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Extremes of branching Ornstein-Uhlenbeck processes

Julien Berestycki* Éric Brunet† Aser Cortines‡ Bastien Mallein§

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Abstract

In this article, we focus on the asymptotic behaviour of extremal particles in a branching Ornstein-Uhlenbeck process: particles move according to an Ornstein-Uhlenbeck process, solution of $dX_s = -\mu X_s ds + dB_s$, and branch at rate 1. We make $\mu = \mu_t$ depend on the time-horizon t at which we observe the particles positions and we suppose that $\mu_t t \rightarrow \gamma \in (0, \infty]$. We show that, properly centred and normalised, the extremal point process continuously interpolates between the extremal point process of the branching Brownian motion (case $\gamma = 0$) and the extremal point process of independent Gaussian random variables (case $\gamma = \infty$). Along the way, we obtain several results on standard branching Brownian motion of intrinsic interest. In particular, we give a probabilistic representation of the main object of study in [DMS16] which is the probability that the maximal position has an abnormally high velocity.

1 Introduction

Spatial branching processes, and in particular, the behaviour of their extremal particles, have been at the centre of an enormous research activity over the past few years, both in the physics [BD09, BDMM06] and in the mathematical literature [Aid13, ABBS13, ABK13, Mad16]. These models have a rich and complex structure that is of intrinsic interest, but they are also representatives of an intriguing “universality” class, the so-called *log-correlated fields* which includes the two-dimensional Gaussian free field [BDZ16, BL18], Gaussian multiplicative chaos [RV14], random matrices [ABB17] and others.

Perhaps the simplest model in this class is the *branching Brownian motion*, in which particles move in \mathbb{R} as Brownian motions, branch into two particles at rate one and are independent of each others. For the system started with a single particle at the origin, let \mathcal{N}_t be the set of particles alive at time t and for $u \in \mathcal{N}_t$ let $X_t(u) \in \mathbb{R}$ be its position. For $s \leq t$ we will also write $X_s(u)$ for the position of the unique ancestor of u at time s so that $(X_s(u), s \leq t)$ is the path followed by the particle u . Then, it was proved in [ABBS13, ABK13] that the point measure

$$\mathcal{E}_t := \sum_{u \in \mathcal{N}_t} \delta_{X_t(u) - \sqrt{2}t + \frac{3}{2\sqrt{2}} \log t} \quad (1.1)$$

converges in law, as $t \rightarrow \infty$ toward a *random intensity decorated Poisson point process* (DPPP for short) \mathcal{E}_∞ .

In general, the law of a DPPP \mathcal{E} is characterized by a pair (ν, \mathfrak{D}) where ν is a random sigma-finite measure on \mathbb{R} and \mathfrak{D} is the law of a random point process on \mathbb{R} . The point measure \mathcal{E} can be constructed, conditionally on ν , by first taking a realisation of a Poisson point process on \mathbb{R} with intensity ν , whose atoms are listed as $(x_i, i \in I)$, and an independent family of i.i.d. point processes $(\mathcal{D}_i, i \in I)$ with law \mathfrak{D} . Then, each atom x_i is replaced by the point process \mathcal{D}_i , shifted by x_i (i.e. we *decorate* x_i with a point process of law \mathfrak{D}). In other words, writing $(d_i^j, j \in J_i)$ the atoms of the point process \mathcal{D}_i , we have

$$\mathcal{E} = \sum_{i \in I} \sum_{j \in J_i} \delta_{x_i + d_i^j}. \quad (1.2)$$

We refer to [SZ15] for an in-depth study of random intensity decorated Poisson point processes, and their occurrences as limit of extremal point measures.

With this notation, $\mathcal{E}_\infty := \lim_{t \rightarrow \infty} \mathcal{E}_t$ is the following DPPP

$$\mathcal{E}_\infty = \text{DPPP}(\kappa Z_\infty e^{-\sqrt{2}x} dx, \mathfrak{D}^1) \quad (1.3)$$

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where κ is an implicit constant, Z_∞ is the a.s. positive limit of the so-called *derivative martingale*

$$Z_t := \sum_{u \in \mathcal{N}_t} (\sqrt{2t} - X_t(u)) e^{\sqrt{2}X_t(u) - 2t}, \quad (1.4)$$

and where the *decoration law* \mathfrak{D}^1 is the law of a point measure supported on $(-\infty, 0]$, with an atom at 0, which belongs to the family $(\mathfrak{D}^\varrho, \varrho \in [1, \infty])$, defined, for $\varrho < \infty$ by the weak limits

$$\mathfrak{D}^\varrho(\cdot) := \lim_{t \rightarrow \infty} \mathbf{P} \left(\sum_{u \in \mathcal{N}_t} \delta_{\{X_t(u) - M_t\}} \in \cdot \mid M_t \geq \sqrt{2\varrho t} \right), \quad (1.5)$$

where $M_t := \max_{u \in \mathcal{N}_t} X_t(u)$, see [BH15] for a proof of the existence of \mathfrak{D}^ϱ . We set \mathfrak{D}^∞ the law of the Dirac mass at 0. Moreover, it is well-known that $\max \mathcal{E}_t$ converges in distribution toward $\max \mathcal{E}_\infty$, where $\max \mathcal{E}$ is the position of the largest atom in a point process \mathcal{E} (see Lalley and Selke [LS87]).

The goal of this article is to study the same question — asymptotic behaviour of the extremal process — for a spatially inhomogeneous branching particle system: the branching Ornstein-Uhlenbeck process. As the name suggests, this is a continuous-time particle system in which particles move according to i.i.d. Ornstein-Uhlenbeck processes and split independently at rate 1.

An Ornstein-Uhlenbeck process X with spring constant μ is the solution of the stochastic differential equation

$$dX_s^\mu = -\mu X_s^\mu ds + dB_s, \quad (1.6)$$

where B is a Brownian motion. It is well-known that Ornstein-Uhlenbeck processes may be represented, if $\mu > 0$, as a space-time scaled Brownian motion: given W a standard Brownian motion, the process defined by

$$\forall s \geq 0, \quad X_s^\mu = X_0 e^{-\mu s} + \frac{e^{-\mu s}}{\sqrt{2\mu}} W_{e^{2\mu s} - 1}, \quad (1.7)$$

is an Ornstein-Uhlenbeck with spring constant μ and initial condition X_0 . Equation (1.7) shows that, if $\mu > 0$, the law of X_s , conditionally on $\{X_0 = x\}$, is $\mathcal{N}(xe^{-\mu s}, \frac{1 - e^{-2\mu s}}{2\mu})$. In particular, X_s is then strongly recurrent and its invariant measure is $\mathcal{N}(0, \frac{1}{2\mu})$.

In a branching Ornstein-Uhlenbeck, since the genealogical structure of the process is independent of the motion of the particles, we continue to denote by \mathcal{N}_t the set of particles alive in a branching Ornstein-Uhlenbeck process with spring constant μ and we write $(X_s^\mu(u), u \in \mathcal{N}_s)$ for the positions of such particles. It will be convenient to work with a normalized version $\widehat{X}_s^\mu(u)$ of $X_s^\mu(u)$ that has variance t so that things happen on the same scale as for the branching Brownian motion. This can be easily obtained by setting

$$\widehat{X}_s^\mu(u) = \sqrt{\frac{2\mu s}{1 - e^{-2\mu s}}} X_s^\mu(u). \quad (1.8)$$

With this notation, we define the extremal point process:

$$\mathcal{E}_t^\mu = \sum_{u \in \mathcal{N}_t} \delta_{\widehat{X}_t^\mu(u) - \sqrt{2t} + \frac{1}{2\sqrt{2}} \log t}. \quad (1.9)$$

Note that here the logarithmic correction is $\frac{1}{2\sqrt{2}}$ instead of $\frac{3}{2\sqrt{2}}$ as in the branching Brownian motion case ($\mu = 0$, see (1.1)).

Throughout this paper, we will choose the spring constant μ as depending on the time-horizon t at which we observe the positions of particles, in the sense that $\mu = \mu_t$ is kept fixed for the evolution of the branching process at all times $s \in [0, t]$. For reasons that will become clear later on in the paper, one should choose μ_t such that $\mu_t t \rightarrow \gamma \in (0, \infty]$ as $t \rightarrow \infty$, which trivially covers the standard case where μ is fixed for all t 's.

Our main result is that $\mathcal{E}_t^{\mu_t}$ converges in the appropriate sense to $\mathcal{E}_\infty^\gamma$, a new random intensity decorated Poisson point process

$$\mathcal{E}_\infty^\gamma := \text{DPPP}(\sqrt{2}C(d_\gamma)W_\infty^{\sqrt{2}c_\gamma} e^{-\sqrt{2}x} dx, d_\gamma \mathfrak{D}^{d_\gamma}), \quad (1.10)$$

the characteristic pair of which is defined as follows: Let $(X_t(u), u \in \mathcal{N}_t)$ be a branching Brownian motion and M_t its maximal displacement at time t . Then,

- W_∞^β is the limit of the *additive martingale*:

$$W_t^\beta = \sum_{u \in \mathcal{N}_t} e^{\beta X_t(u) - \left(\frac{\beta^2}{2} + 1\right)t}, \quad t \geq 0, \quad \beta \in \mathbb{R}. \quad (1.11)$$

As $(W_t^\beta, t \geq 0)$ is a non-negative martingale, it converges a.s. to a limit W_∞^β . Moreover, it is well known that a.s. $W_\infty^\beta > 0$ if, and only if, $\beta \in (-\sqrt{2}, \sqrt{2})$.

- For all $\varrho \in (1, \infty)$

$$C(\varrho) := \varrho \lim_{t \rightarrow \infty} t^{1/2} e^{(\varrho^2 - 1)t} \mathbf{P}(M_t \geq \sqrt{2\varrho t}). \quad (1.12)$$

This can be seen as a precise estimate on the large deviations for the maximal displacement of the branching Brownian motion, see [DMS16, DS17, GH18, BM18] for recent developments on this topic.

- The family of laws $(\mathfrak{D}^\varrho, \varrho \geq 1)$ is the family of point processes introduced in (1.5), the extremal point process in a branching Brownian motion, seen from M_t , conditioned on $M_t > \sqrt{2\varrho t}$, and $c\mathfrak{D}^\varrho$ is the image measure of \mathfrak{D}^ϱ by the application $\mathcal{D} \mapsto \sum_{d_j \in \mathcal{D}} \delta_{cd_j}$, dilating the positions of the atoms by a factor c .
- The constants c_γ and d_γ are given by

$$c_\gamma := \sqrt{\frac{2\gamma}{e^{2\gamma} - 1}} \quad \text{and} \quad d_\gamma := \sqrt{\frac{2\gamma}{1 - e^{-2\gamma}}}. \quad (1.13)$$

In the $\gamma = \infty$ case, we set $c_\infty = 0$ and $d_\infty = \infty$, thus $W_\infty^{\sqrt{2}c_\infty} = W_\infty^0$ is an exponential random variable with mean 1. As is shown in Proposition 1.2, $C(d_\infty) = C(\infty) = \frac{1}{\sqrt{4\pi}}$ and a point measure drawn from $\mathfrak{D}^{d_\infty} = \mathfrak{D}^\infty$ is a.s. δ_0 .

Theorem 1.1. *Assume that $\lim_{t \rightarrow \infty} t\mu_t = \gamma \in (0, \infty]$, then, with the above notations, we have that*

$$\lim_{t \rightarrow \infty} (\mathcal{E}_t^{\mu_t}, \max \mathcal{E}_t^{\mu_t}) = (\mathcal{E}_\infty^\gamma, \max \mathcal{E}_\infty^\gamma) \quad \text{jointly in law,}$$

where the convergence of the point process is in the sense of the topology of vague convergence.

Remark: Recall that a sequence of random point measures $(\mathcal{P}_t)_{t \geq 0}$ on \mathbb{R} converges to \mathcal{P} in law for the topology of vague convergence if and only if, for every compactly supported continuous function φ , the real valued random variables

$$\langle \mathcal{P}_t, \varphi \rangle := \int \varphi(x) \mathcal{P}_t(dx)$$

converge in law to $\langle \mathcal{P}, \varphi \rangle$ as $t \rightarrow \infty$. We prove in the forthcoming Lemma 4.1 that the convergence in law of a random point measure (for the topology of vague convergence) jointly with that of its maximum is equivalent to the convergence in law of $\langle \mathcal{P}_t, \varphi \rangle$ to $\langle \mathcal{P}, \varphi \rangle$ for all continuous functions φ with support bounded from the left. This notion of convergence forms a thinner topology on the space of point measures.

Remark: In the simplest case where $\mu_t = \mu$ is a constant, the Theorem with (1.8) and (1.9) implies the following behaviour for the unnormalised positions $X_t^\mu(u)$: the position of the rightmost particle is almost surely given by

$$\max_{u \in \mathcal{N}_t} X_t^\mu(u) = \sqrt{\frac{t}{\mu}} - \frac{\log t}{4\sqrt{\mu t}} + O(t^{-1/2}),$$

and the next particles are at distance of order $t^{-1/2}$ from the rightmost.

We shall call the case $t\mu_t \rightarrow \infty$ the *uncorrelated case*, because the extremal particles have the same distribution as the extremal particles of an i.i.d. sample of Gaussian random variables. Indeed, in this regime, the dilation factor $\sqrt{2\mu_t t / (1 - e^{-2\mu_t t})}$ diverges as $t \rightarrow \infty$, which prevents the existence of local correlations (decorations) in the limiting picture.

The cases $t\mu_t \rightarrow \gamma \in (0, \infty)$ interpolate between the *uncorrelated case* and the branching Brownian motion regime ($\mu_t = 0$). Notice, though, that the multiplicative factor of the logarithmic correction remains equal to $\frac{1}{2\sqrt{2}}$ (as in the uncorrelated case) and not $\frac{3}{2\sqrt{2}}$ (as in the branching Brownian motion). We believe that there is a second transition when $t\mu \rightarrow 0$ where one gradually goes from the $\frac{1}{2\sqrt{2}} \log t$ correction to $\frac{3}{2\sqrt{2}} \log t$. A similar phenomenon was observed by Bovier and Hartung [BH18] for branching Brownian motion with piecewise constant variance.

Our model is notably different from the one studied by Kiestler and Schmidt [KS15] which yields a different interpolation between the uncorrelated case and the branching Brownian motion.

The function $\varrho \mapsto C(\varrho)$ defined in (1.12) is of intrinsic interest. Indeed, asymptotics of the probability $\mathbf{P}(M_t > \sqrt{2\varrho t})$ were first studied in [CR88] (where the existence of the limit $C(\varrho)$ is implicit). It also plays a key role in [BH15] where it is proven that $C(1) = 0$ and that $\lim_{\varrho \rightarrow \infty} C(\varrho) = (4\pi)^{-1/2}$. More recently, the same function $C(\varrho)$ is the focus of [DMS16] where, in particular, the large ϱ and small $\varrho - 1$ asymptotics are given.

In the present work we will show that $C(\varrho)$ has a probabilistic representation. This will allow us to prove that $\varrho \mapsto C(\varrho)$ is continuous and that the family of limiting point measure $(\mathcal{E}_\infty^\gamma, \gamma > 1)$ is vaguely continuous in γ . This is a key step in the proof of Theorem 1.1.

Let us now describe the representation of $C(\varrho)$ and \mathcal{D}^ϱ in terms of a *spine process*. Let $(B_t, t \geq 0)$ be a standard Brownian motion, $(\sigma_k, k \in \mathbb{N})$ be the ranked atoms of a Poisson point process with intensity $2 dx$ on \mathbb{R}_+ and $(X_t^{(k)}(u), u \in \mathcal{N}_t^{(k)}, t \geq 0)$ for $k \in \mathbb{N}$ be i.i.d. branching Brownian motions. We shall assume that B , the σ_k and the $X^{(k)}$ are independent of one another. Given $\varrho \in (1, \infty)$ and $t \geq 0$, we define the point process

$$\tilde{\mathcal{D}}^\varrho = \delta_0 + \sum_{k \in \mathbb{N}} \sum_{u \in \mathcal{N}_{\sigma_k}^{(k)}} \delta_{B_{\sigma_k} - \sqrt{2}\varrho\sigma_k + X_{\sigma_k}^{(k)}(u)}. \quad (1.14)$$

In words, $\tilde{\mathcal{D}}^\varrho$ is the point process constructed using a Brownian motion with drift $-\sqrt{2}\varrho$, that spawns branching Brownian motions at rate 2. A branching Brownian motion spawned at time σ_k then starts evolving backward in time until it hits time 0, the particles alive at that time are added to the point process.

