we have, for $(x,t,\xi,\tau) \in Q$,*

$$\mathcal{E}_{c^{-1}}(x-\xi,t-\tau) \leq E_{a,0}(x,t;\xi,\tau) \leq \mathcal{E}_c(x-\xi,t-\tau). \tag{2.2}$$

It is worth mentioning the following relationship between $\mathcal{E}_c$ and $E_{c,0}$

$$\mathcal{E}_c(x-\xi,t-\tau) = \frac{(\pi c)^{n/2-1}}{\pi} E_{c/4,0}(x,t,\xi,\tau), \quad (x,t,\xi,\tau) \in Q. \tag{2.3}$$

The following comparaison principle will be useful in the sequel.

Lemma 2.1. *Let $(a,b_1),(a,b_2) \in \mathcal{D}(\lambda,\kappa)$ so that $b_1 \leq b_2$. Then $E_{a,b_2} \leq E_{a,b_1}$.*

Proof. Pick $0 \leq f \in C_0^\infty(\mathbb{R}^n)$. Let u be the solution of the initial value problem

$$P_{a,b_1}u(x,t) = 0 \ \in \mathbb{R}^n \times \{t > \tau\}, \quad u(x,\tau) = f.$$

We have

$$u(x,t) = \int_{\mathbb{R}^n} E_{a,b_1}(x,t;\xi,\tau) f(\xi) d\xi \geq 0. \tag{2.4}$$

On the other hand, as $P_{a,b_1}u(x,t) = 0$ can be rewritten as $P_{a,b_2}u(x,t) = (b_1(x) - b_2(x))u(x,t)$, we obtain

$$\begin{aligned} u(x,t) = &\int_{\mathbb{R}^n} E_{a,b_2}(x,t;\xi,\tau) f(\xi) d\xi \\ &- \int_\tau^t \int_{\mathbb{R}^n} E_{a,b_2}(x,t;\xi,s)(b_1(\xi) - b_2(\xi)) u(\xi,s) d\xi ds. \end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), we get

$$\int_{\mathbb{R}^n} E_{a,b_2}(x,t;\xi,\tau) f(\xi) d\xi \leq \int_{\mathbb{R}^n} E_{a,b_1}(x,t;\xi,\tau) f(\xi) d\xi,$$

which yields in a straightforward manner the expected inequality. □

Consider, for $(a,b) \in \mathcal{D}(\lambda,\kappa)$, the unbounded operator $A_{a,b} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined

$$A_{a,b} = -L_{a,b}, \quad D(A_{a,b}) = H^2(\mathbb{R}^n).$$

It is well known that $A_{a,b}$ generates an analytic semigroup $e^{tA_{a,b}}$. Therefore in light of [7, Theorem 4 on page 30, Theorem 18 on page 44 and the proof in the beginning of Section 1.4.2 on page 35], $k_{a,b}(t,x;\xi)$, the Schwarz kernel of $e^{tA_{a,b}}$, is Hölder continuous with respect to x and ξ and satisfies

$$|k_{a,b}(t,x,\xi)| \leq e^{-\delta t} \mathcal{E}_c(x-\xi,t-\tau) \tag{2.6}$$

and, for $|h| \leq \sqrt{t} + |x-\xi|$,

$$|k_{a,b}(t,x+h,\xi) - k_{a,b}(t,x,\xi)| \leq e^{-\delta t} \left(\frac{|h|}{\sqrt{t}+|x-\xi|} \right)^\eta \mathcal{E}_c(x-\xi,t-\tau), \tag{2.7}$$

$$|k_{a,b}(t,x,\xi+h) - k_{a,b}(t,x,\xi)| \leq e^{-\delta t} \left(\frac{|h|}{\sqrt{t}+|x-\xi|} \right)^\eta \mathcal{E}_c(x-\xi,t-\tau), \tag{2.8}$$

the constants $c > 0$ and $\delta > 0$ only depend on n, λ and κ, and $\eta > 0$.

By the uniqueness of solutions of the Cauchy problem

$$u'(t) = A_{a,b}u(t), \ t > 0 \quad u(0) = f \in C_0^\infty(\mathbb{R}^n) \tag{2.9}$$

we deduce in a straightforward manner that $k_{a,b}(t,x;\xi) = E_{a,b}(x,t;\xi,0)$.

Prior to give the construction of the fundamental solution for the variable coefficients operators, we state a result for operators with constant coefficients. This result is proved in Appendix A.

Lemma 2.2. *Let $\mu > 0$ and $\nu > 0$ be two constants. Then the fundamental solution for the operator $-\mu\Delta + \nu$ is given by $G_{\mu,\nu}(x,\xi) = \mathcal{G}_{\mu,\nu}(x-\xi)$, $x, \xi \in \mathbb{R}^n$, with*

$$\mathcal{G}_{\mu,\nu}(x) = (2\pi\mu)^{-n/2}(\sqrt{\nu\mu}/|x|)^{n/2-1}K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}), \quad x \in \mathbb{R}^n.$$

Here $K_{n/2-1}$ is the usual modified Bessel function of second kind. Moreover the following two-sided inequality holds

$$C^{-1}\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{n-2}} \leq \mathcal{G}_{\mu,\nu}(x) \leq C\frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}}{|x|^{n-2}}, \quad x \in \mathbb{R}^n, \tag{2.10}$$

the constant $C > 1$ only depends on n, μ and ν.

The main result of this section is the following theorem.

Theorem 2.2. *Let $(a,b) \in \mathcal{D}(\lambda,\kappa)$. Then there exists a unique function $G_{a,b}$ satisfying $G_{a,b}(\cdot,\xi) \in C(\mathbb{R}^n \setminus \{\xi\})$, $\xi \in \mathbb{R}^n$, $G_{a,b}(x,\cdot) \in C(\mathbb{R}^n \setminus \{x\})$, $x \in \mathbb{R}^n$ and*
(i) $L_{a,b}G_{a,b}(\cdot,\xi) = 0$ in $\mathscr{D}'(\mathbb{R}^n \setminus \{\xi\})$, $\xi \in \mathbb{R}^n$,
(ii) for any $f \in C_0^\infty(\mathbb{R}^n)$,

$$u(x) = \int_{\mathbb{R}^n} G_{a,b}(x,\xi)f(\xi)d\xi$$

belongs to $H^2(\mathbb{R}^n)$, and is the unique solution of $L_{a,b}u = f$,
(iii) there exist a constant $c > 1$, only depending on n and λ, and a constant $C > 0$, only depending on n, λ and κ, so that

$$C^{-1}\frac{e^{-2\sqrt{c\kappa}|x-\xi|}}{|x-\xi|^{n-2}} \leq G_{a,b}(x,\xi) \leq C\frac{e^{-\frac{|x-\xi|}{\sqrt{c\kappa}}}}{|x-\xi|^{n-2}}. \tag{2.11}$$

Proof. Pick $s \geq 1$ arbitrary. Applying Hölder inequality, we find

$$\int_{\mathbb{R}^n} k_{a,b}(t,x,\xi)|f(\xi)|d\xi \leq \|k_{a,b}(t,x,\cdot)\|_{L^s(\mathbb{R}^n)}\|f\|_{L^{s'}(\mathbb{R}^n)},$$

where s' is the conjugate exponent of s.

But, according to (2.6)

$$\|k_{a,b}(t,x,\cdot)\|^s_{L^s(\mathbb{R}^n)} \leq \left(\frac{c}{t^{n/2}}\right)^s \int_{\mathbb{R}^n} e^{-\frac{s|x-\xi|^2}{ct}}d\xi.$$

Next, making a change of variable $\xi = (\sqrt{ct/s})\eta + x$, we get

$$\|k_{a,b}(t,x,\cdot)\|^s_{L^s(\mathbb{R}^n)} \leq \left(\frac{c}{t^{n/2}}\right)^s \left(\frac{ct}{s}\right)^{n/2} \int_{\mathbb{R}^n} e^{-|\eta|^2}d\eta.$$

Hence

$$\|k_{a,b}(t,x,\cdot)\|_{L^s(\mathbb{R}^n)} \leq t^{n(1/s-1)/2}C_s,$$

with

$$C_s = c\left(\frac{c}{s}\right)^{n/2}\left(\int_{\mathbb{R}^n} e^{-c|\eta|^2}d\eta\right)^{1/s}.$$

Under the choice $1 \le s \le \frac{n}{n-2} < \tilde{s}$, we have

$$\begin{aligned}\int_0^{+\infty} \int_{\mathbb{R}^n} k_{a,b}(t,x,\xi)|f(\xi)|d\xi dt &= \int_0^1 \int_{\mathbb{R}^n} k_{a,b}(t,x,\xi)|f(\xi)|d\xi dt + \int_1^{+\infty} \int_{\mathbb{R}^n} k_{a,b}(t,x,\xi)|f(\xi)|d\xi dt \\ &\le C_s \|f\|_{L^{s'}(\mathbb{R}^n)} \int_0^1 t^{\frac{n}{2}(1/s-1)} dt + C_{\tilde{s}} \|f\|_{L^{\tilde{s}'}(\mathbb{R}^n)} \int_1^{+\infty} t^{\frac{n}{2}(1/\tilde{s}-1)} dt.\end{aligned}$$

In light of Fubini's theorem, we get

$$\int_0^{+\infty} \int_{\mathbb{R}^n} k_{a,b}(t,x,\xi) f(\xi) d\xi dt = \int_{\mathbb{R}^n} \left(\int_0^{+\infty} k_{a,b}(t,x,\xi) dt \right) f(\xi) d\xi. \tag{2.12}$$

Define $G_{a,b}$ as follows

$$G_{a,b}(x,\xi) = \int_0^{+\infty} k_{a,b}(t,x,\xi) dt.$$

Then (2.12) takes the form

$$\int_0^{+\infty} \int_{\mathbb{R}^n} k_{a,b}(t,x,\xi) f(\xi) d\xi dt = \int_{\mathbb{R}^n} G_{a,b}(x,\xi) f(\xi) d\xi. \tag{2.13}$$

Noting that $A_{a,b}$ is invertible, we obtain

$$\begin{aligned}-A_{a,b}^{-1} f(x) &= \left(\int_0^{+\infty} e^{tA_{a,b}} f dt \right)(x) \\ &= \int_0^{+\infty} \int_{\mathbb{R}^n} k_{a,b}(t,x,\xi) f(\xi) d\xi dt, \quad x \in \mathbb{R}^n.\end{aligned}$$

This and (2.13) entail

$$-A_{a,b}^{-1} f(x) = \int_{\mathbb{R}^n} G_{a,b}(x,\xi) f(\xi) d\xi, \quad x \in \mathbb{R}^n.$$

In other words, u defined by

$$u(x) = \int_{\mathbb{R}^n} G_{a,b}(x,\xi) f(\xi) d\xi \quad x \in \mathbb{R}^n$$

belongs to $H^2(\mathbb{R}^n)$ and satisfies $L_{a,b} u = f$.

