



HAL
open science

Kramers-Fokker-Planck operators with homogeneous potentials

Mona Ben Said

► **To cite this version:**

| Mona Ben Said. Kramers-Fokker-Planck operators with homogeneous potentials. 2019. hal-02132097

HAL Id: hal-02132097

<https://hal.science/hal-02132097>

Preprint submitted on 16 May 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Kramers-Fokker-Planck operators with homogeneous potentials

Mona Ben Said
Laboratoire Analyse, Géométrie et Applications
Université Paris 13
99 Avenue Jean Baptiste Clément
93430 Villetaneuse, France
bensaid@univ-paris13.fr

May 16, 2019

Abstract

In this article we establish a global subelliptic estimate for Kramers-Fokker-Planck operators with homogeneous potentials $V(q)$ under some conditions, involving in particular the control of the eigenvalues of the Hessian matrix of the potential. Namely, this work presents a different approach from the one in [Ben], in which the case $V(q_1, q_2) = -q_1^2(q_1^2 + q_2^2)^n$ was already treated only for $n = 1$. With this article, after the former one dealing with non homogeneous polynomial potentials, we conclude the analysis of all the examples of degenerate ellipticity at infinity presented in the framework of Witten Laplacian by Helffer and Nier in [HeNi]. Like in [Ben], our subelliptic lower bounds are the optimal ones up to some logarithmic correction.

Key words: subelliptic estimates, compact resolvent, Kramers-Fokker-Planck operator.
MSC-2010: 35Q84, 35H20, 35P05, 47A10, 14P10

Contents

1	Introduction and main results	1
2	Observations and first inequalities	4
2.1	Dyadic partition of unity	4
2.2	Localisation in a fixed Shell	6
3	Proof of the main result	7

1 Introduction and main results

In this work we study the Kramers-Fokker-Planck operator

$$K_V = p \cdot \partial_q - \partial_q V(q) \cdot \partial_p + \frac{1}{2}(-\Delta_p + p^2), \quad (q, p) \in \mathbb{R}^{2d}, \quad (1.1)$$

where q denotes the space variable, p denotes the velocity variable and the potential $V(q)$ is a real-valued function defined in the whole space \mathbb{R}_q^d .

Setting

$$O_p = \frac{1}{2}(D_p^2 + p^2), \quad \text{and} \quad X_V = p \cdot \partial_q - \partial_q V(q) \cdot \partial_p,$$

the Kramers-Fokker-Planck operator K_V defined in (1.1) reads $K_V = X_V + O_p$.

We firstly list some notations used throughout the paper. We denote for an arbitrary function $V(q)$ in $\mathcal{C}^\infty(\mathbb{R}^d)$

$$\begin{aligned} \text{Tr}_{+,V}(q) &= \sum_{\substack{\nu \in \text{Spec}(\text{Hess } V) \\ \nu > 0}} \nu(q), \\ \text{Tr}_{-,V}(q) &= - \sum_{\substack{\nu \in \text{Spec}(\text{Hess } V) \\ \nu \leq 0}} \nu(q). \end{aligned}$$

In particular for a polynomial V of degree less than 3, $\text{Tr}_{+,V}$ and $\text{Tr}_{-,V}$ are two constants. In this case we define the constants A_V and B_V by

$$\begin{aligned} A_V &= \max\{(1 + \text{Tr}_{+,V})^{2/3}, 1 + \text{Tr}_{-,V}\}, \\ B_V &= \max\left\{\min_{q \in \mathbb{R}^d} |\nabla V(q)|^{4/3}, \frac{1 + \text{Tr}_{-,V}}{\log(2 + \text{Tr}_{-,V})^2}\right\}. \end{aligned}$$

This work is principally based on the publication by Ben Said, Nier, and Viola [BNV], which concerns the study of Kramers-Fokker-Planck operators with polynomials of degree less than three. In [BNV] we proved the existence of a constant $c > 0$, independent of V , such that the following global subelliptic estimate with remainder

$$\begin{aligned} \|K_V u\|_{L^2(\mathbb{R}^{2d})}^2 + A_V \|u\|_{L^2(\mathbb{R}^{2d})}^2 &\geq c \left(\|O_p u\|_{L^2(\mathbb{R}^{2d})}^2 + \|X_V u\|_{L^2(\mathbb{R}^{2d})}^2 \right. \\ &\quad \left. + \|\langle \partial_q V(q) \rangle^{2/3} u\|_{L^2(\mathbb{R}^{2d})}^2 + \|\langle D_q \rangle^{2/3} u\|_{L^2(\mathbb{R}^{2d})}^2 \right) \end{aligned} \quad (1.2)$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. Furthermore, supposing $\text{Tr}_{-,V} + \min_{q \in \mathbb{R}^d} |\nabla V(q)| \neq 0$, there exists a constant $c > 0$, independent of V , such that

$$\|K_V u\|_{L^2(\mathbb{R}^{2d})}^2 \geq c B_V \|u\|_{L^2(\mathbb{R}^{2d})}^2, \quad (1.3)$$

is valid for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. As a consequence collecting (1.3) and (1.2) together, there is a constant $c > 0$, independent of V , so that the global subelliptic estimates without remainder

$$\begin{aligned} \|K_V u\|_{L^2(\mathbb{R}^{2d})}^2 &\geq \frac{c}{1 + \frac{A_V}{B_V}} \left(\|O_p u\|_{L^2(\mathbb{R}^{2d})}^2 + \|X_V u\|_{L^2(\mathbb{R}^{2d})}^2 \right. \\ &\quad \left. + \|\langle \partial_q V(q) \rangle^{2/3} u\|_{L^2(\mathbb{R}^{2d})}^2 + \|\langle D_q \rangle^{2/3} u\|_{L^2(\mathbb{R}^{2d})}^2 \right) \end{aligned} \quad (1.4)$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. Here and throughout the paper we use the notation

$$\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}.$$

Moreover we remind that for an arbitrary potential $V \in \mathcal{C}^\infty(\mathbb{R}^d)$, the Kramers-Fokker-Planck operator K_V is essential maximal accretive when endowed with the domain $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ (see Proposition 5.5, page 44 in [HeNi]). Thanks to this property we deduce that the domain of the closure of K_V is given by

$$D(K_V) = \{u \in L^2(\mathbb{R}^{2d}), K_V u \in L^2(\mathbb{R}^{2d})\}.$$

Resultently, by density of $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ in the domain $D(K_V)$ all estimates written in this article, which are verified with $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ functions, can be extended to $D(K_V)$. By relative bounded perturbation with bound less than 1, this result holds as well when $V \in \mathcal{C}^\infty(\mathbb{R} \setminus \{0\})$ is an homogeneous function of degree $r > 1$.

Our results will require the following assumption after setting

$$\mathcal{S} = \{q \in \mathbb{R}^d, |q| = 1\}. \quad (1.5)$$

Assumption 1. *The potential $V(q)$ is an homogeneous function of degree $r > 2$ in $\mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\})$ and satisfies:*

$$\forall q \in \mathcal{S}, \quad \partial_q V(q) = 0 \Rightarrow \text{Tr}_{-,V}(q) > 0. \quad (1.6)$$

Our main result is the following.

Theorem 1.1. *If the potential $V(q)$ verifies Assumption 1, then there exists a strictly positive constant $C_V > 1$ (which depends on V) such that*

$$\begin{aligned} \|K_V u\|_{L^2}^2 + C_V \|u\|_{L^2}^2 &\geq \frac{1}{C_V} \left(\|L(O_p)u\|_{L^2}^2 + \|L(\langle \nabla V(q) \rangle^{2/3})u\|_{L^2}^2 \right. \\ &\quad \left. + \|L(\langle \text{Hess } V(q) \rangle^{1/2})u\|_{L^2}^2 + \|L(\langle D_q \rangle^{2/3})u\|_{L^2}^2 \right), \end{aligned} \quad (1.7)$$

holds for all $u \in D(K_V)$ where $L(s) = \frac{s+1}{\log(s+1)}$ for any $s \geq 1$.

