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Koszul calculus of preprojective algebras

Roland Berger and Rachel Taillefer

Abstract

We show that the Koszul calculus of a preprojective algebra, whose graph is distinct from A_1 and A_2 , vanishes in any (co)homological degree $p > 2$. Moreover, the (higher) cohomological calculus is isomorphic as a bimodule to the (higher) homological calculus, by exchanging degrees p and $2 - p$, and we prove a generalised version of the 2-Calabi-Yau property. For the ADE Dynkin graphs, the preprojective algebras are not Koszul and they are not Calabi-Yau in the sense of Ginzburg's definition, but they satisfy our generalised Calabi-Yau property and we say that they are *generalised Calabi-Yau*. For generalised Calabi-Yau algebras of any dimension, defined in terms of derived categories, we prove a Poincaré Van den Bergh duality theorem. We compute explicitly the Koszul calculus of preprojective algebras for the ADE Dynkin graphs.

1 Introduction

Preprojective algebras are quiver algebras with quadratic relations, that play an important role in the representation theory of quiver algebras [23, 13, 1], with various applications [11, 12] and many developments [10, 22, 7, 25]. In [25], the reader will find an introduction to the various aspects of the preprojective algebras in representation theory, with an extended bibliography. In our paper, we are interested in some homological properties linked to Hochschild cohomology.

In the two last decades, the Hochschild cohomology of preprojective algebras, as well as some extra algebraic structures, have been computed in several steps, as follows.

1) Erdmann and Snashall [14, 15] determined the Hochschild cohomology and its cup-product in type A.

2) Crawley-Boevey, Etingof and Ginzburg [10] determined the Hochschild cohomology for all preprojective algebras of non-Dynkin type (which are Koszul in this case [29, 26]).

3) In type DE and characteristic zero, Etingof and Eu [17] determined the Hochschild cohomology and Eu [18] the cup product. The cyclic homology was computed in type ADE in [17].

4) Assembling and completing the previous results in characteristic zero, Eu gave an explicit description of the Tamarkin-Tsygan calculus [32] of the preprojective algebras in type ADE, that is, the homology and the cohomology, the cup product, the contraction map and the Lie derivative, the Connes differential and the Gerstenhaber bracket [19].

5) Eu and Schedler extended the ADE results to the case where the base ring is \mathbb{Z} , and obtained the corresponding ADE results in any characteristic [20].

In [6], a Koszul calculus was associated with any quadratic algebra over a field, in order to produce new homological invariants for non-Koszul quadratic algebras. We begin this paper by extending the Koszul calculus to quadratic quiver algebras. We shall compute the Koszul calculus of the preprojective algebras whose graph is Dynkin of type ADE (the preprojective

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algebras are then finite dimensional). Except for types A_1 and A_2 , these quadratic quiver algebras are not Koszul [29, 26], so that the Koszul calculus and the Hochschild calculus provide different information.

Before presenting our computations, we state and develop a Poincaré Van den Bergh duality theorem [33] for the Koszul homology/cohomology of *any preprojective algebra* whose graph is different from A_1 and A_2 . This theorem is formulated as follows and constitutes the first main result of the present paper. The duality is precisely part (ii) in this theorem.

Theorem 1.1. *Let A be the preprojective algebra of a (non-labelled) connected graph Δ distinct from A_1 and A_2 , over a field \mathbb{F} . Let M be an A -bimodule.*

(i) *The Koszul bimodule complex $K(A)$ of A has length 2. In particular, $\mathrm{HK}^p(A, M) = \mathrm{HK}_p(A, M) = 0$ for any $p > 2$.*

(ii) *The $\mathrm{HK}^\bullet(A)$ -bimodules $\mathrm{HK}^\bullet(A, M)$ and $\mathrm{HK}_{2-\bullet}(A, M)$ are isomorphic.*

(iii) *The $\mathrm{HK}_{hi}^\bullet(A)$ -bimodules $\mathrm{HK}_{hi}^\bullet(A, M)$ and $\mathrm{HK}_{2-\bullet}^{hi}(A, M)$ are isomorphic.*

In this statement, following [6], $\mathrm{HK}_p(A, M)$ and $\mathrm{HK}^p(A, M)$ denote the Koszul homology and cohomology spaces with coefficients in M , while $\mathrm{HK}_p^{hi}(A, M)$ and $\mathrm{HK}_{hi}^p(A, M)$ denote the higher Koszul homology and cohomology spaces. When $M = A$, these notations are simplified into $\mathrm{HK}_p(A)$, $\mathrm{HK}^p(A)$, $\mathrm{HK}_p^{hi}(A)$ and $\mathrm{HK}_{hi}^p(A)$.

In the general setting [6], the Koszul calculus of a quadratic algebra A consists of the graded associative algebra $\mathrm{HK}^\bullet(A)$ endowed with the Koszul cup product and, for all A -bimodules M , of the graded $\mathrm{HK}^\bullet(A)$ -bimodules $\mathrm{HK}^\bullet(A, M)$ and $\mathrm{HK}_\bullet(A, M)$, with actions respectively defined by the Koszul cup and cap products. The higher Koszul calculus of A is given by the analogous data, adding the subscript and superscript *hi*. Sometimes (as will be the case with our computations in ADE types), these calculi are *restricted*, meaning that the data is limited to $M = A$, so that the restricted Koszul calculus consists of the graded associative algebra $\mathrm{HK}^\bullet(A)$ and of the graded $\mathrm{HK}^\bullet(A)$ -bimodule $\mathrm{HK}_\bullet(A)$ – similarly for the higher version.

Using Theorem 1.1 for Δ Dynkin of type ADE, we shall deduce the (higher) homological restricted Koszul calculus from the computation of the (higher) cohomological restricted Koszul calculus.

Part (ii) in Theorem 1.1 comes from an explicit isomorphism from the complex defining $\mathrm{HK}^p(A, M)$ to the complex defining $\mathrm{HK}_{2-p}(A, M)$, described as follows.

Proposition 1.2. *Let A be the preprojective algebra of a connected graph Δ distinct from A_1 and A_2 . Let M be an A -bimodule. The Koszul cup and cap products are denoted by \smile_K and \frown_K . Define $\omega_0 = \sum_i e_i \otimes \sigma_i$, where the sum runs over the vertices i of Δ and, for each vertex i , e_i is the idempotent and σ_i is the quadratic relation in A associated with i .*

For each Koszul p -cochain f with coefficients in M , we define the Koszul $(2-p)$ -chain $\theta_M(f)$ with coefficients in M by

$$\theta_M(f) = \omega_0 \frown_K f.$$

Then θ_M is an isomorphism of complexes. Moreover, the equalities

$$\theta_{M \otimes_A N}(f \smile_K g) = \theta_M(f) \frown_K g = f \frown_K \theta_N(g)$$

hold for any Koszul cochains f and g with coefficients in bimodules M and N respectively.

The proof of Proposition 1.2 relies on some manipulations of the defining formula of θ_M with fundamental formulas of Koszul calculus [6], using actions involving \smile_K and \frown_K . The fundamental

formulas of Koszul calculus express the Koszul differential b_K in cohomology and the Koszul differential b^K in homology respectively as a cup bracket and a cap bracket, namely

$$b_K = -[\epsilon_A, -]_{\smile_K}, \quad b^K = -[\epsilon_A, -]_{\frown_K},$$

where ϵ_A is a fundamental Koszul 1-cocycle defined by restricting the Euler derivation of the preprojective algebra A to its arrow space.

In order to extract a generalised version of the 2-Calabi-Yau property from our Poincaré Van den Bergh duality (Theorem 1.1), we apply this theorem to the left A^e -module $M = A^e$ viewed as an A -bimodule. We show that the complex of Koszul chains with coefficients in the left A^e -module A^e is naturally isomorphic, as a right A^e -module, to the Koszul bimodule complex $K(A)$. Using the fact that the homology of $K(A)$ is isomorphic to A in degree 0, and to 0 in degree 1, we obtain a generalisation of the 2-Calabi-Yau property, formulated as follows.

Theorem 1.3. *Let A be the preprojective algebra of a connected graph Δ distinct from A_1 and A_2 , over a field \mathbb{F} . Let us denote by $K(A)$ the Koszul bimodule complex of A . Then the A -bimodule $\mathrm{HK}^p(A, A^e)$ is isomorphic to the A -bimodule $H_{2-p}(K(A))$ for $0 \leq p \leq 2$. In particular, we have the following.*

- (i) *The A -bimodule $\mathrm{HK}^2(A, A^e)$ is isomorphic to the A -bimodule A .*
- (ii) *$\mathrm{HK}^1(A, A^e) = 0$.*
- (iii) *The A -bimodule $\mathrm{HK}^0(A, A^e)$ is isomorphic to the A -bimodule $H_2(K(A))$, which is always non-zero when Δ is Dynkin of type ADE.*

We then say that the preprojective algebra A is a *generalised Calabi-Yau algebra of dimension 2*. We generalise this definition to any dimension n in Definition 1.4 below, formulated in terms of derived categories. Since there is an \mathbb{F} -linear isomorphism

$$H(\theta_M) = \overline{\omega}_0 \frown_K - : \mathrm{HK}^\bullet(A, M) \rightarrow \mathrm{HK}_{2-\bullet}(A, M),$$

we say that the class $\overline{\omega}_0 \in \mathrm{HK}_2(A)$ is the *fundamental class* of the generalised Calabi-Yau algebra A , by analogy with Poincaré's duality in singular homology/cohomology [27]. In Definition 1.7, we give a stronger version of Definition 1.4 in order to obtain a Poincaré-like duality, that is, a duality isomorphism expressed as a cap action by a suitably defined fundamental class.

Let us remark that the A -bimodule structures in Theorem 1.3 are compatible with the Koszul cup and cap actions of $\mathrm{HK}^\bullet(A)$ on $\mathrm{HK}^\bullet(A, A^e)$ and $H(K(A))$. These actions can be viewed as graded actions of left $\mathrm{HK}^\bullet(A)^e$ -modules, while the A -bimodules can be viewed as compatible right A^e -modules. So the isomorphism $\mathrm{HK}^\bullet(A, A^e) \cong H_{2-\bullet}(K(A))$ in Theorem 1.3 is an isomorphism of graded $\mathrm{HK}^\bullet(A)^e$ - A^e -bimodules. This enriched isomorphism is the expression of the stronger version of the generalised Calabi-Yau property, as we shall see in Definition 1.7.

Note that if Δ is not Dynkin ADE, then A is Koszul, so Theorem 1.3 enables us to recover the well-known result that A is 2-Calabi-Yau in the sense of Ginzburg [10, 8]. However, if Δ is Dynkin ADE, then A is not homologically smooth since its minimal projective resolution has infinite length, so that Ginzburg's definition of Calabi-Yau algebras cannot be applied in this case [24]. Moreover, the restricted Hochschild calculus is drastically different from the restricted Koszul calculus, because by [16] there is a cohomological Hochschild periodicity

$$\mathrm{HH}^{p+6}(A) \cong \mathrm{HH}^p(A), \quad p > 0$$

and, consequently, there are non-zero spaces $\mathrm{HH}^p(A)$ for infinitely many values of p . Even taking into account this 6-periodicity, the list of cohomological Koszul invariants consists only of $\mathrm{HK}^0(A)$, $\mathrm{HK}^1(A)$ and $\mathrm{HK}^2(A)$ and is therefore shorter than the list of Hochschild invariants.

In [20], Eu and Schedler define periodic Calabi-Yau Frobenius algebras, for finite dimensional algebras only. Their main example is given by the preprojective algebras of Dynkin ADE graphs [20, Example 2.3.10]. Then the above cohomological Hochschild periodicity is a part of remarkable isomorphisms in Hochschild calculus for any periodic Calabi-Yau Frobenius algebra [20, Theorem 2.3.27 and Theorem 2.3.47].

From Theorem 1.3, we are led to introduce a general definition.

Definition 1.4. *Let $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ be a finite quiver, and let \mathbb{F} be a field. Let A be an \mathbb{F} -algebra defined on the path algebra $\mathbb{F}\mathcal{Q}$ of \mathcal{Q} by homogeneous quadratic relations. Define the ring $k = \mathbb{F}\mathcal{Q}_0$, so that A is regarded as a quadratic k -algebra (not necessarily k -central, but \mathbb{F} -central). We say that A is generalised Calabi-Yau of dimension n , for an integer $n \geq 0$, if*

- (i) *the bimodule Koszul complex $K(A)$ of A has length n , and*
- (ii) *$\mathrm{RHom}_{A^e}(K(A), A^e) \cong K(A)[-n]$ in the bounded derived category of A -bimodules.*

Property (ii) means, in terms of derived categories, that the A -bimodules $\mathrm{HK}^p(A, A^e)$ and $H_{n-p}(K(A))$ are isomorphic for any p .

In our context (that of quadratic algebras), Definition 1.4 is a definition of a new Calabi-Yau property, valid whether A is finite dimensional or not. In this definition, we do not impose that $K(A)$ be a resolution of A , that is, A is not necessarily Koszul, meaning that the bimodules $\mathrm{HK}^p(A, A^e)$ for $0 \leq p \leq n-2$ may be non-zero. Under the assumptions of Definition 1.4, we verify that, if A is Koszul, Definition 1.4 is equivalent to Ginzburg's definition of n -Calabi-Yau algebras [24, 35]. We then prove a new Poincaré Van den Bergh duality for generalised Calabi-Yau algebras, adapted to Koszul (co)homologies.

Theorem 1.5. *Let A be a generalised Calabi-Yau algebra of dimension n . Then for any A -bimodule M , the \mathbb{F} -vector spaces $\mathrm{HK}^p(A, M)$ and $\mathrm{HK}_{n-p}(A, M)$ are isomorphic.*

Definition 1.6. *Let A be a generalised Calabi-Yau algebra of dimension n . The image $c \in \mathrm{HK}_n(A)$ of the unit 1 of the algebra A under the isomorphism $\mathrm{HK}^0(A) \cong \mathrm{HK}_n(A)$ in Theorem 1.5 is called the fundamental class of the generalised Calabi-Yau algebra A .*

In order to describe the duality isomorphism of Theorem 1.5 explicitly as a cap-product by the fundamental class for strong generalised Calabi-Yau algebras, we shall use derived categories in the general context of DG algebras, as presented and detailed in the preprint book by Yekutieli [37]. Let us present briefly what we need in this general context.

We introduce the \mathbb{F} -central DG algebra $\tilde{A} = \mathrm{Hom}_{A^e}(K(A), A)$. The complexes $K(A)$ and $\mathrm{Hom}_{A^e}(K(A), A^e)$ of A -bimodules have an enriched structure since they can be viewed as DG \tilde{A} -bimodules in the abelian category \mathcal{A} of A -bimodules, in the sense of [37].

Denote by $\mathcal{C}(\tilde{A}, \mathcal{A})$ the DG category of DG \tilde{A} -bimodules in \mathcal{A} [37]. Let M be an A -bimodule. For any chain DG \tilde{A} -bimodule C in \mathcal{A} , $\mathrm{Hom}_{A^e}(C, M)$ is a cochain DG \tilde{A} -bimodule in the abelian category \mathcal{E} of \mathbb{F} -vector spaces (in \mathcal{A} when $M = A^e$). For any cochain DG \tilde{A} -bimodule C' in \mathcal{A} , $M \otimes_{A^e} C'$ is a cochain DG \tilde{A} -bimodule in \mathcal{E} . The bounded derived categories $\mathcal{D}^b(\tilde{A}, \mathcal{A})$ and $\mathcal{D}^b(\tilde{A}, \mathcal{E})$ are defined in [37]. However we do not know if the functors $\mathrm{Hom}_{A^e}(-, M)$ and $M \otimes_{A^e} -$ from $\mathcal{C}^b(\tilde{A}, \mathcal{A})$ to $\mathcal{C}^b(\tilde{A}, \mathcal{E})$ are derivable.

Definition 1.7. *Let A be a generalised Calabi-Yau algebra of dimension n . Then A is said to be a strong generalised Calabi-Yau algebra if the derived functor of the endofunctor $\mathrm{Hom}_{A^e}(-, A^e)$ of $\mathcal{C}^b(\tilde{A}, \mathcal{A})$ exists and if $\mathrm{RHom}_{A^e}(K(A), A^e) \cong K(A)[-n]$ in the bounded derived category $\mathcal{D}^b(\tilde{A}, \mathcal{A})$.*

The second property in this definition (assuming the first one) is equivalent to saying that the graded $\mathrm{HK}^\bullet(A)^e$ - A^e -bimodules $\mathrm{HK}^\bullet(A, A^e)$ and $H_{n-\bullet}(K(A))$ are isomorphic [37].

Theorem 1.8. *Let A be a generalised Calabi-Yau algebra of dimension n and let c be its fundamental class. We assume that A is a strong generalised Calabi-Yau algebra and that the derived functors of the functors $\mathrm{Hom}_{A^e}(-, A)$ and $A \otimes_{A^e} -$ from $\mathcal{C}^b(\tilde{A}, \mathcal{A})$ to $\mathcal{C}^b(\tilde{A}, \mathcal{E})$ exist. Then*

$$c \underset{K}{\frown} - : \mathrm{HK}^\bullet(A) \rightarrow \mathrm{HK}_{n-\bullet}(A)$$

is an isomorphism of $\mathrm{HK}^\bullet(A)$ -bimodules, inducing an isomorphism of $\mathrm{HK}_{hi}^\bullet(A)$ -bimodules from $\mathrm{HK}_{hi}^\bullet(A)$ to $\mathrm{HK}_{n-\bullet}^{hi}(A)$. For all $\alpha \in \mathrm{HK}^p(A)$, we have $c \underset{K}{\frown} \alpha = (-1)^{np} \alpha \underset{K}{\frown} c$.

Let us describe the contents of the paper. In Section 2, we extend the general formalism – including some results – of Koszul calculus [6] to quadratic quiver algebras. In Section 3, we introduce a right action which is an important tool in order to adapt the definition of Calabi-Yau algebras to quadratic quiver algebras endowed with the Koszul calculus instead of the Hochschild calculus. The Poincaré Van den Bergh duality for preprojective algebras is presented in Section 4, where Theorem 1.1, Proposition 1.2 and Theorem 1.3 of our introduction are proved. In Section 5, we define our generalisations of Calabi-Yau algebras and we thoroughly explain the new objects and remaining results outlined in the introduction. Section 6 is devoted to the computations of the Koszul calculus in ADE Dynkin types. As an application of the computations, we prove that the spaces $\mathrm{HK}_{hi}^0(A)$, $\mathrm{HK}_{hi}^1(A)$ and $\mathrm{HK}_{hi}^2(A)$ form a minimal complete list of invariants for the ADE preprojective algebras.

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2 Koszul calculus for quiver algebras with quadratic relations

2.1 Setup

Let \mathcal{Q} be a *finite* quiver, meaning that the vertex set \mathcal{Q}_0 and the arrow set \mathcal{Q}_1 are finite. Let \mathbb{F} be a field. The vertex space $k = \mathbb{F}\mathcal{Q}_0$ becomes a commutative ring by associating with \mathcal{Q}_0 a complete set of orthogonal idempotents $\{e_i; i \in \mathcal{Q}_0\}$. The ring k is isomorphic to $\mathbb{F}^{|\mathcal{Q}_0|}$, where $|\mathcal{Q}_0|$ is the cardinal of \mathcal{Q}_0 . Throughout the paper, the case $|\mathcal{Q}_0| = 1$ will be called *the one vertex case*, which is equivalent to saying that k is a field. Koszul calculus over a field k is treated in [6].

For each arrow $\alpha \in \mathcal{Q}_1$, denote its source vertex by $\mathfrak{s}(\alpha)$ and its target vertex by $\mathfrak{t}(\alpha)$. The arrow space $V = \mathbb{F}\mathcal{Q}_1$ is a k -bimodule for the following actions: $e_j \alpha e_i$ is equal to zero if $i \neq \mathfrak{s}(\alpha)$ or $j \neq \mathfrak{t}(\alpha)$, and is equal to α if $i = \mathfrak{s}(\alpha)$ and $j = \mathfrak{t}(\alpha)$. If there is an arrow joining two distinct vertices, the k -bimodule V is not symmetric.

Via the ring morphism $\mathbb{F} \rightarrow k$ that maps 1 to $\sum_{i \in \mathcal{Q}_0} e_i$, the tensor k -algebra $T_k(V)$ of the k -bimodule V is an \mathbb{F} -algebra isomorphic to the path algebra $\mathbb{F}\mathcal{Q}$, so that $V^{\otimes_k m}$ is identified with $\mathbb{F}\mathcal{Q}_m$, where \mathcal{Q}_m is the set of paths of length m . For two arrows α and β , note that $\alpha \otimes_k \beta$ is zero if $\mathfrak{t}(\beta) \neq \mathfrak{s}(\alpha)$, and otherwise $\alpha \otimes_k \beta$ is identified with the composition $\alpha\beta$ of paths (where paths are written from right to left, as in [3]).

Let R be a sub- k -bimodule of $V \otimes_k V \cong \mathbb{F}\mathcal{Q}_2$. The unital associative k -algebra $A = T_k(V)/(R)$, where (R) denotes the two-sided ideal of $T_k(V)$ generated by R , is called a *quadratic k -algebra over the finite quiver \mathcal{Q}* . The degree induced on A by the path length is called the *weight*, so that A is a graded algebra for the weight grading. The component of weight m of A is denoted by A_m . Clearly, $A_0 \cong k$ and $A_1 \cong V$. The algebra A is \mathbb{F} -central, but if there is an arrow joining two distinct vertices, A is not k -central. The A -bimodules are not necessarily

k -symmetric, but they are always assumed to be \mathbb{F} -symmetric. Setting $A^e = A \otimes_{\mathbb{F}} A^{op}$, any A -bimodule can be viewed as a left (or right) A^e -module, as usual.

For brevity, the notation $\otimes_{\mathbb{F}}$ is replaced by the unadorned tensor product \otimes . Similarly for the notations $\text{Hom}_{\mathbb{F}}$ and $\dim_{\mathbb{F}}$ abbreviated to Hom and \dim . If unspecified, a vector space is an \mathbb{F} -vector space and a linear map is \mathbb{F} -linear.

The tensor product \otimes_k is different from the unadorned tensor product \otimes . However, if M is a right A -module and N is a left A -module, then the natural linear map $Me_i \otimes e_i N \rightarrow Me_i \otimes_k e_i N$ is an isomorphism, so that for $a \in Me_i$ and $b \in e_i N$, we can identify $a \otimes_k b = a \otimes b$. Similarly, if M and N are A -bimodules, $e_i Me_i$ and $e_i Ne_i$ are k -bimodules, that may be viewed as left and right k^e -modules, where $k^e = k \otimes k$. The natural linear map $e_i Me_i \otimes e_i Ne_i \rightarrow e_i Me_i \otimes_{k^e} e_i Ne_i$ is an isomorphism, so for $a \in e_i Me_i$ and $b \in e_i Ne_i$, we can identify $a \otimes_{k^e} b = a \otimes b$. We shall freely use these identifications, without explicitly mentioning them.

Although the algebra A is not k -central, we define its bar resolution $B(A)$ following standard texts [36] by $(A \otimes_k A^{\otimes_k \bullet} \otimes_k A, d)$ with

$$\begin{aligned} d(a \otimes_k a_1 \dots a_p \otimes_k a') &= aa_1 \otimes_k a_2 \dots a_p \otimes_k a' + \sum_{1 \leq i \leq p-1} (-1)^i a \otimes_k a_1 \dots (a_i a_{i+1}) \dots a_p \otimes_k a' \\ &\quad + (-1)^p a \otimes_k a_1 \dots a_{p-1} \otimes_k a_p a', \end{aligned}$$

for a, a' and a_1, \dots, a_p in A . When k is not central, the extra degeneracy is defined and is still a contracting homotopy, hence $B(A)$ is a resolution of A by projective A -bimodules. See Lemma 2.1 below for the fact that the A -bimodules $A \otimes_k A^{\otimes_k p} \otimes_k A$ are projective.

For any A -bimodule M , Hochschild homology and cohomology are defined by

$$\begin{aligned} \text{HH}_{\bullet}(A, M) &= \text{Tor}_{\bullet}^{A^e}(M, A) = H_{\bullet}(M \otimes_{A^e} B(A), M \otimes_{A^e} d), \\ \text{HH}^{\bullet}(A, M) &= \text{Ext}_{A^e}^{\bullet}(A, M) = H^{\bullet}(\text{Hom}_{A^e}(B(A), M), \text{Hom}_{A^e}(d, M)). \end{aligned}$$

The linear map

$$\begin{aligned} M \otimes_{A^e} (A \otimes_k A^{\otimes_k p} \otimes_k A) &\rightarrow M \otimes_{k^e} (A^{\otimes_k p}) \\ m \otimes_{A^e} (a_0 \otimes_k \dots \otimes_k a_{p+1}) &\mapsto (a_{p+1} m a_0) \otimes_{k^e} (a_1 \otimes_k \dots \otimes_k a_p) \end{aligned} \quad (2.1)$$

is well-defined and is an isomorphism. Similarly, the linear map

$$\begin{aligned} \text{Hom}_{k^e}(A^{\otimes_k p}, M) &\rightarrow \text{Hom}_{A^e}(A \otimes_k A^{\otimes_k p} \otimes_k A, M) \\ f &\mapsto (a_0 \otimes_k \dots \otimes_k a_{p+1} \mapsto a_0 f(a_1 \otimes_k \dots \otimes_k a_p) a_{p+1}) \end{aligned} \quad (2.2)$$

defines an isomorphism. Transporting $M \otimes_{A^e} d$ and $\text{Hom}_{A^e}(d, M)$ via these isomorphisms, we obtain the Hochschild differentials b^H and b_H , so that

$$\begin{aligned} \text{HH}_{\bullet}(A, M) &\cong H_{\bullet}(M \otimes_{k^e} A^{\otimes_k \bullet}, b^H), \\ \text{HH}^{\bullet}(A, M) &\cong H^{\bullet}(\text{Hom}_{k^e}(A^{\otimes_k \bullet}, M), b_H). \end{aligned}$$

The Hochschild homology differential is then defined, for $m \in M$ and a_1, \dots, a_p in A , by

$$\begin{aligned} b_p^H(m \otimes_{k^e} (a_1 \dots a_p)) &= m a_1 \otimes_{k^e} (a_2 \dots a_p) + \sum_{1 \leq i \leq p-1} (-1)^i m \otimes_{k^e} (a_1 \dots (a_i a_{i+1}) \dots a_p) \\ &\quad + (-1)^p a_p m \otimes_{k^e} (a_1 \dots a_{p-1}). \end{aligned}$$

The Hochschild cohomology differential (including a Koszul sign in $\text{Hom}_{A^e}(d, M)$) is defined, for $f \in \text{Hom}_{k^e}(A^{\otimes_k p}, M)$ and a_1, \dots, a_{p+1} in A , by

$$\begin{aligned} b_H^{p+1}(f)(a_1 \dots a_{p+1}) &= f(a_1 \dots a_p) a_{p+1} - \sum_{1 \leq i \leq p} (-1)^{p+i} f(a_1 \dots (a_i a_{i+1}) \dots a_{p+1}) \\ &\quad - (-1)^p a_1 f(a_2 \dots a_{p+1}). \end{aligned}$$

2.2 Koszul homology and cohomology

Let $A = T_k(V)/(R)$ be a quadratic k -algebra over \mathcal{Q} . Following [6], the Koszul complex $K(A)$ is the subcomplex of the bar resolution $B(A)$ defined by the sub- A -bimodules $A \otimes_k W_p \otimes_k A$ of $A \otimes_k A^{\otimes_k p} \otimes_k A$, where $W_0 = k$, $W_1 = V$ and, for $p \geq 2$,

$$W_p = \bigcap_{i+2+j=p} V^{\otimes_k i} \otimes_k R \otimes_k V^{\otimes_k j}. \quad (2.3)$$

Here W_p is considered as a sub- k -bimodule of $V^{\otimes_k p} \subseteq A^{\otimes_k p}$. It is immediate that the differential d of $K(A)$ is defined on $A \otimes_k W_p \otimes_k A$ by

$$d(a \otimes_k x_1 \dots x_p \otimes_k a') = ax_1 \otimes_k x_2 \dots x_p \otimes_k a' + (-1)^p a \otimes_k x_1 \dots x_{p-1} \otimes_k x_p a', \quad (2.4)$$

for a, a' in A and $x_1 \dots x_p$ in W_p .

In this paper, we systematically follow [6] for the notation of elements of W_p . Let us recall this notation. As in (2.4), an arbitrary element of W_p is denoted by a product $x_1 \dots x_p$ thought of as a sum of such products, where x_1, \dots, x_p are in V . Moreover, regarding W_p as a subspace of $V^{\otimes_k q} \otimes_k W_r \otimes_k V^{\otimes_k s}$ with $q+r+s=p$, the element $x_1 \dots x_p$ viewed in $V^{\otimes_k q} \otimes_k W_r \otimes_k V^{\otimes_k s}$ will be denoted by the *same* notation, meaning that the product $x_{q+1} \dots x_{q+r}$ represents an element of W_r and the other x_i are arbitrary in V .

We pursue along the same lines as [6]. We present the different objects with their fundamental results more quickly. We keep the same notations as in [6] and we leave the details to the reader when they are the same as in the one vertex case.

The homology of $K(A)$ is equal to A in degree 0, and to 0 in degree 1. The quadratic algebra A is said to be *Koszul* if the homology of $K(A)$ is 0 in any degree > 1 . Denote by $\mu : A \otimes_k A \rightarrow A$ the multiplication of A . Then A is Koszul if and only if $\mu : K(A) \rightarrow A$ is a resolution of A . If $R = 0$ and if $R = V \otimes_k V$, then A is Koszul. Besides these extreme examples, many Koszul algebras occur in the literature in the one vertex case [31]. Beyond the one vertex case, it is well-known that preprojective algebras are Koszul when the graph is not Dynkin of type ADE (see Proposition 4.1).

Let us show that the A -bimodules $A \otimes_k W_p \otimes_k A$ forming $K(A)$ are projective and finitely generated. It is an immediate consequence of the following lemma, since W_p is a sub- k -bimodule of $V^{\otimes_k p}$.

