Estimates for the SVD of the truncated Fourier transform on L2(exp(b|×|)) and stable analytic continuation

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ESTIMATES FOR THE SVD OF THE TRUNCATED FOURIER TRANSFORM ON $L^2(\cosh(b|\cdot|))$ AND STABLE ANALYTIC CONTINUATION

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ABSTRACT. The Fourier transform truncated on $[-c, c]$ is usually analyzed when acting on $L^2(-1/b, 1/b)$ and its right-singular vectors are the prolate spheroidal wave functions. This paper considers the operator acting on the larger space $L^2(\cosh(b|\cdot|))$ on which it remains injective. We give nonasymptotic upper and lower bounds on the singular values with similar qualitative behavior in m (the index), b, and c. The lower bounds are used to obtain rates of convergence for stable analytic continuation of possibly nonbandlimited functions whose Fourier transform belongs to $L^2(e^{b|\cdot|})$. We also derive bounds on the sup-norm of the singular functions. Finally, we propose a numerical method to compute the SVD and apply it to stable analytic continuation when the function is observed with error on an interval. In the application we consider cases where the function that extrapolate is not bandlimited and when it is bandlimited but the bandlimits are unknown.

1. INTRODUCTION

Extrapolating an analytic square integrable function f from its observation with error on $[-c, c]$ to $\mathbb{R}$ has a wide range of applications, for example in imaging and signal processing [25], in geostatistics and with big data [18], and finance [24]. A researcher may want to estimate

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a density from censored data. This means that the data is only available on a smaller set than the one of interest (see, *e.g.*, [5, 6]). When the function is a Fourier transform, this is a type of super-resolution in image restoration [7, 8, 22] which can be achieved under auxiliary information such as information on the support of the object. The related problem of out-of-band extrapolation (see, *e.g.*, [2, 7]) consists in recovering a function from partial observation of its Fourier transform. A particular instance of this framework appears in the analysis of the random coefficients linear model (see, *e.g.* [21]). There, the model takes the form

$$Y = \alpha + \beta^\top X,$$

where $(\alpha, \beta^\top) \in \mathbb{R}^{p+1}$ and $X \in \mathbb{R}^p$ are independent random vectors, and the researcher has at her disposal an independent and identically distributed sample of $(Y, X^\top)$ from which she can estimate the Fourier transform of the density of the coefficients $(\alpha, \beta^\top)$ on $\{(t, tx) : \ (t, x) \in \mathbb{R} \times \mathcal{X}\}$, where $\mathcal{X} \subseteq \mathbb{R}^p$ is the support of X and the object of interest is the density the coefficients.

It is customary to rely on analytic functions and use Hilbert space techniques. For the extrapolation problem, one can restrict attention to bandlimited functions which are square integrable functions whose Fourier transforms have support in $[-1/b, 1/b]$. For out-of-band extrapolation, one can work with square-integrable functions whose support is a subset of $[-1/b, 1/b]$ in which case their Fourier transform is analytic by the Paley-Wiener theorem (see [40]). Prolate spheroidal wave functions (henceforth PSWF, see [38, 43]) are the right-singular functions of the truncated Fourier transform restricted to functions with support in $[-1/b, 1/b]$. The truncated Fourier transform maps functions to their Fourier transform on $[-c, c]$. The PSWF form an orthonormal basis of the space $L^2(-c, c)$ of square-integrable functions on $(-c, c)$, are restrictions of square integrable orthogonal analytic functions on $\mathbb{R}$, and form a complete system of the bandlimited functions with bandlimits $[-1/b, 1/b]$. Hence, a bandlimited function on the whole line is simply the series expansion on the PSWF basis, sometimes called Slepian series, whose coefficients only depend on the function on $(-c, c)$, almost everywhere on $\mathbb{R}$. This makes sense if we understand the PSWF functions as their extension to $\mathbb{R}$. In this framework, analytic continuation is an inverse problem in the sense that the solution does not depend continuously on the data, more specifically severely ill-posed (see, *e.g.*, [20, 26, 42, 50]), and many methods have been proposed (see, *e.g.*, [4, 9, 17, 18, 19, 29, 31, 45]). To obtain precise error bounds, it is useful to obtain nonasymptotic upper and lower bounds on the singular

values of the truncated Fourier transform rather than the more usual asymptotic estimates on a logarithmic scale. In several applications, uniform estimates on the right singular functions are useful as well. This occurs for example to show that certain nonparametric statistical procedures involving series are adaptive (see, *e.g.*, [16]). This means that an estimator with a data-driven smoothing parameter reaches the optimal minimax rate of convergence. Importantly, such a program providing nonasymptotic bounds on the singular values and right singular functions has been carried recently in relation to bandlimited functions in [12, 13, 14, 37, 38, 41]. A second important aspect is the access to efficient methods to obtain the singular value decomposition (henceforth SVD). While numerical solutions to the inverse problems have for a long time relied on the Tikhonov or iterative methods such as the Landweber method (Gerchberg method for out-of band extrapolation, see [7]) to avoid using the SVD, recent developments have made it possible to approximate efficiently the PSWF and the SVD (see Section 8 of [38]).

Assuming that the function observed on an interval is the restriction of a bandlimited function can be questionable. For example, in the case of censored data, the observed function is a truncated density and the underlying function a density, and none of the usual families are bandlimited. Moreover, even if the function were bandlimited, one would require an upper bound on $1/b$ which might not be available in practice (see [44]). For this reason, this paper considers the larger class of functions whose Fourier transforms belong to the space $L^2(e^{b|\cdot|})$ of square-integrable functions with weight function $e^{b|\cdot|}$. This is the largest space that we can consider to extrapolate a function with Hilbert space techniques because, for $a > 0$, $\left\{ f \in L^2(\mathbb{R}) : \ \forall b < a, \ \mathcal{F}[f] \in L^2\left(e^{b|\cdot|}\right) \right\}$ is the set of square-integrable functions which have an analytic continuation on $\{z \in \mathbb{C} : \ |\mathrm{Im}(z)| < a/2\}$ (see Theorem IX.13 in [40]). The broader class $L^2(e^{b|\cdot|})$ has rarely been used in this context and, unlike the PSWF, much fewer results are available, with the notable exception of [32, 47]. It is considered in [5] in the case of censored data and in [21] for the problem of estimating the density of random coefficients in the linear random coefficients model. There it is meaningful to assume the Laplace transform of the density is finite near 0 or equivalently that it does not have heavy-tails.

In this paper, we use the weight $\cosh(b\cdot)$ rather than $e^{b|\cdot|}$ because the Fourier transform of $\mathrm{sech} = 1/\cosh$ is essentially itself and, though with a different scalar product, $L^2\left(\cosh(b\cdot)\right) = L^2\left(e^{b|\cdot|}\right)$. Theorem II in [47] provides, for given $b, c > 0$ and a value of the index m going to infinity, an equivalent of the logarithm of the singular values of the truncated Fourier transform

acting on $L^2(\cosh(b\cdot))$. Such an equivalent is important but this result is silent on the polynomial preexponential factors, their dependence with respect to c and b, and to deduce upper and lower bounds on the singular values which hold for all m, c, and b. This behavior is important in [21] where we integrate the bounds over c in intervals $[a, b]$ where a can be arbitrarily close to 0. This paper provides nonasymptotic upper and lower bounds on the singular values, with similar qualitative behavior, and applies the lower bounds to error bounds for stable analytic continuation using the spectral cut-off method. There, the nonasymptotic lower bounds are important to obtain a tight polynomial rates of convergence for "supersmooth functions". We also analyze a differential operator which commutes with a symmetric integral operator obtained by applying the truncated Fourier operator to its adjoint. The corresponding eigenvalue problem involves singular Sturm-Liouville equations. This allows to prove uniform estimates on the right-singular functions. Solving numerically singular differential equations allows to obtain these functions, hence all the SVD. Working with the differential operator is useful because its eigenvalues increase quadratically while those of the integral operator decrease exponentially. Finally, we illustrate numerically the proposed method for stable analytic continuation by spectral cut-off. We propose an adaptive method to select the cut-off. When the function is bandlimited and the researcher knows an interval which contains the bandlimits, we rely on the PSWF and efficient methods to compute the SVD. We also illustrate the proposed method involving numerical schemes for singular differential equations when the researcher does not have prior information on the bandlimits and when she questions the fact that the function can be bandlimited as in the statistical applications presented above.

2. Preliminaries

We use $\mathbb{N}_0$ for the set of nonnegative natural numbers, $a \vee b$ for the maximum of a and b, *a.e.* for almost everywhere, and $f(\cdot)$ for a function f of some generic argument. We denote, for $a > 0$, by $L^2(-a, a)$ and $L^2(\mathbb{R})$ the usual L^2 spaces of complex-valued functions equipped with the Hermitian inner product, for example $\langle f, g\rangle_{L^2(-a,a)} = \int_{-a}^{a} f(x)\overline{g}(x)dx$, by $L^2(W)$ for a positive function W on $\mathbb{R}$ the weighted L^2 spaces equipped with $\langle f, g\rangle_{L^2(W)} = \int_{\mathbb{R}} f(x)\overline{g}(x)W(x)dx$, and by $S^{\perp}$ the orthogonal complement of the set S in a Hilbert space. We denote by $\|f\|_{L^\infty([a,b])}$ the sup-norm of the function f on $[a, b]$. $\mathcal{E}$ is the operator which extends a function in $L^2(-1, 1)$ to $L^2(\mathbb{R})$ by assigning the value 0 outside $[-1, 1]$ and $\mathcal{R}L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is such that $\mathcal{R}f = f(-\cdot)$.

The inverse of a mapping f, when it exists, is denoted by f^I. We denote, for $b, c > 0$, by

$$(1) \qquad \begin{array}{rccc} \mathcal{C}_c: & L^2(\mathbb{R}) & \to & L^2(\mathbb{R}) \\ & f & \mapsto & cf(c\cdot) \end{array}, \qquad \begin{array}{rccc} \mathcal{F}_{b,c}: & L^2(\cosh(b\cdot)) & \to & L^2(-1,1) \\ & f & \mapsto & \mathcal{F}[f](c\cdot) \end{array},$$

by $\mathcal{F}[f] = \int_{\mathbb{R}} e^{ix\cdot} f(x)dx$ the Fourier transform of f in $L^1(\mathbb{R})$ and also use the notation $\mathcal{F}[f]$ for the Fourier transform in $L^2(\mathbb{R})$. $\mathcal{O}^*$ denotes the Hermitian adjoint of an operator $\mathcal{O}$. Recovering f such that for b small enough $f \in L^2(\cosh(b\cdot))$ based on its Fourier transform on $[-c, c]$ amounts to inverting $\mathcal{F}_{b,c}$. This can be achieved using the SVD. Define the finite convolution operator

$$(2) \qquad \begin{array}{rccc} \mathcal{Q}_c: & L^2(-1,1) & \to & L^2(-1,1) \\ & h & \mapsto & \int_{-1}^{1} \pi c \operatorname{sech}\left(\frac{\pi c}{2}(\cdot - y)\right) h(y)dy. \end{array}$$

It is compact, symmetric, and positive on spaces of real and complex valued functions. Denote by $(\rho_m^c)_{m\in\mathbb{N}_0}$ its positive real eigenvalues in decreasing order and repeated according to multiplicity and by $(g_m^c)_{m\in\mathbb{N}_0}$ its eigenfunctions which can be taken to be real valued. The next proposition relies on the fact that, for all $c > 0$, $\mathcal{F}[\operatorname{sech}(c\cdot)](\star) = (\pi/c)\operatorname{sech}(\pi \star /(2c))$.

Proposition 1. For $b, c > 0$, we have $c\mathcal{F}_{b,c}\mathcal{F}_{b,c}^* = \mathcal{Q}_{c/b}$.