Corollary 1.2. *The Kramers-Fokker-Planck operator K_V with a potential $V(q)$ satisfying Assumption 1 has a compact resolvent.*

Proof. Let $0 < \delta < 1$. Define the functions $f_\delta : \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$f_\delta(q) = |\nabla V(q)|^{\frac{4}{3}(1-\delta)} + |\text{Hess } V(q)|^{1-\delta}.$$

As a result of (1.7) in Theorem 1.1 there is a constant $C_V > 1$ such that

$$\|K_V u\|_{L^2}^2 + C_V \|u\|_{L^2}^2 \geq \frac{1}{C_V} \left(\langle u, f_\delta u \rangle + \|L(O_p)u\|_{L^2}^2 + \|L(\langle D_q \rangle^{\frac{2}{3}})u\|_{L^2}^2 \right),$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and all $\delta \in (0, 1)$. In order to show that the operator K_V has a compact resolvent it is sufficient to prove that $\lim_{q \rightarrow +\infty} f_\delta(q) = +\infty$. It is a matter of how different derivatives scale. Consider the unit sphere $S = \{q \in \mathbb{R}^d : |q| = 1\}$. By Assumption (1.6), at every point on S either $\nabla V \neq 0$ or $|\text{Hess } V| \neq 0$. Then the function f_δ is always positive on S . By hypothesis, f_δ is continuous on S and therefore it achieves a positive minimum there, call it $m_\delta > 0$.

For any $y, |y| > 1$ there exists $\lambda > 1$ such that $y = \lambda q$ for some $q \in S$. By homogeneity,

$$V(y) = \lambda^r V\left(\frac{y}{\lambda}\right) = \lambda^r V(q)$$

and therefore, by the chain rule

$$|\nabla V(y)| = \lambda^{r-1} |\nabla V(q)|$$

and

$$|\text{Hess } V(y)| = \lambda^{d(r-2)} |\text{Hess } V(q)|.$$

Adding these up,

$$|\nabla V(y)|^{\frac{4}{3}(1-\delta)} + |\text{Hess } V(y)|^{1-\delta} \geq \lambda^{(1-\delta) \min\{\frac{4}{3}(r-1), d(r-2)\}} f_\delta(q) \geq m_\delta \lambda^{(1-\delta) \min\{\frac{4}{3}(r-1), d(r-2)\}}$$

which goes to infinity as $|y| = \lambda \rightarrow \infty$, since by assumption $r > 2$. \square

Remark 1.3. *The result of Corollary does not hold in the case of homogenous polynomial of degree 2 with degenerate Hessian. Indeed, we already know that in this case, the resolvent of the Kramers-Fokker-Planck operator K_V is not compact since it is not as so for the Witten Laplacian (cf. Proposition 5.19 and Theorem 10.16 in [HeNi]).*

Remark 1.4. *Our results are in agreement with the results of Wei-Xi-Li [Li][Li2] and those of Helffer-Nier on Witten Laplacian with homogeneous potential [HeNi1].*

2 Observations and first inequalities

2.1 Dyadic partition of unity

In this paper, we make use of a locally finite dyadic partition of unity with respect to the position variable $q \in \mathbb{R}^d$. Such a partition is described in the following Proposition. For a detailed proof, we refer to [BCD] (see page 59).

Proposition 2.1. *Let \mathcal{C} be the shell $\{x \in \mathbb{R}^d, \frac{3}{4} < |x| < \frac{8}{3}\}$. There exist radial functions χ and ϕ valued in the interval $[0, 1]$, belonging respectively to $\mathcal{C}_0^\infty(B(0, \frac{4}{3}))$ and to $\mathcal{C}_0^\infty(\mathcal{C})$ such that*

$$\forall x \in \mathbb{R}^d, \quad \chi(x) + \sum_{j \geq 0} \phi(2^{-j}x) = 1,$$

$$\forall x \in \mathbb{R}^d \setminus \{0\}, \quad \sum_{j \in \mathbb{Z}} \phi(2^{-j}x) = 1.$$

Setting for all $q \in \mathbb{R}^d$,

$$\begin{aligned} \chi_{-1}(q) &= \frac{\chi(2q)}{\left(\chi^2(2q) + \sum_{j' \geq 0} \phi^2(2^{-j'}q)\right)^{\frac{1}{2}}} = \frac{\chi(2q)}{\left(\chi^2(2q) + \phi^2(q)\right)^{\frac{1}{2}}}, \\ \chi_j(q) &= \frac{\phi(2^{-j}q)}{\left(\chi^2(2q) + \sum_{j' \geq 0} \phi^2(2^{-j'}q)\right)^{\frac{1}{2}}} \text{ if } j \leq 2, \quad \frac{\phi(2^{-j}q)}{\left(\sum_{j-1 \leq j' \leq j+1} \phi^2(2^{-j'}q)\right)^{\frac{1}{2}}} \end{aligned}$$

we get a locally finite dyadic partition of unity

$$\sum_{j \geq -1} \chi_j^2(q) = \tilde{\chi}_{-1}^2(2|q|) + \tilde{\chi}_0^2(2|q|) + \sum_{j \geq 0} \tilde{\chi}^2(2^{-j}|q|) = 1 \quad (2.1)$$

where for all $j \in \mathbb{N}$, the cutoff functions $\tilde{\chi}_0, \tilde{\chi}$ and $\tilde{\chi}_{-1}$ belong respectively to $\mathcal{C}_0^\infty([\frac{3}{4}, \frac{8}{3}[)$, $\mathcal{C}_0^\infty([\frac{3}{4}, \frac{8}{3}[)$ and $\mathcal{C}_0^\infty([0, \frac{4}{3}[)$.

Lemma 2.2. *Let V be in $\mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\})$. Consider the Kramers-Fokker-Planck operator K_V defined as in (1.1). For a locally finite partition of unity $\sum_{j \geq -1} \chi_j^2(q) = 1$ one has*

$$\|K_V u\|_{L^2(\mathbb{R}^{2d})}^2 = \sum_{j \geq -1} \|K_V(\chi_j u)\|_{L^2(\mathbb{R}^{2d})}^2 - \|(p\partial_q \chi_j)u\|_{L^2(\mathbb{R}^{2d})}^2, \quad (2.2)$$

for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

In particular when the cutoff functions χ_j have the form (2.1), there exists a uniform constant $c > 0$ so that

$$(1 + 4c)\|K_V u\|_{L^2(\mathbb{R}^{2d})}^2 + c\|u\|_{L^2(\mathbb{R}^{2d})}^2 \geq \sum_{j \geq -1} \|K_V(\chi_j u)\|_{L^2(\mathbb{R}^{2d})}^2, \quad (2.3)$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

Proof. The proof of the equality (2.2) is detailed in [Ben]. Now it remains to show the inequality (2.3), after considering a locally finite dyadic partition of unity

$$\sum_{j \geq -1} \chi_j^2(q) = 1, \quad (2.4)$$

where for all $j \in \mathbb{N}$, the cutoff functions χ_j and χ_{-1} are respectively supported in the shell $\{q \in \mathbb{R}^d, 2^j \frac{3}{4} \leq |q| \leq 2^j \frac{8}{4}\}$ and in the ball $B(0, \frac{3}{4})$.

Since the partition is locally finite, for each index $j \geq -1$ there are finitely many j' such that $(\partial_q \chi_j) \chi_{j'}$ is nonzero. Along these lines, there exists a uniform constant $c > 0$ so that

$$\begin{aligned} \sum_{j \geq -1} \|(p \partial_q \chi_j) u\|_{L^2}^2 &= \sum_{j \geq -1} \sum_{j' \geq -1} \|(p \partial_q \chi_j) \chi_{j'} u\|_{L^2}^2 \\ &\leq c \sum_{j \geq -1} \frac{1}{(2^j)^2} \|p \chi_j u\|_{L^2}^2, \end{aligned} \quad (2.5)$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

On the other hand, for every $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$,

$$c \sum_{j \geq -1} \frac{1}{(2^j)^2} \|p \chi_j u\|_{L^2}^2 \leq 4c \|pu\|_{L^2}^2 \leq 8c \operatorname{Re} \langle u, K_V u \rangle \leq 4c (\|u\|_{L^2}^2 + \|K_V u\|_{L^2}^2). \quad (2.6)$$

Collecting the estimates (2.2), (2.5) and (2.6), we establish the desired inequality (2.3). \square

2.2 Localisation in a fixed Shell

Lemma 2.3. *Let $V(q)$ be an homogeneous function in $\mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\})$ of degree r and assume $j \in \mathbb{Z}$. Given $u_j \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$, one has*

$$\|K_V u_j\|_{L^2(\mathbb{R}^{2d})} = \|K_{j,V} v_j\|_{L^2(\mathbb{R}^{2d})},$$

where the operator $K_{j,V}$ is defined by

$$K_{j,V} = \frac{1}{2^j} p \partial_q - (2^j)^{r-1} \partial_q V(q) \partial_p + O_p, \quad (2.7)$$

and $v_j(q, p) = 2^{\frac{jd}{2}} u_j(2^j q, p)$.

