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ON THE PADOVAN SEQUENCE

ALAIN FAISANT

ABSTRACT. The aim of this article is to give some properties of the so-called Padovan sequence $(T_n)_{n \geq 0}$ defined by

$$T_{n+3} = T_{n+1} + T_n \quad \text{for all } n \in \mathbf{N}, \quad T_1 = T_2 = T_3 = 1$$

that is divisibility properties, periods, identities.

1. INTRODUCTION

Let $(T_n)_{n \geq 0}$ be the Padovan sequence, recursively defined by

$$T_{n+3} = T_{n+1} + T_n, \quad n \geq 0$$

and the initial values $T_0 = 0, T_1 = 1, T_2 = 1$. In this paper we consider a prime p , and give information on the index ω_p of first occurrence of p as divisor of any T_n in the sequence. The principal result is that $\omega_p \leq t_p \leq p^{r_p} - 1$, where t_p is the period modulo p of $(T_n)_{n \geq 0}$, and $r_p = 1, 2, 3$ the degree of the extension R_p/\mathbb{F}_p , R_p being the splitting field of the polynomial $T(X) = X^3 - X - 1 \pmod{p}$ associated to the Padovan sequence. We give also some technical precisions (using in particular class field theory), examples, and identities relative to the sequence $(T_n)_{n \geq 0}$. Many questions remain open.

2. DEFINITIONS

We consider the sequences recursively defined by $x_{n+3} = x_{n+1} + x_n$, $n \geq 0$.

The choice of (x_0, x_1, x_2) determines the sequence : usually the Cordonnier-Padovan sequence is determined by the conditions $x_1 = x_2 = x_3 = 1$, corresponding to the choice $x_0 = 0$ denoted here by $(T_n)_{n \geq 0}$: the Padovan sequence. It is convenient for us to extend $T_{-1} = 1, T_{-2} = 0, T_{-3} = 0$. It appears as sequence A000931 in Sloane's Online Encyclopedia of Integer Sequences [4].

This sequence is also cited in [2] as $(R_n)_{n \geq 0}$: $R_n = T_{n-2}$.

The other choice $(3, 0, 2)$ gives the Perrin sequence $(P_n)_{n \geq 0}$ [1].

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The first few terms of the sequence are

n	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
T_n	0	0	1	0	1	1	1	2	2	3	4	5	7	9	12	16	21	28	37	49	65	86	114	151

The associated generating series is

$$\sum_{n \geq 0} T_n X^n = \frac{1 - X}{1 - X^2 - X^3}$$

Let K be a field, and consider

$$\mathcal{T}_K = \{(x_n)_{n \geq 0} : x_n \in K, x_{n+3} = x_{n+1} + x_n, \forall n \geq 0\}$$

This is a 3-dimensional K -vector space, with explicit evident isomorphism :

$$\begin{array}{ccc} K^3 & \xrightarrow{\varphi} & \mathcal{T}_K \\ (x_0, x_1, x_2) & \mapsto & (x_n)_{n \geq 0} \end{array}$$

The K^3 -basis $\{(0, 1, 0), (1, 0, 1), (0, 1, 1)\}$ gives the \mathcal{T}_K -basis $\{(T_{n-2})_{n \geq 0}, (T_{n-1})_{n \geq 0}, (T_n)_{n \geq 0}\}$.

3. BASIS OF \mathcal{T}_K

We want to find a basis of \mathcal{T}_K of geometrical progressions $(x^n)_{n \geq 0}$, i.e.