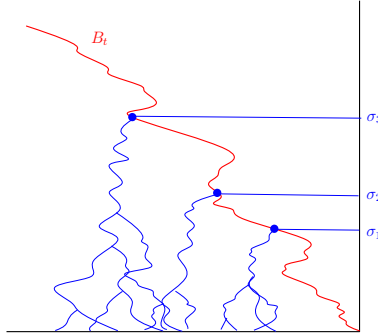


Figure 1: Construction of the point process $\tilde{\mathcal{D}}^\varrho$.

Theorem 1.2. *Let $C : [1, \infty] \mapsto \mathbb{R}_+$ be the function given by (1.12) and for $\varrho \geq 1$ let \mathcal{D}^ϱ be a random point measure of law \mathcal{D}^ϱ as defined in (1.5). Then*

(i) $C(\varrho) = \frac{1}{\sqrt{4\pi}} \mathbf{P}(\tilde{\mathcal{D}}^\varrho((0, \infty)) = 0)$. *The function C is continuous on $[1, \infty]$. It also satisfies $C(1) = 0$, $C(\varrho) > 0$ for $\varrho > 1$ and $C(\infty) = 1/\sqrt{4\pi}$.*

(ii) $\mathbf{P}(\mathcal{D}^\varrho \in \cdot) = \mathbf{P}(\tilde{\mathcal{D}}^\varrho \in \cdot \mid \tilde{\mathcal{D}}^\varrho((0, \infty)) = 0)$. *The family of point processes $(\mathcal{D}^\varrho, \varrho \in (1, \infty])$ is continuous in the space of Radon point measures equipped with the topology of vague convergence.*

The definition (1.5), in the case $\varrho = 1$, is the one given in [ABK12] for the decoration of the extremal process of the branching Brownian motion. The above backbone description, in the case $\varrho = 1$, is similar to the one given in [ABBS13].

The convergences $C(\varrho) \rightarrow 1/\sqrt{4\pi}$ and $\mathcal{D}^\varrho \rightarrow_d \delta_0$ as $\varrho \rightarrow \infty$ were already proved in [BH15, proof of Lemma 3.3]. We believe the family of point measures $(\mathcal{D}^\varrho, \varrho > 1)$ to be vaguely continuous as $\varrho \rightarrow 1$, however this result is not as straightforward and is left open in the present paper.

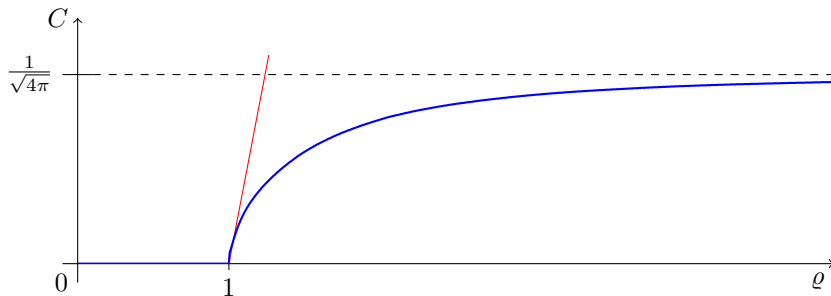


Figure 2: An approximation of the function C , computed using its representation from Theorem 1.2, together with its right derivative at $\varrho = 1$.

1.1 Open questions and future work

One question that we have not explored in the present work is what happens when $\lim_{t \rightarrow 0} t\mu_t = 0$. We conjecture that in that case, the decoration measure is always \mathcal{D}^1 , which is the decoration of the branching Brownian motion. However, we believe that the precise constant in front of the logarithmic correction for the positions of the extremal particles will now depend on how fast $t\mu_t$ decreases towards zero.

More precisely, it is predicted in [DMS16] that $C(\varrho) \sim \kappa(\varrho - 1)$ as $\varrho \rightarrow 1$, with the same constant κ as in (1.3). Note that $\kappa \approx 1.18$ is also the constant such that $\lim_t \mathbf{P}(M_t \geq \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + y) \sim \frac{\kappa}{\sqrt{2}} y e^{-\sqrt{2}y}$, as $y \rightarrow \infty$. This constant is proved to exist for all branching random walks in [Aid13, Proposition 4.1]. Note that in [DMS16] the function Φ defined by

$$u(ct, t) \sim \frac{e^{-t(c^2/4-1)}}{\sqrt{4\pi t}} \Phi(c) \quad \text{as } t \rightarrow \infty$$

where u is the solution of the Fisher-KPP equation $\partial_t u = \partial_x^2 u + u(1 - u)$ started from the Heaviside initial condition is the analogue of C . The exact correspondence between the functions Φ and C is

$$C(\varrho) = \frac{\varrho}{\sqrt{4\pi}} \Phi(2\varrho).$$

Our factor κ is thus given by the constant denoted 2α in [DMS16] (see Equation (73) there).

On the other hand, we also know from [Mad16], that for the additive martingale W^β

$$\lim_{\beta \rightarrow \sqrt{2}-} \frac{W_\infty^\beta}{\sqrt{2} - \beta} = \sqrt{2} Z_\infty,$$

with Z_∞ the limit of the derivative martingale. Since $d_\gamma \simeq 1 + \gamma/2$ and $c_\gamma \simeq 1 - \gamma/2$ when $\gamma \rightarrow 0$, we see that

$$C(d_\gamma) W_\infty^{\sqrt{2}c_\gamma} \simeq \frac{\kappa\gamma^2}{2} Z_\infty \quad \text{as } \gamma \rightarrow 0.$$

Since $\gamma^2 e^{-\sqrt{2}x} = e^{-\sqrt{2}(x - \sqrt{2} \log \gamma)}$, the extremal point process $\mathcal{E}_\infty^\gamma$ is roughly \mathcal{E}_∞ , the centred extremal point process of the standard branching Brownian motion see (1.3), shifted to the left by $\sqrt{2}|\log \gamma| + \mathcal{O}(1)$ (as $\gamma \rightarrow 0$). This might suggest that the above-mentioned intermediate logarithmic corrections between $\frac{1}{2\sqrt{2}}$ and $\frac{3}{2\sqrt{2}}$ should appear for $\mu_t = t^{-\alpha}$ with $\alpha \in (1, 3/2)$, and the extremal point measure would be the same as for the branching Brownian motion as soon as $\mu_t = o(t^{-3/2})$. This would complement the recent work [BH18].

The case $\mu < 0$ is also interesting and is not covered in the present work. Notice that in the case $\mu > 0$ we rely heavily on the results from Bovier and Hartung [BH15]. However we think that the $\mu < 0$ case corresponds to that of decreasing variances for the variable speed branching Brownian motion for which results concerning the position of the maximum are known (see e.g. Maillard and Zeitouni [MZ16]), but not concerning the full extremal point process.

Our interest in the extremal point measure of branching Ornstein-Uhlenbeck processes was sparked by a conjecture in [CM18], that the genealogy of a branching Ornstein-Uhlenbeck with selection is given by a Beta-coalescent whose parameters can be tuned by the spring constant. The study of the extremal point process is a first step towards a better understanding of the relevant objects.

Organisation of the paper. The rest of the article is organised as follows. In the next section, we observe that in the particular case $\mu_t = \gamma/t$ for some $\gamma \in (0, \infty)$, Theorem 1.1 is a direct consequence of the result of Bovier and Hartung [BH15] on the convergence of the extremal process of time-inhomogeneous branching Brownian motions.

In Section 3, we prove Theorem 1.2, i.e. the probabilistic representation of the law of the decoration. Then, in Section ??, we introduce some technical tools: Gaussian tail estimates in Section ?? and comparison theorems in Section ??.

The proof of Theorem 1.1 when $\mu_t \neq \gamma/t$ is in Section 4. Section 5 is devoted to a simple and self-contained proof of the convergence of the extremal point process when the spring force is constant. We conclude in Section 1.1 with some open questions, conjectures and natural ways to extend the present work.

For further reference we gather here the meaning of some of our notations:

- The point process $\mathcal{E}_t^{\mu_t}$ is the normalised positions $\widehat{X}_t^{\mu_t}(u)$ in the branching Ornstein-Uhlenbeck at time t , seen from $\sqrt{2}t - \frac{1}{2\sqrt{2}} \log t$.

- The point process \mathcal{D}^ρ , of law \mathfrak{D}^ρ , is the limit point process of positions in a branching Brownian motion conditioned to have a particle right of $\sqrt{2}\rho t$, seen from the rightmost particle.
- The point process $\mathcal{E}_\infty^\gamma$ is a decorated Poisson point process with random exponential intensity, where the decoration is given by \mathcal{D}^ρ for some ρ .
- The point process $\tilde{\mathcal{D}}^\rho$ is the one obtained by the spine construction : branching Brownian motions started at rate 2 on a drifted Brownian spine.

2 The case $\mu_t = \gamma/t, \gamma \in (0, \infty)$ and variable speed branching Brownian motion

In this section, we prove Theorem 1.1 assuming that $\mu_t = \gamma/t$, where $\gamma \in (0, \infty)$ is a fixed constant. Our approach relies on earlier work on the extremal process of *variable speed branching Brownian motions*. These processes were introduced in [FZ12] and further studied in [BH14, BH15], where the convergence of the extremal point measure for a wide class of variance profiles is established. In particular, if $\mu_t = \gamma/t$ then the convergence stated in Theorem 1.1 above follows readily from [BH15]. We start with a brief introduction to the variable speed branching Brownian motion, and state the convergence result [BH15, Theorem 1.2]. We then explain the connection between a branching Ornstein-Uhlenbeck process with spring constant $\mu_t = \gamma/t$ and variable speed branching Brownian motion, before using this result to prove Theorem 1.1 in that particular case.

Let $A : [0, 1] \rightarrow [0, 1]$ be a twice differentiable increasing function with $A(0) = 0$ and $A(1) = 1$. Then, the variable speed branching Brownian motion with variance profile A and time horizon t is defined in the same way as a branching Brownian motion, except that particles move as Brownian motions with time-dependent variance $\sigma_t^2(s) = A'(s/t)$ where $s \in [0, t]$ is the time of the process. In particular, the position of a particle at time s is a Gaussian random variable with variance $tA(s/t)$.

The following result on the convergence of the extremal point measure of branching Brownian motion with variance profile A is proved in [BH15].

Theorem A (Bovier and Hartung [BH15] Theorem 1.2). *Assume that the twice differentiable increasing function $A : [0, 1] \rightarrow [0, 1]$ satisfies*

1. $A(0) = 0, A(1) = 1$ and $A(x) < x$ for all $x \in (0, 1)$;
2. $\sigma_b^2 := A'(0) < 1$ and $\sigma_e^2 := A'(1) > 1$.

Let $(Y_s(u); u \in \mathcal{N}_s; s \in [0, t])$ denote the variable speed branching Brownian motion with variance profile A and

$$\bar{\mathcal{E}}_t^A = \sum_{u \in \mathcal{N}_t} \delta_{Y_t(u) - \sqrt{2}t + \frac{1}{2\sqrt{2}} \log t}$$

be its extremal point measure at time t . Then

- (i) the extremal process $\bar{\mathcal{E}}_t^A$ converges in law for the topology of the vague convergence to $\bar{\mathcal{E}}_\infty^A$, which is a

$$\text{DPPP}(\sqrt{2}C(\sigma_e)W_\infty^{\sqrt{2}\sigma_b}e^{-\sqrt{2}x}dx, \sigma_e\mathfrak{D}^{\sigma_e});$$

- (ii) the maximal displacement of the process converges in law, and for all $x \in \mathbb{R}$,

$$\lim_{t \rightarrow \infty} \mathbf{P} \left(\max \bar{\mathcal{E}}_t^A \leq x \right) = \mathbf{P} \left(\max \bar{\mathcal{E}}_\infty^A \leq x \right).$$

with C, W_∞ and \mathfrak{D} respectively defined in (1.12), (1.11) and (1.5).

Remark: It is well-known (and we give a fairly general proof of that fact in Lemma 4.1) that if $\bar{\mathcal{E}}_t^A$ and its maximum converge toward $\bar{\mathcal{E}}_\infty^A$ and its maximum in law, then there is joint convergence of the two quantities, just as in our Theorem 1.1.

We will now show how Theorem 1.1 follows from Theorem A in the special case $\mu_t = \gamma/t, \gamma \in (0, \infty)$.

Proof of Theorem 1.1 in the $\mu_t = \gamma/t$ case. Recall from (1.7) that, an Ornstein-Uhlenbeck X_s^μ at time s with spring-constant μ started from 0 can be written as

$$X_s^\mu = \frac{e^{-\gamma s/t}}{\sqrt{2\gamma/t}} W_{e^{2\gamma s/t} - 1}.$$

Recall also that given $\gamma > 0$ and $t > 0$, we denote by $(X_s^{\gamma/t}(u), u \in \mathcal{N}_s)$ a branching Ornstein-Uhlenbeck process on $[0, t]$ with spring constant $\mu_t = \gamma/t$.

For any $u \in \mathcal{N}_t$, we define $(Y_s(u), s \in [0, t])$ by

$$Y_s(u) = \sqrt{\frac{2\gamma}{e^{2\gamma} - 1}} e^{\gamma s/t} X_s^{\gamma/t}(u). \quad (2.1)$$

Clearly, $Y_s(u)$ has variance $t \frac{e^{2\gamma s/t} - 1}{e^{2\gamma} - 1}$. It is easily checked that the whole process $(Y_s(u), s \leq t)_{u \in \mathcal{N}_t}$ is then a variable speed branching Brownian motion, with variance profile $A(x) := \frac{e^{2\gamma x} - 1}{e^{2\gamma} - 1}$, which is a function satisfying the assumptions of Theorem A with

$$\sigma_b^2 = A'(0) = \frac{2\gamma}{e^{2\gamma} - 1} = c_\gamma^2, \quad \sigma_e^2 = A'(1) = \frac{2\gamma}{1 - e^{-2\gamma}} = d_\gamma^2.$$

Therefore, the extremal point process $\sum_{u \in \mathcal{N}_t} \delta_{Y_t(u) - \sqrt{2t} + \frac{1}{2\sqrt{2}} \log t}$ converges in distribution as $t \rightarrow \infty$ to a

$$\text{DPPP}(\sqrt{2}C(\sigma_e)W_\infty^{\sqrt{2}\sigma_b}e^{-\sqrt{2}x}dx, \sigma_e \mathfrak{D}^{\sigma_e}),$$

and the maximal atom converges as well. Since $Y_t(u) = \widehat{X}_t^{\gamma/t}(u)$ by (1.8), and using the forthcoming Lemma 4.1, we conclude in the joint convergence $(\mathcal{E}_t^{\gamma/t}, \max \mathcal{E}_t^{\gamma/t})$ toward $(\mathcal{E}_\infty^\gamma, \max \mathcal{E}_\infty^\gamma)$ in law, completing the proof of Theorem 1.1 when $\mu_t = \gamma/t$. \square

Note that when $t\mu_t \rightarrow \gamma$ but we do not assume that $\mu_t = \gamma/t$, it is not possible to apply directly the same approach since in that case the variance profile of $Y_s(u)$ will be a function of s and t and not just of s/t . Instead, we need to rely on comparison and continuity results.

3 Spine representation of $C(\varrho)$ and weak continuity of the cluster distribution

In this section, we prove Theorem 1.2, that is the weak continuity in ϱ of the cluster point process \mathfrak{D}^ϱ as well as the continuity of the function $\varrho \mapsto C(\varrho)$ and their spine representation.

