Adaptive estimation in the linear random coefficients model when regressors have limited variation

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ADAPTIVE ESTIMATION IN THE LINEAR RANDOM COEFFICIENTS MODEL WHEN REGRESSORS HAVE LIMITED VARIATION

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Abstract. We consider a linear model where the coefficients - intercept and slopes - are random with a distribution in a nonparametric class and independent from the regressors. The main drawback of this model is that identification usually requires the regressors to have a support which is the whole space. This is rarely satisfied in practice. Rather, in this paper, the regressors can have a support which is a proper subset. This is possible by assuming that the slopes do not have heavy tails. Lower bounds on the supremum risk for the estimation of the joint density of the random coefficients density are derived for this model and a related white noise model. We present an estimator, its rates of convergence, and a data-driven rule which delivers adaptive estimators. The corresponding R package is RandomCoefficients.

1. Introduction

For a random variable α and random vectors $\boldsymbol{X}$ and $\boldsymbol{\beta}$ of dimension p, the linear random coefficients model is

$$Y = \alpha + \boldsymbol{\beta}^\top \boldsymbol{X}, \tag{1}$$

$$(\alpha, \boldsymbol{\beta}^\top) \text{ and } \boldsymbol{X} \text{ are independent.} \tag{2}$$

The researcher has at her disposal n observations $(Y_i, \boldsymbol{X}_i^\top)_{i=1}^n$ of $(Y, \boldsymbol{X}^\top)$ but does not observe the realizations $(\alpha_i, \boldsymbol{\beta}_i^\top)_{i=1}^n$ of $(\alpha, \boldsymbol{\beta}^\top)$. α subsumes the intercept and error term and the vector of slope coefficients $\boldsymbol{\beta}$ is heterogeneous (*i.e.*, varies across i). For example, a researcher interested in the effect of the income of the parents on pupils' achievements might want to allow different effects for different pupils and to estimate the density of the effect. $(\alpha, \boldsymbol{\beta}^\top)$ correspond to multidimensional unobserved heterogeneity and $\boldsymbol{X}$ to observed heterogeneity. Restricting

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unobserved heterogeneity to a scalar, as when only α is random, can have undesirable implications such as monotonicity in the literature on policy evaluation (see [25]). Parametric assumptions are often made by convenience and can drive the results (see [30]). For this reason, this paper considers a nonparametric setup. Model (1) is also a type of linear model with homogeneous slopes and heteroscedastic errors, hence the averages of the coefficients are easy to obtain. However, the law of coefficients, their quantiles, prediction intervals for Y given $\boldsymbol{X} = \boldsymbol{x}$ as in [3], welfare measures, treatment and counterfactual effects, which depend on the distribution of the coefficients can be of great interest.

Estimation of the density of random coefficients $f_{\alpha,\boldsymbol{\beta}}$ when the support of $\boldsymbol{X}$ is $\mathbb{R}^p$ and $\boldsymbol{X}$ has heavy enough tails has been studied in [4, 32]. These papers notice that the inverse problem is related to a tomography problem (see, *e.g.*, [10, 11]) involving the Radon transform. Assuming the support of $\boldsymbol{X}$ is $\mathbb{R}^p$ amounts to assuming that the law of angles has full support, moreover a lower bound on the density of $\boldsymbol{X}$ is assumed so that the law of the angles is nondegenerate. When $p = 1$ this is implied by densities of $\boldsymbol{X}$ which follow a Cauchy distribution. The corresponding tomography problem has a nonuniform and estimable density of angles and the dimension can be larger than in tomography due to more than one regressor. More general specifications of random coefficients model are important in econometrics (see, *e.g.*, [26, 31, 41] and references therein) and there has been recent interest in nonparametric tests (see [9, 20]).

This paper considers the case where the support of $\boldsymbol{X}$ is a proper (*i.e.*, strict) subset. This is a much more useful and realistic framework for the random coefficients model. In our motivating example, the income of the parents is positive and probably bounded. When $p = 1$, the problem is related to limited angle tomography (see, *e.g.*, [21, 33]). There, one has measurements over a subset of angles and the unknown density has support in the unit disk. This is too restrictive for a density of random coefficients and implies that α has compact support, ruling out usual parametric assumptions on error terms. Due to (2), the conditional characteristic function of Y given $\boldsymbol{X} = \boldsymbol{x}$ at t is the Fourier transform of $f_{\alpha,\boldsymbol{\beta}}$ at $(t, t\boldsymbol{x}^\top)^\top$. Hence, the family of conditional characteristic functions indexed by $\boldsymbol{x}$ in the support of $\boldsymbol{X}$ gives access to the Fourier transform of $f_{\alpha,\boldsymbol{\beta}}$ on a double cone of axis $(1, 0, \dots, 0) \in \mathbb{R}^{p+1}$ and apex 0. When $\alpha = 0$, $\mathbb{S}_{\boldsymbol{\beta}}$ is compact, and $\mathcal{X} \subseteq \mathbb{S}_{\boldsymbol{X}}$ is an arbitrary compact set of nonempty interior, this is the problem of out-of-band extrapolation or super-resolution (see, *e.g.*, [5] sections 11.4 and 11.5). Because we allow α to be nonzero, we generalize this approach. Estimation of $f_{\alpha,\boldsymbol{\beta}}$ is a statistical inverse problem for which the deterministic problem is the inversion of a truncated Fourier transform (see, *e.g.*, [2] and the references therein). The companion paper [24] presents conditions on the law of $\left(\alpha, \boldsymbol{\beta}^\top\right)^\top$ and the support of $\boldsymbol{X}$ that imply nonparametric identification. It considers weak conditions on α which could have infinite absolute moments and the marginals of $\boldsymbol{\beta}$ could have heavy tails. In this paper, we obtain rates of convergence when the marginals of $\boldsymbol{\beta}$ do not have heavy tails but can have noncompact support.

A related approach is extrapolation. It is used in [42] to perform deconvolution of compactly supported densities while allowing the characteristic function of the error to vanish on a set of positive measure. In this paper, the relevant operator is viewed as a composition of two operators based on partial Fourier transforms. One involves a truncated Fourier transform and we make use of properties of the singular value decomposition rather than extrapolation.

Similar to [27, 34], we study optimality in the minimax sense. We obtain lower bounds under weak to strong integrability in the first argument for this and a white noise model. We present an estimator involving: series based estimation of the partial Fourier transform of the density with respect to the first variable, interpolation around zero, and inversion of the partial Fourier transform. We give rates of convergence and use a Goldenshluger-Lepski type method to obtain data-driven estimators. We consider estimation of $f_{\boldsymbol{\beta}}$ in Appendix B.3. We present a numerical method to compute the estimator which is implemented in the R package RandomCoefficients.

2. Notations

The notations $\cdot$, $\cdot_1$, $\cdot_2$, $\star$ are used to denote a variable in a function. The notation $a \wedge b$ (resp. $a \vee b$) is used for the minimum (resp. maximum) between a and b, $(\cdot)_+$ for $0 \vee \cdot$, and $\mathbb{1}\{A\}$ for the indicator function of a set A. $\mathbb{N}$ and $\mathbb{N}_0$ stand for the positive and nonnegative integers. Bold letters are used for vectors. For all $r \in \mathbb{R}$, $\underline{\boldsymbol{r}}$ is the vector, which dimension will be clear from the text, where each entry is r. The iterated logarithms are $\ln_0(t) = t$ and, for $j \geq 1$ and t large enough, $\ln_j(t) = \ln(\ln_{j-1}(t))$. $\mathcal{W}$ is the inverse of $x \in [0, \infty) \mapsto xe^x$. $|\cdot|_q$ for $q \in [1, \infty]$ stands for the ℓ_q norm of a vector or sequence. For all $\boldsymbol{\beta} \in \mathbb{C}^d$, $(f_m)_{m \in \mathbb{N}_0}$ functions with values in $\mathbb{C}$, and $\boldsymbol{m} \in \mathbb{N}_0^d$, denote by $\boldsymbol{\beta}^{\boldsymbol{m}} = \prod_{k=1}^d \boldsymbol{\beta}_k^{\boldsymbol{m}_k}$, $|\boldsymbol{\beta}|^{\boldsymbol{m}} = \prod_{k=1}^d |\boldsymbol{\beta}_k|^{\boldsymbol{m}_k}$, and $f_{\boldsymbol{m}} = \prod_{k=1}^d f_{\boldsymbol{m}_k}$. For a differentiable function f of real variables, $f^{(\boldsymbol{m})}$ denotes $\prod_{j=1}^d \frac{\partial^{\boldsymbol{m}_j}}{\partial x_j^{\boldsymbol{m}_j}} f$ and $\text{supp}(f)$ its support. $C^\infty\left(\mathbb{R}^d\right)$ is the space of infinitely differentiable functions. The inverse of a mapping f, when it exists, is denoted by f^I. We denote the interior of $\mathcal{S} \subseteq \mathbb{R}^d$ by $\overset{\circ}{\mathcal{S}}$ and its closure by $\overline{\mathcal{S}}$. When $\mathcal{S}$ is measurable and μ a function from $\mathcal{S}$ to $[0, \infty]$, $L^2(\mu)$ is the space of complex-valued square integrable functions equipped with $\langle f, g \rangle_{L^2(\mu)} = \int_{\mathcal{S}} f(\boldsymbol{x}) \overline{g}(\boldsymbol{x}) \mu(\boldsymbol{x}) d\boldsymbol{x}$. This is denoted by $L^2(\mathcal{S})$ when $\mu = 1$. When $W_{\mathcal{S}} = \mathbb{1}\{\mathcal{S}\} + \infty \, \mathbb{1}\{\mathcal{S}^c\}$, we have $L^2(W_{\mathcal{S}}) = \left\{f \in L^2\left(\mathbb{R}^d\right) : \text{supp}(f) \subseteq \mathcal{S}\right\}$ and $\langle f, g \rangle_{L^2(W_{\mathcal{S}})} = \int_{\mathcal{S}} f(\boldsymbol{x}) \overline{g}(\boldsymbol{x}) d\boldsymbol{x}$. Denote by $\mathcal{D}$ the set of densities, by $\Pi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ such that $\Pi f(\boldsymbol{x}) = f(-\boldsymbol{x})$, and by $\otimes$ the product of functions (*e.g.*, $W^{\otimes d}(\boldsymbol{b}) = \prod_{j=1}^d W(\boldsymbol{b}_j)$) or measures. The Fourier transform of $f \in L^1\left(\mathbb{R}^d\right)$ is $\mathcal{F}[f](\boldsymbol{x}) = \int_{\mathbb{R}^d} e^{i\boldsymbol{b}^\top \boldsymbol{x}} f(\boldsymbol{b}) d\boldsymbol{b}$ and $\mathcal{F}[f]$ is also the Fourier transform in $L^2\left(\mathbb{R}^d\right)$. For all $c > 0$, denote the Paley-Wiener space by $PW(c) := \left\{f \in L^2(\mathbb{R}) : \text{supp}\left(\mathcal{F}[f]\right) \subseteq [-c, c]\right\}$, by $\mathcal{P}_c$ the projector from $L^2(\mathbb{R})$ to $PW(c)$ ($\mathcal{P}_c[f] = \mathcal{F}^I\left[\mathbb{1}\{[-c, c]\}\mathcal{F}[f]\right]$), and, for all $c \neq 0$, by

