



**HAL**  
open science

## On the continuous dual of the sequence space $bv$

Mohammed El Azhari

► **To cite this version:**

Mohammed El Azhari. On the continuous dual of the sequence space  $bv$ . Acta Mathematica Universitatis Comenianae, 2020, 89 (2), pp.295-298. hal-02132299

**HAL Id: hal-02132299**

**<https://hal.science/hal-02132299>**

Submitted on 17 Jun 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## On the continuous dual of the sequence space $bv$

M. El Azhari

**Abstract.** Imaninezhad and Miri introduced the sequence space  $d_\infty$  in order to characterize the continuous dual of the sequence space  $bv$ . We show by a counterexample that this claimed characterization is false.

**Mathematics Subject Classification 2010:** 46B10, 46B45.

**Keywords:** continuous dual, sequence space.

### 1. Preliminaries

Let  $\omega$  denote the vector space of all complex sequences, where addition and scalar multiplication are defined pointwise. For each  $k \geq 1$ , let  $e^{(k)} = (e_n^{(k)})_{n \geq 1}$  be the complex sequence defined by  $e_k^{(k)} = 1$  and  $e_n^{(k)} = 0$  for  $n \neq k$ . A sequence space is a vector subspace of  $\omega$  including the set  $\{e^{(k)} : k \geq 1\}$ . By  $l_\infty$ ,  $c$ , and  $l_q$  ( $1 \leq q < \infty$ ), we denote the sequence spaces of all bounded, convergent and absolutely  $q$ -summable sequences, respectively. Also the inclusions  $l_q \subset c \subset l_\infty$  are strict.  $c$  and  $l_\infty$  are Banach spaces with the supremum norm  $\|x\|_\infty = \sup\{|x_k| : k \geq 1\}$ .  $l_q$  is a Banach space with the norm  $\|x\|_q = (\sum_{k=1}^\infty |x_k|^q)^{\frac{1}{q}}$ . Let  $bv_p$  ( $1 \leq p < \infty$ ) denote the sequence space of all sequences of bounded variation defined, in [2], by  $bv_p = \{x = (x_k)_{k \geq 1} \in \omega : \sum_{k=1}^\infty |x_k - x_{k-1}|^p < \infty\}$  with  $x_0 = 0$ .  $bv_p$  is a Banach space with the norm  $\|x\|_{bv_p} = (\sum_{k=1}^\infty |x_k - x_{k-1}|^p)^{\frac{1}{p}}$ . We will write  $bv$  instead of  $bv_1$ . If we define the sequence  $b^{(k)} = (b_n^{(k)})_{n \geq 1}$  of elements of the space  $bv_p$  for every  $k \geq 1$  by  $b_n^{(k)} = 0$  if  $n < k$  and  $b_n^{(k)} = 1$  if  $n \geq k$ , then the sequence  $(b^{(k)})_{k \geq 1}$  is a Schauder basis [2] for  $bv_p$  and every  $x \in bv_p$  has a unique representation of the form  $x = \sum_{k=1}^\infty (x_k - x_{k-1})b^{(k)}$  with  $x_0 = 0$ . Let  $\Delta_\omega$  denote the difference operator on  $\omega$  defined by  $\Delta_\omega x = (x_k - x_{k-1})_{k \geq 1}$  with  $x_0 = 0$  for every  $x = (x_k)_{k \geq 1} \in \omega$ . For each  $1 \leq p < \infty$ , the map  $\Delta : bv_p \rightarrow l_p$ ,  $\Delta x = \Delta_\omega x$ , is an isometric linear isomorphism [2]. If  $(E, \|\cdot\|)$  is a normed sequence space, then  $E^*$  denotes the continuous dual of  $E$  with the norm defined by  $\|f\| = \sup\{|f(x)| : \|x\| = 1\}$  for all  $f \in E^*$ . It is well-known that  $l_1^* \cong l_\infty$  and  $l_p^* \cong l_q$  ( $1 < p, q < \infty$ ) with  $\frac{1}{p} + \frac{1}{q} = 1$ .