Noting that, for $x \neq \xi$,

$$\int_0^{+\infty} \frac{1}{t^{n/2}} e^{-\frac{|x-\xi|^2}{ct}} dt = \left(c^{n/2-1} \int_0^{+\infty} \tau^{n/2-2} e^{-\tau} d\tau \right) \frac{1}{|x-\xi|^{n-2}},$$

we get in light of (2.7)

$$|G_{a,b}(x+h,\xi) - G_{a,b}(x,\xi)| \le \frac{C}{|x-\xi|^{n+2+\eta}} |h|^\eta, \quad x \neq \xi, \ |h| \le |x-\xi|.$$

the constant C only depends on n, λ and κ. In particular $G_{a,b}(\cdot,\xi) \in C(\mathbb{R}^n \setminus \{\xi\})$. Similarly, using (2.8) instead of (2.7), we obtain $G_{a,b}(x,\cdot) \in C(\mathbb{R}^n \setminus \{x\})$. More specifically we have

$$|G_{a,b}(x,\xi+h) - G_{a,b}(x,\xi)| \le \frac{C}{|x-\xi|^{n+2+\eta}} |h|^\eta, \quad x \neq \xi, \ |h| \le |x-\xi|. \tag{2.14}$$

Take $\xi \in \mathbb{R}^n$ and K a compact subset of $\mathbb{R}^n \setminus \xi$. Then set

$$w_{a,b}(y) = \int_K G_{a,b}(x,y)dx, \quad y \in B(\xi, \mathrm{dist}(\xi, K)).$$

It follows from (2.14) that

$$|w_{a,b}(y+h) - w_{a,b}(y)| \leq \frac{C}{\mathrm{dist}(y,K)^{n+2+\eta}}|h|^\eta, \quad x \in K,\ |h| < \mathrm{dist}(y,K).$$

Therefore $w_{a,b} \in C(B(\xi, \mathrm{dist}(\xi, K))$.

Let $\mathcal{M}(\mathbb{R}^n)$ be the space of bounded measures on $\mathbb{R}^n$. Pick a sequence (f_n) of a positive functions of $C_0^\infty(\mathbb{R}^n)$ converging in $\mathcal{M}(\mathbb{R}^n)$ to δ_ξ and let $u_n = -A_{a,b}^{-1} f_n$. In consequence according to Fubini's theorem we have

$$\begin{aligned}\int_K u_n(x)dx &= \int_K \int_{\mathbb{R}^n} G_{a,b}(x,y) f_n(y)dy \\ &= \int_{\mathbb{R}^n} w_{a,b}(y) f_n(y)dy \longrightarrow w_{a,b}(\xi) = \int_K G_{a,b}(x,\xi)dx,\end{aligned}$$

where we used that $\mathrm{supp} f_n \subset B(\xi, \mathrm{dist}(\xi, K))$, provided that n is sufficiently large. That is we proved that u_n converges to $G_{a,b}(\cdot, \xi)$ in $L^1_{loc}(\mathbb{R}^n \setminus \{\xi\})$.

Now, as $L_{a,b}u_n = f_n$, we find $L_{a,b}G_{a,b}(\cdot,\xi) = 0$ in $\mathbb{R}^n \setminus \{\xi\}$ in the distributional sense.

We note that the uniqueness of $G_{a,b}$ is a straightforward consequence of that of u.

As $\kappa^{-1} \leq b \leq \kappa$ we deduce from Lemma 2.1 that

$$E_{a,\kappa}(x,t,\xi,0) \leq E_{a,b}(x,t,\xi,0) \leq E_{a,\kappa^{-1}}(x,t,\xi,0).$$

But a simple change of variable shows

$$E_{a,\kappa^{-1}}(x,t,\xi,0) = e^{-\kappa^{-1}t} E_{a,0}(x,t,\xi,0), \tag{2.15}$$

and

$$E_{a,\kappa}(x,t,\xi,0) = e^{-\kappa t} E_{a,0}(x,t,\xi,0). \tag{2.16}$$

Therefore from Theorem 2.1 and identity (2.3) there exists a constant $c > 1$ depending only on n and λ so that

$$\begin{aligned}e^{-\kappa t}\frac{(\pi c^{-1})^{n/2-1}}{\pi} E_{c^{-1}/4,0}(x,t,\xi,0) &\leq E_{a,b}(x,t,\xi,0) \\ &\leq e^{-\kappa^{-1}t}\frac{(\pi c)^{n/2-1}}{\pi} E_{c/4,0}(x,t,\xi,0),\end{aligned}$$

which combined again with identities (2.15) and (2.16) give

$$\begin{aligned}\frac{(\pi c^{-1})^{n/2-1}}{\pi} E_{c^{-1}/4,\kappa}(x,t,\xi,0) &\leq E_{a,b}(x,t,\xi,0) \\ &\leq \frac{(\pi c)^{n/2-1}}{\pi} E_{c/4,\kappa^{-1}}(x,t,\xi,0).\end{aligned}$$

From the uniqueness of $G_{a,b}$ we obtain by integrating over $(0,+\infty)$ each member of the above inequalities

$$\frac{(\pi c^{-1})^{n/2-1}}{\pi} G_{c^{-1}/4,\kappa}(x,\xi) \leq G_{a,b}(x,\xi) \leq \frac{(\pi c)^{n/2-1}}{\pi} G_{c/4,\kappa^{-1}}(x,\xi).$$

This two-sided inequality together with (2.10) yield in a straightforward manner (2.11). □

The function $G_{a,b}$ given by the previous theorem is usually called a fundamental solution of the operator $L_{a,b}$.

2.2. **Regularity of fundamental solutions.**

Proposition 2.1. *Let $\mathcal{O} \Subset \mathbb{R}^n \setminus \{\xi\}$, $\xi \in \mathbb{R}^n$ and $0 < \beta < 1$. Then $G_{a,b}(\cdot, \xi) \in C^{2,\beta}(\overline{\mathcal{O}})$ and*

$$\|G_{a,b}(\cdot, \xi)\|_{C^{2,\beta}(\overline{\mathcal{O}})} \leq C, \tag{2.17}$$

the constant $C > 0$ only depends on n, $\mathcal{O}$, β, λ, κ and ξ.

Proof. In this proof $C > 0$ is a constant only depending on n, $\mathcal{O}$, β, λ, κ and ξ.

Fix $\mathcal{O} \Subset \mathcal{O}' \Subset \mathbb{R}^n \setminus \{\xi\}$ with $\mathcal{O}'$ of class $C^{1,1}$. As $G_{a,b}(\cdot, \xi) \in C(\partial\mathcal{O}')$ by [16, Theorem 6.18, page 106] $G_{a,b}(\cdot, \xi)$ belongs to $C^{2,\beta}(\mathcal{O}')$. Applying then [16, Corollary 6.3, page 93] in order to get

$$\|G_{a,b}(\cdot, \xi)\|_{C^{2,\beta}(\overline{\mathcal{O}})} \leq C\|G_{a,b}(\cdot, \xi)\|_{C(\overline{\mathcal{O}'})}.$$

This estimate together with the one in Theorem 2.2 (iii) yield (2.17). □

2.3. **Gradient estimates of the quotient of two fundamental solutions.** Fix $\xi_1, \xi_2 \in \mathbb{R}^n$ with $\xi_1 \neq \xi_2$ and set $u_j = G_{a,b}(\cdot, \xi_j)$, $j = 1, 2$, where $G_{a,b}$ is the fundamental solution constructed in Theorem 2.2 corresponding to $(a, b) \in \mathcal{D}(\lambda, \kappa)$.

It is useful in the sequel to observe that $u = u_2/u_1$ is the solution of the equation

$$\operatorname{div}(\sigma(x)w(x)) = 0, \quad \text{in } \mathbb{R}^n \setminus \{\xi_1, \xi_2\},$$

with $\sigma = au_1^2$.

Also, we see that as an immediate consequence of Proposition 2.1 we have

Corollary 2.1. *Let $\mathcal{O}$ be a bounded domain of $\mathbb{R}^n \setminus \{\xi_1, \xi_2\}$ and $0 < \beta < 1$. Then there exists a constant $M > 0$ only depending on $\mathcal{O}$, β, λ, ξ_1 and ξ_2 so that*

$$\left\| \frac{u_2}{u_1} \right\|_{C^{2,\beta}(\overline{\mathcal{O}})} \leq M.$$

Pick $0 < \vartheta < 1/2$ and $1/2 + \vartheta < \beta < 1$. Then by the definition of $W^{s,p}$-spaces in [17, Definition 1.3.2.1, page 16] we deduce that $C^{2,\beta}(\overline{\mathcal{O}})$ is continuously embedded in $H^{2+\vartheta}(\mathcal{O})$. In light of Corollary 2.1 we can state the following result

Lemma 2.3. *Let $\mathcal{O}$ be a bounded domain of $\mathbb{R}^n \setminus \{\xi_1, \xi_2\}$ and $0 < \vartheta < 1/2$. Then there exists a constant $M > 0$ only depending on $\mathcal{O}$, ϑ, λ, ξ_1 and ξ_2, so that*

$$\left\| \frac{u_2}{u_1} \right\|_{H^{2+\vartheta}(\mathcal{O})} \leq M.$$

Lemma 2.4. *There exist $x^* \in \mathbb{R}^n$, $C > 0$ and $\rho > 0$ only depending only on n, λ, κ, ξ_1 and ξ_2 so that $\overline{B}(x^*, \rho) \subset \mathbb{R}^n \setminus \{\xi_1, \xi_2\}$ and*

$$C \leq \left\| \nabla \left(\frac{u_2}{u_1} \right) \right\|_{L^2(B(x^*,\rho))}.$$

Proof. Pick $R > 0$ is such a way that $\xi_1, \xi_2 \in B(0,R)$. Set $w = u_2/u_1$. Then according to Theorem 2.2 we have

$$C^{-1}\frac{|x-\xi_1|^{n-2}}{|x-\xi_2|^{n-2}} \leq w(x) \leq C\frac{|x-\xi_1|^{n-2}}{|x-\xi_2|^{n-2}}, \quad |x| \leq R, \tag{2.18}$$

the constant $C > 1$ only depends on n, λ, κ ξ_1 and ξ_2.