Recall the construction (1.14) of the point process $\widetilde{\mathcal{D}}^\varrho$

$$\widetilde{\mathcal{D}}^\varrho = \delta_0 + \sum_{k \in \mathbb{N}} \sum_{u \in \mathcal{N}_{\sigma_k}^{(k)}} \delta_{B_{\sigma_k} - \sqrt{2}\varrho\sigma_k + X_{\sigma_k}^{(k)}(u)}$$

and define

$$\widetilde{\mathcal{D}}_t^\varrho = \delta_0 + \sum_{k \in \mathbb{N}: \sigma_k \leq t} \sum_{u \in \mathcal{N}_{\sigma_k}^{(k)}} \delta_{B_{\sigma_k} - \sqrt{2}\varrho\sigma_k + X_{\sigma_k}^{(k)}(u)}. \quad (3.1)$$

The measure $\widetilde{\mathcal{D}}^\varrho$ is the increasing limit of $\widetilde{\mathcal{D}}_t^\varrho$. The first thing to show is that $\widetilde{\mathcal{D}}^\varrho$ is a sigma-finite point measure, meaning that for every $a < b \in (-\infty, \infty]$, we have $\widetilde{\mathcal{D}}_t^\varrho([a, b]) \nearrow \widetilde{\mathcal{D}}^\varrho([a, b]) < \infty$ a.s. as $t \rightarrow \infty$. We start with the case $\varrho > 1$.

Lemma 3.1. *For all $\varrho > 1$, $\widetilde{\mathcal{D}}^\varrho$ is a well-defined point measure. Moreover, we have*

$$\lim_{t \rightarrow \infty} \widetilde{\mathcal{D}}_t^\varrho = \widetilde{\mathcal{D}}^\varrho \quad \text{a.s. for the topology of the vague convergence.}$$

Proof. Let $\varrho > 1$, the point measure $\widetilde{\mathcal{D}}^\varrho$ can be rewritten as

$$\widetilde{\mathcal{D}}^\varrho = \delta_0 + \sum_{k \in \mathbb{N}} \sum_{u \in \mathcal{N}_{\sigma_k}^{(k)}} \delta_{(B_{\sigma_k} - \sqrt{2}(\varrho-1)\sigma_k) + (X_{\sigma_k}^{(k)}(u) - \sqrt{2}\sigma_k)}. \quad (3.2)$$

We observe that $(B_{\sigma_k} - \sqrt{2}(\varrho - 1)\sigma_k, k \geq 0)$ is a random walk with negative drift $-\sqrt{2}(\varrho - 1)/2$. Moreover, for all $k \in \mathbb{N}$ the position of the largest atom in the point measure $\sum_{u \in \mathcal{N}_{\sigma_k}^{(k)}} \delta_{X_{\sigma_k}^{(k)}(u) - \sqrt{2}\sigma_k}$ is, for large values of k , typically around position $-\frac{3}{2\sqrt{2}} \log \sigma_k \approx -\frac{3}{2\sqrt{2}} \log k$. Thus, heuristically, if $\varrho > 1$, the random walk drifts to $-\infty$ at positive speed such that only a finite number of branching Brownian motions put particles in any given compact set. On the other hand, when $\varrho = 1$, the random walk B_{σ_k} has drift zero and we show that it implies that an infinite number of particles are to be found in any finite neighbourhood of 0.

To make the above argument rigorous, we write M_t for the maximal displacement at time t in a branching Brownian motion. Setting $m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$, it is well-known [ABBS13, ABK13] that $(M_t - m_t, t \geq 0)$ is tight and has uniform exponential tails. More precisely, it is proved in [Fan12] (in a much more general settings) there exists $C > 0$ and $\lambda > 0$ such that

$$\mathbf{P}(|M_t - m_t| \geq x) \leq Ce^{-\lambda x} \text{ for all } t, x > 0. \quad (3.3)$$

Given $k \in \mathbb{N}$, we denote by $M^{(k)} = \max_{u \in \mathcal{N}_{\sigma_k}^{(k)}} X_{\sigma_k}^{(k)}(u)$ the maximal displacement of $X^{(k)}$ at time σ_k . Using the bounds from (3.3), we observe immediately, using the Borel-Cantelli Lemma and the fact that $\sigma_k \sim_{k \rightarrow \infty} k/2$ that, with probability one,

$$\limsup_{k \rightarrow \infty} \frac{|M^{(k)} - \sqrt{2}\sigma_k|}{\log k} \leq \frac{3}{2\sqrt{2}} + \lambda^{-1}. \quad (3.4)$$

In view of (3.4) and the law of large numbers, we deduce that

$$\lim_{k \rightarrow \infty} \frac{1}{k} (B_{\sigma_k} + M^{(k)} - \sqrt{2}\varrho\sigma_k) = -\frac{\sqrt{2}(\varrho - 1)}{2} < 0 \quad \text{a.s.} \quad (3.5)$$

In particular, it implies that given $A > 0$ one can find a random $T \in \mathbb{R}_+$ such that

$$\forall \sigma_k \geq T, \quad B_{\sigma_k} + M^{(k)} - \sqrt{2}\varrho\sigma_k \leq -A,$$

in which case $\tilde{\mathcal{D}}^{\varrho}((x, \infty)) = \tilde{\mathcal{D}}_t^{\varrho}((x, \infty))$ for all $t > T$ and $x > -A$. This proves that $\tilde{\mathcal{D}}^{\varrho}$ is locally finite a.s. and that $\tilde{\mathcal{D}}_t^{\varrho} \nearrow \tilde{\mathcal{D}}^{\varrho}$ as $t \rightarrow \infty$, as claimed. \square

Next we show the weak continuity of the family $(\tilde{\mathcal{D}}^{\varrho}, \varrho > 1)$.

Lemma 3.2. *The family of point processes $(\tilde{\mathcal{D}}^{\varrho}, \varrho > 1)$ is a.s. continuous in $\varrho > 1$. Moreover, for all $\varrho > 1$, $\mathbf{P}(\tilde{\mathcal{D}}^{\varrho}((0, \infty)) = 0) > 0$ and*

$$\lim_{t \rightarrow \infty} \mathbf{P}(\tilde{\mathcal{D}}_t^1((0, \infty)) = 0) = \lim_{\varrho \rightarrow 1} \mathbf{P}(\tilde{\mathcal{D}}^{\varrho}((0, \infty)) = 0) = 0.$$

Proof. To prove the a.s. continuity of $(\tilde{\mathcal{D}}^{\varrho}, \varrho > 1)$, it is enough to show that for all continuous function φ with compact support, the function $\varrho \mapsto \langle \varphi, \tilde{\mathcal{D}}^{\varrho} \rangle$ is continuous a.s. This is a direct consequence of the fact that there are only finitely atoms in any compact interval, and that the position of these atoms in \mathcal{D}^{ϱ} are decreasing and continuous with ϱ , by (1.14). Hence, for any $\varrho_0 > 1$, there is only a finite number of atoms to follow as ϱ increases to compute $\varrho \in [\varrho_0, \infty) \mapsto \langle \varphi, \tilde{\mathcal{D}}^{\varrho} \rangle$. Hence this function is continuous, which completes the proof of the first statement. For the second statement, it suffices to observe that for $T > 0$ there is positive probability that $\sigma_1 > T$ and that $\tilde{\mathcal{D}}^{\varrho}(0, \infty) - \tilde{\mathcal{D}}_T^{\varrho}(0, \infty) = 0$.

We now focus on the case $\varrho = 1$. By law of iterated logarithms for the random walk, we have that

$$\limsup_{k \rightarrow \infty} k^{-1/2} B_{\sigma_k} = \infty \quad \text{a.s.},$$

which together with (3.4) yields

$$\limsup_{k \rightarrow \infty} \frac{B_{\sigma_k} + M^{(k)} - \sqrt{2}\sigma_k}{k^{1/2}} = \infty \quad \text{a.s.}$$

This shows that the event $\{B_{\sigma_k} + M^{(k)} - \sqrt{2}\sigma_k \geq a \text{ infinitely often}\}$ has probability 1 for every $a > 0$. In particular it implies that $\tilde{\mathcal{D}}_t^1((a, \infty)) \uparrow \infty$ a.s. as $t \rightarrow \infty$.

To conclude the proof, we observe that for all $\varepsilon > 0$, there exists $t > 0$ such that $\mathbf{P}(\tilde{\mathcal{D}}_t^1((0, \infty)) = 0) < \varepsilon$. At the same time it follows from (3.1) that $\tilde{\mathcal{D}}_t^{\varrho}$ is continuous in $\varrho \in \mathbb{R}$, hence for all $\varrho > 1$ small enough, we have

$$\mathbf{P}(\tilde{\mathcal{D}}^{\varrho}((0, \infty)) = 0) \leq \mathbf{P}(\tilde{\mathcal{D}}_t^{\varrho}((0, \infty)) = 0) \leq 2\varepsilon,$$

which shows that $\lim_{\varrho \rightarrow 1} \mathbf{P}(\tilde{\mathcal{D}}^{\varrho}((0, \infty)) = 0) = 0$, completing the proof. \square

We now prove that the cluster law \mathfrak{D}^ϱ associated by Bovier and Hartung [BH14] to the extremal point process of a variable speed branching Brownian motion is related to the point process $\tilde{\mathfrak{D}}^\varrho$ defined in (1.14). This connection is based on the well-known *spinal decomposition* for branching Brownian motions and so-called *probability tilting* techniques based on the additive martingale $W^{\sqrt{2}\varrho}$. Those ideas were pioneered by Lyons, Peamantle and Peres in [LPP95], then generalized to branching random walks by Lyons [Lyo97] and to general branching processes in [BK04].

Let (\mathcal{F}_t) be the filtration associated to the branching Brownian motion, defined by

$$\mathcal{F}_t = \sigma(\mathcal{N}_s, (X_s(u), u \in \mathcal{N}_s), s \leq t).$$

For $\varrho \in \mathbb{R}$ and $t \geq 0$, we introduce the size-biased law as

$$\bar{\mathbf{P}}_\varrho|_{\mathcal{F}_t} = W_t^{\sqrt{2}\varrho} \cdot \mathbf{P}|_{\mathcal{F}_t} \quad (3.6)$$

and call X under $\bar{\mathbf{P}}_\varrho$ the size biased process. The spinal decomposition links the size biased process with the so-called *branching Brownian motion with spine*. It describes the evolution of a branching particle system with a distinguished particle ξ_t , which behaves differently from the others. The system starts with the spine particle at position 0. This particle moves according to a Brownian motion with drift $\sqrt{2}\varrho$ and produces children at rate 2. Each of its children starts an independent (standard) branching Brownian motion from its birth place. We shall use the same notation \mathcal{N}_t for the set of particles alive at time t in this process (it is not a Yule process anymore), and write $\xi_t \in \mathcal{N}_t$ for the label of the spine particle. The law of this branching Brownian motion with spine is denoted by $\hat{\mathbf{P}}_\varrho$. The spinal decomposition can be stated as follows.

Theorem B (Spinal decomposition [LPP95, Lyo97]). *With the above notation, we have $\bar{\mathbf{P}}_\varrho|_{\mathcal{F}_t} = \hat{\mathbf{P}}_\varrho|_{\mathcal{F}_t}$ for all $t \geq 0$. Moreover, for all $u \in \mathcal{N}_t$,*

$$\hat{\mathbf{P}}_\varrho(\xi_t = u | \mathcal{F}_t) = \frac{e^{\sqrt{2}\varrho X_t(u) - t(\varrho^2 + 1)}}{W_t^{\sqrt{2}\varrho}}.$$

In words: the law of the marked tree $((X_s(u), u \in \mathcal{N}_s), s \leq t)$ has same law under probability $\hat{\mathbf{P}}$ and $\bar{\mathbf{P}}$. Moreover, conditionally on this process, one can choose to distinguish at random an individual $u \in \mathcal{N}_t$ with probability proportional to $e^{\sqrt{2}\varrho X_t(u)}$ to construct the law of the branching Brownian motion with spine.

With the spinal decomposition in hands we link the extremal point measure law of the branching Brownian motion \mathfrak{D}^ϱ with the point measure $\tilde{\mathfrak{D}}_t^\varrho$.

Lemma 3.3. *Let $\varrho \geq 1$, $\tilde{\mathfrak{D}}_t^\varrho$ the point measure defined in (3.1) and*

$$\mathcal{E}_t^* = \sum_{u \in \mathcal{N}_t} \delta_{X_t(u) - M_t}$$

the extremal process of the branching Brownian motion seen from the rightmost individual. For all non-negative measurable function F , we have

$$\mathbf{E}\left(F(\mathcal{E}_t^*) \mathbf{1}_{\{M_t \geq \sqrt{2}\varrho t\}}\right) = e^{(1-\varrho^2)t} \mathbf{E}\left(e^{\sqrt{2}\varrho B_t} \mathbf{1}_{\{B_t \leq 0\}} F\left(\tilde{\mathfrak{D}}_t^\varrho\right) \mathbf{1}_{\{\tilde{\mathfrak{D}}_t^\varrho((0, \infty)) = 0\}}\right).$$

Proof. For $t \geq 0$, denote by $u_t^{\text{tip}} \in \mathcal{N}_t$ the label of the largest particle alive at time t (which is a.s. unique). We observe that we can write

$$\mathbf{E}\left(F(\mathcal{E}_t^*) \mathbf{1}_{\{M_t \geq \sqrt{2}\varrho t\}}\right) = \mathbf{E}\left(\sum_{u \in \mathcal{N}_t} F(\mathcal{E}_t^*(u)) \mathbf{1}_{\{u = u_t^{\text{tip}}\}} \mathbf{1}_{\{M_t \geq \sqrt{2}\varrho t\}}\right),$$

where $\mathcal{E}_t^*(u) := \sum_{v \in \mathcal{N}_t} \delta_{X_t(v) - X_t(u)}$ is the extremal point measure seen from particle $u \in \mathcal{N}_t$. Thanks to the spinal decomposition and using (3.6), the above reads

$$\begin{aligned} \mathbf{E}\left(F(\mathcal{E}_t^*) \mathbf{1}_{\{M_t \geq \sqrt{2}\varrho t\}}\right) &= \bar{\mathbf{E}}_\varrho \left(\frac{1}{W_t^{\sqrt{2}\varrho}} \sum_{u \in \mathcal{N}_t} F(\mathcal{E}_t^*(u)) \mathbf{1}_{\{u = u_t^{\text{tip}}\}} \mathbf{1}_{\{M_t \geq \sqrt{2}\varrho t\}} \right) \\ &= \hat{\mathbf{E}}_\varrho \left(e^{-\sqrt{2}\varrho X_t(\xi_t) + (\varrho^2 + 1)t} F(\mathcal{E}_t^*(\xi_t)) \mathbf{1}_{\{\xi_t = u_t^{\text{tip}}\}} \mathbf{1}_{\{X_t(\xi_t) \geq \sqrt{2}\varrho t\}} \right). \end{aligned}$$

Next, we use the definition of the branching Brownian motion with spine to rewrite the above expression. For $s \in [0, t]$ let $B_s = W_{t-s} - W_t$ where $W_t = X_t(\xi_t) - \sqrt{2}\varrho t$ and for all $k \in \mathbb{N}$, σ_k is the k th instant at which

the spine gives birth to a new particle when running time backward from t (i.e. $t - \sigma_1$ is the last time before t at which the spine branches). Then, under $\widehat{\mathbf{E}}_\varrho$, B is a standard Brownian motion and (σ_k) are the atoms of a Poisson point process on \mathbb{R}_+ with intensity measure $2dx$. For each branching event σ_k , the spine gives birth to a standard branching Brownian motion that we call $X^{(k)} \equiv (X_s^{(k)}(u), u \in \mathcal{N}_s^{(k)}; s \in \mathbb{R}_+)$. With these notation we then get

$$\mathcal{E}_t^*(\xi_t) = \sum_{k \in \mathbb{N}: \sigma_k \leq t} \sum_{u \in \mathcal{N}_{\sigma_k}^{(k)}} \delta_{B_{\sigma_k} - \sqrt{2\varrho}\sigma_k + X_{\sigma_k}^{(k)}(u)}. \quad (3.7)$$

All that is left to do is thus to realise that under $\widehat{\mathbf{E}}_\varrho$, the pair of variables $(\mathcal{E}_t^*(\xi_t), X_t(\xi_t))$ jointly have the same law as $(\widetilde{\mathcal{D}}_t^\varrho, (-B_t + \sqrt{2\varrho}t))$ from (3.1). Thus substituting $\mathbf{1}_{\{\xi_t = u^{\text{tip}}\}}$ by $\mathbf{1}_{\{\widetilde{\mathcal{D}}_t^\varrho((0, \infty))=0\}}$ and $\mathbf{1}_{\{X_t(\xi_t) \geq \sqrt{2\varrho}t\}}$ by $\mathbf{1}_{\{B_t \leq 0\}}$ we conclude that

$$\mathbf{E}\left(F(\mathcal{E}_t^*)\mathbf{1}_{\{M_t \geq \sqrt{2\varrho}t\}}\right) = e^{(1-\varrho^2)t} \mathbf{E}\left(e^{\sqrt{2\varrho}B_t} \mathbf{1}_{\{B_t \leq 0\}} F(\widetilde{\mathcal{D}}_t^\varrho) \mathbf{1}_{\{\widetilde{\mathcal{D}}_t^\varrho((0, \infty))=0\}}\right). \quad \square$$

With Lemma 3.3 at hand, we now focus on the asymptotic behaviour of the extremal process of a branching Brownian motion conditioned to have a small minimum.