Proof. Because $\mathcal{F}_{b,c} = \mathcal{F}\mathcal{C}_{c^{-1}} = c^{-1}\mathcal{C}_c\mathcal{F}$, $\mathcal{R}\mathcal{F}_{b,c} = \mathcal{F}_{b,c}\mathcal{R}$,

$$(3) \qquad \mathcal{F}_{b,c}^* = \operatorname{sech}(b\cdot)\mathcal{R}\mathcal{F}_{b,c}\mathcal{E},$$

and $\operatorname{sech}(b\cdot)$ is even, we obtain $\mathcal{F}_{b,c}^* = \mathcal{R}[\operatorname{sech}(b\cdot)\mathcal{F}_{b,c}\mathcal{E}]$ and

$$c\mathcal{F}_{b,c}\mathcal{F}_{b,c}^* = c\mathcal{R}\mathcal{F}_{b,c}[\operatorname{sech}(b\cdot)\mathcal{F}_{b,c}\mathcal{E}] = 2\pi\mathcal{F}^I[\mathcal{C}_{c^{-1}}[\operatorname{sech}(b\cdot)\mathcal{C}_c\mathcal{F}\mathcal{E}]] = 2\pi c\mathcal{F}^I[\mathcal{C}_{c^{-1}}[\operatorname{sech}(b\cdot)]\mathcal{F}\mathcal{E}],$$

where, for *a.e.* $x \in \mathbb{R}$,

$$2\pi c\mathcal{F}^I[\mathcal{C}_{c^{-1}}[\operatorname{sech}(b\cdot)]](x) = \int_{\mathbb{R}} e^{-itx}\operatorname{sech}\left(\frac{bt}{c}\right)dt = \frac{\pi c}{b}\operatorname{sech}\left(\frac{\pi c}{2b}x\right).$$

As a result, we have, for $f \in L^2(-1,1)$,

$$c\mathcal{F}_{b,c}\mathcal{F}_{b,c}^*[f] = \mathcal{C}_{\pi c/(2b)}[2\operatorname{sech}] * \mathcal{E}[f] = \mathcal{Q}_{c/b}[f].$$

□

Proposition 1 yields that $(g_m^{c/b})_{m\in\mathbb{N}_0}$ are the right singular functions of $\mathcal{F}_{b,c}$. The SVD of $\mathcal{F}_{b,c}$, denoted by $\left(\sigma_m^{b,c}, \varphi_m^{b,c}, g_m^{c/b}\right)_{m\in\mathbb{N}_0}$, is such that, for $m \in \mathbb{N}_0$,

$$\sigma_m^{b,c} = \sqrt{\frac{\rho_m^{c/b}}{c}} \tag{4}$$

and $\varphi_m^{b,c} = \mathcal{F}_{b,c}^* g_m^{c/b}/\sigma_m^{b,c}$. It yields, for all $f \in L^2(\cosh(b\cdot))$,

$$f = \sum_{m\in\mathbb{N}_0} \frac{1}{\sigma_m^{b,c}} \left\langle \mathcal{F}_{b,c}[f], g_m^{c/b} \right\rangle_{L^2(-1,1)} \varphi_m^{b,c}. \tag{5}$$

(5) is a core element to approximate a function from partial observations of its Fourier transform when the signal f does not have compact support.

Proposition 2. For all $b, c > 0$, $\mathcal{F}_{b,c}$ is injective and $\left(\varphi_m^{b,c}\right)_{m\in\mathbb{N}_0}$ is a basis of $L^2(\cosh(b\cdot))$.

Proof. We use that, for every $h \in L^2(\cosh(b\cdot))$, if we do not restrict the argument in the definition of $\mathcal{F}_{b,c}[h]$ to $[-1,1]$, $\mathcal{F}_{b,c}[h]$ can be defined as a function in $L^2(\mathbb{R})$. In what follows, for simplicity, we use $\mathcal{F}_{b,c}[h]$ for both the function in $L^2(-1,1)$ and in $L^2(\mathbb{R})$.
Let us show that $\mathcal{F}_{b,c}$ defined in (1) is injective. Take $h \in L^2(\cosh(b\cdot))$ such that $\mathcal{F}_{b,c}[h]$ is zero on $[-1,1]$. Then, using Theorem IX.13 in [40], $\mathcal{F}_{b,c}[h]$ is zero on $\mathbb{R}$. Thus, $\mathcal{F}[h]$ hence h are zero *a.e.* on $\mathbb{R}$.
The second part of Proposition 2 holds by Theorem 15.16 in [28] and the injectivity of $\mathcal{F}_{b,c}$. □

Theorem II in [47] provides the equivalent

$$\log\left(\rho_m^c\right) \underset{m\to\infty}{\sim} -\pi m \frac{K(\operatorname{sech}(\pi c))}{K(\tanh(\pi c))}, \tag{6}$$

where $K(r) = \int_0^{\pi/2} \left(1 - r^2\sin(x)^2\right)^{-1/2} dx$ is the complete elliptic integral of the first kind. This paper provides nonasymptotic upper and lower bounds on the eigenvalues and upper bounds on the sup-norm of the functions $(g_m^c)_{m\in\mathbb{N}_0}$.

The proofs of this paper sometimes rely on the following operator

$$\begin{array}{rccl} \mathcal{F}_c^{W_{[-1,1]}}: & L^2\left(W_{[-1,1]}\right) & \to & L^2(-1,1), \\ & f & \to & \mathcal{F}[f](c\cdot) \end{array} \tag{7}$$

where $W_{[-1,1]} = \mathbb{1}\{[-1,1]\} + \infty\, \mathbb{1}\{[-1,1]^c\}$, for which we use the notations $\rho_m^{W_{[-1,1]},t_m}$ for the m^{th} eigenvalue of $\mathcal{Q}_c^{W_{[-1,1]}} = c\mathcal{F}_c^{W_{[-1,1]}}\left(\mathcal{F}_c^{W_{[-1,1]}}\right)^*$.

3. Lower bounds on the eigenvalues of $\mathcal{Q}_c$ and an application

3.1. Lower bounds on the eigenvalues of $\mathcal{Q}_c$.

Lemma 3. For all $m \in \mathbb{N}_0$, $c \in (0,\infty) \mapsto \rho_m^c$ is nondecreasing.

Proof. Take $m \in \mathbb{N}_0$. Using the maximin principle (see Theorem 5 page 212 in [10]), the $m+1$-st eigenvalue ρ_m^c satisfies

$$\rho_m^c = \max_{V \in S_{m+1}} \min_{f \in V \setminus \{0\}} \frac{\langle \mathcal{Q}_c f, f \rangle_{L^2(-1,1)}}{\|f\|^2_{L^2(-1,1)}},$$

where S_{m+1} is the set of $m+1$-dimensional vector subspaces of $L^2(-1,1)$. Using (3) and Proposition 1, we obtain

$$\begin{aligned} \langle \mathcal{Q}_c f, f \rangle_{L^2(-1,1)} &= c \left\langle \mathcal{F}_{1,c} \mathcal{F}_{1,c}^* [f], f \right\rangle_{L^2(-1,1)} \\ &= c \left\langle \mathcal{F}_{1,c}^* [f], \mathcal{F}_{1,c}^* [f] \right\rangle_{L^2(\cosh)} \\ &= c \left\| \operatorname{sech} \times \mathcal{F}_{1,c} \left[\mathcal{E} [f] \right] \right\|^2_{L^2(\cosh)} \\ &= c \int_{\mathbb{R}} \operatorname{sech}(x) \left| \int_{\mathbb{R}} e^{ictx} \mathcal{E}[f](t) dt \right|^2 dx \\ &= \int_{\mathbb{R}} \operatorname{sech}\left(\frac{x}{c}\right) \left| \mathcal{F}\left[\mathcal{E}[f]\right](x) \right|^2 dx \end{aligned} \tag{8}$$

hence

$$\rho_m^c = \max_{V \in S_{m+1}} \min_{f \in V \setminus \{0\}} \frac{2\pi \int_{\mathbb{R}} \operatorname{sech}(x/c) \left| \mathcal{F}\left[\mathcal{E}[f]\right](x) \right|^2 dx}{\left\| \mathcal{F}\left[\mathcal{E}[f]\right] \right\|^2_{L^2(\mathbb{R})}}. \tag{9}$$

Then, using that $t \mapsto \cosh(t)$ is even, nondecreasing, and positive, we obtain that, for all $0 < c_1 \leq c_2$ and $x \in \mathbb{R}$, $\operatorname{sech}(x/c_2) \geq \operatorname{sech}(x/c_1)$ hence that $\rho_m^{c_1} \leq \rho_m^{c_2}$. □

Theorem 4. For all $c > 0$, we have $\rho_0^c \geq 2\pi$. For all $m \in \mathbb{N}$, we have

$$\forall\ 0 < c \leq \frac{\pi}{4},\ \rho_m^c \geq 2\pi \frac{\sin(2c)^2}{(2ec)^2} \exp\left(-2 \log\left(\frac{7e^2\pi}{2c} \right) m \right) \tag{10}$$

$$\forall c > 0,\ \rho_m^c \geq \pi \exp\left(-\frac{\pi(m+1)}{2c} \right). \tag{11}$$

(10) is valid for $0 < c \leq \pi/4$ and more precise than (11) for c close to 0. (11) is uniformly valid. To prove it, we show that $\rho_m^c \geq \operatorname{sech}(t_m/c)\, \rho_m^{W_{[-1,1]},t_m}$ for well chosen t_m and rely on a lower bound on $\rho_m^{W_{[-1,1]},t_m}$. The proof of (10) uses similar arguments as those in [12] Section

5.1 and a lower bound on the best constant $\Gamma(m,\epsilon)$ such that for all interval $I \subseteq [-\pi,\pi]$ of length $2\epsilon > 0$ and all polynomial of degree at most $m \in \mathbb{N}_0$,

$$\left\|P\left(e^{i\cdot}\right)\right\|^2_{L^2(I)} \geq \Gamma(m,\epsilon)\left\|P\left(e^{i\cdot}\right)\right\|^2_{L^2(-\pi,\pi)}.$$

We use the lower bound in [34] page 240

$$\Gamma(m,\epsilon) \geq \left(\frac{14e\pi}{\epsilon}\right)^{-2m} \tag{12}$$

for $\epsilon = 4c$. It is argued in [34] that it cannot be significantly improved for small ϵ which is precisely the regime for which (10) is used to bound the eigenvalues from below.

Proof. Let $m \in \mathbb{N}_0$, $c > 0$, and $M = (m+1)/(2c)$. For $R > 0$, we denote by $PW(R)$ the Paley-Wiener space of functions whose Fourier transform has support in $[-R,R]$ and by $S_{m+1}(R)$ the set of $m+1$-dimensional subspaces of $PW(R)$. Using (9), we have

$$\rho^c_m = \max_{V\in S_{m+1}(1)} \min_{g\in V\setminus\{0\}} \frac{2\pi\int_{\mathbb{R}} \operatorname{sech}(x/c)\,|g(x)|^2\,dx}{\|g\|^2_{L^2(\mathbb{R})}}.$$

Then, for $g \in PW(1)$, the function $g_{Mc} : x \in \mathbb{R} \mapsto (Mc)^{1/2} g(Mcx)$ satisfies $\|g\|^2_{L^2(\mathbb{R})} = \|g_{Mc}\|^2_{L^2(\mathbb{R})}$ and belongs to $PW(Mc)$. Using

$$\int_{\mathbb{R}} \operatorname{sech}\left(\frac{x}{c}\right)|g(x)|^2\,dx = \int_{\mathbb{R}} \operatorname{sech}(Mx)\,|g_{Mc}(x)|^2\,dx,$$

we have, for $V \in S_{m+1}(Mc)$,

$$\rho^c_m \geq \min_{g\in V\setminus\{0\}} \frac{2\pi\int_{\mathbb{R}} \operatorname{sech}(Mx)\,|g(x)|^2\,dx}{\|g\|^2_{L^2(\mathbb{R})}}. \tag{13}$$

Let us now choose a convenient such space V defined, for $\varphi : t \in \mathbb{R} \mapsto \sin(t/2)/(\pi t)$, as

$$V = \left\{\sum_{k=0}^m P_k e^{i(k-m/2)\cdot}\varphi(\cdot), \quad (P_k)_{k=0}^m \in \mathbb{C}^{m+1}\right\}.$$

The Fourier transform of an element of V is of the form $\sum_{k=0}^m P_k \mathcal{F}[\varphi](\cdot - k + m/2)$ and, because $\mathcal{F}[\varphi](\cdot) = \mathbb{1}\{|\cdot| \leq 1/2\}$, it has support in $[-1/2 - m/2, 1/2 + m/2] = [-Mc, Mc]$. This guarantees that $V \in S_{m+1}(Mc)$.