### 2. The sequence space $d_\infty$

**Proposition 2.1.** A linear functional  $f$  on  $bv$  is continuous if and only if there exists  $a = (a_k)_{k \geq 1} \in l_\infty$  such that  $f(x) = \sum_{k=1}^\infty a_k (\Delta x)_k$  for all  $x \in bv$ . Furthermore,  $f(b^{(i)}) = a_i$  and  $f(e^{(i)}) = a_i - a_{i+1}$  for every  $i \geq 1$ .

**Proof.** The map  $\Delta : bv \rightarrow l_1, \Delta x = (x_k - x_{k-1})_{k \geq 1}$  with  $x_0 = 0$ , is an isometric linear isomorphism [2]. Let  $f$  be a linear functional on  $bv$ . If  $f$  is continuous, then  $f \circ \Delta^{-1}$  is a continuous linear functional on  $l_1$ , so there exists  $a = (a_k)_{k \geq 1} \in l_\infty$  such that  $f \circ \Delta^{-1}(y) = \sum_{k=1}^{\infty} a_k y_k$  for all  $y \in l_1$ . Hence  $f(x) = (f \circ \Delta^{-1})(\Delta x) = \sum_{k=1}^{\infty} a_k (\Delta x)_k$  for all  $x \in bv$ . Conversely, if  $f(x) = \sum_{k=1}^{\infty} a_k (\Delta x)_k$  for all  $x \in bv$  and some  $(a_k)_{k \geq 1} \in l_\infty$ , then  $|f(x)| \leq \sum_{k=1}^{\infty} |a_k| |(\Delta x)_k| \leq \|a\|_\infty \sum_{k=1}^{\infty} |(\Delta x)_k| = \|a\|_\infty \|x\|_{bv}$ . Therefore  $f$  is a continuous linear functional on  $bv$ . Let  $i \geq 1, f(b^{(i)}) = \sum_{k=1}^{\infty} a_k (\Delta b^{(i)})_k = a_i$  and  $f(e^{(i)}) = \sum_{k=1}^{\infty} a_k (\Delta e^{(i)})_k = a_i - a_{i+1}$ .

**Proposition 2.2.**  $bv^*$  is isometrically isomorphic to  $l_\infty$ .

**Proof.** Define  $\varphi : bv^* \rightarrow l_\infty, \varphi(f) = (f(b^{(k)}))_{k \geq 1}$ ,  $\varphi$  is a surjective linear map by Proposition 2.1, and  $\varphi$  is injective since  $(b^{(k)})_{k \geq 1}$  is a basis for  $bv$ . Let  $f \in bv^*$  and  $x \in bv, |f(x)| = |f(\sum_{k=1}^{\infty} (\Delta x)_k b^{(k)})| = |\sum_{k=1}^{\infty} (\Delta x)_k f(b^{(k)})| \leq \sum_{k=1}^{\infty} |(\Delta x)_k| |f(b^{(k)})| \leq \sup_{k \geq 1} |f(b^{(k)})| \sum_{k=1}^{\infty} |(\Delta x)_k| = \|\varphi(f)\|_\infty \|x\|_{bv}$ , then  $\|f\| \leq \|\varphi(f)\|_\infty$ . On the other hand,  $|f(b^{(k)})| \leq \|f\| \|b^{(k)}\|_{bv} = \|f\|$  since  $\|b^{(k)}\|_{bv} = 1$  for all  $k \geq 1$ , then  $\|\varphi(f)\|_\infty = \sup_{k \geq 1} |f(b^{(k)})| \leq \|f\|$ .

Imaninezhad and Miri [3] introduced the sequence space  $d_\infty = \{x = (x_k)_{k \geq 1} \in \omega : \sup_{k \geq 1} |\sum_{i=k}^{\infty} x_i| < \infty\}$ , and claimed that  $T : bv^* \rightarrow d_\infty, T(f) = (f(e^{(k)}))_{k \geq 1}$ , is an isometric linear isomorphism [3, Theorem 3.3] (see also [4, Theorem 4.1]). Here we show that the set  $T(bv^*)$  is not included in  $d_\infty$ .