Lemma 3.4. *Let $\varrho > 1$ and $\varphi : \mathbb{R} \mapsto \mathbb{R}_+$ be a continuous function whose support is bounded from the left. Then*

$$\lim_{t \rightarrow \infty} \varrho t^{1/2} e^{(\varrho^2-1)t} \mathbf{E}\left(e^{-\langle \mathcal{E}_t^*, \varphi \rangle} \mathbf{1}_{\{M_t \geq \sqrt{2\varrho}t\}}\right) = \frac{1}{\sqrt{4\pi}} \mathbf{E}\left(e^{-\langle \widetilde{\mathcal{D}}^\varrho, \varphi \rangle} \mathbf{1}_{\{\widetilde{\mathcal{D}}^\varrho((0, \infty))=0\}}\right).$$

Proof. Fix φ as in the Lemma and $\varrho > 1$. Using Lemma 3.3, we may write for $t > 0$

$$e^{(\varrho^2-1)t} \mathbf{E}\left(e^{-\langle \mathcal{E}_t^*, \varphi \rangle} \mathbf{1}_{\{M_t \geq \sqrt{2\varrho}t\}}\right) = \mathbf{E}\left(e^{\sqrt{2\varrho}B_t} \mathbf{1}_{\{B_t \leq 0\}} e^{-\langle \widetilde{\mathcal{D}}_t^{\varrho, \varphi} \rangle} \mathbf{1}_{\{\widetilde{\mathcal{D}}_t^{\varrho, \varphi}((0, \infty))=0\}}\right).$$

We compute the right hand side by first conditioning on $B_t = x$. Introducing the point measure $\widetilde{\mathcal{D}}_t^{\varrho, x}$ as $\widetilde{\mathcal{D}}_t^\varrho$ conditioned on $\{B_t = x\}$, one gets

$$e^{(\varrho^2-1)t} \mathbf{E}\left(e^{-\langle \mathcal{E}_t^*, \varphi \rangle} \mathbf{1}_{\{M_t \geq \sqrt{2\varrho}t\}}\right) = \int_{-\infty}^0 \frac{dx}{\sqrt{2\pi t}} e^{\sqrt{2\varrho}x - \frac{x^2}{2t}} \mathbf{E}\left(e^{-\langle \widetilde{\mathcal{D}}_t^{\varrho, x}, \varphi \rangle} \mathbf{1}_{\{\widetilde{\mathcal{D}}_t^{\varrho, x}((0, \infty))=0\}}\right).$$

We are going to show that, for any fixed $x < 0$,

$$\lim_{t \rightarrow \infty} \mathbf{E}\left(e^{-\langle \widetilde{\mathcal{D}}_t^{\varrho, x}, \varphi \rangle} \mathbf{1}_{\{\widetilde{\mathcal{D}}_t^{\varrho, x}((0, \infty))=0\}}\right) = \mathbf{E}\left(e^{-\langle \widetilde{\mathcal{D}}^\varrho, \varphi \rangle} \mathbf{1}_{\{\widetilde{\mathcal{D}}^\varrho((0, \infty))=0\}}\right) \quad (3.8)$$

then, the Lemma follows by a simple application of the dominated convergence Theorem.

We shall couple the processes $\widetilde{\mathcal{D}}_t^{\varrho, x}$ and $\widetilde{\mathcal{D}}^\varrho$ in such a way, that for any fixed $x \in \mathbb{R}$,

$$\lim_{t \rightarrow \infty} \langle \widetilde{\mathcal{D}}_t^{\varrho, x}, \varphi \rangle = \langle \widetilde{\mathcal{D}}^\varrho, \varphi \rangle \quad \text{a.s.} \quad (3.9)$$

Then, this gives (3.8) (and the Lemma) by dominated convergence.

Fix $x \in \mathbb{R}$, recall that B is the Brownian underlying the construction of $\widetilde{\mathcal{D}}^\varrho$ and introduce for $0 \leq s \leq t$

$$\beta_s^{(t)} := B_s + \frac{s}{t}(x - B_t).$$

It is well-known that $(\beta_s^{(t)}; s \in [0, t])$ is a Brownian bridge from $\beta_0^{(t)} = 0$ to $\beta_t^{(t)} = x$.

Almost surely, there exists a random constant C such that

$$|B_s| \leq 1 + Cs^{0.51} \quad \text{for all } s \geq 0.$$

Then, with the same constant C , one checks that we have the following uniform bound:

$$|\beta_s^{(t)}| \leq 2 + |x| + Cs^{0.51} + sCt^{-0.49} \quad \text{for all } t \geq 0 \text{ and all } s \in [0, t]. \quad (3.10)$$

Let $a \in \mathbb{R}$ be such that $\varphi(x) = 0$ for $x < a$. Let us fix $0 < \varepsilon < \sqrt{2}(\varrho - 1)$. For all t large enough so that $Ct^{-0.49} < \varepsilon$, observe that

$$\beta_s^{(t)} - \sqrt{2\varrho}s \leq -(\sqrt{2\varrho} - \varepsilon)s + 1 + |x| + Cs^{0.51} \quad \text{for all } s \in [0, t].$$

As in the proof of Lemma 3.1, since $\sqrt{2}\varrho - \varepsilon > \sqrt{2}$, we conclude that there exists $T' < \infty$ a.s. such that *uniformly in t* , all the points in $\tilde{\mathcal{D}}_t^{\varrho,x}$ on the right of a come from branching events on the spine that occurred at times $\sigma_k \leq T'$.

Therefore, in computing $\langle \tilde{\mathcal{D}}_t^{\varrho,x}, \varphi \rangle$, one only needs to consider finitely many points: those that branched from the spine at a time smaller than T' . These points converge, as $t \rightarrow \infty$ to the corresponding points in $\tilde{\mathcal{D}}^\varrho$ (because $\beta_s^{(t)} \rightarrow B_s$ as $t \rightarrow \infty$) and, as φ is continuous, (3.9) holds and the Lemma is proved. \square

Using that last result, we now prove Theorem 1.2.

Proof of Theorem 1.2. We recall from (1.12) that for all $\varrho > 1$, we have

$$C(\varrho) = \varrho \lim_{t \rightarrow \infty} t^{1/2} e^{(\varrho^2 - 1)t} \mathbf{P}(M_t \geq \sqrt{2}\varrho t).$$

Therefore applying Lemma 3.4 with $\varphi \equiv 0$, we can rewrite $C(\varrho)$ as

$$C(\varrho) = \frac{1}{\sqrt{4\pi}} \mathbf{P}\left(\tilde{\mathcal{D}}^\varrho((0, \infty)) = 0\right). \quad (3.11)$$

We deduce from Lemma 3.2 that C is a continuous function on $[1, \infty]$ such that $C(1) = 0$ and $C(\infty) = 1/\sqrt{4\pi}$, proving the first part of Theorem 1.2.

We now turn to the proof of the second part. We recall that by the definition (1.5), given \mathcal{D}^ϱ a point process of law \mathfrak{D}^ϱ , for all continuous function φ with compact support, we have

$$\mathbf{E}\left(e^{-\langle \mathcal{D}^\varrho, \varphi \rangle}\right) = \lim_{t \rightarrow \infty} \mathbf{E}\left(e^{-\langle \tilde{\mathcal{E}}_t, \varphi \rangle} \Big| M_t \geq \sqrt{2}\varrho t\right).$$

At the same time, by Lemma 3.1 and Theorem 1.2(i) we get

$$\lim_{t \rightarrow \infty} \mathbf{E}\left(e^{-\langle \tilde{\mathcal{E}}_t, \varphi \rangle} \Big| M_t \geq \sqrt{2}\varrho t\right) = \lim_{t \rightarrow \infty} \frac{\mathbf{E}\left(e^{-\langle \tilde{\mathcal{E}}_t, \varphi \rangle} \mathbf{1}_{\{M_t \geq \sqrt{2}\varrho t\}}\right)}{\mathbf{P}(M_t \geq \sqrt{2}\varrho t)} = \frac{\mathbf{E}\left(e^{-\langle \tilde{\mathcal{D}}^\varrho, \varphi \rangle} \mathbf{1}_{\{\tilde{\mathcal{D}}^\varrho((0, \infty)) = 0\}}\right)}{\mathbf{P}(\tilde{\mathcal{D}}^\varrho((0, \infty)) = 0)}.$$

This shows that for all good test function $\varphi : \mathbb{R} \mapsto \mathbb{R}_+$

$$\mathbf{E}\left(e^{-\langle \mathcal{D}^\varrho, \varphi \rangle}\right) = \mathbf{E}\left(e^{-\langle \tilde{\mathcal{D}}^\varrho, \varphi \rangle} \Big| \tilde{\mathcal{D}}^\varrho((0, \infty)) = 0\right),$$

proving that $\mathbb{P}(\mathcal{D}^\varrho \in \cdot) = \mathbb{P}(\tilde{\mathcal{D}}^\varrho \in \cdot \mid \tilde{\mathcal{D}}^\varrho((0, \infty)) = 0)$. The weak continuity of \mathfrak{D}^ϱ for $\varrho \in (1, \infty)$ follows readily from Lemma 3.1 and the continuity of C . This concludes the proof. \square

3.1 An alternative proof for the first part of Theorem 1.2

In this section, we sketch an alternative proof for the representation of $C(\varrho)$ in terms of the point processes $\tilde{\mathcal{D}}^\varrho$ defined in (1.14). This proof is based on PDE analysis rather than tight probabilistic estimates.

Let M_t be the maximum at time t in a branching Brownian motion, and set $u(x, t) = \mathbf{P}(M_t > x)$ its tail distribution. We recall that u is solution to the Fisher-KPP equation $\partial_t u = \frac{1}{2} \partial_x^2 u + u - u^2$ with step initial condition $u(x, 0) = \mathbf{1}_{\{x < 0\}}$. We can thus compute $C(\varrho)$ from its definition (1.12) using the Feynman-Kac representation to evaluate $\mathbf{P}(M_t > \sqrt{2}\varrho t) = u(\sqrt{2}\varrho t, t)$.

Recall from Feynman-Kac that, given a function $K(x, t)$, the solution to $\partial_t h = \frac{1}{2} \partial_x^2 h + Kh$ can be written as

$$h(x, t) = \mathbf{E}^x \left[h(B_t, 0) \exp \left(\int_0^t ds K(B_s, t-s) \right) \right].$$

We apply this not to $u(x, t)$, but to $\partial_x u(x, t)$, the *derivative* of the solution to the Fisher-KPP equation, which is solution to $\partial_t [\partial_x u] = \frac{1}{2} \partial_x^2 [\partial_x u] + (1 - 2u) [\partial_x u]$ with initial condition $\partial_x u(x, 0) = -\delta(x)$. This gives

$$\begin{aligned} \partial_x u(x, t) &= -e^t \mathbf{E}^x \left[\delta(B_t) e^{-2 \int_0^t ds u(B_s, t-s)} \right] \\ &= -\frac{1}{\sqrt{2\pi t}} e^{t - \frac{x^2}{2t}} \mathbf{E}^{x \rightarrow 0} \left[e^{-2 \int_0^t ds u(B_s, t-s)} \right] \end{aligned}$$

where in the last expression B is a Brownian bridge from x to 0. We write $B_s = x(1 - \frac{s}{t}) - \tilde{B}_{t-s}$, so that \tilde{B} is a Brownian bridge from 0 to 0, we make the change of variable $\tilde{s} = t - s$ and we drop the tildas:

$$\partial_x u(x, t) = -\frac{1}{\sqrt{2\pi t}} e^{t - \frac{x^2}{2t}} \mathbf{E}^{0 \rightarrow 0} \left[e^{-2 \int_0^t ds u(x \frac{s}{t} - B_s, s)} \right]$$

Then, by setting $x = \sqrt{2} \varrho t + z$ and integrating over $z > 0$, one gets

$$u(\sqrt{2} \varrho t, t) = \frac{e^{(1-\varrho^2)t}}{\sqrt{2\pi t}} \int_0^\infty dz e^{-\sqrt{2} \varrho z - \frac{z^2}{2t}} \mathbf{E}^{0 \rightarrow 0} \left[e^{-2 \int_0^t ds u(z \frac{s}{t} + \sqrt{2} \varrho s - B_s, s)} \right]$$

For $\varrho > 1$, the quantity $u(z \frac{s}{t} + \sqrt{2} \varrho s - B_s, s)$ goes exponentially fast to 0 as $s \rightarrow \infty$, (unless B has wild fluctuations, but these events have a vanishingly small probability). Then, using the fact that B_s (the value at time s of a Brownian bridge over a time t) looks, as $t \rightarrow \infty$ for fixed s , more and more like a Brownian motion at time s , it is not very difficult (and akin to what was done in the proof of Lemma 3.4) to show that

$$\mathbf{E}^{0 \rightarrow 0} \left[e^{-2 \int_0^t ds u(z \frac{s}{t} + \sqrt{2} \varrho s - B_s, s)} \right] \xrightarrow[t \rightarrow \infty]{} \mathbf{E}^0 \left[e^{-2 \int_0^\infty ds u(\sqrt{2} \varrho s - B_s, s)} \right] \quad \text{for } \varrho > 1,$$

where B on the right hand side is a Brownian motion. In fact, the convergence also holds for $\varrho = 1$, as one can check that the quantities on either side are then equal to zero. Then, by dominated convergence,

$$\begin{aligned} \int_0^\infty dz e^{-\sqrt{2} \varrho z - \frac{z^2}{2t}} \mathbf{E}^{0 \rightarrow 0} \left[e^{-2 \int_0^t ds u(z \frac{s}{t} + \sqrt{2} \varrho s - B_s, s)} \right] \\ \xrightarrow[t \rightarrow \infty]{} \frac{1}{\sqrt{2} \varrho} \mathbf{E}^0 \left[e^{-2 \int_0^\infty ds u(\sqrt{2} \varrho s - B_s, s)} \right] \quad \text{for } \varrho \geq 1 \end{aligned}$$

and

$$C(\varrho) = \frac{1}{\sqrt{4\pi}} \mathbf{E}^0 \left[e^{-2 \int_0^\infty ds u(\sqrt{2} \varrho s - B_s, s)} \right]$$

Observe that in the point process (1.14) the probability that there are no particles on the right of 0 is then

$$\mathbf{P}(\tilde{\mathcal{D}}^\varrho((0, \infty)) = 0) = \mathbf{E} \left[\prod_k [1 - u(\sqrt{2} \varrho \sigma_k - B_{\sigma_k}, \sigma_k)] \right] = \mathbf{E} \left[e^{-2 \int_0^\infty ds u(\sqrt{2} \varrho s - B_s, s)} \right]$$

and therefore $C(\varrho) = \frac{1}{\sqrt{4\pi}} \mathbf{P}(\tilde{\mathcal{D}}^\varrho((0, \infty)) = 0)$, as claimed.

4 Proof of Theorem 1.1

In Section 2, Theorem 1.1 was proved in the special case $\mu_t = \gamma/t$. We need to extend this result to any function $t \mapsto \mu_t$ such that $\lim_{t \rightarrow \infty} t\mu_t$ exists and is positive. The main idea is to compare the Laplace transforms of the limiting point measures when $\lim_{t \rightarrow \infty} t\mu_t = \gamma$ to the one we get when $t\mu_t$ is a well-chosen constant.