In particular when u_j is supported in $\{q \in \mathbb{R}^d, 2^j \frac{3}{4} \leq |q| \leq 2^j \frac{8}{3}\}$, the support of v_j is a fixed shell $\bar{\mathcal{C}} = \{q \in \mathbb{R}^d, \frac{3}{4} \leq |q| \leq \frac{8}{3}\}$.

Proof. Let $j \in \mathbb{Z}$ be an index. Assume $u_j \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and state

$$v_j(q, p) = 2^{\frac{jd}{2}} u_j(2^j q, p). \quad (2.8)$$

On the grounds that the function V is homogeneous of degree r we deduce that respectively its gradient $\partial_q V(q)$ is homogeneous of degree $r - 1$. As follows, we can write

$$\begin{aligned} K_V u_j(q, p) &= K_V \left(2^{-\frac{j d}{2}} v_j(2^{-j} q, p) \right) \\ &= 2^{-\frac{j d}{2}} \left((2^{-j} p \partial_q - (2^j)^{r-1} \partial_q V(q) \partial_p + O_p) v_j \right) (2^{-j} q, p) . \end{aligned}$$

Notice that if

$$\text{supp } u_j \subset \left\{ q \in \mathbb{R}^d, 2^j \frac{3}{4} \leq |q| \leq 2^j \frac{8}{3} \right\} ,$$

the cutoff functions v_j , defined in (2.8), are all supported in the fixed shell

$$\bar{\mathcal{C}} = \left\{ q \in \mathbb{R}^d, \frac{3}{4} \leq |q| \leq \frac{8}{3} \right\} .$$

□

Remark 2.4. Assume $j \in \mathbb{N}$. If we introduce a small parameter $h = 2^{-2(r-1)j}$ then the operator $K_{j,V}$, defined in (2.7), can be rewritten as

$$K_{j,V} = \frac{1}{h} \left(\sqrt{h} p (h^{\frac{1}{2} + \frac{1}{2(r-1)}} \partial_q) - \sqrt{h} \partial_q V(q) \partial_p + \frac{h}{2} (-\Delta_p + p^2) \right) .$$

Now owing to a dilation with respect to the velocity variable p , which for $(\sqrt{h} p, \sqrt{h} \partial_p)$ associates $(p, h \partial_p)$, we deduce that the operator $K_{j,V}$ is unitary equivalent to

$$\widehat{K}_{j,V} = \frac{1}{h} \left(p (h^{\frac{1}{2} + \frac{1}{2(r-1)}} \partial_q) - \partial_q V(q) h \partial_p + \frac{1}{2} (-h^2 \Delta_p + p^2) \right) .$$

In particular, taking $r = 2$,

$$\widehat{K}_{j,V} = \frac{1}{h} \left(p (h \partial_q) - \partial_q V(q) h \partial_p + \frac{1}{2} (-h^2 \Delta_p + p^2) \right) ,$$

is clearly a semiclassical operator with respect to the variables q and p . However if $r > 2$, the operator $\widehat{K}_{j,V}$ is semiclassical only with respect to the velocity variable p (since $h^{\frac{1}{2} + \frac{1}{2(r-1)}} > h$). For a polynomial $V(q)$, the case $r = 2$ corresponds to the quadratic situation. Extensive works have been done concerned with this case (see [Hor]/[HiPr]/[Vio]/[Vio1]/[AlVi]/[BNV]).

3 Proof of the main result

In this section we present the proof of Theorem 1.1.

Proof. In the whole proof we denote

$$\bar{\mathcal{C}} = \left\{ q \in \mathbb{R}^d, \frac{3}{4} \leq |q| \leq \frac{8}{3} \right\}.$$

Assume $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and consider a locally finite dyadic partition of unity defined as in (2.1). By Lemma 2.2 (see (2.3)), there is a uniform constant c such that

$$(1 + 4c)\|K_V u\|_{L^2(\mathbb{R}^{2d})}^2 + c\|u\|_{L^2(\mathbb{R}^{2d})}^2 \geq \sum_{j \geq -1} \|K_V u_j\|_{L^2(\mathbb{R}^{2d})}^2. \quad (3.1)$$

where we denote $u_j = \chi_j u$. We obtain by Lemma 2.3 and the estimate (3.1)

$$(1 + 4c)\|K_V u\|_{L^2(\mathbb{R}^{2d})}^2 + c\|u\|_{L^2(\mathbb{R}^{2d})}^2 \geq \sum_{j \geq -1} \|K_{j,V} v_j\|_{L^2(\mathbb{R}^{2d})}^2, \quad (3.2)$$

where the operator

$$K_{j,V} = \frac{1}{2^j} p \partial_q - (2^j)^{r-1} \partial_q V(q) \partial_p + O_p,$$

and $v_j(q, p) = 2^{\frac{id}{2}} u_j(2^j q, p)$. Setting $h = 2^{-2(r-1)j}$, one has

$$K_{j,V} = p(h^{\frac{1}{2(r-1)}} \partial_q) - h^{-\frac{1}{2}} \partial_q V(q) \partial_p + \frac{1}{2}(-\Delta_p + p^2).$$

Now, fix $\nu > 0$ such that

$$\max\left(\frac{1}{6}, \frac{1}{8} + \frac{3}{8(r-1)}\right) < \nu < \frac{1}{4} + \frac{1}{4(r-1)}. \quad (3.3)$$

Such a choice is always possible:

- In the case $r \geq 10$, $\max(\frac{1}{6}, \frac{1}{8} + \frac{3}{8(r-1)})$ equals $\frac{1}{6}$ while $\frac{1}{4} + \frac{1}{4(r-1)}$ is always greater than $\frac{1}{4}$. So we can choose a value ν independent of r between $\frac{1}{6}$ and $\frac{1}{4}$.
- in the case $2 < r < 10$, $\max(\frac{1}{6}, \frac{1}{8} + \frac{3}{8(r-1)})$ equals $\frac{1}{8} + \frac{3}{8(r-1)} < \frac{1}{4} + \frac{1}{4(r-1)}$. Hence, we can choose for example $\nu = \frac{3}{16} + \frac{5}{16(r-1)}$.

Taking $\nu > 0$, satisfying (3.3), we consider a locally finite partition of unity with respect to $q \in \mathbb{R}^d$ given by

$$\begin{aligned} \sum_{k \geq -1} (\theta_{k,h}(q))^2 &= \sum_{k \geq -1} \left(\theta\left(\frac{1}{|\ln(h)|h^\nu} q - q_k\right) \right)^2 \\ &= \sum_{k \geq -1} \left(\theta\left(\frac{1}{|\ln(h)|h^\nu} (q - q_{k,h})\right) \right)^2 = 1, \end{aligned}$$

where for any index k

$$q_{k,h} = |\ln(h)|h^\nu q_k, \quad \text{supp } \theta_{k,h} \subset B(q_{k,h}, |\ln(h)|h^\nu), \quad \theta_{k,h} \equiv 1 \text{ in } B(q_{k,h}, \frac{1}{2}|\ln(h)|h^\nu).$$

Using this partition we get through Lemma 2.2 (see (2.2)),

$$\|K_{j,V}v_j\|_{L^2}^2 \geq \sum_{k \geq -1} \|K_{j,V}\theta_{k,h}v_j\|_{L^2}^2 - |\ln(h)|^{-2}h^{\frac{1}{r-1}-2\nu} \|p\theta_{k,h}v_j\|_{L^2}^2. \quad (3.4)$$

In order to reduce the written expressions we denote in the whole of the proof

$$w_{k,j} = \theta_{k,h}v_j.$$

Taking into account (3.4),

$$\begin{aligned} \|K_{j,V}v_j\|_{L^2}^2 &\geq \sum_{k \geq -1} \|K_{j,V}w_{k,j}\|_{L^2}^2 - |\ln(h)|^{-2}h^{\frac{1}{r-1}-2\nu} \|w_{k,j}\|_{L^2} \|K_{j,V}w_{k,j}\|_{L^2} \\ &\geq \sum_{k \geq -1} \frac{3}{4} \|K_{j,V}w_{k,j}\|_{L^2}^2 - 2|\ln(h)|^{-2}h^{\frac{1}{r-1}-2\nu} \|w_{k,j}\|_{L^2}^2. \end{aligned} \quad (3.5)$$

Notice that in the last inequality we simply use respectively the fact that

$$\|pw_{k,j}\|_{L^2}^2 \leq 2\text{Re}\langle w_{k,j}, K_{j,V}w_{k,j} \rangle \leq \|w_{k,j}\|_{L^2} \|K_{j,V}w_{k,j}\|_{L^2},$$

and the Cauchy inequality with epsilon ($ab \leq \epsilon a^2 + \frac{1}{4\epsilon}b^2$).