Set $\tilde{t} = \operatorname{dist}(\xi_2, \partial B(0,R))/2$ and define

$$\varphi(t,\theta) = w(\xi_2 + t\theta), \quad (t,\theta) \in (0,\tilde{t}] \times \mathbb{S}^{n-1}.$$

We deduce from Corollary (2.1) that $\varphi \in C^{2,\beta}_{loc}((0,\tilde{t}] \times \mathbb{S}^{n-1})$ and consequently

$$\varphi(\tilde{t},\theta) - \varphi(t,\theta) = \int_t^{\tilde{t}} \nabla w(\xi_2 + s\theta)\cdot\theta ds,$$

which in turn gives

$$\begin{aligned}
|\varphi(\tilde{t},\theta) - \varphi(t,\theta)|^2 &\leq (\tilde{t}-t)\int_t^{\tilde{t}} |\nabla w(\xi_2+s\theta)|^2\, ds \\
&\leq \tilde{t}\int_t^{\tilde{t}} |\nabla w(\xi_2+s\theta)|^2\, ds \\
&\leq \tilde{t}\int_t^{\tilde{t}} \frac{s^{n-1}}{t^{n-1}} |\nabla w(\xi_2+s\theta)|^2\, ds, \quad (t,\theta) \in (0,\tilde{t}] \times \mathbb{S}^{n-1}.
\end{aligned}$$

Whence, where $t \in (0,\tilde{t}]$,

$$t^{n-1}\int_{\mathbb{S}^{n-1}} |\varphi(\tilde{t},\theta) - \varphi(t,\theta)|^2 d\theta \leq \tilde{t}\int_{\mathscr{C}_t} |\nabla w(x)|^2\, dx. \tag{2.19}$$

Here

$$\mathscr{C}_t = \left\{x \in \mathbb{R}^n : \ t < |x-\xi_2| < \tilde{t}\right\}.$$

On the other hand inequalities (2.18) imply, where $(t,\theta) \in (0,\tilde{t}] \times \mathbb{S}^{n-1}$,

$$C^{-1}\frac{|\xi_2 - \xi_1 + t\theta|^{n-2}}{t^{n-2}} \leq \varphi(t,\theta) \leq C\frac{|\xi_2 - \xi_1 + t\theta|^{n-2}}{t^{n-2}}.$$

Hence

$$C^{-1}\left(\frac{3\,|\xi_1-\xi_2|}{4t}\right)^{n-2} \leq \varphi(t,\theta) \leq C\left(\frac{5\,|\xi_1-\xi_2|}{4t}\right)^{n-2},$$

provided that $t \in (0,\hat{t})$ and $\theta \in \mathbb{S}^{n-1}$ with $\hat{t} \leq \min(\tilde{t}, |\xi_1-\xi_2|/4)$. Let us then choose $\hat{t}$ sufficiently small in such a way that

$$C^{-1}\left(\frac{3\,|\xi_1-\xi_2|}{4t}\right)^{n-2} - C\left(\frac{5|\xi_1-\xi_2|}{4\tilde{t}}\right)^{n-2} \geq 0, \quad t \in (0,\hat{t}).$$

Therefore

$$\left(C^{-1}\left(\frac{3\,|\xi_1-\xi_2|}{4t}\right)^{n-2} - C\left(\frac{5|\xi_1-\xi_2|}{4\tilde{t}}\right)^{n-2}\right)^2 \leq |\varphi(\tilde{t},\theta) - \varphi(t,\theta)|^2 \tag{2.20}$$

if $(t,\theta) \in (0,\hat{t}) \times \theta \in \mathbb{S}^{n-1}$.

We then obtain by combining inequalities (2.19) and (2.20)

$$|\mathbb{S}^{n-1}|\left(C^{-1}\left(\frac{3\,|\xi_1-\xi_2|}{4t}\right)^{n-2}-C\left(\frac{5|\xi_1-\xi_2|}{4\tilde{t}}\right)^{n-2}\right)^2\leq \tilde{t}\int_{\mathscr{C}_t}|\nabla w(x)|^2\,dx,$$

for $t\in(0,\hat{t})$.

As the left hand side of this inequality converges to ∞ when t tends to 0, we find $t_0\in(0,\hat{t})$, depending only on λ, κ, ξ_1 and ξ_2, so that

$$C_0\leq\int_{\mathscr{C}_{t_0}}|\nabla w(x)|^2\,dx.$$

Here and until the end of this proof $C_0>0$ is a generic constant only depending on n, λ, κ ξ_1 and ξ_2.

Set $\rho=t_0/4$. Then it is straightforward to check that, for any $x\in\overline{\mathscr{C}_{t_0}}$,

$$\overline{B}(x,\rho)\subset\{y\in\mathbb{R}^n;\ 3t_0/4\leq|y-\xi_2|\leq 5\tilde{t}/4\}\subset B(0,R).$$

Since $\overline{\mathscr{C}_{t_0}}$ is a compact we find a positive integer N, only depending on λ, κ, ξ_1 and ξ_2, and $x_j\in\overline{\mathscr{C}_{t_0}}$, $j=1,\cdots,N$, so that

$$\overline{\mathscr{C}_{t_0}}\subset\bigcup_{j=1}^N B(x_j,\rho).$$

Hence

$$C_0\leq\int_{\cup_{j=1}^N B(x_j,\rho)}|\nabla w(x)|^2\,dx.$$

Pick then $x^*\in\{x_j,\ 1\leq j\leq N\}$ in such a way that

$$\int_{B(x^*,\rho)}|\nabla w(x)|^2\,dx=\max_{1\leq j\leq N}\int_{B(x_j,\rho)}|\nabla w(x)|^2\,dx.$$

Therefore

$$C_0\leq\int_{B(x^*,\rho)}|\nabla w(x)|^2\,dx,$$

which finishes the proof. □

3. Uniform lower bound for the gradient

In this section we derive a polynomial lower bound of the local L^2-norm of the gradient of solutions of $L_{\sigma,0}u=0$. In a first step we derive, via a three-ball inequality for the gradient, a uniform lower bound of negative exponential type. We use then in a second step an argument based on the so-called frequency function in order to improve this lower bound.

Let $\mathcal{O}$ be a Lipschitz bounded domain of $\mathbb{R}^n$ and $\sigma\in C^{0,1}(\overline{\mathcal{O}})$ satisfying

$$\varkappa^{-1}\leq\sigma\quad\text{and}\quad\|\sigma\|_{C^{0,1}(\overline{\mathcal{O}})}\leq\varkappa,\tag{3.1}$$

for some fixed constant $\varkappa>1$.

For simplicity convenience we use in the sequel the notation L_σ instead of $L_{\sigma,0}$.

3.1. **Preliminary lower bound.** We need hereafter the following three-ball inequality for the gradient.

Theorem 3.1. *Let $0 < k < \ell < m$ be real. There exist constants $C > 0$ and $0 < \gamma < 1$, only depending on n, k, ℓ, m and $\varkappa$, so that, for any $v \in H^1(\mathcal{O})$ satisfying $L_\sigma v = 0$, $y \in \mathcal{O}$ and $0 < r < dist(y, \partial\mathcal{O})/m$, we have*

$$C\|\nabla v\|_{L^2(B(y,\ell r))} \leq \|\nabla v\|^{\gamma}_{L^2(B(y,kr))}\|\nabla v\|^{1-\gamma}_{L^2(B(y,mr))}.$$

A proof of this theorem can be found in [10] .

Define the geometric distance d_g^D on the bounded domain D of $\mathbb{R}^n$ by

$$d_g^D(x,y) = \inf\left\{\ell(\psi);\ \psi : [0,1] \to D \text{ Lipschitz path joining } x \text{ to } y\right\},$$

where

$$\ell(\psi) = \int_0^1 |\dot{\psi}(t)|dt$$

is the length of ψ.

Note that according to Rademacher's theorem any Lipschitz continuous function $\psi : [0,1] \to D$ is almost everywhere differentiable with $|\dot{\psi}(t)| \leq k$ a.e. $t \in [0,1]$, where k is the Lipschitz constant of ψ.

Lemma 3.1. *Let D be a bounded Lipschitz domain of $\mathbb{R}^n$. Then $d_g^D \in L^\infty(D \times D)$ and there exists a constant $\mathfrak{c}_D > 0$ so that*

$$|x - y| \leq d_g^D(x,y) \leq \mathfrak{c}_D|x - y|, \quad x, y \in D. \tag{3.2}$$

This lemma is due to Tom ter Elst. We provided its proof in [11, Appendix A].

In this subsection we use the following notations

$$\mathcal{O}^\delta = \{x \in \mathcal{O};\ \mathrm{dist}(x, \partial\mathcal{O}) > \delta\}$$

and

$$\chi(\mathcal{O}) = \max\{\delta > 0;\ \mathcal{O}^\delta \neq \emptyset\}.$$

Let $\delta \in (0, \chi(\mathcal{O})/3)$, $x_0 \in \mathcal{O}^{3\delta}$. Let η and M satisfy $0 < \eta < M$. Define then

$$\begin{aligned}\mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta) = \big\{u \in H^1(\mathcal{O});\ L_\sigma u = 0 \text{ in } \mathcal{O},\\ \|\nabla u\|_{L^2(\mathcal{O})} \leq M,\ \|\nabla u\|_{L^2(B(x_0,\delta))} \geq \eta\big\}.\end{aligned} \tag{3.3}$$

Lemma 3.2. *There exists a constant $c > 0$ only depending on n, η, $\varkappa$ and M so that, for any $u \in \mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta)$ and $x \in \mathcal{O}^{3\delta}$, we have*

$$e^{-c|x-x_0|/\delta} \leq \|\nabla u\|_{L^2(B(x,\delta))}. \tag{3.4}$$

Proof. Pick $u \in \mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta)$. Let $x \in \mathcal{O}^{3\delta}$ and $\psi : [0,1] \to \mathcal{O}$ be a Lipschitz path joining $x = \psi(0)$ to $x_0 = \psi(1)$, so that $\ell(\psi) \leq 2d_g(x_0, x)$. Here and henceforth for simplicity convenience we use $d_g(x_0, x)$ $\mathbf{d}_g$ and $\mathfrak{c}$ instead of $d_g^{\mathcal{O}}(x_0, x)$, $\mathbf{d}_g^{\mathcal{O}}$ and $\mathfrak{c}_{\mathcal{O}}$.

Let $t_0 = 0$ and $t_{k+1} = \inf\{t \in [t_k, 1];\ \psi(t) \notin B(\psi(t_k), \delta)\}$, $k \geq 0$. We claim that there exists an integer $N \geq 1$ verifying $\psi(1) \in B(\psi(t_N), \delta)$. If not, we would have $\psi(1) \notin B(\psi(t_k), \delta)$ for any $k \geq 0$. As the sequence (t_k) is non decreasing and bounded from above by 1, it converges to $\hat{t} \leq 1$. In particular, there exists an integer $k_0 \geq 1$ so that $\psi(t_k) \in B\left(\psi(\hat{t}), \delta/2\right)$, $k \geq k_0$. But this contradicts the fact that $|\psi(t_{k+1}) - \psi(t_k)| \geq \delta$, $k \geq 0$.