$$x^{n+3} = x^{n+1} + x^n \quad \forall n \geq 0$$

this is equivalent to $x^3 = x + 1$; so we have to solve the equation $T(x) = 0$ in K , where

$$T(X) = X^3 - X - 1 \in K[X]$$

The general case is when T admits 3 distinct roots $\alpha, \beta, \gamma \in K$: there exist $c_\alpha, c_\beta, c_\gamma \in K$ such that

$$(3.1) \quad T_n = c_\alpha \alpha^n + c_\beta \beta^n + c_\gamma \gamma^n, \quad n \geq 0$$

corresponding to a K -linear system whose (Van der Monde) determinant is

$$\delta = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} = (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)$$

and whose solution is

$$(3.2) \quad T_n = \frac{\alpha + 1}{(\alpha - \beta)(\alpha - \gamma)} \alpha^n + \frac{\beta + 1}{(\beta - \alpha)(\beta - \gamma)} \beta^n + \frac{\gamma + 1}{(\gamma - \alpha)(\gamma - \beta)} \gamma^n$$

Since $1 + \alpha = \alpha^3$ we have the useful equivalent expression

$$(3.3) \quad -\delta T_n = (\beta - \gamma) \alpha^{n+3} + (\gamma - \alpha) \beta^{n+3} + (\alpha - \beta) \gamma^{n+3}$$

Remark 3.1. If T has a double root $\alpha = \beta$, a basis of \mathcal{T}_K is

$$\{(\alpha^n)_{n \geq 0}, (n\alpha^{n-1})_{n \geq 0}, (\gamma^n)_{n \geq 0}\}$$

obtained by derivation of α^n with respect to the “variable” n :

$$n\alpha^{n-1} + (n-1)\alpha^{n-2} = n\alpha^{n+1} - \alpha^{n-2} = \alpha^{n+1}(n - \alpha^{-3}) = \alpha^{n+1}(n - (1 - \alpha^{-2})).$$

But $T'(\alpha) = 3\alpha^2 - 1 = 0$ so $\alpha^{-2} = 3$: $n\alpha^{n-1} + (n-1)\alpha^{n-2} = \alpha^{n+1}(n+2)$ proving that $(n\alpha^{n-1})_{n \geq 0}$ lies in \mathcal{T}_K .

Remark 3.2. For the Perrin sequence $(P_n)_{n \geq 0}$ mentioned above we have $P_n = \alpha^n + \beta^n + \gamma^n$: if p is prime we have $P_p = \alpha^p + \beta^p + \gamma^p = (\alpha + \beta + \gamma)^p = 0$ in the splitting field of $T \pmod p$: so that

$$n \text{ prime} \implies n | P_n$$

Perrin [1] observed that, reciprocally, for many, many, non prime n we have $n \nmid P_n$: indeed the first counterexample is the “pseudoprime” $n = 271441$: $n = 521^2$.

4. CALCULUS OF THE ROOTS α, β, γ OF $T(X) = X^3 - X - 1$

4.1. Rational case.

If $K = \mathbb{Q}$, then T is irreducible and separable. Its discriminant is $Disc(T) = -23 < 0$, so there exists one real root, and two complexes. Cardano’s method gives explicitly the real root :

$$\psi = \sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} + \sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}}$$

Here $\psi \simeq 1,324718\dots$ is the **plastic number**, a Pisot number (greater than 1, and conjugate less than 1 : here $\beta = \bar{\gamma}$ and $\beta\gamma = 1/\psi < 1$). Consequently by 3.1 $\lim_{n \rightarrow +\infty} \frac{T_{n+1}}{T_n} = \psi$.

4.2. Finite case.

Here $K = \mathbb{F}_p$ is the Galois field with p elements, and we read modulo p :

$$T(X) = X^3 - X - 1 \in \mathbb{F}_p[X]$$

We need the following lemma, which expresses two roots rationally from the third and the discriminant.

Lemma 4.1.

Let $\alpha, \beta, \gamma \in \overline{\mathbb{F}_p}$ be the roots of $T(X) = X^3 - X - 1$

i) $\delta = (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)$ verifies $\delta^2 = Disc(T) = -23 \pmod p$,

ii)

- if $p \neq 2, 23$:

$$(4.1) \quad \beta, \gamma = \frac{1}{2}[-\alpha \pm \frac{\alpha}{2\alpha + 3}\delta]$$

- if $p = 2$: $\beta = \alpha^2, \gamma = \alpha^2 + \alpha$,

- if $p = 23$: $\alpha = 3, \beta = \gamma = 10$.