$$\begin{array}{cccccccccc} \mathcal{F}_c : & L^2\left(W^{\otimes d}\right) & \to & L^2\left([-1,1]^d\right) & \text{and} & \mathcal{C}_c : & L^2\left(\mathbb{R}^d\right) & \to & L^2\left(\mathbb{R}^d\right) \\ & f & \to & \mathcal{F}[f](c\,\cdot) & & & f & \to & |c|^d f(c\,\cdot). \end{array} \tag{3}$$

Abusing notations, we sometimes use $\mathcal{F}_c[f]$ for the function in $L^2(\mathbb{R})$. $\mathcal{E}xt[f]$ assigns the value 0 outside $[-1, 1]^d$ and $\mathcal{F}_{1\text{st}}[f](t, \cdot_2)$ is the partial Fourier transform of f with respect to the first variable. For a random vector $\boldsymbol{X}$, $\mathbb{P}_{\boldsymbol{X}}$ is its law, $f_{\boldsymbol{X}}$ its density, $f_{\boldsymbol{X}|\mathcal{X}}$ the truncated density of $\boldsymbol{X}$ given $\boldsymbol{X} \in \mathcal{X}$, $\mathbb{S}_{\boldsymbol{X}}$ its support, and $f_{Y|\boldsymbol{X}=\boldsymbol{x}}$ the conditional density. For a sequence of random variables $(X_{n_0,n})_{(n_0,n) \in \mathbb{N}_0^2}$, $X_{n_0,n} = \underset{\mathcal{U}}{O_p}(1)$ means that, for all $\epsilon > 0$, there exists M such that $\mathbb{P}(|X_{n_0,n}| \geq M) \leq \epsilon$ for all $(n_0, n) \in \mathbb{N}_0^2$ such that $\mathcal{U}$ holds. In the absence of constraint,

we drop the notation $\mathcal{U}$. With a single index the $O_p(1)$ notation requires a bound holding for all value of the index (the usual notation if the random variables are bounded in probability).

3. Preliminaries

Assumption 1. (H1.1) $f_{\boldsymbol{X}}$ and $f_{\alpha,\boldsymbol{\beta}}$ exist;

(H1.2) $f_{\alpha,\boldsymbol{\beta}} \in L^2\left(w \otimes W^{\otimes p}\right)$, where $w \geq 1$ and W is even, nondecreasing on $[0,\infty)$, such that $W(0) > 0$ and $\lim_{x\to\infty} W(x) = \infty$, $\sum_{k\in\mathbb{N}} M_k^{-1/k} = \infty$ with $M_k = \left(\int_{\mathbb{R}} b^{2k}(1/W)(b)db\right)^{1/2}$;

(H1.3) There exists $x_0 > 0$ and $\mathcal{X} = [-x_0, x_0]^p \subseteq \mathbb{S}_{\boldsymbol{X}}$ and we have at our disposal i.i.d $(Y_i, \boldsymbol{X}_i)_{i=1}^n$ and an estimator $\widehat{f}_{\boldsymbol{X}|\mathcal{X}}$ based on $\mathcal{G}_{n_0} = (\boldsymbol{X}_i)_{i=-n_0+1}^0$ independent of $(Y_i, \boldsymbol{X}_i)_{i=1}^n$;

(H1.4) $\mathcal{E}$ is a set of densities on $\mathcal{X}$ such that, for $c_{\boldsymbol{X}}, C_{\boldsymbol{X}} \in (0,\infty)$, for all $f \in \mathcal{E}$, $\|f\|_{L^\infty(\mathcal{X})} \leq C_{\boldsymbol{X}}$ and $\|1/f\|_{L^\infty(\mathcal{X})} \leq c_{\boldsymbol{X}}$, and, for $(v(n_0,\mathcal{E}))_{n_0\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ which tends to 0, we have

$$\frac{1}{v(n_0,\mathcal{E})} \sup_{f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}} \left\|\widehat{f}_{\boldsymbol{X}|\mathcal{X}} - f_{\boldsymbol{X}|\mathcal{X}}\right\|_{L^\infty(\mathcal{X})}^2 = O_p\left(1\right).$$

We maintain this assumption for all results presenting upper bounds. When $w = 1$, $\mathbb{E}\left[\alpha^k\right]$, for $k \in \mathbb{N}$, might not exist. Due to Theorem 3.14 in [18], if there exist $R > 0$, $(a_j)_{j\in\mathbb{N}_0} \in (0,\infty)^{\mathbb{N}_0}$, and $(p_j)_{j\in\mathbb{N}_0} \in (-\infty,1]^{\mathbb{N}_0}$ equal to 0 for j large enough, such that, for all $x \in \mathbb{R}$,

$$W(x) \geq \exp\left(\frac{x^2}{\prod_{j=0}^\infty \ln_j^{p_j}(a_j|x|)}\right) \mathbb{1}\left\{|x| \geq R\right\} \quad \left(e.g.,\, W(x) = \exp\left(\frac{|x|}{a_0 \ln(a_1|x|)}\right)\mathbb{1}\left\{|x| \geq R\right\}\right),$$

then $\sum_{m\in\mathbb{N}} 1/\left\|\cdot^{2m}/W(\cdot)\right\|_{L^\infty(\mathbb{R})}^{1/(2m)} = \infty$ which implies (H1.2). Marginal distributions can have infinite moment generating functions hence be heavy-tailed and their Fourier transforms belong to a quasi-analytic class but not be analytic. Now on, we use $W_{[-R,R]}$ or $\cosh(\cdot/R)$ for W. This rules out heavy tails and nonanalytic Fourier transforms. When $W = W_{[-R,R]}$, integrability in $\boldsymbol{b}$ amounts to $\mathbb{S}_{\boldsymbol{\beta}} \subseteq [-R,R]^p$, but $\cosh(\cdot/R)$ allows for noncompact $\mathbb{S}_{\boldsymbol{\beta}}$. Though with a different scalar product, we have $L^2\left(\cosh(b\cdot)\right) = L^2\left(e^{b|\cdot|}\right)$ and (see Theorem IX.13 in [47]), for $a > 0$, $\left\{f \in L^2(\mathbb{R}) :\ \forall b < a,\ f \in L^2\left(e^{b|\cdot|}\right)\right\}$ is the set of square-integrable functions which Fourier transform have an analytic continuation on $\{z \in \mathbb{C} :\ |\mathrm{Im}(z)| < a/2\}$. Incidentally, the Laplace transform is finite near 0 and, if f is a density, it does not have heavy-tails. The condition $\mathcal{X} = [-x_0, x_0]^p \subseteq \mathbb{S}_{\boldsymbol{X}}$ in (H1.4) is not restrictive because $Y = \alpha + \boldsymbol{\beta}^\top \underline{\boldsymbol{x}} + \boldsymbol{\beta}^\top(\boldsymbol{X} - \underline{\boldsymbol{x}})$, we can take $\underline{\boldsymbol{x}}$ and x_0 such that $\mathcal{X} \subseteq \mathbb{S}_{\boldsymbol{X}-\underline{\boldsymbol{x}}}$, and there is a one-to-one mapping between $f_{\alpha+\boldsymbol{\beta}^\top\underline{\boldsymbol{x}},\boldsymbol{\beta}}$ and $f_{\alpha,\boldsymbol{\beta}}$. We assume (H1.4) because the estimator involves estimators of $f_{\boldsymbol{X}|\mathcal{X}}$ in denominators. Alternative solutions exist when $p = 1$ (see, *e.g.*, [37]) only. Assuming the availability of an estimator of $f_{\boldsymbol{X}|\mathcal{X}}$ using the preliminary sample $\mathcal{G}_{n_0}$ is common in the deconvolution literature (see, *e.g.*, [16]). By using estimators of $f_{\boldsymbol{X}|\mathcal{X}}$ for a well chosen $\mathcal{X}$ rather than of $f_{\boldsymbol{X}}$, the assumption $\left\|f_{\boldsymbol{X}|\mathcal{X}}\right\|_{L^\infty(\mathcal{X})} \leq C_{\boldsymbol{X}}$ and $\left\|1/f_{\boldsymbol{X}|\mathcal{X}}\right\|_{L^\infty(\mathcal{X})} \leq c_{\boldsymbol{X}}$ in (H1.4) becomes mild.