From now on, set

$$K_0 = \{q \in \bar{\mathcal{C}}, \quad \partial_q V(q) = 0\}.$$

Clearly, by continuity of the map $q \mapsto \partial_q V(q)$ on the shell $\bar{\mathcal{C}}$ (which is a compact set of \mathbb{R}^d), we deduce the compactness of K_0 .

Since $q \mapsto \frac{\text{Tr}_{-,V}(q)}{1+\text{Tr}_{+,V}(q)}$ is uniformly continuous on any compact neighborhood of K_0 , there exists $\epsilon_1 > 0$ such that

$$d(q, K_0) \leq \epsilon_1 \Rightarrow \frac{\text{Tr}_{-,V}(q)}{1+\text{Tr}_{+,V}(q)} \geq \frac{\epsilon_0}{2}, \quad (3.6)$$

where $\epsilon_0 := \min_{q \in K_0} \frac{\text{Tr}_{-,V}(q)}{1+\text{Tr}_{+,V}(q)}$.

On the other hand, in vue of the definition of K_0 and by continuity of $q \mapsto \partial_q V(q)$ on $\bar{\mathcal{C}}$, there is a constant $\epsilon_2 > 0$ (that depends on ϵ_1) such that

$$\forall q \in \bar{\mathcal{C}}, \quad d(q, K_0) \geq \epsilon_1 \Rightarrow |\partial_q V(q)| \geq \epsilon_2. \quad (3.7)$$

Now let us introduce

$$\Sigma(\epsilon_1) = \{q \in \mathcal{C}, \quad d(q, K_0) \geq \epsilon_1\},$$

$$I(\epsilon_1) = \{k \in \mathbb{Z}, \quad \text{supp } \theta_{k,h} \subset \Sigma(\epsilon_1)\}.$$

In order to establish a subelliptic estimate for $K_{j,V}$, we distinguish the two following cases.

Case 1 $k \notin I(\epsilon_1)$. In this case the support of the cutoff function $\theta_{k,h}$ might intersect the set of zeros of the gradient of V .

Case 2 $k \in I(\epsilon_1)$. Here the gradient of V does not vanish for all q in the support of $\theta_{k,h}$.

The idea is to use, in the suitable situation, either quadratic or linear approximating polynomial \tilde{V} near some $q'_{k,h} \in \text{supp } \theta_{k,h}$ to write

$$\sum_{k \geq -1} \|K_{j,V} w_{k,j}\|_{L^2}^2 \geq \frac{1}{2} \sum_{k \geq -1} \|K_{j,\tilde{V}} w_{k,j}\|_{L^2}^2 - \|(K_{j,V} - K_{j,\tilde{V}}) w_{k,j}\|_{L^2}^2,$$

or equivalently

$$\sum_{k \geq -1} \|K_{j,V} w_{k,j}\|_{L^2}^2 \geq \frac{1}{2} \sum_{k \geq -1} \|K_{j,\tilde{V}} w_{k,j}\|_{L^2}^2 - \left\| \frac{1}{\sqrt{h}} (\partial_q V(q) - \partial_q \tilde{V}(q)) \partial_p w_{k,j} \right\|_{L^2}^2. \quad (3.8)$$

Then based on the estimates written in [BNV], which are valid for the operator $K_{\tilde{V}}$, we deduce a subelliptic estimate for $K_{\tilde{V}}$, after a careful control of the errors which appear in (3.5) and (3.8).

Case 1. In this situation, we use the quadratic approximation near some element $q'_{k,h} \in \text{supp } \theta_{k,h} \cap (\mathbb{R}^d \setminus \Sigma(\epsilon_1))$,

$$V_{k,h}^2(q) = \sum_{|\alpha| \leq 2} \frac{\partial_q^\alpha V(q'_{k,h})}{\alpha!} (q - q'_{k,h})^\alpha.$$

Notice that one has for all $q \in \mathbb{R}^d$,

$$|V(q) - V_{k,h}^2(q)| = \mathcal{O}(|q - q'_{k,h}|^3). \quad (3.9)$$

Accordingly, for every q in the support of $w_{k,j}$,

$$\begin{aligned} |\partial_q V(q) - \partial_q V_{k,h}^2(q)| &= \mathcal{O}(|q - q'_{k,h}|^2) \\ &= \mathcal{O}(|\ln(h)|^2 h^{2\nu}). \end{aligned} \quad (3.10)$$

Combining (3.8) and (3.10), there is a constant $c > 0$ such that

$$\begin{aligned} \sum_{k \geq -1} \|K_{j,V} w_{k,j}\|_{L^2}^2 &\geq \frac{1}{2} \sum_{k \geq -1} \|K_{j,V_{k,h}^2} w_{k,j}\|_{L^2}^2 - c \frac{(|\ln(h)|^2 h^{2\nu})^2}{h} \|\partial_p w_{k,j}\|_{L^2}^2 \\ &\geq \frac{1}{2} \sum_{k \geq -1} \|K_{j,V_{k,h}^2} w_{k,j}\|_{L^2}^2 - c \frac{(|\ln(h)|^2 h^{2\nu})^2}{h} \|w_{k,j}\|_{L^2} \|K_{j,V_{k,h}^2} w_{k,j}\|_{L^2} \\ &\geq \frac{3}{16} \sum_{k \geq -1} \|K_{j,V_{k,h}^2} w_{k,j}\|_{L^2}^2 - 2c \frac{(|\ln(h)|^2 h^{2\nu})^2}{h} \|w_{k,j}\|_{L^2}^2. \end{aligned} \quad (3.11)$$

Putting (3.5) and (3.11) together,

$$\|K_{j,V}v_j\|^2 \geq \frac{9}{64} \sum_{k \geq -1} \|K_{j,V_{k,h}^2} w_{k,j}\|^2 - \frac{3}{2} c \frac{(|\ln(h)|^2 h^{2\nu})^2}{h} \|w_{k,j}\|^2 - 2|\ln(h)|^{-2} h^{\frac{1}{r-1} - 2\nu} \|w_{k,j}\|^2 . \quad (3.12)$$

On the other hand, owing to a change of variables $q'' = qh^{\frac{1}{2(r-1)}}$, one can write

$$\|K_{j,V_{k,h}^2} w_{k,j}\|_{L^2} = \|\tilde{K}_{j,V_{k,h}^2} \tilde{w}_{k,j}\|_{L^2} , \quad (3.13)$$

where the operator $\tilde{K}_{j,V_{k,h}^2}$ reads

$$\begin{aligned} \tilde{K}_{j,V_{k,h}^2} &= p\partial_q - h^{-\frac{1}{2}} \partial_q V_{k,h}^2 (h^{\frac{1}{2(r-1)}} q) \partial_p + \frac{1}{2} (-\Delta_p + p^2) \\ &= p\partial_q - \underbrace{h^{-\frac{1}{2} + \frac{1}{2(r-1)}}}_{=:H} \partial_q V_{k,h}^2(q) \partial_p + \frac{1}{2} (-\Delta_p + p^2) , \end{aligned}$$

and

$$w_{k,j}(q, p) = \frac{1}{h^{\frac{d}{4(r-1)}}} \tilde{w}\left(\frac{q}{h^{\frac{1}{2(r-1)}}}, p\right) .$$