Let us check that $N \leq N_0$ where N_0 depends only on $|x - x_0|$ and δ. Pick $1 \leq j \leq n$ so that

$$\max_{1\leq i\leq n} |\psi_i(t_{k+1}) - \psi_i(t_k)| = |\psi_j(t_{k+1}) - \psi_j(t_k)|\,,$$

where ψ_i is the ith component of ψ. Then

$$\delta \leq n\,|\psi_j(t_{k+1}) - \psi_j(t_k)| = n\left|\int_{t_k}^{t_{k+1}} \dot{\psi}_j(t)dt\right| \leq n\int_{t_k}^{t_{k+1}} |\dot{\psi}(t)|dt.$$

Consequently, where $t_{N+1} = 1$,

$$(N+1)\delta \leq n\sum_{k=0}^{N}\int_{t_k}^{t_{k+1}} |\dot{\psi}(t)|dt = n\ell(\psi) \leq 2nd_g(x_0, x) \leq 2n\mathfrak{c}|x - x_0|.$$

Therefore

$$N \leq N_0 = \left[\frac{2n\mathfrak{c}|x - x_0|}{\delta}\right].$$

Let $y_0 = x$ and $y_k = \psi(t_k)$, $1 \leq k \leq N$. If $|z - y_{k+1}| < \delta$, then $|z - y_k| \leq |z - y_{k+1}| + |y_{k+1} - y_k| < 2\delta$. In other words $B(y_{k+1}, \delta) \subset B(y_k, 2\delta)$.

We get from Theorem 3.1

$$\|\nabla u\|_{L^2(B(y_j,2\delta))} \leq C\|\nabla u\|^{1-\gamma}_{L^2(B(y_j,3\delta))}\|\nabla u\|^{\gamma}_{L^2(B(y_j,\delta))}, \quad 0 \leq j \leq N, \tag{3.5}$$

the constants $C > 0$ and $0 < \gamma < 1$ only depend on n and $\varkappa$.

Set $I_j = \|\nabla u\|_{L^2(B(y_j,\delta))}$, $0 \leq j \leq N$ and $I_{N+1} = \|\nabla u\|_{L^2(B(x_0,\delta))}$. Since $B(y_{j+1}, \delta) \subset B(y_j, 2\delta)$, $1 \leq j \leq N-1$, estimate (3.5) implies

$$I_{j+1} \leq CM_0^{1-\gamma} I_j^{\gamma}, \;\; 0 \leq j \leq N, \tag{3.6}$$

where we set $M_0 = \max(M, 1)$.

Let $C_1 = C^{1+\gamma+\ldots+\gamma^{N+1}}$ and $\beta = \gamma^{N+1}$. Then by a simple induction argument estimate (3.6) yields

$$I_{N+1} \leq C_1 M_0^{1-\beta} I_0^{\beta}. \tag{3.7}$$

Without loss of generality, we assume in the sequel that $C \geq 1$ in (3.6). Using that $N \leq N_0$, we have

$$\beta \geq \beta_0 = s^{N_0+1},$$
$$C_1 \leq C^{\frac{1}{1-s}},$$
$$\left(\frac{I_0}{M_0}\right)^{\beta} \leq \left(\frac{I_0}{M_0}\right)^{\beta_0}.$$

These estimates in (3.7) give

$$\frac{I_{N+1}}{M_0} \leq C^{\frac{1}{1-\gamma}}\left(\frac{I_0}{M_0}\right)^{\gamma^{N_0+1}},$$

from which we deduce that

$$\|\nabla u\|_{L^2(B(x_0,\delta))} \leq C^{\frac{1}{1-\gamma}} M_0^{1-\gamma^{N_0+1}}\|\nabla u\|^{\gamma^{N_0+1}}_{L^2(B(x,\delta))}.$$

But $M_0 \geq 1$. Whence

$$\eta \leq \|\nabla u\|_{L^2(B(x_0,\delta))} \leq C^{\frac{1}{1-\gamma}} M_0\|\nabla u\|^{\gamma^{N_0+1}}_{L^2(B(x,\delta))}.$$

The expected inequality follows readily from this last estimate. □

3.2. **An estimate for the frequency function.** Some tools in the present section are borrowed from [14, 15, 18]. Let $u \in H^1(\mathcal{O})$ and $\sigma \in C^{0,1}(\overline{\mathcal{O}})$ satisfying the bounds (3.1). We recall that the usual frequency function, relative to the operator L_σ, associated to u is defined by

$$N(u)(x_0, r) = \frac{rD(u)(x_0, r)}{H(u)(x_0, r)},$$

provided that $\overline{B}(x_0, r) \subset \mathcal{O}$, with

$$D(u)(x_0, r) = \int_{B(x_0,r)} \sigma(x)|\nabla u(x)|^2 dx,$$
$$H(u)(x_0, r) = \int_{\partial B(x_0,r)} \sigma(x)u^2(x)dS(x).$$

Define also

$$K(u)(x_0, r) = \int_{B(x,r)} \sigma(x)u(x)^2 dx.$$

Prior to study the properties of the frequency function, we prove some technical lemmas.

Fix $u \in H^2(\mathcal{O})$ so that $L_\sigma u = 0$ in $\mathcal{O}$ and, for simplicity convenience, we drop in the sequel the dependence on u of N, D, H and K.

Lemma 3.3. *For $x_0 \in \mathcal{O}^\delta$ and $0 < r < \delta$ we have*

$$\partial_r H(x_0, r) = \frac{n-1}{r} H(x_0, r) + \tilde{H}(x_0, r) + 2D(x_0, r), \tag{3.8}$$

$$\partial_r D(x_0, r) = \frac{n-2}{r} D(x_0, r) + \tilde{D}(x_0, r) + 2\hat{H}(x_0, r). \tag{3.9}$$

Here

$$\tilde{H}(x_0, r) = \int_{\partial B(x_0,r)} u^2 \nabla\sigma(x) \cdot \nu(x) dS(x),$$
$$\hat{H}(x_0, r) = \int_{\partial B(x_0,r)} \sigma(x)(\partial_\nu u(x))^2 dS(x),$$
$$\tilde{D}(x_0, r) = \int_{B(x_0,r)} |\nabla u(x)|^2 \nabla\sigma(x) \cdot (x - x_0) dx.$$

Proof. Pick $x_0 \in \mathcal{O}^\delta$ and $0 < r < \delta$. A simple change of variable yields

$$H(x_0, r) = \int_{B(0,1)} \sigma(x_0 + ry)u^2(x_0 + ry)r^{n-1} dS(y).$$

Hence

$$\begin{aligned}\partial_r H(x_0,r) &= \frac{n-1}{r}H(x_0,r) + \int_{B(0,1)} \nabla(\sigma u^2)(x_0+ry)\cdot y r^{n-1} dS(y)\\ &= \frac{n-1}{r}H(x_0,r) + \int_{B(0,1)} u^2\nabla\sigma(x_0+ry)\cdot y r^{n-1} dS(y)\\ &\qquad + \int_{\partial B(0,1)} \sigma\nabla(u^2)(x_0+ry)\cdot y r^{n-1} dS(y)\\ &= \frac{n-1}{r}H(x_0,r) + \int_{\partial B(x_0,r)} u^2\nabla\sigma(x)\cdot\nu(x) dS(x)\\ &\qquad + \int_{\partial B(x_0,r)} \sigma(x)\nabla(u^2)(x)\cdot\nu(x) dS(x)\\ &= \frac{n-1}{r}H(x_0,r) + \tilde{H}(x_0,r) + \int_{\partial B(x_0,r)} \sigma\nabla(u^2)(x)\cdot\nu(x) dS(x).\end{aligned}$$

Identity (3.8) will follow if we prove

$$2D(x_0,r) = \int_{\partial B(x_0,r)} \sigma\nabla(u^2)(x)\cdot\nu(x) dS(x).$$

To do so we note that, since $\operatorname{div}(\sigma\nabla u) = 0$, we have

$$\operatorname{div}(\sigma\nabla(u^2)) = 2u\operatorname{div}(\sigma\nabla u) + 2\sigma|\nabla u|^2 = 2\sigma|\nabla u|^2.$$

Applying the divergence theorem we get

$$\begin{aligned}2D(x_0,r) &= \int_{B(x_0,r)} \operatorname{div}(\sigma(x)\nabla(u^2)(x)) dx\\ &= \int_{\partial B(x_0,r)} \sigma(x)\nabla(u^2)(x)\cdot\nu(x) dS(x).\end{aligned} \tag{3.10}$$

By a change of variable we have

$$D(x_0,r) = \int_0^r \int_{\partial B(0,1)} \sigma(x_0+ty)|\nabla u(x_0+ty)|^2 t^{n-1} dS(y) dt.$$

Hence

$$\begin{aligned}\partial_r D(x_0,r) &= \int_{\partial B(0,1)} \sigma(x_0+ry)|\nabla u(x_0+ty)|^2 r^{n-1} dS(y)\\ &= \int_{\partial B(x_0,r)} \sigma(x)|\nabla u(x)|^2 dS(x)\\ &= \frac{1}{r}\int_{\partial B(x_0,r)} \sigma(x)|\nabla u(x)|^2 (x-x_0)\cdot\nu(x) dS(x).\end{aligned}$$

Then an application of the divergence theorem gives

$$\partial_r D(x_0,r) = \frac{1}{r}\int_{B(x_0,r)} \operatorname{div}(\sigma(x)|\nabla u(x)|^2(x-x_0)) dx.$$

Therefore

$$\partial_r D(x_0,r) = \frac{1}{r}\int_{B(x_0,r)} |\nabla u(x)|^2 \mathrm{div}(\sigma(x)(x-x_0))dx + \frac{1}{r}\int_{B(x_0,r)} \sigma(x)(x-x_0)\cdot\nabla(|\nabla u(x)|^2)dx$$

implying

$$\partial_r D(x_0,r) = \frac{n}{r}D(x_0,r) + \frac{1}{r}\tilde{D}(x_0,r) + \frac{1}{r}\int_{B(x_0,r)} \sigma(x)(x-x_0)\cdot\nabla(|\nabla u(x)|^2)dx. \tag{3.11}$$