3.1. Inverse problem in Hilbert spaces.

Estimation of $f_{\alpha,\boldsymbol{\beta}}$ is a statistical ill-posed inverse problem. The operator depends on w and W. Now on, the functions w and W are those of

(H1.2). We have, for all $t \in \mathbb{R}$ and $\boldsymbol{u} \in [-1,1]^p$, $\mathcal{K} f_{\alpha,\boldsymbol{\beta}}(t,\boldsymbol{u}) = \mathcal{F}\left[f_{Y|\boldsymbol{X}=x_0\boldsymbol{u}}\right](t) x_0 |t|^{p/2}$, where

$$\begin{array}{rccc} \mathcal{K}: & L^2\left(w \otimes W^{\otimes p}\right) & \to & L^2(\mathbb{R} \times [-1,1]^p) \\ & f & \to & (t,\boldsymbol{u}) \mapsto \mathcal{F}[f](t, x_0 t \boldsymbol{u}) x_0 |t|^{p/2}. \end{array} \tag{4}$$

Proposition 1. $L^2\left(w \otimes W^{\otimes p}\right)$ is continuously embedded into $L^2(\mathbb{R}^{p+1})$. Moreover, $\mathcal{K}$ is injective and continuous, and not compact if $w = 1$.

The case $w = 1$ corresponds to mild integrability assumptions in the first variable when the SVD of $\mathcal{K}$ does not exist. This makes it difficult to prove rates of convergence even for estimators which do not rely explicitly on the SVD such as the Tikhonov and Landweber method (Gerchberg algorithm in out-of-band extrapolation, see, *e.g.*, [5]). Rather than work with $\mathcal{K}$ directly, we use that $\mathcal{K}$ is the composition of operators which are easier to analyze

$$\text{for } t \in \mathbb{R},\ \mathcal{K}[f](t,\star) = \mathcal{F}_{tx_0}\left[\mathcal{F}_{1\text{st}}[f](t,\cdot_2)\right](\star) x_0 |t|^{p/2} \text{ in } L^2([-1,1]^p). \tag{5}$$

For all $f \in L^2\left(w \otimes W^{\otimes p}\right)$, W either $W_{[-R,R]}$ or $\cosh(\cdot/R)$, and $t \in \mathbb{R}$, $\mathcal{F}_{1\text{st}}[f](t,\cdot_2)$ belongs to $L^2(W^{\otimes p})$ and, for $c \neq 0$, $\mathcal{F}_c:\ L^2(W^{\otimes p}) \to L^2([-1,1]^p)$ admits a SVD, where both orthonormal systems are complete. This is a tensor product of the SVD when $p = 1$ that we denote by $\left(\sigma_m^{W,c}, \varphi_m^{W,c}, g_m^{W,c}\right)_{m\in\mathbb{N}_0}$, where $\left(\sigma_m^{W,c}\right)_{m\in\mathbb{N}_0} \in (0,\infty)^{\mathbb{N}_0}$ is in decreasing order repeated according to multiplicity, $\left(\varphi_m^{W,c}\right)_{m\in\mathbb{N}_0}$ and $\left(g_m^{W,c}\right)_{m\in\mathbb{N}_0}$ are orthonormal systems of, respectively, $L^2(W)$ and $L^2([-1,1])$. This holds for the following reason. Because $\mathcal{F}_c = \mathcal{F}\mathcal{C}_{1/c} = (1/|c|)\mathcal{C}_c\mathcal{F}$, $\Pi\mathcal{F}_c = \mathcal{F}_c\Pi$, $\mathcal{F}_c^* = (1/W)\Pi\mathcal{F}_c\mathcal{E}xt$, and W is even, we obtain $\mathcal{F}_c^* = \Pi\left((1/W)\mathcal{F}_c\mathcal{E}xt\right)$ and $\mathcal{F}_c\mathcal{F}_c^* = \Pi\mathcal{F}_c\left((1/W)\mathcal{F}_c\mathcal{E}xt\right) = (2\pi/|c|)\mathcal{F}^I\left(\mathcal{C}_{1/c}\left((1/W)\mathcal{C}_c\mathcal{F}\mathcal{E}xt\right)\right) = 2\pi\mathcal{F}^I\left(\mathcal{C}_{1/c}\left(1/W\right)\mathcal{F}\mathcal{E}xt\right)$. The operator $\mathcal{Q}_c^W = \left(|c|/(2\pi)\right)\mathcal{F}_c\mathcal{F}_c^*$ is a compact positive definite self-adjoint operator (see [46] and [51] for the two choices of W). Its eigenvalues in decreasing order repeated according to multiplicity are denoted by $\left(\rho_m^{W,c}\right)_{m\in\mathbb{N}_0}$ and a basis of eigenfunctions by $\left(g_m^{W,c}\right)_{m\in\mathbb{N}_0}$. The other elements of the SVD are $\sigma_m^{W,c} = \sqrt{2\pi\rho_m^{W,c}/|c|}$ and $\varphi_m^{W,c} = \mathcal{F}_c^* g_m^{W,c}/\sigma_m^{W,c}$.

Proposition 2. For all $c \neq 0$, $\left(\varphi_m^{W,c}\right)_{m\in\mathbb{N}_0}$ is a basis of $L^2(W)$.

The singular vectors $\left(g_m^{W_{[-1,1]},c}\right)_{m\in\mathbb{N}_0}$ are the Prolate Spheroidal Wave Functions (hereafter PSWF, see, *e.g.*, [46]). They can be extended as entire functions in $L^2(\mathbb{R})$ and form a complete orthogonal system of $PW(c)$ for which we use the same notation. They are useful to carry interpolation and extrapolation (see, *e.g.*, [40]) with Hilbertian techniques. In this paper, for all $t \neq 0$, $\mathcal{F}_{1\text{st}}[f_{\alpha,\boldsymbol{\beta}}](t,\cdot_2)$ plays the role of the Fourier transform in the definition of $PW(c)$. The weight $\cosh(\cdot/R)$ allows for larger classes than $PW(c)$ and noncompact $\mathbb{S}_{\boldsymbol{\beta}}$. This is useful even if $\mathbb{S}_{\boldsymbol{\beta}}$ is compact when the researcher does not know a superset containing $\mathbb{S}_{\boldsymbol{\beta}}$. The useful results on the corresponding SVD and a numerical algorithm to compute it are given in [23].

3.2. **Sets of smooth and integrable functions.** Define, for all $(\phi(t))_{t\geq 0}$ and $(\omega_m)_{m\in\mathbb{N}_0}$ increasing, $\phi(0)=\omega_0=1$, $l,M>0$, $q\in\{1,\infty\}$, $t\in\mathbb{R}$, $\boldsymbol{m}\in\mathbb{N}_0^p$, $k\in\mathbb{N}_0$,

$$\mathcal{H}_{w,W}^{q,\phi,\omega}(l,M):=\left\{f:\sum_{k\in\mathbb{N}_0}\int_{\mathbb{R}}\phi^2(|t|)\theta_{q,k}^2(t)dt\bigvee\sum_{k\in\mathbb{N}_0}\omega_k^2\|\theta_{q,k}\|_{L^2(\mathbb{R})}^2\leq 2\pi l^2,\ \|f\|_{L^2(w\otimes W^{\otimes p})}\leq M\right\}$$

and $\mathcal{H}_{w,W}^{q,\phi,\omega}(l)$ when we replace $\|f\|_{L^2(w\otimes W^{\otimes p})}\leq M$ by $\|f\|_{L^2(w\otimes W^{\otimes p})}<\infty$, where

$$b_{\boldsymbol{m}}(t):=\left\langle\mathcal{F}_{1\text{st}}\left[f\right](t,\cdot_2),\varphi_{\boldsymbol{m}}^{W,x_0t}\right\rangle_{L^2(W^{\otimes p})},\ \theta_{q,k}(t):=\left(\sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_q=k}|b_{\boldsymbol{m}}(t)|^2\right)^{1/2}. \tag{6}$$

The first inequality in the definition of $\mathcal{H}_{w,W}^{q,\phi,\omega}(l,M)$ defines the notion of smoothness analyzed in this paper. It involves a maximum, thus two inequalities: the first for smoothness in the first variable and the second for smoothness in the other variables. The additional inequality imposes integrability (see Theorem 1). The asymmetry in the treatment of the first and remaining variables is due to the fact that only the random slopes are multiplied by regressors which have limited variation and we make integrability assumptions in the first variable which are as mild as possible. The use of the Fourier transform to express smoothness in the first variable is classical. For the remaining variables, we allow for functions with compact and noncompact support and work with the bases $\left(\varphi_{\boldsymbol{m}}^{W,x_0t}\right)_{\boldsymbol{m}\in\mathbb{N}_0^p}$ for $t\neq 0$. For functions with compact support, we show in Section B.5 that it is also possible to use instead the basis giving rise to Fourier series to define the smoothness in the remaining variables. We analyze all types of smoothness and because smoothness is unknown, we provide an adaptive estimator. We analyze two values of q and show that its value matters for the rates of convergence for supersmooth functions.

