

Variational Bayesian Approach in Model-Based Iterative Reconstruction for 3D X-Ray Computed Tomography with Gauss-Markov-Potts prior

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MaxEnt 2019 : Workshop on Bayesian Inference and Maximum Entropy Methods in Science and Engineering, June 30-July 5, Munich, Germany

Summary: 3D X-ray Computed Tomography (CT) is used in medicine and non-destructive testing (NDT) for industry to visualize the interior of a volume and control its healthiness. Compared to analytical reconstruction methods, model-based iterative reconstruction (MBIR) methods obtain high-quality reconstructions while reducing the dose. Nevertheless, usual Maximum-A-Posteriori (MAP) estimation does not enable to quantify the uncertainties on the reconstruction, which can be useful for the control performed afterwards. Herein, we propose to estimate these uncertainties jointly with the reconstruction by computing Posterior Mean (PM) thanks to Variational Bayesian Approach (VBA). We present our reconstruction algorithm using a Gauss-Markov-Potts prior model on the volume to reconstruct. For PM calculation in VBA, the uncertainties on the reconstruction are given by the variances of the posterior distribution of the volume. To estimate these variances in our algorithm, we need to compute diagonal coefficients of the posterior covariance matrix. Since this matrix is not available in 3D X-ray CT, we propose an efficient solution to tackle this difficulty, based on the use of a matched pair of projector and backprojector, which is here the Separable Footprint (SF) pair.

Forward model

- **Objectives:** inspect the interior of an object using X-ray tomography. The object is rotated to have several projections.
- **Difficulties:** ill-posed problem, huge dimension
- **Forward model:** linear model based on X-ray transform discretization and accounting for errors

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \zeta$$

where, $\forall i$,

$$p(\zeta_i | \rho_{\zeta_i}) = \mathcal{N}(\zeta_i | 0, \rho_{\zeta_i}^{-1})$$

and

$$p(\rho_{\zeta_i} | \alpha_{\zeta_0}, \beta_{\zeta_0}) = \mathcal{G}(\rho_{\zeta_i} | \alpha_{\zeta_0}, \beta_{\zeta_0})$$

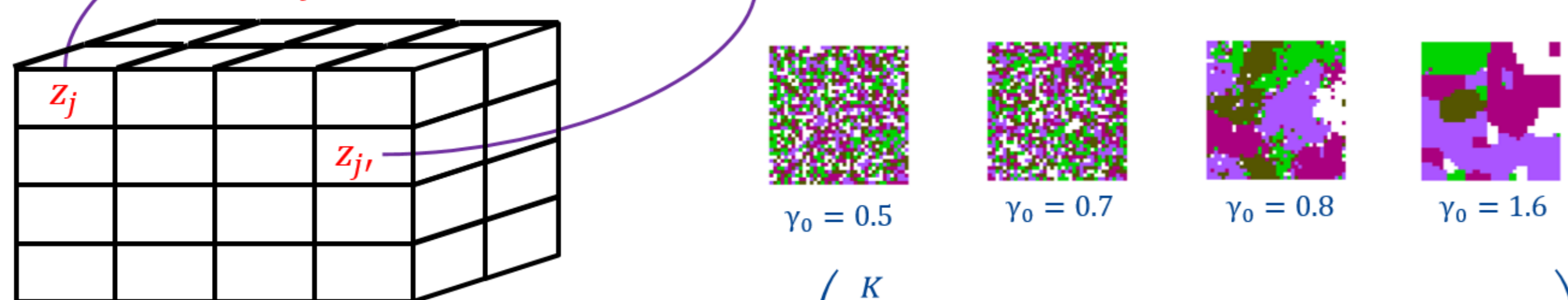
Gauss-Markov-Potts prior model

$$f_j \sim \mathcal{N}(m_k, \rho_k^{-1}) \text{ if } z_j = k$$

$$m_k \sim \mathcal{N}(m_0, v_0)$$

$$\rho_k \sim \mathcal{G}(\alpha_0, \beta_0)$$

$$z_j = k, k \in \{1, \dots, K\}$$



$$p(z_j | z_i, i \in \mathcal{V}(j); \alpha, \gamma_0) \propto \exp \left(\sum_{k=1}^K \alpha_k \delta(z_j - k) + \gamma_0 \sum_{i \in \mathcal{V}(j)} \delta(z_j - z_i) \right)$$

- K materials in the volume
- Hidden Markovian labels depending on the material in each voxel : \mathbf{z}

Variational Bayesian Approach (VBA)