First however, we need to provide a necessary and sufficient condition for the joint convergence in law of a point process and its maximum. The sequence of point processes $(\mathcal{P}_t)_{t \geq 0}$ on \mathbb{R} converges to \mathcal{P}_∞ in law for the topology of vague convergence on the space of locally finite point measure if and only if, for every compactly supported continuous function φ , the real valued random variables

$$\langle \mathcal{P}_t, \varphi \rangle := \int \varphi(x) \mathcal{P}_t(dx)$$

converge in law to $\langle \mathcal{P}_\infty, \varphi \rangle$ as $t \rightarrow \infty$ (see for instance [Kal02, Theorem 14.16]).

Lemma 4.1. *Let $(\mathcal{P}_t, \mathcal{P}_\infty)$ be point processes on \mathbb{R} such that $\mathcal{P}_\infty((0, \infty)) < \infty$ a.s. The four following statements are equivalent: as $t \rightarrow \infty$,*

- (i) $(\mathcal{P}_t, \max \mathcal{P}_t) \rightarrow_d (\mathcal{P}_\infty, \max \mathcal{P}_\infty)$ jointly;
- (ii) $\mathcal{P}_t \rightarrow_d \mathcal{P}_\infty$ and $\max \mathcal{P}_t \rightarrow_d \max \mathcal{P}_\infty$;
- (iii) $\mathbf{E}(e^{-\langle \mathcal{P}_t, \varphi \rangle}) \rightarrow \mathbf{E}(e^{-\langle \mathcal{P}_\infty, \varphi \rangle})$ for all continuous function φ with support bounded from the left.
- (iv) $\mathbf{E}(e^{-\langle \mathcal{P}_t, \varphi \rangle}) \rightarrow \mathbf{E}(e^{-\langle \mathcal{P}_\infty, \varphi \rangle})$ for all smooth non-decreasing function φ with support bounded from the left [and such that for some $a \in \mathbb{R}$ $\varphi(x)$ is constant for $x > a$].

Since this is rather classical and not the main object of the present work, the proof is postponed to an appendix.

The next proposition shows that $\gamma \mapsto \mathcal{E}_\infty^\gamma$, is continuous in distribution.

Proposition 4.2. *The family $((\mathcal{E}_\infty^\gamma, \max \mathcal{E}_\infty^\gamma); \gamma \in (0, \infty])$ defined in (1.10) is continuous in law. Otherwise said, as per Lemma 4.1, for all $\varphi : \mathbb{R} \rightarrow \mathbb{R}_+$ non-decreasing with bounded support from the left, the function*

$$\gamma \in (0, \infty] \mapsto \mathbf{E} \left(e^{-\langle \mathcal{E}_\infty^\gamma, \varphi \rangle} \right) \quad \text{is continuous.}$$

Proof. Let φ be a continuous non-decreasing function, with support bounded from the left. For any $\gamma > 0$, by Campbell's formula, we have

$$\mathbf{E} \left(e^{-\langle \mathcal{E}_\infty^\gamma, \varphi \rangle} \right) = \mathbf{E} \left(\exp \left(- \int_{\mathbb{R}} \mathbf{E} \left(1 - e^{-\langle \mathcal{D}^{d_\gamma, \varphi}(d_\gamma \cdot + z) \rangle} \right) \sqrt{2} C(d_\gamma) W_\infty^{\sqrt{2}c_\gamma} e^{-\sqrt{2}z} dz \right) \right). \quad (4.1)$$

We observe that $C(d_\gamma), W_\infty^{\sqrt{2}c_\gamma}$ as well as $\mathbf{E} \left(1 - e^{-\langle \mathcal{D}^{d_\gamma, \varphi}(d_\gamma \cdot + z) \rangle} \right)$ are non-negative for all $\gamma > 0$ and hence the exponential term on the right-hand side of (4.1) is bounded by 1. Therefore, by dominated convergence, it is enough to prove that each of the above functions is continuous.

It is obvious from the definition that both functions $\gamma \mapsto c_\gamma$ and $\gamma \mapsto d_\gamma$ are continuous in γ with $c_\gamma \in (0, 1)$ and $d_\gamma > 1$ for all $\gamma > 0$. At the same time, Theorem 1.2 says that both

$$\varrho \mapsto C(\varrho) \quad \text{and} \quad \varrho \mapsto \mathbf{E} \left(1 - e^{-\langle \mathcal{D}^\varrho, \varphi(d_\gamma \cdot + z) \rangle} \right)$$

are continuous in $\varrho > 1$. Finally, Biggins [Big92] proved that the convergence of the additive martingales in branching random walks is uniform on compact sets, *i.e.* for all $\varepsilon \in (0, 1/2)$, we have

$$\lim_{t \rightarrow \infty} \sup_{\varrho \in (\varepsilon-1, 1-\varepsilon)} \left| W_t^{\sqrt{2}\varrho} - W_\infty^{\sqrt{2}\varrho} \right| = 0 \quad \text{a.s.}$$

As a result, we deduce that $W_\infty^{\sqrt{2}\varrho}$ is continuous in ϱ , completing the proof. \square

To prove Theorem 1.1, we will need to compare the laws of extremal point processes of branching Ornstein-Uhlenbeck processes with different spring force functions μ_t . To that end, we shall use a more general version of Slepian's Lemma due to Kahane [Kah86] (see also [Bov16, Chapter 3.1] for a proof of Kahane's Theorem, as well as other Gaussian comparison estimates).

Theorem C (Kahane's Theorem [Kah86]). *Let $(X_j, j \leq n), (Y_j, j \leq n)$ be two centred Gaussian vectors. Let F be a twice differentiable function on \mathbb{R}^n with bounded second derivatives, that satisfies*

$$\frac{\partial^2 F}{\partial x_i \partial x_j}(x) \geq 0 \quad \text{if } \mathbf{E}(X_i X_j) > \mathbf{E}(Y_i Y_j), \quad \text{and} \quad \frac{\partial^2 F}{\partial x_i \partial x_j}(x) \leq 0 \quad \text{if } \mathbf{E}(X_i X_j) < \mathbf{E}(Y_i Y_j).$$

Then we have $\mathbf{E}(F(X)) \geq \mathbf{E}(F(Y))$.

From Kahane's Theorem C, we obtain Lemma 4.3 below, which allows to compare the Laplace transform of the extremal point measures of branching Ornstein-Uhlenbeck processes with different spring constants. Theorem 1.1 will then follow by comparison to the $\mu_t = \gamma/t$ case.

Lemma 4.3. *Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous non-negative non-decreasing function. Then, for all $\mu \leq \nu \leq \infty$ and $t > 0$, we have*

$$\mathbf{E} \left(\exp \left(-\langle \varphi, \mathcal{E}_t^\mu \rangle \right) \right) \geq \mathbf{E} \left(\exp \left(-\langle \varphi, \mathcal{E}_t^\nu \rangle \right) \right),$$

where \mathcal{E}_t^μ and \mathcal{E}_t^ν are the normalized, centred extremal point measures of branching Ornstein-Uhlenbeck processes as defined in (1.9) when $\nu < \infty$, and \mathcal{E}_t^∞ is the point measure defined as

$$\mathcal{E}_t^\infty = \sum_{u \in \mathcal{N}_t} \delta_{\widehat{X}_t^\infty(u) - \sqrt{2}t + \frac{1}{2\sqrt{2}} \log t},$$

where $(\widehat{X}_t^\infty(u), u \in \mathcal{N}_t)$ is a family of i.i.d. Gaussian random variables with variance t .

Remark: Note that as the spring constant μ increases toward ∞ , the vector of normalized leaves $(\widehat{X}_t^\mu(u), u \in \mathcal{N}_t)$ converges in law toward i.i.d. Gaussian random variables with variance t . This can be checked by computing the covariance function of this vector, conditionally on \mathcal{N}_t . Therefore, we have $\lim_{\mu \rightarrow \infty} \mathcal{E}_t^\mu = \mathcal{E}_t^\infty$ in law, for the topology of weak convergence, justifying the notation.

Proof. Remember that, since the branching events are independent of the spatial displacements, one can construct a branching Ornstein-Uhlenbeck with spring constant μ by first drawing its genealogical Yule tree $(\mathcal{N}_s, s \geq 0)$ then, conditionally on $(\mathcal{N}_s, s \geq 0)$ the spatial positions $(X_s^\mu(u), u \in \mathcal{N}_s, s \geq 0)$. Thus, given two spring constants μ, ν , we can construct the two branching Ornstein-Uhlenbeck processes X^μ and X^ν using the same $(\mathcal{N}_s, s \geq 0)$. In the rest of the proof we work conditionally on $(\mathcal{N}_s, s \geq 0)$ to study the extremal processes.

For $u, v \in \mathcal{N}_t$, we denote by $\tau_{u,v}$ the time of the most recent common ancestor of u and v . The covariance matrix of the Gaussian vectors X^μ is given by

$$\mathbf{Var}(X_t^\mu(u)) = \frac{1 - e^{-2\mu t}}{2\mu} \text{ and } \mathbf{Cov}(X_t^\mu(u), X_t^\mu(v)) = e^{-2\mu(t-\tau_{u,v})} \frac{1 - e^{-2\mu\tau_{u,v}}}{2\mu}.$$

Recall that we normalize positions to have variance t , setting as in (1.8)

$$\widehat{X}_t^\mu(u) = X_t^\mu(u) \sqrt{\frac{2\mu t}{1 - e^{-2\mu t}}}.$$

As a result, we have that

$$\mathbf{Cov}(\widehat{X}_t^\mu(u), \widehat{X}_t^\mu(v)) = t \frac{e^{2\mu\tau_{u,v}} - 1}{e^{2\mu t} - 1}. \quad (4.2)$$

Observe that when $\mu \leq \nu$ (including the case $\nu = \infty$), we have that

$$\mathbf{Cov}(\widehat{X}_t^\mu(u), \widehat{X}_t^\mu(v)) \geq \mathbf{Cov}(\widehat{X}_t^\nu(u), \widehat{X}_t^\nu(v)),$$

for all $u, v \in \mathcal{N}_t$. Indeed, it is easy to verify that for all $0 < s < t$ fixed the function $\mu \mapsto \frac{e^{2\mu s} - 1}{e^{2\mu t} - 1}$ is non-increasing in $\mu \in \mathbb{R}$.

We start by showing the result for $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, a smooth non-negative non-decreasing function, such that φ' has compact support. Then the function

$$F : x \in \mathbb{R}^{\mathcal{N}_t} \mapsto \exp\left(-\sum_{u \in \mathcal{N}_t} \varphi(x_u)\right),$$

is twice differentiable and constant outside of a compact, hence its second derivatives are bounded. It satisfies

$$\frac{\partial^2 F}{\partial x_i \partial x_j}(x) = \varphi'(x_i) \varphi'(x_j) \exp\left(-\sum_{u \in \mathcal{N}_t} \varphi(x_u)\right) \geq 0, \quad \text{for all } i \neq j \in \mathcal{N}_t,$$

by monotonicity of φ . Thus, we can apply Kahane's Theorem C, and we have that for all $\mu \leq \nu \leq \infty$,

$$\mathbf{E}\left(F(\widehat{X}_t^\mu) \mid \mathcal{N}_t\right) \geq \mathbf{E}\left(F(\widehat{X}_t^\nu) \mid \mathcal{N}_t\right).$$

Therefore, averaging over the genealogical tree $(\mathcal{N}_t, t \geq 0)$, we obtain that

$$\mu \in (-\infty, \infty] \mapsto \mathbf{E}\left(\exp(-\langle \mathcal{E}_t^\mu, \varphi \rangle)\right)$$

is non-increasing.

To conclude, note that any continuous non-decreasing non-negative function φ can be approached from below a sequence $(\varphi_n, n \geq 1)$ of smooth non-decreasing functions with derivatives having compact support. Moreover,

$$\lim_{n \rightarrow \infty} \mathbf{E}\left(\exp(-\langle \mathcal{E}_t^\mu, \varphi_n \rangle)\right) = \mathbf{E}\left(\exp(-\langle \mathcal{E}_t^\mu, \varphi \rangle)\right)$$

by monotone convergence. Hence, we conclude that $\mu \mapsto \mathbf{E}\left(\exp(-\langle \mathcal{E}_t^\mu, \varphi \rangle)\right)$ is non-increasing. \square

We now show that the point process \mathcal{E}_t^∞ converges in law, as $t \rightarrow \infty$, toward the Poisson point process $\mathcal{E}_\infty^\infty$ defined in (1.10), jointly with its maximum.

Lemma 4.4. *We have*

$$\lim_{t \rightarrow \infty} (\mathcal{E}_t^\infty, \max \mathcal{E}_t^\infty) = (\mathcal{E}_\infty^\infty, \max \mathcal{E}_\infty^\infty) \quad \text{in law.}$$

Proof. Note this result can be straightforwardly deduced from standard extreme values theory for Gaussian processes. We include a direct self-contained proof which furthermore demonstrates how our toolbox can be used. Recall from Lemma 4.1 that to prove the joint convergence of \mathcal{E}_t^∞ and its maximum, it is enough to prove the convergence of $\mathbf{E}(\exp(-\langle \mathcal{E}_t^\infty, f \rangle))$ for all non-decreasing continuous functions f with support bounded from the left.

Observe, by Campbell's formula for Poisson point processes, that

$$\mathbf{E}(\exp(-\langle \mathcal{E}_\infty^\infty, f \rangle) | W_\infty^0) = \exp\left(-W_\infty^0 \int (1 - e^{-f(y)}) \frac{\sqrt{2}e^{-\sqrt{2}y}}{\sqrt{4\pi}} dy\right).$$

Therefore, as W_∞^0 is distributed as a standard exponential random variable, we have

$$\mathbf{E}(\exp(-\langle \mathcal{E}_\infty^\infty, f \rangle)) = \left(1 + \frac{1}{\sqrt{2\pi}} \int (1 - e^{-f(y)}) e^{-\sqrt{2}y} dy\right)^{-1} \quad (4.3)$$

On the other hand, conditioning with respect to $\#\mathcal{N}_t$ the number of leaves at time t , and writing X_t for a Gaussian random variable with variance t and $m_t = \sqrt{2t} - \frac{1}{2\sqrt{2}} \log t$, we have

$$\mathbf{E}(\exp(-\langle \mathcal{E}_t^\infty, f \rangle)) = \mathbf{E}\left(\mathbf{E}\left(e^{-f(X_t - m_t)}\right)^{\#\mathcal{N}_t}\right)$$

As $\#\mathcal{N}_t$ is a geometric random variable with parameter e^{-t} , we have

$$\begin{aligned} \mathbf{E}(\exp(-\langle \mathcal{E}_t^\infty, f \rangle)) &= \frac{e^{-t} \mathbf{E}(e^{-f(X_t - m_t)})}{1 - (1 - e^{-t}) \mathbf{E}(e^{-f(X_t - m_t)})} \\ &= \frac{\mathbf{E}(e^{-f(X_t - m_t)})}{e^t \mathbf{E}((1 - e^{-f(X_t - m_t)})) + \mathbf{E}(e^{-f(X_t - m_t)})} \end{aligned} \quad (4.4)$$

Therefore, to complete the proof, it is enough to prove that

$$\lim_{t \rightarrow \infty} e^t \mathbf{E}\left(\left(1 - e^{-f(X_t - m_t)}\right)\right) = \frac{1}{\sqrt{2\pi}} \int (1 - e^{-f(y)}) e^{-\sqrt{2}y} dy, \quad (4.5)$$

which implies that (4.4) converges to (4.3) as $t \rightarrow \infty$.

We now turn to the proof of (4.5). By change of variables, we have

$$\begin{aligned} \mathbf{E}\left(1 - e^{-f(X_t - m_t)}\right) &= \int (1 - e^{-f(x - m_t)}) \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} dx \\ &= \int \frac{(1 - e^{-f(y)})}{\sqrt{2\pi t}} e^{-\sqrt{2}y} e^{-t + \frac{1}{2} \log t} e^{-\frac{y^2}{2t} - y \frac{\sqrt{2} \log t}{2t} - \frac{(\log t)^2}{8t}} dy. \end{aligned}$$

Hence, as the support of $y \mapsto 1 - e^{-f(y)}$ is bounded from the left, we can apply the dominated convergence theorem in the above equation yielding, as $t \rightarrow \infty$,

$$\mathbf{E}\left(1 - e^{-f(X_t - m_t)}\right) \sim \frac{e^{-t}}{\sqrt{2\pi}} \int (1 - e^{-f(y)}) e^{-\sqrt{2}y} dy$$

concluding the proof. \square

We can now prove Theorem 1.1.