In the rest of the proof we denote

$$H = h^{-\frac{1}{2}} h^{\frac{1}{2(r-1)}} .$$

From now on assume $j \in \mathbb{N}$. In view of (3.6), $\text{Tr}_{-,V_{k,h}^2} = \text{Tr}_{-,V}(q'_{k,h}) \neq 0$. Hence by (1.3),

$$\|\tilde{K}_{j,V_{k,h}^2} \tilde{w}_{k,j}\|_{L^2}^2 \geq c \frac{1 + H \text{Tr}_{-,V_{k,h}^2}}{\log(2 + H \text{Tr}_{-,V_{k,h}^2})^2} \|\tilde{w}_{k,j}\|_{L^2}^2 . \quad (3.14)$$

Or samely

$$\|\tilde{K}_{j,V_{k,h}^2} \tilde{w}_{k,j}\|_{L^2}^2 \geq c \frac{1 + H \text{Tr}_{-,V}(q'_{k,h})}{\log(2 + H \text{Tr}_{-,V}(q'_{k,h}))^2} \|\tilde{w}_{k,j}\|_{L^2}^2 . \quad (3.15)$$

Using once more (3.6),

$$\text{Tr}_{-,V}(q'_{k,h}) \geq \frac{\epsilon_0}{2} (1 + \text{Tr}_{+,V}(q'_{k,h})) , \quad (3.16)$$

where we remind that $\epsilon_0 = \min_{q \in K_0} \frac{\text{Tr}_{-,V}(q)}{1 + \text{Tr}_{+,V}(q)}$. Consequently

$$|\text{Hess } V(q'_{k,h})| \geq \text{Tr}_{-,V}(q'_{k,h}) \geq \frac{\epsilon_0}{2} , \quad (3.17)$$

and

$$\begin{aligned}
\mathrm{Tr}_{-,V}(q'_{k,h}) &\geq \frac{1}{2}\mathrm{Tr}_{-,V}(q'_{k,h}) + \frac{\epsilon_0}{4}(1 + \mathrm{Tr}_{+,V}(q'_{k,h})) \\
&\geq \frac{1}{2}\min(1, \frac{\epsilon_0}{2})(\mathrm{Tr}_{-,V}(q'_{k,h}) + \mathrm{Tr}_{+,V}(q'_{k,h})) \\
&\geq \frac{1}{2}\min(1, \frac{\epsilon_0}{2})|\mathrm{Hess} V(q'_{k,h})|.
\end{aligned} \tag{3.18}$$

Furthermore by continuity of the map $q \mapsto \mathrm{Tr}_{-,V}(q)$ on the compact set $\bar{\mathcal{C}}$, there exists a constant $\epsilon_3 > 0$ such that $\mathrm{Tr}_{-,V}(q) \leq \epsilon_3$ for all $q \in \bar{\mathcal{C}}$. Hence

$$\frac{\epsilon_0}{2} \leq \mathrm{Tr}_{-,V}(q'_{k,h}) \leq \epsilon_3. \tag{3.19}$$

From (3.15), (3.18) and (3.19),

$$\|\tilde{K}_{j,V_{k,h}^2} \tilde{w}_{k,j}\|_{L^2}^2 \geq c \frac{H}{\log(H)^2} \|\tilde{w}_{k,j}\|_{L^2}^2.$$

It follows from the above inequality and (3.13),

$$\|K_{j,V_{k,h}^2} w_{k,j}\|_{L^2}^2 \geq c \frac{H}{\log(H)^2} \|w_{k,j}\|_{L^2}^2. \tag{3.20}$$

Now using the estimate (3.20), we should control the errors coming from the partition of unity and the quadratic approximation. For this reason, notice that our choice of exponent ν in (3.3) implies

$$\begin{cases} \frac{(|\ln(h)|^2 h^{2\nu})^2}{h} \ll \frac{H}{\log(H)^2} \\ |\ln(h)|^{-2} h^{\frac{1}{r-1}-2\nu} \ll \frac{H}{\log(H)^2}. \end{cases}$$

As a result, collectting the estimates (3.12) and (3.20), we deduce the existence of a constant $c > 0$ such that

$$\|K_{j,V} v_j\|_{L^2}^2 \geq c \sum_{k \geq -1} \|K_{j,V_{k,h}^2} w_{k,j}\|_{L^2}^2. \tag{3.21}$$

Via (1.2), there is a constant $c > 0$ so that

$$\begin{aligned}
\|\tilde{K}_{j,V_{k,h}^2} \tilde{w}_{k,j}\|^2 + (1 + 10c)H|\mathrm{Hess} V(q'_{k,h})| \|\tilde{w}_{k,j}\|^2 &\geq c \left(\|O_p \tilde{w}_{k,j}\|^2 + \|\langle D_q \rangle^{\frac{2}{3}} \tilde{w}_{k,j}\|^2 \right. \\
&\quad \left. + H|\mathrm{Hess} V(q'_{k,h})| \|\tilde{w}_{k,j}\|^2 \right).
\end{aligned} \tag{3.22}$$

Hence using the reverse change of variables $q'' = \frac{q}{h^{\frac{1}{2(\tau-1)}}}$, we obtain in view of the above estimate and (3.13),

$$\begin{aligned} \|K_{j,V_{k,h}^2} w_{k,j}\|^2 + (1+10c)H|\text{Hess } V(q'_{k,h})| \|w_{k,j}\|^2 &\geq c \left(\|O_p w_{k,j}\|^2 + \|\langle h^{\frac{1}{2(\tau-1)}} D_q \rangle^{\frac{2}{3}} w_{k,j}\|^2 \right. \\ &\quad \left. + H|\text{Hess } V(q'_{k,h})| \|w_{k,j}\|^2 \right). \end{aligned} \quad (3.23)$$

Or by (3.18) and (3.19),

$$\frac{\epsilon_0}{2} \leq |\text{Hess } V(q'_{k,h})| \leq \frac{2\epsilon_3}{\min(1, \frac{\epsilon_0}{2})}, \quad (3.24)$$

Putting (3.23) and (3.24) together, there is a constant $c > 0$ so that

$$\begin{aligned} \|K_{j,V_{k,h}^2} w_{k,j}\|^2 + H\|w_{k,j}\|^2 &\geq c \left(\|O_p w_{k,j}\|^2 + \|\langle h^{\frac{1}{2(\tau-1)}} D_q \rangle^{\frac{2}{3}} w_{k,j}\|^2 \right. \\ &\quad \left. + H\|w_{k,j}\|^2 + \|\langle H|\text{Hess } V(q'_{k,h})| \rangle^{\frac{1}{2}} w_{k,j}\|^2 \right). \end{aligned} \quad (3.25)$$

On the other hand, for all $q \in \text{supp } w_{k,j}$,

$$|\text{Hess } V(q) - \text{Hess } V(q'_{k,h})| = \mathcal{O}(|q - q'_{k,h}|) = \mathcal{O}(|\ln(h)|h^\nu) \quad (3.26)$$

Therefore by (3.24) and (3.26), we obtain for every $q \in \text{supp } w_{k,j}$ and all j sufficiently large.

$$\frac{1}{2}|\text{Hess } V(q'_{k,h})| \leq |\text{Hess } V(q)| \leq \frac{3}{2}|\text{Hess } V(q'_{k,h})|. \quad (3.27)$$

From (3.25) and (3.47), there exists a constant $c > 0$ so that

$$\begin{aligned} \|K_{j,V_{k,h}^2} w_{k,j}\|^2 + H\|w_{k,j}\|^2 &\geq c \left(\|O_p w_{k,j}\|^2 + \|\langle h^{\frac{1}{2(\tau-1)}} D_q \rangle^{\frac{2}{3}} w_{k,j}\|^2 \right. \\ &\quad \left. + H\|w_{k,j}\|^2 + \|\langle H|\text{Hess } V(q)| \rangle^{\frac{1}{2}} w_{k,j}\|^2 \right), \end{aligned} \quad (3.28)$$

is valid for all j large enough.