- Approximate joint posterior distribution

$$q(\mathbf{f}, \mathbf{z}, \rho_{\zeta}, \mathbf{m}, \rho) = \prod_{j=1}^N q_f(f_j | z_j) \times \prod_{j=1}^N q_{z_j}(z_j) \times \prod_{i=1}^M q_{\rho_{\zeta_i}}(\rho_{\zeta_i}) \times \prod_{k=1}^K q_{m_k}(m_k) \times \prod_{k=1}^K q_{\rho_k}(\rho_k)$$

- The dependency between f_j and z_j is preserved to avoid a too gross approximation
- Due to the non-fully-factorized approximation, we have for q_{z_j} :

$$q_{z_j}(z_j) \propto \exp \left[\mathbb{E}_{q_{f_j}} (\ln (p(\mathbf{f}, \mathbf{z}, \rho_{\zeta}, \mathbf{m}, \rho | \mathbf{g}))) + \mathcal{H}(q_{f_j | z_j}) \right]$$

where $\mathcal{H}(q_{f_j | z_j})$ is the entropy of $q_{f_j}(f_j | z_j)$.

Computation of diagonal coefficients

- Diagonal coefficients in the updating formulae :

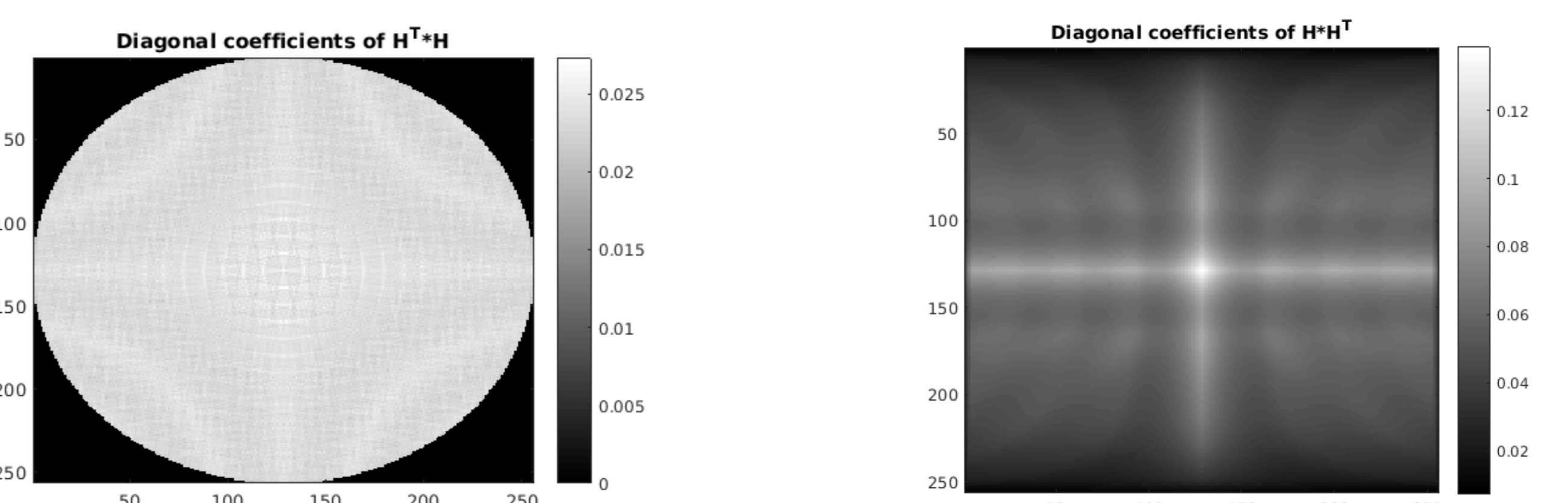
$$q_{f_j}(f_j | z_j = k) = \mathcal{N}(f_j | \tilde{m}_{jk}, \tilde{v}_{jk}), \tilde{v}_{jk}^{(t)} = \left(\frac{\tilde{\alpha}_{0k}^{(t-1)}}{\tilde{\beta}_{0k}^{(t-1)}} + [\mathbf{H}^T \tilde{\mathbf{V}}_{\zeta}^{-1} \mathbf{H}]_{jj} \right)^{-1}$$

- Difficulty : \mathbf{H} is not stored in memory, its coefficients are computed on-the-fly on GPU