Proof.

i) Classically $Disc(X^3 + pX + q) = [(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)]^2 = \delta^2 = -4p^3 - 27q^2$.

ii) For $p = 2, 23$: direct calculation. Else one uses the relations

$$(4.2) \quad \begin{aligned} \alpha + \beta + \gamma &= 0 \\ \alpha\beta + \beta\gamma + \gamma\alpha &= -1 \\ \alpha\beta\gamma &= 1 \\ \alpha^3 &= \alpha + 1 \end{aligned}$$

On the one hand $\beta + \gamma = -\alpha$, on the other hand $\delta \neq 0$, so $\beta - \gamma = \frac{\delta}{(\alpha - \beta)(\gamma - \alpha)} = \dots = \frac{-\delta}{2\alpha + 3}$; since $p \neq 2, \beta = \frac{1}{2}[(\beta + \gamma) + (\beta - \gamma)]$ and the formula follows. \square

Define $R_p = \mathbb{F}_p(\alpha, \beta, \gamma)$ as the splitting field of T over \mathbb{F}_p , and $r_p = [R_p : \mathbb{F}_p]$ his degree. There exist three cases :

• if T has a root $\alpha \in \mathbb{F}_p$

1. either -23 is a square in \mathbb{F}_p : $\delta \in \mathbb{F}_p$, and by the lemma 4.1 $\beta, \gamma \in \mathbb{F}_p$: $R_p = \mathbb{F}_p$ and $r_p = 1$ **CASE 1** (examples $p = 23, 59, 101, 173, 211$)

2. or -23 is not a square : by the lemma 4.1 $\beta, \gamma \in \mathbb{F}_p(\delta) = \mathbb{F}_{p^2}$; $R_p = \mathbb{F}_{p^2}$ and $r_p = 2$ **CASE 2** (examples $p = 5, 7, 11, 17$)

• if T is irreducible in $\mathbb{F}_p[X]$

let $\alpha \in \overline{\mathbb{F}_p}$ be a root of T , then $R_p = \mathbb{F}_p(\alpha) = \mathbb{F}_{p^3}$ and $r_p = 3$: indeed $\mathbb{F}_p(\beta), \mathbb{F}_p(\gamma)$ are cubic extensions of \mathbb{F}_p , so are equal, to \mathbb{F}_{p^3} ; moreover in this case -23 is a square in \mathbb{F}_p : $-23 = \delta^2$ is a square in \mathbb{F}_{p^3} , so in \mathbb{F}_p :

CASE 3 (examples : $p = 2, 3, 13, 29, 31$).

By quadratique reciprocity -23 is a square in \mathbb{F}_p if and only if p is a square mod 23, that is

$$p \equiv 0, 1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18 \pmod{23}$$

In short

$$\begin{array}{l} \text{if } -23 \text{ square mod } p \\ \text{if } -23 \text{ not square mod } p \end{array} \quad \begin{array}{l} \swarrow \\ \searrow \end{array} \quad \begin{array}{ll} \alpha \in \mathbb{F}_p & R_p = \mathbb{F}_p \quad r_p = 1 \\ T \text{ irreducible} & R_p = \mathbb{F}_{p^3} \quad r_p = 3 \\ \alpha \in \mathbb{F}_p & R_p = \mathbb{F}_{p^2} \quad r_p = 2 \end{array}$$

5. PRIME DIVISORS IN THE SEQUENCE $(T_n)_{n \geq 0}$

We are interested in the first occurrence of the prime p in the Padovan sequence. In the case of the Fibonacci sequence $(F_n)_{n \geq 0}$, the answer is easy, linked with the order of $\Phi/\overline{\Phi}$ mod p (where Φ is the golden ratio, $\overline{\Phi}$ the conjugate of Φ) : this is a divisor of $p - 1$ or $p + 1$ according to

$p \equiv \pm 1, p \equiv \pm 2 \pmod{5}$. Here it is much more complicated.