Furthermore, by continuity of the map $q \mapsto |\partial_q V(q)|^{\frac{4}{3}}$ on the fixed shell $\bar{\mathcal{C}}$, for all $q \in \text{supp } w_{k,j}$

$$\frac{1}{4}H \geq c|h^{-\frac{1}{2}}\partial_q V(q)|^{\frac{4}{3}}, \quad (3.29)$$

holds for all j sufficiently large.

In such a way, considering (3.28) and (3.29)

$$\begin{aligned} \|K_{j,V_{k,h}^2} w_{k,j}\|^2 + (2+H)\|w_{k,j}\|^2 &\geq c \left(\|O_p w_{k,j}\|^2 + \|\langle h^{\frac{1}{2(\tau-1)}} D_q \rangle^{\frac{2}{3}} w_{k,j}\|^2 + (2+H)\|w_{k,j}\|^2 \right. \\ &\quad \left. + \|\langle H|\text{Hess } V(q)| \rangle^{\frac{1}{2}} w_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}}\partial_q V(q)| \rangle^{\frac{2}{3}} w_{k,j}\|^2 \right). \end{aligned} \quad (3.30)$$

Putting (3.20) and (3.30) together,

$$\begin{aligned} \|K_{j,V_{k,h}^2} w_{k,j}\|^2 &\geq c \left(\left\| \frac{O_p}{\log(2+H)} w_{k,j} \right\|^2 + \left\| \frac{\langle h^{\frac{1}{2(r-1)}} D_q \rangle^{\frac{2}{3}}}{\log(2+H)} w_{k,j} \right\|^2 + \left\| \frac{(2+H)^{\frac{1}{2}}}{\log(2+H)} w_{k,j} \right\|^2 \right. \\ &\quad \left. + \left\| \frac{\langle H |\text{Hess } V(q)| \rangle^{\frac{1}{2}}}{\log(2+H)} w_{k,j} \right\|^2 + \left\| \frac{\langle h^{-\frac{1}{2}} |\partial_q V(q)| \rangle^{\frac{2}{3}}}{\log(2+H)} w_{k,j} \right\|^2 \right), \end{aligned} \quad (3.31)$$

holds for all $j \geq j_0$, for some $j_0 \geq 1$ large enough.

Now let us collect the finite remaining terms for $-1 \leq j \leq j_0$. After recalling $h = 2^{-j}$ and $H = h^{-\frac{1}{2} + \frac{1}{2(r-1)}}$ we define

$$\begin{aligned} c_V^{(1)} &= \max_{-1 \leq j \leq j_0} \left[A_{V_{k,h}^2} + \sup_{q \in \text{supp}(x_j \theta_{k,h})} \left(\langle H |\text{Hess } V(q)| \rangle + \langle h^{-\frac{1}{2}} |\partial_q V(q)| \rangle^{4/3} \right) \right. \\ &\quad \left. + \frac{(2+H)}{\log(2+H)^2} + \frac{3}{2} c \frac{(|\ln(h)|^2 h^{2\nu})^2}{h} + 2 |\ln(h)|^{-2} h^{\frac{1}{r-1} - 2\nu} \right]. \end{aligned}$$

From the lower bound (1.2), we deduce the existence of a constant $c > 0$ so that

$$\begin{aligned} \frac{9}{64} \|K_{V_{k,h}^2} w_{k,j}\|^2 + (c_V^{(1)} - \frac{3}{2} c \frac{(|\ln(h)|^2 h^{2\nu})^2}{h} - 2 |\ln(h)|^{-2} h^{\frac{1}{r-1} - 2\nu}) \|w_{k,j}\|^2 \\ \geq c \left(\|O_p w_{k,j}\|^2 + \left\| \frac{\langle h^{\frac{1}{2(r-1)}} D_q \rangle^{2/3}}{\log(2+H)} w_{k,j} \right\|^2 + \left\| \frac{\langle h^{-\frac{1}{2}} |\partial_q V(q)| \rangle^{2/3}}{\log(2+H)} w_{k,j} \right\|^2 \right. \\ \left. + \left\| \frac{\langle H |\text{Hess } V(q)| \rangle^{1/2}}{\log(2+H)} w_{k,j} \right\|^2 + \left\| \frac{(2+H)^{\frac{1}{2}}}{\log(2+H)} w_{k,j} \right\|^2 \right), \end{aligned} \quad (3.32)$$

holds for all $-1 \leq j \leq j_0$.

Finally, collecting (3.21), (3.31) and (3.32),

$$\begin{aligned} \|K_{j,V} v_j\|^2 + c_V^{(2)} \|v_j\|^2 &\geq c \sum_{k \notin I(\epsilon_1)} \left(\left\| \frac{O_p}{\log(2+H)} w_{k,j} \right\|^2 + \left\| \frac{\langle h^{\frac{1}{2(r-1)}} D_q \rangle^{\frac{2}{3}}}{\log(2+H)} w_{k,j} \right\|^2 + \left\| \frac{(2+H)^{\frac{1}{2}}}{\log(2+H)} w_{k,j} \right\|^2 \right. \\ &\quad \left. + \left\| \frac{\langle H |\text{Hess } V(q)| \rangle^{\frac{1}{2}}}{\log(2+H)} w_{k,j} \right\|^2 + \left\| \frac{\langle h^{-\frac{1}{2}} |\partial_q V(q)| \rangle^{\frac{2}{3}}}{\log(2+H)} w_{k,j} \right\|^2 \right), \end{aligned} \quad (3.33)$$

is valid for every $j \geq -1$.

Case 2. We consider in this case the linear approximating polynomial

$$V_{k,h}^1(q) = \sum_{|\alpha|=1} \frac{\partial_q^\alpha V(q_{k,h})}{\alpha!} (q - q_{k,h})^\alpha.$$

Note that for any $q \in \mathbb{R}^d$,

$$|V(q) - V_{k,h}^1(q)| = \mathcal{O}(|q - q_{k,h}|^2), \quad (3.34)$$

and for every $q \in \text{supp } w_{k,j}$,

$$\begin{aligned} |\partial_q V(q) - \partial_q V_{k,h}^1(q)| &= \mathcal{O}(|q - q_{k,h}|) \\ &= \mathcal{O}(|\ln(h)|h^\nu). \end{aligned} \quad (3.35)$$

Due to (3.8) and (3.35),

$$\begin{aligned} \sum_{k \geq -1} \|K_{j,V} w_{k,j}\|^2 &\geq \frac{1}{2} \sum_{k \geq -1} \|K_{j,V_{k,h}^1} w_{k,j}\|^2 - c \frac{(|\ln(h)|h^\nu)^2}{h} \|\partial_p w_{k,j}\|^2 \\ &\geq \frac{1}{2} \sum_{k \geq -1} \|K_{j,V_{k,h}^1} w_{k,j}\|^2 - c \frac{(|\ln(h)|h^\nu)^2}{h} \|w_{k,j}\| \|K_{j,V_{k,h}^1} w_{k,j}\| \\ &\geq \frac{3}{16} \sum_{k \geq -1} \|K_{j,V_{k,h}^1} w_{k,j}\|^2 - 2c \frac{(|\ln(h)|h^\nu)^2}{h} \|w_{k,j}\|^2. \end{aligned} \quad (3.36)$$

Assembling (3.5) and (3.36),

$$\|K_{j,V} v_j\|^2 \geq \frac{9}{64} \sum_{k \geq -1} \|K_{j,V_{k,h}^1} w_{k,j}\|^2 - \frac{3}{2} c \frac{(|\ln(h)|^2 h^{2\nu})^2}{h} \|w_{k,j}\|^2 - 2|\ln(h)|^{-2} h^{\frac{1}{r-1}-2\nu} \|w_{k,j}\|^2. \quad (3.37)$$