On the other hand,

$$\begin{aligned}
&\int_{B(x_0,r)} \sigma(x)(x_j-x_{0,j})\partial_j(\partial_i u(x))^2 dx \\
&\quad = 2\int_{B(x_0,r)} \sigma(x)(x_j-x_{0,j})\partial^2_{ij}u\partial_i u(x)dx \\
&\quad = -2\int_{B(x_0,r)} \partial_i\left[\partial_i u(x)\sigma(x)(x_j-x_{0,j})\right]\partial_j u(x)dx \\
&\qquad\qquad + 2\int_{\partial B(x_0,r)} \sigma(x)\partial_i u(x)(x_j-x_{0,j})\partial_j u(x)\nu_i(x)dS(x) \\
&\quad = -2\int_{B(x_0,r)} \partial^2_{ii}u(x)\sigma(x)(x_j-x_{0,j})\partial_j u(x)dx \\
&\qquad - 2\int_{B(x_0,r)} \partial_i u(x)\partial_j u(x)\partial_i\left[\sigma(x)(x_j-x_{0,j})\right]dx \\
&\qquad\qquad + 2\int_{\partial B(x_0,r)} \sigma(x)\partial_i u(x)(x_j-x_{0,j})\partial_j u(x)\nu_i(x)dS(x).
\end{aligned}$$

Thus taking into account that $\sigma\Delta u = -\nabla\sigma\cdot\nabla u$ we get

$$\int_{B(x_0,r)} \sigma(x)(x-x_0)\cdot\nabla(|\nabla u(x)|^2)dx = -2\int_{B(x_0,r)}\sigma(x)|\nabla u(x)|^2dx + 2r\int_{\partial B(x_0,r)}\sigma(x)(\partial_\nu u(x))^2 dS(x).$$

This identity in (3.11) yields

$$\partial_r D(x_0,r) = \frac{n-2}{r}D(x_0,r) + \frac{1}{r}\tilde{D}(x_0,r) + 2\hat{H}(x_0,r).$$

That is we proved (3.9). □

Lemma 3.4. *We have*

$$K(x_0,r) \le \frac{\delta^n e^{\delta\varkappa^2}}{n}H(x_0,r), \quad x_0\in\mathcal{O}^\delta,\ 0<r<\delta.$$

Proof. Since

$$H(x_0,r) = \frac{1}{r}\int_{\partial B(x_0,r)}\sigma(x)u^2(x)(x-x_0)\cdot\nu(x)dS(x),$$

we get by applying the divergence theorem

$$H(x_0,r)=\frac{1}{r}\int_{B(x_0,r)}\operatorname{div}\left(\sigma(x)u^2(x)(x-x_0)\right)dx. \tag{3.12}$$

Hence

$$\begin{aligned}H'(x_0,r)&=-\frac{1}{r}H(x_0,r)+\frac{1}{r}\int_{\partial B(x_0,r)}\operatorname{div}\left(\sigma(x)u^2(x)(x-x_0)\right)dS(x)\\&=\frac{n-1}{r}H(r)+\int_{\partial B(x_0,r)}\partial_\nu\sigma(x)u^2(x)dS(x)\\&\qquad+2\int_{\partial B(x_0,r)}\sigma(x)\partial_\nu u(x)u(x)dS(x).\end{aligned}$$

But

$$\begin{aligned}&\int_{\partial B(x_0,r)}\sigma(x)\partial_\nu u(x)u(x)dS(x)\\&\qquad=\int_{B(x_0,r)}\operatorname{div}(\sigma(x)\nabla u(x))u+\int_{B(x_0,r)}\sigma(x)|\nabla u|^2dx\\&\qquad=\int_{B(x_0,r)}\sigma(x)|\nabla u(x)|^2dx=D(x_0,r).\end{aligned}$$

Therefore

$$\begin{aligned}H'(x_0,r)&=\frac{n-1}{r}H(x_0,r)+2D(x_0,r)+\int_{\partial B(x_0,r)}\partial_\nu\sigma(x)u^2(x)dS(x)\\&\geq\int_{\partial B(x_0,r)}\partial_\nu\sigma(x)u^2(x)dS(x)\\&\geq\int_{\partial B(x_0,r)}\frac{\partial_\nu\sigma(x)}{\sigma(x)}\sigma(x)u^2(x)dS(x)\geq-\varkappa^2H(x_0,r),\end{aligned}$$

where we used that $H(x_0,r)\geq 0$ and $D(x_0,r)\geq 0$.

Consequently $r\to e^{r\varkappa^2}H(x_0,r)$ is non decreasing and then

$$\begin{aligned}\int_0^r H(x_0,t)t^{n-1}dt&\leq\int_0^r e^{t\varkappa^2}H(x_0,t)t^{n-1}dt\\&\leq\int_0^r e^{r\varkappa^2}H(x_0,r)t^{n-1}dt\leq\frac{r^n}{n}e^{r\varkappa^2}H(x_0,r).\end{aligned}$$

As

$$K(x_0,r)=\int_0^r H(x_0,t)t^{n-1}dt,$$

we end up getting

$$K(x_0,t)\leq\frac{\delta^n e^{\delta\varkappa^2}}{n}H(x_0,r).$$

This completes the proof. □

Now straightforward computations yield, for $x_0\in\mathcal{O}^\delta$ and $0<r<\delta$,

$$\frac{\partial_r N(x_0,r)}{N(x_0,r)}=\frac{1}{r}+\frac{\partial_r D(x_0,r)}{D(x_0,r)}-\frac{\partial_r H(x_0,r)}{H(x_0,r)}. \tag{3.13}$$

Lemma 3.5. *For $x_0 \in \mathcal{O}^\delta$ and $0 < r < \delta$, we have*

$$N(x_0, r) \leq e^{\mu\delta} N(x_0, \delta),$$

with $\mu = \varkappa^2 (1 + \chi(\mathcal{O}))$.

Proof. We have from formulas (3.8) and (3.9) and identity (3.13)

$$\begin{aligned} \frac{\partial_r N(x_0, r)}{N(x_0, r)} &= \frac{\tilde{D}(x_0, r)}{D(x_0, r)} - \frac{\tilde{H}(x_0, r)}{H(x_0, r)} + 2\frac{\hat{H}(x_0, r)}{D(x_0, r)} - 2\frac{D(x_0, r)}{H(x_0, r)} \\ &= \frac{\tilde{D}(x_0, r)}{D(x_0, r)} - \frac{\tilde{H}(x_0, r)}{H(x_0, r)} + 2\frac{\hat{H}(x_0, r)H(x_0, r) - D(x_0, r)^2}{D(x_0, r)H(x_0, r)}. \end{aligned} \tag{3.14}$$

But from (3.10) we have

$$D(x_0, r) = \int_{\partial B(x_0, r)} \sigma(x) u(x) \partial_\nu u(x) dS(x).$$

Then we find by applying Cauchy-Schwarz's inequality

$$D(x_0, r)^2 \leq \left(\int_{\partial B(x_0, r)} \sigma(x) u^2(x) dS(x) \right) \left(\int_{\partial B(x_0, r)} \sigma(x) (\partial_\nu u)^2(x) dS(x) \right).$$

That is

$$D^2(x_0, r) \leq H(x_0, r) \hat{H}(x_0, r). \tag{3.15}$$

This and (3.14) lead

$$\frac{\partial_r N(x_0, r)}{N(x_0, r)} \geq \frac{\tilde{D}(x_0, r)}{D(x_0, r)} - \frac{\tilde{H}(x_0, r)}{H(x_0, r)}. \tag{3.16}$$

On the other hand

$$\left| \tilde{H}(x_0, r) \right| \leq \varkappa \|\nabla a\|_\infty H(x_0, r) \leq \varkappa^2 H(x_0, r), \tag{3.17}$$

and similarly

$$\left| \tilde{D}(x_0, r) \right| \leq \varkappa^2 \delta D(x_0, r). \tag{3.18}$$

In light of (3.16), (3.17) and (3.18), we derive

$$\frac{\partial_r N(x_0, r)}{N(x_0, r)} \geq -\mu,$$

that is to say

$$\partial_r (e^{\mu r} N(x_0, r)) \geq 0.$$

Consequently

$$N(x_0, r) \leq e^{\mu(\delta - r)} N(x_0, \delta) \leq e^{\mu\delta} N(x_0, \delta),$$

as expected. □

3.3. **Polynomial lower bound.**

Lemma 3.6. *There exist a constant $c > 0$, only depending on n, $\varkappa$, η, δ and M, so that, for any $u \in \mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$, we have*

$$\|N(u)(x,\cdot)\|_{L^\infty(0,\delta)} \leq ce^{c|x-x_0|}, \quad x \in \mathcal{O}^\delta,$$

Proof. In this proof $c > 0$ is a generic constant only depending on n, $\varkappa$, η, δ and M.

Pick $x \in \mathcal{O}^\delta$. Then from Lemma 3.2

$$\|\nabla u\|_{L^2(B(x,\delta/3))} \geq e^{-c|x-x_0|}.$$

Whence

$$K(u)(x,\delta) = \|u\|^2_{L^2(B(x,\delta))} \geq e^{-c|x-x_0|} \tag{3.19}$$

by Caccioppoli's inequality.

In light of Lemma 3.4, we derive from (3.19)

$$H(u)(x,\delta) \geq e^{-c|x-x_0|}. \tag{3.20}$$

But we have from Lemma 3.5

$$N(x,r) \leq \tilde{c}\frac{\|\nabla u\|_{L^2(\mathcal{O})}}{H(u)(x,\delta)}, \quad 0 < r < \delta,$$

the constant $\tilde{c}$ only depends on $\varkappa$ and $\mathcal{O}$. This inequality and (3.20) give the expected one. □

Proposition 3.1. *There exist two constants $c > 0$ and $\tilde{c} > 0$, only depending on n, $\varkappa$, η, δ and M, so that if $u \in \mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$ then*

$$\tilde{c}^{-1}e^{-\tilde{c}|x-x_0|}\left(\frac{r}{\delta}\right)^{ce^{c|x-x_0|}+n-1} \leq \|u\|^2_{L^2(B(x,r))}, \quad x \in \mathcal{O}^\delta,\ 0 < r < \delta.$$

Proof. In this proof $c > 0$ is a generic constant only depending on n, $\varkappa$, η, δ and M.

Observing that, where $H = H(u)$,

$$\partial_r\left(\ln\frac{H(x,r)}{r^{n-1}}\right) = \frac{\partial_r H(x,r)}{H(x,r)} - \frac{n-1}{r},$$

we get from Lemma 3.6, (3.8) and the fact that $|\tilde{H}(x,r)| \leq \varkappa^2 H(x,r)$,

$$\partial_r\left(\ln\frac{H(x,r)}{r^{n-1}}\right) \leq \varkappa^2 + \frac{N(x,r)}{r} \leq \varkappa^2 + \frac{ce^{c|x-x_0|}}{r}, \quad 0 < r < \delta,$$

Thus

$$\int_{sr}^{s\delta}\partial_t\left(\ln\frac{H(x,t)}{t^{n-1}}\right)dt = \ln\frac{H(x,s\delta)r^{n-1}}{H(x,sr)\delta^{n-1}} \leq \varkappa^2(\delta-r)s + ce^{c|x-x_0|}\ln\frac{\delta}{r},$$

for $0 < s < 1$ and $0 < r < \delta$.