Remark 1. The next model is related to (1) under Assumption 1 when $f_{\boldsymbol{X}}$ is known:

$$dZ(t)=\mathcal{K}\left[f\right](t,\cdot_2)dt+\frac{\sigma}{\sqrt{n}}dG(t),\quad t\in\mathbb{R}, \tag{7}$$

where f plays the role of $f_{\alpha,\boldsymbol{\beta}}$, $\sigma>0$ is known, and $(G(t))_{t\in\mathbb{R}}$ is a complex two-sided cylindrical Gaussian process on $L^2([-1,1]^p)$. This means, for Φ Hilbert-Schmidt from $L^2([-1,1]^p)$ to a separable Hilbert space H, $(\Phi G(t))_{t\in\mathbb{R}}$ is a Gaussian process in H of covariance $\Phi\Phi^*$ (see [17]). Taking $\Phi G(t)=\sum_{\boldsymbol{m}\in\mathbb{N}_0^p}\Phi\left[g_{\boldsymbol{m}}^{W,x_0t}\right]B_{\boldsymbol{m}}(t)$, where $B_{\boldsymbol{m}}(t)=B_{\boldsymbol{m}}^{\mathfrak{R}}(t)+iB_{\boldsymbol{m}}^{\mathfrak{I}}(t)$, $(B_{\boldsymbol{m}}^{\mathfrak{R}}(t))_{t\in\mathbb{R}}$ and $\left(B_{\boldsymbol{m}}^{\mathfrak{I}}(t)\right)_{t\in\mathbb{R}}$ are independent two-sided Brownian motions, the system of independent equations

$$Z_{\boldsymbol{m}}(t)=\int_0^t\sigma_{\boldsymbol{m}}^{W,x_0s}b_{\boldsymbol{m}}(s)ds+\frac{\sigma}{\sqrt{n}}B_{\boldsymbol{m}}(t),\quad t\in\mathbb{R}, \tag{8}$$

where, $Z_{\boldsymbol{m}}:=\left\langle Z(\star),g_{\boldsymbol{m}}^{W,x_0\star}\right\rangle_{L^2([-1,1]^p)}$ and $\boldsymbol{m}\in\mathbb{N}_0^p$, is equivalent to (7). Because $\sigma_{\boldsymbol{m}}^{W,x_0s}$ is small when $|\boldsymbol{m}|_q$ is large or s is small (see Lemma B.4), the estimator of Section 5.1 truncates large values of $|\boldsymbol{m}|_q$ and does not rely on small values of $|s|$ but uses interpolation.

Remark 2. [33] considers a Gaussian sequence model corresponding to (7), $\mathcal{K}$ is the Radon transform, $p=1$, G is a two-sided cylindrical Wiener process, and $L^2(w\otimes W)$ is a weighted L^2 space of functions with support in the unit disk of $\mathbb{R}^2$ for which $\mathcal{K}$ has a SVD with a known rate of decay of the singular values.

3.3. **Interpolation.** Define, for all $\underline{a},\epsilon>0$, the operator on $L^2(\mathbb{R})$ with domain $PW(\underline{a})$

$$\mathcal{I}_{\underline{a},\epsilon}[f] := \sum_{m\in\mathbb{N}_0} \frac{\rho_m^{W_{[-1,1]},\underline{a}\epsilon}}{\left(1-\rho_m^{W_{[-1,1]},\underline{a}}\right)\epsilon} \left\langle f, \mathcal{C}_{1/\epsilon}\left[g_m^{W_{[-1,1]},\underline{a}\epsilon}\right]\right\rangle_{L^2(\mathbb{R}\setminus(-\epsilon,\epsilon))} \mathcal{C}_{1/\epsilon}\left[g_m^{W_{[-1,1]},\underline{a}\epsilon}\right]. \tag{9}$$

Proposition 3. For all $\underline{a},\epsilon>0$, we have $\mathcal{I}_{\underline{a},\epsilon}\left(L^2(\mathbb{R})\right)\subseteq L^2([-\epsilon,\epsilon])$ and, for all $g\in PW(\underline{a})$, $\mathcal{I}_{\underline{a},\epsilon}[g]=g$ in $L^2(\mathbb{R})$ and, for $C_0:=4\cdot/\left(\pi\left(1-\rho_0^{W_{[-1,1]},\cdot}\right)^2\right)$ and all $f,h\in L^2(\mathbb{R})$,

$$\left\|f-\mathcal{I}_{\underline{a},\epsilon}[h]\right\|^2_{L^2([-\epsilon,\epsilon])} \le 2(1+C_0(\underline{a}\epsilon))\left\|f-\mathcal{P}_{\underline{a}}[f]\right\|^2_{L^2(\mathbb{R})} + 2C_0(\underline{a}\epsilon)\left\|f-h\right\|^2_{L^2(\mathbb{R}\setminus(-\epsilon,\epsilon))}. \tag{10}$$

If $f\in PW(\underline{a})$, $\mathcal{I}_{\underline{a},\epsilon}[f]$ only relies on $f\mathbb{1}\{\mathbb{R}\setminus(-\epsilon,\epsilon)\}$ and $\mathcal{I}_{\underline{a},\epsilon}[f]=f$ on $\mathbb{R}\setminus(-\epsilon,\epsilon)$, so (9) provides an analytic formula to carry interpolation on $[-\epsilon,\epsilon]$ of functions in $PW(\underline{a})$. Else, (10) provides an upper bound on the error made by approximating f by $\mathcal{I}_{\underline{a},\epsilon}[h]$ on $[-\epsilon,\epsilon]$ when h approximates f outside $[-\epsilon,\epsilon]$. We use interpolation when the variance of an initial estimator $\widehat{f}^0$ of f is large due to its values near 0 but $\left\|f-\widehat{f}^0\right\|^2_{L^2(\mathbb{R}\setminus(-\epsilon,\epsilon))}$ is small and work with $\widehat{f}(t)=\widehat{f}^0(t)\mathbb{1}\{|t|\ge\epsilon\}+\mathcal{I}_{\underline{a},\epsilon}\left[\widehat{f}^0\right](t)\mathbb{1}\{|t|<\epsilon\}$. Then, (10) yields

$$\left\|f-\widehat{f}\right\|^2_{L^2(\mathbb{R})} \le (1+2C_0(\underline{a}\epsilon))\left\|f-\widehat{f}^0\right\|^2_{L^2(\mathbb{R}\setminus(-\epsilon,\epsilon))} + 2(1+C_0(\underline{a}\epsilon))\left\|f-\mathcal{P}_{\underline{a}}[f]\right\|^2_{L^2(\mathbb{R})}. \tag{11}$$

When $\mathrm{supp}\left(\mathcal{F}[f]\right)$ is compact, $\underline{a}$ is taken such that $\mathrm{supp}\left(\mathcal{F}[f]\right)\subseteq[-\underline{a},\underline{a}]$. Else, $\underline{a}$ goes to infinity so the second term in (11) goes to 0. ϵ is taken such that $\underline{a}\epsilon$ is constant because, due to (3.87) in [46], $\lim_{t\to\infty}C_0(t)=\infty$ and (10) and (11) become useless. We set $C=2\left(1+C_0(\underline{a}\epsilon)\right)$.

3.4. **Risk.** The risk of an estimator $\widehat{f}_{\alpha,\boldsymbol{\beta}}$ is the mean integrated squared error (MISE)

$$\mathcal{R}^W_{n_0}\left(\widehat{f}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right) := \mathbb{E}\left[\left\|\widehat{f}_{\alpha,\boldsymbol{\beta}}-f_{\alpha,\boldsymbol{\beta}}\right\|^2_{L^2(1\otimes W^{\otimes p})}\middle|\mathcal{G}_{n_0}\right].$$

When $W=W_{[-R,R]}$ and $\mathrm{supp}\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}\right)\subseteq\mathbb{R}\times[-R,R]^p$, it is $\mathbb{E}\left[\left\|\widehat{f}_{\alpha,\boldsymbol{\beta}}-f_{\alpha,\boldsymbol{\beta}}\right\|^2_{L^2(\mathbb{R}^{p+1})}\middle|\mathcal{G}_{n_0}\right]$, else,

$$\mathbb{E}\left[\left\|\widehat{f}_{\alpha,\boldsymbol{\beta}}-f_{\alpha,\boldsymbol{\beta}}\right\|^2_{L^2(\mathbb{R}^{p+1})}\middle|\mathcal{G}_{n_0}\right] \le \left\|1/W\right\|^p_{L^\infty(\mathbb{R})}\mathcal{R}^W_{n_0}\left(\widehat{f}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right). \tag{12}$$

4. Lower bounds

The lower bounds involve a function r (for rate) and take the form

$$\exists\nu>0:\ \underline{\lim}_{n\to\infty}\frac{1}{r(n)^2}\inf_{\widehat{f}_{\alpha,\boldsymbol{\beta}}}\sup_{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l)\cap\mathcal{D}}\mathbb{E}\left[\left\|\widehat{f}_{\alpha,\boldsymbol{\beta}}-f_{\alpha,\boldsymbol{\beta}}\right\|^2_{L^2(\mathbb{R}^{p+1})}\right]\ge\nu. \tag{13}$$

When we replace $f_{\alpha,\boldsymbol{\beta}}$ by f, $\widehat{f}_{\alpha,\boldsymbol{\beta}}$ by $\widehat{f}$, remove $\mathcal{D}$ from the nonparametric class, and consider model (8), we refer to (13').

Theorem 1. For all $q \in \{1,\infty\}$, ϕ increasing on $[0,\infty)$, w such that $\int_1^\infty w(a)/a^4 < \infty$, and $0 < l, s, R, \kappa < \infty$, if $W = W_{[-R,R]}$,

(T1.1a) $(\omega_k)_{k\in\mathbb{N}_0} = (k^\sigma)_{k\in\mathbb{N}_0}$, ϕ is such that $\lim_{\tau\to\infty}\int_0^\infty \phi(t)^2 e^{-2\tau t}dt = 0$, $f_{\boldsymbol{X}}$ is known, $\mathbb{S}_{\boldsymbol{X}} = \mathcal{X}$, and $\|f_{\boldsymbol{X}}\|_{L^\infty(\mathcal{X})} < \infty$, then (13) holds with $r(n) = (\ln(n)/\ln_2(n))^{-(2+k_q/2)\vee\sigma}$,

(T1.1b) we consider model (8), $(\omega_k)_{k\in\mathbb{N}_0} = \left(e^{\kappa k \ln(k+1)}\right)_{k\in\mathbb{N}_0}$, then (13') holds with $r(n) = n^{-\kappa/(2\kappa+2k_q)}/\ln(n)$,

else if $W = \cosh(\cdot/R)$, we consider model (8),

(T1.2a) $(\omega_k)_{k\in\mathbb{N}_0} = (k^\sigma)_{k\in\mathbb{N}_0}$, for all $\overline{\sigma} > 1/2$, then (13') holds with $r(n) = \ln(n/\ln(n))^{-\overline{\sigma}\vee\sigma}$

(T1.2b) $(\omega_k)_{k\in\mathbb{N}_0} = \left(e^{\kappa k}\right)_{k\in\mathbb{N}_0}$, then (13') holds with $r(n) = n^{-\kappa/(2\kappa+2k_q)}$.