**Counterexample.** Let  $a = (a_k)_{k \geq 1} \in l_\infty \setminus c$ , and consider the map  $f(x) = \sum_{k=1}^{\infty} a_k (\Delta x)_k$  for all  $x \in bv$ . By Proposition 2.1,  $f \in bv^*$  and  $f(e^{(i)}) = a_i - a_{i+1}$  for every  $i \geq 1$ . Let  $k \geq 1$  and  $n \geq k, \sum_{i=k}^n f(e^{(i)}) = \sum_{i=k}^n (a_i - a_{i+1}) = a_k - a_{n+1}$ . If  $(f(e^{(k)}))_{k \geq 1} \in d_\infty$ , then  $\lim_{n \rightarrow \infty} \sum_{i=k}^n f(e^{(i)}) = \sum_{i=k}^{\infty} f(e^{(i)}) \in \mathbb{C}$  and consequently  $\lim_{n \rightarrow \infty} a_{n+1} = a_k - \sum_{i=k}^{\infty} f(e^{(i)})$ , which contradicts the fact that the sequence  $(a_k)_{k \geq 1}$  is not convergent.

### 3. The sequence space $d_q$ ( $1 < q < \infty$ )

**Proposition 3.1.** Let  $1 < p, q < \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . A linear functional  $f$  on  $bv_p$  is continuous if and only if there exists  $a = (a_k)_{k \geq 1} \in l_q$  such that  $f(x) = \sum_{k=1}^{\infty} a_k (\Delta x)_k$  for all  $x \in bv_p$ . Furthermore,  $f(b^{(i)}) = a_i$  and  $f(e^{(i)}) = a_i - a_{i+1}$  for every  $i \geq 1$ .

**Proof.** The map  $\Delta : bv_p \rightarrow l_p, \Delta x = (x_k - x_{k-1})_{k \geq 1}$  with  $x_0 = 0$ , is an isometric linear isomorphism [2]. Let  $f$  be a linear functional on  $bv_p$ . If  $f$  is continuous, then  $f \circ \Delta^{-1}$  is a continuous linear functional on  $l_p$ , so there exists  $a = (a_k)_{k \geq 1} \in l_q$  such that  $f \circ \Delta^{-1}(y) = \sum_{k=1}^{\infty} a_k y_k$  for all  $y \in l_p$ . Hence  $f(x) = (f \circ \Delta^{-1})(\Delta x) = \sum_{k=1}^{\infty} a_k (\Delta x)_k$  for all  $x \in bv_p$ . Conversely, if  $f(x) = \sum_{k=1}^{\infty} a_k (\Delta x)_k$  for all  $x \in bv_p$  and some  $a = (a_k)_{k \geq 1} \in l_q$ ,

then  $|f(x)| \leq \sum_{k=1}^{\infty} |a_k| |(\Delta x)_k| \leq (\sum_{k=1}^{\infty} |a_k|^q)^{\frac{1}{q}} (\sum_{k=1}^{\infty} |(\Delta x)_k|^p)^{\frac{1}{p}} = \|a\|_q \|x\|_{bv_p}$  by Holder inequality. Therefore  $f$  is a continuous linear functional on  $bv_p$ .

**Proposition 3.2.** Let  $1 < p, q < \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . Then  $bv_p^*$  is isometrically isomorphic to  $l_q$ .