We now obtain a lower bound on the right-hand side of (13). Let $g \in V$, defined via the

coefficients $(P_k)_{k=0}^m$, and, for $x \in \mathbb{R}$, let $P(x) = \sum_{k=0}^m P_k x^k$. Let $0 < x_0 \leq \pi/2$. We have, using $\forall x \in [0, 2x_0)$, $\sin(x/2)/x \geq \sin(x_0)/(2x_0)$ for the last display,

$$\begin{aligned}\int_{\mathbb{R}} \operatorname{sech}(Mx) |g(x)|^2 dx &\geq \int_{-2x_0}^{2x_0} \operatorname{sech}(Mx) \left|\sum_{k=0}^m P_k e^{ikx}\right|^2 |\varphi(x)|^2 dx \\ &\geq \frac{1}{\cosh(2Mx_0)} \min_{x\in[-2x_0,2x_0]} |\varphi(x)|^2 \int_{-2x_0}^{2x_0} \left|\sum_{k=0}^m P_k e^{ikx}\right|^2 dx \\ &\geq \frac{\sin(x_0)^2}{(2\pi x_0)^2} e^{-2Mx_0} \left\|P\left(e^{i\cdot}\right)\right\|^2_{L^2(-2x_0,2x_0)}.\end{aligned}$$

Now, using that, for $k \in \mathbb{N}_0$, $t \mapsto \mathcal{F}[\varphi](t - k + m/2)$ have disjoint supports, we obtain

$$\begin{aligned}\|g\|^2_{L^2(\mathbb{R})} &= \frac{1}{2\pi} \|\mathcal{F}[g]\|^2_{L^2(\mathbb{R})} \\ &= \frac{1}{2\pi} \sum_{k=0}^m |P_k|^2 \|\mathcal{F}[\varphi]\|^2_{L^2(\mathbb{R})} \\ &= \frac{1}{(2\pi)^2} \left\|P\left(e^{i\cdot}\right)\right\|^2_{L^2(-\pi,\pi)},\end{aligned}$$

hence, by (12),

$$\rho_m^c \geq 2\pi \frac{\sin(x_0)^2}{x_0^2} e^{-2Mx_0} \left(\frac{7e\pi}{x_0}\right)^{-2m}.$$

We obtain, for $0 < x_0 \leq \pi/2$ and $m \in \mathbb{N}_0$,

$$\rho_m^c \geq 2\pi \frac{\sin(x_0)^2}{x_0^2} \exp\left(-\frac{x_0}{c}(m+1) - 2\log\left(\frac{7e\pi}{x_0}\right) m\right).$$

Thus, for all $c > 0$, we have $\rho_0^c \geq 1$ (by letting x_0 tend to 0) and, for all $m \in \mathbb{N}$,

$$\rho_m^c \geq 2\pi e^{-2\log(7e\pi)m} \sup_{x_0\in(0,\pi/2]} \left(\frac{\sin(x_0)}{x_0}\right)^2 e^{-x_0/c} \exp\left(-\left(\frac{x_0}{c} - 2\log(x_0)\right) m\right). \tag{14}$$

Using that if $2c < \pi$, $x_0 \mapsto x_0/c - 2\log(x_0)$ admits a minimum at $x_0 = 2c$, we obtain, for all $0 < c \leq \pi/4$,

$$\rho_m^c \geq 2\pi \frac{\sin(2c)^2}{(2ec)^2} \exp\left(-2\log\left(\frac{7e^2\pi}{2c}\right) m\right).$$

We now prove the second bound on ρ_m^c. Let $m \in \mathbb{N}_0$ and $t_m = \pi(m+1)/2$. For all $x \in \mathbb{R}$, we have $\operatorname{sech}(x/c) \geq \operatorname{sech}(t_m/c)\, \mathbb{1}\{|x| \leq t_m\}$, hence, by (9), we have

$$\begin{aligned}
\rho_m^c &= \max_{V \in S_{m+1}} \min_{f \in V \setminus \{0\}} \int_{\mathbb{R}} \operatorname{sech}\left(\frac{x}{c}\right) \left|\mathcal{F}\left[\mathcal{E}\left[f\right]\right](x)\right|^2 dx \frac{1}{\|f\|^2_{L^2(-1,1)}} \\
&\geq \operatorname{sech}\left(\frac{t_m}{c}\right) \max_{V \in S_{m+1}} \min_{f \in V \setminus \{0\}} \int_{\mathbb{R}} \mathbb{1}\{|x| \leq t_m\} \left|\mathcal{F}\left[\mathcal{E}\left[f\right]\right](x)\right|^2 dx \frac{1}{\|f\|^2_{L^2(-1,1)}} \\
&\geq \operatorname{sech}\left(\frac{t_m}{c}\right) \rho_m^{W_{[-1,1]}, t_m}. \qquad (15)
\end{aligned}$$

Using that $m = 2t_m/\pi - 1$ and (5.2) in [12] (with a difference by a factor $1/(2\pi)$ in the normalisation of $\mathcal{Q}_c^{W_{[-1,1]}}$), we have $\rho_m^{W_{[-1,1]}, t_m} \geq \pi$ hence, for all $m \in \mathbb{N}_0$,

$$\begin{aligned}
\rho_m^c &\geq \exp\left(-\frac{t_m}{c}\right) \rho_m^{W_{[-1,1]}, t_m} \quad \text{(by (15))} \\
&\geq \pi \exp\left(-\frac{\pi(m+1)}{2c}\right).
\end{aligned}$$

□

The best lower bound in terms of the factor in the exponential is (10) for $c \leq c_0$, where $c_0 = 0.12059$, and (11) for larger c (see Figure 1). This yields

Corollary 5. For all $c > 0$,

$$\forall m \in \mathbb{N},\ \rho_m^c \geq \theta(c) e^{-2\beta(c) m}, \qquad (16)$$
$$\rho_0^c \geq 2\pi$$

where

$$\begin{aligned}
\beta &: c \mapsto \log\left(\frac{7e^2\pi}{2c}\right) \mathbb{1}\{c \leq c_0\} + \frac{\pi}{4c} \mathbb{1}\{c > c_0\}, \\
\theta &: c \mapsto \frac{2\pi \sin(2c)^2}{(2ec)^2} \mathbb{1}\{c \leq c_0\} + \frac{\pi}{e^{\pi/(2c)}} \mathbb{1}\{c > c_0\}.
\end{aligned}$$

Clearly, because $c_0 \leq \pi/4$ and $x \mapsto \sin(x)/x$ is decreasing on $(0, \pi/2)$, the lower bound holds when we replace θ by

$$\widetilde{\theta} :\ c \mapsto \frac{2\pi \sin(2c_0)^2}{(2ec_0)^2} \mathbb{1}\{c \leq c_0\} + \frac{\pi}{e^{\pi/(2c)}} \mathbb{1}\{c > c_0\}.$$

3.2. **Application: Error bounds for stable analytic continuation of functions whose Fourier transform belongs to $L^2(\cosh(b\cdot))$.** In this section, we consider the problem where we observe the function f with error on (x_0-c, x_0+c), for $c>0$ and $x_0 \in \mathbb{R}$,

$$f_\delta(cx+x_0) = f(cx+x_0) + \delta\xi(x), \quad \text{for } a.e.\ x \in (-1,1), \quad \mathcal{F}[f] \in L^2(\cosh(b\cdot)), \tag{17}$$

where $\xi \in L^2(-1,1)$, $\|\xi\|_{L^2(-1,1)} \leq 1$, and $\delta > 0$. We consider the problem of approximating $f_0 = f$ on $L^2(\mathbb{R})$ from f_δ on (x_0-c, x_0+c). This is a classical problem for which an approach based on PSWF is prone to criticism when the researcher does not have a priori information on the bandlimits or when she questions the bandlimited assumption. As we have stressed before such an assumption makes little sense for probability densities.

Noting that, for a.e. $x \in (-1,1)$,

$$\frac{1}{2\pi}\mathcal{F}_{b,c}\left[\mathcal{F}\left[f(x_0-\cdot)\right]\right](x) = f(cx+x_0) \tag{18}$$

suggests the two steps regularising procedure:

(1) approximate $\mathcal{F}\left[f(x_0-\cdot)\right]/(2\pi) \in L^2(\cosh(b\cdot))$ by the spectral cut-off regularization,

$$F_\delta^N = \sum_{m\leq N} \frac{1}{\sigma_m^{b,c}} \left\langle f_\delta(c\cdot+x_0), g_m^{c/b}(\cdot)\right\rangle_{L^2(-1,1)} \varphi_m^{b,c}, \tag{19}$$

(2) take the inverse Fourier transform and define

$$f_\delta^N(\cdot) = 2\pi\mathcal{F}^I\left[F_\delta^N\right](x_0-\cdot). \tag{20}$$

These steps require numerical approximations of an inner product, of an inverse Fourier transform over $\mathbb{R}$, and of the singular functions. Sections 7 and 8 address these issues. The lower bounds on the eigenvalues of $\mathcal{Q}_{c/b}$ of Theorem 4 are useful to obtain rates of convergence when $\mathcal{F}[f]$, which appears on the left-hand side of (18), satisfies a source condition: $f \in \mathcal{H}_{\omega,x_0}^{b,c}(M)$, where

$$\mathcal{H}_{\omega,x_0}^{b,c}(M) = \left\{ f : \quad \sum_{m\in\mathbb{N}_0} \omega_m^2 \left| \left\langle \mathcal{F}[f(x_0-\cdot)], \varphi_m^{b,c}\right\rangle_{L^2(\cosh(b\cdot))}\right|^2 \leq M^2 \right\} \tag{21}$$

for a given sequence $(\omega_m)_{m\in\mathbb{N}_0}$. The set can also be written as

$$\mathcal{H}_{\omega,x_0}^{b,c}(M) = \left\{ f : \quad \sum_{m\in\mathbb{N}_0} \left(2\pi\frac{\omega_m}{\sigma_m^{b,c}}\right)^2 \left| \left\langle f(c\cdot+x_0), g_m^{b,c}\right\rangle_{L^2(-1,1)}\right|^2 \leq M^2 \right\}. \tag{22}$$

This amounts to smoothness of $f(c\cdot+x_0)$ on $(-1,1)$. When $\omega_m = 1$ for all m this corresponds to analyticity of f in $\mathbb{R}$. We consider below the case where we have a preexponential polynomial or exponential sequence ω_m. Theorem 1 in [14] provides a comparison between the smoothness in terms of a summability condition involving the coefficients on the PSWF basis and Sobolev smoothness on $(-1,1)$. Such a result is not available when the PSWF basis is replaced by $(g_m^{b,c})_{m\in\mathbb{N}_0}$ and requires further investigation.