Additionally, one has

$$\|K_{j,V_{k,h}^1} w_{k,j}\|_{L^2} = \|\tilde{K}_{j,V_{k,h}^1} \tilde{w}_{k,j}\|_{L^2}, \quad (3.38)$$

where the operator $\tilde{K}_{j,V_{k,h}^1}$ is given by

$$\begin{aligned} \tilde{K}_{j,V_{k,h}^1} &= p\partial_q - h^{-\frac{1}{2}} \partial_q V_{k,h}^1(h^{\frac{1}{2(r-1)}} q) \partial_p + \frac{1}{2}(-\Delta_p + p^2) \\ &= p\partial_q - h^{-\frac{1}{2}} \partial_q V(q_{k,h}) \partial_p + \frac{1}{2}(-\Delta_p + p^2), \end{aligned} \quad (3.39)$$

and

$$w_{k,j}(q, p) = \frac{1}{h^{\frac{d}{4(r-1)}}} \tilde{w}\left(\frac{q}{h^{\frac{1}{2(r-1)}}}, p\right). \quad (3.40)$$

Now, in order to absorb the errors in (3.37) we need the following estimates showed in [BNV] (see (1.3)),

$$\|\tilde{K}_{j,V_{k,h}^1} \tilde{w}_{k,j}\|_{L^2}^2 \geq c \|(h^{-\frac{1}{2}} |\partial_q V(q_{k,h})|)^{\frac{2}{3}} \tilde{w}_{k,j}\|_{L^2}^2. \quad (3.41)$$

From now on assume $j \in \mathbb{N}$. Taking into account (3.7) and (3.41),

$$\|K_{j,V_{k,h}^1} \tilde{w}_{k,j}\|_{L^2}^2 \geq c \|(h^{-\frac{1}{2}})^{\frac{2}{3}} \tilde{w}_{k,j}\|_{L^2}^2. \quad (3.42)$$

Owing to (3.38) and (3.41),

$$\|K_{j,V_{k,h}^1} w_{k,j}\|^2 \geq c \|(h^{-\frac{1}{2}})^{\frac{2}{3}} w_{k,j}\|^2. \quad (3.43)$$

Note that one has Therefore, combining (3.37) and (3.43), there is a constant $c > 0$ so that

$$\|K_{j,V} v_j\|^2 \geq c \sum_{k \geq -1} \|K_{j,V_{k,h}^1} w_{k,j}\|^2. \quad (3.44)$$

Using once more [BNV] (see (1.2)), there is a constant $c > 0$ such that

$$\|\tilde{K}_{j,V_{k,h}^1} \tilde{w}_{k,j}\|^2 \geq c \left(\|O_p \tilde{w}_{k,j}\|^2 + \|\langle D_q \rangle^{\frac{2}{3}} \tilde{w}_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}} |\partial_q V(q_{k,h})| \rangle^{\frac{2}{3}} \tilde{w}_{k,j}\|^2 \right). \quad (3.45)$$

As a consequence of (3.38) and (3.45),

$$\|K_{j,V_{k,h}^1} w_{k,j}\|^2 \geq c \left(\|O_p w_{k,j}\|^2 + \|\langle h^{\frac{1}{2(r-1)}} D_q \rangle^{\frac{2}{3}} w_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}} |\partial_q V(q_{k,h})| \rangle^{\frac{2}{3}} w_{k,j}\|^2 \right). \quad (3.46)$$

By (3.7) and (3.35),

$$\frac{1}{2} |\partial_q V(q)| \leq |\partial_q V(q_{k,h})| \leq \frac{3}{2} |\partial_q V(q)|, \quad (3.47)$$

holds for all $q \in \text{supp } w_{k,j}$ and any j large. Then, it follows from (3.47) and (3.46),

$$\|K_{j,V_{k,h}^1} w_{k,j}\|^2 \geq c \left(\|O_p w_{k,j}\|^2 + \|\langle h^{\frac{1}{2(r-1)}} D_q \rangle^{\frac{2}{3}} w_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}} |\partial_q V(q)| \rangle^{\frac{2}{3}} w_{k,j}\|^2 \right). \quad (3.48)$$

Or in this case, in vue of the (3.7), one has $|\partial_q V(q)| \geq \epsilon_2$ for all $q \in \text{supp } w_{k,j}$. Hence it results from the above inequality

$$\|K_{j,V_{k,h}^1} w_{k,j}\|^2 \geq c \left(\|O_p w_{k,j}\|^2 + \|\langle h^{\frac{1}{2(r-1)}} D_q \rangle^{\frac{2}{3}} w_{k,j}\|^2 + \|(h^{-\frac{1}{2}})^{\frac{2}{3}} w_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}} |\partial_q V(q)| \rangle^{\frac{2}{3}} w_{k,j}\|^2 \right). \quad (3.49)$$

Furthermore, by continuity of $q \mapsto |\text{Hess } V(q)|$ on the compact set $\bar{\mathcal{C}}$, one has for all $q \in \text{supp } w_{k,j}$ and any j large

$$\frac{1}{4} (h^{-\frac{1}{2}})^{\frac{4}{3}} \geq c H |\text{Hess } V(q)|. \quad (3.50)$$

Then by the above inequality and (3.49), we get

$$\begin{aligned} \|K_{j,V_{k,h}^1} w_{k,j}\|^2 &\geq c \left(\|O_p w_{k,j}\|^2 + \|\langle h^{\frac{1}{2(r-1)}} D_q \rangle^{\frac{2}{3}} w_{k,j}\|^2 + \|(2+H)^{\frac{1}{2}} w_{k,j}\|^2 \right. \\ &\quad \left. + \|\langle H |\text{Hess } V(q)| \rangle^{\frac{1}{2}} w_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}} |\partial_q V(q)| \rangle^{\frac{2}{3}} w_{k,j}\|^2 \right), \end{aligned} \quad (3.51)$$

for every $j \geq j_1$ for some $j_1 \geq 1$ large. Now set

$$c_V^{(3)} = \max_{-1 \leq j \leq j_1} \left[\sup_{q \in \text{supp}(\chi_j \theta_{k,h})} \left(\langle H | \text{Hess } V(q) \rangle + \langle h^{-\frac{1}{2}} | \partial_q V(q) \rangle^{4/3} \right) + \frac{(2+H)}{\log(2+H)^2} + \frac{3}{2} c \frac{(|\ln(h)|^2 h^{2\nu})^2}{h} + 2 |\ln(h)|^{-2} h^{\frac{1}{r-1} - 2\nu} \right].$$

Seeing (1.2), we deduce the existence of a constant $c > 0$ so that

$$\begin{aligned} \frac{9}{64} \|K_{V_{k,h}^1} w_{k,j}\|^2 + (c_V^{(3)} - \frac{3}{2} c \frac{(|\ln(h)|^2 h^{2\nu})^2}{h} - 2 |\ln(h)|^{-2} h^{\frac{1}{r-1} - 2\nu}) \|w_{k,j}\|^2 \\ \geq c (\|O_p w_{k,j}\|^2 + \|\langle h^{\frac{1}{2(r-1)}} D_q \rangle^{2/3} w_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}} | \partial_q V(q) \rangle^{2/3} w_{k,j}\|^2 \\ + \|\langle H | \text{Hess } V(q) \rangle^{1/2} w_{k,j}\|^2 + \|\frac{(2+H)^{\frac{1}{2}}}{\log(2+H)} w_{k,j}\|^2), \end{aligned} \quad (3.52)$$

holds for all $-1 \leq j \leq j_1$.

Thus, combining the estimates (3.44), (3.51) and (3.52)

$$\begin{aligned} \|K_{j,V} v_j\|^2 + c_V^{(4)} \|v_j\|^2 \geq c \sum_{k \in I(\epsilon_1)} \left(\|O_p w_{k,j}\|^2 + \|\langle h^{\frac{1}{2(r-1)}} D_q \rangle^{\frac{2}{3}} w_{k,j}\|^2 + \|\frac{(2+H)^{\frac{1}{2}}}{\log(2+H)} w_{k,j}\|^2 \right. \\ \left. + \|\langle H | \text{Hess } V(q) \rangle^{\frac{1}{2}} w_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}} | \partial_q V(q) \rangle^{\frac{2}{3}} w_{k,j}\|^2 \right), \end{aligned} \quad (3.53)$$

holds for all $j \geq -1$.