Proof of Theorem 1.1. We denote by $(X_t^{\mu_t}(u), u \in \mathcal{N}_t)$ a branching Ornstein-Uhlenbeck process with spring constant μ_t and assume that

$$\lim_{t \rightarrow \infty} t\mu_t = \gamma \in (0, \infty].$$

We first consider the case $\gamma < \infty$. Let $0 < \underline{\gamma} < \gamma < \bar{\gamma}$. For t large enough $\underline{\gamma}/t < \mu_t < \bar{\gamma}/t$. Thus, by Lemma 4.3,

$$\mathbf{E}\left(\exp\left(-\langle \varphi, \mathcal{E}_t^{\underline{\gamma}/t} \rangle\right)\right) \leq \mathbf{E}\left(\exp\left(-\langle \varphi, \mathcal{E}_t^{\mu_t} \rangle\right)\right) \leq \mathbf{E}\left(\exp\left(-\langle \varphi, \mathcal{E}_t^{\bar{\gamma}/t} \rangle\right)\right),$$

for all φ continuous non-decreasing functions $\mathbb{R} \rightarrow \mathbb{R}_+$.

As a result, taking $t \rightarrow \infty$, and supposing furthermore that φ has bounded support on the left, combining Lemma 4.1 and Theorem A, we obtain that

$$\begin{aligned} \liminf_{t \rightarrow \infty} \mathbf{E} \left(\exp \left(-\langle \varphi, \mathcal{E}_t^{\mu_t} \rangle \right) \right) &\geq \mathbf{E} \left(\exp \left(-\langle \varphi, \mathcal{E}_\infty^\gamma \rangle \right) \right) \\ \limsup_{t \rightarrow \infty} \mathbf{E} \left(\exp \left(-\langle \varphi, \mathcal{E}_t^{\mu_t} \rangle \right) \right) &\leq \mathbf{E} \left(\exp \left(-\langle \varphi, \mathcal{E}_\infty^{\bar{\gamma}} \rangle \right) \right) \end{aligned}$$

Now, letting $\underline{\gamma} \uparrow \gamma$ and $\bar{\gamma} \downarrow \gamma$, using Proposition 4.2 we obtain

$$\lim_{t \rightarrow \infty} \mathbf{E} \left(\exp \left(-\langle \varphi, \mathcal{E}_t^{\mu_t} \rangle \right) \right) = \mathbf{E} \left(\exp \left(-\langle \varphi, \mathcal{E}^\gamma \rangle \right) \right).$$

We conclude by Lemma 4.1 that $(\mathcal{E}_t^{\mu_t}, \max \mathcal{E}_t^{\mu_t})$ converge toward $(\mathcal{E}^\gamma, \max \mathcal{E}^\gamma)$.

We now consider the case $\gamma = \infty$. If $\lim_{t \rightarrow \infty} t\mu_t = \infty$, then for all $\underline{\gamma} > 0$, one has $\mu_t \geq \underline{\gamma}/t$ for all t large enough. On the other hand, $(X_t^{\mu_t}(u), u \in \mathcal{N}_t)$ is straightforwardly “more correlated” than i.i.d. Gaussian random variables (formally corresponding to the case $\gamma = \infty$). Hence, using again Lemma 4.3, then Lemma 4.1 and Theorem A for the lower bound, and Lemma 4.4 for the upper bound, we obtain

$$\begin{aligned} \liminf_{t \rightarrow \infty} \mathbf{E} \left(\exp \left(-\langle \varphi, \mathcal{E}_t^{\mu_t} \rangle \right) \right) &\geq \mathbf{E} \left(\exp \left(-\langle \varphi, \mathcal{E}_\infty^\gamma \rangle \right) \right), \\ \limsup_{t \rightarrow \infty} \mathbf{E} \left(\exp \left(-\langle \varphi, \mathcal{E}_t^{\mu_t} \rangle \right) \right) &\leq \mathbf{E} \left(\exp \left(-\langle \varphi, \mathcal{E}_\infty^\infty \rangle \right) \right) \end{aligned}$$

for all smooth increasing function $\varphi : \mathbb{R} \rightarrow [0, 1]$ such that φ' has compact support. Letting $\underline{\gamma} \rightarrow \infty$ concludes the proof of Theorem 1.1. \square

5 Large spring constant

This section is devoted to an alternative and self-contained proof of Theorem 1.1 in the case μ constant.

The argument is based on a precise estimate on the tail of the maximal displacement and follows closely the standard strategy used e.g. in the case of the branching Brownian motion. However, parts of the argument are made simpler by the fact that in this *strong disorder* regime particles are much less correlated.

It will be convenient to change the centring function in (1.9). Precisely, we set

$$\tilde{m}_t := m_t - \frac{\log(4\pi)}{2\sqrt{2}} = \sqrt{2}t - \frac{\log(4\pi t)}{2\sqrt{2}}.$$

Recall that $(X_t^\mu(u), u \in \mathcal{N}_t)$ are the positions of the particles, and that for all $u \in \mathcal{N}_t$ we define the normalized positions as

$$\hat{X}_t^\mu(u) = \sqrt{\frac{2\mu t}{1 - e^{-2\mu t}}} X_t^\mu(u).$$

We defined the point process as

$$\mathcal{E}_t^\mu = \sum_{u \in \mathcal{N}_t} \delta_{\{\hat{X}_t^\mu(u) - m_t\}}.$$

Here, $\gamma = \infty$ and thus the content of Theorem 1.1 is that \mathcal{E}_t^μ converges weakly as $t \rightarrow \infty$ to a Poisson point process (PPP) with intensity $((2\pi)^{-1/2} W_\infty^0 e^{-\sqrt{2}x} dx)$. Notice that this is equivalent to the fact that

$$\tilde{\mathcal{E}}_t^\mu = \sum_{u \in \mathcal{N}_t} \delta_{\{\hat{X}_t^\mu(u) - \tilde{m}_t\}}$$

converges weakly as $t \rightarrow \infty$ to a PPP $(\sqrt{2}W_\infty^0 e^{-\sqrt{2}x} dx)$.

The main element in the proof is the following proposition on the right tail probabilities of $\max_{u \in \mathcal{N}_t} \hat{X}_t^\mu(u)$ the rightmost normalized position at time t .

Proposition 5.1. *Assume that $\mu > 0$ is fixed, then for all $\varepsilon > 0$ there exists $K > 0$ such that*

$$\limsup_{t \rightarrow \infty} \mathbf{P} \left(\max_{u \in \mathcal{N}_t} \hat{X}_t^\mu(u) \geq \tilde{m}_t + z \right) - e^{-\sqrt{2}z} \leq \varepsilon e^{-\sqrt{2}z}, \quad \text{for all } z > K. \quad (5.1)$$

The proof of Proposition 5.1 is based on the second moment method. Precisely, we obtain bounds on the first and second moments of

$$Z_t(z) = \tilde{\mathcal{E}}_t^\mu([z, \infty)) := \sum_{u \in \mathcal{N}_t} \mathbf{1}_{\{\hat{X}_t^\mu(u) - \tilde{m}_t \geq z\}},$$

the number of particles in the centred normalized extremal process at time t lying in the set $[z, \infty)$. We then use these bounds to estimate the right tail of $\max_{u \in \mathcal{N}_t} \widehat{X}_t^\mu(u)$ via Markov and Paley-Zygmund inequalities.

For the first moment of $Z_t(z)$ we shall use the many-to-one lemma. This is a widely used formula for branching Markov processes, whose history can be traced back at least to the early work of Kahane and Peyrière [Pey74, KP76]. The version we use in this article for branching Ornstein-Uhlenbeck processes (see e.g. [HR17, Lemma 1]) is the following.

Lemma 5.2 (Many-to-one formula). *For all $t > 0$ and non-negative measurable function f , we have*

$$\mathbf{E} \left(\sum_{u \in \mathcal{N}_t} f(X_t^\mu(u)) \right) = e^t \mathbf{E} (f(X_t^\mu)),$$

where X^μ is an Ornstein-Uhlenbeck process with spring constant μ .

This simple version of the many-to-one Lemma is rather straightforward, as there are on average e^t particles at time t , and the position of each given particle has the same law as a single Ornstein-Uhlenbeck. With the many-to-one lemma, we can now bound the mean of $Z_t(z)$.

Lemma 5.3. *For all $\mu > 0$, we have $\lim_{t \rightarrow \infty} \sup_{|z| \leq t^{0.49}} \left| e^{\sqrt{2}z} \mathbf{E}[Z_t(z)] - 1 \right| = 0$.*

Proof. By the many-to-one lemma, we have

$$\mathbf{E}[Z_t(z)] = e^t \mathbf{P} \left(\widehat{X}_t^\mu \geq \widetilde{m}_t + z \right).$$

As \widehat{X}_t^μ is a centred Gaussian random variable with variance t , the classical tail bound (A.2) yields

$$\frac{t^{1/2}}{x\sqrt{2\pi}} \left(1 - 2\frac{t}{x^2} \right) e^{-\frac{x^2}{2t}} \leq \mathbf{P} \left(\widehat{X}_t^\mu \geq x \right) \leq \frac{t^{1/2}}{x\sqrt{2\pi}} e^{-\frac{x^2}{2t}}, \quad \text{for all } x \geq 0.$$

Set now $x = \widetilde{m}_t + z$ with $|z| \leq t^{0.49}$ to obtain

$$\begin{aligned} \frac{x^2}{2t} &= t - \frac{1}{2} \log(4\pi t) + \sqrt{2}z + \frac{\log(4\pi t)^2}{16t} + \frac{-z \log(4\pi t) + \sqrt{2}z^2}{2\sqrt{2}t} \\ &= t - \frac{1}{2} \log(4\pi t) + \sqrt{2}z + \mathcal{O}(t^{-0.02}) \text{ as } t \rightarrow \infty. \end{aligned}$$

Above, the $\mathcal{O}(t^{-0.02})$ term is uniformly bounded in $|z| \leq t^{0.49}$. Therefore, we obtain

$$\lim_{t \rightarrow +\infty} \sup_{|z| \leq t^{0.49}} \left| e^{t+\sqrt{2}z} \sqrt{\frac{t}{2\pi}} \frac{1}{\widetilde{m}_t + z} e^{-\frac{(\widetilde{m}_t+z)^2}{2t}} - 1 \right| = 0,$$

which implies that $\lim_{t \rightarrow +\infty} \sup_{|z| \leq t^{0.49}} \left| e^{\sqrt{2}z} \mathbf{E}[Z_t(z)] - 1 \right| = 0$ and proves the statement. \square

For the second moment of $Z_t(z)$, we use the many-to-two lemma. We refer again to [HR17, Lemma 1] for a proof.

Lemma 5.4 (Many-to-two formula). *Let f be non-negative measurable function, then*

$$\mathbf{E} \left(\left(\sum_{u \in \mathcal{N}_t} f(X_t^\mu(u)) \right)^2 \right) = e^t \mathbf{E} (f(X_t^\mu)^2) + 2 \int_0^t e^{2t-s} \mathbf{E} (f(X_{s,t}^{\mu,1}) f(X_{s,t}^{\mu,2})) ds.$$

Above, $X_{s,t}^{\mu,1}$ and $X_{s,t}^{\mu,2}$ are Ornstein-Uhlenbeck processes with spring constant μ such that $X_{s,t}^{\mu,1} = X_{s,t}^{\mu,2}$ if $t \leq s$ and the evolution is independent for $t > s$.

We obtain an uniform bound on the second moment of $Z_t(z)$.

Lemma 5.5. *There exists a constant C such that for every $K > 0$ the following inequality holds:*

$$\limsup_{t \rightarrow \infty} \sup_{0 < z < K} \mathbf{E}[Z_t(z)^2] - \mathbf{E}[Z_t(z)] \leq C e^{-2\sqrt{2}z}. \quad (5.2)$$

The proof of this result is only a technical computation and hence has been postponed to the appendix. We now use the above first and second moment estimates to obtain Proposition 5.1.

Proof of Proposition 5.1. We have $\mathbf{P}(\max \widehat{X}_t^\mu(u) \geq \widetilde{m}_t + z) \leq \mathbf{E}[Z_t(z)]$ by the Markov inequality, hence by Lemma 5.3, we have

$$\limsup_{z \rightarrow \infty} \limsup_{t \rightarrow \infty} e^{\sqrt{2}z} \mathbf{P} \left(\max \widehat{X}_t^\mu(u) \geq \widetilde{m}_t + z \right) \leq 1.$$

For the lower bound, we use Paley-Zugmund inequality to bound the probability by $\mathbf{E}(Z_t(z))^2 / \mathbf{E}(Z_t(z)^2)$. Therefore, by Lemma 5.5 we have that

$$\liminf_{z \rightarrow \infty} \liminf_{t \rightarrow \infty} e^{\sqrt{2}z} \mathbf{P} \left(\max \widehat{X}_t^\mu(u) \geq \widetilde{m}_t + z \right) \geq \liminf_{z \rightarrow \infty} \liminf_{t \rightarrow \infty} e^{\sqrt{2}z} \frac{\mathbf{E}(Z_t(z))^2}{\mathbf{E}(Z_t(z)^2)} \geq 1,$$

which concludes the proof. \square

Thanks to Proposition 5.1, we can now rely on standard branching techniques to obtain the full asymptotic behaviour of the maximal displacement of a branching Ornstein-Uhlenbeck process with fixed spring constant.

Lemma 5.6. *For all $\mu > 0$ and $z \in \mathbb{R}$, we have*

$$\lim_{t \rightarrow \infty} \mathbf{P} \left(\max \widehat{X}_t^\mu(u) \leq \widetilde{m}_t + z \right) = \mathbf{E} \left(\exp(-W_\infty^0 e^{-\sqrt{2}z}) \right) = (1 + e^{-\sqrt{2}z})^{-1}.$$

Proof. The second equality follows readily from the well-known fact that W_∞^0 , the limit of the martingale associated to the Yule process, is exponentially distributed. Therefore, we only prove the first equality above.