By (12), (T1.2a), and (T1.2b), we obtain lower bounds involving $\mathcal{R}^W_{n_0}$.

5. Estimation

5.1. **Estimator.** For all $q \in \{1,\infty\}$, $0 < \epsilon < 1 < T$, $\underline{N} \in \mathbb{R}^{\mathbb{R}}$, $N(t) = \lfloor \underline{N}(t) \rfloor$ for $\epsilon \le |t| \le T$, $N(t) = N(\epsilon)$ for $|t| \le \epsilon$ and $N(t) = N(T)$ for $|t| > T$, a regularized inverse is obtained by:

(S.1) for all $t \neq 0$, obtain a preliminary approximation of $F_1(t,\cdot) := \mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot)$

$$F_1^{q,N,T,0}(t,\cdot_2) := \mathbb{1}\{|t| \le T\} \sum_{|\boldsymbol{m}|_q \le N(t)} \frac{c_{\boldsymbol{m}}(t)}{\sigma^{W,x_0 t}_{\boldsymbol{m}}} \varphi^{W,x_0 t}_{\boldsymbol{m}}, \; c_{\boldsymbol{m}}(t) := \left\langle \mathcal{F}\left[f_{Y|\boldsymbol{X}=x_0\cdot}\right](t), g^{W,x_0 t}_{\boldsymbol{m}} \right\rangle_{L^2([-1,1]^p)},$$

(S.2) for all $t \in [-\epsilon,\epsilon]$, $F_1^{q,N,T,\epsilon}(t,\cdot) := F_1^{q,N,T,0}(t,\cdot)\mathbb{1}\{|t| \ge \epsilon\} + \mathcal{I}_{\underline{a},\epsilon}\left[F_1^{q,N,T,0}(\star,\cdot)\right](t)\mathbb{1}\{|t| < \epsilon\}$,

(S.3) $f^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}(\cdot_1,\cdot_2) := \mathcal{F}^I_{1\text{st}}\left[F_1^{q,N,T,\epsilon}(\star,\cdot_2)\right](\cdot_1)$.

To deal with the statistical problem, we replace $c_{\boldsymbol{m}}$ by

$$\widehat{c}_{\boldsymbol{m}} := \frac{1}{n}\sum_{j=1}^n \frac{e^{i\star Y_j}}{x_0^p \widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{X}_j)} \overline{g^{W,x_0\star}_{\boldsymbol{m}}}\left(\frac{\boldsymbol{X}_j}{x_0}\right) \mathbb{1}\left\{\boldsymbol{X}_j \in \mathcal{X}\right\}, \tag{14}$$

where $\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{X}_j) := \widehat{f}_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{X}_j) \vee \sqrt{\delta(n_0)}$ and $\delta(n_0)$ is a trimming factor converging to zero with n_0. This yields the estimators $\widehat{F}_1^{q,N,T,0}$, $\widehat{F}_1^{q,N,T,\epsilon}$, and $\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}$. We use $\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}\right)_+$ as a final estimator of $f_{\alpha,\boldsymbol{\beta}}$ which always has a smaller risk than $\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}$ (see [26, 50]). We use $n_e = n \wedge \lfloor \delta(n_0)/v(n_0,\mathcal{E}) \rfloor$ for the sample size required for an ideal estimator where $f_{\boldsymbol{X}|\mathcal{X}}$ is known to achieve the rate of the plug-in estimator. The upper bounds below take the form

$$\frac{1}{r(n_e)^2} \sup_{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)\cap\mathcal{D},\; f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}} \mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) = O_p(1). \tag{15}$$

With the restriction $f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}^{q,\phi,\omega}_{w,W}(l) \cap \mathcal{D}$, we refer to (15'). We use $k_q := \mathbb{1}\{q=1\} + p\mathbb{1}\{q=\infty\}$ and $k'_q = p + 1 - k_q$.

5.2. **Upper bounds.** We use $T = \phi^I\left(\omega_{\underline{N}}\right)$, $\underline{a} = w^I\left(\omega^2_{\underline{N}}\right)$ when $w \neq W_{[-\underline{a},\underline{a}]}$, for $u > 0$, $K_{\underline{a}}(u) := \underline{a}\mathbb{1}\{w \neq W_{[-\underline{a},\underline{a}]}\} + u\mathbb{1}\{w = W_{[-\underline{a},\underline{a}]}\}$. Below $\underline{N}$, hence N, is constant.

Theorem 2. Let $W = W_{[-R,R]}$. For all $q \in \{1,\infty\}$, $l, M, s, R, \sigma, \kappa, \mu, \gamma, \nu > 0$, $\mathbb{S}_{\boldsymbol{\beta}} \subseteq [-R,R]^p$, $\underline{N}$ solution of $2k_q\underline{N}\ln\left(\underline{N}K_{\underline{a}}(1)\right) + \ln\left(\omega^2_{\underline{N}}\right) + (p-1)\ln\left(\underline{N}\right) = \ln(n_e)$, and $\epsilon = 7e/\left(2Rx_0K_{\underline{a}}(1)\right)$,

(T2.1) if $\phi = 1 \vee |\cdot|^s$, $(\omega_k)_{k\in\mathbb{N}_0} = \left(k^\sigma\right)_{k\in\mathbb{N}_0}$, and $w = 1 \vee |\cdot|^\mu$, then (15) holds with $r(n_e) = \left(\ln\left(n_e\right)/\ln_2\left(n_e\right)\right)^{-\sigma}$,

(T2.2) if $\phi = 1 \vee |\cdot|^s$, $(\omega_k)_{k\in\mathbb{N}_0} = \left(e^{\kappa k\ln(k+1)}\right)_{k\in\mathbb{N}_0}$, $s \geq \kappa(p+1)/(2k_q(\nu\mathbb{1}\{W \neq W_{[-\underline{a},\underline{a}]}\}+1))$, and $\Lambda := (p-1)(1-(\kappa(p+1)/\left(2s\left(k_q(\cdot+1)+\kappa\right)\right))/2$,

(T2.2a) $w^I\left(e^{2\kappa|\cdot|\ln(|\cdot|+1)}\right) = \cdot^\nu$, then (15) holds with $r(n_e) = n_e^{-\kappa/(2\kappa+2(\nu+1)k_q)}\ln(n_e)^{\Lambda(\nu)}$,

(T2.2b) $\underline{a}$ such that $\mathbb{S}_\alpha \subseteq [-\underline{a},\underline{a}]$, $w = W_{[-\underline{a},\underline{a}]}$, then (15') holds with $r(n_e) = n_e^{-\kappa/(2\kappa+2k_q)}\ln(n_e)^{\Lambda(0)}$,

(T2.3) if $\phi = e^{\gamma|\cdot|}$, $r > 1$, $(\omega_k)_{k\in\mathbb{N}_0} = \left(e^{\kappa(k\ln(k+1))^r}\right)_{k\in\mathbb{N}_0}$, w such that $w^I\left(e^{2\kappa(|\cdot|\ln(|\cdot|+1))^r}\right) = \cdot^\nu$, $d_0 = 2\kappa\left(1+(p-1)/(p+1)^r + 2k_q(1+\nu)/((p+1)\ln(p+2))^{r-1}\right)$, and for $k \in \{1,\ldots,k_0\}$,

$$d_k := \left(\frac{k_q(1+\nu)(2\kappa)^{1-1/r}\mathbb{1}\{k\equiv 0(\text{mod }2)\}}{\kappa(1+1/((p+1)\ln(p+2)))^r} + \left(\frac{(k_q+1)p+k_q-1}{p+1} + k_q\nu\right)\frac{\mathbb{1}\{k\equiv 1(\text{mod }2)\}}{\kappa d_0^{1/r-1}}\right)^k,$$

$k_0 := \lfloor r/(r-1)\rfloor$, and $\varphi := \exp\left(-\sum_{k=1}^{k_0}(-1)^k d_k\ln(\cdot)^{(1/r-1)k+1}\right)\bigvee \ln(\cdot)^{p+1+(p-1)/r}$ then (15) holds with $r(n_e) = \sqrt{\varphi\left(n_e\right)/n_e}$.

Theorem 1 shows the rate in (T2.2) is optimal when $f_{\boldsymbol{X}}$ is known and $\mathbb{S}_{\boldsymbol{X}} = \mathcal{X}$. It is the same as in [42] for deconvolution with a known characteristic function of the noise on an interval when the signal has compact support. The rates in Theorem 2 are independent of p as common for severely ill-posed problems (see [13, 23]).