Theorem 6. Take $M > 0$ and define β as in (16), then we have

(1) for $(\omega_m)_{m\in\mathbb{N}_0} = (m^\sigma)_{m\in\mathbb{N}_0}$, $\sigma > 1/2$, $N = \lfloor \overline{N} \rfloor$, and $\overline{N} = \ln(1/\delta)/(2\beta(c/b))$,

$$\sup_{f\in\mathcal{H}^{b,c}_{\omega,x_0}(M),\|\xi\|_{L^2(-1,1)}\leq 1} \left\|f_\delta^N - f\right\|_{L^2(\mathbb{R})} = \underset{\delta\to 0}{O}((-\log(\delta))^{-\sigma}), \tag{23}$$

(2) for $(\omega_m)_{m\in\mathbb{N}_0} = (e^{\kappa m})_{m\in\mathbb{N}_0}$, $\kappa > 0$, $N = \lfloor \overline{N} \rfloor$, and $\overline{N} = \ln(1/\delta)/(\kappa + \beta(c/b))$,

$$\sup_{f\in\mathcal{H}^{b,c}_{\omega,x_0}(M),\|\xi\|_{L^2(-1,1)}\leq 1} \left\|f_\delta^N - f\right\|_{L^2(\mathbb{R})} = \underset{\delta\to 0}{O}\left(\delta^{\kappa/(\kappa+\beta(c/b))}\right). \tag{24}$$

Proof. We have, using the Plancherel equality for the first equality,

$$\begin{aligned}
\left\|f_\delta^N - f\right\|^2_{L^2(\mathbb{R})} =& \frac{1}{2\pi}\left\|\mathcal{F}\left[f_\delta^N\right] - \mathcal{F}\left[f\right]\right\|^2_{L^2(\mathbb{R})} \\
=& \frac{1}{2\pi}\left\|\mathcal{F}\left[f_\delta^N(x_0-\cdot)\right] - \mathcal{F}\left[f(x_0-\cdot)\right]\right\|^2_{L^2(\mathbb{R})} \\
\leq& \frac{1}{2\pi}\left\|\mathcal{F}\left[f_\delta^N(x_0-\cdot)\right] - \mathcal{F}\left[f(x_0-\cdot)\right]\right\|^2_{L^2(\cosh(b\cdot))} \\
\leq& \frac{1}{\pi}\left\|\mathcal{F}\left[f_\delta^N(x_0-\cdot)\right] - \mathcal{F}\left[f_0^N(x_0-\cdot)\right]\right\|^2_{L^2(\cosh(b\cdot))} \\
&+ \frac{1}{\pi}\left\|\mathcal{F}\left[f_0^N(x_0-\cdot)\right] - \mathcal{F}\left[f(x_0-\cdot)\right]\right\|^2_{L^2(\cosh(b\cdot))}.
\end{aligned} \tag{25}$$

Using (19) for the first equality, the Cauchy-Schwarz inequality and (4) for the first inequality, and (16) for the second inequality, we obtain

$$\begin{aligned}
&\left\|\mathcal{F}\left[f_\delta^N(x_0-\cdot)\right] - \mathcal{F}\left[f_0^N(x_0-\cdot)\right]\right\|^2_{L^2(\cosh(b\cdot))} \\
&= \left\|\sum_{m\leq N}\frac{2\pi}{\sigma_m^{b,c}}\left\langle (f_\delta - f)(c\cdot+x_0), g_m^{c/b}(\cdot)\right\rangle_{L^2(-1,1)}\varphi_m^{b,c}(\cdot)\right\|^2_{L^2(\cosh(b\cdot))} \\
&= \sum_{m\leq N}\left(\frac{2\pi}{\sigma_m^{b,c}}\right)^2\left|\left\langle (f_\delta - f)(c\cdot+x_0), g_m^{c/b}(\cdot)\right\rangle_{L^2(-1,1)}\right|^2
\end{aligned}$$

$$\leq (2\pi)^2 \left\|(f_\delta - f)(c\cdot + x_0)\right\|^2_{L^2(-1,1)} \sum_{m\leq N} \frac{c}{\rho_m^{c/b}}$$

$$\leq \frac{(2\pi)^2 c\delta^2}{\theta(c)} \left\|\xi\right\|^2_{L^2(-1,1)} \sum_{m\leq N} e^{2\beta(c/b)m}$$

$$\leq \frac{(2\pi)^2 c\delta^2}{\theta(c)\left(1-e^{-2\beta(c/b)}\right)} e^{2\beta(c/b)N}. \tag{26}$$

Using (20), we have

$$\begin{aligned}
\mathcal{F}\left[f_0^N(x_0-\cdot)\right](\star) &= \sum_{m\leq N} \frac{2\pi}{\sigma_m^{b,c}} \left\langle f(c\cdot+x_0), g_m^{c/b}(\cdot)\right\rangle_{L^2(-1,1)} \varphi_m^{b,c}(\star) \\
&= \sum_{m\leq N} \frac{2\pi}{\sigma_m^{b,c}} \left\langle \mathcal{F}_{b,c}\left[\frac{1}{2\pi}\mathcal{F}\left[f(x_0-\cdot)\right]\right], g_m^{c/b}\right\rangle_{L^2(-1,1)} \varphi_m^{b,c}(\star) \\
&= \sum_{m\leq N} \frac{1}{\sigma_m^{b,c}} \left\langle \mathcal{F}\left[f(x_0-\cdot)\right], \mathcal{F}^*_{b,c}\left[g_m^{c/b}\right]\right\rangle_{L^2(\cosh(b\cdot))} \varphi_m^{b,c}(\star) \\
&= \sum_{m\leq N} \left\langle \mathcal{F}\left[f(x_0-\cdot)\right], \varphi_m^{c/b}\right\rangle_{L^2(\cosh(b\cdot))} \varphi_m^{b,c}(\star).
\end{aligned}$$

Thus, using Proposition 2 and Pythagoras' theorem, we obtain

$$\begin{aligned}
\left\|\mathcal{F}\left[f_0^N(x_0-\cdot)\right] - \mathcal{F}\left[f(x_0-\cdot)\right]\right\|^2_{L^2(\cosh(b\cdot))} &= \sum_{m>N} \left|\left\langle \mathcal{F}[f(x_0-\cdot)], \varphi_m^{b,c}(\cdot)\right\rangle_{L^2(\cosh(b\cdot))}\right|^2 \\
&\leq \sum_{m\in\mathbb{N}_0} \left(\frac{\omega_m}{\omega_N}\right)^2 \left|\left\langle \mathcal{F}[f(x_0-\cdot)], \varphi_m^{b,c}(\cdot)\right\rangle_{L^2(\cosh(b\cdot))}\right|^2 \\
&\leq \frac{M^2}{\omega_N^2} \quad (\text{using } f\in\mathcal{H}^{b,c}_{\omega,x_0}(M)). \qquad (27)
\end{aligned}$$

Finally, using (25)-(27) yields

$$\left\|f_\delta^N - f\right\|^2_{L^2(\mathbb{R})} \leq \frac{1}{\pi}\left(\frac{(2\pi)^2 c}{\theta(c)\left(1-e^{-2\beta(c/b)}\right)}\delta^2 e^{2\beta(c/b)N} + \frac{M^2}{\omega_N^2}\right). \tag{28}$$

Consider case (1). Take δ small enough so that $\overline{N}\geq 2$ and $\log\left(\delta\log(1/\delta)^{2\sigma}\right)\leq 0$. By (28) and the definition of $(\omega_N)_{n\in\mathbb{N}_0}$ in the first display below, $\overline{N}-1\leq N\leq\overline{N}$ in the second display, and $\overline{N}\geq 2$ in the third display, we obtain

$$\left\|f_\delta^N - f\right\|^2_{L^2(\mathbb{R})} \leq \frac{N^{-2\sigma}}{\pi}\left(\frac{(2\pi)^2 c}{\theta(c)\left(1-e^{-2\beta(c/b)}\right)}\delta^2 e^{2\beta(c/b)N} N^{2\sigma} + M^2\right)$$

$$\leq \frac{\overline{N}^{-2\sigma}\left(1-1/\overline{N}\right)^{-2\sigma}}{\pi}\left(\frac{(2\pi)^2 c}{\theta(c)\left(1-e^{-2\beta(c/b)}\right)}\delta^2 e^{2\beta(c/b)\overline{N}}\overline{N}^{2\sigma}+M^2\right)$$
$$\leq \frac{\overline{N}^{-2\sigma}2^{2\sigma}}{\pi}\left(\frac{(2\pi)^2 c}{\theta(c)\left(1-e^{-2\beta(c/b)}\right)}\delta^2 e^{2\beta(c/b)\overline{N}}\overline{N}^{2\sigma}+M^2\right).$$

Using that

$$\delta^2\exp\left(2\beta\left(\frac{c}{b}\right)\overline{N}\right)\overline{N}^{2\sigma}=\exp\left(2\sigma\log\left(\frac{1}{2\beta(c/b)}\right)+\log\left(\delta\log\left(\frac{1}{\delta}\right)^{2\sigma}\right)\right)\leq\left(\frac{1}{\beta(c/b)}\right)^{2\sigma},$$

yields

(29)
$$\left\|f_\delta^N-f\right\|_{L^2(\mathbb{R})}^2\leq\frac{1}{\pi}\left(4\beta\left(\frac{c}{b}\right)\right)^{2\sigma}\left(\frac{(2\pi)^2 c}{\theta(c)\left(1-e^{-2\beta(c/b)}\right)}\left(\frac{1}{\beta(c/b)}\frac{\sigma}{e}\right)^{2\sigma}+M^2\right)(-\log(\delta))^{-2\sigma},$$

hence the result.
Consider now case (2). Using $\overline{N}-1\leq N\leq\overline{N}$ in the first display and $\delta^2\exp\left(2\left(\beta\left(\frac{c}{b}\right)+\kappa\right)\overline{N}\right)=1$ and the definition of $\overline{N}$ in the second display, yields

$$\left\|f_\delta^N-f\right\|_{L^2(\mathbb{R})}^2\leq\frac{e^{-2\kappa\left(\overline{N}-1\right)}}{\pi}\left(\frac{(2\pi)^2 c}{\theta(c)\left(1-e^{-2\beta(c/b)}\right)}\delta^2 e^{2(\beta(c/b)+\kappa)\overline{N}}+M^2\right)$$
$$\leq\frac{e^{2\kappa}}{\pi}\left(\frac{(2\pi)^2 c}{\theta(c)\left(1-e^{-2\beta(c/b)}\right)}+M^2\right)\delta^{2\kappa/(\kappa+\beta(c/b))},$$

hence the result. □

The rate in (23) does not depend on c but the constant blows up as $c\to 0$ (see (29)). In contrast, the rate in (24) deteriorates for small values of c. The result (24) is related to those obtained for the so-called "2exp-severely ill-posed problems" (see [15] for a survey and [46] which obtains similar polynomial rates) where the singular values decay exponentially and the functions are supersmooth.

The proof of Theorem 6 requires an upper bound on a sum involving the singular values for small m in the denominator. Theorem 4 allows to obtain (26). Without it, one could at best obtain, instead of (26), the upper bound $(2\pi)^2 c\delta^2(N+1)/\rho_N^{b,c}$. Because (6) is an equivalent of the logarithm we are unable to obtain a polynomial rate of convergence as sharp as in (24).

4. Upper bounds on the eigenvalues of $\mathcal{Q}_c$

Theorem 7. For $m \in \mathbb{N}_0$ and $0 < c < 1$, we have

$$\rho_m^c \leq \frac{2ec^{2m+1}}{\sqrt{2m+1}(1-c^2)}.$$

The proof of this result uses arguments which have been used in the proof of Theorem 3.1 in [12] for the PSWF.