We adopt the following notations

- $t_p =$ period of the sequence $(T_n)_{n \geq 0}$
- $\Omega_p = \{n \geq 1 : p \text{ divide } T_n\}$
- $\omega_p = \min \Omega_p$: index of first occurrence of p in the sequence $(T_n)_{n \geq 1}$
- $A_p = \Omega_p \cap \{1, 2, \dots, t_p\}$
- $a = o(\alpha), b = o(\beta), c = o(\gamma)$ (orders in R_p^\times : divisors of $p^{r_p} - 1$)

5.1. Principal results.

Theorem 5.1.

Let $R_p = \mathbb{F}_p(\alpha, \beta, \gamma), r_p = [R_p : \mathbb{F}_p], r_p = 1, 2, 3$

1) for all $p : t_p - 3, t_p - 2, t_p \in A_p$, hence $\omega_p \leq t_p - 3$

2) $\Omega_p = A_p + \mathbf{N}t_p$

3) if $p \neq 23 : t_p = \text{LCM}\{a, b, c\}$, and it divides $p^{r_p} - 1$

4) if $p = 23 : t_{23} = 506 = p(p - 1)$, and $r_p = 1$

5) moreover :

- ♣ $r_p = 1$ if and only if $p = x^2 + 23y^2$
- ♣ if $r_p = 2$:
 - $\alpha \in \mathbb{F}_p$, and $t_p = b = c$
 - b divide $(p + 1)a$
 - $\omega_p \leq (p + 1)o(\alpha^3) - 3$
- ♣ if $r_p = 3$: $t_p = a = b = c$ divide $\frac{p^3 - 1}{p - 1} = p^2 + p + 1$.

Corollary 5.2. For every prime p there exists $n \geq 1$ such that p divides T_n , and the index ω_p verifies : $\omega_p \leq p^3 - 4$.

Proof. By definition $t_p = \min\{n \geq 1 : T_n \equiv 0, T_{n+1} \equiv 1, T_{n+2} \equiv 1 \pmod{p}\}$. Along the proof we denote $t_p = t$.

1) Since $T_t = 0, T_{t+1} = 1, T_{t+2} = 1$, by ‘‘redshift’’ we deduce $T_{t-1} = 1, T_{t-2} = 0, T_{t-3} = 0$.

2) Let $n \in \Omega_p : n = ut + v, 0 \leq v \leq t - 1$; we have $T_t = T_0, T_{t+1} = T_1, T_{t+2} = T_2$, so induction gives $T_{t+m} = T_m$, then (induction on u) $T_{ut+m} = T_m$; so here $T_n = T_v$; $n \in \Omega_p \Rightarrow T_n = 0$, so $T_v = 0$ and $v \in \Omega_p \cap \{1, \dots, t\} = A_p$.

3) Let $s = \text{LCM}\{a, b, c\}$; $p \neq 23 \Rightarrow \delta \neq 0$. Apply 3.3 : $-\delta T_{s-3} = \beta - \gamma + \gamma - \alpha + \alpha - \beta = 0$, so $T_{s-3} = 0$.

Similarly $-\delta T_{s-2} = (\beta - \gamma)\alpha + (\gamma - \alpha)\beta + (\alpha - \beta)\gamma = 0$, so $T_{s-2} = 0$, and $T_s = T_{s-2} + T_{s-3} = 0$.