Theorem 3. Let $W = \cosh(\cdot/R)$. For all $q \in \{1,\infty\}$, $l, M, s, R, \sigma, \kappa, \mu > 0$, $\phi = 1\vee|\cdot|^s$, $\underline{N}$ solution of $2k_q\underline{N}\ln\left(K_{\underline{a}}(e)\right) + \ln\left(\omega^2_{\underline{N}}\right) + (p-1)\ln\left(\underline{N}\right)/q = \ln(n_e)$, and $\epsilon = 7e^2/\left(4Rx_0K_{\underline{a}}(7e^2/\pi)\right)$,

(T3.1) if $(\omega_k)_{k\in\mathbb{N}_0} = \left(k^\sigma\right)_{k\in\mathbb{N}_0}$, and $w = 1\vee|\cdot|^\mu$, then (15) holds with $r(n_e) = \left(\ln\left(n_e\right)/\ln_2\left(n_e\right)\right)^{-\sigma}$,

(T3.2) if $(\omega_k)_{k\in\mathbb{N}_0} = \left(e^{\kappa k}\right)_{k\in\mathbb{N}_0}$, $\underline{a}$ such that $\mathbb{S}_\alpha \subseteq [-\underline{a},\underline{a}]$, and $w = W_{[-\underline{a},\underline{a}]}$, then (15') holds with $r(n_e) = n_e^{-\kappa/(2\kappa+2k_q)}\ln(n_e)^{(p-1)\kappa/(2q(\kappa+k_q))}$.

In (T3.2), we relax the assumption that $\mathbb{S}_{\boldsymbol{\beta}}$ is compact maintained in (T2.2a). The discussion after Theorem 2 in [42] considers densities with noncompact support and a pointwise bound outside $[-1,1]$. By inspection of the proof, Rx_0 can vary with n. The results of theorems 2 and 3 are related to those for "2exp-severely ill-posed problems" (see [12] and [49] which obtains the same polynomial rates up to logarithmic factor as in (T2.2b) when $1/v(n_0,\mathcal{E}) \geq n$ and $p = 1$). When $1/v(n_0,\mathcal{E}) \geq n$, the rate in (T3.2) matches the lower bound in model (8).

5.3. **Data-driven estimator.** We use a Goldenshluger-Lepski method (see [29, 39]). Let $R, \epsilon > 0$, $q \in \{1,\infty\}$, $\zeta_0 = 1/\left(1+4p\left(1+\mathbb{1}\left\{W = \cosh(\cdot/R)\right\}\right)\right)$, $K_{\max} := \lfloor\zeta_0\ln(n)/\ln(2)\rfloor$,

$T_{\max} := 2^{K_{\max}}$, $\mathcal{T}_n := \left\{2^k : k = 1, \dots, K_{\max}, 2^k \geq \epsilon\right\}$, $p_n := 3 \bigvee 6(1+\zeta_0)\ln(n)$, and, for all $N \in \mathbb{N}_0^{\mathbb{R}}$, $N_0, T \in \mathbb{N}_0$, $t \neq 0$, $N^W_{\max,q} = \left\lfloor \underline{N}^W_{\max,q} \right\rfloor$,

$$B_1(t, N_0) := \max_{N_0 \leq N' \leq N^W_{\max,q}} \left(\sum_{N_0 \leq |\boldsymbol{m}|_q \leq N'} \left(\frac{|\widehat{c}_{\boldsymbol{m}}(t)|}{\sigma^{W,x_0 t}_{\boldsymbol{m}}} \right)^2 - \Sigma\left(t, N'\right) \right)_+,$$

$$B_2(T, N) := \max_{T' \in \mathcal{T}_n, T' \geq T} \left(\int_{T \leq |t| \leq T'} \sum_{|\boldsymbol{m}|_q \leq N(t)} \left(\frac{|\widehat{c}_{\boldsymbol{m}}(t)|}{\sigma^{W,x_0 t}_{\boldsymbol{m}}} \right)^2 - \Sigma\left(t, N(t)\right) dt \right)_+,$$

$$\Sigma(t, N_0) := 8(2+\sqrt{5})(1+2p_n)\frac{c_{\boldsymbol{X}}}{n}\left(\frac{|t|}{2\pi}\right)^p \nu^W_q(x_0 t, N_0),\ \Sigma_2(T, N) := \int_{\epsilon \leq |t| \leq T} \Sigma(t, N(t)) dt;$$

(N.1) when $W = W_{[-R,R]}$, $\underline{N}^W_{\max,q}$ solution of $2k_q N_0 \ln(7eN_0/(2Rx_0\epsilon)) = \ln(n)$,

$$\nu^W_q(t, N_0) = (N_0+1)^{k_q}\left[\frac{(N_0+p-1)^{p-1}}{(p-1)!}\mathbb{1}\{q=1\} + 2^p \mathbb{1}\{q=\infty\}\right]\left(1 \bigvee \frac{7e(N_0+1)}{2R|t|}\right)^{2k_q N_0}.$$

(N.2) when $W = \cosh(\cdot/R)$,

$$\underline{N}^W_{\max,q} = \frac{\ln(n)}{2k_q}\mathbb{1}\left\{\epsilon = \frac{\pi}{4Rx_0}\right\} + \frac{\ln(n)}{2k_q \ln\left(7e^2/(4Rx_0\epsilon)\right)}\mathbb{1}\left\{\epsilon < \frac{\pi}{4Rx_0}\right\},$$

$$\begin{aligned}\nu^W_q(t, N_0) = 2^p\left(\frac{2eR|t|}{\pi}\right)^{k_q}\left(\frac{(N_0+p-1)^{p-1}}{(p-1)!q} + \mathbb{1}\{q=\infty\}\right)\exp\left(\frac{\pi k_q(N_0+k'_q)}{2R|t|}\right)\mathbb{1}\left\{|t| > \frac{\pi}{4R}\right\} \\ + 2^{k_q}\left(\frac{e\pi}{2}\right)^{2p}\left(\frac{(N_0+p-1)^{p-1}}{(p-1)!q} + \mathbb{1}\{q=\infty\}\right)\left(\frac{7e^2}{4R|t|}\right)^{2k_q N_0}\mathbb{1}\left\{|t| \leq \frac{\pi}{4R}\right\};\end{aligned}$$

$\widehat{N}$ and $\widehat{T}$ are defined, using $c_1 = 1 + 1/(2+\sqrt{5})^2$ (to handle the estimation of $f_{\boldsymbol{X}|\mathcal{X}}$), as

$$\forall t \in \mathbb{R} \setminus (-\epsilon, \epsilon), \quad \widehat{N}(t) \in \operatorname*{argmin}_{0 \leq N \leq N^W_{\max,q}} \left(B_1(t, N) + c_1 \Sigma(t, N)\right), \tag{16}$$

$$\widehat{T} \in \operatorname*{argmin}_{T \in \mathcal{T}_n} \left(B_2\left(T, \widehat{N}\right) + \Sigma_2\left(T, \widehat{N}\right)\right). \tag{17}$$

The upper bounds below take the form

$$\frac{1}{r(n)^2} \sup_{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}^{q,\phi,\omega}_{w,W}(l,M) \cap \mathcal{D},\ f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}} \mathcal{R}^W_{n_0}\left(\widehat{f}^{q,\widehat{N},\widehat{T},\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) = \underset{v(n_0,\mathcal{E})/\delta(n_0) \leq n^{-2}\ln(n)^{-p},\ n_e \geq 3}{O_p}(1), \tag{18}$$

and we refer to (18') when we use the restriction $f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}^{q,\phi,\omega}_{w,W}(l) \cap \mathcal{D}$.

Theorem 4. For all $l, M, s, R, \mu, \sigma > 0$, $H \in \mathbb{N}$, $q \in \{1, \infty\}$, $\phi = 1 \vee |\cdot|^s$, if

(T4.1) $(\omega_k)_{k \in \mathbb{N}_0} = (k^\sigma)_{k \in \mathbb{N}_0}$, $w = 1 \vee |\cdot|$, (18) holds with $r(n) = \left(\ln(n)/\ln_2(n)\right)^{-\sigma}$ when
(T4.1a) $W = W_{[-R,R]}$, $\mathbb{S}_{\boldsymbol{\beta}} \subseteq [-R, R]^p$, $\underline{a} = 1/\epsilon$, and $\epsilon = 7e/(2Rx_0\ln(n))$,
(T4.1b) $W = \cosh(\cdot/R)$, $\underline{a} = 1/\epsilon$ and $\epsilon = 7e^2/(4Rx_0\ln(n))$,

(T4.2) $(\omega_k)_{k\in\mathbb{N}_0} = (e^{\kappa k \ln(1+k)})_{k\in\mathbb{N}_0}$, $\underline{a}$ such that $\mathbb{S}_\alpha \subseteq [-\underline{a}, \underline{a}]$, $w = W_{[-\underline{a},\underline{a}]}$, $W = W_{[-R,R]}$, $\mathbb{S}_{\boldsymbol{\beta}} \subseteq [-R,R]^p$, $\epsilon = 7e/(2Rx_0)$, and $s > (2p+1/2) \vee (\kappa(p+1)/(2k_q))$, (18') holds with $r(n) = n^{-\kappa/(2\kappa+2k_q)} \ln(n)^{1/2+\Lambda(0)}$ and Λ defined in (T2.2),

(T4.3) $(\omega_k)_{k\in\mathbb{N}_0} = (e^{\kappa k})_{k\in\mathbb{N}_0}$, $\underline{a}$ such that $\mathbb{S}_\alpha \subseteq [-\underline{a}, \underline{a}]$, $w = W_{[-\underline{a},\underline{a}]}$, $W = \cosh(\cdot/R)$, $\epsilon = \pi/(4Rx_0)$, and $s > 4p+1/2$, (18') holds with $r(n) = n^{-\kappa/(2\kappa+2k_q)} \ln(n)^{1/2+(p-1)\kappa/(2q(\kappa+k_q))}$.