Proof. Using the minimax principle (see Theorem 4 page 212 in [10]), the $m+1$-th eigenvalue ρ_m^c satisfies

$$\rho_m^c = \min_{V \in S_m} \max_{f \in V^\perp} \frac{\langle \mathcal{Q}_c f, f \rangle_{L^2(-1,1)}}{\|f\|^2_{L^2(-1,1)}},$$

where S_m is the set of m-dimensional vector subspaces of $L^2(-1,1)$. We use (8), which yields

$$\rho_m^c = \min_{V \in S_m} \max_{f \in V^\perp} \frac{c \left\langle \mathcal{F}^*_{1,c}[f], \mathcal{F}^*_{1,c}[f] \right\rangle_{L^2(\cosh)}}{\|f\|^2_{L^2(-1,1)}}.$$

We denote by $(P_m)_{m\in\mathbb{N}_0}$ the Legendre polynomials with the normalization $P_m(1) = 1$. They are such that $\left(\sqrt{m+1/2}P_m\right)_{m\in\mathbb{N}_0}$ is an orthonormal basis of $L^2(-1,1)$. Let V be the vector space spanned by $P_0, \ldots, P_{m-1}$. Take $f \in V^\perp$ of norm 1. It is of the form $f = \sum_{k=m}^\infty a_k \sqrt{k+1/2} P_k$, where $\sum_{k=m}^\infty |a_k|^2 = 1$. The Cauchy-Schwarz inequality yields, for *a.e.* $x \in \mathbb{R}$,

$$\left|\mathcal{F}^*_{1,c} f(x)\right|^2 \leq \left(\sum_{k=m}^\infty |a_k|^2\right) \left(\sum_{k=m}^\infty \left(k+\frac{1}{2}\right) \left|\mathcal{F}^*_{1,c} P_k(x)\right|^2\right)$$

and after integration

$$\left\|\mathcal{F}^*_{1,c} f\right\|^2_{L^2(\cosh)} \leq \sum_{k=m}^\infty \left(k+\frac{1}{2}\right) \left\|\mathcal{F}^*_{1,c} P_k\right\|^2_{L^2(\cosh)}.$$

Thus, we have

$$\rho_m^c \leq c \sum_{k=m}^\infty \left(k+\frac{1}{2}\right) \left\|\mathcal{F}^*_{1,c} P_k\right\|^2_{L^2(\cosh)}. \tag{30}$$

Then, using (18.17.19) in [36], we obtain, for *a.e.* x and $c > 0$,

$$\begin{aligned}
\mathcal{F}^*_{1,c}\left[P_k\right](x) &= \operatorname{sech}(x)\, \mathcal{F}_{1,c}\left[\mathcal{E}\left[P_k\right]\right](-x) \\
&= \operatorname{sech}(x)\, i^{-k} \sqrt{\frac{2\pi}{c\,|x|}} J_{k+1/2}(c\,|x|),
\end{aligned}$$

where $J_{k+1/2}$ is the Bessel function of order $k+1/2$. Using (9.1.62) in [1] in the first display, $\Gamma(k+3/2) = (k+1/2)\Gamma(k+1/2)$ and $\Gamma(k+1/2) \geq \sqrt{2\pi}e^{-k-1/2}(k+1/2)^k$ (see (1.4) in [33]) in the second display, we obtain

$$\begin{aligned}\left|\mathcal{F}_{1,c}^*\left[P_k\right](x)\right| &\leq \operatorname{sech}(x)\sqrt{\pi}\frac{|cx/2|^k}{\Gamma(k+3/2)}.\\ &\leq \operatorname{sech}(x)\sqrt{\frac{e}{2}}\frac{1}{k+1/2}\left(\frac{ec}{2(k+1/2)}\right)^k|x|^k.\end{aligned}$$

Thus, we have

$$\begin{aligned}\left\|\mathcal{F}_{1,c}^*\left[P_k\right]\right\|_{L^2(\cosh)}^2 &\leq \frac{e}{2(k+1/2)^2}\left(\frac{ec}{2(k+1/2)}\right)^{2k}\int_{\mathbb{R}} x^{2k}\operatorname{sech}(x)\,dx\\ &\leq \frac{e}{(k+1/2)^2}\left(\frac{ec}{2(k+1/2)}\right)^{2k}\int_0^\infty x^{2k}e^{-x}dx\\ &\leq \frac{e}{(k+1/2)^2}\left(\frac{ec}{2(k+1/2)}\right)^{2k}\Gamma(2k+1).\end{aligned}$$

Then, by (30), we have

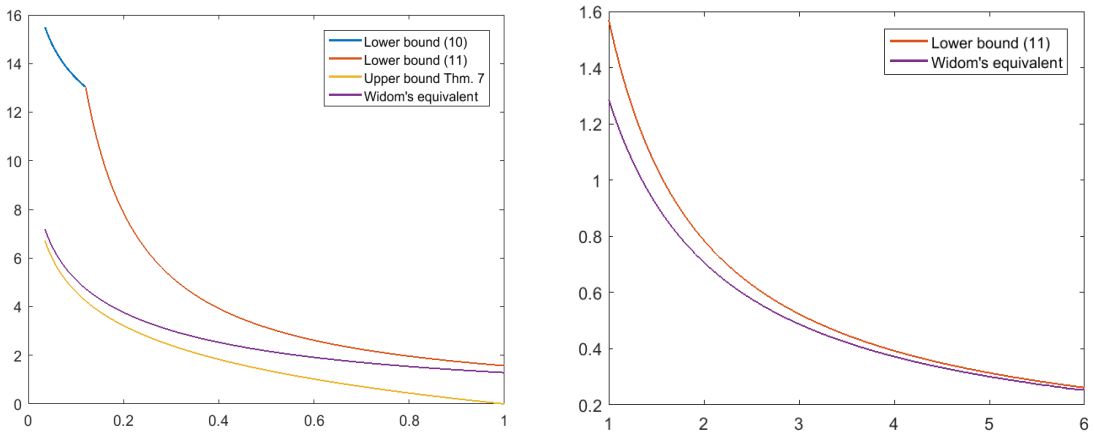
$$\begin{aligned}\rho_m^c &\leq ec\sum_{k=m}^\infty \frac{1}{k+1/2}\left(\frac{ec}{2(k+1/2)}\right)^{2k}\Gamma(2k+1)\\ &\leq ec\sum_{k=m}^\infty \frac{1}{k+1/2}\left(\frac{ec}{2(k+1/2)}\right)^{2k}(2k+1)^{2k+1/2}e^{-2k} \quad \text{(using (1.3) in [33])}\\ &= 2ec\sum_{k=m}^\infty \frac{1}{\sqrt{2k+1}}c^{2k}\\ &\leq \frac{2ec}{\sqrt{2m+1}}\sum_{k=m}^\infty c^{2k}\end{aligned}$$

hence, for $0<c<1$, this yields the result. □

Theorem 7 holds for a limited range of values of c but this range is enough to construct the so-called test functions to prove the minimax lower bounds in [21]. Note also that, by (10) and Theorem 7, we have, for all $0<c\leq c_0<1$ and $m\in\mathbb{N}$,

$$\frac{2\pi\sin(2c_0)^2}{(2ec_0)^2}\exp\left(-2\left(\log\left(\frac{1}{c}\right)+2+\log\left(\frac{7\pi}{2}\right)\right)m\right) \leq \rho_m^c \leq \frac{2ec_0}{\sqrt{2m+1}(1-c_0^2)}\exp\left(-2\log\left(\frac{1}{c}\right)m\right).$$

These upper and lower bounds have a similar behavior as c approaches 0 (see also Figure 1).



(a) $c < 1$ (b) $c \geq 1$

FIGURE 1. Bounds on $\lim_{m\to\infty} -\log(\rho_m^c)/m$ and Widom's equivalent (6) as a function of c.

5. Properties of a differential operator which commutes with $\mathcal{Q}_c$

In this section, we consider differential operators $\mathcal{L}[\psi] = -(p\psi')' + q\psi$ on $L^2(-1,1)$, with (1) $p(x) = \cosh(4c) - \cosh(4cx)$ and $q(x) = 3c^2\cosh(4cx)$, (2) $p(x) = 1-x^2$ and $q(x) = q^c(x)$ where, for $Y(x) = \sin(X(x))$, $X(x) = (\pi/U(c))\int_0^x p(\xi)^{-1/2}d\xi$, and

$$U(c) = \int_{-1}^{1} p(\xi)^{-1/2}d\xi,$$

$$q^c(Y(x)) = \frac{1}{2} + \frac{1}{4}\tan(X(x))^2 - \left(\frac{U(c)c}{\pi}\right)^2\left(\cosh(4cx) + \frac{\sinh^2(4cx)}{p(x)}\right), \tag{31}$$

and (3) $p(x) = 1-x^2$ and $q(x) = 0$. By [47] (see also [32]), the eigenfunctions of $\mathcal{Q}_c$ are those of the differential operator in case (1) with domain $\mathcal{D} \subset \mathcal{D}_{\max} = \{\psi \in L^2(-1,1) : \mathcal{L}[\psi] \in L^2(-1,1)\}$ with boundary conditions of continuity at ± 1. This is an important property for the asymptotic analysis in [47] and to obtain bounds on the sup-norm of these functions in Section 6 and numerical approximations of them in Section 7. To study $\mathcal{L}$ in case (1), [47] uses the changes of variable and function, for all $x \in (-1,1)$ and $\psi \in \mathcal{D}_{\max}$,

$$y = Y(x), \tag{32}$$

$$\forall y \in (-1,1),\ \Gamma(y) = F(y)\psi\left(Y^{-1}(y)\right),\ F(y) = \left(\frac{p(Y^{-1}(y))}{1-y^2}\right)^{1/4}, \tag{33}$$

where Y is a $C^\infty-$diffeomorphism on $(-1,1)$. This relates an eigenvalue problem for (1) to an eigenvalue problem for (2) and it is useful to view the operator in case (2) as a perturbation of the operator in case (3). In the three cases, $1/p$ and q are holomorphic on a simply connected open set $(-1,1) \subseteq E \subseteq \mathbb{C}$. The spectral analysis involves the solutions to (H_λ): $-(p\psi')' + (q-\lambda)\psi = 0$ with $\lambda \in \mathbb{C}$ which are holomorphic on E and span a vector space of dimension 2 (see Sections IV 1 and 10 in [27]). So they are infinitely differentiable on $(-1,1)$, have isolated zeros in $(-1,1)$, and the condition of continuity (or boundedness) at ± 1 makes sense.

We now present a few useful estimates.