3.3: $-\delta T_{s-1} = (\beta - \gamma)\alpha^2 + (\gamma - \alpha)\beta^2 + (\alpha - \beta)\gamma^2 = -\delta$, so $T_{s-1} = 1$. Hence

$$T_s = 0, T_{s+1} = 0, T_{s+2} = 1 \Rightarrow t \leq s$$

On the other hand, following 3.1

$$\begin{aligned} T_t &= c_\alpha \alpha^t &+& c_\beta \beta^t &+& c_\gamma \gamma^t &= 0 \\ T_{t+1} &= c_\alpha \alpha^{t+1} &+& c_\beta \beta^{t+1} &+& c_\gamma \gamma^{t+1} &= 1 \\ T_{t+2} &= c_\alpha \alpha^{t+2} &+& c_\beta \beta^{t+2} &+& c_\gamma \gamma^{t+2} &= 1 \end{aligned}$$

This proves that $(c_\alpha \alpha^t, c_\beta \beta^t, c_\gamma \gamma^t)$ is solution of the system

$$\begin{cases} x + y + z = 0 \\ \alpha x + \beta y + \gamma z = 1 \\ \alpha^2 x + \beta^2 y + \gamma^2 z = 1 \end{cases}$$

Its determinant is $(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha) = \delta \neq 0$ and its unique solution is $(c_\alpha, c_\beta, c_\gamma)$; so necessarily

$$\begin{aligned} c_\alpha \alpha^t &= c_\alpha \\ c_\beta \beta^t &= c_\beta \\ c_\gamma \gamma^t &= c_\gamma \end{aligned}$$

$c_\alpha \neq 0$ since $\alpha \neq -1 : T(-1) = -1 \neq 0$; we conclude that $\alpha^t = \beta^t = \gamma^t = 1$, and $s = LCM\{o(\alpha), o(\beta), o(\gamma)\}$ divides t , and finally $s = t = t_p$.

4) The case $p = 23$ is different (cf 3.1) : here $3T_n = 4.3^n - 4.10^n + 8.n.10^{n-1}$; we have to solve the equation $(-2)^n = 9n + 1$ giving $n = 503$ and $t_{23} = 506 = 22 \times 23$.

5)

♣ $r_p = 1 \Leftrightarrow p = x^2 + 23y^2$ see below 5.3

♣ $r_p = 2$: $\alpha \in \mathbb{F}_p, \beta, \gamma = \frac{1}{2}[-\alpha \pm \frac{\alpha}{2\alpha+3}\sqrt{-23}]$ are conjugate in $\mathbb{F}_p(\sqrt{-23})$ hence have same order $b = c$; moreover $\alpha = \frac{1}{\beta\gamma} \Rightarrow \alpha^b = 1 : a|b$ and $t_p = b = c$.

The Frobenius $x \mapsto x^p$ coincides with the conjugation : $\beta^p = \gamma, \beta^{p+1} = \beta\gamma = \frac{1}{\alpha}$, and $\beta^{(p+1)a} = 1 : b|(p+1)a$.

Let $m \geq 1$: $\alpha^{m(p+1)} = \alpha^{2m}(\alpha^{p-1} = 1); \beta^{m(p+1)} = \alpha^{-m}, \gamma^{m(p+1)} = \alpha^{-m}$; by 3.3 :

$-\delta T_{m(p+1)-3} = (\beta - \gamma)\alpha^{2m} + (\gamma - \alpha)\alpha^{-m} + (\alpha - \beta)\alpha^{-m} = (\beta - \gamma)\alpha^{-m}(\alpha^{3m} - 1)$. The choice $m = o(\alpha^3)$ gives $m(p+1) - 3 \in \Omega_p$. Example : $p=7, \alpha = 5, \alpha^3 = -1$, hence $2(7+1) - 3 = 13 \in \Omega_p$.

♣ $r_p = 3$: α, β, γ are conjugate so $a = b = c$; by Frobenius $x \mapsto x^p : \alpha^p = \beta$ or $\gamma, \alpha^{p^2} = \gamma$ or β , and $\alpha\beta\gamma = 1 = \alpha\alpha^p\alpha^{p^2} : a = b = c|1 + p + p^2$. \square

Remark 5.3. The analog of the theorem may be done for the Tribonacci sequence : $\omega_p \leq p^3 - 2; r_p = 1$ for $p = 47, 53, 269, \dots$; $r_p = 2$ for $p = 13, 17, 19, \dots$; $r_p = 3$ for $p = 3, 5, 23 \dots$

It may happen that two consecutive terms are divisible by p .