Lemma 2.1. *Let E be a k -bimodule.*

- (i) *The A -bimodule $A \otimes_k E \otimes_k A$ is projective.*
- (ii) *If E is finite dimensional, then the A -bimodule $A \otimes_k E \otimes_k A$ is finitely generated.*

Proof. Clearly $E = \bigoplus_{i,j \in \mathcal{Q}_0} e_j E e_i$. From $A = \bigoplus_{i \in \mathcal{Q}_0} A e_i = \bigoplus_{j \in \mathcal{Q}_0} e_j A$, we deduce that the A -bimodule $A \otimes_k E \otimes_k A$ is isomorphic to the A -bimodule

$$F_1 = \bigoplus_{i,j \in \mathcal{Q}_0} A e_j \otimes e_j E e_i \otimes e_i A.$$

Considering F_1 as a sub- A -bimodule of $F = A \otimes E \otimes A$, we see that $F = F_1 \oplus F_2$, where

$$F_2 = \bigoplus_{i_1, i_2, i_3, i_4 \in \mathcal{Q}_0} A e_{i_1} \otimes e_{i_2} E e_{i_3} \otimes e_{i_4} A$$

in which the sum is taken over the set of indices with $i_1 \neq i_2$ and $i_3 \neq i_4$. As the A -bimodule F is free, we conclude that F_1 is projective. Part (ii) follows from the fact that E is finite dimensional if and only if all the $(e_j E e_i)$ are finite dimensional and is left to the reader. ■

Definition 2.2. For any A -bimodule M , Koszul homology and cohomology are defined by

$$\mathrm{HK}_\bullet(A, M) = H_\bullet(M \otimes_{A^e} K(A)) \text{ and } \mathrm{HK}^\bullet(A, M) = H^\bullet(\mathrm{Hom}_{A^e}(K(A), M)).$$

We set $\mathrm{HK}_\bullet(A) = \mathrm{HK}_\bullet(A, A)$ and $\mathrm{HK}^\bullet(A) = \mathrm{HK}^\bullet(A, A)$.

Since $K(A)$ is a complex of projective A -bimodules, $M \mapsto \mathrm{HK}_\bullet(A, M)$ and $M \mapsto \mathrm{HK}^\bullet(A, M)$ define δ -functors from the category of A -bimodules to the category of vector spaces, that is, a short exact sequence of A -bimodules gives rise to a long exact sequence in Koszul homology and in Koszul cohomology [36]. As in [6], $\mathrm{HK}_p(A, M)$ (respectively $\mathrm{HK}^p(A, M)$) is isomorphic to a Hochschild hyperhomology (respectively hypercohomology) space.

The inclusion $\chi : K(A) \rightarrow B(A)$ is a morphism of complexes that induces the following morphisms of complexes

$$\tilde{\chi} = M \otimes_{A^e} \chi : M \otimes_{A^e} K(A) \rightarrow M \otimes_{A^e} B(A),$$

$$\chi^* = \mathrm{Hom}_{A^e}(\chi, M) : \mathrm{Hom}_{A^e}(B(A), M) \rightarrow \mathrm{Hom}_{A^e}(K(A), M).$$

The linear maps $H(\tilde{\chi}) : \mathrm{HK}_p(A, M) \rightarrow \mathrm{HH}_p(A, M)$ and $H(\chi^*) : \mathrm{HH}^p(A, M) \rightarrow \mathrm{HK}^p(A, M)$ are always isomorphisms for $p = 0$ and $p = 1$, and if A is Koszul they are isomorphisms for any p .

Replacing $B(A)$ by $K(A)$ in the isomorphisms (2.1) and (2.2) gives the isomorphisms

$$\begin{aligned} \mathrm{HK}_\bullet(A, M) &\cong H_\bullet(M \otimes_{k^e} W_\bullet, b^K), \\ \mathrm{HK}^\bullet(A, M) &\cong H^\bullet(\mathrm{Hom}_{k^e}(W_\bullet, M), b_K). \end{aligned}$$

The Koszul differentials b^K and b_K are given by

$$b_p^K(m \otimes_{k^e} x_1 \dots x_p) = m \cdot x_1 \otimes_{k^e} x_2 \dots x_p + (-1)^p x_p \cdot m \otimes_{k^e} x_1 \dots x_{p-1}, \quad (2.5)$$

$$b_K^{p+1}(f)(x_1 \dots x_{p+1}) = f(x_1 \dots x_p) \cdot x_{p+1} - (-1)^p x_1 \cdot f(x_2 \dots x_{p+1}), \quad (2.6)$$

where $m \in M$ and $x_1 \dots x_p \in W_p$, respectively $f \in \mathrm{Hom}_{k^e}(W_p, M)$ and $x_1 \dots x_{p+1} \in W_{p+1}$.

Note that the k -algebra A is *augmented* by the natural projection $\epsilon_A : A \rightarrow A_0 \cong k$. Let us examine now the particular case $M = k$, where k is the A -bimodule defined by ϵ_A . The action on k of an element of A_p with $p > 0$ is zero, so that the Koszul differentials vanish when $M = k$. Consequently, we have the linear isomorphisms

$$\mathrm{HK}_p(A, k) \cong k \otimes_{k^e} W_p \cong \bigoplus_{i \in \mathcal{Q}_0} e_i W_p e_i,$$

$$\mathrm{HK}^p(A, k) \cong \mathrm{Hom}_{k^e}(W_p, k) \cong \bigoplus_{i \in \mathcal{Q}_0} \mathrm{Hom}(e_i W_p e_i, \mathbb{F}).$$

In particular $\mathrm{HK}^p(A, k) \cong \mathrm{Hom}(\mathrm{HK}_p(A, k), \mathbb{F})$, generalising [6].

2.3 Koszul cup and cap products

Let $A = T_k(V)/(R)$ be a quadratic k -algebra over \mathcal{Q} . We proceed as in [6]. The usual cup and cap products \smile and \frown in Hochschild cohomology and homology provide, by restriction from $B(A)$ to $K(A)$, the Koszul cup and cap products \smile_K and \frown_K in Koszul cohomology and homology. Let us give these products, expressed on Koszul cochains and chains. Let P, Q and M be A -bimodules. For $f \in \mathrm{Hom}_{k^e}(W_p, P)$, $g \in \mathrm{Hom}_{k^e}(W_q, Q)$ and $z = m \otimes_{k^e} x_1 \dots x_q \in$

$M \otimes_{k^e} W_q$, we define $f \underset{K}{\smile} g \in \text{Hom}_{k^e}(W_p, P \otimes_A Q)$, $f \underset{K}{\frown} z \in (P \otimes_A M) \otimes_{k^e} W_{q-p}$ and $z \underset{K}{\frown} f \in (M \otimes_A P) \otimes_{k^e} W_{q-p}$ by

$$(f \underset{K}{\smile} g)(x_1 \dots x_{p+q}) = (-1)^{pq} f(x_1 \dots x_p) \otimes_A g(x_{p+1} \dots x_{p+q}), \quad (2.7)$$

$$f \underset{K}{\frown} z = (-1)^{(q-p)p} (f(x_{q-p+1} \dots x_q) \otimes_A m) \otimes_{k^e} x_1 \dots x_{q-p}, \quad (2.8)$$

$$z \underset{K}{\frown} f = (-1)^{pq} (m \otimes_A f(x_1 \dots x_p)) \otimes_{k^e} x_{p+1} \dots x_q. \quad (2.9)$$

For any Koszul cochains f, g, h and any Koszul chain z , we have the *associativity relations*

$$\begin{aligned} (f \underset{K}{\smile} g) \underset{K}{\smile} h &= f \underset{K}{\smile} (g \underset{K}{\smile} h), \\ f \underset{K}{\frown} (g \underset{K}{\frown} z) &= (f \underset{K}{\smile} g) \underset{K}{\frown} z, \\ (z \underset{K}{\frown} g) \underset{K}{\frown} f &= z \underset{K}{\frown} (g \underset{K}{\smile} f), \\ f \underset{K}{\frown} (z \underset{K}{\frown} g) &= (f \underset{K}{\smile} z) \underset{K}{\frown} g, \end{aligned}$$

inducing the same relations on Koszul classes.

We define the *general Koszul calculus of A* as being the datum of all the spaces $\text{HK}^\bullet(A, P)$ and $\text{HK}_\bullet(A, M)$ endowed with $\underset{K}{\smile}$ and $\underset{K}{\frown}$, when the A -bimodules P and M vary.

For any A -bimodule M , $\text{Hom}_{k^e}(W_\bullet, M)$ and $M \otimes_{k^e} W_\bullet$ are DG bimodules over the DG algebra $(\text{Hom}_{k^e}(W_\bullet, A), b_K, \underset{K}{\smile})$ for the actions of $\underset{K}{\smile}$ and $\underset{K}{\frown}$ respectively. The *Koszul calculus of A* consists of the graded associative algebra $(\text{HK}^\bullet(A), \underset{K}{\smile})$ and of the graded $\text{HK}^\bullet(A)$ -bimodules $\text{HK}^\bullet(A, M)$ and $\text{HK}_\bullet(A, M)$. Since $\text{HK}^0(A) = Z(A)$ is the centre of the algebra A , the spaces $\text{HK}^p(A, M)$ and $\text{HK}_p(A, M)$ are symmetric $Z(A)$ -bimodules. However $\text{HK}^p(A, M)$ and $\text{HK}_p(A, M)$ are not k -bimodules in general. For example, $e_i \notin Z(A)$ if there is an arrow joining i to another vertex.

The *restricted Koszul calculus of A* consists of the graded associative algebra $\text{HK}^\bullet(A)$ and of the graded $\text{HK}^\bullet(A)$ -bimodule $\text{HK}_\bullet(A)$. The *scalar Koszul calculus of A* consists of the graded associative algebra $\text{HK}^\bullet(A, k)$ and of the graded $\text{HK}^\bullet(A, k)$ -bimodule $\text{HK}_\bullet(A, k)$.

Example 2.3. If $\mathcal{Q}_1 = \emptyset$, then $V = 0$ and A is reduced to k . The (general) Koszul calculus of k coincides with the (tensor) category of k -bimodules.

2.4 Fundamental formulas of Koszul calculus

Let $A = T_k(V)/(R)$ be a quadratic k -algebra over \mathcal{Q} . We continue to follow the one vertex case [6]. First, we define the Koszul cup and cap brackets. Let P, Q and M be A -bimodules, and take $f \in \text{Hom}_{k^e}(W_p, P)$, $g \in \text{Hom}_{k^e}(W_q, Q)$, $z = m \otimes_{k^e} x_1 \dots x_q \in M \otimes_{k^e} W_q$. When P or Q is equal to A , we set

$$[f, g]_{\underset{K}{\smile}} = f \underset{K}{\smile} g - (-1)^{pq} g \underset{K}{\smile} f. \quad (2.10)$$

When P or M is equal to A , we set

$$[f, z]_{\underset{K}{\frown}} = f \underset{K}{\frown} z - (-1)^{pq} z \underset{K}{\frown} f. \quad (2.11)$$

These brackets induce brackets on the Koszul classes.

The Koszul 1-cocycles $f : V \rightarrow M$ are called Koszul derivations with coefficients in M . Such an f extends to a unique derivation from the k -algebra A to the A -bimodule M , realising

an isomorphism from the space of Koszul derivations with coefficients in M to the space of derivations from A to M . In particular, the Koszul 1-cocycle from V to A coinciding with the identity map on V , is sent to the Euler derivation D_A of the graded algebra A . This Koszul 1-cocycle is denoted by \mathfrak{e}_A and is called the *fundamental 1-cocycle*. Its Koszul class is denoted by $\bar{\mathfrak{e}}_A$ and is called the *fundamental 1-class*. In the one vertex case, \mathfrak{e}_A is not a coboundary if $V \neq 0$ [6], but this property does not hold in general.

Lemma 2.4. *Let $A = T_k(V)/(R)$ be a quadratic k -algebra over \mathcal{Q} with $\mathcal{Q}_1 \neq \emptyset$. If the underlying graph of \mathcal{Q} is simple, that is, it contains neither loops nor multiple edges, then \mathfrak{e}_A is a coboundary.*

Proof. The 1-cocycle \mathfrak{e}_A is a coboundary if and only if there exists a k^e -linear map $c : k \rightarrow k$ such that $\mathfrak{e}_A = b_K(c)$. Such a map is of the form $c(e_i) = \lambda_i e_i$ with $\lambda_i \in \mathbb{F}$, for any $i \in \mathcal{Q}_0$. Then $\mathfrak{e}_A = b_K(c)$ if and only if $\lambda_{t(\alpha)} - \lambda_{s(\alpha)} = 1$ for any $\alpha \in \mathcal{Q}_1$. The assumption on the graph means that \mathcal{Q} has no loop and that given two distinct vertices, there is at most one arrow joining them. Then we can choose $\lambda_{t(\alpha)} = 1$ and $\lambda_{s(\alpha)} = 0$. ■

This proof shows that if \mathcal{Q} has a loop, \mathfrak{e}_A is not a coboundary. The same conclusion holds if $\text{char } \mathbb{F} \neq 2$ and \mathcal{Q} contains an oriented 2-cycle. The following propositions are proved as in the one vertex case. Formulas (2.12) and (2.13) are the fundamental formulas of Koszul calculus.

Proposition 2.5. *Let $A = T_k(V)/(R)$ be a quadratic k -algebra over \mathcal{Q} . For any Koszul cochain f and any Koszul chain z with coefficients in an A -bimodule M , we have*

$$[\mathfrak{e}_A, f]_{\underset{K}{\smile}} = -b_K(f), \quad (2.12)$$

$$[\mathfrak{e}_A, z]_{\underset{K}{\frown}} = -b_K(z). \quad (2.13)$$

Proposition 2.6. *Let $A = T_k(V)/(R)$ be a quadratic k -algebra over \mathcal{Q} and let M be an A -bimodule. For any $\alpha \in \text{HK}^p(A, M)$ with $p = 0$ or $p = 1$, $\beta \in \text{HK}^q(A)$ and $\gamma \in \text{HK}_q(A)$, we have the identities*

$$[\alpha, \beta]_{\underset{K}{\smile}} = 0, \quad (2.14)$$

$$[\alpha, \gamma]_{\underset{K}{\frown}} = 0. \quad (2.15)$$

Identity (2.15) also holds if $p = q \notin \{0, 1\}$.

2.5 Higher Koszul calculus

Let $A = T_k(V)/(R)$ be a quadratic k -algebra over \mathcal{Q} . Formula (2.7) shows that $\mathfrak{e}_A \underset{K}{\smile} \mathfrak{e}_A = 0$. Therefore, $\mathfrak{e}_A \underset{K}{\smile} -$ is a cochain differential on $\text{Hom}_{k^e}(W_\bullet, M)$, and $\bar{\mathfrak{e}}_A \underset{K}{\smile} -$ is a cochain differential on $\text{HK}^\bullet(A, M)$. Similarly, $\mathfrak{e}_A \underset{K}{\frown} -$ is a chain differential on $M \otimes_{k^e} W_\bullet$, and $\bar{\mathfrak{e}}_A \underset{K}{\frown} -$ is a chain differential on $\text{HK}_\bullet(A, M)$. For a p -cocycle $f : W_p \rightarrow M$ and $x_1 \dots x_{p+1}$ in W_{p+1} , we have

$$(\mathfrak{e}_A \underset{K}{\smile} f)(x_1 \dots x_{p+1}) = f(x_1 \dots x_p) \cdot x_{p+1}.$$

For a p -cycle $z = m \otimes_{k^e} x_1 \dots x_p$ in $M \otimes_{k^e} W_p$, we have

$$\mathfrak{e}_A \underset{K}{\frown} z = m x_1 \otimes_{k^e} x_2 \dots x_p.$$

Definition 2.7. Let $A = T_k(V)/(R)$ be a quadratic k -algebra over a finite quiver \mathcal{Q} and let M be an A -bimodule. The differentials $\bar{\epsilon}_A \underset{K}{\smile} -$ and $\bar{\epsilon}_A \underset{K}{\frown} -$ are denoted by ∂_{\smile} and ∂_{\frown} . The homologies of the complexes $(\mathrm{HK}^\bullet(A, M), \partial_{\smile})$ and $(\mathrm{HK}_\bullet(A, M), \partial_{\frown})$ are called the higher Koszul cohomology and homology of A with coefficients in M and are denoted by $\mathrm{HK}_{hi}^\bullet(A, M)$ and $\mathrm{HK}_\bullet^{hi}(A, M)$. We set $\mathrm{HK}_{hi}^\bullet(A) = \mathrm{HK}_{hi}^\bullet(A, A)$ and $\mathrm{HK}_\bullet^{hi}(A) = \mathrm{HK}_\bullet^{hi}(A, A)$.

The higher classes of Koszul classes will be denoted between square brackets. For example, the unit 1 of A is still the unit of $\mathrm{HK}^\bullet(A)$, and $\partial_{\smile}(1) = \bar{\epsilon}_A$ implies that $[\bar{\epsilon}_A] = 0$. If $\bar{\epsilon}_A \neq 0$, the unit of $\mathrm{HK}^\bullet(A)$ does not survive in higher Koszul cohomology.

As in the one vertex case, the actions of the Koszul cup and cap products of $\mathrm{HK}^\bullet(A)$ on $\mathrm{HK}^\bullet(A, M)$ and $\mathrm{HK}_\bullet(A, M)$ induce actions on higher cohomology and homology. Thus $\mathrm{HK}_{hi}^\bullet(A)$ is a graded algebra, and $\mathrm{HK}_{hi}^\bullet(A, M)$, $\mathrm{HK}_\bullet^{hi}(A, M)$ are graded $\mathrm{HK}_{hi}^\bullet(A)$ -bimodules, constituting the higher Koszul calculus of A . If $\bar{\epsilon}_A = 0$, the higher Koszul calculus coincides with the Koszul calculus. It is the case when $A = k$ as in Example 2.3.

For $M = k$, $\epsilon_A \underset{K}{\smile} -$ and $\epsilon_A \underset{K}{\frown} -$ vanish, so that the higher scalar Koszul calculus coincides with the scalar Koszul calculus. Proposition 3.12 in [6] generalises immediately as follows.

Proposition 2.8. Let $A = T_k(V)/(R)$ be a quadratic k -algebra over \mathcal{Q} and let M be an A -bimodule. Then $\mathrm{HK}_{hi}^0(A, M)$ is the space of elements u in $Z(M)$ such that there exists $v \in M$ satisfying $u.a = v.a - a.v$ for any a in \mathcal{Q}_1 .

2.6 Grading the restricted Koszul calculus by the weight

A Koszul p -cochain $f : W_p \rightarrow A_m$ is said to be homogeneous of weight m . Since \mathcal{Q}_1 is finite, the spaces W_p are finite dimensional, thus the space of Koszul cochains $\mathrm{Hom}_{k^e}(W_\bullet, A)$ is $\mathbb{N} \times \mathbb{N}$ -graded by the biweight (p, m) , where p is called the *homological weight* and m is called the *coefficient weight*. If $f : W_p \rightarrow A_m$ and $g : W_q \rightarrow A_n$ are homogeneous of biweights (p, m) and (q, n) respectively, then $f \underset{K}{\smile} g : W_{p+q} \rightarrow A_{m+n}$ is homogeneous of biweight $(p+q, m+n)$. Moreover b_K is homogeneous of biweight $(1, 1)$ and the algebra $\mathrm{HK}^\bullet(A)$ is $\mathbb{N} \times \mathbb{N}$ -graded by the biweight. The homogeneous component of biweight (p, m) of $\mathrm{HK}^\bullet(A)$ is denoted by $\mathrm{HK}^p(A)_m$. Since

$$\partial_{\smile} : \mathrm{HK}^p(A)_m \rightarrow \mathrm{HK}^{p+1}(A)_{m+1},$$

the algebra $\mathrm{HK}_{hi}^\bullet(A)$ is $\mathbb{N} \times \mathbb{N}$ -graded by the biweight, and its (p, m) -component is denoted by $\mathrm{HK}_{hi}^p(A)_m$. From Proposition 2.8, we deduce the following.

Proposition 2.9. Let $A = T_k(V)/(R)$ be a quadratic k -algebra over \mathcal{Q} . Assume that A is finite dimensional. Let \max be the highest m such that $A_m \neq 0$. Then $\mathrm{HK}_{hi}^0(A)_{\max}$ is isomorphic to the space spanned by the cycles of \mathcal{Q} of length \max .

Similarly, a Koszul q -chain z in $A_n \otimes_{k^e} W_q$ is said to be homogeneous of weight n . The space of Koszul chains $A \otimes_{k^e} W_\bullet$ is $\mathbb{N} \times \mathbb{N}$ -graded by the biweight (q, n) , where q is called the *homological weight* and n is called the *coefficient weight*. Moreover b^K is homogeneous of biweight $(-1, 1)$ and the space $\mathrm{HK}_\bullet(A)$ is $\mathbb{N} \times \mathbb{N}$ -graded by the biweight. The homogeneous component of biweight (q, n) of $\mathrm{HK}_\bullet(A)$ is denoted by $\mathrm{HK}_q(A)_n$. Since

$$\partial_{\frown} : \mathrm{HK}_q(A)_n \rightarrow \mathrm{HK}_{q-1}(A)_{n+1},$$

the space $\mathrm{HK}_\bullet^{hi}(A)$ is $\mathbb{N} \times \mathbb{N}$ -graded by the biweight, and its (q, n) -component is denoted by $\mathrm{HK}_q^{hi}(A)_n$.

If $f : W_p \rightarrow A_m$ and $z \in A_n \otimes_{k^e} W_q$ are homogeneous of biweights (p, m) and (q, n) respectively, then $f \underset{K}{\frown} z$ and $z \underset{K}{\frown} f$ are homogeneous of biweight $(q - p, m + n)$ where

$$f \underset{K}{\frown} z = (-1)^{(q-p)p} f(x_{q-p+1} \dots x_q) a \otimes_{k^e} x_1 \dots x_{q-p}, \quad (2.16)$$

$$z \underset{K}{\frown} f = (-1)^{pq} a f(x_1 \dots x_p) \otimes_{k^e} x_{p+1} \dots x_q, \quad (2.17)$$

and $z = a \otimes_{k^e} x_1 \dots x_q$. The $\text{Hom}_{k^e}(W_\bullet, A)$ -bimodule $A \otimes_{k^e} W_\bullet$, the $\text{HK}^\bullet(A)$ -bimodule $\text{HK}_\bullet(A)$ and the $\text{HK}_{hi}^\bullet(A)$ -bimodule $\text{HK}_\bullet^{hi}(A)$ are thus $\mathbb{N} \times \mathbb{N}$ -graded by the biweight. The proof of the following is left to the reader.

Proposition 2.10. *Let $A = T_k(V)/(R)$ be a quadratic k -algebra over \mathcal{Q} . We have*

$$\text{HK}_0^{hi}(A)_0 \cong \text{HK}_0(A)_0 \cong k.$$

Moreover $\text{HK}_0(A)_1 \cong \text{HK}_1(A)_0$ is isomorphic to the space spanned by the loops of \mathcal{Q} , and $\partial_- : \text{HK}_1(A)_0 \rightarrow \text{HK}_0(A)_1$ identifies with the identity map of this space. As a consequence,

$$\text{HK}_0^{hi}(A)_1 \cong \text{HK}_1^{hi}(A)_0 \cong 0.$$

2.7 Invariance of Koszul calculus

In [5], the first author proved that the Koszul calculus of an N -homogeneous algebra A over a field k only depends on the structure of associative algebra of A , independently of any presentation $A = T_k(V)/(R)$ of A as an N -homogeneous algebra. This result was based on an isomorphism lemma due to Bell and Zhang [2]. In the quadratic case $N = 2$, we are going to extend this Koszul calculus invariance to any quadratic quiver algebra. For that, we shall use an extension of the isomorphism lemma to quiver algebras with homogeneous relations, due to Gaddis [21].

Let \mathcal{Q} and \mathcal{Q}' be finite quivers, and \mathbb{F} be a field. We introduce the commutative rings $k = \mathbb{F}\mathcal{Q}_0$ and $k' = \mathbb{F}\mathcal{Q}'_0$, the k -bimodule $V = \mathbb{F}\mathcal{Q}_1$ and the k' -bimodule $V' = \mathbb{F}\mathcal{Q}'_1$. As explained in Subsection 2.1, we make the identifications of graded algebras $T_k(V) \cong \mathbb{F}\mathcal{Q}$ and $T_{k'}(V') \cong \mathbb{F}\mathcal{Q}'$. We are interested in the graded \mathbb{F} -algebra isomorphisms $u : T_k(V) \rightarrow T_{k'}(V')$ given by a ring isomorphism $u_0 : k \rightarrow k'$ and by a k -bimodule isomorphism $u_1 : V \rightarrow V'$, where V' is a k -bimodule via u_0 . This implies that u_0 maps \mathcal{Q}_0 to \mathcal{Q}'_0 , and the bijection $\mathcal{Q}_0 \rightarrow \mathcal{Q}'_0$ induced by u_0 transforms the adjacency matrix of \mathcal{Q} into the adjacency matrix of \mathcal{Q}' [21].

Let us fix a sub- k -bimodule R of $V \otimes_k V$ and a sub- k' -bimodule R' of $V' \otimes_{k'} V'$. We define the graded k -algebra $A = T_k(V)/(R)$ and the graded k' -algebra $A' = T_{k'}(V')/(R')$. Following the terminology of the one vertex case, a graded \mathbb{F} -algebra isomorphism $u : A \rightarrow A'$ is called a *Manin isomorphism* if u is defined by a ring isomorphism $u_0 : k \rightarrow k'$, and by a k -bimodule isomorphism $u_1 : V \rightarrow V'$, such that the k -bimodule isomorphism $u_1^{\otimes_{k^e} 2} : V^{\otimes_{k^e} 2} \rightarrow V'^{\otimes_{k^e} 2}$ (that actually takes values in $V'^{\otimes_{k^e} 2}$ via u_0) satisfies $u_1^{\otimes_{k^e} 2}(R) = R'$. In particular, u is an isomorphism of the augmented k -algebra A to the augmented k' -algebra A' , the augmentations being the projections $A \rightarrow A_0 \cong k$ and $A' \rightarrow A'_0 \cong k'$.

As in [5], for any A -bimodule M , the Manin isomorphism u naturally defines an isomorphism of complexes from $(M \otimes_{k^e} W_\bullet, b^K)$ to $(M \otimes_{k'^e} W'_\bullet, b^K)$, where M is an A' -bimodule via u , inducing natural isomorphisms $\text{HK}_\bullet(A, M) \cong \text{HK}_\bullet(A', M)$. Similarly, u induces natural isomorphisms $\text{HK}^\bullet(A', M) \cong \text{HK}^\bullet(A, M)$. It is clear from the definitions in Subsection 2.3 that these isomorphisms respect the Koszul cup and cap products. To summarise all these properties, we say that a Manin isomorphism induces isomorphic (general) Koszul calculi. Since $u_1(\epsilon_A) = \epsilon_{A'}$ by functoriality, it also induces isomorphic higher Koszul calculi.

As an application, let us show that the various Koszul calculi of a quadratic k -algebra $A = T_k(V)/(R)$ over a finite quiver \mathcal{Q} only depend on the graph Δ underlying \mathcal{Q} , not on an orientation of Δ . In fact, if a quiver \mathcal{Q}' has the same underlying graph Δ , then $\mathcal{Q}_0 = \mathcal{Q}'_0$ and there is a natural bijection $\mathcal{Q}_1 \rightarrow \mathcal{Q}'_1$ inducing a k -bimodule isomorphism $u_1 : V \rightarrow V' = \mathbb{F}\mathcal{Q}'_1$, so that R maps to a sub- k -bimodule R' of $\mathbb{F}\mathcal{Q}'_2$. We obtain a Manin isomorphism u from A to $A' = T_k(V')/(R')$. Thus the (general, higher) Koszul calculi of A and A' are isomorphic.

Using Gaddis's result, we can now prove ungraded invariance. Let C be a commutative ring. Let A be an augmented associative C -algebra (not necessarily C -central) having a quadratic quiver algebra presentation B , meaning that the augmented C -algebra A is isomorphic to a quadratic k -algebra $B = T_k(V)/(R)$ over a finite quiver \mathcal{Q} , naturally augmented over k by the projection $B \rightarrow B_0 \cong k$. This implies that the ring C is isomorphic to $k \cong \mathbb{F}\mathcal{Q}_0$. Then *we can define the (general) Koszul calculus of A as being the (general) Koszul calculus of B .*

In fact, if $B' = T_{k'}(V')/(R')$ over a finite quiver \mathcal{Q}' is another quadratic quiver algebra presentation of A , then the ungraded augmented k -algebra B is isomorphic to the ungraded augmented k' -algebra B' . By Gaddis's theorem [21], there exists a Manin isomorphism from B to B' , thus the (general) Koszul calculi of B and B' are isomorphic by Manin invariance. *The higher Koszul calculus of A is also defined as being the higher Koszul calculus of B .*

2.8 Comparing Koszul (co)homology with Hochschild (co)homology in degree 2

Let $A = T_k(V)/(R)$ be a quadratic k -algebra over \mathcal{Q} . Recall that, for $p = 0$ and $p = 1$, we have linear isomorphisms $\mathrm{HK}_p(A, M) \cong \mathrm{HH}_p(A, M)$ and $\mathrm{HH}^p(A, M) \cong \mathrm{HK}^p(A, M)$ (Subsection 2.2). It is no longer true if $p \geq 2$ and if A is an arbitrary non-Koszul algebra. Preprojective algebras of Dynkin type will give infinitely many counterexamples when $p = 2$. However, in general, we can compare the Koszul and Hochschild space when $p = 2$, by providing a surjection $\mathrm{HK}_2(A, M) \rightarrow \mathrm{HH}_2(A, M)$ and an injection $\mathrm{HH}^2(A, M) \rightarrow \mathrm{HK}^2(A, M)$. To prove that, we use a minimal projective resolution of the graded k -algebra A , described as follows.

As in the one vertex case [6], we know that, in the category of graded A -bimodules, A has a minimal projective resolution $P(A)$ whose component of homological degree p can be written as $A \otimes_k E_p \otimes_k A$, where E_p is a weight-graded k -bimodule. Then $P(A) \otimes_A k$ (respectively $k \otimes_A P(A)$) is a minimal projective resolution of the graded left (respectively right) A -module k (see for instance [4]), so that the weight-graded k -bimodules E_p can be obtained inductively on p either by the construction of a left minimal projective resolution of k or by the construction of a right one. The differential δ of $P(A)$ is the graded sum of the left (respectively right) differentials $\delta \otimes_A k$ (respectively $k \otimes_A \delta$) extended by right (respectively left) A -linearity to each $A \otimes_k E_p \otimes_k A$.