Hence

$$H(x,s\delta) \leq e^{\varkappa^2\delta}\left(\frac{\delta}{r}\right)^{ce^{c|x-x_0|}+n-1} H(x,sr),$$

and then

$$\begin{aligned}\|u\|^2_{L^2(B(x,\delta))} &= \delta^{n-1}\int_0^1 H(x,s\delta)s^{n-1}ds\\ &\leq e^{\varkappa^2\delta}\left(\frac{\delta}{r}\right)^{ce^{c|x-x_0|}+n-1} r^{n-1}\int_0^1 H(x,rs)s^{n-1}ds\\ &\leq \delta^{n-1}e^{\varkappa^2\delta}\left(\frac{\delta}{r}\right)^{ce^{c|x-x_0|}+n-1}\|u\|^2_{L^2(B(x,r))}.\end{aligned}$$

Combined with (3.19) this estimate yields in a straighforward manner

$$\tilde{c}^{-1}e^{-\tilde{c}|x-x_0|}\left(\frac{r}{\delta}\right)^{ce^{c|x-x_0|}+n-1}\leq \|u\|^2_{L^2(B(x,r))}.$$

This is the expected inequality. □

For a bounded domain D, we denote the first non zero eigenvalue of the Laplace-Neumann operator on D by $\mu_2(D)$. Since $\mu_2(B(x_0,r))=\mu_2(B(0,1))/r^2$, we get by applying Poincaré-Wirtinger's inequality

$$\begin{aligned}\|w-\{w\}\|^2_{L^2(B(x,r)} &\leq \frac{1}{\mu_2(B(x,r))}\|\nabla w\|^2_{L^2(B(x,r))}\\ &\leq \frac{r^2}{\mu_2(B(0,1))}\|\nabla w\|^2_{L^2(B(x,r))},\end{aligned}\tag{3.21}$$

for any $w\in H^1(B(x,r))$, where $\{w\}=\frac{1}{|B(x,r)|}\int_{B(x,r)}w(x)dx$.

Noting that $\mathscr{S}(\mathcal{O},x_0,M,\eta,\delta/3)$ is invariant under the transformation $u\to u-\{u\}$, we can state the following consequence of Proposition 3.1

Corollary 3.1. *There exist two constants $c>0$ and $\tilde{c}>0$, only depending on n, $\varkappa$, η, δ and M, so that if $u\in\mathscr{S}(\mathcal{O},x_0,M,\eta,\delta/3)$ then*

$$\tilde{c}^{-1}e^{-\tilde{c}|x-x_0|}\left(\frac{r}{\delta}\right)^{ce^{c|x-x_0|}+n-1}\leq\|\nabla u\|^2_{L^2(B(x,r))},\quad x\in\mathcal{O}^\delta,\ 0<r<\delta.$$

The following consequence of the preceding corollary will be useful in the proof of Theorem 1.1.

Lemma 3.7. *Let $\omega\Subset\mathcal{O}$, $0<\beta<1$ and set $\delta=dist(\omega,\partial\mathcal{O})$. Then there exist two constants $C>0$ and $c>0$ only depending on n, $\varkappa$, η, δ and M so that, for any $u\in\mathscr{S}(\mathcal{O},x_0,M,\eta,\delta/3)$ and $f\in C^{0,\beta}(\mathcal{O})$, we have*
(i) for any $x\in\omega$,

$$|f(x)|\leq\tilde{c}e^{\tilde{c}|x-x_0|}\|f\|^{1-\mu_{c,\beta}(|x-x_0|)}_{C^{0,\beta}(\overline{\mathcal{O}})}\|f|\nabla u|^2\|^{\mu_{c,\beta}(|x-x_0|)}_{L^1(\mathcal{O})},\tag{3.22}$$

where

$$\mu_{c,\beta}(s)=\frac{\beta}{ce^{c|x-x_0|}+\beta+n-1},\quad s\geq 0.$$

(ii) if

$$\gamma=\frac{\beta}{ce^{c\mathfrak{m}}+\beta+n-1},$$

with $\mathfrak{m}=\max\{|x-x_0|;\ x\in\overline{\omega}\}$, then

$$\|f\|_{L^\infty(\omega)}\leq\tilde{c}e^{\tilde{c}\mathfrak{m}}\|f\|^{1-\gamma}_{C^{0,\beta}(\overline{\mathcal{O}})}\|f|\nabla u|^2\|^{\gamma}_{L^1(\mathcal{O})}\tag{3.23}$$

Proof. By homogeneity it is enough to consider those functions $f \in C^{0,\beta}(\mathcal{O})$ satisfying $\|f\|_{C^{0,\beta}(\mathcal{O})} = 1$.

Let $u \in \mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$ and $f \in C^{0,\beta}(\mathcal{O})$ satisfying $\|f\|_{C^{0,\beta}(\mathcal{O})} = 1$. Pick then $x \in \overline{\omega}$. From Corollary 3.1 there exist two constants $c > 0$ and $\tilde{c} > 0$, only depending on n, $\varkappa$, η, δ and M, so that

$$\tilde{c}^{-1} e^{-\tilde{c}|x-x_0|} \left(\frac{r}{\delta}\right)^{ce^{c|x-x_0|}+n-1} \leq \|\nabla u\|_{L^2(B(x,r))}, \quad 0 < r < \delta. \tag{3.24}$$

On the other hand it is straightforward to check that

$$|f(x)| \leq |f(y)| + r^\beta, \quad y \in B(x,r).$$

Whence

$$|f(x)| \int_{B(x,r)} |\nabla u(y)|^2 dy \leq \int_{B(x,r)} |f(y)||\nabla u(y)|^2 dy + r^\beta \int_{B(x,r)} |\nabla u(y)|^2 dy.$$

That is we have

$$|f(x)| \|\nabla u\|^2_{L^2(B(x,r)} \leq \|f|\nabla u|^2\|_{L^1(B(x,r))} + r^\beta \|\nabla u\|^2_{L^2(B(x,r))}.$$

Since u is non constant, $\|\nabla u\|^2_{L^2(B(x,r))} \neq 0$ for any $0 < r < \delta$ by the unique continuation property. Therefore

$$|f(x)| \leq \frac{\|f|\nabla u|^2\|_{L^1(B(x,r))}}{\|\nabla u\|^2_{L^2(B(x,r))}} + r^\beta, \quad 0 < r < \delta.$$

This and (3.24) entail

$$|f(x)| \leq \tilde{c} e^{\tilde{c}|x-x_0|} \left(\frac{\delta}{r}\right)^{ce^{c|x-x_0|}+n-1} \|f|\nabla u|^2\|_{L^1(B(x,r))} + r^\beta, \quad 0 < r < \delta.$$

Equivalently, we have

$$|f(x)| \leq \tilde{c} e^{\tilde{c}|x-x_0|} \left(\frac{1}{s}\right)^{ce^{c|x-x_0|}+n-1} \|f|\nabla u|^2\|_{L^1(B(x,r))} + \delta^\beta s^\beta, \quad 0 < s < 1. \tag{3.25}$$

Changing $\tilde{c}$ if necessary, we mat assume that $\tilde{c} e^{\tilde{c}|x-x_0|} \geq \delta^\beta$.

We introduce the following temporary notations

$$\aleph = \|f|\nabla u|^2\|_{L^1(B(x,r))}, \quad \alpha = ce^{c|x-x_0|}$$

and

$$\Lambda = \tilde{c} e^{\tilde{c}|x-x_0|}.$$

Then (3.25) takes the form

$$|f(x)| \leq \Lambda \left(\frac{\aleph}{s^\alpha} + s^\beta\right), \quad 0 < s < 1. \tag{3.26}$$

If $0 < \aleph < 1$ we can take in (3.26) $s = \aleph^{1/(\alpha+\beta)}$. This choice yields

$$|f(x)| \leq 2\Lambda \aleph^{\beta/(\alpha+\beta)}. \tag{3.27}$$

When $\aleph \geq 1$ we have

$$|f(x)| \leq 1 \leq \aleph \leq \aleph \aleph^{\beta/(\alpha+\beta)}. \tag{3.28}$$

Inequality (3.23) follows then from (3.27) and (3.28).

To prove (ii), we see that (3.25) yields

$$\|f\|_{L^\infty(\omega)} \leq \tilde{c}e^{\tilde{c}\mathfrak{m}} \left(\frac{1}{s}\right)^{ce^{c\mathfrak{m}}+n-1} \|f|\nabla u|^2\|_{L^1(B(x,r))} + \delta^\beta s^\beta, \quad 0<s<1.$$

We then mimic the preceding proof to derive inequality (3.23). □

4. Proof Theorem 1.1

Pick $(a,b),(\tilde{a},\tilde{b}) \in \mathcal{D}(\lambda,\kappa)$ and let $u_j = G_{a,b}(\cdot,\xi_j)$ and $\tilde{u}_j = G_{\tilde{a},\tilde{b}}(\cdot,\xi_j)$, $j=1,2$. As we have seen before $w = u_2/u_1$ is the solution of the equation

$$\operatorname{div}(\sigma\nabla w) = 0 \quad \text{in } \mathbb{R}^n \setminus \{\xi_1,\xi_2\},$$

where

$$\sigma = au_1^2 = \frac{av_1^2}{b^2}.$$

Similarly, $\tilde{w} = \tilde{u}_2/\tilde{u}_1$ is the solution of the equation

$$\operatorname{div}(\tilde{\sigma}\nabla \tilde{w}) = 0 \quad \text{in } \mathbb{R}^n \setminus \{\xi_1,\xi_2\},$$

where

$$\tilde{\sigma} = \tilde{a}\tilde{u}_1^2 = \frac{\tilde{a}\tilde{u}_1^2}{\tilde{b}^2}.$$

We know from Lemma 2.4 that there exist $\rho>0$, $x^* \in \mathbb{R}\setminus\{\xi_1,\xi_2\}$ and $\eta>0$ only depending on ξ_1, ξ_2, λ, κ, and n so that $B(x^*,\rho) \Subset \mathbb{R}\setminus\{\xi_1,\xi_2\}$ and

$$\eta \leq \|\nabla w\|_{L^2(B(x^*,\rho))}. \tag{4.1}$$

Fix then a bounded domain $\mathcal{Q}$ of $\mathbb{R}^n\setminus\{\xi_1,\xi_2\}$ is such a way that $\Omega\cup B(x^*,\rho) \Subset \mathcal{Q}$, and set

$$\delta = \operatorname{dist}(\Omega\cup B(x^*,\rho), \partial\mathcal{Q}).$$

According to Lemma 2.3

$$\|\nabla w\|_{L^2(\mathcal{O})} \leq M, \tag{4.2}$$

the constant M only depending on λ, κ, ξ_1, ξ_2, and Ω.