Lemma 8. We have, for all $c > 0$,

$$\frac{\sqrt{2}e^{2c}}{\sinh(4c)} < U(c) < \pi\frac{\sqrt{2}e^{2c}}{\sinh(4c)}.$$

Proof. By the second equation page 229 of [47]

$$U(c) = \frac{1}{c(1+\cosh(4c))^{1/2}} K\left(\frac{e^{4c}-1}{e^{4c}+1}\right),$$

and the result follows from the fact that, by Corollary 3.3 in [3],

$$\frac{ce^{2c}}{\sinh(2c)} < K\left(\frac{e^{4c}-1}{e^{4c}+1}\right) < \frac{\pi c e^{2c}}{\sinh(2c)}.$$

We obtain the final expressions by classical relations between hyperbolic functions. □

We make use of the identity, for all $x \in [-1,1]$,

$$p(x) = 4c\sinh(4c)(1-x)(1+u(x)), \tag{34}$$

$$u(x) = \int_x^1 \frac{4c\cosh(4ct)}{\sinh(4c)(1-x)}(x-t)dt, \tag{35}$$

which is obtained by Taylor's theorem with remainder in integral form

$$\cosh(4cx) = \cosh(4c) + (x-1)4c\sinh(4c) + \int_1^x 16c^2\cosh(4ct)(x-t)dt.$$

Also, u is increasing on $[-1,1]$ because, for all $x \in [-1,1]$,

$$u'(x) = \frac{4c}{\sinh(4c)(1-x)^2}\int_x^1 \cosh(4ct)(1-t)dt > 0 \tag{36}$$

and, for all $x \in [0,1]$,

$$-1 + \frac{1}{4c\sinh(4c)}\left(\cosh(4c)-1\right) \le u(x) \le 0. \tag{37}$$

Lemma 9. We have, for all $c>0$ and $x \in [0,1]$,

$$\frac{c\sinh(4c)}{1-x} - \frac{8c^3\sinh(4c)\cosh(4c)}{3\left(\cosh(4c)-1\right)} \le \left(\int_x^1 p(\xi)^{-1/2}d\xi\right)^{-2} \le \frac{c\sinh(4c)}{1-x}.$$

Proof. We have

$$\left(\int_x^1 p(\xi)^{-1/2}d\xi\right)^{-2} = \left(\frac{1}{(4c\sinh(4c))^{1/2}}\left(2(1-x)^{1/2} - \int_x^1\int_1^\xi \frac{u'(t)}{2\sqrt{1-\xi}(1+u(t))^{3/2}}d\xi dt\right)\right)^{-2}$$

$$= \frac{c\sinh(4c)}{1-x}\frac{1}{(1+\widetilde{u}(x))^2} \tag{38}$$

$$= \frac{c\sinh(4c)}{1-x} - \frac{c\sinh(4c)(2+\widetilde{u}(x))\widetilde{u}(x)}{(1-x)(1+\widetilde{u}(x))^2}, \tag{39}$$

where

$$\widetilde{u}(x) = \int_x^1 \frac{1}{4\sqrt{(1-\xi)(1-x)}}\left(\int_\xi^1 \frac{u'(t)}{(1+u(t))^{3/2}}dt\right)d\xi. \tag{40}$$

The upper bound in Lemma 9 uses that, for all $x \in [0,1]$, $\widetilde{u}(x) \ge 0$. We now consider the lower bound. By (36), u is a C^1-diffeomorphism and

$$\int_x^1 \frac{u'(t)dt}{(1+u(t))^2}dt = -\frac{u(x)}{1+u(x)}. \tag{41}$$

Now, by (35), we have, for all $x \in [0,1]$,

$$-u(x) \le \frac{4c\cosh(4c)}{\sinh(4c)(1-x)}\int_x^1 (t-x)dt$$

$$= \frac{2c\cosh(4c)}{\sinh(4c)}(1-x),$$

and, by (37),

$$\int_x^1 \frac{u'(t)dt}{(1+u(t))^2}dt \le 8c^2\frac{\cosh(4c)}{\cosh(4c)-1}(1-x). \tag{42}$$

Now, using that $\widetilde{u}(x) \ge 0$ and that $g : t \mapsto (2+t)/(1+t)^2$ is decreasing on $[0,\infty)$ hence $g(t)t \le 2t$ for $t \ge 0$, we have

$$\frac{(2+\widetilde{u}(x))\widetilde{u}(x)}{(1+\widetilde{u}(x))^2} \le \int_x^1 \frac{1}{2\sqrt{(1-\xi)(1-x)}}\left(\int_\xi^1 \frac{u'(t)}{(1+u(t))^2}dt\right)d\xi \quad \text{(by (40))}$$

$$\leq \frac{4c^2\cosh(4c)}{\cosh(4c)-1}\int_x^1 \sqrt{\frac{1-\xi}{1-x}}d\xi \quad \text{(by (42))}$$

$$\leq \frac{8c^2\cosh(4c)}{3\left(\cosh(4c)-1\right)}(1-x). \tag{43}$$

□

Proposition 10. *F is such that*

$$\|F\|^4_{L^\infty([-1,1])} \leq 2\pi^2 e^{4c}c^2 \tag{44}$$

$$\|1/F\|^4_{L^\infty([-1,1])} \leq \frac{\pi^2 e^{-4c}}{4c}\left(1+\frac{4c^2}{3}\right)^2 \coth(2c). \tag{45}$$

For all $c>0$ and $\lambda\in\mathbb{C}$, the change of variables and function (32)-(33) maps a solution of $\left(H_{U(c)^2\lambda/\pi^2}\right)$ in case (2) to a solution of (H_λ) in case (1) and reciprocally the inverse transformation maps a solution of (H_λ) in case (1) to a solution of $\left(H_{U(c)^2\lambda/\pi^2}\right)$ in case (2) and is a bijection of $\mathcal{D}$. Also, q^c can be extended by continuity to $[-1,1]$ and, for all $y\in[-1,1]$,

$$\frac{1}{2}-\left(\frac{U(c)c}{\pi}\right)^2 - R(c) \leq q^c(y) \leq \frac{1}{2}-\left(\frac{U(c)c}{\pi}\right)^2, \tag{46}$$

where

$$R(c) = \frac{2}{\pi^2}+\left(\frac{U(c)c}{\pi}\right)^2\left(\left(\cosh(4c)\left(1+\frac{c}{3}\coth(2c)\right)-1\right)+2c\sinh(4c)\right).$$

Proof. To prove (44) and (45) it is sufficient, by parity, to consider $x\in[0,1]$.
(44) is a obtained by the following sequence of inequalities

$$\begin{aligned}
\frac{p(x)}{1-Y(x)^2} =& \frac{p(x)}{\sin^2\left(\pi\int_x^1 p(\xi)^{-1/2}d\xi/U(c)\right)}\\
\leq& \left(\frac{U(c)}{2}\right)^2\frac{p(x)}{\left(\int_x^1 p(\xi)^{-1/2}d\xi\right)^2} \quad \text{(because } \sin(x)\geq 2x/\pi\text{)}\\
\leq& \left(\frac{U(c)}{2}\right)^2 p(x)\frac{c\sinh(4c)}{1-x} \quad \text{(by Lemma 9)}\\
\leq& \left(\frac{U(c)}{2}\right)^2 4c^2\sinh(4c)^2(1+u(x)) \quad \text{(by (34))}\\
\leq& \frac{\pi^2 e^{4c}c^2\sinh(4c)^2}{\sinh(2c)^2(1+\cosh(4c))} \quad \text{(by Lemma 8 and (37)).}
\end{aligned}$$

We obtain (45) by the inequalities below. Using for the first display that, for $x \in [0, \pi/2]$, $\sin(x) \leq x$, (34) and (38) for the second display, (37) and (43) for the third, and Lemma 8 for the fourth, we obtain, for all $x \in [0, 1)$,

$$\begin{aligned}\frac{1 - Y(x)^2}{p(x)} &\leq \left(\frac{\pi}{U(c)}\right)^2 \frac{\left(\int_x^1 p(\xi)^{-1/2} d\xi\right)^2}{p(x)} \\ &\leq \left(\frac{\pi}{U(c)}\right)^2 \frac{2\left(1 + \widetilde{u}(x)\right)^2}{(4c\sinh(4c))^2} \frac{1}{1 + u(x)} \\ &\leq \left(\frac{\pi}{U(c)}\right)^2 \frac{(1 + 4c^2/3)^2}{2c\sinh(4c)^2} \frac{\sinh(4c)}{\cosh(4c) - 1} \\ &\leq \frac{\pi^2 e^{-4c}}{4c} \frac{(1 + 4c^2/3)^2 \sinh(4c)}{\cosh(4c) - 1}.\end{aligned}$$

Classical relations between hyperbolic functions yield the final expressions for (44) and (45). Let Γ and ψ related via (33). Buy the above if one function is in $\mathcal{D}$ the other is s well. Moreover, by (33), we have

$$F'(y) = \frac{F(y)}{4}\left(\frac{p'}{pY'}\left(Y^{-1}(y)\right) + \frac{2y}{1 - y^2}\right)$$

$$(1 - y^2)\Gamma'(y) = \frac{F(y)}{4}\left((1 - y^2)\left(\frac{p'\psi}{pY'} + 4\frac{\psi'}{Y'}\right)\left(Y^{-1}(y)\right) + 2y\psi\left(Y^{-1}(y)\right)\right)$$

so differentiating a second time and injecting the above inequality, yields

$$\begin{aligned}\left((1 - y^2)\Gamma'\right)'(y) = \frac{F(y)}{4}\Big(&\frac{1}{4}\left(\frac{p'}{pY'}\left(Y^{-1}(y)\right) + \frac{2y}{1 - y^2}\right)\left((1 - y^2)\left(\frac{p'\psi}{pY'} + 4\frac{\psi'}{Y'}\right)\left(Y^{-1}(y)\right) + 2y\psi\left(Y^{-1}(y)\right)\right) \\ &- 2y\left(\frac{p'\psi}{pY'} + 4\frac{\psi'}{Y'}\right)\left(Y^{-1}(y)\right) + (1 - y^2)\left[\frac{1}{Y'}\left(\frac{p'\psi}{pY'} + 4\frac{\psi'}{Y'}\right)'\right]\left(Y^{-1}(y)\right) \\ &+ 2\left(\psi\left(Y^{-1}(y)\right) + y\frac{\psi'}{Y'}\left(Y^{-1}(y)\right)\right)\Big).\end{aligned}$$

Dividing by $F(y)/4$ and using (32), Γ is solution of $\left(H_{U(c)^2\lambda/\pi^2}\right)$ iff ψ is solution on $(-1, 1)$ of

$$\begin{aligned}&\frac{1}{4p(x)}\left(p'(x) + \frac{2YY'p}{1 - Y^2}(x)\right)\left(\frac{1 - Y^2}{(Y')^2 p}(x)\left(p'\psi + 4p\psi'\right)(x) + 2\frac{Y}{Y'}(x)\psi(x)\right) \\ &- 2\frac{Y}{Y'}(x)\left(\frac{p'\psi}{p} + 4\psi'\right)(x) + \frac{1 - Y^2}{Y'}(x)\left(\frac{p'\psi}{pY'} + 4\frac{\psi'}{Y'}\right)'(x) + 2\left(\psi(x) + \frac{Y}{Y'}(x)\psi'(x)\right) \\ &= 4\left(q^c\left(Y(x)\right) - \frac{U(c)^2\lambda}{\pi^2}\right)\psi(x).\end{aligned}$$

We now use, for all $x \in (-1,1)$,

$$Y'(x) = \frac{\pi}{U(c)p(x)^{1/2}}\cos(X(x)), \tag{47}$$

which yields the equality between C^∞ functions: $(1-Y^2)/\left((Y')^2 p\right) = (U(c)/\pi)^2$ and

$$\begin{aligned}
&\left(1+2\frac{Y}{p'Y'}\left(\frac{\pi}{U(c)}\right)^2\right)\left(\left(\frac{U(c)}{\pi}\right)^2\left(\frac{(p')^2}{4p}\psi+p'\psi'\right)+\frac{Y}{2pp'Y'}\left((p')^2\psi\right)\right)\\
&-2\frac{Y}{p'Y'}\left(\frac{(p')^2}{p}\psi+4p'\psi'\right)+\left(\frac{U(c)}{\pi}\right)^2 pY'\left(\frac{p'\psi}{pY'}+4\frac{\psi'}{Y'}\right)'+2\frac{Y}{p'Y'}p'\psi'\\
&=4\left(q^c(Y)-\frac{1}{2}-\frac{U(c)^2\lambda}{\pi^2}\right)\psi.
\end{aligned}$$

The term in factor of ψ on the left-hand side of the above equality is

$$\left(\frac{U(c)}{\pi}\right)^2\left(1+2\frac{Y}{p'Y'}\left(\frac{\pi}{U(c)}\right)^2\right)^2\frac{(p')^2}{4p}-2\frac{Y}{p'Y'}\frac{(p')^2}{p}+\left(\frac{U(c)}{\pi}\right)^2\frac{pp''Y'-(p')^2Y'-pp'Y''}{pY'}$$

Using $-2pY''=p'Y'+2\left(\pi/U(c)\right)^2 Y$ which is obtained from (47), this becomes

$$\begin{aligned}
&\left(\frac{U(c)}{\pi}\right)^2\left(1+2\frac{Y}{p'Y'}\left(\frac{\pi}{U(c)}\right)^2\right)^2\frac{(p')^2}{4p}-\frac{Y}{Y'}\frac{p'}{p}+\left(\frac{U(c)}{\pi}\right)^2\left(p''-\frac{(p')^2}{2p}\right)\\
&=\left(\frac{Y}{Y'}\right)^2\left(\frac{\pi}{U(c)}\right)^2\frac{1}{p}+\left(\frac{U(c)}{\pi}\right)^2\left(p''-\frac{(p')^2}{4p}\right)\\
&=\tan\left(X(x)\right)^2+\left(\frac{U(c)}{\pi}\right)^2\left(p''-\frac{(p')^2}{4p}\right).
\end{aligned}$$

hence

$$4\left(\frac{U(c)}{\pi}\right)^2(p\psi')'=4\left(q^c(Y)-\frac{1}{2}-\frac{1}{4}\tan\left(X(x)\right)^2+\frac{1}{4}\left(\frac{U(c)}{\pi}\right)^2\left(\frac{(p')^2}{4p}-p''\right)-\left(\frac{U(c)}{\pi}\right)^2\lambda\right)\psi$$

and ψ is solution of (H_λ) in case (1).