The left or right construction of the weight-graded k -bimodules E_p shows that the minimal weight of E_p is equal to p and the homogeneous component of weight p in E_p is equal to W_p . The inclusions $A \otimes_k W_p \otimes_k A \hookrightarrow A \otimes_k E_p \otimes_k A$ constitute an inclusion map $\iota : K(A) \hookrightarrow P(A)$ of weight-graded A -bimodule complexes. So we can view the complex $K(A)$ as the diagonal part of the weight-graded resolution $P(A)$, and A is Koszul if and only if $P(A) = K(A)$. The beginning of $P(A)$ coincides with $K(A)$, that is, $E_0 = k$, $E_1 = V$, $E_2 = R$, and the differential δ of $P(A)$ coincides with the differential d of $K(A)$ in degrees 1 and 2.

For any A -bimodule M , ι induces $\tilde{\iota} = M \otimes_{A^e} \iota$ and $\iota^* = \mathrm{Hom}_{A^e}(\iota, M)$ decomposed in $\tilde{\iota}_p : M \otimes_{k^e} W_p \rightarrow M \otimes_{k^e} E_p$ and $\iota_p^* : \mathrm{Hom}_{k^e}(E_p, M) \rightarrow \mathrm{Hom}_{k^e}(W_p, M)$. The linear maps

$$H(\tilde{\iota})_p : \mathrm{HK}_p(A, M) \rightarrow \mathrm{HH}_p(A, M) \quad \text{and} \quad H(\iota^*)_p : \mathrm{HH}^p(A, M) \rightarrow \mathrm{HK}^p(A, M) \quad (2.18)$$

are isomorphisms for $p = 0$ and $p = 1$, and for any p if A is Koszul. Since ι_p is an identity map

for $p = 0, 1, 2$, $\tilde{\iota}_p$ and ι_p^* are also identity maps for the same p 's. Therefore

$$\begin{array}{ccccc} M \otimes_{k^e} W_3 & \xrightarrow{b_3^K} & M \otimes_{k^e} R & \xrightarrow{b_2^K} & M \otimes_{k^e} V \\ \tilde{\iota}_3 \downarrow & & id \downarrow & & id \downarrow \\ M \otimes_{k^e} E_3 & \xrightarrow{\tilde{\delta}_3} & M \otimes_{k^e} R & \xrightarrow{\tilde{\delta}_2} & M \otimes_{k^e} V \end{array} \quad (2.19)$$

$$\begin{array}{ccccc} \mathrm{Hom}_{k^e}(V, M) & \xrightarrow{\delta_2^*} & \mathrm{Hom}_{k^e}(R, M) & \xrightarrow{\delta_3^*} & \mathrm{Hom}_{k^e}(E_3, M) \\ id \downarrow & & id \downarrow & & \iota_3^* \downarrow \\ \mathrm{Hom}_{k^e}(V, M) & \xrightarrow{b_K^2} & \mathrm{Hom}_{k^e}(R, M) & \xrightarrow{b_K^3} & \mathrm{Hom}_{k^e}(W_3, M) \end{array} \quad (2.20)$$

are commutative diagrams in which $\tilde{\iota}_3$ is injective and ι_3^* is surjective (the ring k^e is semi-simple), so that we obtain.

Proposition 2.11. *Let $A = T_k(V)/(R)$ be a quadratic k -algebra over \mathcal{Q} . For any M ,*

- (i) $H(\tilde{\iota})_2 : \mathrm{HK}_2(A, M) \rightarrow \mathrm{HH}_2(A, M)$ *is surjective with kernel isomorphic to $\mathrm{im}(\tilde{\delta}_3)/\mathrm{im}(b_3^K)$,*
- (ii) $H(\iota^*)_2 : \mathrm{HH}^2(A, M) \rightarrow \mathrm{HK}^2(A, M)$ *is injective with image isomorphic to $\ker(\delta_3^*)/\mathrm{im}(b_K^2)$.*

We can be more specific when $M = A$, by using the weight grading (Subsection 2.6). Unlike the Koszul differentials b^K and b_K , the Hochschild differentials b^H and b_H are not homogeneous for the coefficient weight, but only for the total weight. The grading of $\mathrm{HH}_p(A)$ and $\mathrm{HH}^p(A)$ for the total weight t is denoted by $\mathrm{HH}_p(A)_t$ and $\mathrm{HH}^p(A)_t$. Denote the weight of a homogeneous element a of A by $|a|$. Recall that the total weight of a homogeneous p -chain $z = a \otimes_{k^e} (a_1 \dots a_p)$ is equal to $t = |a| + |a_1| + \dots + |a_p|$, and the total weight of a homogeneous p -cochain f mapping $a_1 \dots a_p$ to an element of A_m is equal to $t = m - |a_1| - \dots - |a_p|$. Then $H(\tilde{\iota})_2$ is homogeneous from the coefficient weight r to the total weight $r + 2$, while $H(\iota^*)_2$ is homogeneous from the total weight $r - 2$ to the coefficient weight r .

Corollary 2.12. *Let $A = T_k(V)/(R)$ be a quadratic k -algebra over \mathcal{Q} .*

- (i) $H(\tilde{\iota})_2$ *is an isomorphism from $\mathrm{HK}_2(A)_r$ to $\mathrm{HH}_2(A)_{r+2}$ if $r = 0$ and $r = 1$.*
- (ii) *Assume that A is finite dimensional. Let \max be the highest m such that $A_m \neq 0$. Then $H(\iota^*)_2$ is an isomorphism from $\mathrm{HH}^2(A)_{r-2}$ to $\mathrm{HK}^2(A)_r$ if $r = \max$ and $r = \max - 1$.*

Proof. Denote by $E_{p,m}$ the homogeneous component of weight m of E_p . Since $E_{3,2} = 0$ and $E_{3,3} = W_3$, both maps $\tilde{\delta}_3$ and b_3^K vanish on the component of total weight 2 of $A \otimes_{k^e} E_3$, while on that of total weight 3, they coincide with the inclusion map of W_3 into $V \otimes_{k^e} R$. Then we deduce (i) from (i) of the proposition.

Under the assumptions of (ii), if $f : R \rightarrow A_{\max}$, then $b_K^3(f) = 0$. Moreover, any other component of $\delta_3^*(f)$ mapping $E_{3,m}$ to $A_{\max+m-2} = 0$ vanishes as well. Thus $\delta_3^*(f) = 0$, and we conclude by (ii) of the proposition. The same proof works if $f : R \rightarrow A_{\max-1}$ since $\delta_3^*(f)$ is then reduced to a map $W_3 \rightarrow A_{\max}$ coinciding with $b_K^3(f)$. ■

3 A right action on the Koszul calculus

3.1 Compatibility

Lemma 3.1. *Let A and B be unital associative \mathbb{F} -algebras. Let M be an A -bimodule (hence the induced \mathbb{F} -bimodule is symmetric). Assume that M is a right B -module such that the actions*

of \mathbb{F} induced on M by A and by B are the same. Let $A^e = A \otimes_{\mathbb{F}} A^{op}$ be the enveloping algebra. The following are equivalent.

- (i) Viewing M as a left A^e -module, M is an A^e - B -bimodule.
- (ii) Viewing M as a right A^e -module, the right actions of A^e and B on M commute.
- (iii) M is an A - B -bimodule and the right actions of A and B on M commute.

The proof is straightforward. Under the assumptions of the lemma and if the equivalent assertions hold, we say that the right action of B on M is *compatible* with the A -bimodule M .

Example 3.2. With $B = M = A^e$, A^e is a natural A^e - A^e -bimodule for the multiplication of the \mathbb{F} -algebra A^e . Recall that the left A^e -module A^e is isomorphic to the A -bimodule $A \overset{o}{\otimes} A$ for the outer action $(a \otimes b).(\alpha \otimes \beta) = (a\alpha) \otimes (b\beta)$, while the right A^e -module A^e is isomorphic to the A -bimodule $A \overset{i}{\otimes} A$ for the inner action $(\alpha \otimes \beta).(a \otimes b) = (\alpha a) \otimes (b\beta)$.

3.2 DG bimodules over a DG algebra

Let $A = T_k(V)/(R)$ be a quadratic k -algebra over \mathcal{Q} . Fix a unital associative \mathbb{F} -algebra B and an A -bimodule M . We assume that M is a right B -module compatible with the A -bimodule structure. Then the space $M \otimes_{k^e} W_q$ is a right B -module for the action of $b \in B$ on $z = m \otimes_{k^e} x_1 \dots x_q \in M \otimes_{k^e} W_q$ defined by

$$z.b = (m.b) \otimes_{k^e} x_1 \dots x_q.$$

It is well-defined since $(\lambda m \mu).b = \lambda(m.b)\mu$ for any λ and μ in k . From (2.5) and (2.8), we check that b^K and $\epsilon_A \underset{K}{\frown} -$ are B -linear. Thus $\mathrm{HK}_{\bullet}(A, M)$ and $\mathrm{HK}_{\bullet}^{hi}(A, M)$ are graded right B -modules.

Moreover, for any k -bimodule morphism $f : W_p \rightarrow A$, we verify that

$$f \underset{K}{\frown} (z.b) = (f \underset{K}{\frown} z).b \text{ and } (z.b) \underset{K}{\frown} f = (z \underset{K}{\frown} f).b,$$

so that the right action of B on $M \otimes_{k^e} W_{\bullet}$ is compatible with the $\mathrm{Hom}_{k^e}(W_{\bullet}, A)$ -bimodule structure underlying the DG bimodule $M \otimes_{k^e} W_{\bullet}$ (Subsection 2.3). Therefore the right action of B on $\mathrm{HK}_{\bullet}(A, M)$ and on $\mathrm{HK}_{\bullet}^{hi}(A, M)$ is respectively compatible with the structure of $\mathrm{HK}^{\bullet}(A)$ -bimodule and of $\mathrm{HK}_{hi}^{\bullet}(A)$ -bimodule.

It is convenient to translate what we have obtained in terms of DG bimodules over a DG algebra. We refer to [37] for these notions. We introduce the \mathbb{F} -central DG algebra

$$\tilde{A} = \mathrm{Hom}_{A^e}(K(A), A) \cong \mathrm{Hom}_{k^e}(W_{\bullet}, A)$$

whose grading is given by the cohomological degree of cochains, whose differential is b_K and whose multiplication is $\underset{K}{\frown}$. Note that $H(\tilde{A}) = \mathrm{HK}^{\bullet}(A)$. Denote by \mathcal{B} the category of right B -modules. Then $M \otimes_{k^e} W_{\bullet}$ is a DG \tilde{A} -bimodule in the abelian category \mathcal{B} , in the sense of [37]. Moreover, $\mathrm{HK}_{\bullet}(A, M)$ is a graded $\mathrm{HK}^{\bullet}(A)$ -bimodule in \mathcal{B} , and $\mathrm{HK}_{\bullet}^{hi}(A, M)$ is a graded $\mathrm{HK}_{hi}^{\bullet}(A)$ -bimodule in \mathcal{B} .

Similarly, $\mathrm{Hom}_{k^e}(W_{\bullet}, M)$ is a right B -module for the action of b on $f : W_p \rightarrow M$ defined by

$$(f.b)(x_1 \dots x_p) = f(x_1 \dots x_p).b.$$

Then b_K and $\epsilon_A \underset{K}{\frown} -$ are B -linear, so that $\mathrm{HK}^{\bullet}(A, M)$ and $\mathrm{HK}_{hi}^{\bullet}(A, M)$ are graded right B -modules. For $g : W_q \rightarrow A$, we have

$$g \underset{K}{\frown} (f.b) = (g \underset{K}{\frown} f).b \text{ and } (f.b) \underset{K}{\frown} g = (f \underset{K}{\frown} g).b.$$

The right B -module $\text{Hom}_{k^e}(W_\bullet, M)$ is compatible with the $\text{Hom}_{k^e}(W_\bullet, A)$ -bimodule structure underlying the DG bimodule $\text{Hom}_{k^e}(W_\bullet, M)$. The right action of B on $\text{HK}^\bullet(A, M)$ and on $\text{HK}_{hi}^\bullet(A, M)$ are compatible respectively with the structure of $\text{HK}^\bullet(A)$ -bimodule and of $\text{HK}_{hi}^\bullet(A)$ -bimodule. In terms of DG bimodules, $\text{Hom}_{k^e}(W_\bullet, M)$ is a DG \tilde{A} -bimodule in \mathcal{B} , $\text{HK}^\bullet(A, M)$ is a graded $\text{HK}^\bullet(A)$ -bimodule in \mathcal{B} , and $\text{HK}_{hi}^\bullet(A, M)$ is a graded $\text{HK}_{hi}^\bullet(A)$ -bimodule in \mathcal{B} .

3.3 Application to the Koszul complex $K(A)$

Let us specialise to $B = M = A^e$ as in Example 3.2. So $M = A \overset{o}{\otimes} A$ is a left A^e -module for the outer structure, and a right A^e -module for the inner structure. Our aim is to identify the A -bimodule complex $K(A)$ with the complex $((A \overset{o}{\otimes} A) \otimes_{k^e} W_\bullet, b^K)$ endowed with the right action of A^e . The statement is the following.

Proposition 3.3. *Let $A = T_k(V)/(R)$ be a quadratic k -algebra over \mathcal{Q} .*

(i) *For any $q \geq 0$, the bilinear map $\varphi_q : (A \overset{o}{\otimes} A) \times W_q \rightarrow A \otimes_k W_q \otimes_k A$ defined by*

$$\varphi_q(\alpha \otimes \beta, x_1 \dots x_q) = \beta \otimes_k (x_1 \dots x_q) \otimes_k \alpha$$

induces an isomorphism $\tilde{\varphi}_q : (A \overset{o}{\otimes} A) \otimes_{k^e} W_q \rightarrow A \otimes_k W_q \otimes_k A$.

(ii) *The direct sum $\tilde{\varphi}$ of the maps $\tilde{\varphi}_q$ is an isomorphism from the complex $((A \overset{o}{\otimes} A) \otimes_{k^e} W_\bullet, b^K)$ to the Koszul complex $(K(A), d)$.*

(iii) *The isomorphism $\tilde{\varphi}$ is right A^e -linear.*

Proof. The A -bimodule $A \overset{o}{\otimes} A$ is a k -bimodule for the actions $\lambda(\alpha \otimes \beta)\mu = \lambda\alpha \otimes \beta\mu$, with α and β in A , λ and μ in k , thus it is a right k^e -module for $(\alpha \otimes \beta)(\lambda \otimes \mu) = \mu\alpha \otimes \beta\lambda$. Then it is easy to check that

$$\varphi_q(\mu\alpha \otimes \beta\lambda, x_1 \dots x_q) = \varphi_q(\alpha \otimes \beta, \lambda x_1 \dots x_q \mu),$$

proving the existence of $\tilde{\varphi}_q$. We define similarly an inverse linear map, therefore $\tilde{\varphi}_q$ is an isomorphism, which gives (i).

Let us show that $\tilde{\varphi}$ is a morphism of complexes. From

$$b^K((\alpha \otimes \beta) \otimes_{k^e} x_1 \dots x_q) = (\alpha \otimes (\beta x_1)) \otimes_{k^e} x_2 \dots x_q + (-1)^q ((x_q \alpha) \otimes \beta) \otimes_{k^e} x_1 \dots x_{q-1},$$

we get

$$\tilde{\varphi} \circ b^K((\alpha \otimes \beta) \otimes_{k^e} x_1 \dots x_q) = \beta x_1 \otimes_k x_2 \dots x_q \otimes_k \alpha + (-1)^q \beta \otimes_k x_1 \dots x_{q-1} \otimes_k x_q \alpha$$

whose right-hand side is equal to $d(\beta \otimes_k x_1 \dots x_p \otimes_k \alpha)$, as expected.

Let us prove (iii). Here the A -bimodule $A \otimes_k W_q \otimes_k A$ is seen as a right A^e -module. For $z = (\alpha \otimes \beta) \otimes_{k^e} x_1 \dots x_q$ and a, b in A , we have

$$\tilde{\varphi}_q(z \cdot (a \otimes b)) = \tilde{\varphi}_q((\alpha a \otimes b \beta) \otimes_{k^e} x_1 \dots x_q)$$

whose right-hand side is equal to the left-hand side of

$$b\beta \otimes_k (x_1 \dots x_q) \otimes_k \alpha a = \tilde{\varphi}_q(z) \cdot (a \otimes b). \quad \blacksquare$$

So $\tilde{\varphi}$ is an isomorphism from the A -bimodule complex $((A \overset{o}{\otimes} A) \otimes_{k^e} W_\bullet, b^K)$ whose A -bimodule structure is the inner one, to the A -bimodule complex $K(A)$. Denote by \mathcal{A} the

category of A -bimodules. According to Subsection 3.2, $(A \overset{\circ}{\otimes} A) \otimes_{k^e} W_\bullet$ is a DG \tilde{A} -bimodule in \mathcal{A} . Transporting the $\text{Hom}_{k^e}(W_\bullet, A)$ -bimodule structure via $\tilde{\varphi}$, we obtain that *the Koszul complex $K(A)$ is a DG \tilde{A} -bimodule in the abelian category \mathcal{A}* . This DG bimodule will play an essential role in the generalisations of Calabi-Yau algebras (Sections 4 and 5). Next, $H(K(A))$ is a graded $\text{HK}^\bullet(A)$ -bimodule in \mathcal{A} , so that $H(\tilde{\varphi}) : \text{HK}_\bullet(A, A \overset{\circ}{\otimes} A) \rightarrow H(K(A))$ is an isomorphism of graded $\text{HK}^\bullet(A)$ -bimodules in \mathcal{A} .

Let us compute explicitly the $\text{Hom}_{k^e}(W_\bullet, A)$ -bimodule structure obtained on $K(A)$ in this way. Consider $z = (\alpha \otimes \beta) \otimes_{k^e} x_1 \dots x_q$ in $(A \overset{\circ}{\otimes} A) \otimes_{k^e} W_q$ and f in $\text{Hom}_{k^e}(W_p, A)$, we easily derive from (2.8) that the left action of f on $K(A)$ is defined by

$$f \underset{K}{\frown} (\beta \otimes_k x_1 \dots x_q \otimes_k \alpha) = (-1)^{(q-p)p} \beta \otimes_k x_1 \dots x_{q-p} \otimes_k f(x_{q-p+1} \dots x_q) \alpha. \quad (3.1)$$

Analogously, using (2.9), we define the right action of f on $K(A)$ by

$$(\beta \otimes_k x_1 \dots x_q \otimes_k \alpha) \underset{K}{\frown} f = (-1)^{pq} \beta f(x_1 \dots x_p) \otimes_k x_{p+1} \dots x_q \otimes_k \alpha. \quad (3.2)$$

The fundamental formula (2.13) reduces to

$$d(z') = -\epsilon_A \underset{K}{\frown} z' + (-1)^q z' \underset{K}{\frown} \epsilon_A \quad (3.3)$$

on $K(A)$, where $z' = \beta \otimes_k x_1 \dots x_q \otimes_k \alpha$, and

$$\epsilon_A \underset{K}{\frown} z' = (-1)^{q-1} \beta \otimes_k x_1 \dots x_{q-1} \otimes_k x_q \alpha, \quad (3.4)$$

$$z' \underset{K}{\frown} \epsilon_A = (-1)^q \beta x_1 \otimes_k x_2 \dots x_q \otimes_k \alpha. \quad (3.5)$$

The differential $\epsilon_A \underset{K}{\frown} -$ induces a differential, still denoted by ∂_\frown , on $H(K(A))$. The homology of $(H(K(A)), \partial_\frown)$ is denoted by $H^{hi}(K(A))$ and is called the higher homology of $K(A)$. Then $H^{hi}(K(A))$ is a graded $\text{HK}_{hi}^\bullet(A)$ -bimodule in \mathcal{A} and $H(H(\tilde{\varphi})) : \text{HK}_\bullet^{hi}(A, A \overset{\circ}{\otimes} A) \rightarrow H^{hi}(K(A))$ is an isomorphism of graded $\text{HK}_{hi}^\bullet(A)$ -bimodules in \mathcal{A} .

4 Poincaré Van den Bergh duality of preprojective algebras

4.1 Preprojective algebras

Throughout this section, Δ is a connected graph whose vertex set and edge set are finite. Following a usual presupposition in the papers devoted to Hochschild (co)homology of preprojective algebras, we assume that the graph Δ is not labelled, that is, the labels of the edges are all equal to $(1, 1)$ [3, Definition 4.1.9]. In particular, the Dynkin graphs are limited to types ADE, and the Euclidean (or extended) Dynkin graphs are limited to types $\tilde{A}\tilde{D}\tilde{E}$ [3, Definition 4.5.1].

Let Q be a quiver whose underlying graph is Δ . Define a quiver Q^* whose vertex set is Q_0 and whose arrow set is $Q_1^* = \{a^*; a \in Q_1\}$ where $\mathfrak{s}(a^*) = \mathfrak{t}(a)$ and $\mathfrak{t}(a^*) = \mathfrak{s}(a)$. Let \overline{Q} be the *double quiver* of Q , that is, the quiver whose vertex set is $\overline{Q}_0 = Q_0$ and whose arrow set is the disjoint union $\overline{Q}_1 = Q_1 \cup Q_1^*$. We shall view $(-)^*$ as an involution of \overline{Q}_1 .

Let \mathbb{F} be a field. As before, we denote the ring $\mathbb{F}Q_0$ by k and the k -bimodule $\mathbb{F}\overline{Q}_1$ by V and we identify the graded k -algebras $T_k(V) \cong \mathbb{F}\overline{Q}$ (see Subsection 2.1).

The *preprojective algebra associated with the graph Δ over the field \mathbb{F}* is the quadratic k -algebra $A(\Delta)$ over \overline{Q} defined by $A(\Delta) = \mathbb{F}\overline{Q}/(R)$, where the sub- k -bimodule R of $\mathbb{F}\overline{Q}_2$ is generated by

$$\sigma_i := \sum_{\substack{a \in Q_1 \\ \mathfrak{t}(a)=i}} aa^* - \sum_{\substack{a \in Q_1 \\ \mathfrak{s}(a)=i}} a^*a = \sum_{\substack{a \in \overline{Q}_1 \\ \mathfrak{t}(a)=i}} \varepsilon(a)aa^* \quad \text{for all } i \in Q_0,$$

where $\varepsilon(a) = 1$ if $a \in Q_1$, $\varepsilon(a) = -1$ if $a \in Q_1^*$.

If Q' is another quiver whose underlying graph is Δ , then the underlying graphs of \overline{Q} and \overline{Q}' are the same, equal to the double of Δ . Therefore, according to Subsection 2.7, the quadratic k -algebra $A(\Delta)$ and the (general, higher) Koszul calculus of $A(\Delta)$ depend only on the graph Δ and not on Q , justifying the notation $A(\Delta)$. If Δ is a tree, $A(\Delta)$ is isomorphic to the preprojective algebra defined without signs (as in [14, 15, 16]).

If $\Delta = A_1$, then $A(\Delta) = k$. If $\Delta = A_2$, then $R = \mathbb{F}\overline{Q}_2$ and $A(\Delta) = \mathbb{F}Q_0 \oplus \mathbb{F}\overline{Q}_1$. These quadratic k -algebras are Koszul, but they are the only exceptions among the Dynkin graphs. More precisely, the following standard result holds, for which we just give proof references.

Proposition 4.1. *Assume that the graph Δ is distinct from A_1 and A_2 . The following are equivalent.*

- (i) Δ is Dynkin of type ADE.
- (ii) $A(\Delta)$ is not Koszul.
- (iii) $A(\Delta)$ is finite dimensional.

Proof. The equivalence (i) \Leftrightarrow (ii) is treated in [29] if Δ is a tree, in [26] otherwise. The equivalence (i) \Leftrightarrow (iii) for any Dynkin graph is cited in [30] as a result by Gelfand and Ponomarev [23]. ■

Sections 2 and 3 can be applied to preprojective algebras. For example, according to the remark following Lemma 2.4, the fundamental 1-cocycle $\mathbf{e}_{A(\Delta)}$ is not a coboundary if Δ has a loop or if $\text{char } \mathbb{F} \neq 2$ and $\Delta \neq A_1$. In the remainder of this section, we often abbreviate $A(\Delta)$ to A and we freely use notations and results from Sections 2 and 3.

4.2 The Koszul complex $K(A)$ has length 2

If $\Delta = A_1$, then $K(A)$ has length 0. If $\Delta = A_2$, then $K(A)$ has infinite length. Apart from these cases, the length of $K(A)$ is always equal to 2.

Theorem 4.2. *Let $A = A(\Delta)$ be a preprojective algebra over \mathbb{F} with $\Delta \neq A_1$ and $\Delta \neq A_2$. Then the Koszul complex $K(A)$ of A has length 2. Consequently, $\text{HK}^p(A, M) \cong \text{HK}_p(A, M) \cong 0$ for all A -bimodules M and all $p \geq 3$.*

Proof. From the defining equality (2.3) of W_p , we have $W_p = (W_{p-1} \otimes_k V) \cap (V \otimes_k W_{p-1})$ for all $p \geq 3$. Moreover $R \neq 0$, therefore it is enough to prove that $W_3 = 0$, that is, $(R \otimes_k V) \cap (V \otimes_k R) = 0$. For that, we only assume that $\Delta \neq A_1$. Our goal is to prove that $W_3 \neq 0$ implies $\Delta = A_2$.

Let u be a non-zero element in W_3 , viewed as an element in $\mathbb{F}\overline{Q}_3$. There exist vertices e, f in Q_0 such that $euf \neq 0$, therefore we may assume that u is in eW_3f . Then u can be written uniquely as

$$u = \sum_{\substack{a \in e\overline{Q}_1 \\ \alpha \in e\overline{Q}_1f}} \lambda_\alpha \varepsilon(a) a a^* \alpha = \sum_{\substack{b \in f\overline{Q}_1 \\ \beta \in f\overline{Q}_1e}} \mu_\beta \varepsilon(b) \beta b b^*.$$

We now use the fact that \overline{Q}_1 is the disjoint union of Q_1 and Q_1^* and the definition of ε to write

$$\begin{aligned} u &= \sum_{\substack{\alpha \in eQ_1f \\ a \in eQ_1}} \lambda_\alpha a a^* \alpha - \sum_{\substack{\alpha \in eQ_1f \\ a \in eQ_1e}} \lambda_\alpha a^* a \alpha + \sum_{\substack{\alpha \in fQ_1e \\ a \in eQ_1}} \lambda_\alpha a^* a a^* \alpha^* - \sum_{\substack{\alpha \in fQ_1e \\ a \in Q_1e}} \lambda_\alpha a^* a a^* \alpha^* \\ &= \sum_{\substack{\beta \in eQ_1f \\ b \in fQ_1}} \mu_\beta \beta b b^* - \sum_{\substack{\beta \in eQ_1f \\ b \in Q_1f}} \mu_\beta \beta b^* b + \sum_{\substack{\beta \in fQ_1e \\ b \in fQ_1}} \mu_\beta \beta^* b b^* - \sum_{\substack{\beta \in fQ_1e \\ b \in Q_1f}} \mu_\beta \beta^* b^* b. \end{aligned}$$

From these expressions, we obtain the following relations in the path algebra $\mathbb{F}\overline{Q}$:

$$\sum_{\substack{\alpha \in fQ_1e \\ a \in eQ_1}} \lambda_{\alpha^*} a a^* \alpha^* = 0 \quad (4.1)$$

$$\sum_{\substack{\alpha \in eQ_1f \\ a \in Q_1e}} \lambda_{\alpha} a^* a \alpha = 0 \quad (4.2)$$

$$\sum_{\substack{\beta \in eQ_1f \\ b \in fQ_1}} \mu_{\beta} \beta b b^* = 0 \quad (4.3)$$

$$\sum_{\substack{\beta \in fQ_1e \\ b \in Q_1f}} \mu_{\beta^*} \beta^* b^* b = 0 \quad (4.4)$$

$$\sum_{\substack{\alpha \in eQ_1f \\ a \in eQ_1}} \lambda_{\alpha} a a^* \alpha = - \sum_{\substack{\beta \in eQ_1f \\ b \in Q_1f}} \mu_{\beta} \beta b^* b \quad (4.5)$$

$$\sum_{\substack{\alpha \in fQ_1e \\ a \in Q_1e}} \lambda_{\alpha^*} a^* a \alpha^* = - \sum_{\substack{\beta \in fQ_1e \\ b \in fQ_1}} \mu_{\beta^*} \beta^* b b^* \quad (4.6)$$

Indeed, relation (4.1) follows from the fact that no other path that occurs in the expressions of u ends with two arrows in Q^* , and the other relations are obtained from similar arguments.

In the path algebra $\mathbb{F}\overline{Q}$, the relations (4.1) to (4.4) above are equivalent to

$$\forall \alpha \in fQ_1e, \forall a \in eQ_1, \lambda_{\alpha^*} = 0 \quad (4.7)$$

$$\forall \alpha \in eQ_1f, \forall a \in Q_1e, \lambda_{\alpha} = 0 \quad (4.8)$$

$$\forall \beta \in eQ_1f, \forall a \in fQ_1, \lambda_{\beta} = 0 \quad (4.9)$$

$$\forall \beta \in fQ_1e, \forall a \in Q_1f, \lambda_{\beta^*} = 0. \quad (4.10)$$

We have assumed that $u \neq 0$, so that either there exists $\alpha \in eQ_1f$ such that $\lambda_{\alpha} \neq 0$ or there exists $\alpha \in fQ_1e$ such that $\lambda_{\alpha^*} \neq 0$, using the first expression of u . We separate the two cases.

Assume that there exists $\alpha \in eQ_1f$ such that $\lambda_{\alpha} \neq 0$. Then it follows from relation (4.8) that Q_1e is empty. From (4.5), for all $a \in eQ_1$, there exist $\beta \in eQ_1f$ and $b \in Q_1f$ such that $\lambda_{\alpha} a a^* \alpha = -\mu_{\beta} \beta b^* b$. Hence $\beta = a = b = \alpha$ and therefore $eQ_1 = \{\alpha\} = eQ_1f$ and $\mu_{\beta} = -\lambda_{\alpha} \neq 0$. From (4.9), it follows that fQ_1 is empty. Finally, (4.5) becomes

$$\lambda_{\alpha} \alpha \alpha^* \alpha = \lambda_{\alpha} \alpha \alpha^* \alpha + \lambda_{\alpha} \sum_{\substack{b \in Q_1f \\ b \neq \alpha}} \alpha b^* b$$

so that $\sum_{\substack{b \in Q_1f \\ b \neq \alpha}} \alpha b^* b = 0$ and hence $Q_1f = \{\alpha\}$.