Now if $\rho \leq \delta/3$ then (4.1) yields obviously

$$\eta \leq \|\nabla w\|_{L^2(B(x_0,\delta/3))}, \tag{4.3}$$

with η as in (4.1).

When $\rho > \delta/3$ we can use the three-ball inequality in order to get

$$\tilde{C}\|\nabla w\|_{L^2(B(x^*,\rho))} \leq \|\nabla w\|^s_{L^2(B(x_0,\delta/3))} \|\nabla w\|^{1-s}_{L^2(B(x^*,\rho+\delta/3))},$$

the constants $\tilde{C}$ and $0<s<1$ only depend on λ, κ, Ω, δ, ξ_1 and ξ_2. This and (4.1) imply that that (4.1) holds again in the present case with a constant C that can depend also on δ, ξ_1, and ξ_2.

In light of (4.2) and (4.3), we can assert that $w \in \mathscr{S}(\mathcal{Q},x^*,M,\eta,\delta/3)$, where $\mathscr{S}(\mathcal{Q},x^*,M,\eta,\delta/3)$ was introduced in (3.3).

Lemma 4.1. *Let $0<\theta<1/2$. Then*

$$C\|(\sigma-\tilde{\sigma})|\nabla w|^2\|_{L^1(\Omega)} \leq \|w-\tilde{w}\|^{\theta/(2+\theta)}_{L^2(\Omega)} + \|\sigma-\tilde{\sigma}\|_{L^\infty(\Gamma)}, \tag{4.4}$$

the constant $C>0$ only depends on n, Ω, κ, λ, θ, ξ_1 and ξ_2.

Proof. Clearly, if $\zeta = \sigma - \tilde{\sigma}$ and $u = w - \tilde{w}$, then

$$\mathrm{div}(\zeta \nabla u) = \mathrm{div}(\tilde{\sigma} \nabla u).$$

Recall that sgn_0 is the sign function defined on $\mathbb{R}$ by: $\mathrm{sgn}_0(t) = -1$ if $t < 1$, $\mathrm{sgn}_0(0) = 0$ and $\mathrm{sgn}_0(t) = 1$ if $t > 0$. Since

$$\begin{aligned} \mathrm{div}(|\zeta| \nabla w) &= \nabla |\zeta| \cdot \nabla w + |\zeta| \Delta w \\ &= \mathrm{sgn}_0(\zeta) \nabla \zeta \cdot \nabla w + \mathrm{sgn}_0(\zeta) \zeta \Delta w \\ &= \mathrm{sgn}_0(\zeta) \mathrm{div}(\zeta \nabla w) = \mathrm{sgn}_0(\zeta) \mathrm{div}(\tilde{\sigma} \nabla u), \end{aligned}$$

we get by integrating by parts

$$\begin{aligned} \int_\Omega |\zeta| \nabla w|^2 dx &= - \int_\Omega \mathrm{div}(|\zeta| \nabla w) w dx + \int_\Gamma |\zeta| w \partial_\nu w dS(x) \\ &= - \int_\Omega \mathrm{sgn}_0(\zeta) \mathrm{div}(\tilde{\sigma} \nabla u) w dx + \int_\Gamma |\zeta| w \partial_\nu w dS(x). \end{aligned} \tag{4.5}$$

Thus

$$\int_\Omega |\zeta| \nabla w|^2 dx \leq C \left(\|u\|_{H^2(\Omega)} + \|\zeta\|_{L^\infty(\Gamma)} \right).$$

The previous inequality, the following interpolation inequality

$$\|u\|_{H^2(\Omega)} \leq c_\Omega \|u\|_{L^2(\Omega)}^{\theta/(2+\theta)} \|u\|_{H^{2+\theta}(\Omega)}^{2/(2+\theta)},$$

and Lemma 2.3 give (4.4). □

Fix $x \in \Omega$ and for simplicity convenience we set

$$\mu = \mu_{c,\beta}(|x - x^*|) = \frac{\beta}{c e^{c|x-x^*|} + \beta + n - 1}.$$

Thereafter $\tilde{c}$ is generic constant only depending on n, κ, λ, β, θ, Ω, ξ_1, and ξ_2, and $c > 0$ is a generic constant only depending on n, κ, λ, Ω, ξ_1, and ξ_2.

We have from (3.22) in Lemma 3.7

$$|\tilde{\sigma}(x) - \sigma(x)| \leq \tilde{c} e^{\tilde{c}|x-x^*|} \|\tilde{\sigma} - \sigma\|_{C^{0,\beta}(\overline{\Omega})}^{1-\mu} \|(\sigma - \tilde{\sigma}) |\nabla w|^2\|_{L^1(\Omega)}^{\mu},$$

from which we derive

$$|\tilde{\sigma}(x) - \sigma(x)| \leq \tilde{c} e^{\tilde{c}|x-x^*|} \max\left(1, \|\tilde{\sigma} - \sigma\|_{C^{0,\beta}(\overline{\Omega})}\right) \|(\sigma - \tilde{\sigma}) |\nabla w|^2\|_{L^1(\Omega)}^{\mu}.$$

Combined with Proposition 2.1, this inequality gives

$$|\tilde{\sigma}(x) - \sigma(x)| \leq \tilde{c} e^{\tilde{c}|x-x^*|} \|(\sigma - \tilde{\sigma}) |\nabla w|^2\|_{L^1(\Omega)}^{\mu}.$$

Therefore we obtain in light of Lemma 4.1

$$|\tilde{\sigma}(x) - \sigma(x)| \leq \tilde{c} e^{\tilde{c}|x-x^*|} \left(\|w - \tilde{w}\|_{L^2(\Omega)}^{\theta/(2+\theta)} + \|\sigma - \tilde{\sigma}\|_{L^\infty(\Gamma)} \right)^{\mu}.$$

Since $\tilde{a} = a$ and $\tilde{b} = b$ on Γ and regarding the regularity of u_i and $\tilde{u}_i$, $i = 1, 2$, we finally get

$$|\tilde{\sigma}(x) - \sigma(x)| \leq \tilde{c} e^{\tilde{c}|x-x^*|} \left(\|v_1 - \tilde{v}_1\|_{C(\overline{\Omega})} + \|v_2 - \tilde{v}_2\|_{C(\overline{\Omega})} \right)^{\mu_0}, \tag{4.6}$$

with

$$\mu_0 = \frac{\theta \mu}{2 + \theta}.$$

Lemma 4.2. *Let $0<\theta<1/2$ and $0<\beta<1$. There exist two constants $0<\gamma_1<1$ and $C>0$, only depending on n, β, Ω, κ, λ, θ, Ω, ξ_1 and ξ_2, so that*

$$\|u_1^{-1}-\tilde{u}_1^{-1}\|_{C^{2,\beta}(\overline{\Omega})}\leq C\left(\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})}+\|v_2-\tilde{v}_2\|_{C(\overline{\Omega})}\right)^{\gamma_1}. \tag{4.7}$$

Proof. In this proof $C>0$ is a generic constant only depending on n, Ω, κ, λ, θ, Ω, ξ_1 and ξ_2.

We firstly note that is not hard to check that

$$\begin{aligned}-\operatorname{div}(\sigma\nabla u_1^{-1})&=v_1 \quad \text{in } \Omega,\\ -\operatorname{div}(\tilde{\sigma}\nabla \tilde{u}_1^{-1})&=\tilde{v}_1 \quad \text{in } \Omega.\end{aligned}$$

Hence

$$-\operatorname{div}(\sigma\nabla(u_1^{-1}-\tilde{u}_1^{-1}))=(v_1-\tilde{v}_1)+\operatorname{div}((\sigma-\tilde{\sigma})\nabla\widetilde{u}_1^{-1}) \quad \text{in } \Omega.$$

By the usual Hölder a priori estimate (see [16, Theorem 6.6, page 98])

$$\begin{aligned}C\|u_1^{-1}-\tilde{u}_1^{-1}\|_{C^{2,\beta}(\overline{\Omega})}&\leq \|v_1-\tilde{v}_1\|_{C^{0,\beta}(\overline{\Omega})}\\ &\quad+\|\operatorname{div}((\sigma-\tilde{\sigma})\nabla\widetilde{u}_1^{-1})\|_{C^{0,\beta}(\overline{\Omega})}+\|u_1^{-1}-\tilde{u}_1^{-1}\|_{C^{0,\beta}(\Gamma)}.\end{aligned}$$

Consequently

$$\|u_1^{-1}-\tilde{u}_1^{-1}\|_{C^{2,\beta}(\overline{\Omega})}\leq C\left(\|v_1-\tilde{v}_1\|_{C^{0,\beta}(\overline{\Omega})}+\|\sigma-\tilde{\sigma}\|_{C^{1,\beta}(\overline{\Omega})}\right), \tag{4.8}$$

where we used that

$$\|u_1^{-1}-\tilde{u}_1^{-1}\|_{C^{0,\beta}(\Gamma)}=\|b(v_1^{-1}-\tilde{v}_1^{-1})\|_{C^{0,\beta}(\Gamma)}.$$

On the other hand, since

$$\|\sigma-\tilde{\sigma}\|_{C^{1,1}(\overline{\Omega})}\leq C, \quad \|v_1-\tilde{v}_1\|_{C^{0,\beta}(\overline{\Omega})}\leq C$$

and Ω is $C^{1,1}$, by the interpolation inequality in [16, Lemma 6.35, page 135] we get

$$\|\sigma-\tilde{\sigma}\|_{C^{1,\beta}(\overline{\Omega})}\leq C\|\sigma-\tilde{\sigma}\|^{\tau}_{C(\overline{\Omega})}, \quad \|v_1-\tilde{v}_1\|_{C^{0,\beta}(\overline{\Omega})}\leq \|v_1-\tilde{v}_1\|^{\tau}_{C(\overline{\Omega})}, \tag{4.9}$$

the constant $0<\tau<1$ only depends on β and Ω.

Inequality (4.9) in (4.8) yields

$$\|u_1^{-1}-\tilde{u}_1^{-1}\|_{C^{2,\beta}(\overline{\Omega})}\leq C\left(\|v_1-\tilde{v}_1\|^{\tau}_{C(\overline{\Omega})}+\|\sigma-\tilde{\sigma}\|^{\tau}_{C(\overline{\Omega})}\right). \tag{4.10}$$

On the other hand similarly to (4.6) we have, by using inequality (3.23) instead of (3.22) in Lemma 3.7,

$$\|\tilde{\sigma}-\sigma\|_{L^\infty(\Omega)}\leq C\left(\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})}+\|v_2-\tilde{v}_2\|_{C(\overline{\Omega})}\right)^{\gamma_0}, \tag{4.11}$$

with

$$\gamma_0=\frac{\theta\gamma}{2+\theta},$$

the constant γ is the same as in (ii) of Lemma 3.7.