We denote by (\mathcal{F}_t) the natural filtration of the branching Ornstein-Uhlenbeck process. Thanks to the branching property, we have for all $t_0 > 0$ (to be determined latter) and $t > t_0$

$$\mathbf{P} \left(\max \widehat{X}_t^\mu(u) \leq \widetilde{m}_t + z \mid \mathcal{F}_{t_0} \right) = \prod_{u \in \mathcal{N}_{t_0}} \mathbf{P}_{X_{t_0}(u)} \left(\sqrt{\frac{2\mu t}{1-e^{-2\mu t}}} \max_{u \in \mathcal{N}_{t-t_0}} X_{t-t_0}^\mu(u) \leq \widetilde{m}_t + z \right), \quad (5.3)$$

where \mathbf{P}_x stands for the law of the branching Ornstein-Uhlenbeck process starting with one particle at position x . Observe that by (1.7), we can rewrite

$$\mathbf{P}_x \left(\sqrt{\frac{2\mu t}{1-e^{-2\mu t}}} \max_{u \in \mathcal{N}_{t-t_0}} X_{t-t_0}^\mu(u) \leq \widetilde{m}_t + z \right) = \mathbf{P} \left(\sqrt{\frac{2\mu t}{1-e^{-2\mu t}}} \max_{u \in \mathcal{N}_{t-t_0}} (X_{t-t_0}^\mu(u) + x e^{-\mu(t-t_0)}) \leq \widetilde{m}_t + z \right).$$

Writing $\lambda_{\mu t} = \sqrt{\frac{2\mu t}{1-e^{-2\mu t}}}$, fixing t_0 and letting $t \rightarrow \infty$ we have

$$\lambda_{\mu(t-t_0)} = \lambda_{\mu t} \left(1 - \frac{t_0}{2t} + o(t^{-1}) \right) \quad \text{and} \quad \widetilde{m}_t - \widetilde{m}_{t-t_0} - \sqrt{2}t_0 = o(1).$$

Moreover, for a fixed $x \in \mathbb{R}$, the term $x e^{-\mu(t-t_0)}$ is also $o(1)$ as $t \rightarrow \infty$. Thus

$$\begin{aligned} & \mathbf{P}_x \left(\lambda_{\mu t} \max_{u \in \mathcal{N}_{t-t_0}} X_{t-t_0}^\mu(u) \leq \widetilde{m}_t + z \right) \\ &= \mathbf{P} \left(\lambda_{\mu(t-t_0)} \max_{u \in \mathcal{N}_{t-t_0}} X_{t-t_0}^\mu(u) \leq (\widetilde{m}_{t-t_0} + \sqrt{2}t_0 + z + o(1)) \left(1 - \frac{t_0}{2t} + o(t^{-1}) \right) \right) \\ &= \mathbf{P} \left(\max_{u \in \mathcal{N}_{t-t_0}} \widehat{X}_{t-t_0}^\mu(u) \leq \widetilde{m}_{t-t_0} + z + \frac{\sqrt{2}}{2}t_0 + o(1) \right). \end{aligned}$$

Now we use Proposition 5.1 to estimate the above. In particular, we conclude that given $\varepsilon > 0$, $x \in \mathbb{R}$ and $z \in \mathbb{R}$, if t_0 is large enough

$$\begin{aligned} \limsup_{t \rightarrow \infty} \mathbf{P}_x \left(\lambda_{\mu t} \max_{u \in \mathcal{N}_{t-t_0}} X_{t-t_0}^\mu(u) \leq \widetilde{m}_t + z \right) &\leq 1 - (1 - \varepsilon) e^{-\sqrt{2}z - t_0}, \\ \liminf_{t \rightarrow \infty} \mathbf{P}_x \left(\lambda_{\mu t} \max_{u \in \mathcal{N}_{t-t_0}} X_{t-t_0}^\mu(u) \leq \widetilde{m}_t + z \right) &\geq 1 - (1 + \varepsilon) e^{-\sqrt{2}z - t_0}. \end{aligned}$$

In (5.3), the lim sup leads to

$$\limsup_{t \rightarrow \infty} \mathbf{P} \left(\max \widehat{X}_t^\mu(u) \leq \widetilde{m}_t + z \mid \mathcal{F}_{t_0} \right) \leq [1 - (1 - \varepsilon) e^{-\sqrt{2}z - t_0}]^{\#\mathcal{N}_{t_0}}$$

and

$$\limsup_{t_0 \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbf{P} \left(\max \widehat{X}_t^\mu(u) \leq \widetilde{m}_t + z \mid \mathcal{F}_{t_0} \right) \leq e^{-(1-\varepsilon)W_\infty^0 e^{-\sqrt{2}z}}$$

where we used $\lim_{t_0 \rightarrow \infty} e^{-t_0} (\#\mathcal{N}_{t_0}^0) = W_\infty^0$ a.s. Taking the expectation gives

$$\limsup_{t \rightarrow \infty} \mathbf{P} \left(\max \widehat{X}_t^\mu(u) \leq \widetilde{m}_t + z \right) \leq \mathbf{E} \left[e^{-(1-\varepsilon)W_\infty^0 e^{-\sqrt{2}z}} \right]$$

Doing the same on the lim inf leads to the conclusion that $\mathbf{P}(\max \widehat{X}_t^\mu(u) \leq \widetilde{m}_t + z)$ has a limit as $t \rightarrow \infty$ and, then,

$$\lim_{t_0 \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbf{P} \left(\lambda_{\mu t} \max_{u \in \mathcal{N}_{t-t_0}} X_t^\mu(u) \leq \widetilde{m}_t + z \mid \mathcal{F}_{t_0} \right) = e^{-W_\infty^0 e^{-\sqrt{2}z}} \quad \text{a.s.} \quad \square$$

Using Lemma 5.6, we are finally ready to prove Theorems 1.1 for $\mu > 0$ fixed.

Proof of Theorem 1.1 for $\mu > 0$ fixed. We shall prove the equivalent statement: $\widetilde{\mathcal{E}}_t$ converges in distribution to $\widetilde{\mathcal{E}}_\infty := \text{PPP}(\sqrt{2}W_\infty^0 e^{-\sqrt{2}z} dz)$.

Introduce

$$\widetilde{\mathcal{E}}_t^{t_0} = \sum_{u \in \mathcal{N}_t} \delta_{\widehat{X}_t^\mu(u) - \widetilde{m}_t} \mathbf{1}_{\{X_t(u) = \max_{v \in \mathcal{N}_t: \tau_{u,v} \geq t_0} X_t(v)\}},$$

where $\tau_{u,v}$ is the last time at which the most recent common ancestor of u and v was alive. Note that $\widetilde{\mathcal{E}}_t^{t_0}$ corresponds to the set of *local leaders* in $\widetilde{\mathcal{E}}_t$, i.e. individuals that are higher than all their close relatives.

The proof is then in two steps:

- Let $a \in \mathbb{R}$ and φ a smooth positive bounded function with support in (a, ∞) . Then

$$\lim_{t_0 \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbf{E} \left(e^{-\langle \widetilde{\mathcal{E}}_t^{t_0}, \varphi \rangle} \right) = \mathbf{E} \left(e^{-\langle \widetilde{\mathcal{E}}_\infty, \varphi \rangle} \right)$$

- $\lim_{t_0 \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbf{P} \left(\widetilde{\mathcal{E}}_t^{t_0}((a, \infty)) \neq \widetilde{\mathcal{E}}_t((a, \infty)) \right) = 0$

As the points in $\widetilde{\mathcal{E}}_t^{t_0}$ are a subset of the points in $\widetilde{\mathcal{E}}_t$, the last item means that the probability the two point processes are different on the right of a goes to 0. This allows to show with the first item that $\lim_{t \rightarrow \infty} \mathbf{E} \left(e^{-\langle \widetilde{\mathcal{E}}_t, \varphi \rangle} \right) = \mathbf{E} \left(e^{-\langle \widetilde{\mathcal{E}}_\infty, \varphi \rangle} \right)$, and the proof is completed by applying Lemma 4.1.

Proof of the first item. We rewrite $\widetilde{\mathcal{E}}_t^{t_0}$ in a more tractable way: for $u \in \mathcal{N}_{t_0}$, we set

$$\widehat{M}_t(u) = \max \{ \widehat{X}_t^\mu(v) : v \in \mathcal{N}_t \text{ such that } v \text{ is a descendant of } u \},$$

the maximal displacement among the descendants at time t of u . It is then straightforward from the definition that

$$\widetilde{\mathcal{E}}_t^{t_0} = \sum_{u \in \mathcal{N}_{t_0}} \delta_{\widehat{M}_t(u) - \widetilde{m}_t}$$

It follows from (1.7) that conditionally to $\mathcal{F}_{t_0} := \sigma\{\widehat{X}_s(u), u \in \mathcal{N}_s, s \leq t_0\}$,

$$(\widehat{M}_t(u) - \widehat{X}_{t_0}^\mu(u) e^{-\mu(t-t_0)}, u \in \mathcal{N}_{t_0}) = (R_{t-t_0}^u, u \in \mathcal{N}_{t_0}) \text{ in law,}$$

where $(R_{t-t_0}^u; u \in \mathcal{N}_{t_0})$ are i.i.d. copies independent from \mathcal{F}_{t_0} of $\max \widehat{X}_{t-t_0}^\mu$, the rightmost position at time $t - t_0$ of a branching Ornstein-Uhlenbeck process started from 0.

Moreover, Lemma 5.6 proves

$$\lim_{t \rightarrow \infty} R_{t-t_0}^u - \widetilde{m}_t = G_{t_0}^u \quad \text{in law,}$$

with $(G_{t_0}^u, u \in \mathcal{N}_{t_0})$ i.i.d. random variables with cumulative distribution function $\mathbf{P}(G_{t_0}^u \leq z) = (1 + e^{-\sqrt{2}z - t_0})^{-1}$.

Now we keep t_0 fixed and take $t \rightarrow \infty$, so the above shows that

$$\lim_{t \rightarrow \infty} \langle \widetilde{\mathcal{E}}_t^{t_0}, \varphi \rangle = \sum_{u \in \mathcal{N}_{t_0}} \varphi(G_{t_0}^u), \quad \text{in law, conditionnaly on } \mathcal{F}_{t_0}.$$

and

$$\lim_{t \rightarrow \infty} \mathbf{E} \left[e^{-\langle \widetilde{\mathcal{E}}_t^{t_0}, \varphi \rangle} \mid \mathcal{F}_{t_0} \right] = \mathbf{E} \left[e^{-\varphi(G_{t_0}^u)} \right]^{\#\mathcal{N}_{t_0}} = \left[1 - \int dz (1 - e^{-\varphi(z)}) \frac{\sqrt{2}e^{-\sqrt{2}z - t_0}}{(1 + e^{-\sqrt{2}z - t_0})^2} \right]^{\#\mathcal{N}_{t_0}}$$

Next we take $t_0 \rightarrow \infty$ and use the fact that $e^{-t_0} \# \mathcal{N}_{t_0}$ is a uniformly bounded martingale converging to W_0^∞ to obtain

$$\lim_{t_0 \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbf{E}[e^{-\langle \tilde{\mathcal{E}}_t^{t_0}, \varphi \rangle} | \mathcal{F}_{t_0}] = \exp\left(-\sqrt{2}W_0^\infty \int dz (1 - e^{-\varphi(z)})e^{-\sqrt{2}z}\right) \quad \text{a.s.}$$

Taking the expectation yields the desired result.

Proof of the second item. In the double limit, the expected number of particles on the right of a in $\tilde{\mathcal{E}}_t^{t_0}$ is

$$\lim_{t_0 \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbf{E}[\tilde{\mathcal{E}}_t^{t_0}([a, \infty))] = \mathbf{E}\left[\int_a^\infty \sqrt{2}W_0^\infty e^{-\sqrt{2}z} dz\right] = e^{-\sqrt{2}a}.$$

At the same time, Lemma 5.3 yields

$$\lim_{t \rightarrow \infty} \mathbf{E}[\tilde{\mathcal{E}}_t([a, \infty))] = e^{-\sqrt{2}a}.$$

As $\mathbf{E}[\tilde{\mathcal{E}}_t([a, \infty))] \geq \mathbf{E}[\tilde{\mathcal{E}}_t^{t_0}([a, \infty))]$, the above yields

$$\lim_{t_0 \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbf{P}\left(\tilde{\mathcal{E}}_t([a, \infty)) \neq \tilde{\mathcal{E}}_t^{t_0}([a, \infty))\right) \leq \lim_{t_0 \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbf{E}\left(\tilde{\mathcal{E}}_t([a, \infty)) - \tilde{\mathcal{E}}_t^{t_0}([a, \infty))\right) = 0,$$

which concludes the proof. \square

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A Some technical estimates

In this appendix, we give the proofs of some technical estimates, which are non-obvious, but whose proofs would break the readability of the present paper. Precisely, we bound the second moment of the number of particle $Z_t(z)$ to the right of position $\tilde{m}_t + z$ in a branching Ornstein-Uhlenbeck process with fixed spring constant $\mu > 0$.

But first we recall a bound on the joint tail probabilities of a two-dimensional Gaussian random variable which is summarized in the next Lemma. It is a consequence of Savage's bound on Mill's ratio of multivariate normal distributions [Sav62].

Lemma A.1. *Let (X_1, X_2) be a centred Gaussian vector with covariance matrix $\begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$ where $\alpha \in [-1, 1]$. Then for any $x > 0$ we have*

$$\mathbf{P}(X_1 \geq x, X_2 \geq x) \leq \frac{(1 + \alpha)^2}{2\pi x^2 \sqrt{1 - \alpha^2}} e^{-\frac{x^2}{1 + \alpha}}.$$

Proof. We start noting that

$$\begin{aligned} \mathbf{P}(X_1 \geq x, X_2 \geq x) &= \frac{1}{2\pi(1 - \alpha^2)^{1/2}} \iint_{[x, \infty)^2} \exp\left(-\frac{y_1^2 + y_2^2 - 2\alpha y_1 y_2}{2(1 - \alpha^2)}\right) dy_1 dy_2 \\ &= \frac{1}{2\pi(1 - \alpha^2)^{1/2}} \iint_{\mathbb{R}_+^2} e^{-\frac{2(1 - \alpha)x^2}{2(1 - \alpha^2)}} e^{-\frac{2(1 - \alpha)x(y_1 + y_2)}{2(1 - \alpha^2)}} e^{-\frac{y_1^2 + y_2^2 - 2\alpha y_1 y_2}{2(1 - \alpha^2)}} dy_1 dy_2 \\ &\leq \frac{e^{-\frac{x^2}{1 + \alpha}}}{2\pi(1 - \alpha^2)^{1/2}} \iint_{\mathbb{R}_+^2} \exp\left(-\frac{x(y_1 + y_2)}{1 + \alpha}\right) dy_1 dy_2 = \frac{(1 + \alpha)^2}{2\pi x^2 \sqrt{1 - \alpha^2}} e^{-\frac{x^2}{1 + \alpha}}. \quad \square \end{aligned}$$

The estimate obtained in this Lemma is sharp for $x \gg (1 - \alpha)^{-1/2}$. On the other hand, when $x \ll (1 - \alpha)^{-1/2}$ a better bound is given by (A.2), precisely

$$\mathbf{P}(X_1 \geq x, X_2 \geq x) \leq \mathbf{P}(X_1 \geq x) \leq \frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Proof of Lemma 5.5. Let $K > 0$, and $z \in [0, K]$. Throughout this proof, c and C are general positive constants, that might change from line to line and depend on K and μ , but not on $t \geq 2$ or z .

By the many-to-two lemma, we have

$$\mathbf{E}[Z_t(z)^2] = \mathbf{E}[Z_t(z)] + 2 \int_0^t e^{2t-s} \mathbf{P} \left(\lambda_{\mu t} X_{s,t}^{(1)} \geq \tilde{m}_t + z, \lambda_{\mu t} X_{s,t}^{(2)} \geq \tilde{m}_t + z \right) ds,$$

where $(X_{s,\cdot}^{(1)}, X_{s,\cdot}^{(2)})$ are Ornstein-Uhlenbeck processes with spring constant μ_t , that remain equal until time s , at which time they split and evolve independently afterwards. Therefore it is enough to prove that

$$\limsup_{t \rightarrow \infty} \sup_{0 < z < K} \int_0^t e^{2t-s} \mathbf{P} \left(\lambda_{\mu t} X_{s,t}^{(1)} \geq \tilde{m}_t + z, \lambda_{\mu t} X_{s,t}^{(2)} \geq \tilde{m}_t + z \right) ds \leq C e^{-2\sqrt{2}z}. \quad (\text{A.1})$$

We split the proof in two parts. We first bound the integral in a small neighbourhood of t , then study the behaviour of the reminder of the integral.

Let $h > 0$, we first control the integral running from $t - h$ to t via the rough bound

$$\mathbf{P} \left(\lambda_{\mu t} X_{s,t}^{(1)} \geq \tilde{m}_t + z, \lambda_{\mu t} X_{s,t}^{(2)} \geq \tilde{m}_t + z \right) \leq \mathbf{P} \left(\lambda_{\mu t} X_{s,t}^{(1)} \geq \tilde{m}_t + z \right).$$

Using the classical bounds

$$\forall x > 0, \quad \frac{1 - x^{-2}}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}} \leq \mathbf{P}(Z \geq x) \leq \frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (\text{A.2})$$

where Z is a standard Gaussian variable, we have

$$\begin{aligned} \mathbf{P} \left(\lambda_{\mu t} X_{s,t}^{(1)} \geq m_t + z \right) &\leq \frac{t^{1/2}}{\sqrt{2\pi}(\tilde{m}_t + z)} \exp \left(-\frac{(\tilde{m}_t + z)^2}{2t} \right) \\ &\leq C t^{-1/2} \exp \left(-\left(t - \frac{1}{2} \log t + \sqrt{2}z \right) \right). \end{aligned}$$

As a result, we have

$$\begin{aligned} \int_{t-h}^t e^{2t-s} \mathbf{P} \left(\lambda_{\mu t} X_{s,t}^{(1)} \geq \tilde{m}_t + z, \lambda_{\mu t} X_{s,t}^{(2)} \geq \tilde{m}_t + z \right) ds &\leq C e^{-\sqrt{2}z} \int_{t-h}^t e^{t-s} ds \\ &\leq C e^{-\sqrt{2}z} h, \end{aligned} \quad (\text{A.3})$$

which vanishes as $h \rightarrow 0$ uniformly in $z \in [0, K]$.