We have proved that $Q_1e = \emptyset = fQ_1$ so that in particular $e \neq f$, and that $Q_1f = \{\alpha\} = eQ_1 = eQ_1f$. Finally, $Q = e \xrightarrow{\alpha} f$ and $\Delta = A_2$.

In the case where there exists $\alpha \in fQ_1e$ such that $\lambda_{\alpha^*} \neq 0$, a similar proof using (4.7), (4.10) and (4.6) shows that $Q = f \xrightarrow{\alpha} e$ and $\Delta = A_2$. \blacksquare

As an immediate consequence of Proposition 2.6 and Theorem 4.2, we obtain that in the Koszul calculus of $A(\Delta)$, the Koszul cup product is graded commutative and the Koszul cap product is graded symmetric. The precise statement is the following.

Corollary 4.3. *Let $A = A(\Delta)$ be a preprojective algebra over \mathbb{F} with $\Delta \neq A_1$ and $\Delta \neq A_2$. We consider an A -bimodule M . For any $\alpha \in \mathrm{HK}^\bullet(A, M)$, $\beta \in \mathrm{HK}^\bullet(A)$ and $\gamma \in \mathrm{HK}_\bullet(A)$, we have the identities*

$$[\alpha, \beta]_{\underset{K}{\smile}} = 0, \quad (4.11)$$

$$[\alpha, \gamma]_{\underset{K}{\frown}} = 0. \quad (4.12)$$

The same conclusion holds if $\Delta = A_1$ (obvious) and if $\Delta = A_2$ (because A is Koszul).

4.3 Duality in Koszul (co)homology of preprojective algebras

There is a remarkable duality between Koszul homology and cohomology for preprojective algebras. This duality is realised as a cap action by a Koszul 2-chain $\omega_0 \in A \otimes_{k^e} R$ defined for any graph Δ by

$$\omega_0 = \sum_{i \in Q_0} e_i \otimes_{k^e} \sigma_i = \sum_{i \in Q_0} e_i \otimes \sigma_i.$$

From $\sigma_i = \sum_{a \in \overline{Q}_1, t(a)=i} \varepsilon(a) a a^*$, we get

$$\omega_0 = \sum_{a \in \overline{Q}_1} 1 \otimes_{k^e} \varepsilon(a) a a^* = - \sum_{a \in \overline{Q}_1} 1 \otimes_{k^e} \varepsilon(a) a^* a. \quad (4.13)$$

Then it is easy to check that ω_0 is a Koszul 2-cycle. Being homogeneous of weight 0, ω_0 is not a 2-boundary whenever $\Delta \neq A_1$.

Theorem 4.4. *Let $A = A(\Delta)$ be a preprojective algebra over \mathbb{F} with $\Delta \neq A_1$ and $\Delta \neq A_2$. Consider the Koszul 2-cycle $\omega_0 = \sum_{i \in Q_0} e_i \otimes \sigma_i \in A \otimes_{k^e} R$. For each Koszul p -cochain f with coefficients in an A -bimodule M , we define the Koszul $(2-p)$ -chain $\theta_M(f)$ with coefficients in M by*

$$\theta_M(f) = \omega_0 \underset{K}{\frown} f. \quad (4.14)$$

Then the equalities

$$\theta_{M \otimes_A N}(f \underset{K}{\smile} g) = \theta_M(f) \underset{K}{\frown} g = f \underset{K}{\frown} \theta_N(g) \quad (4.15)$$

hold for any Koszul cochains f and g with coefficients in A -bimodules M and N respectively.

Moreover the linear map $\theta_M : \mathrm{Hom}_{k^e}(W_\bullet, M) \rightarrow M \otimes_{k^e} W_{2-\bullet}$ is an isomorphism of DG bimodules over the DG algebra $\tilde{A} = \mathrm{Hom}_{k^e}(W_\bullet, A)$.

It follows that $H(\theta_M) : \mathrm{HK}^\bullet(A, M) \rightarrow \mathrm{HK}_{2-\bullet}(A, M)$ is an isomorphism of graded $\mathrm{HK}^\bullet(A)$ -bimodules and that $H(H(\theta_M)) : \mathrm{HK}_{hi}^\bullet(A, M) \rightarrow \mathrm{HK}_{2-\bullet}^{hi}(A, M)$ is an isomorphism of graded $\mathrm{HK}_{hi}^\bullet(A)$ -bimodules.

Proof. First we show that $f \underset{K}{\frown} \omega_0 = \omega_0 \underset{K}{\frown} f$ for all $f \in \mathrm{Hom}_{k^e}(W_p, M)$. Using the definition of ω_0 and the equalities (4.13), (2.8) and (2.9), we obtain for $p = 0, 1, 2$,

$$\begin{aligned} f \underset{K}{\frown} \omega_0 &= \sum_{i \in Q_0} (f(1)e_i) \otimes_{k^e} \sigma_i = \sum_{i \in Q_0} (e_i f(1)) \otimes_{k^e} \sigma_i = \omega_0 \underset{K}{\frown} f, \\ f \underset{K}{\frown} \omega_0 &= - \sum_{a \in \overline{Q}_1} \varepsilon(a) (f(a^*)1) \otimes_{k^e} a = \sum_{a \in \overline{Q}_1} \varepsilon(a) (1f(a)) \otimes_{k^e} a^* = \omega_0 \underset{K}{\frown} f, \\ f \underset{K}{\frown} \omega_0 &= \sum_{i \in Q_0} (e_i f(\sigma_i)) \otimes_{k^e} 1 = \sum_{i \in Q_0} (f(\sigma_i)e_i) \otimes_{k^e} 1 = \omega_0 \underset{K}{\frown} f. \end{aligned}$$

Next, for $f \in \text{Hom}_{k^e}(W_p, M)$ and $g \in \text{Hom}_{k^e}(W_q, M)$, we have

$$\theta_{M \otimes_A N}(f \underset{K}{\smile} g) = \omega_0 \underset{K}{\frown} (f \underset{K}{\smile} g) = (\omega_0 \underset{K}{\frown} f) \underset{K}{\frown} g = (f \underset{K}{\frown} \omega_0) \underset{K}{\frown} g = f \underset{K}{\frown} (\omega_0 \underset{K}{\frown} g),$$

providing equalities (4.15). Therefore $\theta_M : \text{Hom}_{k^e}(W_\bullet, M) \rightarrow M \otimes_{k^e} W_{2-\bullet}$ is a morphism of graded $\text{Hom}_{k^e}(W_\bullet, A)$ -bimodules for the actions of $\underset{K}{\smile}$ and $\underset{K}{\frown}$ respectively.

Assuming $M = A$ in the equalities (4.15), we derive

$$\theta_N([f, g] \underset{K}{\smile}) = [\theta_A(f), g] \underset{K}{\frown} = [f, \theta_N(g)] \underset{K}{\frown}.$$

Combining $\theta_N([\epsilon_A, g] \underset{K}{\smile}) = [\epsilon_A, \theta_N(g)] \underset{K}{\frown}$ with $b_K = -[\epsilon_A, -] \underset{K}{\smile}$ and $b^K = -[\epsilon_A, -] \underset{K}{\frown}$, we deduce that $\theta_M : \text{Hom}_{k^e}(W_\bullet, M) \rightarrow M \otimes_{k^e} W_{2-\bullet}$ is a morphism of complexes, thus a morphism of DG bimodules over the DG algebra $\tilde{A} = (\text{Hom}_{k^e}(W_\bullet, A), b_K, \underset{K}{\smile})$.

We prove that θ_M is an isomorphism by giving an inverse map $\eta : M \otimes_{k^e} W_{2-\bullet} \rightarrow \text{Hom}_{k^e}(W_\bullet, M)$. We define $\eta_p : M \otimes_{k^e} W_{2-p} \rightarrow \text{Hom}_{k^e}(W_p, M)$ for $p = 0, 1, 2$, by

$$\begin{aligned} \eta_0(m \otimes_{k^e} \sigma_i)(e_j) &= \delta_{ij} e_j m e_i \\ \eta_1(m \otimes_{k^e} a)(b) &= \delta_{ba^*} \varepsilon(b) \mathfrak{t}(b) m \mathfrak{s}(b) \quad \text{for any arrows } a \text{ and } b \text{ of } \overline{Q} \\ \eta_2(m \otimes_{k^e} e_i)(\sigma_j) &= \delta_{ij} e_j m e_i \end{aligned}$$

where δ is the Kronecker symbol. It is routine to verify that these linear maps are well-defined and form an inverse map for θ_M .

Finally the isomorphism $H(\theta_M)$ of graded $\text{HK}^\bullet(A)$ -bimodules satisfies

$$H(\theta_M)(\bar{\epsilon}_A \underset{K}{\smile} \alpha) = \bar{\epsilon}_A \underset{K}{\frown} H(\theta_M)(\alpha)$$

for all $\alpha \in \text{HK}^\bullet(A, M)$. Therefore $H(\theta_M)$ is a morphism of complexes for higher (co)homologies. Taking higher (co)homologies, we get a $\text{HK}_{hi}^\bullet(A)$ -bimodule isomorphism

$$H(H(\theta_M)) : \text{HK}_{hi}^\bullet(A, M) \rightarrow \text{HK}_{2-\bullet}^{hi}(A, M). \quad \blacksquare$$

By analogy with the Poincaré duality in singular (co)homology [27] and with the Van den Bergh duality in Hochschild (co)homology [33, 28], we say that the isomorphism

$$H(\theta_M) = \bar{\omega}_0 \underset{K}{\frown} - : \text{HK}^\bullet(A, M) \rightarrow \text{HK}_{2-\bullet}(A, M)$$

is a Poincaré Van den Bergh duality for Koszul (co)homology, of fundamental class $\bar{\omega}_0$, where $\bar{\omega}_0 \in \text{HK}_2(A)_0$. In the next subsection, we extract from this duality a generalisation of the 2-Calabi-Yau property.

Apart from the cases where Δ has no loop and $\text{char } \mathbb{F} = 2$, the class $\bar{\epsilon}_A \in \text{HK}^1(A)_1$ is non-zero, hence

$$H(\theta_A)(\bar{\epsilon}_A) = \bar{\omega}_0 \underset{K}{\frown} \bar{\epsilon}_A = \partial_\frown(\bar{\omega}_0)$$

is non-zero in $\text{HK}_1(A)_1$. Consequently, the fundamental class $\bar{\omega}_0$ of the Poincaré Van den Bergh duality is not a cycle for the higher Koszul homology, so that the isomorphism $H(H(\theta_M))$ cannot be naturally expressed as a cap action.

The class $H(\theta_A)(\bar{\epsilon}_A)$ is the class of the Koszul 1-cycle $\omega_0 \underset{K}{\frown} \epsilon_A$ where

$$\omega_0 \underset{K}{\frown} \epsilon_A = \sum_{a \in \overline{Q}_1} \varepsilon(a) a \otimes_{k^e} a^* = \sum_{a \in Q_1} (a \otimes_{k^e} a^* - a^* \otimes_{k^e} a).$$

It is interesting to view the last element as the image by the canonical linear map $can : V \otimes_k V \rightarrow V \otimes_{k^e} V$ of the element

$$w = \sum_{a \in Q_1} (a \otimes_k a^* - a^* \otimes_k a) = \sum_{a \in \overline{Q}_1} \varepsilon(a) a a^* \in R \subseteq V \otimes_k V.$$

In the identification $V \otimes_k V \cong \mathbb{F}\overline{Q}_2$, $V \otimes_{k^e} V$ is identified with the subspace of cycles of length 2 and the map can is identified with the projection whose kernel is the space spanned by the non-cyclic paths. Since R is generated by the cycles σ_i , we can make the identification $\omega_0 \underset{K}{\frown} \mathfrak{e}_A = w$. The element w was defined in [10, Proposition 8.1.1] as a representative of a bi-symplectic 2-form ω . Bi-symplectic 2-forms were introduced by Crawley-Boevey, Etingof and Ginzburg as an essential ingredient of the Hamiltonian reduction in noncommutative geometry [10]; they are related to the double Poisson algebras defined by Van den Bergh [34].

4.4 Deriving an adapted 2-Calabi-Yau property

Let $A = A(\Delta)$ be a preprojective algebra over \mathbb{F} with $\Delta \neq A_1$ and $\Delta \neq A_2$. Let M be an A -bimodule. Assume that B is a unital associative algebra such that M is a right B -module compatible with the A -bimodule structure (Subsection 3.1). Denote by \mathcal{B} the category of right B -modules. Recall that \tilde{A} denotes the DG algebra $(\text{Hom}_{k^e}(W_\bullet, A), b_K, \underset{K}{\frown})$.

According to Subsection 3.2, $M \otimes_{k^e} W_\bullet$ and $\text{Hom}_{k^e}(W_\bullet, M)$ are DG \tilde{A} -bimodules in the abelian category \mathcal{B} . Moreover, $\text{HK}_\bullet(A, M)$ and $\text{HK}^\bullet(A, M)$ are graded $\text{HK}^\bullet(A)$ -bimodules in \mathcal{B} , that is, graded $\text{HK}^\bullet(A)^e$ - B -bimodules. Finally, $\text{HK}_{hi}^\bullet(A, M)$ and $\text{HK}_{hi}^\bullet(A, M)$ are graded $\text{HK}_{hi}^\bullet(A)$ -bimodules in \mathcal{B} , that is, graded $\text{HK}_{hi}^\bullet(A)^e$ - B -bimodules.

Lemma 4.5. *The map $\theta_M : \text{Hom}_{k^e}(W_\bullet, M) \rightarrow M \otimes_{k^e} W_{2-\bullet}$ is an isomorphism of DG \tilde{A} -bimodules in \mathcal{B} . Moreover, $H(\theta_M) : \text{HK}^\bullet(A, M) \rightarrow \text{HK}_{2-\bullet}(A, M)$ is an isomorphism of graded $\text{HK}^\bullet(A)$ -bimodules in \mathcal{B} , and $H(H(\theta_M)) : \text{HK}_{hi}^\bullet(A, M) \rightarrow \text{HK}_{2-\bullet}^{hi}(A, M)$ is an isomorphism of graded $\text{HK}_{hi}^\bullet(A)$ -bimodules in \mathcal{B} .*

Proof. It is enough to prove that $\theta_M : f \mapsto \omega_0 \underset{K}{\frown} f$ is B -linear. For a k -bimodule morphism $f : W_p \rightarrow M$, $z = a \otimes_{k^e} x_1 \dots x_q \in A \otimes_{k^e} W_q$ and $b \in B$, we verify the identities

$$(f.b) \underset{K}{\frown} z = (f \underset{K}{\frown} z).b \text{ and } z \underset{K}{\frown} (f.b) = (z \underset{K}{\frown} f).b.$$

The first one uses the fact that the right actions of A and B on M commute, while the second one uses the fact that M is an A - B -bimodule (see (iii) in Lemma 3.1). Applying the second one to $z = \omega_0$, we obtain that θ_M is B -linear. \blacksquare

We specialise this lemma to $M = B = A^e$ and apply Subsection 3.3 to get the next proposition (recall that \mathcal{A} denotes the abelian category of A -bimodules).

Proposition 4.6. *Let $A = A(\Delta)$ be a preprojective algebra over \mathbb{F} with $\Delta \neq A_1$ and $\Delta \neq A_2$. The map*

$$\theta_{A^e} : \text{Hom}_{k^e}(W_\bullet, A^e) \rightarrow K(A)_{2-\bullet}$$

is an isomorphism of DG \tilde{A} -bimodules in \mathcal{A} . Moreover,

$$H(\theta_{A^e}) : \text{HK}^\bullet(A, A^e) \rightarrow H_{2-\bullet}(K(A))$$

is an isomorphism of graded $\text{HK}^\bullet(A)$ -bimodules in \mathcal{A} .

The homology of $K(A)$ is isomorphic to A in degree 0, and to 0 in degree 1, hence we obtain a generalisation of the 2-Calabi-Yau property, formulated as follows.

Theorem 4.7. *Let $A = A(\Delta)$ be a preprojective algebra over \mathbb{F} with $\Delta \neq A_1$ and $\Delta \neq A_2$. Then the $\mathrm{HK}^\bullet(A)^e$ - A^e -bimodules $\mathrm{HK}^\bullet(A, A^e)$ and $H_{2-\bullet}(K(A))$ are isomorphic. In particular, we have the following.*

- (i) *The A -bimodule $\mathrm{HK}^2(A, A^e)$ is isomorphic to the A -bimodule A .*
- (ii) *$\mathrm{HK}^1(A, A^e) = 0$.*
- (iii) *The A -bimodule $\mathrm{HK}^0(A, A^e)$ is isomorphic to the A -bimodule $H_2(K(A))$.*

Since $H_1(K(A)) \cong \mathrm{HK}^1(A, A^e) = 0$, the higher Koszul differentials vanish. Therefore $H_p^{hi}(K(A)) \cong H_p(K(A))$, $\mathrm{HK}_{hi}^p(A, A^e) \cong \mathrm{HK}^p(A, A^e)$ and $H(H(\theta_{A^e})) \cong H(\theta_{A^e})$.

From the generator $1 \otimes_k 1$ of the A -bimodule $H_0(K(A))$, we draw from (i) a generator of the free A -bimodule $\mathrm{HK}^2(A, A^e)$ defined as the class of $f : R \rightarrow A \overset{o}{\otimes} A$ with $f(\sigma_i) = e_i \otimes e_i$ for any i .

In (iii), the A -bimodules are never 0 when Δ is Dynkin of types ADE since A is not Koszul in this case. This situation is drastically different from the 2-Calabi-Yau property defined by Ginzburg in terms of the Hochschild cohomology spaces $\mathrm{HH}^p(A, A^e)$ [24]. In Ginzburg's definition, these spaces are zero for all $p < 2$.

5 Generalisations of Calabi-Yau algebras

5.1 Duality for generalised Calabi-Yau algebras

From Theorem 4.7, we are led to introduce a general definition in the framework of quiver algebras with homogeneous quadratic relations (see Section 2). Notations of Section 2 stand throughout. We are interested in quadratic k -algebras $A = T_k(V)/(R)$ with finite quiver \mathcal{Q} as defined in Subsection 2.1, and in the Koszul calculus of A as presented in the remainder of Section 2.

Definition 5.1. *Let $A = T_k(V)/(R)$ be a quadratic k -algebra over \mathcal{Q} . Let $n \geq 0$ be an integer. We say that A is generalised Calabi-Yau of dimension n if*

- (i) *the Koszul bimodule complex $K(A)$ of A has length n , and*
- (ii) *$\mathrm{RHom}_{A^e}(K(A), A^e) \cong K(A)[-n]$ in the bounded derived category of A -bimodules.*

It is a fundamental fact of derived categories that the homology functor is *conservative* [37, Corollary 7.2.13]. This means in our situation that property (ii) is equivalent to saying that the A -bimodules $\mathrm{HK}^p(A, A^e)$ and $H_{n-p}(K(A))$ are isomorphic for any p . According to Theorem 4.2 and Theorem 4.7, a preprojective algebra $A(\Delta)$ over \mathbb{F} with $\Delta \neq A_1$ and $\Delta \neq A_2$ is generalised Calabi-Yau of dimension 2.

Definition 5.1 is a true generalisation of Ginzburg's definition [24, 35]. If Δ is Dynkin of type ADE, $A(\Delta)$ is not Calabi-Yau in Ginzburg's definition since $A(\Delta)$ is not homologically smooth in this case (the minimal projective resolution of $A(\Delta)$ has infinite length). However, the two definitions coincide if A is Koszul.

Proposition 5.2. *Let $A = T_k(V)/(R)$ be a quadratic k -algebra over \mathcal{Q} . Assume that A is Koszul. Then A is generalised n -Calabi-Yau if and only if A is n -Calabi-Yau.*

Proof. Assume that A is generalised n -Calabi-Yau. Since A is Koszul, A is homologically smooth, that is, A has a finite projective resolution by finitely generated A -bimodules. Furthermore, $K(A) \cong A$ in $\mathcal{D}^b(\mathcal{A})$. Thus $\mathrm{RHom}_{A^e}(A, A^e) \cong A[-n]$ in $\mathcal{D}^b(\mathcal{A})$, and we recover Ginzburg's definition in terms of derived categories [35, Definition 8.2].

Assume that A is n -Calabi-Yau. We know that n is equal to the projective dimension of the A -bimodule A [33] which in turn is equal to the length of a minimal projective resolution of A (see for instance [4]). Hence $K(A)$ has length n and $K(A) \cong A$ in $\mathcal{D}^b(\mathcal{A})$, which allows us to conclude that A is generalised n -Calabi-Yau. ■

Consequently, if the graph Δ is not Dynkin ADE, we know that $A(\Delta)$ is Koszul (Proposition 4.1), thus we recover the fact that $A(\Delta)$ is 2-Calabi-Yau [10, 8].

In Subsection 2.8, we have seen that $K(A)$ and the minimal projective resolution $P(A)$ coincide up to the homological degree 2. Therefore, if $n \in \{0, 1\}$ and if A is n -Calabi Yau or generalised n -Calabi-Yau, then $P(A) = K(A)$ so that A is Koszul, and it follows that the two definitions are equivalent when $n \in \{0, 1\}$.

Proposition 5.3. *Let $A = T_k(V)/(R)$ be a quadratic k -algebra over \mathcal{Q} . Then A is Calabi-Yau of dimension 0 if and only if $\mathcal{Q}_1 = \emptyset$.*

We leave the proof as an exercise. If A is Calabi-Yau of dimension 1, then $R = 0$, that is, $A = T_k(V) \cong \mathbb{F}\mathcal{Q}$ with $\mathcal{Q}_1 \neq \emptyset$. It is indeed 1-Calabi-Yau if \mathcal{Q} has only one vertex and one loop, but we have not yet found other examples when \mathcal{Q} is connected.

If A is n -Calabi-Yau, the Van den Bergh duality theorem states that the vector spaces $\mathrm{HH}^p(A, M)$ and $\mathrm{HH}_{n-p}(A, M)$ are isomorphic [33]. From Definition 5.1, we draw an analogous duality theorem for Koszul homology/cohomology.

Theorem 5.4. *Let A be a generalised Calabi-Yau algebra of dimension n . Then for any A -bimodule M , the vector spaces $\mathrm{HK}^p(A, M)$ and $\mathrm{HK}_{n-p}(A, M)$ are isomorphic.*

Proof. Denote by \mathcal{A} (respectively \mathcal{E}) the abelian category of A -bimodules (resp. vector spaces). For any A -bimodule M , the left derived functor $M \overset{L}{\otimes}_{A^e} -$ and the right derived functor $\mathrm{RHom}_{A^e}(-, M)$ are defined from the bounded derived category $\mathcal{D}^b(\mathcal{A})$ to the bounded derived category $\mathcal{D}^b(\mathcal{E})$ [36]. For any bounded complex C of projective A -bimodules, we have

$$H_p(M \otimes_{A^e} C) \cong H_p(M \overset{L}{\otimes}_{A^e} C), \quad H^p(\mathrm{Hom}_{A^e}(C, M)) \cong H^p(\mathrm{RHom}_{A^e}(C, M)).$$

For $C = K(A)$, we obtain

$$\mathrm{HK}_p(A, M) \cong H_p(M \overset{L}{\otimes}_{A^e} K(A)), \quad \mathrm{HK}^p(A, M) \cong H^p(\mathrm{RHom}_{A^e}(K(A), M)).$$

Using Lemma 5.5 below for $C = K(A)$, we have

$$M \overset{L}{\otimes}_{A^e} \mathrm{RHom}_{A^e}(K(A), A^e) \cong \mathrm{RHom}_{A^e}(K(A), M)$$

in $\mathcal{D}^b(\mathcal{E})$. Thus (ii) in Definition 5.1 implies that

$$\mathrm{RHom}_{A^e}(K(A), M) \cong M \overset{L}{\otimes}_{A^e} K(A)[-n]$$

in $\mathcal{D}^b(\mathcal{E})$. Passing to homology, we deduce $\mathrm{HK}^p(A, M) \cong \mathrm{HK}_{n-p}(A, M)$ as vector spaces. ■

Lemma 5.5 is based on a natural transformation $\phi_M : H \circ G \Rightarrow F$ depending on an A -bimodule M . Let $F : \mathcal{A} \rightarrow \mathcal{E}$ be the functor $F : N \mapsto \mathrm{Hom}_{A^e}(N, M)$ where M and N are seen as right A^e -modules. Specialising to $M = A^e$ in F , we define a functor $G : \mathcal{A} \rightarrow \mathcal{A}$. Let $H : \mathcal{A} \rightarrow \mathcal{E}$ be the functor $H : N' \mapsto M \otimes_{A^e} N'$ where N' is viewed as a left A^e -module. Then we define a linear map

$$\phi_M : M \otimes_{A^e} \mathrm{Hom}_{A^e}(N, A^e) \rightarrow \mathrm{Hom}_{A^e}(N, M)$$

by $\phi_M(m \otimes_{A^e} g)(x) = m.g(x)$ for $m \in M$, $g \in \text{Hom}_{A^e}(N, A^e)$ and $x \in N$. This map is functorial in N , defining a natural transformation $\phi_M : H \circ G \Rightarrow F$.

For any bounded chain complex C of A -bimodules, ϕ_M induces a morphism of bounded cochain complexes of vector spaces

$$\phi_M : M \otimes_{A^e} \text{Hom}_{A^e}(C, A^e) \rightarrow \text{Hom}_{A^e}(C, M) \quad (5.1)$$

still denoted by ϕ_M . Recall that if $d : C_{p+1} \rightarrow C_p$ is the differential of C , the differential

$$d^* : \text{Hom}_{A^e}(C_p, M) \rightarrow \text{Hom}_{A^e}(C_{p+1}, M)$$

of $\text{Hom}_{A^e}(C, M)$ is defined by $d^*(u) = -(-1)^p u \circ d$ for $u \in \text{Hom}_{A^e}(C_p, M)$. Extending F , G and H as functors on bounded complexes of A -bimodules, we view $\phi_M : H \circ G \Rightarrow F$ as a natural transformation of functors on bounded complexes. The last natural transformation induces a natural transformation $\phi_M : LH \circ RG \Rightarrow RF$ of derived functors [36, 37].

Lemma 5.5. *Let M and P be A -bimodules viewed as right A^e -modules. Assume that P is projective and finitely generated. Then the linear map*

$$\phi_M : M \otimes_{A^e} \text{Hom}_{A^e}(P, A^e) \rightarrow \text{Hom}_{A^e}(P, M) \quad (5.2)$$

is an isomorphism. For any bounded chain complex C of finitely generated projective A -bimodules, ϕ_M induces in $\mathcal{D}^b(\mathcal{E})$ an isomorphism

$$M \overset{L}{\otimes}_{A^e} \text{RHom}_{A^e}(C, A^e) \cong \text{RHom}_{A^e}(C, M). \quad (5.3)$$

Proof. The linear isomorphism (5.2) comes from [9, Proposition (8.3) (c)], in which $R = A^e$ and P is replaced by $\text{Hom}_{A^e}(P, A^e)$. On the category of finitely generated projective A -bimodules, we have thus a natural isomorphism $\phi_M : H \circ G \cong F$, inducing $\phi_M : LH \circ RG \cong RF$ on the bounded chain complexes of finitely generated projective A -bimodules. Then (5.3) follows. ■

5.2 Generalised Calabi-Yau algebras versus Calabi-Yau algebras

Recall that if $\Delta \neq A_1$ and $\Delta \neq A_2$, then $A(\Delta)$ is generalised 2-Calabi-Yau. But observe that if $A(\Delta)$ is moreover 2-Calabi-Yau, then $A(\Delta)$ is Koszul. We are led to the following conjecture.

Conjecture 5.6. *Let $A = T_k(V)/(R)$ be a quadratic k -algebra. If A is n -Calabi-Yau and generalised n -Calabi-Yau, then A is Koszul. In other words, if A is not Koszul, the properties n -Calabi-Yau and generalised n -Calabi-Yau are not simultaneously true.*

Proposition 5.7. *Let $A = T_k(V)/(R)$ be a quadratic k -algebra. Conjecture 5.6 holds if $n \leq 3$.*

Proof. Assume that A is n -Calabi-Yau and generalised n -Calabi-Yau. We can assume that $n \geq 2$. Since $\text{HK}^p(A, A^e) \cong \text{HH}^p(A, A^e) = 0$ when $p = 0$ and $p = 1$, we have $H_n(K(A)) \cong H_{n-1}(K(A)) = 0$, hence A is Koszul if $n = 2$ and $n = 3$. ■

5.3 Strong generalised Calabi-Yau algebras

Definition 5.8. *Let A be a generalised n -Calabi-Yau algebra. The image $c \in \text{HK}_n(A)$ of the unit 1 of the algebra A by the isomorphism $\text{HK}^0(A) \cong \text{HK}_n(A)$ in Theorem 5.4 is called the fundamental class of the generalised n -Calabi-Yau algebra A .*

We shall now define strong generalised Calabi-Yau algebras, and for this we need to work with DG \tilde{A} -bimodules in \mathcal{A} . Recall that \mathcal{A} (resp. \mathcal{E}) denotes the category of A -bimodules (resp. vector spaces) and that \tilde{A} denotes the DG algebra $\text{Hom}_{A^e}(K(A), A)$. Denote by $\mathcal{C}(\tilde{A}, \mathcal{A})$ the DG category of DG \tilde{A} -bimodules in \mathcal{A} . Following [37], a DG \tilde{A} -bimodule in \mathcal{A} is a chain complex C in \mathcal{A} (as usual, C can be viewed as a cochain complex), together with two morphisms of \mathbb{F} -central DG algebras

$$\eta_C^\ell : \tilde{A} \rightarrow \text{End}_{\mathcal{A}}(C), \quad \eta_C^r : \tilde{A}^{op} \rightarrow \text{End}_{\mathcal{A}}(C),$$

encoding the left and right graded actions (assumed to commute). According to Subsection 3.3, $K(A)$ is a DG \tilde{A} -bimodule in \mathcal{A} .

For any A -bimodule M , $\text{Hom}_{A^e}(C, M)$ is a (cochain) DG \tilde{A} -bimodule in \mathcal{E} (in \mathcal{A} when $M = A^e$) for the following actions

$$(f.u)(x) = (-1)^p u(x.f), \quad (u.f)(x) = u(f.x)$$

where $f : A \otimes_k W_p \otimes_k A \rightarrow A$, $u : C_q \rightarrow M$ and $x \in C_{p+q}$. Note that $x.f$ and $f.x$ are in C_q by the graded actions of \tilde{A} on C . If $C = K(A)$, we recover the cup actions, that is, $f.u = f \smile_K u$ and $u.f = u \smile_K f$. In particular, $\text{Hom}_{A^e}(K(A), A^e)$ is a DG \tilde{A} -bimodule in \mathcal{A} .