We now obtain upper and lower bounds on the even function $q^c(Y(x))$, for $x\in[0,1]$, and start with the lower bound. To bound $\tan(X)^2$ in (31), we use

$$\tan\left(\frac{\pi}{U(c)}\int_0^x p(\xi)^{-1/2}d\xi\right)^2=\left(\tan\left(\frac{\pi}{U(c)}\int_x^1 p(\xi)^{-1/2}d\xi\right)\right)^{-2}, \tag{48}$$

and (96) in [48] in the first display and Lemma 9 and the fact that $(a-b)^2 \geq a^2 - 2ab$ for $a, b > 0$ in the second display. We obtain

$$\begin{aligned}\tan(X(x))^2 &\geq \left(\frac{U(c)}{\pi}\left(\int_x^1 p(\xi)^{-1/2}d\xi\right)^{-1} - \frac{4}{\pi U(c)}\int_x^1 p(\xi)^{-1/2}d\xi\right)^2 \\ &\geq \left(\frac{U(c)c}{\pi}\right)^2\left(\frac{\sinh(4c)}{c(1-x)} - \frac{8c\sinh(4c)\cosh(4c)}{3\left(\cosh(4c)-1\right)}\right) - \frac{8}{\pi^2}.\end{aligned}$$

To bound the second term in the bracket in (31) we proceed as follows. We have

$$\begin{aligned}\frac{4c\sinh(4cx)^2}{p(x)} &= \frac{\sinh(4c)}{1-x}\frac{1}{1+u(x)}\frac{\sinh(4cx)^2}{\sinh(4c)^2} \quad \text{(by (34))} \\ &= \frac{\sinh(4c)}{1-x}\left(1 + \int_x^1 \frac{u'(t)dt}{(1+u(t))^2}\right) \quad \text{(by (41))} \\ &\leq \frac{\sinh(4c)}{1-x}\left(1 + 8c^2(1-x)\right) \quad \text{(by (42))},\end{aligned} \tag{49}$$

hence

$$\begin{aligned}q^c(Y(x)) \geq& \frac{1}{2} - \frac{2}{\pi^2} - \left(\frac{U(c)c}{\pi}\right)^2\left(\cosh(4c)\left(1 + \frac{2c\sinh(4c)}{3\left(\cosh(4c)-1\right)}\right) + 2c\sinh(4c)\right) \\ \geq& \frac{1}{2} - \left(\frac{U(c)c}{\pi}\right)^2 - R(c).\end{aligned}$$

Consider the upper bound on q^c. For $x \in [0,1]$, by (48) and $0 < z \leq \tan(z)$ on $(0, \pi/2]$, we have

$$q^c(Y(x)) \leq \frac{1}{2} + \left(\frac{U(c)c}{\pi}\right)^2\left(\frac{1}{4c^2\left(\int_x^1 p(\xi)^{-1/2}d\xi\right)^2} - \frac{\sinh(4cx)^2}{p(x)} - \cosh(4cx)\right).$$

Using Lemma 9, (49), and (36), we have

$$q^c(Y(x)) \leq \frac{1}{2} - \left(\frac{U(c)c}{\pi}\right)^2. \tag{50}$$

□

The unbounded operator $\mathcal{L}$ on domain $\mathcal{D}$ in case (3) is self-adjoint. Indeed, it is shown page 571 of [35] that $\mathcal{D}$ is the domain of the self-adjoint Friedrichs extension of the minimal operator corresponding to the differential operator on $L^2(-1,1)$ on the domain $\mathcal{D}_{\min}$ (the subset of $\mathcal{D}_{\max}$ of functions with support in $(-1,1)$, see page 173 in [49], we removed one condition on $\mathcal{D}_{\max}$ which is automatically satisfied). By Proposition 10, the multiplication defined, for

$\psi \in \mathcal{D}_{\max}$, by $\psi \to q^c\psi$ is bounded and symmetric on $L^2(-1,1)$. Thus, by the Kato-Rellich theorem (see, *e.g.*, [39]), the unbounded operator $\mathcal{L}$ on domain $\mathcal{D}$ in case (2) is self-adjoint. Denote by $\left((U(c)/\pi)^2\chi_m^c\right)_{m\in\mathbb{N}_0}$ the eigenvalues of the unbounded operator $\mathcal{L}$ on domain $\mathcal{D}$ in case (2) arranged in increasing order and repeated according to multiplicity. They are real and, because the operator is bounded below, they are bounded below by the same constant. Moreover, Proposition 10 yields that $(\chi_m^c)_{m\in\mathbb{N}_0}$ are the eigenvalues of the unbounded operator $\mathcal{L}$ on domain $\mathcal{D}$ in case (1). The following result gives exact constants and a behavior uniform over m which is coherent with the asymptotic result on page 14 of [47].

Theorem 11. We have, for all $m \in \mathbb{N}_0$ and $c > 0$,

$$\left(\frac{\pi}{U(c)}\right)^2\left(m(m+1)+\frac{1}{2}-R(c)\right)+c^2 \le \chi_m^c \le \left(\frac{\pi}{U(c)}\right)^2\left(m(m+1)+\frac{1}{2}\right)-c^2.$$

Proof. This follows from the min-max theorem and (46). □

6. Uniform estimates on the singular functions g_m^c

Theorem 12. We have, for all $m \in \mathbb{N}_0$ and $c > 0$,

$$\|g_m^c\|_{L^\infty([-1,1])} \le \frac{\pi}{e^{2c}}\left(1+\frac{4c^2}{3}\right)^{1/2}\left(\frac{\sinh(4c)}{4c}\right)^{1/4}\cosh(2c)^{1/2}\left(\frac{2R(c)}{m+1/2}+\left(1+\sqrt{\frac{2}{3}}\frac{R(c)}{m+1/2}\right)\sqrt{m+\frac{1}{2}}\right).$$

The proof of this result uses similar ideas as in the proof of Proposition 5 in [13]. The important additional ingredients are the change of variables and functions and Proposition 10.

Proof. Using in the first display the change of variables (32) and the change of functions (33) with $\psi = g_m^c$, and denoting by $\Gamma_m^c(\cdot) = F(\cdot)g_m^c\left(Y^{-1}(\cdot)\right)$ and $\widetilde{\Gamma}_m^c = \Gamma_m^c\sqrt{U(c)/\pi}$, which is real valued, and (47) and (33) in the second display, we obtain

$$\begin{aligned}\int_{-1}^1 \left|\widetilde{\Gamma}_m^c(y)\right|^2 dy &= \frac{U(c)}{\pi}\int_{-1}^1 Y'(x)\left|F(Y(x))\right|^2\left|g_m^c(x)\right|^2 dx \\ &= \int_{-1}^1 \frac{\cos(X(x))}{\sqrt{1-\sin(X(x))^2}}\left|g_m^c(x)\right|^2 dx = 1.\end{aligned}$$

Also, by Proposition 10, for all $y \in (-1,1)$,

$$\left((1-y^2)\left(\widetilde{\Gamma}_m^c\right)'\right)'(y) + m(m+1)\widetilde{\Gamma}_m^c(y) = \left(m(m+1)-\left(\frac{U(c)}{\pi}\right)^2\chi_m^c + q^c(y)\right)\widetilde{\Gamma}_m^c(y). \tag{51}$$

We obtain, by the method of variation of constants and knowledge of the solutions to the homogenous equation corresponding to the left-hand side of (51), that there exist $A, B \in \mathbb{R}$ such that, for $y \in (-1,1)$,

$$\widetilde{\Gamma}^c_m(y) = A\overline{P}_m(y) + BQ_m(y) + \frac{1}{m+1/2}\int_y^1 L_m(y,z)\sqrt{1-z^2}G_c(z)\widetilde{\Gamma}^c_m(z)dz, \tag{52}$$

where $\overline{P}_m$ is the Legendre polynomial of degree m and norm 1 in $L^2(-1,1)$, Q_m is the Legendre function of the second kind, $G_c(y) = m(m+1) - (U(c)/\pi)^2\chi^c_m + q^c(y)$, and $L_m(y,z) = \sqrt{1-z^2}\left(\overline{P}_m(y)Q_m(z) - \overline{P}_m(z)Q_m(y)\right)$. By Theorem 11 and Proposition 10, we have $\|G_c\|_{L^\infty([-1,1])} \leq R(c)$. Because $\Gamma^c_m(1)$ is finite, $\overline{P}_m$ is bounded but $\lim_{y\to 1} Q_m(y) = \infty$, we know that $B = 0$. By the result after Lemma 9 in [13], for all $0 \leq y \leq z \leq 1$, $|L_m(y,z)| \leq 1$. Hence, by the Cauchy-Schwarz inequality, we have, for all $y \in (1,1)$,

$$\begin{aligned}\left|\widetilde{\Gamma}^c_m(y) - A\overline{P}_m(y)\right| &\leq \frac{1}{m+1/2}\left(\int_y^1 (L_m(y,z))^2(1-z^2)dz\right)^{1/2}\left(\int_y^1 G_c(z)^2\widetilde{\Gamma}^c_m(z)^2dz\right)^{1/2},\\ &\leq \frac{R(c)}{m+1/2}(1-y)\end{aligned} \tag{53}$$

so

$$\int_{-1}^1 \left|\widetilde{\Gamma}^c_m(y) - A\overline{P}_m(y)\right|^2 dy \leq \frac{2R(c)^2}{3(m+1/2)^2}$$

and, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\int_{-1}^1 \left|\widetilde{\Gamma}^c_m(y) - A\overline{P}_m(y)\right|^2 dy \geq& 1 + A^2 - 2|A|\int_{-1}^1 \left|\widetilde{\Gamma}^c_m(y)\right|^2 dy \int_{-1}^1 \left|\overline{P}_m(y)\right|^2 dy\\ \geq& (1-|A|)^2,\end{aligned}$$

hence

$$|A| \leq 1 + \sqrt{\frac{2}{3}}\frac{R(c)}{m+1/2}. \tag{54}$$

Also, by (45) and Lemma 8, we have

$$\|1/F\|_{L^\infty([-1,1])}\sqrt{\frac{\pi}{U(c)}} \leq \pi e^{-2c}\left(1 + \frac{4c^2}{3}\right)^{1/2}\left(\frac{\sinh(4c)}{4c}\right)^{1/4}\cosh(2c)^{1/2},$$

and we obtain the result by (53), (54), and $\left\|\overline{P}_m\right\|_{L^\infty([-1,1])} \leq \sqrt{m+1/2}$. □