The results in Theorem 4 are for $v(n_0,\mathcal{E})/\delta(n_0) \leq n^{-2}\ln(n)^{-p}$, in which case $n_e = n$. Theorem 1 and (T4.1a) show that $\widehat{f}^{q,\widehat{N},\widehat{T},\epsilon}_{\alpha,\boldsymbol{\beta}}$ is adaptive. The rate in (T4.2) matches, up to a logarithmic factor, the lower bound in Theorem (T.1.1b) for model (8). For the other cases, the risk is different for the lower bounds and the upper bounds in Theorem 4, but using (12) we obtain the same rate up to logarithmic factors for the risk involving the weight W. (T4.2) and (T4.2) rely on $\mathbb{S}_\alpha \subseteq [-\underline{a},\underline{a}]$ because, else, the choice $\underline{a} = w^I\left(\omega^2_{\underline{N}}\right)$ in Section 5.2 dependents on the parameters of the smoothness class. However, it is possible to check that we can obtain the rate in (T2.2a) up to a $\sqrt{\ln(n)}$ factor when $\nu = 1$ for a choice of $\underline{a}$ independent of the parameters of the smoothness class.

To gain insight, let us sketch the proof when $\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}} = f_{\boldsymbol{X}|\mathcal{X}}$ (hence we simply write $\mathcal{R}^W$). Let

$$\forall t\in\mathbb{R},\ N\in\mathbb{N}_0^{\mathbb{R}},\ T\in[0,\infty),\ \mathcal{L}^W_q(t,N,T) := \left\|\left(\widehat{F}^{q,N,T,0}_1 - \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})},$$

$\widetilde{w} := \mathbb{1}\{w \neq W_{[-\underline{a},\underline{a}]}\}/w$, and $T \in \mathcal{T}_n$. The Plancherel identity and (A.24) yield

$$\mathcal{R}^W\left(\widehat{f}^{q,\widehat{N},\widehat{T},\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) \leq \frac{C}{2\pi}\int_{\epsilon\leq|t|}\mathbb{E}\left[\mathcal{L}^W_q\left(t,\widehat{N}(t),\widehat{T}\right)\right]dt + CM^2\widetilde{w}(\underline{a}). \tag{19}$$

The upper bound in (19) with nonrandom $\widehat{N}$ and $\widehat{T}$ is the one we use to obtain theorems 2 and 3. The idea behind (17) is that it allows to obtain an upper bound with a similar quantity but with arbitrary nonrandom $\widehat{T}$. By arguments in the proof of Lemma A.5 for the first inequality and (17) for the second,

$$\begin{aligned}
&\int_{\epsilon\leq|t|}\mathbb{E}\left[\mathcal{L}^W_q\left(t,\widehat{N}(t),\widehat{T}\right)\right]dt\\
&\leq\left(1+\frac{2}{\sqrt{5}}\right)\int_{\epsilon\leq|t|}\mathbb{E}\left[\mathcal{L}^W_q\left(t,\widehat{N}(t),T\right)\right]dt\\
&\quad+(2+\sqrt{5})\left(\mathbb{E}\left[B_2\left(\widehat{T},\widehat{N}\right)+\Sigma_2\left(T,\widehat{N}\right)\right]+\mathbb{E}\left[B_2\left(T,\widehat{N}\right)+\Sigma_2\left(\widehat{T},\widehat{N}\right)\right]\right)\\
&\leq\left(1+\frac{2}{\sqrt{5}}\right)\int_{\epsilon\leq|t|}\mathbb{E}\left[\mathcal{L}^W_q\left(t,\widehat{N}(t),T\right)\right]dt+2(2+\sqrt{5})\mathbb{E}\left[B_2\left(T,\widehat{N}\right)+\Sigma_2\left(T,\widehat{N}\right)\right]
\end{aligned}$$

and, by a Talagrand's inequality,

$$\mathbb{E}\left[B_2\left(T,\widehat{N}\right)\right] \leq \left(1+\frac{2}{\sqrt{5}}\right)\int_{\epsilon\leq|t|}\mathbb{E}\left[\mathcal{L}^W_q(t,\widehat{N}(t),T)\right]dt + O\left(\frac{1}{n}\right),$$

	$W = W_{[-7.5,7.5]}$, Case 1			$W = \cosh(\cdot/7.5)$, Case 2		
	$n = 300$	$n = 500$	$n = 1000$	$n = 300$	$n = 500$	$n = 1000$
MISE (data-driven)	0.092	0.086	0.083	0.089	0.087	0.085
MISE (oracle)	0.091	0.086	0.082	0.088	0.087	0.085

TABLE 1. Risk

where the $O(1/n)$ term is independent of T and $\widehat{N}$. (16) allows to obtain yet another upper bound which replaces $\widehat{N}$ by an arbitrary nonrandom N. We conclude because the final upper bound (A.38) has a similar form as (A.32) when we deal with nonrandom $\widehat{N}$ and $\widehat{T}$.

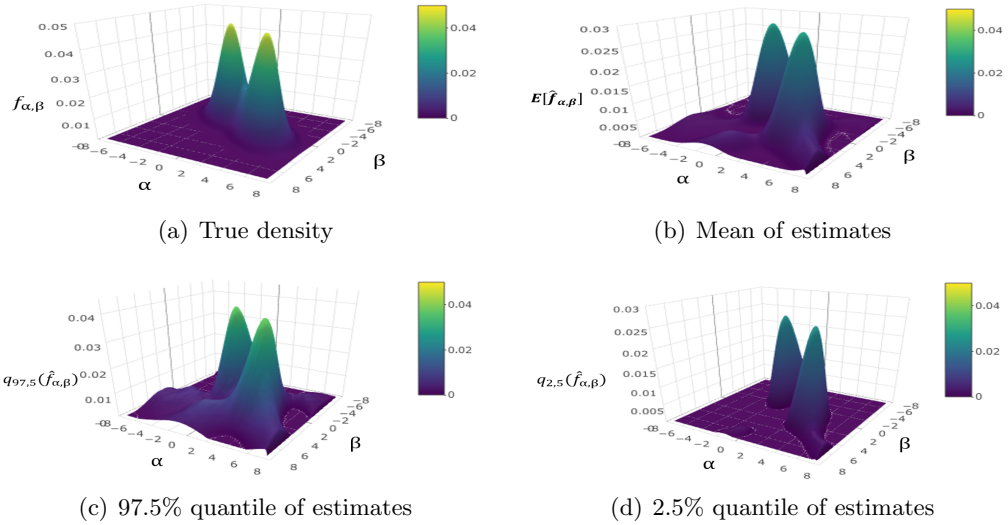
6. Simulations

Let $p = 1$, $q = \infty$, and $(\alpha, \beta)^\top = \xi_1 D + \xi_2(1 - D)$ with $\mathbb{P}(D = 1) = \mathbb{P}(D = 0) = 0.5$. The law of X is a truncated normal based on a normal of mean 0 and variance 2.5 and truncated to $\mathcal{X}$ with $x_0 = 1.5$. The laws of ξ_1 and ξ_2 are either: (Case 1) truncated normals based on normals with means $\mu_1 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\mu_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, same covariance $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, and truncated to $[-6, 6]^{p+1}$ or (Case 2) not truncated. Table 1 compares $\mathbb{E}\left[\left\|\widehat{f}_{\alpha,\beta}^{\infty,\widehat{N},\widehat{T},\epsilon} - f_{\alpha,\beta}\right\|^2_{L^2([-7.5,7.5]^2)}\right]$ and the risk of the oracle $\min_{T\in\mathcal{T}_n, N\in\mathcal{N}_{n,H}} \mathbb{E}\left[\left\|\widehat{f}_{\alpha,\beta}^{\infty,N,T,\epsilon} - f_{\alpha,\beta}\right\|^2_{L^2([-7.5,7.5]^2)}\right]$ for cases 1 and 2. The Monte-Carlo use 1000 simulations. Figure 1 (resp. Figure 2) displays summaries of the law of the estimator for $W = W_{[-7.5,7.5]}$ (resp. $W = \cosh(\cdot/7.5)$) in Case 1 (resp. Case 2) and $n = 1000$. As standard in the literature (see, *e.g.*, [15, 19]), the multiplicative constant appearing in Σ is in practice calibrated from a simulation study. $\widehat{f}_{X|X\in\mathcal{X}}$ is obtained with the same data. The estimator requires the SVD of $\mathcal{F}_c$. By Proposition B.1, we have $g_m^{W(\cdot/R),c} = g_m^{W,Rc}$ for all $m \in \mathbb{N}_0$. When $W = W_{[-1,1]}$, the first coefficients of the decomposition on the Legendre polynomials are obtained by solving for the eigenvectors of two tridiagonal symmetric Toeplitz matrices (see Section 2.6 in [46]). When $W = \cosh$, we refer to Section 7 in [23]. We use $\mathcal{F}_c^*\left(g_m^{W,Rc}\right) = \sigma_m^{W,Rc}\varphi_m^{W,Rc}$ and that $\varphi_m^{W,Rc}$ has norm 1 to get the rest of the SVD. The Fourier inverse is obtained by fast Fourier transform.