Similarly, for any cochain DG \tilde{A} -bimodule C' in \mathcal{A} , $M \otimes_{A^e} C'$ is a cochain DG \tilde{A} -bimodule in \mathcal{E} for the following actions

$$f.(m \otimes_{A^e} u) = m \otimes_{A^e} (f.u), \quad (m \otimes_{A^e} u).f = m \otimes_{A^e} (u.f)$$

where $f \in \tilde{A}$, $m \in M$ and $u \in C'$.

The bounded derived categories $\mathcal{D}^b(\tilde{A}, \mathcal{A})$ and $\mathcal{D}^b(\tilde{A}, \mathcal{E})$ are defined in [37]. Unfortunately, it is not clear for us if the functors $\text{Hom}_{A^e}(-, M) : \mathcal{C}^b(\tilde{A}, \mathcal{A}) \rightarrow \mathcal{C}^b(\tilde{A}, \mathcal{E})$ and $M \otimes_{A^e} - : \mathcal{C}^b(\tilde{A}, \mathcal{A}) \rightarrow \mathcal{C}^b(\tilde{A}, \mathcal{E})$ can be derived. Note that the first one takes values in $\mathcal{C}^b(\tilde{A}, \mathcal{A})$ when $M = A^e$.

Definition 5.9. *Let A be a generalised n -Calabi-Yau algebra. Then A is said to be a strong generalised n -Calabi-Yau algebra if the derived functor of the endofunctor $\text{Hom}_{A^e}(-, A^e)$ of $\mathcal{C}^b(\tilde{A}, \mathcal{A})$ exists and if $\text{RHom}_{A^e}(K(A), A^e) \cong K(A)[-n]$ in $\mathcal{D}^b(\tilde{A}, \mathcal{A})$.*

Since the homology functor is conservative [37, Corollary 7.2.13], the second property in this definition (assuming the first one) is equivalent to saying that the graded $\text{HK}^\bullet(A)^e\text{-}A^e$ -bimodules $\text{HK}^\bullet(A, A^e)$ and $H_{n-\bullet}(K(A))$ are isomorphic. In particular, Theorem 4.7 shows that the preprojective algebras of connected graphs distinct from A_1 and A_2 are strong generalised 2-Calabi-Yau algebras if they satisfy the first property.

Theorem 5.10. *Let A be a generalised n -Calabi-Yau algebra with fundamental class c . We assume that A is a strong generalised n -Calabi-Yau algebra and that the derived functors of the functors $\text{Hom}_{A^e}(-, A)$ and $A \otimes_{A^e} -$ from $\mathcal{C}^b(\tilde{A}, \mathcal{A})$ to $\mathcal{C}^b(\tilde{A}, \mathcal{E})$ exist. Then*

$$c \frown_K - : \text{HK}^\bullet(A) \rightarrow \text{HK}_{n-\bullet}(A)$$

is an isomorphism of $\text{HK}^\bullet(A)$ -bimodules, inducing an isomorphism of $\text{HK}_{hi}^\bullet(A)$ -bimodules from $\text{HK}_{hi}^\bullet(A)$ to $\text{HK}_{n-\bullet}^{hi}(A)$. For all $\alpha \in \text{HK}^p(A)$, we have $c \frown_K \alpha = (-1)^{np} \alpha \frown_K c$.

Proof. Following the proof of Theorem 5.4, we are interested in the morphism (5.1) of cochain complexes

$$\phi_M : M \otimes_{A^e} \text{Hom}_{A^e}(C, A^e) \rightarrow \text{Hom}_{A^e}(C, M),$$

when the bounded chain complex C of A -bimodules is moreover a DG \tilde{A} -bimodule in \mathcal{A} . We prove now that ϕ_M is a morphism of DG \tilde{A} -bimodules in \mathcal{E} , that is, a morphism of the category $\mathcal{C}^b(\tilde{A}, \mathcal{E})$ whose objects are viewed as cochain complexes. For this, we need only prove that ϕ_M is a morphism of \tilde{A} -bimodules. Let us prove that ϕ_M is left \tilde{A} -linear, the right linearity being similar. For $f : A \otimes_k W_p \otimes_k A \rightarrow A$, $u : C_q \rightarrow A^e$ and $x \in C_{p+q}$, we have

$$\phi_M(f.(m \otimes_{A^e} u))(x) = \phi_M(m \otimes_{A^e} (f.u))(x) = m.((f.u)(x)) = (-1)^p m.(u(x.f)),$$

while $f.(\phi_M(m \otimes_{A^e} u))(x) = (-1)^p \phi_M(m \otimes_{A^e} u)(x.f) = (-1)^p m.(u(x.f))$, which is what we want.

Continuing as in the proof of Theorem 5.4, the functors F , G and H induce functors on the complexes with enriched structures. Precisely, F and G are now functors from $\mathcal{C}^b(\tilde{A}, \mathcal{A})$ to $\mathcal{C}^b(\tilde{A}, \mathcal{E})$, and H is now an endofunctor of $\mathcal{C}^b(\tilde{A}, \mathcal{A})$. Under these notations, ϕ_M defines a natural transformation $\phi_M : H \circ G \Rightarrow F$.

We specialise to $M = A$. The assumptions in the theorem show that the derived functors of F , G and H exist, so that we can derive the natural transformation ϕ_A [37]. Thus we obtain a natural transformation $LH \circ RG \Rightarrow RF$ still denoted by ϕ_A . Equivalently, we write

$$\phi_A : A \otimes_{A^e}^L \mathrm{RHom}_{A^e}(C, A^e) \Rightarrow \mathrm{RHom}_{A^e}(C, A). \quad (5.4)$$

If the bounded chain complex underlying the DG \tilde{A} -bimodule C in \mathcal{A} is formed by finitely generated projective A -bimodules, Lemma 5.5 shows that we have an isomorphism

$$\phi_A : A \otimes_{A^e}^L \mathrm{RHom}_{A^e}(C, A^e) \cong \mathrm{RHom}_{A^e}(C, A) \quad (5.5)$$

in $\mathcal{D}^b(\tilde{A}, \mathcal{E})$. Applying this to $C = K(A)$ and using Definition 5.9, we get

$$\mathrm{RHom}_{A^e}(K(A), M) \cong M \otimes_{A^e}^L K(A)[-n]$$

in $\mathcal{D}^b(\tilde{A}, \mathcal{E})$. Taking homology, we deduce that $\mathrm{HK}^\bullet(A) \cong \mathrm{HK}_{n-\bullet}(A)$ is an isomorphism of graded bimodules over the graded algebra $H(\tilde{A}) = \mathrm{HK}^\bullet(A)$. Denote this isomorphism by ψ .

The fact that ψ is a morphism of $\mathrm{HK}^\bullet(A)$ -bimodules translates as

$$\psi(\alpha \underset{K}{\frown} \beta) = \psi(\alpha) \underset{K}{\frown} \psi(\beta) = (-1)^{np} \alpha \underset{K}{\frown} \psi(\beta) \quad (5.6)$$

for any $\alpha \in \mathrm{HK}^p(A)$ and $\beta \in \mathrm{HK}^\bullet(A)$. In accordance with Definition 5.8, define $c \in \mathrm{HK}_n(A)$ by $c = \psi(1)$ where $1 \in \mathrm{HK}^0(A)$ is the unit of A . Applying identities (5.6) to the trivial equalities $\alpha = 1 \underset{K}{\frown} \alpha = \alpha \underset{K}{\frown} 1$, we obtain

$$\psi(\alpha) = c \underset{K}{\frown} \alpha = (-1)^{np} \alpha \underset{K}{\frown} c. \quad (5.7)$$

Finally ψ is a morphism of complexes for higher (co)homologies since we have

$$\psi(\bar{\epsilon}_A \underset{K}{\frown} \alpha) = (-1)^n \bar{\epsilon}_A \underset{K}{\frown} \psi(\alpha).$$

Then $H(\psi) : \mathrm{HK}_{hi}^\bullet(A) \rightarrow \mathrm{HK}_{n-\bullet}^{hi}(A)$ is an isomorphism of $\mathrm{HK}_{hi}^\bullet(A)$ -bimodules. ■

Except in some particular cases, $\bar{\epsilon}_A \in \mathrm{HK}^1(A)$ is non-zero, so that

$$\psi(\bar{\epsilon}_A) = (-1)^n \bar{\epsilon}_A \underset{K}{\frown} c = (-1)^n \partial_{\frown}(c)$$

is non-zero in $\mathrm{HK}_{n-1}(A)$. Therefore $c \in \mathrm{HK}_n(A)$ is not a cycle for higher Koszul homology and the isomorphism $H(\psi)$ cannot be naturally expressed as a cap action. As suggested by the preprojective algebras (Subsection 4.3), the class $\psi(\bar{\epsilon}_A)$ should be of interest for further investigations.

It is also interesting to remark that the identities (5.6) involving the isomorphism ψ imply that the graded algebra $\mathrm{HK}^\bullet(A)$ is commutative if and only if the graded $\mathrm{HK}^\bullet(A)$ -bimodule $\mathrm{HK}_\bullet(A)$ is symmetric. As seen in Corollary 4.3, we have a stronger result for the preprojective algebras.

6 Koszul calculus of the preprojective algebras of Dynkin ADE type

We shall determine in this section the Koszul calculus and the higher Koszul calculus of any non-Koszul preprojective algebra A , that is, an algebra of type A, D or E with at least 3 vertices.

We first give some general facts and notation.

(N1) We shall use the dimensions of the Hochschild cohomology and homology spaces of A which can be obtained in all characteristics as a consequence of the work of Etingof, Eu and Schedler in [20, Theorem 3.2.7] and [17]. In particular, by [20, Lemma 3.2.17] the centre of A is independent of the characteristic of \mathbb{F} . Bases of the Hochschild (co)homology spaces in characteristic zero induce free subsets of the Hochschild (co)homology spaces in positive characteristic, but there may be some extra basis elements in some cases.

(N2) We know from Corollary 4.3 that the cup product on the Koszul cohomology of a preprojective algebra is graded commutative and that the cap product is graded symmetric. Moreover, it follows from Theorem 4.4 that the cap product can be obtained from the cup product. Indeed, if $f \in \mathrm{HK}^p(A)$ and $x \in \mathrm{HK}_q(A)$, we have

$$f \underset{K}{\frown} x = \theta_A(f \underset{K}{\smile} \theta_A^{-1}(x)) = \theta_A((-1)^{pq} \theta_A^{-1}(x) \underset{K}{\smile} f) = (-1)^{pq} x \underset{K}{\frown} f. \quad (6.8)$$

(N3) Let X and Y be \mathbb{N} -graded spaces and let $f : X \rightarrow Y$ be a homogeneous map of degree 1. Let y_1, \dots, y_p be elements of pairwise different degrees. Then if $\sum_{i=1}^p y_i \in \mathrm{Im} f$, at least one of the y_i is in $\mathrm{Im} f$. We shall use this in the following context. The differentials b_K^1 and b_K^2 are homogeneous of weight 1. If we have a set of cocycles of pairwise different coefficient weights, that are not coboundaries, then they are linearly independent up to coboundaries, that is, they represent linearly independent cohomology classes. This also applies if some of the elements have the same weight but we already know that these elements are linearly independent up to coboundaries.

(N4) We shall use the map $\kappa : A \rightarrow A$ constructed as follows. Let A be a preprojective algebra over a graph Δ ; let \bar{Q} be its quiver. Consider the map $\bar{Q}_1 \rightarrow \bar{Q}_1$ that sends a to a^* . It induces an anti-automorphism κ of A (since κ sends the relation $\sigma_i = \sum_{\substack{a \in \bar{Q}_1 \\ t(a)=i}} \varepsilon(a) a a^*$ to itself).

(N5) When we define a cochain $f \in \mathrm{Hom}_{k^e}(X, A)$ with $X \in \{k, V, A\}$, it will be implicit that if $f(x)$ is not defined for some $x \in X$ then $f(x) = 0$.

(N6) For any cochain $f \in \mathrm{Hom}_{k^e}(W_p, A)$, we shall set $\check{f} = \theta_A(f) \in A \otimes_{k^e} W_{2-p}$.

(N7) Finally, given a Dynkin graph Δ and a ring L , we shall denote by Λ_L the preprojective algebra of Δ over L , so that $A = \Lambda_{\mathbb{F}}$.

6.1 Koszul calculus for preprojective algebras of type A

The preprojective algebra A of type A_n is defined by the quiver

$$\overline{Q} \quad 0 \begin{array}{c} \xrightarrow{a_0} \\ \xleftarrow{a_0^*} \end{array} 1 \begin{array}{c} \xrightarrow{a_1} \\ \xleftarrow{a_1^*} \end{array} 2 \begin{array}{c} \xrightarrow{a_2} \\ \xleftarrow{a_2^*} \end{array} \cdots \begin{array}{c} \xrightarrow{a_{n-3}} \\ \xleftarrow{a_{n-3}^*} \end{array} n-2 \begin{array}{c} \xrightarrow{a_{n-2}} \\ \xleftarrow{a_{n-2}^*} \end{array} n-1$$

subject to the relations

$$\begin{cases} \sigma_0 = -a_0^* a_0 \\ \sigma_i = a_{i-1} a_{i-1}^* - a_i^* a_i & 1 \leq i \leq n-2 \\ \sigma_{n-1} = a_{n-2} a_{n-2}^* \end{cases}$$

Erdmann and Snashall have given in [14] a basis B of A . We shall only need the sets $e_i B e_i$, $e_i B e_{i+1}$ and $e_{i+1} B e_i$, which can be rewritten as follows: set $m_A = \lfloor \frac{n-1}{2} \rfloor$; then

$$\begin{aligned} e_i B e_i &= \left\{ (a_i^* a_i)^\ell; 0 \leq \ell \leq \min(i, n-1-i) \right\} \text{ for } 0 \leq i \leq n-1, \text{ with } (a_i^* a_i)^0 = e_i \text{ for all } i, \\ e_{i+1} B e_i &= \left\{ a_i (a_i^* a_i)^\ell; 0 \leq \ell \leq \min(i, n-2-i) \right\} \quad \text{and} \quad e_i B e_{i+1} = \kappa(e_{i+1} B e_i). \end{aligned}$$

For each i , $A e_i$ contains precisely one basis element of maximal length $n-1$, which is

$$\pi_i = \begin{cases} a_{n-2} \cdots a_i (a_i^* a_i)^i & \text{if } i < m_A \\ a_{m_A} (a_{m_A}^* a_{m_A})^{m_A} & \text{if } i = m_A \text{ and } n \text{ is even} \\ (a_{m_A}^* a_{m_A})^{m_A} & \text{if } i = m_A \text{ and } n \text{ is odd} \\ a_{n-i-1}^* \cdots a_{i-1}^* (a_{i-1} a_{i-1}^*)^{n-1-i} & \text{if } i > m_A. \end{cases}$$

They form a basis of the socle of A .

It can be checked using [38, Proposition 3.3] that the Nakayama automorphism ν of A is given by $\nu(e_i) = e_{n-1-i}$, $\nu(a_i) = a_{n-2-i}^*$ and $\nu(a_i^*) = a_{n-2-i}$.

6.1.1 The Koszul cohomology and homology spaces in type A

The spaces $\mathrm{HK}^0(A) = \mathrm{HH}^0(A) = Z(A)$ and $\mathrm{HK}^1(A) = \mathrm{HH}^1(A)$ are known from [14]. Therefore we only need to compute $\mathrm{HK}^2(A)$. Recall our assumption that $n \geq 3$; then by Theorem 4.2 all the elements in $\mathrm{Hom}_{k^e}(R, A)$ are cocycles. Moreover, using Theorem 4.4 and [20, Theorem 3.2.7], we have $\dim \mathrm{HK}^2(A) = \dim \mathrm{HK}_0(A) = \dim \mathrm{HH}_0(A) = n$. Since every element in $\mathrm{Im} b_K^2$ has coefficient weight at least 1, the n cocycles h_i defined by $h_i(\sigma_j) = \delta_{ij} e_i$ for all j are linearly independent modulo $\mathrm{Im} b_K^2$. It follows that they form a basis of $\mathrm{HK}^2(A)$.

Combining with the results from [14], we have the following result.

Proposition 6.1. *Let A be a preprojective algebra of type A_n .*

A basis of $\mathrm{HK}^0(A)$ is given by the set $\{z_\ell; 0 \leq \ell \leq m_A\}$ with $z_0 = 1$ and $z_\ell = \sum_{i=1}^{n-2} (a_i^ a_i)^\ell = \sum_{i=\ell}^{n-1-\ell} (a_i^* a_i)^\ell = z_1^\ell$ for $1 \leq \ell \leq m_A$.*

A basis of $\mathrm{HK}^1(A)$ is given by the set $\{\bar{\zeta}_\ell; 0 \leq \ell \leq n-2-m_A\}$, where $\zeta_\ell \in \mathrm{Hom}_{k^e}(V, A)$ is defined by $\zeta_\ell(a_i) = a_i (a_i^ a_i)^\ell$ for all i (or for $\ell \leq i \leq n-2-\ell$).*

A basis of $\mathrm{HK}^2(A)$ is given by the set $\{\bar{h}_i; 0 \leq i \leq n-1\}$ where $h_i \in \mathrm{Hom}_{k^e}(R, A)$ defined by $h_i(\sigma_j) = \delta_{ij} e_i$ for all j .

As a consequence of Theorem 4.4, we obtain bases of the Koszul homology spaces.

Corollary 6.2. *A basis of $\mathrm{HK}_0(A)$ is given by the set $\{\bar{h}_i; 0 \leq i \leq n-1\}$ where $\check{h}_i = e_i \otimes e_i$.*

A basis of $\mathrm{HK}_1(A)$ is given by the set $\{\bar{\zeta}_\ell; 0 \leq \ell \leq n - m_A - 2\}$ where $\check{\zeta}_\ell = \sum_{i=0}^{n-2} a_i(a_i^ a_i)^\ell \otimes a_i^* = \sum_{i=\ell}^{n-2-\ell} a_i(a_i^* a_i)^\ell \otimes a_i^*$. A basis of $\mathrm{HK}_2(A)$ is given by the set $\{\check{z}_\ell; 0 \leq \ell \leq m_A\}$ where $\check{z}_\ell = \sum_{i=0}^{n-1} (a_i^* a_i)^\ell \otimes \sigma_i$.*

Note that $\check{z}_0 = \omega_0$ is the fundamental class.

6.1.2 Cup and cap products

We know from Corollary 4.3 that the cup product on $\mathrm{HK}^\bullet(A)$ is graded-commutative. The following result gives all the non zero cup products of elements in $\mathrm{HK}^\bullet(A)$.

Proposition 6.3. *Let A be a preprojective algebra of type A_n . Up to graded commutativity, the non zero cup products in $\mathrm{HK}^\bullet(A)$ are given by*

$$\begin{aligned} z_0 \smile_K \bar{f} &= \bar{f} \quad \text{for all } \bar{f} \in \mathrm{HK}^\bullet(A) \\ z_{\ell_1} \smile_K z_{\ell_2} &= z_{\ell_1+\ell_2} \quad \text{if } \ell_1 + \ell_2 \leq m_A \\ z_{\ell_1} \smile_K \bar{\zeta}_{\ell_2} &= \bar{\zeta}_{\ell_1+\ell_2} \quad \text{if } \ell_1 + \ell_2 \leq n - m_A - 2 \end{aligned}$$

Proof. The first cup product is clear and the other cup products in the statement only involve in $\mathrm{HH}^0(A)$ and $\mathrm{HH}^1(A)$, therefore they are known from [14].

The basis elements of $\mathrm{HK}^2(A)$ have coefficient weight 0, and b_K^2 is homogeneous of weight 1, therefore any element that has positive coefficient weight must be a coboundary. The other cup products (that all vanish) follow from this. ■

We now deduce the cap products from (6.8).

Corollary 6.4. *Up to graded symmetry, the non zero cap products are the following.*

$$\begin{aligned} z_0 \frown_K x &= x \quad \text{for all } x \in \mathrm{HK}_\bullet(A) \\ z_{\ell_1} \frown_K \check{z}_{\ell_2} &= \check{z}_{\ell_1+\ell_2} \quad \text{if } \ell_1 + \ell_2 \leq m_A \\ z_{\ell_1} \frown_K \bar{\zeta}_{\ell_2} &= \bar{\zeta}_{\ell_1+\ell_2} \quad \text{if } \ell_1 + \ell_2 \leq n - m_A - 2 \end{aligned}$$

6.1.3 Higher Koszul cohomology and homology

We start with a lemma giving the cohomology class of the fundamental 1-cocycle.

Lemma 6.5. *The cohomology class of ϵ_A is equal to the cohomology class of $2\zeta_0$.*

Proof. Let $\zeta_0^* \in \mathrm{Hom}_{k^e}(V, A)$ be the cocycle defined by $\zeta_0^*(a_i^*) = a_i^*$ for all $i \in Q_0$. Since $\epsilon_A = \zeta_0 + \zeta_0^*$, we must prove that $\zeta_0^* - \zeta_0$ is a coboundary.

Consider $v = \sum_{i=0}^{n-2} \sum_{j=i}^{n-2} e_i \in \bigoplus_{i \in Q_0} e_i A e_i \cong \mathrm{Hom}_{k^e}(k, A)$. Then $b_K^1(v) = \zeta_0^* - \zeta_0$, as required. ■

As a consequence, the complex defining the higher Koszul cohomology is

$$0 \rightarrow \mathrm{HK}^0(A) \xrightarrow{\partial^1} \mathrm{HK}^1(A) \xrightarrow{\partial^2} \mathrm{HK}^2(A) \rightarrow 0 \cdots$$

with $\partial^1(z_\ell) = 2\bar{\zeta}_\ell$ for $0 \leq \ell \leq n - m_A - 2$ and $\partial^2 = 0$. We then have the following higher Koszul cohomology.

Proposition 6.6. *Let A be a preprojective algebra of type A_n . If $\text{char}(\mathbb{F}) = 2$, then $\text{HK}_{hi}^\bullet(A) = \text{HK}^\bullet(A)$.*

If $\text{char}(\mathbb{F}) \neq 2$ and n is even, then

$$\begin{aligned}\text{HK}_{hi}^2(A) &= \text{HK}^2(A) \\ \text{HK}_{hi}^p(A) &= 0 \text{ if } p \neq 2.\end{aligned}$$

Finally, if $\text{char}(\mathbb{F}) \neq 2$ and n is odd, then

$$\begin{aligned}\text{HK}_{hi}^0(A) &= \text{HK}^0(A)_{2m_A} \text{ has dimension 1 and is spanned by } z_{m_A} \\ \text{HK}_{hi}^2(A) &= \text{HK}^2(A) \\ \text{HK}_{hi}^p(A) &= 0 \text{ if } p \neq 0 \text{ and } p \neq 2.\end{aligned}$$

Higher Koszul homology can then be deduced using duality (Theorem 4.4).

Corollary 6.7. *If $\text{char}(\mathbb{F}) = 2$, then $\text{HK}_\bullet^{hi}(A) = \text{HK}_\bullet(A)$.*

If $\text{char}(\mathbb{F}) \neq 2$ and n is even, then

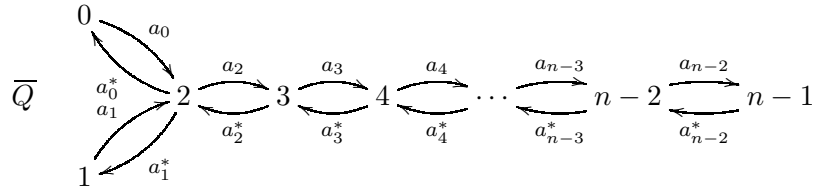
$$\begin{aligned}\text{HK}_0^{hi}(A) &= \text{HK}_0(A) \\ \text{HK}_p^{hi}(A) &= 0 \text{ if } p \neq 0.\end{aligned}$$

Finally, if $\text{char}(\mathbb{F}) \neq 2$ and n is odd, then

$$\begin{aligned}\text{HK}_0^{hi}(A) &= \text{HK}_0(A) \\ \text{HK}_2^{hi}(A) &= \text{HK}_2(A)_{2m_A} \text{ has dimension 1 and is spanned by } \check{z}_{m_A} \\ \text{HK}_p^{hi}(A) &= 0 \text{ if } p \neq 0 \text{ and } p \neq 2.\end{aligned}$$

6.2 Koszul calculus for preprojective algebras of type D

The preprojective algebra A of type D_n is defined by the quiver



subject to the relations

$$\begin{aligned}\sigma_0 &= -a_0^* a_0 & \sigma_i &= a_{i-1} a_{i-1}^* - a_i^* a_i & 3 \leq i \leq n-2 \\ \sigma_1 &= -a_1^* a_1 & \sigma_{n-1} &= a_{n-2} a_{n-2}^* \\ \sigma_2 &= a_0 a_0^* + a_1 a_1^* - a_2^* a_2\end{aligned}$$

Eu has given in [18] a basis B of A . Set $m_D = \lfloor \frac{n-2}{2} \rfloor$ and $u = n - m_D - 2$. We shall only need bases of the $e_j A e_i$ when i and j are equal or adjacent vertices, which can be rewritten as follows:

$$\begin{aligned}e_0 B e_0 &= \left\{ (a_0^* a_1 a_1^* a_0)^\ell; 0 \leq \ell \leq m_D \right\} \\ e_1 B e_1 &= \left\{ (a_1^* a_0 a_0^* a_1)^\ell; 0 \leq \ell \leq m_D \right\}\end{aligned}$$

$$\begin{aligned}
e_i B e_i &= \left\{ (a_i^* a_i)^\ell; 0 \leq \ell \leq n-i-1 \right\} \\
&\cup \left\{ (a_i^* a_i)^\ell a_{i-1} \cdots a_2 a_1 a_1^* a_2^* \cdots a_{i-1}^*; 0 \leq \ell \leq n-i-1 \right\} \quad \text{if } i \geq 2 \\
e_2 B e_i &= \left\{ (a_2^* a_2)^\ell a_i; 0 \leq \ell \leq n-3 \right\} \quad \text{for } i \in \{0, 1\} \\
e_{i+1} B e_i &= \left\{ (a_{i+1}^* a_{i+1})^\ell a_i; 0 \leq \ell \leq n-i-2 \right\} \\
&\cup \left\{ (a_{i+1}^* a_{i+1})^\ell a_i \cdots a_2 a_1 a_1^* a_2^* \cdots a_{i-1}^*; 0 \leq \ell \leq n-i-2 \right\} \quad \text{if } i \geq 2 \\
e_i B e_{i+1} &= \kappa(e_{i+1} B e_i), \quad e_i B e_2 = \kappa(e_2 B e_0) \quad \text{for } i \in \{0, 1\}.
\end{aligned}$$

For each i , Ae_i contains precisely one basis element of maximal length $2(n-2)$, which is

$$\begin{aligned}
\pi_0 &= a_0^* a_2^* a_3^* \cdots a_{n-2}^* a_{n-2} \cdots a_3 a_2 a_0 = \begin{cases} (a_0^* a_1 a_1^* a_0)^{m_D} & \text{if } n \text{ is even} \\ a_1^* a_0 (a_0^* a_1 a_1^* a_0)^{m_D} & \text{if } n \text{ is odd} \end{cases} \\
\pi_1 &= a_1^* a_2^* a_3^* \cdots a_{n-2}^* a_{n-2} \cdots a_3 a_2 a_1 = \begin{cases} (a_1^* a_0 a_0^* a_1)^{m_D} & \text{if } n \text{ is even} \\ a_0^* a_1 (a_1^* a_0 a_0^* a_1)^{m_D} & \text{if } n \text{ is odd} \end{cases} \\
\pi_i &= (a_i^* a_i)^{n-i-1} a_{i-1} \cdots a_2 a_1 a_1^* a_2^* \cdots a_{i-1}^* \quad \text{if } 2 \leq i \leq n-2 \\
\pi_{n-1} &= a_{n-2} a_{n-3} \cdots a_1 a_1^* a_2^* \cdots a_{n-2}^*.
\end{aligned}$$

They form a basis of the socle of A .

It can be checked using [38, Proposition 3.3] that the Nakayama automorphism ν of A is id_A if n is even, and that when n is odd, it exchanges e_0 and e_1 , a_0 and a_1 and a_0^* and a_1^* , and fixes all the other vertices and arrows.

6.2.1 The Koszul cohomology and homology spaces in type D

The centre of A does not depend on the characteristic of \mathbb{F} by fact (N1) and was computed in [18] in characteristic 0. Moreover, $\dim \text{HK}^1(A) = \dim \text{HH}^1(A)$, which is equal to $\dim \text{HH}^1(\Lambda_{\mathbb{C}}) = n$ if $\text{char}(\mathbb{F}) \neq 2$ and to $\dim \text{HH}^1(\Lambda_{\mathbb{C}}) + m$ if $\text{char}(\mathbb{F}) = 2$ by [20].

It also follows from Theorem 4.4 and [20] that $\dim \text{HK}^2(A) = \dim \text{HK}_0(A) = \dim \text{HH}_0(A)$, which is equal to $n - m_D - 2$ if $\text{char}(\mathbb{F}) \neq 2$ and to $n - 2$ if $\text{char}(\mathbb{F}) = 2$.

In order to give bases of the $\text{HK}^p(A)$ for $p = 0, 1, 2$, we define the following cochains:

- the elements $z_0 = 1$ and $z_\ell = (a_0^* a_1 a_1^* a_0)^\ell + (a_1^* a_0 a_0^* a_1)^\ell + \sum_{i=2}^{n-2} (a_i^* a_i)^{2\ell} = z_1^\ell$ for $\ell > 0$ in A . Note that if n is even, then $z_1^u = z_1^{m_D} = \pi_0 + \pi_1$, but if n is odd then $z_1^u = 0$;
- the elements $\zeta_\ell \in \text{Hom}_{k^e}(V, A)$ with $0 \leq \ell \leq u-1$ defined by $\zeta_\ell(a_i) = a_i z_\ell$ for all i ;
- the elements $\rho_\ell \in \text{Hom}_{k^e}(V, A)$ with $0 \leq \ell \leq m_D - 1$ where $\rho_\ell(a_i) = (a_2^* a_2)^{2\ell+1} a_i$ for $i = 0, 1$ and $\rho_\ell(a_i^*) = a_i^* (a_2^* a_2)^{2\ell+1}$ for $i = 0, 1$;
- the elements $h_j \in \text{Hom}_{k^e}(R, A)$ for $0 \leq j \leq n-1$, where $h_j(\sigma_i) = \delta_{ij} e_j$;
- the elements $\gamma_\ell \in \text{Hom}_{k^e}(R, A)$ for $1 \leq \ell \leq m_D$ where $\gamma_\ell(\sigma_0) = (a_0^* a_1 a_1^* a_0)^\ell$.