Corollary 13. For all $m \in \mathbb{N}_0$ and $c > 0$,

$$\|g_m^c\|_{L^\infty([-1,1])} \leq H(c)\sqrt{m+\frac{1}{2}}, \tag{55}$$

where

$$H(c) = \pi\sqrt{1+\frac{4c^2}{3}}\left(1+2\sqrt{2}\left(2+\frac{1}{\sqrt{3}}\right)\left(\frac{2}{\pi^2}+\frac{8}{3}(1+2c)\left(c^2+\frac{9c}{8}+\frac{1}{2}\right)\right)\right).$$

Proof. By the above results, (55) holds with

$$\pi e^{-2c}\left(1+\frac{4c^2}{3}\right)^{1/2}\left(\frac{\sinh(4c)}{4c}\right)^{1/4}\cosh(2c)^{1/2}\left(1+2\sqrt{2}R(c)\left(2+\frac{1}{\sqrt{3}}\right)\right)$$

in place of $H(c)$ and

$$R(c) < \frac{2}{\pi^2}+2\left(\frac{ce^{2c}}{\sinh(4c)}\right)^2\left(\left(\cosh(4c)\left(1+\frac{c}{3}\coth(2c)\right)-1\right)+2c\sinh(4c)\right).$$

hence, using that $e^c \geq 1+c$ which implies $c\coth(c) \leq c+2$,

$$R(c) < \frac{2}{\pi^2}+\frac{8ce^{4c}}{3\sinh(4c)}\left(c^2+\frac{9c}{8}+\frac{1}{2}\right) < \frac{2}{\pi^2}+\frac{8}{3}(1+2c)\left(c^2+\frac{9c}{8}+\frac{1}{2}\right).$$

We obtain the result, using

$$\begin{aligned} e^{-2c}\left(\frac{\sinh(4c)}{4c}\right)^{1/4}\cosh(2c)^{1/2} &= e^{-2c}\left(\frac{\sinh(2c)}{2c}\right)^{1/4}\cosh(2c)^{3/4} \\ &= \left(\frac{1-e^{-4c}}{4c}\right)^{1/4}\left(\frac{1+e^{-4c}}{2}\right)^{3/4} \leq 1. \end{aligned}$$

□

As a result we have, for a constant C_0,

$$\|g_m^c\|_{L^\infty([-1,1])} \leq C_0\,(c\vee 1)^4\sqrt{m+\frac{1}{2}}. \tag{56}$$

7. Numerical method to obtain the SVD of $\mathcal{F}_{b,c}$

In recent years, efficient numerical methods to obtain the SVD of the truncated Fourier transform acting on the space of bandlimited functions have been developped. This allows to go beyond the usual toolbox based on the Tikhonov or iterative methods such as the Landweber method (Gerchberg method for out-of-band extrapolation, see [7]). The strategy that we implement in the next section is to first compute a numerical approximation of the right singular functions (the PSWF). We use that the first coefficients of the decomposition of the PSWF on

the Legendre polynomials can be obtained by solving for the eigenvectors of two tridiagonal symmetric Toeplitz matrices (for even and odd values of m, see Section 2.6 in [38]). We can then compute their image by $\mathcal{F}_c^{W_{[-1,1]}*}$ (see (7) for the definition of $\mathcal{F}_c^{W_{[-1,1]}}$) because, by [21], $\mathcal{F}_c^{W_{[-1,1]}*} = \mathcal{R}\left[\mathbb{1}\{[-1,1]\}\mathcal{F}_c^{W_{[-1,1]}}\mathcal{E}\right]$ applied to the Legendre polynomials has a closed form involving the Bessel functions of the first kind (see (18.17.19) in [36]).

For nonbandlimited functions, we propose to rely on the differential operator $\mathcal{L}$ in case (1) at the beginning of Section 5. We have used that because $\mathcal{Q}_c$ commutes with $\mathcal{L}$, $(g_m^c)_{m\in\mathbb{N}_0}$ are the eigenfunctions of $\mathcal{L}$. To obtain a numerical approximation of these functions, we use $\mathcal{L}$, whose eigenvalues are of the order of m^2 (see Theorem 11), rather than $\mathcal{Q}_c$, whose eigenvalues decay to zero exponentially. This is achieved by solving numerically for the eigenfunctions of a singular Sturm-Liouville operator. We approximate the values of the eigenfunctions on a grid on $[-1,1]$ using the MATLAB package MATSLISE 2 (it implements constant perturbation methods for limit point nonoscillatory singular problems, see [30] chapters 6 and 7 for the method and an analysis of the numerical approximation error). By Proposition A.1 in [21], we have $\varphi_m^{b,c}(\cdot) = \varphi_m^{1,c/b}(b\cdot)\sqrt{b}$ for all $m \in \mathbb{N}_0$. Finally, we use $\mathcal{F}_{1,c/b}^*\left[g_m^{c/b}\right] = \sigma_m^{1,c/b}\varphi_m^{1,c/b}$ and that $\varphi_m^{1,c/b}$ has norm 1 to obtain the remaining of the SVD. $\mathcal{F}_{1,c/b}^*\left[g_m^{c/b}\right]$ is computed using the fast Fourier transform.

8. Illustration: application to analytic continuation

We solve for f in (17) in Case (a) $f = 0.5/\cosh(2\cdot)$, which is not bandlimited, and Case (b) $f = \mathrm{sinc}(2\cdot)/6$ which is bandlimited, when $c = 0.5$, $x_0 = 0$, and $\xi = \cos(50\cdot)$. We use approximation f_δ^N described in Section 3.2 with $b = 1$ for Case (a), $b = 1/6.5$ for Case (b).

By analogy with the statistical problem where $\delta\xi$ is random rather than bounded, we use the terminology estimator. We select the value for the parameter $N = \widehat{N}$ based on a type of Goldenshluger-Lepski method (see [23]):

$$\widehat{N} \in \underset{N'\in\{0,\ldots,N_{\max}\}}{\operatorname{argmin}} B(N) + \Sigma(N),$$

$$B(N) = \sup_{N\le N'\le N_{\max}} \left(\left\| F_\delta^{N'\vee N} - F_\delta^N \right\|^2_{L^2(\cosh(b\cdot))} + \Sigma(N') \right)_+, \quad \Sigma(N) = \frac{2\pi c\delta^2 e^{2\beta(c/b)N}}{1 - e^{-2\beta(c/b)}},$$

and $N_{\max} = \lfloor \log(1/\delta) \rfloor$. Performing analytic continuation using (19) requires the approximation of the scalar products on $[-1,1]$ of the observed function f_δ with $g_m^{c/b}$. We use the package MATSLISE 2 to compute the value of the functions $\left(g_m^{c/b}\right)_{m=0}^{N_{\max}}$ at the n first Gauss-Legendre

quadrature nodes. Results are presented in figures 2 and 3, where we use a 2^{12} resolution in the Fast Fourier transform, $n = 15000$, and precision of 10^{-10} for the computation of the eigenvalues in MATSLISE 2, which also controls the precision of the computation of the eigenfunctions in the function computeEigenfunction of MATSLISE 2 despite that this is not explicitely computed (see sections 7.2.3 and 5.2 in [30] for examples).

We compare f_δ^N to a similar estimator based on (19) but with the PSWF instead of g_m^c in Case (b). This approach can only be used to perform analytic continuation of bandlimited functions when the researcher knows an interval which contains the bandlimits. In contrast, even for bandlimited functions, using the estimator based on $g_m^{c/b}$ allows to perform analytic continuation without the knowledge of an interval containing the support of the Fourier transform of the function. Importantly, Figure 4 shows that f_δ^N performs almost as well as the unfeasible method using the PSWF.

For the sake of conciseness, this paper does not study the effect of the various discretizations which can be carried out with arbitrary precision. Rather, we used in the numerical illustration conservative choices for those. This paper also does not consider the statistical problem, prove minimax lower bounds for it, and the adaptivity of the data-driven rule giving $N = \widehat{N}$. This is the object of future work. The interested reader can refer to [21] for the full statistical analysis for estimation of the density of random coefficients in the linear random coefficients model.

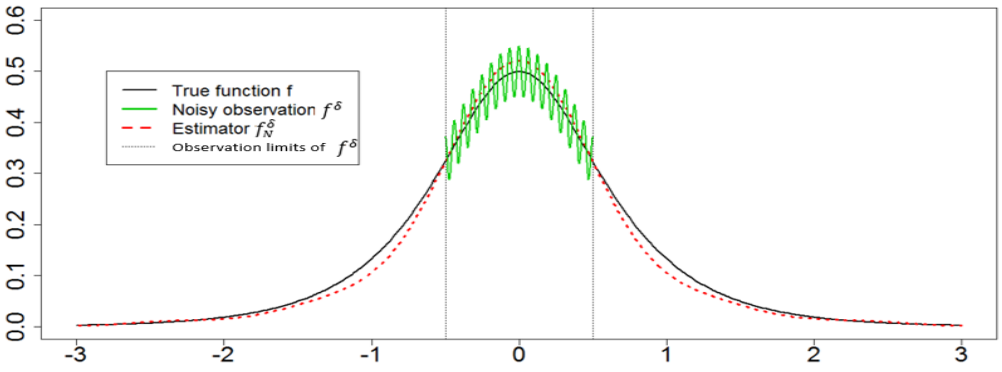


FIGURE 2. Case (a) with noise ($\delta = 0.05$), where F_δ^N in (19) uses $g_m^{c/b}$.

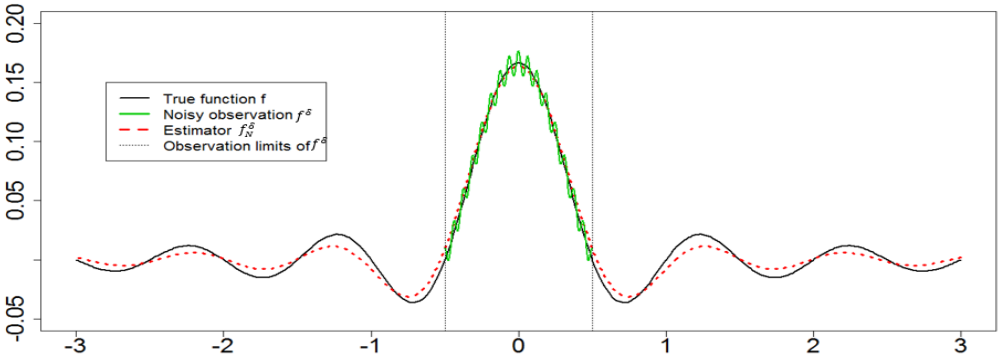


FIGURE 3. Case (b) with noise ($\delta = 0.01$), where F_δ^N in (19) uses $g_m^{c/b}$.

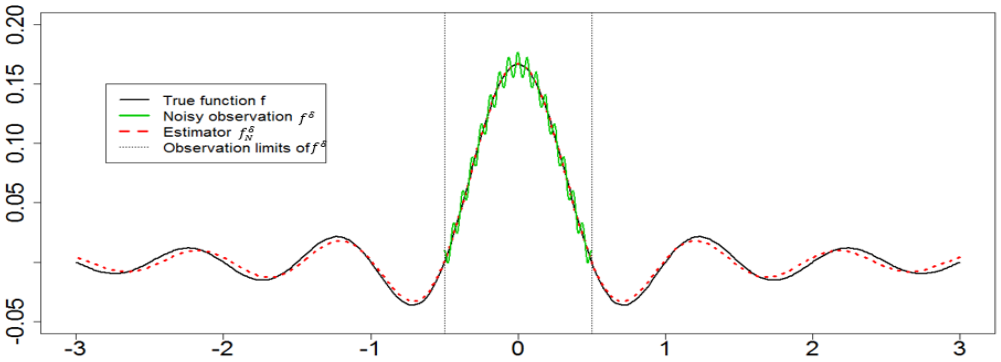


FIGURE 4. Case (b) with noise ($\delta = 0.01$), where F_δ^N in (19) uses the PSWF.

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