Whence we get in light of inequalities (4.10) and (4.11), where $\gamma_1=\tau\gamma_0$.

$$\|u_1^{-1}-\tilde{u}_1^{-1}\|_{C^{2,\beta}(\overline{\Omega})}\leq C\left(\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})}+\|v_2-\tilde{v}_2\|_{C(\overline{\Omega})}\right)^{\gamma_1}.$$

This is the expected inequality. □

From the identity

$$a-\tilde{a}=\sigma u_1^{-2}-\tilde{\sigma}\tilde{u}_1{}^{-2}=(\sigma-\tilde{\sigma})u_1^{-2}+\tilde{\sigma}(u_1^{-2}-\tilde{u}_1^{-2}) \\ =(\sigma-\tilde{\sigma})u_1^{-2}+\tilde{\sigma}(u_1^{-1}+\tilde{u}_1^{-1})(u_1^{-1}-\tilde{u}_1^{-1}), \tag{4.12}$$

we get

$$|a(x)-\tilde{a}(x)|\leq C\left(|\sigma(x)-\tilde{\sigma}(x)|+\|u_1-\tilde{u}_1\|_{C(\overline{\Omega})}\right),\quad x\in\Omega.$$

In light of inequalities (4.6) and (4.7) we find

$$\tilde{c}^{-1}e^{-\tilde{c}|x-x^*|}|a(x)-\tilde{a}(x)|\leq\left(\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})}+\|v_2-\tilde{v}_2\|_{C(\overline{\Omega})}\right)^{\mu_0} \\ +\left(\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})}+\|v_2-\tilde{v}_2\|_{C(\overline{\Omega})}\right)^{\gamma_1},\quad x\in\Omega.$$

We have

$$\mu_0=\left(\frac{\theta}{2+\theta}\right)\frac{\beta}{ce^{c|x-x^*|}+\beta+n-1}\geq\gamma_1=\tau\left(\frac{\theta}{2+\theta}\right)\frac{\beta}{ce^{c\mathfrak{m}}+\beta+n-1},$$

where $\mathfrak{m}=\max\{|x-x^*|;\ x\in\overline{\Omega}\}$.

Therefore if $\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})}+\|v_2-\tilde{v}_2\|_{C(\overline{\Omega})}\leq 1$ then

$$|a(x)-\tilde{a}(x)|\leq\tilde{c}e^{\tilde{c}|x-x^*|}\left(\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})}+\|v_2-\tilde{v}_2\|_{C(\overline{\Omega})}\right)^{\mu_0},\ x\in\Omega.$$

Such an estimate is obviously satisfied when $\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})}+\|v_2-\tilde{v}_2\|_{C(\overline{\Omega})}\geq 1$.

We can proceed similarly for $b-\tilde{b}$ since

$$b-\tilde{b}=v_1u_1^{-1}-\tilde{v}_1\tilde{u}_1^{-1}=(v_1-\tilde{v}_1)u_1^{-1}+\tilde{v}_1(u_1^{-1}-\tilde{u}_1^{-1}),$$

The proof of Theorem 1.1 is then complete.

Remark 4.1. The notations in this remark are those of the preceding proof.

We have according to identity (4.12)

$$\|a-\tilde{a}\|_{C^{1,\beta}(\overline{\Omega})}\leq C\left(\|v_1-\tilde{v}_1\|_{C^{0,\beta}(\overline{\Omega})}+\|\sigma-\tilde{\sigma}\|_{C^{1,\beta}(\overline{\Omega})}\right). \tag{4.13}$$

This inequality together with the interpolation inequality (4.9) entail

$$\|a-\tilde{a}\|_{C^{1,\beta}(\overline{\Omega})}\leq C\left(\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})}+\|\sigma-\tilde{\sigma}\|_{C(\overline{\Omega})}\right)^{\tau}.$$

In light of (4.11) this inequality yields

$$\|a-\tilde{a}\|_{C^{1,\beta}(\overline{\Omega})}\leq C\left(\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})}+\|v_2-\tilde{v}_2\|_{C(\overline{\Omega})}\right)^{\tau\gamma_0}.$$

We have similarly

$$\|b-\tilde{b}\|_{C^{0,\beta}(\overline{\Omega})}\leq C\left(\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})}+\|v_2-\tilde{v}_2\|_{C(\overline{\Omega})}\right)^{\tau\gamma_0}.$$

In other words we proved the following result, where we fixed $0<\theta<1/2$ appearing in the preceding proof .

Theorem 4.1. *Let $0<\beta<1$. Then there exists two constants $C>0$ and $0<\mu<1$, only depending on n, ξ_1, ξ_2, κ, λ, Ω, and β so that, for $(a,b),(\tilde{a},\tilde{b})\in\mathcal{D}(\lambda,\kappa)$ satisfying $(a,b)=(\tilde{a},\tilde{b})$ on Γ, we have*

$$\|a-\tilde{a}\|_{C^{1,\beta}(\overline{\Omega})}+\|b-\tilde{b}\|_{C^{0,\beta}(\overline{\Omega})}\leq C\left(\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})}+\|v_2-\tilde{v}_2\|_{C(\overline{\Omega})}\right)^{\mu}.$$

Appendix A. Proof of Lemma 2.2

In this appendix $C > 1$ is a generic constant only depending on n, μ and ν.

For a given constant $\nu > 0$, it is well known that $G_{1,\nu}$, the fundamental solution of the operator $-\Delta + \nu$, is given by $G_{1,\nu}(x,\xi) = \mathcal{G}_{1,\nu}(x-\xi)$, $x, \xi \in \mathbb{R}^n$, with

$$\mathcal{G}_{1,\nu}(x) = (2\pi)^{-n/2}(\sqrt{\nu}/|x|)^{n/2-1}K_{n/2-1}(\sqrt{\nu}|x|).$$

In the particular case $n = 3$, we have $K_{1/2}(z) = \sqrt{\pi/(2z)}e^{-z}$ and therefore

$$\mathcal{G}_{1,\nu}(x) = \frac{e^{-\sqrt{\nu}|x|}}{4\pi|x|},$$

in dimension three.

Let $f \in C_0^\infty(\mathbb{R}^n)$, $\mu > 0$, and $\nu > 0$ be two constants, and denote by u the solution of the equation

$$(-\mu\Delta + \nu)u = f \quad \text{in } \mathbb{R}^n.$$

Then

$$u(x) = \int_{\mathbb{R}^n} G_{\mu,\nu}(x,\xi)f(\xi)d\xi, \quad x \in \mathbb{R}^n. \tag{A.1}$$

We remark that $v(x) = u(\sqrt{\mu}x)$, $x \in \mathbb{R}^n$ satisfies $(-\Delta + \nu)v = f(\sqrt{\mu}\,\cdot)$. Whence

$$\begin{aligned} u(\sqrt{\mu}x) = v(x) &= \int_{\mathbb{R}^n} \mathcal{G}_{1,\kappa}(x-\xi)f(\sqrt{\mu}\xi)d\xi \\ &= \mu^{-n/2}\int_{\mathbb{R}^n} \mathcal{G}_{1,\nu}(x-\xi/\sqrt{\mu})f(\xi)d\xi, \quad x \in \mathbb{R}^n. \end{aligned}$$

Hence

$$u(x) = \mu^{-n/2}\int_{\mathbb{R}^n} \mathcal{G}_{1,\nu}((x-\xi)/\sqrt{c})f(\xi)d\xi, \quad x \in \mathbb{R}^n. \tag{A.2}$$

Comparing (A.1) and (A.2) we find

$$G_{\mu,\nu}(x,\xi) = c^{-n/2}\mathcal{G}_{1,\nu}((x-\xi)/\sqrt{c}), \quad x, \xi \in \mathbb{R}^n.$$

Consequently $G_{\mu,\nu}(x,\xi) = \mathcal{G}_{\mu,\nu}(x-\xi)$ with

$$\mathcal{G}_{\mu,\nu}(x) = (2\pi\mu)^{-n/2}(\sqrt{\nu\mu}/|x|)^{n/2-1}K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}), \quad x \in \mathbb{R}^n. \tag{A.3}$$

By the usual asymptotic formula for modified Bessel functions of the second kind (see for instance [5, 9.7.2, page 378]) we have, when $|x| \to \infty$,

$$K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}) = \left(\frac{\pi\sqrt{\mu}}{2\sqrt{\nu}|x|}\right)^{1/2} e^{-\sqrt{\nu}|x|/\sqrt{\mu}}\left(1 + O(1/|x|)\right),$$

where $O(1/|x|)$ only depends on n, μ and ν.

Consequently, there exits $R > 0$, only depending on n, μ and ν, so that

$$C^{-1}\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{1/2}} \le K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}) \le C\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{1/2}}, \quad |x| \ge R. \tag{A.4}$$

Substituting if necessary R by $\max(R, 1)$, we have

$$\frac{1}{|x|^{n/2-1}} \le \frac{1}{|x|^{1/2}}, \quad |x| \ge R. \tag{A.5}$$

Moreover, we have

$$\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{1/2}} = \left[|x|^{(n-3)/2}e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}\right]\frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}}{|x|^{n/2-1}}, \quad |x| \geq R.$$

Since the function $x \to |x|^{(n-3)/2}e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}$ is bounded in $\mathbb{R}^n$, we deduce

$$\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{1/2}} \leq C\frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}}{|x|^{n/2-1}}, \quad |x| \geq R. \tag{A.6}$$

Using (A.5) and (A.6) in (A.4) in order to obtain

$$C^{-1}\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{n/2-1}} \leq K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}) \leq C\frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}}{|x|^{n/2-1}}, \quad |x| \geq R. \tag{A.7}$$

We now establish a similar estimate when $|x| \to 0$. To this end we recall that according to formula [5, 9.6.9, page 375] we have

$$K_{n/2-1}(\rho) \sim \frac{1}{2}\Gamma(n/2-1)\left(\frac{2}{\rho}\right)^{n/2-1} \quad \text{as } \rho \to 0,$$

from which we deduce in a straightforward manner that there exists $0 < r \leq R$, so that

$$C^{-1}\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{n/2-1}} \leq K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}) \leq C\frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\nu})}}{|x|^{n/2-1}}, \quad |x| \leq r. \tag{A.8}$$

The expected two sided inequality (2.10) follows by combining (A.4), (A.7) and (A.8).

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