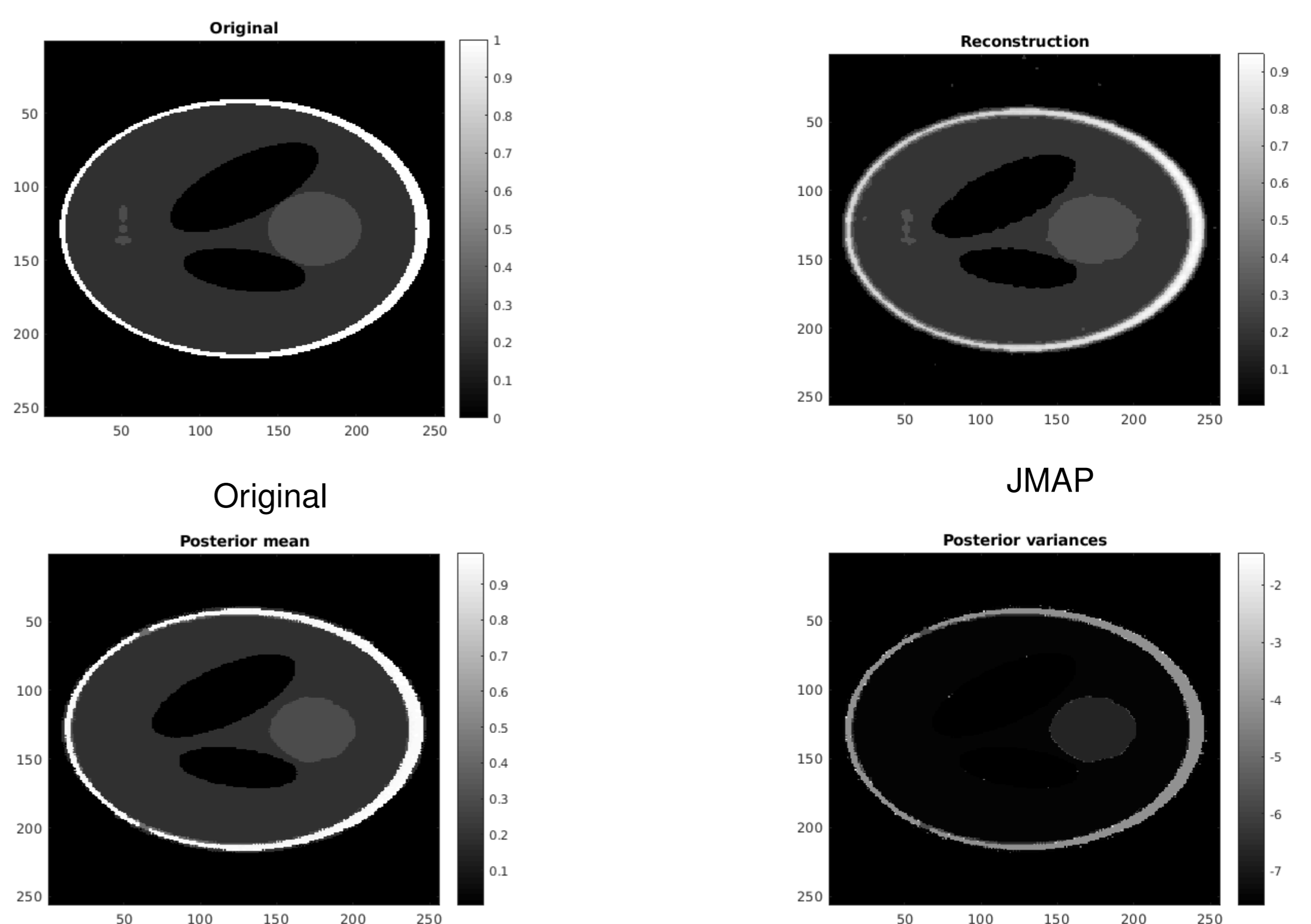
- Implement *squared* projector $\mathbf{H}^{(2)}$ and backprojector $(\mathbf{H}^{(2)})^T$:

$$[\mathbf{H}^T \tilde{\mathbf{V}}_{\zeta}^{-1} \mathbf{H}]_{jj} = \sum_{i=1}^M H_{ij}^2 \tilde{v}_{\zeta_i}^{-1} = \left[(\mathbf{H}^{(2)})^T \tilde{\mathbf{v}}_{\zeta}^{-1} \right]_j$$

- Use the matched SF pair to ensure the validity of the formula



Results in simulation and conclusion



- Loss of details, compared to JMAP. This loss is not highlighted in the estimated posterior variances
- Under-estimation of the posterior variances due to the use of Kullback-Leibler divergence
- Improve this estimation in future works