In conclusion, in view of (3.33) and (3.53), there is a constant $c > 0$ such that

$$\begin{aligned} \|K_{j,V} v_j\|^2 + c_V^{(5)} \|v_j\|^2 \geq c \sum_{k \geq -1} \left(\|\frac{O_p}{\log(2+H)} w_{k,j}\|^2 + \|\frac{\langle h^{\frac{1}{2(r-1)}} D_q \rangle^{\frac{2}{3}}}{\log(2+H)} w_{k,j}\|^2 + \|\frac{(2+H)^{\frac{1}{2}}}{\log(2+H)} w_{k,j}\|^2 \right. \\ \left. + \|\frac{\langle H | \text{Hess } V(q) \rangle^{\frac{1}{2}}}{\log(2+H)} w_{k,j}\|^2 + \|\frac{\langle h^{-\frac{1}{2}} | \partial_q V(q) \rangle^{\frac{2}{3}}}{\log(2+H)} w_{k,j}\|^2 \right), \end{aligned} \quad (3.54)$$

holds for all $j \geq -1$.

Finally setting $L(s) = \frac{s+1}{\log(s+1)}$ for all $s \geq 1$, notice that there is a constant $c > 0$ such that for all $x \geq 1$,

$$\inf_{t \geq 2} \frac{x}{\log(t)} + t \geq \frac{1}{c} L(x). \quad (3.55)$$

After setting the quantities

$$\begin{aligned}\Lambda_{1,j} &= \frac{O_p}{\log(2+H)}, & \Lambda_{2,j} &= \frac{\langle H|\text{Hess } V(q)|\rangle^{1/2}}{\log(2+H)}, & \Lambda_{3,j} &= \frac{\langle h^{-\frac{1}{2}}|\partial_q V(q)|\rangle^{\frac{2}{3}}}{\log(2+H)}, \\ \Lambda_{4,j} &= \frac{2+H}{\log(2+H)^2}, & \Lambda_{5,j} &= \frac{\langle h^{\frac{1}{2(r-1)}}D_q\rangle^{\frac{2}{3}}}{\log(2+H)},\end{aligned}$$

we get through the estimate (3.55), for every $j, k \geq -1$

$$\|\Lambda_{1,j}w_{k,j}\|_{L^2}^2 + \frac{1}{4}\|\Lambda_{4,j}w_{k,j}\|_{L^2}^2 \geq c_1\|L(O_p)w_{k,j}\|_{L^2}^2,$$

$$\|\Lambda_{5,j}w_{k,j}\|_{L^2}^2 + \frac{1}{4}\|\Lambda_{4,j}w_{k,j}\|_{L^2}^2 \geq c_2\|L(\langle h^{\frac{1}{2(r-1)}}D_q\rangle^{\frac{2}{3}})w_{k,j}\|_{L^2}^2,$$

$$\|\Lambda_{2,j}w_{k,j}\|_{L^2}^2 + \frac{1}{4}\|\Lambda_{4,j}w_{k,j}\|_{L^2}^2 \geq c_3\|L(\langle H|\text{Hess } V(q)|\rangle^{\frac{1}{2}})w_{k,j}\|_{L^2}^2,$$

$$\|\Lambda_{3,j}w_{k,j}\|_{L^2}^2 + \frac{1}{4}\|\Lambda_{4,j}w_{k,j}\|_{L^2}^2 \geq c_4\|L(\langle h^{-\frac{1}{2}}|\partial_q V(q)|\rangle^{\frac{2}{3}})w_{k,j}\|_{L^2}^2.$$

From the above estimates and (3.54),

$$\begin{aligned}\|K_{j,V}v_j\|^2 + c_V^{(6)}\|v_j\|^2 &\geq c \sum_{k \geq -1} \left(\|L(O_p)w_{k,j}\|^2 + \|L(\langle h^{\frac{1}{2(r-1)}}D_q\rangle^{\frac{2}{3}})w_{k,j}\|^2 \right. \\ &\quad \left. + \|L(\langle H|\text{Hess } V(q)|\rangle^{\frac{1}{2}})w_{k,j}\|^2 + \|L(\langle h^{-\frac{1}{2}}|\partial_q V(q)|\rangle^{\frac{2}{3}})w_{k,j}\|^2 \right).\end{aligned}\tag{3.56}$$

Therefore in view of Lemma 2.5 in [Ben] conjugated by the unitary transformation of the change of scale,

$$\begin{aligned}\|K_{j,V}v_j\|^2 + c_V^{(7)}\|v_j\|^2 &\geq c \left(\|L(O_p)v_j\|^2 + \|L(\langle h^{\frac{1}{2(r-1)}}D_q\rangle^{\frac{2}{3}})v_j\|^2 \right. \\ &\quad \left. + \|L(\langle H|\text{Hess } V(q)|\rangle^{\frac{1}{2}})v_j\|^2 + \|L(\langle h^{-\frac{1}{2}}|\partial_q V(q)|\rangle^{\frac{2}{3}})v_j\|^2 \right),\end{aligned}\tag{3.57}$$

or equivalently

$$\begin{aligned}\|K_V u_j\|^2 + c_V^{(7)}\|u_j\|^2 &\geq c \left(\|L(O_p)u_j\|^2 + \|L(\langle D_q\rangle^{\frac{2}{3}})u_j\|^2 \right. \\ &\quad \left. + \|L(\langle \text{Hess } V(q)|\rangle^{\frac{1}{2}})u_j\|^2 + \|L(\langle \partial_q V(q)\rangle^{\frac{2}{3}})u_j\|^2 \right),\end{aligned}\tag{3.58}$$

for every $j \geq -1$.

Therefore, combining the last estimate and (3.1), there is a constant $C_V > 1$ so that

$$\begin{aligned} \|K_V u\|_{L^2(\mathbb{R}^{2d})}^2 + C_V \|u\|_{L^2(\mathbb{R}^{2d})}^2 &\geq \frac{1}{C_V} \left(\|L(O_p)u\|^2 + \|L(\langle D_q \rangle^{\frac{2}{3}})u\|^2 \right. \\ &\quad \left. + \|L(\langle \text{Hess } V(q) \rangle^{\frac{1}{2}})u\|^2 + \|L(\langle \partial_q V(q) \rangle^{\frac{2}{3}})u\|^2 \right) \end{aligned} \quad (3.59)$$

holds for all $u \in C_0^\infty(\mathbb{R}^{2d})$. □

Acknowledgement I would like to thank my supervisor Francis Nier for his support and guidance throughout this work.

References

- [AlVi] A. Aleman, J. Viola: On weak and strong solution operators for evolution equations coming from quadratic operators. *J. Spectr. Theory* 8, no. 1, 33-121, (2018).
- [BCD] H. Bahouri, J.-Y. Chemin, R. Danchin: Fourier analysis and nonlinear partial differential equations. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, 343. Springer, Heidelberg, xvi+523 pp, (2011).
- [Ben] M. Ben Said: Global subelliptic estimates for Kramers-Fokker-Planck operators with some class of polynomial. *ArXiv:1812.06645*, (2018).
- [BNV] M. Ben Said, F. Nier, J. Viola : Quaternionic structure and analysis of some Kramers-Fokker-Planck. *Arxiv1807.01881*, (2018).
- [HeNi] B. Helffer, F. Nier: *Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians*. *Lecture Notes in Mathematics*, 1862. Springer-Verlag, x+209 pp, (2005).
- [HeNi1] B. Helffer, F. Nier: Criteria to the Poincare Inequality Associated with Dirichlet Forms in \mathbb{R}^d , $d \geq 2$. *Int. Math. Res. Not.*, 22, 1199-1223, (2003).
- [HiPr] M. Hitrik, K. Pravda-Starov: Spectra and semigroup smoothing for non-elliptic quadratic operators. *Math. Ann.* 344, no. 4, 801846, (2009).
- [Hor] L. Hörmander: Symplectic classification of quadratic forms, and general Mehler formulas. *Math. Z.*, 219:413–449, (1995).
- [Li] W.-X. Li: Global hypoellipticity and compactness of resolvent for Fokker-Planck operator. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 11 , no. 4, 789815, (2012).
- [Li2] W.-X. Li: Compactness criteria for the resolvent of Fokker-Planck operator. prepublication. *ArXiv1510.01567* (2015).

[Vio] J. Viola: Spectral projections and resolvent bounds for partially elliptic quadratic differential operators. *J. Pseudo-Diff. Oper. Appl.* 4, no. 2, 145–221, (2013).

[Vio1] J. Viola: The elliptic evolution of non-self-adjoint degree-2 Hamiltonians. *ArXiv* 1701.00801, (2017).