Proposition 5.4. *If $p \neq 23$, the following assertions are equivalent :*

- i) $\alpha^n = \beta^n = \gamma^n$
- ii) $p|T_{n-3}$ and $p|T_{n-2}$.

Proof.

i) \Rightarrow ii) : by 3.3.

ii) \Rightarrow i) : by hypothesis and 3.3 :

$$(5.1) \quad (\alpha - \beta)\gamma^n = (\gamma - \beta)\alpha^n + (\alpha - \gamma)\beta^n \text{ and}$$

$$(5.2) \quad (\alpha - \beta)\gamma^{n+1} = (\gamma - \beta)\alpha^{n+1} + (\alpha - \gamma)\beta^{n+1}$$

5.1 \Rightarrow $(\alpha - \beta)\gamma^{n+1} = (\gamma - \beta)\alpha^n\gamma + (\alpha - \gamma)\beta^n\gamma = (\text{by 5.2})(\gamma - \beta)\alpha^{n+1} + (\alpha - \gamma)\beta^{n+1}$; so $(\gamma - \beta)\alpha^n(\alpha - \gamma) + (\alpha - \gamma)\beta^n(\beta - \gamma) = 0$, and $(\gamma - \beta)(\alpha - \gamma)(\alpha^n - \beta^n) = 0$; $p \neq 23 \Rightarrow \alpha^n = \beta^n$. In the same way $\beta^n = \gamma^n$. \square

For example if $p = 7$ we have $\alpha^{16} = \beta^{16} = \gamma^{16} = 2$, so $7|T_{13}$ and $7|T_{14}$.

5.2. Examples.

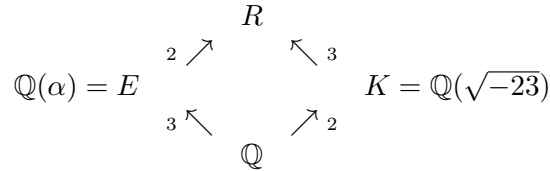
p	roots and orders	t_p	ω_p	A_p	$\sqrt{-23}$	r_p	
3	α	$a = 13$	13	6	6, 10, 11, 13	± 1	3
	$\beta = \alpha + 1$	$b = 13$					
	$\gamma = \alpha - 1$	$c = 13$					
7	$\alpha = 5$	$a = 6$	48	9	9, 13, 14, 16, 25, 29 30, 32, 41, 45, 46, 48	$\pm\sqrt{-2}$	2
	$\beta = 1 + 2\sqrt{-2}$	$b = 48$					
	$\gamma = 1 - 2\sqrt{-2}$	$c = 48$					
11	$\alpha = 6$	$a = 10$	120	25	25, 35, 43, 64, 87, 98 104, 113, 117, 118, 120	$\pm\sqrt{-1}$	2
	$\beta = -3 + 2\sqrt{-1}$	$b = 120$					
	$\gamma = -3 - 2\sqrt{-1}$	$c = 120$					
59	$\alpha = 4$	$a = 29$	58	42	42, 51, 55, 56, 58	± 6	1
	$\beta = 13$	$b = 58$					
	$\gamma = 42$	$c = 58$					

The last three elements of A_p are predicted by the theorem.

5.3. The sequence $(T_n)_{n \geq 0}$ and class field theory.

We refer to the text [3] which gives similar results for the Tribonacci sequence.

Here $K = \mathbb{Q}(\sqrt{-23})$: the class number is $h_K = 3$, the ring of integers is $\mathcal{O}_K = \mathbb{Z}[\sqrt{-23}]$. Denote by $R = \mathbb{Q}(X^3 - X - 1) \subset \mathbb{C}$ the splitting field of $T(X) = X^3 - X - 1$, and $\alpha \in \mathbb{C}$ a root of T .



1- $\mathcal{O}_E = \mathbb{Z}[\alpha]$: indeed $Disc(\alpha) = Disc(T) = -23$ is squarefree.