Proposition 6.8. *Let A be a preprojective algebra of type D_n .*

- (i) *The elements in $\{\pi_i; 0 \leq i \leq n-1 \text{ and } \nu(e_i) = e_i\} \cup \{z_\ell; 0 \leq \ell \leq u-1\}$ form a basis of $\text{HK}^0(A)$.*

(ii) If $\text{char}(\mathbb{F}) \neq 2$, the $\bar{\zeta}_\ell$, for $0 \leq \ell \leq u-1$, form a basis of $\text{HK}^1(A)$.

If $\text{char}(\mathbb{F}) = 2$, the $\bar{\zeta}_\ell$ for $0 \leq \ell \leq u-1$ and the $\bar{\rho}_\ell$ for $0 \leq \ell \leq m_D-1$ form a basis of $\text{HK}^1(A)$.

(iii) If $\text{char}(\mathbb{F}) \neq 2$, the \bar{h}_j for $0 \leq j \leq n-1$ form a basis of $\text{HK}^2(A)$.

If $\text{char}(\mathbb{F}) = 2$, the \bar{h}_j for $0 \leq j \leq n-1$ and the $\bar{\gamma}_\ell$ for $1 \leq \ell \leq m_D$ form a basis of $\text{HK}^2(A)$.

Proof. The results for $\text{HK}^0(A)$ and, when $\text{char}(\mathbb{F}) \neq 2$, for $\text{HK}^1(A)$ follow from the comments before the proposition.

Assume that $\text{char}(\mathbb{F}) = 2$. In order to prove the result for $\text{HK}^1(A)$, we must prove that the elements we have considered in $\text{Hom}_{k^e}(V, A)$ are cocycles that are linearly independent modulo coboundaries. It is in fact enough to prove that the ρ_ℓ are cocycles that are not coboundaries by fact (N3).

First note that, at the level of cochains, $\rho_\ell = \rho_0 \underset{K}{\smile} z_\ell$. Therefore, to prove that ρ_ℓ is a cocycle, it is enough to prove that ρ_0 is a cocycle, and this is easy to check.

Since $\rho_\ell \underset{K}{\smile} z_{m_D-1-\ell} = \rho_{m_D-1}$, in order to prove that ρ_ℓ is not a coboundary for all ℓ , it is enough to prove that ρ_{m_D-1} is not a coboundary. The map ρ_{m_D-1} has coefficient weight $4m_D-1$. If ρ_{m_D-1} is a coboundary, then it is the image of a morphism in $\text{Hom}_{k^e}(k, A) \cong \bigoplus_{i \in Q_0} e_i A e_i$ whose coefficients are linear combinations of cycles in A of weight $4m_D-2$, which are known. It is then straightforward to show that the image of any such morphism under b_K^1 is not equal to ρ_{m_D-1} .

For (iii), we first observe that every cochain in $\text{Hom}_{k^e}(R, A)$ is a cocycle. Moreover, the h_j are n cocycles that are clearly linearly independent modulo coboundaries (all coboundaries have coefficient weight at least equal to 1). Therefore if $\text{char}(\mathbb{F}) \neq 2$, the result follows.

If $\text{char}(\mathbb{F}) = 2$, it is enough to prove that the γ_ℓ are not coboundaries by fact (N3). At the level of cochains, $\gamma_\ell \underset{K}{\smile} z_{m_D-\ell} = \gamma_{m_D}$ for $1 \leq \ell \leq m_D$, therefore it is enough to prove that γ_{m_D} is not a coboundary. The map γ_{m_D} has coefficient weight $4m_D$, therefore if γ_{m_D} is a coboundary, then it is the image of a morphism in $\text{Hom}_{k^e}(V, A)$ whose coefficients are linear combinations of elements of weight $4m_D-2$ in A between two adjacent vertices in Q_0 , which are known. Here again, checking that the image of any such morphism under b_K^2 is not equal to γ_{m_D} is straightforward. ■

Koszul homology follows using duality (Theorem 4.4), as in Corollary 6.2.

6.2.2 Cup and cap products

We now determine the cup products of the elements in the bases of the Koszul cohomology spaces given above.

Lemma 6.9. For $1 \leq i \leq n-2$, consider the cochains u_i , v_i and w_i in $\text{Hom}_{k^e}(R, A)$ defined by $u_i(\sigma_j) = \delta_{ij} e_i z_1$, $v_i(\sigma_j) = \delta_{ij} \pi_i$ and $w_i(\sigma_j) = \delta_{ij} a_i^* a_i z_1$ for all $j \in Q_0$.

If $\text{char}(\mathbb{F}) \neq 2$, the u_i , v_i and w_i are all coboundaries.

If $\text{char}(\mathbb{F}) = 2$, then the u_i for $i \geq 2$ are coboundaries, and $\bar{u}_0 = \bar{u}_1 = \bar{\gamma}_1$. Moreover, if n is odd, all the v_i are coboundaries and if n is even, then $\bar{v}_i = \bar{\gamma}_{m_D}$ for all i . Finally, the w_i are all coboundaries.

Proof. Every element in $\text{Hom}_{k^e}(R, A)$ is a cocycle. Moreover, the differential b_K^2 is homogeneous of degree 1 with respect to the coefficient weight, and the coefficient weight of all the basis elements in $\text{HK}^2(A)$ is a multiple of 4, and is 0 if $\text{char}(\mathbb{F}) \neq 2$.

It follows that if $\text{char}(\mathbb{F}) \neq 2$, all the u_i , v_i and w_i must be coboundaries, and if $\text{char}(\mathbb{F}) = 2$, the w_i are coboundaries and so are the v_i if n is odd.

Assume that $\text{char}(\mathbb{F}) = 2$. We must now study the u_i , as well as the v_i when n is even.

Note that $u_{n-2} = 0$ and $u_0 = \gamma_1$. For $0 \leq i \leq n-3$, define $p_i \in \text{Hom}_{k^e}(V, A)$ by $p_i(a_0) = a_1 a_1^* a_0$, $p_i(a_1) = a_0 a_0^* a_1$ and $p_i(a_i) = a_i a_i^* a_i$ if $i \geq 2$. Then, for $2 \leq i \leq n-3$, we have $u_i = b_K^2 \left(\sum_{j=i}^{n-3} p_j \right)$. Moreover, $b_K^2 \left(\sum_{j=0}^{n-3} p_j \right) = u_0 + u_1$. It follows that the cohomology classes of u_0 and u_1 are both equal to that of γ_1 .

We now turn to the v_i . Note that since n is even and $\text{char}(\mathbb{F}) = 2$, the map v_0 is the map γ_{m_D} , which is not a coboundary.

Define $q_i \in \text{Hom}_{k^e}(V, A)$ by $q_i(a_i) = (a_{i+1}^* a_{i+1})^{n-i-2} a_i \cdots a_2 a_1 a_1^* a_2^* \cdots a_{i-1}^*$ for $1 \leq i \leq n-2$ and $q_0(a_0) = (a_2^* a_2)^{n-3}$. Then, for $2 \leq i \leq n-2$, we have $v_i - v_1 = b_K^2 \left(\sum_{j=1}^{i-1} q_j \right)$, and $v_1 - v_0 = b_K^2(q_0 - q_1)$. Therefore $\bar{v}_i = \bar{v}_0 = \bar{\gamma}_{m_D}$ for all i . ■

We now give all the non zero cup products.

Proposition 6.10. *Let A be a preprojective algebra of type D_n . Up to graded commutativity, the non zero cup products of elements in $\text{HK}^\bullet(A)$ are:*

$$\begin{aligned} z_0 \smile_K \bar{f} &= \bar{f} \text{ for all } \bar{f} \in \text{HK}^\bullet(A); & z_{\ell_1} \smile_K z_{\ell_2} &= \begin{cases} z_{\ell_1+\ell_2} & \text{if } \ell_1 + \ell_2 \leq u-1; \\ \pi_0 + \pi_1 & \text{if } n \text{ is even and } \ell_1 + \ell_2 = m_D; \end{cases} \\ z_{\ell_1} \smile_K \bar{\zeta}_{\ell_2} &= \bar{\zeta}_{\ell_1+\ell_2} \text{ if } \ell_1 + \ell_2 \leq u-1; \\ z_{\ell_1} \smile_K \bar{\rho}_{\ell_2} &= \bar{\rho}_{\ell_1+\ell_2} \text{ if } \ell_1 + \ell_2 \leq u-1; & z_\ell \smile_K \bar{h}_i &= \begin{cases} \bar{\gamma}_\ell & \text{if } \ell \geq 1, \text{ char}(\mathbb{F}) = 2 \text{ and } i \in \{0, 1\} \\ h_i & \text{if } \ell = 0; \end{cases} \\ z_{\ell_1} \smile_K \bar{\gamma}_{\ell_2} &= \bar{\gamma}_{\ell_1+\ell_2} \text{ if } \ell_1 + \ell_2 \leq m_D; & \pi_i \smile_K \bar{h}_j &= \gamma_{m_D} \text{ if } i = j, n \text{ is even and } \text{char}(\mathbb{F}) = 2. \end{aligned}$$

Proof. We use the notation in Lemma 6.9.

For $\ell \geq 1$, we have $z_\ell \smile_K h_i = z_{\ell-1} \smile_K z_1 \smile_K h_i = z_{\ell-1} \smile_K u_i$ and the result follows from Lemma 6.9.

Next, $\pi_i \smile_K h_j = \delta_{ij} v_j$ and again the result is a consequence of Lemma 6.9.

Now assume that $\text{char}(\mathbb{F}) = 2$, so that the $\bar{\rho}_\ell$ occur in the basis of $\text{HK}^1(A)$. At the level of cochains, we have $\rho_{\ell_1} \smile_K \rho_{\ell_2} = w_2 \smile_K z_{\ell_1+\ell_2}$, which is a coboundary.

The map $\rho_{\ell_1} \smile_K \zeta_{\ell_2} = u_2 \smile_K z_{\ell_1+\ell_2+1}$ is also a coboundary, as required.

The remaining cup products are easy to compute. Note that the cup product in $\text{HK}_0(A) \cong Z(A)$ is the ordinary product, and that the elements π_i are in the socle of A , hence are annihilated by the radical of A . ■

The cap products follow using duality, as in Corollary 6.4.

6.2.3 Higher Koszul (co)homology

As in the case of a preprojective algebra of type A , the cohomology class of the fundamental 1-cocycle is equal to $2\bar{\zeta}_0$ so that $\partial_\smile^1(z_\ell) = 2\bar{\zeta}_\ell$ for $0 \leq \ell \leq u-1$, $\partial_\smile^1(\pi_i) = 0$ and $\partial_\smile^2 = 0$. We then have the following higher Koszul cohomology.

Proposition 6.11. *Let A be a preprojective algebra of type D_n .*

(i) *If $\text{char}(\mathbb{F}) = 2$, then $\text{HK}_{hi}^\bullet(A) = \text{HK}^\bullet(A)$.*

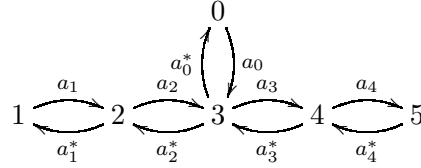
(ii) If $\text{char}(\mathbb{F}) \neq 2$, then

$$\begin{aligned} \text{HK}_{hi}^0(A) &= \text{HK}^0(A)_{>0} \text{ has basis the } \pi_i \text{ that are in } Z(A) \\ \text{HK}_{hi}^2(A) &= \text{HK}^2(A) \\ \text{HK}_{hi}^p(A) &= 0 \text{ if } p \neq 0 \text{ and } p \neq 2. \end{aligned}$$

Higher Koszul homology follows from Theorem 4.4 as in Corollary 6.7.

6.3 Koszul calculus for preprojective algebras of type E_6

The preprojective algebra A of type E_6 is defined by the quiver



subject to the relations

$$\begin{aligned} \sigma_0 &= -a_0^*a_0 & \sigma_3 &= a_0a_0^* + a_2a_2^* - a_3^*a_3 \\ \sigma_1 &= -a_1^*a_1 & \sigma_4 &= a_3a_3^* - a_4^*a_4 \\ \sigma_2 &= a_1a_1^* - a_2^*a_2 & \sigma_5 &= a_4a_4^* \end{aligned}$$

To simplify notation, we shall denote by $c_0 = a_0a_0^*$, $c_2 = a_2a_2^*$ and $c_3 = a_3^*a_3$ the three 2-cycles at the vertex 3.

The socle is the part of weight 10 of A , and the set $\{\pi_i; i \in Q_0\}$ where $\pi_0 = a_0^*c_3^2c_0c_3a_0$, $\pi_1 = a_4a_3c_0c_3c_0a_2a_1$, $\pi_2 = a_2^*(c_3c_0)^2a_3^*$, $\pi_3 = c_3(c_0c_3)^2$, $\pi_4 = \kappa(\pi_2)$ and $\pi_5 = \kappa(\pi_1)$.

The Nakayama automorphism is defined by $\nu(a_i) = -a_i$ and $\nu(a_i^*) = a_i^*$ for all $i \in Q_0$.

6.3.1 The Koszul cohomology and homology spaces in type E_6

We shall follow the same method as in types A and D, using the results from [20] and Theorem 4.4 to determine the dimensions of the spaces, and using results from [18] for the parts that are the same as in characteristic 0.

We define the following elements

- in A : $z_0 = 1$, $z_6 = a_1^*a_2^*a_3^*a_3a_2a_1 + a_2^*c_3^2a_2 - c_0c_3c_0 + a_3c_2^2a_3^* + a_4a_3a_2a_2^*a_3^*a_4^*$ and $z_8 = a_2^*c_0c_3c_0a_2 + c_0c_3^2c_0 + a_3c_0c_3c_0a_3^*$;
- in $\text{Hom}_{k^e}(V, A)$: the maps ζ_ℓ defined by $\zeta_\ell(a_i) = a_i z_\ell$ for $\ell \in \{0, 6, 8\}$, the map ρ_3 defined by $\rho_3(a_2) = c_0a_2$, $\rho_3(a_3) = a_3c_3$ and $\rho_3(a_2^*) = a_2^*c_3$, and the map ρ_5 defined by $\rho_5(a_0) = c_2c_3a_0$, $\rho_5(a_1) = a_2^*c_0a_2a_1$, $\rho_5(a_2) = c_2^2a_2$, $\rho_5(a_0^*) = -a_0^*c_3^2$, $\rho_5(a_1^*) = -a_1^*a_2^*c_0a_2$ and $\rho_5(a_2^*) = a_2^*c_3^2$;
- in $\text{Hom}_{k^e}(R, A)$: the maps h_j defined for $0 \leq j \leq 5$ by $h_j(\sigma_i) = \delta_{ij}e_j$ for all i , the map γ_4 defined by $\gamma_4(\sigma_0) = a_0^*c_3a_0$ and the map γ_6 defined by $\gamma_6(\sigma_0) = a_0^*c_3^2a_0$.

We shall use the following lemma.

Lemma 6.12. Assume that $\text{char}(\mathbb{F}) = 3$. Let $\gamma \in \text{Hom}_{k^e}(R, A)$ be an element of coefficient weight 6, so that

$$\begin{aligned}\gamma(\sigma_0) &= \lambda_0 a_0^* c_3^2 a_0 & \gamma(\sigma_3) &= \lambda_3 c_3^2 c_0 + \lambda'_3 c_3 c_0 c_3 + \lambda''_3 c_0 c_3^2 \\ \gamma(\sigma_1) &= \lambda_1 a_1^* a_2^* c_0 a_2 a_1 & \gamma(\sigma_4) &= \lambda_4 a_3 c_0 c_3 a_3^* + \lambda'_4 a_3 c_3 c_0 a_3^* \\ \gamma(\sigma_2) &= \lambda_2 a_2^* c_0 c_3 a_2 + \lambda'_2 a_2^* c_3 c_0 a_2 & \gamma(\sigma_5) &= \lambda_5 a_4 a_3 c_0 a_3^* a_4^*.\end{aligned}$$

Then γ is a coboundary if, and only if, $\sum_{i=0}^5 \lambda_i + \sum_{i=2}^4 \lambda'_i + \lambda''_3 = 0$.

Proof. The proof is straightforward, once we know that a cochain of weight 5 takes its values in $A_5 = E \oplus \kappa(E)$, where E is the space spanned by $c_3^2 a_0, c_0 c_3 a_0, a_2^* c_0 a_2 a_1, c_3^2 a_2, c_0 c_3 a_2, a_3 c_0 c_3, a_3 c_3 c_0, a_4 a_3 c_0 a_3^*$. ■

Proposition 6.13. Let A be a preprojective algebra of type E_6 .

- (i) The elements in $\{z_0, z_6, z_8, \pi_0, \pi_3\}$ form a basis of $\text{HK}^0(A)$.
- (ii) If $\text{char}(\mathbb{F}) \notin \{2, 3\}$, the elements in $\{\bar{\zeta}_\ell; \ell = 0, 6, 8\}$ form a basis of $\text{HK}^1(A)$.
If $\text{char}(\mathbb{F}) = 2$, the elements in $\{\bar{\zeta}_\ell; \ell = 0, 6, 8\} \cup \{\bar{\rho}_3\}$ form a basis of $\text{HK}^1(A)$.
If $\text{char}(\mathbb{F}) = 3$, the elements in $\{\bar{\zeta}_\ell; \ell = 0, 6, 8\} \cup \{\bar{\rho}_5\}$ form a basis of $\text{HK}^1(A)$.
- (iii) If $\text{char}(\mathbb{F}) \notin \{2, 3\}$, the elements in $\{\bar{h}_j; j \in Q_0\}$ form a basis of $\text{HK}^2(A)$.
If $\text{char}(\mathbb{F}) = 2$, the elements in $\{\bar{h}_j; j \in Q_0\} \cup \{\bar{\gamma}_4\}$ form a basis of $\text{HK}^2(A)$.
If $\text{char}(\mathbb{F}) = 3$, the elements in $\{\bar{h}_j; j \in Q_0\} \cup \{\bar{\gamma}_6\}$ form a basis of $\text{HK}^2(A)$.

Proof. The centre was given in [18], so we have (i).

For $\text{HK}^1(A)$ and $\text{HK}^2(A)$, the number of elements in the statement is equal to the dimension of the corresponding cohomology space. Moreover, all the elements in the statement are indeed cocycles.

If $\text{char}(\mathbb{F})$ is not 2 or 3, a basis of $\text{HK}^1(A) = \text{HH}^1(A)$ was given in [18]. It consists of the classes of the ζ'_ℓ with $\ell \in \{0, 6, 8\}$ where $\zeta'_\ell(a_i) = a_i z_\ell$ for $0 \leq i \leq 2$ and $\zeta'_\ell(a_i^*) = a_i^* z_\ell$ for $3 \leq i \leq 4$. Since $\zeta_0 - \zeta'_0$ is equal to $b_K^1(e_3 + 2e_5)$, and $\zeta_\ell - \zeta'_\ell = (\zeta_0 - \zeta'_0) \underset{K}{\smile} z_\ell$ is also a coboundary, ζ_ℓ and ζ'_ℓ represent the same cohomology class for $\ell \in \{0, 6, 8\}$. Moreover, as in types A and D, the elements \bar{h}_j form a basis of $\text{HK}^2(A)$.

If $\text{char}(\mathbb{F}) \in \{2, 3\}$, we need only prove that the extra elements are not coboundaries by fact (N3).

If $\text{char}(\mathbb{F}) = 2$, we have $z_6 \underset{K}{\smile} \rho_3 - \zeta_8 = b_K^1(g)$ where g is defined by $g(e_2) = a_2^* c_3 c_0 c_3 a_2$, and ζ_8 is not a coboundary, therefore ρ_3 cannot be a coboundary. Moreover, assume that $\gamma_4 = b_K^1(g')$ is a coboundary. Then g' would be of coefficient weight 3, and we would necessarily take values in $A_3 = E \oplus \kappa(E)$ where E is spanned by $c_3 a_0, c_0 a_2, c_3 a_2, a_3 c_0, a_3 c_3$. This leads to a contradiction.

If $\text{char}(\mathbb{F}) = 3$, assume that ρ_5 is a coboundary $b_K^1(h)$, then h is of weight 4, and necessarily $h(e_0) = \lambda_0 a_0^* c_3 a_0$, $h(e_2) = \lambda_2 a_2^* c_3 a_2$ and $h(e_3) = \lambda_3 c_3 c_0 + \lambda'_3 c_3^2 + \lambda''_3 c_0 c_3^2$, and by considering $b_K^1(h)(a_0)$, $b_K^1(h)(a_0^*)$, $b_K^1(h)(a_2)$ and $b_K^1(h)(a_2^*)$ we get a contradiction. Finally, the fact that γ_6 is not a coboundary follows from Lemma 6.12. ■

Koszul homology follows using duality (Theorem 4.4), as in Corollary 6.2.

6.3.2 Cup and cap products

We now determine the cup products of the elements in the bases of the Koszul cohomology spaces given above.

Proposition 6.14. *Let A be a preprojective algebra of type E_6 . Up to graded commutativity, the non zero cup products of elements in $\mathrm{HK}^\bullet(A)$ are:*

$$\begin{aligned} z_0 \smile_K \bar{f} &= \bar{f} \text{ for all } \bar{f} \in \mathrm{HK}^\bullet(A) & z_\ell \smile_K \bar{\zeta}_0 &= \bar{\zeta}_\ell \text{ for } \ell \in \{0, 6, 8\} \\ z_6 \smile_K \bar{h}_i &= \bar{\gamma}_6 \text{ if } i \in \{1, 4\} & z_6 \smile_K \bar{h}_i &= -\bar{\gamma}_6 \text{ if } i \in \{2, 5\} \\ z_6 \smile_K \bar{\rho}_3 &= \bar{\zeta}_8 & \bar{\zeta}_0 \smile_K \bar{\rho}_3 &= \bar{\gamma}_4. \end{aligned}$$

Proof. The first two cup products are clear.

For the cup products of z_6 with the \bar{h}_i and for $\bar{\zeta}_0 \smile_K \bar{\rho}_5$, we use Lemma 6.12.

The last cup product follows from the fact that we have $\zeta_0 \smile_K \rho_3 - \gamma_4 = b_K^1(g')$ where $g'(a_0^*) = a_0^*c_3$ and $g'(a_3) = a_3c_3$. The cup product $z_6 \smile_K \bar{\rho}_3$ was already in the proof of Proposition 6.13.

Consideration of the coefficient weights yields the vanishing of the other cup products. ■

The cap products follow using duality, as in Corollary 6.4.

6.3.3 Higher Koszul cohomology and homology

As in type A, the differential ∂_\smile^1 sends z_ℓ to $2\bar{\zeta}_\ell$ for $\ell \in \{0, 6, 8\}$ and the differential ∂_\smile^2 is zero.

We then have the following higher Koszul cohomology.

Proposition 6.15. *Let A be a preprojective algebra of type E_6 .*

If $\mathrm{char}(\mathbb{F}) = 2$, then $\mathrm{HK}_{hi}^\bullet(A) = \mathrm{HK}^\bullet(A)$.

If $\mathrm{char}(\mathbb{F}) = 3$, then

$$\begin{aligned} \mathrm{HK}_{hi}^0(A) &= \mathrm{HK}^0(A)_{10} \text{ has dimension 2 and is spanned by } \pi_0 \text{ and } \pi_3 \\ \mathrm{HK}_{hi}^2(A) &= \mathrm{HK}^2(A) \\ \mathrm{HK}_{hi}^1(A) &= \mathrm{span}\{[\bar{\rho}_5]\} \\ \mathrm{HK}_{hi}^p(A) &= 0 \text{ if } p > 2. \end{aligned}$$

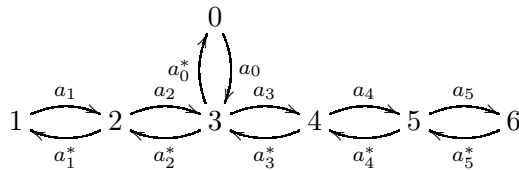
If $\mathrm{char}(\mathbb{F}) \notin \{2, 3\}$, then

$$\begin{aligned} \mathrm{HK}_{hi}^0(A) &= \mathrm{HK}^0(A)_{10} \text{ has dimension 2 and is spanned by } \pi_0 \text{ and } \pi_3 \\ \mathrm{HK}_{hi}^2(A) &= \mathrm{HK}^2(A) \\ \mathrm{HK}_{hi}^p(A) &= 0 \text{ if } p \neq 0 \text{ and } p \neq 2. \end{aligned}$$

Higher Koszul homology follows from Theorem 4.4.

6.4 Koszul calculus for preprojective algebras of type E_7

The preprojective algebra A of type E_7 is defined by the quiver



subject to the relations

$$\begin{aligned}
\sigma_0 &= -a_0^*a_0 & \sigma_4 &= a_3a_3^* - a_4^*a_4 \\
\sigma_1 &= -a_1^*a_1 & \sigma_5 &= a_4a_4^* - a_5^*a_5 \\
\sigma_2 &= a_1a_1^* - a_2^*a_2 & \sigma_6 &= a_5a_5^* \\
\sigma_3 &= a_0a_0^* + a_2a_2^* - a_3^*a_3
\end{aligned}$$

To simplify notation, we shall denote by $c_0 = a_0a_0^*$, $c_2 = a_2a_2^*$ and $c_3 = a_3^*a_3$ the three 2-cycles at the vertex 3.

The socle of A is the part of weight 16 of A . A basis of the socle is given by $\pi_0 = (a_0^*c_3a_0)^4$, $\pi_1 = -a_1^*a_2^*c_0c_3(c_3c_0)^2a_2a_1$, $\pi_2 = -(a_2^*c_0a_2)^4$, $\pi_3 = (c_3c_0)^3c_3^2$, $\pi_4 = -(a_3c_0a_3^*)^4$, $\pi_5 = -a_4a_3(c_3c_0)^3a_3^*a_4^*$ and $\pi_6 = -a_5a_4(a_3c_0a_3^*)^3a_4^*a_5^*$.

The Nakayama automorphism is given by $\nu(a_i) = -a_i$ and $\nu(a_i^*) = a_i^*$ for $i \in Q_0$.

6.4.1 The Koszul cohomology and homology spaces in type E_7

We define the following elements

- in A : $z_0 = 1$, $z_8 = a_0^*c_2c_0c_2a_0 - a_2^*c_2c_0c_2a_2 - c_2c_3^2c_2 + a_3c_0c_2c_0a_3^* - a_4a_3c_2^2a_3^*a_4^* + a_5a_4a_3c_0a_3^*a_4^*a_5^*$ and $z_{12} = a_0^*(c_2c_0)^2c_2a_0^* + a_2^*(c_0c_2)^2c_0a_2 - (c_3c_0c_3)^2 + a_3c_3(c_0c_3)^2a_3^*$;
- in $\text{Hom}_{k^e}(V, A)$: the maps ζ_ℓ defined by $\zeta_\ell(a_i) = a_iz_\ell$ for $\ell \in \{0, 8, 12\}$, the map ρ_3 defined by $\rho_3(a_2) = c_0a_2$, $\rho_3(a_3) = a_3c_3$, $\rho_3(a_4) = a_4a_3a_3^*$ and $\rho_3(a_2^*) = a_2^*c_3$, the map ρ_7 defined by $\rho_7(a_0) = c_3^3a_0 + c_3c_0c_3a_0$, $\rho_7(a_3) = a_3c_3c_0c_3$ and $\rho_7(a_3^*) = c_3c_0c_3a_3^*$, the map ρ_{15} defined by $\rho_{15}(a_0) = (c_2c_0)^3c_2a_0$ and $\rho_{15}(a_0^*) = a_0^*c_2(c_0c_2)^3$ and the map ρ_5 defined by $\rho_5(a_0) = -c_2c_3a_0$, $\rho_5(a_2) = c_3c_0a_2$, $\rho_5(a_3) = a_3c_3^2$, $\rho_5(a_0^*) = a_0^*c_3^2$ and $\rho_5(a_2^*) = -a_2^*c_2c_0$;
- in $\text{Hom}_{k^e}(R, A)$: the maps h_j defined for $0 \leq j \leq 6$ by $h_j(\sigma_i) = \delta_{ij}e_j$ for all i , the map γ_4 defined by $\gamma_4(\sigma_0) = a_0^*c_3a_0$, the map γ_8 defined by $\gamma_8(\sigma_0) = a_0^*c_3^3a_0$, the map γ_{16} defined by $\gamma_{16}(\sigma_0) = \pi_0$ and the map and γ_6 defined by $\gamma_6(\sigma_0) = a_0^*c_3^2a_0$.

Lemma 6.16. *First assume that $\text{char}(\mathbb{F}) = 2$.*

- Let $u_{16} \in \text{Hom}_{k^e}(R, A)$ be an element of weight 16 so that $u_{16}(\sigma_i) = \lambda_i\pi_i$ for $i \in Q_0$. Then u_{16} is a coboundary if, and only if, $\sum_{i=0}^6 \lambda_i = 0$.
- Let $u_8 \in \text{Hom}_{k^e}(R, A)$ be an element of weight 8 so that $u_8(\sigma_0) = \lambda_0a_0^*c_3^3a_0$, $u_8(\sigma_2) = \lambda_2a_2^*c_3c_0c_3a_2 + \lambda_2'a_2^*c_3^2c_0a_2$, $u_8(\sigma_3) = \lambda_3c_3^3c_0 + \lambda_3'c_3c_3^2 + \lambda_3''c_0c_3^2c_0 + \lambda_3'''c_3^2c_0c_3$, $u_8(\sigma_4) = \lambda_4a_3c_3^2c_0a_3^* + \lambda_4'a_3c_3c_0c_3a_3^* + \lambda_4''a_3c_0c_3^2a_3^*$, $u_8(\sigma_5) = \lambda_5a_4a_3c_3c_0a_3^*a_4^* + \lambda_5'a_4a_3c_0c_3a_3^*a_4^*$ and $u_8(\sigma_6) = a_5a_4a_3a_0a_0^*a_3^*a_4^*a_5^*$. Then u_8 is a coboundary if, and only if, $\lambda_0 + \lambda_2 + \lambda_2' + \lambda_3 + \lambda_3' + \lambda_3'' + \lambda_3''' + \lambda_4 + \lambda_4' + \lambda_4'' + \lambda_5 + \lambda_5' + \lambda_6 = 0$.