**Proof.** Define  $\psi : bv_p^* \rightarrow l_q$ ,  $\psi(f) = (f(b^{(k)}))_{k \geq 1}$ ,  $\psi$  is a surjective linear map by Proposition 3.1, and  $\psi$  is injective since  $(b^{(k)})_{k \geq 1}$  is a basis for  $bv_p$ . Let  $f \in bv_p^*$  and  $x \in bv_p$ ,  $|f(x)| = |f(\sum_{k=1}^{\infty} (\Delta x)_k b^{(k)})| = |\sum_{k=1}^{\infty} (\Delta x)_k f(b^{(k)})| \leq \sum_{k=1}^{\infty} |(\Delta x)_k| |f(b^{(k)})| \leq (\sum_{k=1}^{\infty} |(\Delta x)_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |f(b^{(k)})|^q)^{\frac{1}{q}} = \|\psi(f)\|_q \|x\|_{bv_p}$  by Holder inequality, then  $\|f\| \leq \|\psi(f)\|_q$ . On the other hand, let  $f \in bv_p^*$  and define the sequence  $(x^{(n)})_{n \geq 1}$  of the space  $bv_p$  by  $(\Delta x^{(n)})_k = |f(b^{(k)})|^q f(b^{(k)})^{-1}$  if  $1 \leq k \leq n$  and  $f(b^{(k)}) \neq 0$ ,  $(\Delta x^{(n)})_k = 0$  otherwise. Let  $n \geq 1$ ,  $f(x^{(n)}) = f(\sum_{k=1}^{\infty} (\Delta x^{(n)})_k b^{(k)}) = \sum_{k=1}^{\infty} (\Delta x^{(n)})_k f(b^{(k)}) = \sum_{k=1}^n |f(b^{(k)})|^q$ . Then  $\sum_{k=1}^n |f(b^{(k)})|^q = f(x^{(n)}) = |f(x^{(n)})| \leq \|f\| \|x^{(n)}\|_{bv_p} = \|f\| (\sum_{k=1}^n |f(b^{(k)})|^{(q-1)p})^{\frac{1}{p}} = \|f\| (\sum_{k=1}^n |f(b^{(k)})|^q)^{\frac{1}{p}}$  since  $(q-1)p = q$ . Therefore  $(\sum_{k=1}^n |f(b^{(k)})|^q)^{\frac{1}{q}} = (\sum_{k=1}^n |f(b^{(k)})|^q)^{1-\frac{1}{p}} \leq \|f\|$  for all  $n \geq 1$ . Letting  $n \rightarrow \infty$ , we get  $\|\psi(f)\|_q = (\sum_{k=1}^{\infty} |f(b^{(k)})|^q)^{\frac{1}{q}} \leq \|f\|$ .

Akhmedov and Basar [1] introduced the sequence space  $d_q = \{x = (x_k)_{k \geq 1} \in \omega : \sum_{k=1}^{\infty} |\sum_{i=k}^{\infty} x_i|^q < \infty\}$  ( $1 < q < \infty$ ), with the norm  $\|x\|_{d_q} = (\sum_{k=1}^{\infty} |\sum_{i=k}^{\infty} x_i|^q)^{\frac{1}{q}}$ , and proved that the sequence space  $d_q$  is isometrically isomorphic to  $bv_p^*$  ( $1 < p < \infty$ ) with  $\frac{1}{p} + \frac{1}{q} = 1$  (see [1, Theorem 2.3] or [3, Theorem 3.6]). Here we deduce this result as a consequence of Proposition 3.2.

**Corollary 3.3.** Let  $1 < p, q < \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . Then  $bv_p^*$  is isometrically isomorphic to  $d_q$ .

**Proof.** By Proposition 3.2 and the fact that the map  $D : d_q \rightarrow l_q$ ,  $D((x_k)_{k \geq 1}) = (\sum_{i=k}^{\infty} x_i)_{k \geq 1}$ , is an isometric linear isomorphism.

## REFERENCES

- [1] A. M. Akhmedov and F. Basar, The fine spectra of the difference operator  $\Delta$  over the sequence space  $bv_p$  ( $1 \leq p < \infty$ ), Acta Math. Sin. Eng. Ser., 23(10)(2007), 1757-1768.
- [2] F. Basar and B. Altay, On the space of sequences of p-bounded variation and related matrix mappings, Ukrainian Math. J., 55(1)(2003), 136-147.
- [3] M. Imaninezhad and M. Miri, The dual space of the sequence space  $bv_p$  ( $1 \leq p < \infty$ ), Acta Math. Univ. Comenianae, 79(1)(2010), 143-149.
- [4] M. Imaninezhad and M. Miri, The continuous dual of the sequence space  $l_p(\Delta^n)$ , Acta Math. Univ. Comenianae, 79(2)(2010), 273-280.

Ecole Normale Supérieure  
Avenue Oued Akreuch  
Takaddoum, BP 5118, Rabat  
Morocco

E-mail: [mohammed.elazhari@yahoo.fr](mailto:mohammed.elazhari@yahoo.fr)