2- Every $p \neq 23$ is unramified in R/\mathbb{Q} : by Kummer–Dedekind theorem ($\mathcal{O}_E = \mathbb{Z}[\alpha]$) p in unramified in E , also in the Galois closure R of E .

3- $P_{23} = \sqrt{-23}\mathcal{O}_K$ is unramified in R/K : we have $T \pmod{23} = (X - 3)(X - 10)^2$; By Kummer–Dedekind $23\mathcal{O}_E = P_1P_2^2$ with $3 = e_1f_1 + e_2f_2$, so $e = 1 = f_1 = 1, e_2 = 2, f_2 = 1$.

R/K is Galois; we have $P_{23}\mathcal{O}_R = (\mathfrak{Q}_1 \dots \mathfrak{Q}_g)^e, efg = 3$; then $e = 1$: else $e = 3$; let $\mathfrak{P}|P_{23}$ in R :

$$e_{R/\mathbb{Q}}(\mathfrak{P}) = e_{R/K}(\mathfrak{P}).e_{K/\mathbb{Q}}(P_{23}) = 3 \times 2 = 6, \text{ and}$$

$$e_{R/\mathbb{Q}}(\mathfrak{P}) = e_{R/E}(\mathfrak{P}).e_{E/\mathbb{Q}}(P_1 \text{ or } P_2) = [1 \text{ or } 2] \times [\leq 3]$$

necessarily $e_{R/E}(\mathfrak{P}) = 2$, and $e_{E/\mathbb{Q}}(P_i) = 3(i = 1, 2)$, but $e_{E/\mathbb{Q}}(P_i) = 1$ or 2 : contradiction.

4- K being imaginary, R/K is unramified at infinity : so R/K is abelian unramified and $[R : K] = 3 = h_K$; we conclude that R is the Hilbert class field of K .

5- Application :

- if $p \neq 23$ with $r_p = 1$ we have $T \bmod p = (X - \alpha)(X - \beta)(X - \gamma) \in \mathbb{F}_p[X]$: Kummer-Dedekind shows that $p\mathcal{O}_E = P_1P_2P_3$: p is totally decomposed in E , and idem in the Galois closure R of E , a fortiori in K : $p\mathcal{O}_K = \mathfrak{P}_1\mathfrak{P}_2$, and $\mathfrak{P}_1, \mathfrak{P}_2$ are totally decomposed in the Hilbert class field R ; by the Artin reciprocity law \mathfrak{P}_1 is a principal ideal : $\mathfrak{P}_1 = (x + y\sqrt{-23})\mathcal{O}_K$; norm : $p = x^2 + 23y^2$.
- Reciprocally if $p = x^2 + 23y^2$ we have $p\mathcal{O}_K = (x + y\sqrt{-23})\mathcal{O}_K.(x - y\sqrt{-23})\mathcal{O}_K$: p is totally decomposed in K . By Artin reciprocity law , $(x + y\sqrt{-23})\mathcal{O}_K$ being principal, we conclude it is totally decomposed in R , and also in E : $p\mathcal{O}_E = P_1P_2P_3$ and Kummer-Dedekind show that $r_p = 1$.

We have proved

Theorem 5.5. $r_p = 1 \iff p = x^2 + 23y^2$

Clearly $p = 23$ is not a problem for the assertion!

6. SOME IDENTITIES FOR $(T_n)_{n \geq 0}$

6.1.

For all m $(T_{m+n})_{n \geq 0} \in \mathcal{T}_K$, so there exist $u, v, w \in K$ such that $T_{m+n} = uT_n + vT_{n-1} + wT_{n-2}$. Applying this for $n = 0, n = 1, n = 2$, we obtain the system

$$\begin{cases} v & = & T_m \\ u + w & = & T_{m+1} \\ u + v & = & T_{m+2} \end{cases}$$

Easy calculations give :

$$(6.1) \quad T_{m+n} = T_{m-1}.T_n + T_m.T_{n-1} + (T_{m+1} - T_{m-1}).T_{n-2}$$

As application let $m = n$:

$$T_{2n} = 2T_{n-1}T_n + (T_{n+1} - T_{n-1})T_{n-2}$$

if $p|T_{n-2}$ and $p|T_{n-1}$ also $p|T_{2n}$. On can verify that $p = 7|T_{13}$ and $p|T_{14}$, so $p|T_{30}$.