We next focus on $s \leq t - h$, in which case we use the bounds from Lemma A.1 to estimate the probability in (A.1). It can be checked that $(\lambda_{\mu t} X_{s,t}^{(1)}, \lambda_{\mu t} X_{s,t}^{(2)})$ is a mean zero Gaussian vector with covariance matrix

$$\Sigma = \begin{pmatrix} t & t \frac{e^{2\mu(t-s)} - 1}{e^{2\mu t} - 1} \\ t \frac{e^{2\mu s} - 1}{e^{2\mu t} - 1} & t \end{pmatrix}.$$

Therefore, setting $\alpha = \frac{e^{2\mu s} - 1}{e^{2\mu t} - 1}$, by Lemma A.1 we have

$$\mathbf{P} \left(\lambda_{\mu t} X_{s,t}^{(1)} \geq \tilde{m}_t + z, \lambda_{\mu t} X_{s,t}^{(2)} \geq \tilde{m}_t + z \right) \leq \frac{(1 + \alpha)^{3/2} t}{2\pi (\tilde{m}_t + z)^2 \sqrt{1 - \alpha}} \exp \left(-\frac{(\tilde{m}_t + z)^2}{t(1 + \alpha)} \right).$$

As $s < t - h$, we have that

$$\alpha = \frac{e^{-2\mu(t-s)} - e^{-2\mu t}}{1 - e^{-2\mu t}} \leq \frac{e^{-2\mu h} - e^{-2\mu t}}{1 - e^{-2\mu t}},$$

is asymptotically bounded from above by $e^{-2\mu h}$, uniformly in $s \in [0, t - h]$. We have,

$$\mathbf{P} \left(\lambda_{\mu t} X_{s,t}^{(1)} \geq m_t + z, \lambda_{\mu t} X_{s,t}^{(2)} \geq \tilde{m}_t + z \right) \leq C (t + \sqrt{2}z)^{-1} (1 - \alpha)^{-1/2} \exp \left(-\frac{2}{1 + \alpha} \left(t - \frac{1}{2} \log t + \sqrt{2}z \right) \right) \quad (\text{A.4})$$

Integrating this bound over $[0, t - h]$, we obtain

$$\begin{aligned} &\int_0^{t-h} e^{2t-s} \mathbf{P} \left(\lambda_{\mu t} X_{s,t}^{(1)} \geq \tilde{m}_t + z, \lambda_{\mu t} X_{s,t}^{(2)} \geq \tilde{m}_t + z \right) ds \\ &\leq C \int_0^{t-h} (1 - \alpha)^{-1/2} \exp \left(2\left(t - \frac{1}{2} \log t\right) \frac{\alpha}{1 + \alpha} - s \right) \exp \left(-\frac{2\sqrt{2}}{1 + \alpha} z \right) ds \\ &\leq C e^{-\frac{3\sqrt{2}}{2}z} \int_h^t \sqrt{\frac{1 - e^{-2\mu t}}{1 - e^{-2\mu u}}} e^{2t \frac{e^{-2\mu u} - e^{-2\mu t}}{1 + e^{-2\mu u} - 2e^{-2\mu t}} - t + u} du, \end{aligned} \quad (\text{A.5})$$

for h large enough. We now bound the integral in (A.5). We note that $\sqrt{\frac{1-e^{-2\mu t}}{1-e^{-2\mu u}}} \leq \frac{1}{\sqrt{1-e^{-2\mu u}}} \leq C(\frac{1}{\sqrt{u}} + 1)$ for all $u \in [0, t]$. Moreover, for all $0 < u < t$, we set $f_t(u) = 2t \frac{e^{-2\mu u} - e^{-2\mu t}}{1+e^{-2\mu u} - 2e^{-2\mu t}} - t + u$. Then

$$f'_t(u) = 1 - 4t\mu e^{-2\mu u} \frac{1 - e^{-2\mu t}}{(1 + e^{-2\mu u} - 2e^{-2\mu t})^2}.$$

In particular, we have

$$\begin{aligned} f'_t(u) \leq 0 &\iff (1 + e^{-2\mu u} - 2e^{-2\mu t})^2 \leq 4t\mu e^{-2\mu u} (1 - e^{-2\mu t}) \\ &\iff (e^{-2\mu u})^2 + 2e^{-2\mu u} (1 - 2e^{-2\mu t} - 2t\mu(1 - e^{-2\mu t})) + (1 - 2e^{-2\mu t})^2 \leq 0. \end{aligned}$$

The roots of $X^2 + 2X(1 - 2e^{-2\mu t} - 2t\mu(1 - e^{-2\mu t})) + (1 - 2e^{-2\mu t})^2$ being

$$X_{\pm} = 2t\mu(1 - e^{-2\mu t}) + 2e^{-2\mu t} - 1 \pm \sqrt{(1 - 2e^{-2\mu t} - 2t\mu(1 - e^{-2\mu t}))^2 - (1 - 2e^{-2\mu t})^2},$$

we deduce that for all t large enough, f'_t is positive on $[0, c_t)$ and negative on $(c_t, t]$, with

$$c_t = \frac{-\log X_-}{2\mu} \sim \frac{\log t}{2\mu} \quad \text{as } t \rightarrow \infty.$$

We now bound the integral in (A.5), first for $u \leq c_t$. From the above computations, we observe that f_t reaches its maximum on $[h, c_t]$ in h . Moreover, yielding

$$\begin{aligned} &\int_h^{c_t} \sqrt{\frac{1 - e^{-2\mu t}}{1 - e^{-2\mu u}}} e^{2t \frac{e^{-2\mu u} - e^{-2\mu t}}{1+e^{-2\mu u} - 2e^{-2\mu t}} - t + u} du \\ &\leq C \int_h^{c_t} \left(\frac{1}{\sqrt{u}} + 1 \right) e^{2t \frac{e^{-2\mu h} - e^{-2\mu t}}{1+e^{-2\mu h} - 2e^{-2\mu t}} - t + h} du \\ &\leq C c_t e^h e^{t \frac{e^{-2\mu h} - 1}{(e^{-2\mu h} - 1) - 2(e^{-2\mu t} - 1)}}. \end{aligned}$$

Hence for all $h > 0$ small enough, there exists a constant c such that for all t large enough,

$$\int_h^{c_t} \sqrt{\frac{1 - e^{-2\mu t}}{1 - e^{-2\mu u}}} e^{2t \frac{e^{-2\mu u} - e^{-2\mu t}}{1+e^{-2\mu u} - 2e^{-2\mu t}} - t + u} du \leq C(\log t) e^{-cht}. \quad (\text{A.6})$$

Similarly, we observe that f_t reaches its maximum on $[c_t, t]$ at time t , and that $\inf_{t \geq 1} c_t > 0$, therefore

$$\int_{c_t}^t \sqrt{\frac{1 - e^{-2\mu t}}{1 - e^{-2\mu u}}} e^{2t \frac{e^{-2\mu u} - e^{-2\mu t}}{1+e^{-2\mu u} - 2e^{-2\mu t}} - t + u} du \leq C \int_{c_t}^t e^{u-t} du \leq C. \quad (\text{A.7})$$

As a result, plugging (A.4) into (A.1) (remembering (A.3)), and using (A.6) and (A.7), we obtain

$$\mathbf{E}(Z_t(z)^2) - \mathbf{E}(Z_t(z)) \leq C e^{-\sqrt{2}z} h + C e^{-2\sqrt{2}z} \left((\log t) e^{-\tilde{C}ht} + 1 \right).$$

Hence, choosing $h = t^{-1/2}$ and letting $t \rightarrow \infty$, we conclude the proof. \square

B Proof of Lemma 4.1

Proof. Obviously, (i) implies (ii) and (iii) implies (iv). It remains to prove that (ii) implies (iii) and (iv) implies (i).

We start by proving that (ii) implies (iii). First consider the case of a non-negative continuous function φ with support bounded from the left, and introduce for $A \in \mathbb{R}$

$$\varphi^A : x \mapsto \begin{cases} \varphi(x) & \text{if } x < A \\ (A + 1 - x)\varphi(A) & \text{if } x \in [A, A + 1] \\ 0 & \text{if } x > A + 1. \end{cases}$$

The function φ^A is continuous compactly supported, hence by (ii) we have

$$\lim_{t \rightarrow \infty} \mathbf{E}(e^{-\langle \mathcal{P}_t, \varphi^A \rangle}) = \mathbf{E}(e^{-\langle \mathcal{P}_\infty, \varphi^A \rangle}).$$

By triangular inequality,

$$\begin{aligned} \left| \mathbf{E}(e^{-\langle \mathcal{P}_t, \varphi \rangle}) - \mathbf{E}(e^{-\langle \mathcal{P}_\infty, \varphi \rangle}) \right| &\leq \left| \mathbf{E}(e^{-\langle \mathcal{P}_t, \varphi \rangle}) - \mathbf{E}(e^{-\langle \mathcal{P}_t, \varphi^A \rangle}) \right| \\ &\quad + \left| \mathbf{E}(e^{-\langle \mathcal{P}_t, \varphi^A \rangle}) - \mathbf{E}(e^{-\langle \mathcal{P}_\infty, \varphi^A \rangle}) \right| + \left| \mathbf{E}(e^{-\langle \mathcal{P}_\infty, \varphi^A \rangle}) - \mathbf{E}(e^{-\langle \mathcal{P}_\infty, \varphi \rangle}) \right| \end{aligned}$$

Moreover, as φ is non-negative, we have for all $t \geq 0$ and also for $t = \infty$:

$$\left| \mathbf{E}(e^{-\langle \mathcal{P}_t, \varphi \rangle}) - \mathbf{E}(e^{-\langle \mathcal{P}_t, \varphi^A \rangle}) \right| \leq \mathbf{P}(\max \mathcal{P}_t \geq A).$$

Hence, by convergence of $\max \mathcal{P}_t$, we have

$$\limsup_{t \rightarrow \infty} \left| \mathbf{E}(e^{-\langle \mathcal{P}_t, \varphi \rangle}) - \mathbf{E}(e^{-\langle \mathcal{P}_\infty, \varphi \rangle}) \right| \leq 2\mathbf{P}(\max \mathcal{P}_\infty \geq A)$$

As the right hand side goes to zero as $A \rightarrow \infty$, we have proved (iii) for non-negative functions. Now consider an arbitrary continuous function φ with support bounded on the left, and write

$$\varphi = \varphi_+ - \varphi_- \quad \text{where } \varphi_+(x) = \max(\varphi(x), 0) \text{ and } \varphi_-(x) = \max(-\varphi(x), 0).$$

Then, for any $\alpha, \beta \geq 0$, the function $\alpha\varphi_+ + \beta\varphi_-$ is continuous non-negative with support bounded on the left and, therefore,

$$\lim_{t \rightarrow \infty} \mathbf{E}(e^{-\alpha\langle \mathcal{P}_t, \varphi_+ \rangle - \beta\langle \mathcal{P}_t, \varphi_- \rangle}) = \mathbf{E}(e^{-\alpha\langle \mathcal{P}_\infty, \varphi_+ \rangle - \beta\langle \mathcal{P}_\infty, \varphi_- \rangle})$$

We conclude that $(\langle \mathcal{P}_t, \varphi_+ \rangle, \langle \mathcal{P}_t, \varphi_- \rangle)$ jointly converge in law toward $(\langle \mathcal{P}_\infty, \varphi_+ \rangle, \langle \mathcal{P}_\infty, \varphi_- \rangle)$. Therefore, $\langle \mathcal{P}_t, \varphi \rangle$ converges as well toward $\langle \mathcal{P}_\infty, \varphi \rangle$, which implies that (iii) holds

We now prove that (iv) implies (i). Let f be a smooth increasing function such that $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x > 1$. For any $y \in \mathbb{R}$ and $\varepsilon > 0$, we set $f_{\varepsilon, y}(x) = f(\varepsilon^{-1}(x - y))$.

Noting that $f_{\varepsilon, y}(x) \leq \mathbf{1}_{\{x > y\}} \leq f_{\varepsilon, y - \varepsilon}(x)$, we have for all $(y_1, \dots, y_n) \in \mathbb{R}^n$, $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}_+^n$ and $\varepsilon > 0$:

$$\mathbf{E}(e^{-\sum_i \lambda_i \langle \mathcal{P}_t, f_{\varepsilon, y_i - \varepsilon} \rangle}) \leq \mathbf{E}(e^{-\sum_i \lambda_i \mathcal{P}_t((y_i, \infty))}) \leq \mathbf{E}(e^{-\sum_i \lambda_i \langle \mathcal{P}_t, f_{\varepsilon, y_i} \rangle})$$

As $t \rightarrow \infty$, the two bounds converge by (iv) applied to the functions $\sum_i \lambda_i f_{\varepsilon, y_i}$ and $\sum_i \lambda_i f_{\varepsilon, y_i - \varepsilon}$

$$\begin{aligned} \mathbf{E}(e^{-\sum_i \lambda_i \langle \mathcal{P}_\infty, f_{\varepsilon, y_i - \varepsilon} \rangle}) &\leq \liminf_{t \rightarrow \infty} \mathbf{E}(e^{-\sum_i \lambda_i \mathcal{P}_t((y_i, \infty))}) \\ &\leq \limsup_{t \rightarrow \infty} \mathbf{E}(e^{-\sum_i \lambda_i \mathcal{P}_t((y_i, \infty))}) \leq \mathbf{E}(e^{-\sum_i \lambda_i \langle \mathcal{P}_\infty, f_{\varepsilon, y_i} \rangle}) \end{aligned}$$

Note that $f_{\varepsilon, y}(x) \rightarrow \mathbf{1}_{\{x > y\}}$ and $f_{\varepsilon, y - \varepsilon}(x) \rightarrow \mathbf{1}_{\{x \geq y\}}$ as $\varepsilon \rightarrow 0$. Hence one gets

$$\begin{aligned} \mathbf{E}(e^{-\sum_i \lambda_i \mathcal{P}_\infty((y_i, \infty))}) &\leq \liminf_{t \rightarrow \infty} \mathbf{E}(e^{-\sum_i \lambda_i \mathcal{P}_t((y_i, \infty))}) \\ &\leq \limsup_{t \rightarrow \infty} \mathbf{E}(e^{-\sum_i \lambda_i \mathcal{P}_t((y_i, \infty))}) \leq \mathbf{E}(e^{-\sum_i \lambda_i \mathcal{P}_\infty((y_i, \infty))}) \end{aligned} \tag{B.1}$$

We conclude that $(\mathcal{P}_t((y_i, \infty)), i \leq n)$ jointly converge in law to $(\mathcal{P}_\infty((y_i, \infty)), i \leq n)$ as $t \rightarrow \infty$, except at discontinuity points y_i where $\mathcal{P}_\infty(\{y_i\}) > 0$ with positive probability. Hence, \mathcal{P}_t converges in law to \mathcal{P}_∞ for the topology of vague convergence.

In (B.1), add one extra pair (λ, y) to the λ_i, y_i , and send λ to infinity. Noticing that for $A \geq 0$ that

$$\mathbf{E}(A \mathbf{1}_{\{\max \mathcal{P}_t \leq y\}}) \leq \mathbf{E}(A e^{-\lambda \mathcal{P}_t((y, \infty))}) \leq \mathbf{E}(A \mathbf{1}_{\{\max \mathcal{P}_t \leq y\}}) + e^{-\lambda} \mathbf{E}(A),$$

one gets

$$\begin{aligned} \mathbf{E}(e^{-\sum_i \lambda_i \mathcal{P}_\infty((y_i, \infty))} \mathbf{1}_{\{\max \mathcal{P}_\infty \leq y\}}) &\leq \liminf_{t \rightarrow \infty} \mathbf{E}(e^{-\sum_i \lambda_i \mathcal{P}_t((y_i, \infty))} \mathbf{1}_{\{\max \mathcal{P}_t \leq y\}}) \\ &\leq \limsup_{t \rightarrow \infty} \mathbf{E}(e^{-\sum_i \lambda_i \mathcal{P}_t((y_i, \infty))} \mathbf{1}_{\{\max \mathcal{P}_t \leq y\}}) \\ &\leq \mathbf{E}(e^{-\sum_i \lambda_i \mathcal{P}_\infty((y_i, \infty))} \mathbf{1}_{\{\max \mathcal{P}_\infty \leq y\}}) \end{aligned}$$

Hence $(\mathcal{P}_t, \max \mathcal{P}_t)$ converges to $(\mathcal{P}_\infty, \max \mathcal{P}_\infty)$ in law jointly. \square

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