Now assume that $\text{char}(\mathbb{F}) = 3$.

- Let $u_6 \in \text{Hom}_{k^e}(R, A)$ be an element of weight 6 so that $u_6(\sigma_0) = \lambda_0a_0^*c_3^2a_0$, $u_6(\sigma_1) = \lambda_1a_1^*a_2^*c_0a_2^*a_1^*$, $u_6(\sigma_2) = \lambda_2a_2^*c_3c_0a_2 + \lambda_2'a_2^*c_3^2a_2$, $u_6(\sigma_3) = \lambda_3c_3c_0c_3 + \lambda_3'c_3^2c_0 + \lambda_3''c_0c_3c_0 + \lambda_3'''c_3^3$, $u_6(\sigma_4) = \lambda_4a_3c_0c_3a_3^* + \lambda_4'a_3c_3c_0a_3^*$ and $u_6(\sigma_5) = \lambda_5a_4a_3c_0a_3^*a_4^*$. Then u_6 is a coboundary if, and only if, $\sum_{i=0}^5 \lambda_i - \lambda_2' + \lambda_3' - \lambda_4' = 0$.

Proof. For each $\ell \in \{16, 6\}$, if the map u_ℓ were a coboundary, it would be the image of a map $g_\ell \in \text{Hom}_{k^e}(V, A)$ whose coefficients would be in the space generated by the paths between adjacent vertices with weight $\ell - 1$. The proof is then straightforward once we know bases of

these spaces. Note that once we have a basis of $\bigoplus_{\alpha \in Q_1} e_{t(\alpha)} A_w e_{s(\alpha)}$, applying κ gives a basis of $e_{s(\alpha)} A_w e_{t(\alpha)}$ for a given weight w .

In weight 15, a basis of $\bigoplus_{\alpha \in Q_1} e_{t(\alpha)} A_{15} e_{s(\alpha)}$ is given by $\bar{a}_0 = (a_0^* c_3 a_0)^3 a_0^* c_3$, $\bar{a}_1 = -a_1^* a_2^* c_0 c_3 (c_3 c_0)^2 a_2$, $\bar{a}_2 = -a_2^* c_0 (c_2 c_0)^3$, $\bar{a}_3 = (c_3 c_0)^3 c_3 a_3^*$, $\bar{a}_4 = -a_3 (c_3 c_0)^3 a_3^* a_4^*$, $\bar{a}_5 = a_4 (a_3 c_0 a_3^*)^3 a_4^* a_5^*$.

In weight 7, a basis of $\bigoplus_{\alpha \in Q_1} e_{t(\alpha)} A_7 e_{s(\alpha)}$ is given by $c_3^3 a_0$, $c_3 c_0 c_3 a_0$, $a_2^* c_3^2 a_2 a_1$, $c_3^2 c_0 a_2$, $c_0 c_3^2 a_2$, $c_3 c_0 c_3 a_2$, $a_3 c_3 c_0 c_3$, $a_3 c_0 c_3^2$, $a_3 c_3^2 c_0$, $a_4 a_3 c_3 c_0 a_3^*$, $a_4 a_3 c_0 c_3 a_3^*$, $a_5 a_4 a_3 c_0 a_3^* a_4^*$.

In weight 5, a basis of $\bigoplus_{\alpha \in Q_1} e_{t(\alpha)} A_5 e_{s(\alpha)}$ is given by $c_3^2 a_0$, $c_0 c_3 a_0$, $a_2^* c_0 a_2 a_1$, $c_3 c_0 a_2$, $c_0 c_3 a_2$, $a_3 c_3 c_0$, $a_3 c_0 c_3$, $a_3 c_3^2$, $a_4 a_3 c_0 a_3^*$. ■

Proposition 6.17. *Let A be a preprojective algebra of type E_7 .*

- (i) *The elements in $\{z_0, z_8, z_{12}\} \cup \{\pi_i; i \in Q_0\}$ form a basis of $\text{HK}^0(A)$.*
- (ii) *If $\text{char}(\mathbb{F}) \notin \{2, 3\}$, the elements in $\{\bar{\zeta}_\ell; \ell = 0, 8, 12\}$ form a basis of $\text{HK}^1(A)$.*
If $\text{char}(\mathbb{F}) = 2$, the elements in $\{\bar{\zeta}_\ell; \ell = 0, 8, 12\} \cup \{\bar{\rho}_3, \bar{\rho}_7, \bar{\rho}_{15}\}$ form a basis of $\text{HK}^1(A)$.
If $\text{char}(\mathbb{F}) = 3$, the elements in $\{\bar{\zeta}_\ell; \ell = 0, 8, 12\} \cup \{\bar{\rho}_5\}$ form a basis of $\text{HK}^1(A)$.
- (iii) *If $\text{char}(\mathbb{F}) \notin \{2, 3\}$, the elements in $\{\bar{h}_j; j \in Q_0\}$ form a basis of $\text{HK}^2(A)$.*
If $\text{char}(\mathbb{F}) = 2$, the elements in $\{\bar{h}_j; j \in Q_0\} \cup \{\bar{\gamma}_4, \bar{\gamma}_8, \bar{\gamma}_{16}\}$ form a basis of $\text{HK}^2(A)$.
If $\text{char}(\mathbb{F}) = 3$, the elements in $\{\bar{h}_j; j \in Q_0\} \cup \{\bar{\gamma}_6\}$ form a basis of $\text{HK}^2(A)$.

Proof. The centre was given in [18], so we have (i).

For $\text{HK}^1(A)$ and $\text{HK}^2(A)$, the number of elements in the statement is equal to the dimension of the corresponding cohomology space. Moreover, all the elements in the statement are indeed cocycles.

If $\text{char}(\mathbb{F})$ is not 2 or 3, a basis of $\text{HK}^1(A) = \text{HH}^1(A)$ was given in [18]. It consists of the classes of the ζ'_ℓ with $\ell \in \{0, 8, 12\}$ where $\zeta'_\ell(a_i) = a_i z_\ell$ for $0 \leq i \leq 2$ and $\zeta'_\ell(a_i^*) = a_i^* z_\ell$ for $3 \leq i \leq 5$. Since $\zeta_0 - \zeta'_0$ is equal to $b_K^1(e_3 + 2e_5 + 3e_6)$, and $\zeta_\ell - \zeta'_\ell = (\zeta_0 - \zeta'_0) \smile_K z_\ell$ is also a coboundary, ζ_ℓ and ζ'_ℓ represent the same cohomology class for $\ell \in \{0, 8, 12\}$. Moreover, as in types A, D and E_6 , the elements \bar{h}_j form a basis of $\text{HK}^2(A)$.

If $\text{char}(\mathbb{F}) \in \{2, 3\}$, we need only prove that the extra elements are not coboundaries by fact (N3).

If $\text{char}(\mathbb{F}) = 2$, it follows from Lemma 6.16 that $\zeta_0 \smile_K \rho_{15} - \gamma_{16}$, $\zeta_8 \smile_K \rho_7 - \gamma_{16}$, $\zeta_{12} \smile_K \rho_3 - \gamma_{16}$, $z_8 \smile_K \gamma_8 - \gamma_{16}$ and $z_{12} \smile_K \gamma_4 - \gamma_{16}$ are coboundaries. Therefore it is enough to check that γ_{16} is not a coboundary, and this also follows from Lemma 6.16.

If $\text{char}(\mathbb{F}) = 3$, again using Lemma 6.16, $\zeta_0 \smile_K \rho_5 - \gamma_6$ is a coboundary and γ_6 is not a coboundary, therefore ρ_5 is not a coboundary either. ■

Koszul homology follows using duality (Theorem 4.4), as in Corollary 6.2.

6.4.2 Cup and cap products

We now determine the cup products of the elements in the bases of the Koszul cohomology spaces given above.

Proposition 6.18. *Let A be a preprojective algebra of type E_7 . Up to graded commutativity, the non zero cup products of elements in $\text{HK}^\bullet(A)$ are:*

$$z_0 \smile_K \bar{f} = \bar{f} \text{ for all } \bar{f} \in \text{HK}^\bullet(A) \qquad z_8^2 = \pi_0 + \pi_4 - \pi_6$$

$$\begin{array}{ll}
z_\ell \smile_K \bar{\zeta}_0 = \bar{\zeta}_\ell \text{ for } \ell \in \{0, 8, 12\} & z_8 \smile_K \bar{\rho}_7 = \bar{\rho}_{15} \\
z_{12} \smile_K \bar{\rho}_3 = \bar{\rho}_{15} & \pi_i \smile_K \bar{h}_i = \bar{\gamma}_{16} \text{ for } i \in Q_0 \\
z_8 \smile_K \bar{h}_i = \bar{\gamma}_8 \text{ if } i \in \{0, 4, 6\} & z_8 \smile_K \bar{\gamma}_8 = \bar{\gamma}_{16} \\
z_{12} \smile_K \bar{\gamma}_4 = \bar{\gamma}_{16} & \bar{\zeta}_0 \smile_K \bar{\rho}_\ell = -\bar{\gamma}_{\ell+1} \text{ for } \ell \in \{3, 7, 15, 6\} \\
\bar{\zeta}_8 \smile_K \bar{\rho}_7 = \bar{\gamma}_{16} & \bar{\zeta}_{12} \smile_K \bar{\rho}_3 = \bar{\gamma}_{16}
\end{array}$$

Proof. Most of the cup-products are easy to compute, follow from Lemma 6.16 or vanish for weight reasons. The remaining ones are obtained as follows (at the level of cochains):

$$\begin{aligned}
z_8 \smile_K \rho_7 &= \rho_{15} + b_K^1([e_3 \mapsto (c_3 c_0)^3 c_3]) \\
z_{12} \smile_K \rho_3 &= \rho_{15} + b_K^1([e_3 \mapsto (c_3 c_0)^3 c_2]) \\
\zeta_0 \smile_K \rho_3 &= \gamma_4 + b_K^1(h)
\end{aligned}$$

where $h \in \text{Hom}_{k^e}(V, A_3)$ is defined by $h(a_0) = c_3 a_0$ and $h(a_2) = c_2 a_2$. ■

The cap products follow using duality, as in Corollary 6.4.

6.4.3 Higher Koszul cohomology and homology

As in types A, D and E_6 , the differential ∂^1_\smile sends z_ℓ to $2\bar{\zeta}_\ell$ for $\ell \in \{0, 8, 12\}$ and the differential ∂^2_\smile is zero except when $\text{char}(\mathbb{F}) = 3$ where $\partial^2_\smile(\bar{\rho}_5) = \bar{\gamma}_6$.

We then have the following higher Koszul cohomology.

Proposition 6.19. *Let A be a preprojective algebra of type E_7 .*

If $\text{char}(\mathbb{F}) = 2$, then $\text{HK}_{hi}^\bullet(A) = \text{HK}^\bullet(A)$.

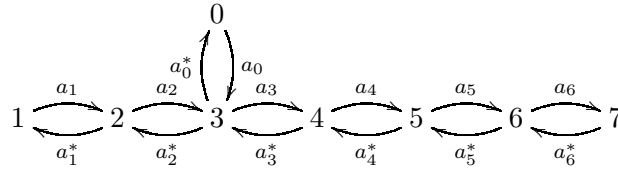
If $\text{char}(\mathbb{F}) \neq 2$, then

$$\begin{aligned}
\text{HK}_{hi}^0(A) &= \text{HK}^0(A)_{16} \text{ has dimension 7 and is spanned by the } \pi_i \text{ for } i \in Q_0 \\
\text{HK}_{hi}^2(A) &= \text{HK}^2(A)_0 \text{ has dimension 7 and is spanned by the } [\bar{h}_i] \text{ for } i \in Q_0 \\
\text{HK}_{hi}^p(A) &= 0 \text{ if } p \neq 0 \text{ and } p \neq 2.
\end{aligned}$$

Higher Koszul homology follows from Theorem 4.4.

6.5 Koszul calculus for preprojective algebras of type E_8

The preprojective algebra A of type E_8 is defined by the quiver



subject to the relations

$$\begin{array}{ll}
\sigma_0 = -a_0^* a_0 & \sigma_4 = a_3 a_3^* - a_4^* a_4 \\
\sigma_1 = -a_1^* a_1 & \sigma_5 = a_4 a_4^* - a_5^* a_5 \\
\sigma_2 = a_1 a_1^* - a_2^* a_2 & \sigma_6 = a_5 a_5^* - a_6^* a_6
\end{array}$$

$$\sigma_3 = a_0 a_0^* + a_2 a_2^* - a_3^* a_3$$

$$\sigma_7 = a_6 a_6^*$$

To simplify notation, we shall denote by $c_0 = a_0 a_0^*$, $c_2 = a_2 a_2^*$ and $c_3 = a_3^* a_3$ the three 2-cycles at the vertex 3.

The socle of A is the part of weight 28 of A . A basis of the socle is given by $\pi_0 = (a_0^* c_2 a_0)^7$, $\pi_1 = -a_1^* a_2^* (c_0 c_2)^5 c_2 c_0 a_2 a_1$, $\pi_2 = -(a_2^* c_0 a_2)^7$, $\pi_3 = (c_2 c_0)^7$, $\pi_4 = -a_3 (c_2 c_0)^6 c_2 a_3^*$, $\pi_5 = a_4 a_3 (c_2 c_0)^6 a_3^* a_4^*$, $\pi_6 = -a_5 a_4 a_3 (c_0 c_2)^5 c_0 a_3^* a_4^* a_5^*$ and $\pi_7 = a_6 a_5 a_4 a_3 (c_0 c_2)^3 (c_2 c_0)^2 a_3^* a_4^* a_5^* a_6^*$.

The Nakayama automorphism is given by $\nu(a_i) = -a_i$ and $\nu(a_i^*) = a_i^*$ for $i \in Q_0$.

6.5.1 The Koszul cohomology and homology spaces in type E_8

We define the following elements

• in A :

- $z_0 = 1$,
- $z_{12} = a_5 a_4 a_3 c_0 c_2 c_0 a_3^* a_4^* a_5^* + a_4 a_3 (c_2 c_0)^2 a_3^* a_4^* + a_3 (c_2 c_0)^2 c_2 a_3^* - (c_3 c_2 c_3)^2 + a_2^* (c_0 c_2)^2 c_0 a_2 - a_1^* a_2^* c_0 c_2^2 c_0 a_2 a_1 + a_0^* (c_2 c_0)^2 c_2 a_0$,
- $z_{20} = a_5 a_4 a_3 (c_0 c_2)^3 c_0 a_3^* a_4^* a_5^* + a_4 a_3 (c_0 c_2)^2 (c_2 c_0)^2 a_3^* a_4^* + a_3 c_0 (c_2 c_0)^4 a_3^* + (c_2 c_0)^5 - (c_0 c_2^2)^3 c_0 + (c_0 c_2)^5 + a_2^* (c_2 c_0 c_2)^3 a_2 + a_0^* (c_2 c_0)^4 c_2 a_0$ and
- $z_{24} = z_{12}^2$;

• in $\text{Hom}_{k^e}(V, A)$:

- the maps ζ_ℓ defined by $\zeta_\ell(a_i) = a_i z_\ell$ for $\ell \in \{0, 12, 20, 24\}$,
- the map ρ_3 defined by $\rho_3(a_2) = c_0 a_2$, $\rho_3(a_3) = a_3 c_3$, $\rho_3(a_4) = a_4 a_3 a_3^*$, $\rho_3(a_5) = a_5 a_4 a_4^*$ and $\rho_3(a_2^*) = a_2^* c_3$,
- the map ρ_7 defined by $\rho_7(a_0) = c_0 c_3^2 a_0$, $\rho_7(a_3) = a_3 c_0 c_3^2 + a_3 c_3^2 c_0 + a_3 c_0 c_3 c_0$ and $\rho_7(a_3^*) = c_3 c_0 c_3 a_3^*$,
- the map ρ_{15} defined by $\rho_{15}(a_3) = a_3 c_2 (c_2 c_0)^2$, $\rho_{15}(a_4) = a_4 a_3 (c_0 c_3)^2 a_3^* + a_4 a_3 (c_0 c_3^2)^2 a_3^* + a_4 a_3 (c_3 c_0)^3 a_3^*$, $\rho_{15}(a_5) = a_5 a_4 a_3 c_0 (c_3 c_0)^2 a_3^* a_4^*$, $\rho_{15}(a_0^*) = a_0^* c_3 (c_3 c_0)^3$ and $\rho_{15}(a_5^*) = a_4 a_3 (c_0 c_3)^2 c_0 a_3^* a_4^* a_5^*$,
- the map ρ_{27} defined by $\rho_{27}(a_3) = a_3 c_0 (c_3 c_0)^6$ and $\rho_{27}(a_3^*) = (c_3 c_0)^6 c_3 a_3^*$,
- the map ρ_5 defined by $\rho_5(a_0) = -c_0 c_3 a_0 - c_3^2 a_0$, $\rho_5(a_3) = a_3 c_3 c_0 + a_3 c_3^2$, $\rho_5(a_4) = a_4 a_3 c_3 a_3^*$, $\rho_5(a_0^*) = a_0^* c_3^2$ and $\rho_5(a_3^*) = -c_3 c_0 a_3^*$,
- the map ρ_{17} defined by defined by $\rho_{17}(a_0) = -c_3^2 (c_0 c_3)^3 a_0 - (c_0 c_3)^4 a_0$, $\rho_{17}(a_3) = a_3 (c_3 c_0)^4 + a_3 (c_0 c_3)^4 + a_3 (c_3 c_0)^3 c_3^2$, $\rho_{17}(a_0^*) = a_0^* (c_2 c_0)^3 c_3^2$ and $\rho_{17}(a_3^*) = -(c_3 c_0)^4 a_3^*$, and
- the map ρ_9 defined by $\rho_9(a_0) = -2c_3^2 c_0 c_3 a_0 + 2c_3 c_0 c_3^2 a_0 + (c_0 c_3)^2 a_0$, $\rho_9(a_2) = c_2^2 c_0 c_2 a_2 + (c_2 c_0)^2 a_2$, $\rho_9(a_3) = -a_3 (c_0 c_3)^2$, $\rho_9(a_0^*) = 2a_0^* c_3^2 c_0 c_3 - 2a_0^* c_3 c_0 c_3^2 + a_0^* (c_3 c_0)^2$, $\rho_9(a_2^*) = a_2^* c_2 c_0 c_2^2 - a_2^* (c_2 c_0)^2$ and $\rho_9(a_3^*) = (c_0 c_3)^2 a_3^*$;

• in $\text{Hom}_{k^e}(R, A)$: the maps h_j defined for $0 \leq j \leq 7$ by $h_j(\sigma_i) = \delta_{ij} e_j$ for all i , and

$$\begin{aligned} \gamma_4: \sigma_0 &\mapsto a_0^* c_3 a_0 & \gamma_8: \sigma_0 &\mapsto a_0^* c_3^3 a_0 \\ \gamma_{16}: \sigma_0 &\mapsto a_0^* (c_2 c_0)^3 c_2 a_0 & \gamma_{28}: \sigma_0 &\mapsto \pi_0 \\ \gamma_6: \sigma_0 &\mapsto a_0^* c_3^2 a_0 & \gamma_{18}: \sigma_0 &\mapsto a_0^* c_3^2 (c_0 c_3)^3 a_0 \\ \gamma_{10}: \sigma_0 &\mapsto a_0^* c_3^2 c_0 c_3 a_0 \end{aligned}$$

Lemma 6.20. *First assume that $\text{char}(\mathbb{F}) = 2$.*

- (i) Let $u_{28} \in \text{Hom}_{k^e}(R, A)$ be an element of weight 28, so that $u_{16}(\sigma_i) = \lambda_i \pi_i$ for all $i \in Q_0$. Then u_{16} is a coboundary if, and only if, $\sum_{i \in Q_0} \lambda_i = 0$.

Now assume that $\text{char}(\mathbb{F}) = 3$.

- (ii) Let $u_{18} \in \text{Hom}_{k^e}(R, A)$ be an element of weight 18, so that $u_{18}(\sigma_0) = \lambda_0 a_0^* c_3^2 (c_0 c_3)^3 a_0 + \lambda'_0 a_0^* c_3 c_0 (c_3 c_0 c_3)^2 a_0$, $u_{18}(\sigma_1) = \lambda_1 a_1^* a_2^* c_0 (c_2 c_0)^3 a_2 a_1$, $u_{18}(\sigma_2) = \lambda_2 a_2^* (c_2 c_0 c_2)^2 c_0 c_2 a_2 + \lambda'_2 a_2^* (c_0 c_2)^4 a_2 + \lambda''_2 a_2^* (c_2 c_0)^4 a_2$, $u_{18}(\sigma_3) = \lambda_3 c_0 (c_3 c_0)^4 + \lambda'_3 (c_3 c_0)^4 c_3 + \lambda''_3 (c_3 c_0)^2 c_3 (c_3 c_0)^2 + \lambda_3^{(3)} (c_0 c_3)^2 c_3 (c_0 c_3)^2 + \lambda_3^{(4)} (c_3 c_0)^3 c_3^2 c_0 + \lambda_3^{(5)} c_0 c_3^2 (c_0 c_3)^3$, $u_{18}(\sigma_4) = \lambda_4 a_3 (c_3 c_0)^4 a_3^* + \lambda'_4 a_3 (c_0 c_3)^4 a_3^* + \lambda''_4 a_3 c_3^2 (c_3 c_0)^3 a_3^* + \lambda_4^{(3)} a_3 (c_0 c_3)^3 c_3^2 a_3^*$, $u_{18}(\sigma_5) = \lambda_5 a_4 a_3 (c_3 c_0)^3 c_3 a_3^* a_4^* + \lambda'_5 a_4 a_3 c_3 (c_3 c_0)^3 a_3^* a_4^* + \lambda''_5 a_4 a_3 (c_0 c_3)^3 c_3 a_3^* a_4^*$, $u_{18}(\sigma_6) = \lambda_6 a_5 a_4 a_3 (c_3 c_0)^3 a_3^* a_4^* a_5^* + \lambda'_6 a_5 a_4 a_3 (c_0 c_3)^3 a_3^* a_4^* a_5^*$, $u_{18}(\sigma_7) = \lambda_7 a_6 a_5 a_4 a_3 (c_0 c_3)^2 c_0 a_3^* a_4^* a_5^* a_6^*$. Then u_{18} is a coboundary if, and only if, $\lambda_0 + \lambda'_0 + \lambda_1 + \lambda'_1 + \lambda_2 + \lambda'_2 + \lambda_3 + \lambda'_3 + \lambda_3^{(3)} + \lambda_3^{(4)} + \lambda_3^{(5)} + \lambda_4 + \lambda'_4 - \lambda_4'' - \lambda_4^{(3)} - \lambda_5 - \lambda'_5 - \lambda_5'' - \lambda_6 - \lambda'_6 - \lambda_7 = 0$.

- (iii) The map $\rho_{17} \in \text{Hom}_{k^e}(V, A)$ is not a coboundary.

Now assume that $\text{char}(\mathbb{F}) = 5$.

- (iv) Let $u_{10} \in \text{Hom}_{k^e}(R, A)$ be an element of weight 10, so that $u_{10}(\sigma_0) = \lambda_0 a_0^* c_3^2 c_0 c_3 a_0 + \lambda'_0 a_0^* c_3 c_0 c_3^2 a_0$, $u_{10}(\sigma_1) = \lambda_1 a_1^* a_2^* c_0 c_2 c_0 a_2 a_1$, $u_{10}(\sigma_2) = \lambda_2 a_2^* (c_0 c_2)^2 a_2 + \lambda'_2 a_2^* (c_2 c_0)^2 a_2 + \lambda''_2 a_2^* c_0 c_2^2 c_0 a_2$, $u_{10}(\sigma_3) = \lambda_3 c_0 (c_3 c_0)^2 + \lambda'_3 c_3 (c_0 c_3)^2 + \lambda''_3 c_3 (c_3 c_0)^2 + \lambda_3^{(3)} (c_0 c_3)^2 c_3 + \lambda_3^{(4)} c_3 c_0 c_3^2 c_0 + \lambda_3^{(5)} c_0 c_3^2 c_0 c_3$, $u_{10}(\sigma_4) = \lambda_4 a_3 (c_0 c_3)^2 a_3^* + \lambda'_4 a_3 (c_3 c_0)^2 a_3^* + \lambda''_4 a_3 c_3 c_0 c_3^2 a_3^* + \lambda_4^{(3)} a_3 c_3^2 c_0 c_3 a_3^*$, $u_{10}(\sigma_5) = \lambda_5 a_4 a_3 c_0 c_3 c_0 a_3^* a_4^* + \lambda'_5 a_4 a_3 c_3^2 c_0 a_3^* a_4^* + \lambda''_5 a_4 a_3 c_0 c_3^2 a_3^* a_4^*$, $u_{10}(\sigma_6) = \lambda_6 a_5 a_4 a_3 c_3 c_0 a_3^* a_4^* a_5^* + \lambda'_6 a_5 a_4 a_3 c_0 c_3 a_3^* a_4^* a_5^*$, $u_{10}(\sigma_7) = \lambda_7 a_6 a_5 a_4 a_3 c_0 a_3^* a_4^* a_5^* a_6^*$. Then u_{10} is a coboundary if, and only if, $\lambda_0 + \lambda'_0 + \lambda_1 + \lambda'_1 + \lambda_2 + \lambda'_2 + \lambda_3 + \lambda'_3 + \lambda_3^{(3)} + \lambda_3^{(4)} + \lambda_3^{(5)} + \lambda_4 - \lambda'_4 + 2\lambda_4'' + 2\lambda_4^{(3)} + \lambda_5 + 2\lambda_5' + 2\lambda_5'' + 2\lambda_6 + 2\lambda_6' + 2\lambda_7 = 0$.

Proof. For each $\ell \in \{28, 18, 10\}$, if the map u_ℓ were a coboundary, it would be the image of a map $g_\ell \in \text{Hom}_{k^e}(V, A)$ whose coefficients would be in the space generated by the paths between adjacent vertices with weight $\ell - 1$. The proof of (i), (ii) and (iv) is then straightforward once we know bases of these spaces.

In weight 27, a basis is given by $\bar{a}_0 = a_0^* c_2 (c_0 c_2)^6$, $\bar{a}_1 = -a_1^* a_2^* (c_0 c_2)^5 c_3^2 a_2$, $\bar{a}_2 = -a_2^* c_0 (c_2 c_0)^6$, $\bar{a}_3 = (c_2 c_0)^6 c_2 a_3^*$, $\bar{a}_4 = -a_3 (c_2 c_0)^6 a_3^* a_4^*$, $\bar{a}_5 = -a_4 a_3 c_0 (c_2 c_0)^5 a_3^* a_4^* a_5^*$, $\bar{a}_6 = a_5 a_4 a_3 (c_0 c_2)^3 (c_2 c_0)^2 a_3^* a_4^* a_5^* a_6^*$, and the $\kappa(\bar{a}_i)$ for all $i \in Q_0$.

In weight 17, the space $\bigoplus_{\alpha \in Q_1} e_{t(\alpha)} A_{18} e_{s(\alpha)}$ has basis $(c_0 c_3)^4 a_0$, $c_3^2 (c_0 c_3)^3 a_0$, $c_3 c_0 (c_3 c_0 c_3)^2 a_0$, $a_2^* c_2 c_0 c_2 (c_2 c_0)^2 a_2 a_1$, $a_2^* (c_0 c_2)^3 c_0 a_2 a_1$, $(c_3 c_0)^4 a_2$, $(c_0 c_3)^4 a_2$, $(c_0 c_3)^3 c_3 c_0 a_2$, $c_3 c_0 (c_3 c_0 c_3)^2 a_2$, $a_3 (c_3 c_0)^4$, $a_3 (c_0 c_3)^4$, $a_3 (c_0 c_3)^3 c_3 c_0$, $a_3 c_0 c_3 (c_3 c_0)^3$, $a_3 (c_3 c_0)^3 c_3^2$, $a_4 a_3 (c_0 c_3)^3 c_0 a_3^*$, $a_4 a_3 (c_0 c_3)^3 c_3 a_3^*$, $a_4 a_3 c_3 (c_3 c_0)^3 a_3^*$, $a_5 a_4 a_3 (c_3 c_0)^3 a_3^* a_4^*$, $a_5 a_4 a_3 (c_0 c_3)^3 a_3^* a_4^*$, $a_6 a_5 a_4 a_3 (c_0 c_3)^2 c_0 a_3^* a_4^* a_5^*$ and $\bigoplus_{\alpha \in Q_1} e_{s(\alpha)} A_{18} e_{t(\alpha)} = \kappa \left(\bigoplus_{\alpha \in Q_1} e_{t(\alpha)} A_{18} e_{s(\alpha)} \right)$.

In weight 9, the space $\bigoplus_{\alpha \in Q_1} e_{t(\alpha)} A_9 e_{s(\alpha)}$ has basis $c_3^2 c_0 c_3 a_0$, $c_3 c_0 c_3^2 a_0$, $(c_0 c_3)^2 a_0$, $a_2^* c_0 c_2 c_0 a_2 a_1$, $c_2^2 c_0 c_2 a_2$, $(c_2 c_0)^2 a_2$, $c_0 c_2^2 c_0 a_2$, $a_3 c_3 c_0 c_3^2$, $a_3 (c_0 c_3)^2$, $a_3 (c_3 c_0)^2$, $a_3 c_0 c_3^2 c_0$, $a_4 a_3 c_3^2 c_0 a_3^*$, $a_4 a_3 c_0 c_3^2 a_3^*$, $a_4 a_3 c_0 c_3 c_0 a_3^*$, $a_5 a_4 a_3 c_3 c_0 a_3^* a_4^*$, $a_5 a_4 a_3 c_0 c_3 a_3^* a_4^*$, $a_6 a_5 a_4 a_3 c_0 a_3^* a_4^* a_5^*$ and $\bigoplus_{\alpha \in Q_1} e_{s(\alpha)} A_9 e_{t(\alpha)} = \kappa \left(\bigoplus_{\alpha \in Q_1} e_{t(\alpha)} A_9 e_{s(\alpha)} \right)$.