6.2.

The well-know formula related to the Fibonacci sequence $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ ("Fibonacci-puzzle") can be proved like this : one prove first that $\Phi^n = F_n\Phi + F_{n-1}$, and the analog with $\bar{\Phi}$ conjugate of Φ ; since $\Phi.\bar{\Phi} = -1$, one has $(\Phi.\bar{\Phi})^n = (-1)^n = (F_n\Phi + F_{n-1})(F_n\bar{\Phi} + F_{n-1})$ and the identity follows. We use here the same idea.

As $(\alpha^n)_{n \geq 0} \in \mathcal{T}_K$, there exist $u, v, w \in K$ such that $\alpha^n = uT_n + vT_{n-1} + wT_{n-2}$: easy calculations give

$$(6.2) \quad \alpha^n = (\alpha^2 - 1)T_n + T_{n-1} + (1 + \alpha - \alpha^2)T_{n-2}$$

and the analog for β^n, γ^n . Write now $\alpha^n\beta^n\gamma^n = 1$:

$$[(\alpha^2 - 1)T_n + T_{n-1} + (1 + \alpha - \alpha^2)T_{n-2}] \dots$$

$$\dots [(\beta^2 - 1)T_n + T_{n-1} + (1 + \beta - \beta^2)T_{n-2}] \cdot [(\gamma^2 - 1)T_n + T_{n-1} + (1 + \gamma - \gamma^2)T_{n-2}] = 1$$

The left member is a polynomial $P(T_n, T_{n-1}, T_{n-2}) = \sum_{(a,b,c)} c_{abc} T_n^a T_{n-1}^b T_{n-2}^c$ homogeneous of degree 3 : $a + b + c = 3$, and the c_{abc} are symmetric polynomials in α, β, γ , so polynomials in the elementary symmetric functions $s_1 = \alpha + \beta + \gamma = 0$, $s_2 = \alpha\beta + \beta\gamma + \gamma\alpha = -1$, $s_3 = \alpha\beta\gamma = 1$. After long calculations we find

$$(6.3) \quad T_n^3 + T_{n-1}^3 + T_{n-2}^3 - T_n T_{n-1}^2 - T_n^2 T_{n-2} + T_{n-1}^2 T_{n-2} + 2T_{n-1} T_{n-2}^2 - 3T_n T_{n-1} T_{n-2} = 1$$

7. QUESTIONS

We have many questions !

Q_1 – In the case $r_p = 1$ it seems that $b = c = 2a$: is it true ?

Q_2 – In the case $r_p = 2$ it seems that $b = c = (p + 1)a$: is it true ?

Q_3 – When $r_p = 3$: $a = b = c$; some examples show that $a = 1 + p + p^2$: is it true ?

Q_4 – What is the exact arithmetical nature of ω_p , of A_p ?

Q_5 – Primitive divisors : the computations give many exceptions :

$$n = 5, 7, 10, 11, 12, 13, 14, 16, 21, 23, 32, 33, 45, \dots$$

Are there finitely many ?

Q_6 – T_n is prime for

$$n = 4, 5, 6, 8, 9, 15, 20, 31, 38, \dots$$

Are there infinitely many ?

Q_7 – We find the following square in $(T_n)_{n \geq 0}$:

$$T_{10} = 9, T_{12} = 16, T_{16} = 49$$

Are there finitely many squares ?

Observe that $\sqrt{9} = 3 = T_6$, $\sqrt{16} = 4 = T_7$, $\sqrt{49} = 7 = T_9$: is it a coincidence ?

Finally the author thanks F. Nuccio for his (im)pertinent galoisian observations !



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