Finally, if ρ_{17} were a coboundary, it would be equal to $b_k^1(g)$ for some $g \in \text{Hom}_{k^e}(k, A_{16})$. Such a map g would necessarily satisfy $g(e_0) \in \text{span}\{a_0^* (c_3 c_0)^3 c_3 a_0\}$, $g(e_3) \in \text{span}\{(c_3 c_0)^4, (c_0 c_3)^4, c_0 c_3 (c_3 c_0)^3, (c_0 c_3)^3 c_3 c_0, c_3^2 (c_0 c_3)^3, c_3 c_0 (c_3 c_0 c_3)^2\}$, and $g(e_4) \in \text{span}\{a_3 c_0 (c_3 c_0)^3 a_3^*, a_3 c_3 (c_0 c_3)^3 a_3^*, a_3 c_3 (c_3 c_0)^3 a_3^*, a_3 (c_0 c_3)^3 c_3 a_3^*\}$. Then, by considering $b_k^1(g)(a_0)$, $b_k^1(g)(a_0^*)$ and $b_k^1(g)(a_3)$ we get a contradiction. ■

Proposition 6.21. *Let A be a preprojective algebra of type E_8 .*

- (i) *The elements in $\{z_0, z_{12}, z_{20}, z_{24}\} \cup \{\pi_i; i \in Q_0\}$ form a basis of $\mathrm{HK}^0(A)$.*
- (ii) *If $\mathrm{char}(\mathbb{F}) \notin \{2, 3, 5\}$, the elements in $\{\bar{\zeta}_\ell; \ell = 0, 12, 20, 24\}$ form a basis of $\mathrm{HK}^1(A)$.*
If $\mathrm{char}(\mathbb{F}) = 2$, the elements in $\{\bar{\zeta}_\ell; \ell = 0, 12, 20, 24\} \cup \{\bar{\rho}_3, \bar{\rho}_7, \bar{\rho}_{15}, \bar{\rho}_{27}\}$ form a basis of $\mathrm{HK}^1(A)$.
If $\mathrm{char}(\mathbb{F}) = 3$, the elements in $\{\bar{\zeta}_\ell; \ell = 0, 12, 20, 24\} \cup \{\bar{\rho}_5, \bar{\rho}_{17}\}$ form a basis of $\mathrm{HK}^1(A)$.
If $\mathrm{char}(\mathbb{F}) = 5$, the elements in $\{\bar{\zeta}_\ell; \ell = 0, 12, 20, 24\} \cup \{\bar{\rho}_9\}$ form a basis of $\mathrm{HK}^1(A)$.
- (iii) *If $\mathrm{char}(\mathbb{F}) \notin \{2, 3, 5\}$, the elements in $\{\bar{h}_j; j \in Q_0\}$ form a basis of $\mathrm{HK}^2(A)$.*
If $\mathrm{char}(\mathbb{F}) = 2$, the elements in $\{\bar{h}_j; j \in Q_0\} \cup \{\bar{\gamma}_4, \bar{\gamma}_8, \bar{\gamma}_{16}, \bar{\gamma}_{28}\}$ form a basis of $\mathrm{HK}^2(A)$.
If $\mathrm{char}(\mathbb{F}) = 3$, the elements in $\{\bar{h}_j; j \in Q_0\} \cup \{\bar{\gamma}_6, \bar{\gamma}_{18}\}$ form a basis of $\mathrm{HK}^2(A)$.
If $\mathrm{char}(\mathbb{F}) = 5$, the elements in $\{\bar{h}_j; j \in Q_0\} \cup \{\bar{\gamma}_{10}\}$ form a basis of $\mathrm{HK}^2(A)$.

Proof. The centre was given in [18], so we have (i).

For $\mathrm{HK}^1(A)$ and $\mathrm{HK}^2(A)$, the number of elements in the statement is equal to the dimension of the corresponding cohomology space. Moreover, all the elements in the statement are indeed cocycles.

If $\mathrm{char}(\mathbb{F})$ is not 2, 3 or 5, a basis of $\mathrm{HK}^1(A) = \mathrm{HH}^1(A)$ was given in [18]. It consists of the classes of the ζ'_ℓ with $\ell \in \{0, 12, 20, 24\}$ where $\zeta'_\ell(a_i) = a_i z_\ell$ for $0 \leq i \leq 2$ and $\zeta'_\ell(a_i^*) = a_i^* z_\ell$ for $3 \leq i \leq 6$. Since $\zeta_0 - \zeta'_0$ is equal to $b_K^1(e_3 + 2e_5 + 3e_6 + 4e_7)$, and $\zeta_\ell - \zeta'_\ell = (\zeta_0 - \zeta'_0) \smile_K z_\ell$ is also a coboundary, ζ_ℓ and ζ'_ℓ represent the same cohomology class for $\ell \in \{0, 12, 20, 24\}$. Moreover, as in types A, D, E_6 and E_7 , the elements \bar{h}_j form a basis of $\mathrm{HK}^2(A)$.

The rest of the proof is the same as that of Proposition 6.17, based on Lemma 6.20 and the fact that the following cup products at the level of cochains are all coboundaries: $\zeta_0 \smile_K \rho_{27} - \gamma_{28}$, $\zeta_{12} \smile_K \rho_{15} - \gamma_{28}$, $\zeta_{24} \smile_K \rho_3 - \gamma_{28}$, $\zeta_{20} \smile_K \rho_7 - \gamma_{28}$, $z_{24} \smile_K \gamma_4 - \gamma_{28}$, $z_{12} \smile_K \gamma_{16} - \gamma_{28}$, $z_{20} \smile_K \gamma_8 - \gamma_{28}$, $z_{12} \smile_K \rho_5 - \rho_{17}$, $z_{12} \smile_K \gamma_6 - \gamma_{18}$, $\zeta_0 \smile_K \rho_9 - \gamma_{10}$, whereas γ_{28} , γ_{18} , γ_{10} and ρ_{17} are not. ■

Koszul homology follows using duality (Theorem 4.4), as in Corollary 6.2.

6.5.2 Cup and cap products

We now determine the cup products of the elements in the bases of the Koszul cohomology spaces given above.

Proposition 6.22. *Let A be a preprojective algebra of type E_8 . Up to graded commutativity, the non zero cup products of elements in $\mathrm{HK}^\bullet(A)$ are:*

$$\begin{array}{ll}
 z_0 \smile_K \bar{f} = \bar{f} \text{ for all } \bar{f} \in \mathrm{HK}^\bullet(A) & z_{12}^2 = z_{24} \\
 z_\ell \smile_K \bar{\zeta}_0 = \bar{\zeta}_\ell \text{ for } \ell \in \{0, 12, 20, 24\} & z_{12} \smile_K \bar{\rho}_3 = \bar{\rho}_{15} \\
 z_{24} \smile_K \bar{\rho}_3 = \bar{\rho}_{27} & z_{12} \smile_K \bar{\rho}_{15} = \bar{\rho}_{27} \\
 z_{20} \smile_K \bar{\rho}_7 = \bar{\rho}_{27} & z_{12} \smile_K \bar{\rho}_5 = \bar{\rho}_{17} \\
 z_{12} \smile_K \bar{\rho}_9 = -\bar{\zeta}_{20} & z_{12} \smile_K \bar{\zeta}_{12} = \bar{\zeta}_{24} \\
 \pi_i \smile_K \bar{h}_i = \bar{\gamma}_{28} \text{ for } i \in Q_0 & z_{12} \smile_K \bar{\gamma}_4 = \bar{\gamma}_{16}
 \end{array}$$

$$\begin{aligned}
z_{12} \smile_K \bar{\gamma}_{16} &= \bar{\gamma}_{28} & z_{24} \smile_K \bar{\gamma}_4 &= \bar{\gamma}_{28} \\
z_{20} \smile_K \bar{\gamma}_8 &= \bar{\gamma}_{28} & z_{12} \smile_K \bar{\gamma}_6 &= \bar{\gamma}_{18} \\
\bar{\zeta}_0 \smile_K \bar{\rho}_\ell &= -\bar{\gamma}_{\ell+1} \text{ for } \ell \in \{3, 7, 15, 27, 9\} & \bar{\zeta}_{12} \smile_K \bar{\rho}_3 &= \bar{\gamma}_{16} \\
\bar{\zeta}_{12} \smile_K \bar{\rho}_{15} &= \bar{\gamma}_{28} & \bar{\zeta}_{20} \smile_K \bar{\rho}_7 &= \bar{\gamma}_{28} \\
\bar{\zeta}_{24} \smile_K \bar{\rho}_3 &= \bar{\gamma}_{28}.
\end{aligned}$$

Proof. Most of the cup-products are easy to compute, follow from Lemma 6.20 or vanish for weight reasons. The remaining ones are obtained as follows (at the level of cochains): $z_{24} \smile_K \rho_3 = \rho_{27} + b_K^1([e_3 \mapsto c_0(c_3c_0)^6])$, we have $z_{12} \smile_K \rho_3 = \rho_{15} + b_K^1(h)$ where h is defined by $h(a_2) = (c_0c_2)^3c_0a_2$, $h(a_3) = a_3c_0(c_3c_0)^3 + a_3c_3(c_0c_3)^3$, $h(a_4) = a_4a_3(c_0c_3^2)^2a_3^*$ and $h(a_2^*) = a_2^*c_0(c_2c_0)^3 + a_2^*c_2(c_0c_2)^3$, and finally $z_{12} \smile_K \rho_9 + \zeta_{20} = b_K(h')$ where h' is defined by $h'(e_0) = a_0^*(c_3c_0)^4c_3a_0$, $h'(e_2) = 2a_2^*(c_2c_0)^4c_2a_2$, $h'(e_3) = -2(c_0c_3)^5 + 2c_3^2(c_0c_3)^4 + (c_3c_0)^4c_3^2$, $h'(e_4) = -a_3(c_0c_3)^4c_0a_3^* + a_3c_3(c_3c_0)^4a_3^* - a_3(c_0c_3)^4c_3a_3^*$ and $h'(e_5) = -a_4a_3(c_0c_3)^4a_3^*a_4^*$. ■

The cap products follow using duality, as in Corollary 6.4.

6.5.3 Higher Koszul cohomology and homology

As in types A, D, E₆ and E₇, the differential ∂_\smile^1 sends z_ℓ to $2\zeta_\ell$ for $\ell \in \{0, 8, 12\}$ and the differential ∂_\smile^2 is zero except when $\text{char}(\mathbb{F}) = 5$ where $\partial_\smile^2(\bar{\rho}_9) = 2\bar{\gamma}_{10}$.

We then have the following higher Koszul cohomology.

Proposition 6.23. *Let A be a preprojective algebra of type E₈.*

If $\text{char}(\mathbb{F}) = 2$, then $\text{HK}_{hi}^\bullet(A) = \text{HK}^\bullet(A)$.

If $\text{char}(\mathbb{F}) = 3$, then

$$\begin{aligned}
\text{HK}_{hi}^0(A) &= \text{HK}^0(A)_{28} \text{ and the } \pi_i \text{ for } i \in Q_0 \text{ form a basis} \\
\text{HK}_{hi}^1(A) &= \text{span}\{[\bar{\rho}_5], [\bar{\rho}_{17}]\} \\
\text{HK}_{hi}^2(A) &= \text{HK}^2(A)_0 \text{ and the } [\bar{h}_i] \text{ for } i \in Q_0 \text{ form a basis} \\
\text{HK}_{hi}^p(A) &= 0 \text{ if } p \neq 0 \text{ and } p \neq 2.
\end{aligned}$$

If $\text{char}(\mathbb{F}) \notin \{2, 3\}$, then

$$\begin{aligned}
\text{HK}_{hi}^0(A) &= \text{HK}^0(A)_{28} \text{ has dimension 8 and is spanned by the } \pi_i \text{ for } i \in Q_0 \\
\text{HK}_{hi}^2(A) &= \text{HK}^2(A)_0 \text{ has dimension 8 and is spanned by the } [\bar{h}_i] \text{ for } i \in Q_0 \\
\text{HK}_{hi}^p(A) &= 0 \text{ if } p \neq 0 \text{ and } p \neq 2.
\end{aligned}$$

Higher Koszul homology follows from Theorem 4.4.

6.6 Comparison of Koszul and Hochschild (co)homology for preprojective algebras of type ADE

Let A be a preprojective algebra over a Dynkin graph of type ADE. Schofield constructed a minimal projective resolution $(P^\bullet, \partial^\bullet)$ of A as a bimodule over itself, that is periodic (of period at most 6), which was described in [16, 18]. Following Proposition 2.11, the embedding $H(\iota^*)_2$ sends the Hochschild cohomology class of an element in $\text{Ker}(\partial^3 \circ -)$ to its

Koszul cohomology class, and the surjection $H(\tilde{l})_2$ induces an isomorphism between $\mathrm{HH}_2(A)$ and $\mathrm{HK}_2(A)/\mathrm{Im}(\mathrm{id}_A \otimes \partial^3)$.

We first transport the maps $\partial^3 \circ -$ and $\mathrm{id}_A \otimes \partial^3$ via the natural isomorphisms $A \otimes_{A^e} (Ae_j \otimes e_i A) \rightarrow e_i Ae_j$ that sends $\lambda \otimes (a \otimes b)$ to $b\lambda a$ and $\mathrm{Hom}_{A^e}(Ae_i \otimes e_j A, A) \cong e_i Ae_j$ that sends f to $f(e_i \otimes e_j)$.

The associative non degenerate bilinear form on the selfinjective preprojective A can be defined as follows, see [16] and [38, Proposition 3.15]: let B be a basis of A consisting of homogeneous elements, that contains the idempotents e_i , $i \in Q_0$, and such that each $v \in B$ belongs to $e_j Ae_i$ for some i, j in Q_0 . Then if $x \in Ae_i$, (y, x) is the coefficient of π_i in the expression of yx as a linear combination of elements in B . The Nakayama automorphism ν of A satisfies $(y, x) = (\nu(x), y)$ for all x, y in A , and induces a permutation of the indices, the Nakayama permutation $\bar{\nu}$, that is, a permutation of Q_0 such that $\mathrm{top}(Ae_i) \cong \mathrm{soc}(Ae_{\bar{\nu}(i)})$, characterised by $\nu(e_i) = e_{\bar{\nu}(i)}$.

Let \hat{B} be the dual basis of B with respect to the non degenerate form $(-, -)$, so that $(\hat{w}, v) = \delta_{vw}$ for all v, w in B . In particular, if $v \in Be_i$, the coefficient of π_i in \hat{v} is 1. Note that $v \in e_j Be_i$ if and only if $\hat{v} \in e_{\bar{\nu}(i)} \hat{B} e_j$.

Then the maps $\partial^3 \circ -$ and $\mathrm{id}_A \otimes_{A^e} \partial^3$ become respectively

$$\begin{aligned} \delta^3: \bigoplus_{i \in Q_0} e_i Ae_i &\rightarrow \bigoplus_{i \in Q_0} e_i Ae_{\bar{\nu}(i)} \text{ defined by } y \mapsto \sum_{x \in B} \hat{x} y x \\ \text{and } \delta_3: \bigoplus_{i \in Q_0} e_i Ae_{\bar{\nu}(i)} &\rightarrow \bigoplus_{i \in Q_0} e_i Ae_i \text{ defined by } y \mapsto \sum_{x \in B} x y \hat{x}. \end{aligned}$$

It then follows as in [20, Proposition 3.2.25] that $\mathrm{Im} \delta_3$ is the span of the $\delta_3(e_i)$ such that $\bar{\nu}(i) = i$, and that for such an i we have $\delta_3(e_i) = \sum_{j \in Q_0, \bar{\nu}(j)=i} \mathrm{tr}(\nu|_{e_j Ae_i}) \pi_j$. Moreover, the matrix whose coefficients are the $\mathrm{tr}(\nu|_{e_j Ae_i})$ is either easy to compute or given in [18] for types D and E_7 . It is also known from [20] that the set of elements of weight 0 in $\mathrm{Ker} \delta^3$ identifies with the kernel of the Cartan matrix of A . Moreover, for any element of positive weight $a \in A$, we have $\delta^3(a) = 0$. Therefore $\mathrm{HH}^2(A)$ is obtained by taking all the elements of positive weight in $\mathrm{HK}^2(A)$ and adding the kernel of the Cartan matrix.

We shall use this as well as the dimensions of the Hochschild and Koszul (co)homology spaces to compare $\mathrm{HH}^2(A)$ with $\mathrm{HK}^2(A)$ and $\mathrm{HH}_2(A)$ with $\mathrm{HK}_2(A)$ in each case.

6.6.1 Comparison of the second Koszul and Hochschild cohomology groups

In type A, the space $\mathrm{HH}^2(A)$ was completely described by Erdmann and Snashall in [14], and they proved that $\dim \mathrm{HH}^2(A) = n - m_A - 1$ and gave a basis $\{\tilde{h}_i; 0 \leq i \leq n - m_A - 2\}$ with $\tilde{h}_i = \bar{h}_i + \bar{h}_{n-1-i}$. The morphism of complexes ι_2^* sends \tilde{h}_i to $h_i + h_{n-1-i}$, and this describes the injection $\mathrm{HH}^2(A) \rightarrow \mathrm{HK}^2(A)$.

In type D, if $\mathrm{char}(\mathbb{F}) \neq 2$ and n is even, there is nothing to do since $\mathrm{HH}^2(A) = 0$. If $\mathrm{char}(\mathbb{F}) \neq 2$ and n is odd, then $\dim \mathrm{HH}^2(A) = 1$, the basis given in [18] for $\mathrm{HH}^2(\Lambda_{\mathbb{C}})$ (where $\Lambda_{\mathbb{C}}$ is the preprojective \mathbb{C} -algebra with the same Dynkin graph as A) also gives a basis of $\mathrm{HH}^2(A)$, and it is the cohomology class of the map ψ_0 defined by $\varphi_0(\sigma_0) = e_0$ and $\psi_0(\sigma_1) = -e_1$. The embedding $\mathrm{HH}^2(A) \rightarrow \mathrm{HK}^2(A)$ is therefore given by $\psi_0 \mapsto h_0 - h_1$.

Now assume that $\mathrm{char}(\mathbb{F}) = 2$. Then $\dim \mathrm{HH}^2(A) = n + m_D - 2$. As we explained above, a basis of $\mathrm{HH}^2(A)$ may be obtained from a basis of the set of elements of positive coefficient weight in $\mathrm{HK}^2(A)$ to which we add elements obtained by determining a basis of the kernel of the Cartan matrix of A . It follows that a basis of $\mathrm{HH}^2(A)$ is given by

$$\{\bar{\gamma}_\ell; 1 \leq \ell \leq m\} \cup \{\bar{\varphi}_i; 2 \leq i \leq n - 1\} \quad \text{if } n \text{ is even}$$

$$\{\bar{\gamma}_\ell; 1 \leq \ell \leq m\} \cup \{\bar{\psi}_0\} \cup \{\psi_i; 3 \leq i \leq n-1\} \quad \text{if } n \text{ is odd}$$

where $\psi_{2p+1}(\sigma_{2p+1}) = e_{2p+1}$, $\psi_{2p+2}(\sigma_{2p+2}) = e_{2p+2}$ and $\psi_{2p+2}(\sigma_2) = e_2$, $\varphi_{2p}(\sigma_{2p}) = e_{2p}$, $\varphi_{2p+1}(\sigma_{2p+3}) = e_{2p+3}$ and $\varphi_{2p+1}(\sigma_3) = e_3$, for $p \geq 1$. Therefore, the embedding $\text{HH}^2(A) \rightarrow \text{HK}^2(A)$ fixes the $\bar{\gamma}_\ell$ and sends $\bar{\varphi}_{2p}$ to \bar{h}_{2p} and $\bar{\varphi}_{2p+3}$ to $\bar{h}_3 + \bar{h}_{2p+3}$ when n is even, and $\bar{\psi}_0$ to $\bar{h}_0 + \bar{h}_1$, $\bar{\psi}_{2p+1}$ to \bar{h}_{2p+1} and $\bar{\psi}_{2p+2}$ to $\bar{h}_2 + \bar{h}_{2p+2}$ when n is odd.

In type E_6 , the Cartan matrix is equivalent, through row operations, to

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{pmatrix}$$

Let $\varphi_0, \varphi_1, \varphi_2$ and φ_3 be the maps in $\text{Hom}_{k^e}(R, A)$ defined by $\varphi_0(\sigma_0) = e_0$, $\varphi_1(\sigma_1) = e_1$, $\varphi_1(\sigma_5) = -e_5$, $\varphi_2(\sigma_2) = e_2$, $\varphi_2(\sigma_4) = -e_4$ and $\varphi_3(\sigma_3) = e_3$. Then $\text{HH}^2(A)$ is the subspace of $\text{HK}^2(A)$ spanned by

$$\begin{aligned} &\bar{\varphi}_1, \bar{\varphi}_2 \text{ if } \text{char}(\mathbb{F}) \notin \{2, 3\} \\ &\bar{\varphi}_0, \bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3, \bar{\gamma}_4 \text{ if } \text{char}(\mathbb{F}) = 2 \\ &\bar{\varphi}_1, \bar{\varphi}_2, \bar{\gamma}_6 \text{ if } \text{char}(\mathbb{F}) = 3. \end{aligned}$$

In type E_7 , if $\text{char}(\mathbb{F}) \neq 2$, we have $\dim \text{HH}^2(A) = \dim \text{HK}^2(A)_{>0}$, so that $\text{HH}^2(A)$ is precisely the subspace of $\text{HK}^2(A)$ of elements of positive weight (which is zero unless $\text{char}(\mathbb{F}) \in \{2, 3\}$). If $\text{char}(\mathbb{F}) = 2$, then the Cartan matrix of A is equivalent, through row operations, to the matrix $\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$ so that $\text{HH}^2(A)$ is the subspace of $\text{HK}^2(A)$ spanned by $\bar{\gamma}_4, \bar{\gamma}_8, \bar{\gamma}_{16}, \bar{h}_1, \bar{h}_2, \bar{h}_3, \bar{h}_5, \bar{h}_0 + \bar{h}_4$ and $\bar{h}_0 + \bar{h}_6$.

In type E_8 , we only need to look at dimensions. Indeed, if $\text{char}(\mathbb{F}) = 2$, then $\dim \text{HH}^2(A) = \dim \text{HK}^2(A)$ so that $\text{HH}^2(A) \cong \text{HK}^2(A)$, and if $\text{char}(\mathbb{F}) \neq 2$ then $\dim \text{HH}^2(A) = \dim \text{HK}^2(A)_{>0}$ so that $\text{HH}^2(A)$ is precisely the subspace of $\text{HK}^2(A)$ of elements of positive weight (which is zero unless $\text{char}(\mathbb{F}) \in \{2, 3, 5\}$).

6.6.2 Comparison of the second Koszul and Hochschild homology groups

In type A , if n is even or if n is odd and $\text{char}(\mathbb{F})$ divides $(m_A + 1)$, we have seen that $\dim \text{HK}_2(A) = \dim \text{HH}_2(A)$ and therefore $\text{HK}_2(A) \cong \text{HH}_2(A)$. Now assume that n is odd and $\text{char}(\mathbb{F}) \nmid (m_A + 1)$. Then $\dim \text{HK}_2(A) = \dim \text{HH}_2(A) - 1$. We must determine the image of the map δ_3 given in Subsection 6.6.

The Nakayama permutation $\bar{\nu}$ has precisely one fixed point, which is m_A . The matrix of ν restricted to $e_{m_A} A e_{m_A}$ is the identity matrix I_{m_A+1} . Consequently, $\delta_3(e_{m_A}) = (m_A + 1)\pi_{m_A}$ spans the image of δ_3 . Via the isomorphism $\bigoplus_{i \in Q_0} e_i A e_i \cong A \otimes_{k^e} R$, π_{m_A} corresponds to \check{z}_{m_A} so that we have $\text{HH}_2(A) \cong \text{HK}_2(A) / \text{span}\{\check{z}_{m_A}\}$.

In type D , if n is odd and $\text{char}(\mathbb{F}) = 2$, then we know that $\dim \text{HK}_2(A) = \dim \text{HH}_2(A)$ and therefore $\text{HK}_2(A) \cong \text{HH}_2(A)$. In the other cases, we need to determine the image of the map δ_3 .

If n is odd and $\text{char}(\mathbb{F}) \neq 2$, the fixed points of the Nakayama permutation are the integers i with $2 \leq i \leq n-1$. The matrix H^ν whose (i, j) -coefficient is the trace of ν restricted to $e_j A e_i$ was given in [18, paragraph 11.2.3], so that $\text{tr}(\nu|_{e_j A e_i}) = 2$ if i and j are even and is equal to 0 otherwise. Therefore for all i fixed by $\bar{\nu}$, we have $\delta_3(e_i) = \sum_{p=0}^{m_D+1} 2\pi_{2p}$. Using the isomorphism $\bigoplus_{i \in Q_0} e_i A e_i \cong A \otimes_{k^e} R$, we obtain $\text{HH}_2(A) \cong \text{HK}_2(A) / \text{span}\left\{\sum_{p=0}^{m_D+1} \check{\pi}_{2p}\right\}$.

If n is even, all the integers i with $0 \leq i \leq n-1$ are fixed points of $\bar{\nu}$. The matrix H^ν was given in [18, paragraph 11.2.2] and is equivalent, through column operations, to the matrix

whose last $n - 2$ columns are 0 and whose first two columns are

$$\begin{pmatrix} m_D+1 & 1-n \\ -m_D & n-1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It follows that, if n is even,

$$\mathrm{HH}_2(A) \cong \begin{cases} \mathrm{HK}_2(A) / \mathrm{span} \left\{ (m_D + 1)\tilde{\pi}_0 - m_D\tilde{\pi}_1 + \sum_{p=0}^{m_D} \tilde{\pi}_{2p+1} \right\} & \text{if } \mathrm{char}(\mathbb{F}) \mid (n-1) \\ \mathrm{HK}_2(A) / \mathrm{span} \left\{ \tilde{\pi}_1 - \tilde{\pi}_2; \tilde{\pi}_0 + \sum_{p=0}^{m_D+1} \tilde{\pi}_{2p+1} \right\} & \text{if } \mathrm{char}(\mathbb{F}) \nmid (n-1) \end{cases}$$

In types E_6 and E_8 , since they have the same dimensions, the homology spaces $\mathrm{HH}_2(A)$ and $\mathrm{HK}_2(A)$ are isomorphic.

Finally, in type E_7 , if $\mathrm{char}(\mathbb{F}) = 3$, then $\mathrm{HH}_2(A) \cong \mathrm{HK}_2(A)$. Now assume that $\mathrm{char}(\mathbb{F}) \neq 3$. The matrix H^ν was given in [19] and its non zero rows are those corresponding to vertices 0, 4 and 6 and are all equal to $(3 \ 0 \ 0 \ 0 \ 3 \ 0 \ 3)$. It follows that $\mathrm{HH}_2(A) \cong \mathrm{HK}_2(A) / \mathrm{span} \{ \tilde{\pi}_0 + \tilde{\pi}_4 + \tilde{\pi}_6 \}$.

6.7 A minimal complete list of invariants

Theorem 6.24. *Let A be the preprojective algebra of a Dynkin graph Δ over \mathbb{F} . Assume that A has type either A_n with $n \geq 3$, or D_n with $n \geq 4$, or E_n with $n = 6, 7, 8$. Let A' be a preprojective algebra of type ADE, where the integer n' concerning A' is subjected to the same assumptions. Denote by (d_p) the equality $\dim \mathrm{HK}_{hi}^p(A) = \dim \mathrm{HK}_{hi}^p(A')$. If (d_p) holds for $p = 0, 1$ and 2, then $n = n'$, and A and A' have the same type. The conclusion of this implication does not hold if (d_2) is removed from the assumption.*

Proof. We apply the results contained in Propositions 6.6, 6.11, 6.15, 6.19 and 6.23. The implication is a consequence of the following items.

- (1) Assume that A and A' have types A or D. If $\mathrm{char} \mathbb{F} \neq 2$, then $n = n'$ by (d_2) , and A and A' have the same type by (d_0) . If $\mathrm{char} \mathbb{F} = 2$ and A and A' both have type A, then $n = n'$ by (d_2) . If $\mathrm{char} \mathbb{F} = 2$, A is of type A_n and A' is of type $D_{n'}$, then the sum of (d_0) and (d_1) shows that $n = 2n' + m'_D - 2$ or -3 which contradicts (d_2) : $n = n' + m'_D$. If $\mathrm{char} \mathbb{F} = 2$ and if A and A' both have type D, then $n = n'$ by (d_1) .
- (2) If A and A' have type E, then $n = n'$ by (d_0) .
- (3) If A is of type A_n and A' of type E_6 , then (d_2) implies either that $n = 6$ if $\mathrm{char} \mathbb{F} \neq 2, 3$ or that $n = 7$ if $\mathrm{char} \mathbb{F} = 2, 3$, but each case is excluded by (d_0) . Similarly when A' is of type E_7 and E_8 .
- (4) If A is of type D_n and A' is of type E_6 , then (d_2) implies one of the three following cases: $n = 6$ if $\mathrm{char} \mathbb{F} \neq 2, 3$, $n = 7$ if $\mathrm{char} \mathbb{F} = 3$ or $n + m_D = 7$ if $\mathrm{char} \mathbb{F} = 2$, but each case is excluded by (d_0) . Similarly when A' is of type E_7 and E_8 (we also use (d_1) for E_8).

Let us show that we cannot remove assumption (d_2) . It is clear if $\mathrm{char} \mathbb{F} \neq 2$ because when A is of type A_3 and A' is of type A_5 , both (d_0) and (d_1) hold. If $\mathrm{char} \mathbb{F} = 2$, we check that when A is of type A_9 and A' is of type E_6 then (d_0) and (d_1) hold. ■

It is obvious that $\mathrm{HK}_{hi}^\bullet(A) \cong \mathbb{F}$ if A is of type A_1 and it is easy to check that $\mathrm{HK}_{hi}^\bullet(A) = 0$ when A is of type A_2 , therefore we have obtained a minimal complete list of invariants for all the ADE preprojective algebras.

Another direct application of our computations is the following. If A is as in Theorem 6.24 and if $\mathrm{char} \mathbb{F} \neq 2$, the product of the algebra $\mathrm{HK}_{hi}^\bullet(A)$ is identically zero. If $\mathrm{char} \mathbb{F} = 2$, $\mathrm{HK}_{hi}^\bullet(A) = \mathrm{HK}^\bullet(A)$ is a unital algebra whose product is fully described in our results.

Remark 6.25. *In the one vertex case, the higher Koszul homology and cohomology play an essential role in a specific formulation of a Koszul Poincaré Lemma (a Koszul Poincaré duality), see Conjectures 6.5 and 7.2 in [6]. For this reason, we have formulated Theorem 6.24 in terms of the higher Koszul cohomology. However, the analogous statement with Koszul cohomology is also true and follows in the same way from our results.*

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