APPENDIX - PROOFS

$\Re$ and $\Im$ denote the real and imaginary parts. We denote, for all $m \in \mathbb{N}_0$, by ψ_m^c the function $g_m^{W_{[-1,1]},c}$ and $\mu_m^c = i^m \sigma_m^{W_{[-1,1]},c}$. Because $\psi_m^c = \mathcal{F}_c(\mathcal{E}xt[\psi_m^c])/\mu_m^c$ in $L^2([-1, 1])$, ψ_m^c can be extended as an entire function which we denote with the same notation. Using the injectivity of $\mathcal{F}_c$ (see the proof of Proposition 1), we have $\varphi_m^{W_{[-1,1]},c} = i^{-m}\mathcal{E}xt[\psi_m^c]$. We make use of

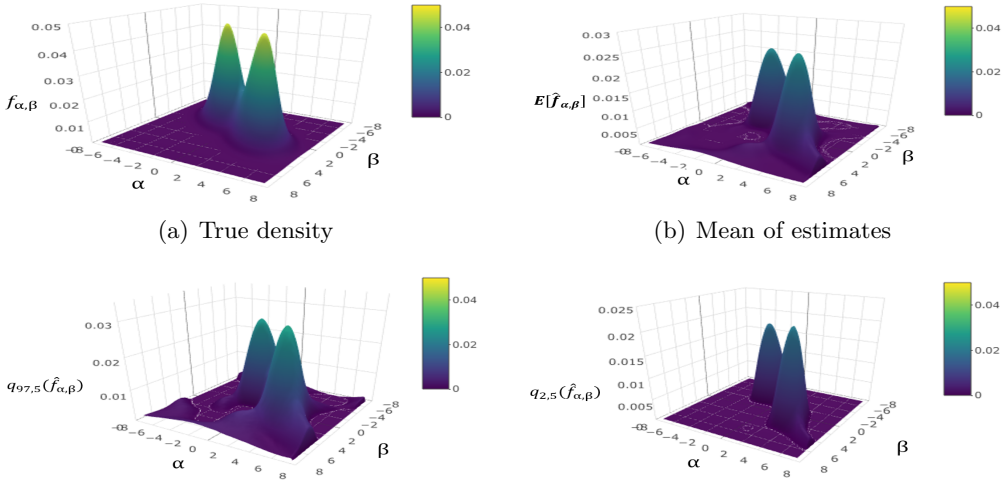
$$\forall a, b > 0,\ \sup_{t\geq 1} \frac{\ln(t)^a}{t^b} = \left(\frac{a}{eb}\right)^a, \tag{A.1}$$



(a) True density (b) Mean of estimates

(c) 97.5% quantile of estimates (d) 2.5% quantile of estimates

FIGURE 1. Case 1, $W = W_{[-7.5,7.5]}$



(a) True density (b) Mean of estimates

(c) 97.5% quantile of estimates (d) 2.5% quantile of estimates

FIGURE 2. Case 2, $W = \cosh(\cdot/7.5)$

$$\forall c > 0,\ \forall a, b \in \mathbb{R},\ ab \leq \frac{a^2}{2c} + \frac{b^2 c}{2}. \tag{A.2}$$

All expectations are conditional on $\mathcal{G}_{n_0}$ when $f_{\boldsymbol{X}|\mathcal{X}}$ is unknown and we rely on $\mathcal{G}_{n_0}$ to estimate it. We remove the conditioning in the notations for simplicity.

A.1. **Proofs of Proposition 1, 2 and 3.**

Proof of Proposition 1. The first assertion comes from the fact that W is nondecreasing on $[0,\infty)$ and $W(0)>0$. For the rest, we use that, for every $h\in L^2(W^{\otimes p})$, if we do not restrict the argument in the definition of $\mathcal{F}_c[h]$ to $[-1,1]^p$, $\mathcal{F}_c[h]$ can be defined as a function in $L^2(\mathbb{R}^p)$. In what follows, for simplicity, we use $\mathcal{F}_c[h]$ for both the function in $L^2([-1,1]^p)$ and in $L^2(\mathbb{R}^p)$. Let us now show that, for all $c\neq 0$, $\mathcal{F}_c$ defined in (3) is injective. Take $h\in L^2\left(W^{\otimes p}\right)\subseteq L^2(\mathbb{R}^p)$ such that $\mathcal{F}_c[h]=0$ in $L^2\left([-1,1]^p\right)$. When $(1/W)$ vanishes at one point, h is compactly supported, thus, by the Paley-Wiener theorem its Fourier transform can be extended as an entire function which restriction to $\mathbb{R}^p$ belongs to $L^2\left(\mathbb{R}^p\right)$. Because the Fourier transform vanishes on a subset with nonempty interior, then $\mathcal{F}[h]=0$ on $\mathbb{R}^p$, thus $h=0$ in $L^2\left(\mathbb{R}^p\right)$. Now, consider the case where $(1/W)(x)>0$ for all $x\in\mathbb{R}$. $\mathcal{F}_c[h]$ belongs to $C^\infty(\mathbb{R}^p)$ by the Lebesgue dominated convergence theorem because, for all $(\boldsymbol{k},\boldsymbol{u})\in\mathbb{N}_0^p\times\mathbb{R}^p$, $\int_{\mathbb{R}^p}\left|c^{|\boldsymbol{k}|_1}\boldsymbol{b}^{\boldsymbol{k}}e^{ic\boldsymbol{b}^T\boldsymbol{u}}h(\boldsymbol{b})\right|d\boldsymbol{b}\leq c^{|\boldsymbol{k}|_1}\left\|h\right\|_{L^2(W^{\otimes p})}\prod_{j=1}^p M_{\boldsymbol{k}_j}$ and, for all $(\boldsymbol{k},\boldsymbol{u})\in\mathbb{N}_0^p\times\mathbb{R}^p$, $\left|\mathcal{F}_c[h]^{(\boldsymbol{k})}(\boldsymbol{u})\right|\leq c^{|\boldsymbol{k}|_1}\left\|h\right\|_{L^2(W^{\otimes p})}\prod_{j=1}^p M_{\boldsymbol{k}_j}$. Theorem B.1 in [18] and the fact that, by the Cauchy-Schwarz inequality, for all $j\in\{1,\dots,p\}$, $\boldsymbol{k}\in\mathbb{N}_0^p$, $M_{\boldsymbol{k}_j}\leq M_{\boldsymbol{k}_j-1}M_{\boldsymbol{k}_j+1}$ yield that $\mathcal{F}_c[h]$ is zero on $\mathbb{R}^p$. Thus, $\mathcal{F}[h]$ and h are zero a.e.

We now show that $\mathcal{K}$ is injective. Take $f\in L^2\left(w\otimes W^{\otimes p}\right)$ such that $\mathcal{K}[f]=0$. By the Plancherel identity and the fact that $w\geq 1$, we have

$$\int_{\mathbb{R}^{p+1}}\left|\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\boldsymbol{b})\right|^2 W^{\otimes p}(\boldsymbol{b})dtd\boldsymbol{b}\leq 2\pi\int_{\mathbb{R}^{p+1}}\left|f(a,\boldsymbol{b})\right|^2 w(a)W^{\otimes p}(\boldsymbol{b})dad\boldsymbol{b}<\infty$$

thus, there exists $\Omega_1\subseteq\mathbb{R}$ of Lebesgue measure 1, such that, for all $t\in\Omega_1$, $\boldsymbol{b}\mapsto\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\boldsymbol{b})\in L^2\left(W^{\otimes p}\right)$. Hence, by the above, for all $t\in\Omega_1$ and $c\in\mathbb{R}$, $\boldsymbol{u}\mapsto\mathcal{F}_c\left[\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\cdot_2)\right](\boldsymbol{u})$ is continuous. Also, because $\left\|\mathcal{K}[f]\right\|_{L^2(\mathbb{R}\times[-1,1]^p)}=0$, there exists $\Omega_2\subseteq\mathbb{R}$ of Lebesgue measure 1, such that, for all $t\in\Omega_2$, $\left\|\mathcal{K}[f](t,\cdot_2)\right\|_{L^2([-1,1]^p)}=0$. As a result, using (5), we have , for all $(t,\boldsymbol{u})\in\Omega_1\cap\Omega_2\times[-1,1]^p$, $\mathcal{K}[f](t,\boldsymbol{u})=0$. Using again (5) and the injectivity of $\mathcal{F}_c$ for all $c\neq 0$, we obtain that for all $t\in(\Omega_1\cap\Omega_2)\setminus\{0\}$, $\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\cdot_2)=0$ in $L^2\left(W^{\otimes p}\right)$, thus $\mathcal{F}_{1\mathrm{st}}\left[f\right]=0$ in $L^2\left(1\otimes W^{\otimes p}\right)$ and $f=0$ in $L^2\left(1\otimes W^{\otimes p}\right)$, hence in $L^2\left(w\otimes W^{\otimes p}\right)$.

We show that $\mathcal{K}$ is continuous at 0. Let $f\in L^2\left(w\otimes W^{\otimes p}\right)$. By the change of variables, the Plancherel identity, and the lower bounds on the weights, we have

$$\left\|\mathcal{K}[f]\right\|^2_{L^2(\mathbb{R}\times[-1,1]^p)}\leq\int_{\mathbb{R}^{p+1}}\left|\mathcal{F}[f](t,\boldsymbol{v})\right|^2(t,\boldsymbol{v})dtd\boldsymbol{v}\leq\left(\frac{2\pi}{W(0)}\right)^p\left\|f\right\|^2_{L^2(w\otimes W^{\otimes p})}.$$

Let $w=1$. We exhibit a bounded sequence $(f_k)_{k\in\mathbb{N}_0}$ in $L^2(1\otimes W^{\otimes p})$ for which there does not exist a convergent subsequence of $\left(\mathcal{K}\left[f_k\right]\right)_{k\in\mathbb{N}_0}$. Take v_0 such that $\mathrm{supp}(v_0)\subset[1,2]$, $\left\|v_0\right\|_{L^2(\mathbb{R})}=1$ and, for all $k\in\mathbb{N}_0$ and $(a,\boldsymbol{b}^\top)^\top\in\mathbb{R}^{p+1}$, $v_k(\cdot)=2^{-k/2}v_0(2^{-k}\cdot)$ and $f_k(a,\boldsymbol{b})=\mathcal{F}^I\left[v_k(\cdot)\varphi_{\underline{\boldsymbol{0}}}^{W,x_0\cdot}(\boldsymbol{b})\right](a)$. $(f_k)_{k\in\mathbb{N}_0}$ is bounded by the Plancherel identity and

$$\left\|f_k\right\|^2_{L^2(1\otimes W^{\otimes p})}=\frac{1}{2\pi}\int_{\mathbb{R}}v_k(t)^2\int_{\mathbb{R}^p}\left|\varphi_{\underline{\boldsymbol{0}}}^{W,x_0t}(\boldsymbol{b})\right|^2 W^{\otimes p}(\boldsymbol{b})dtd\boldsymbol{b}\leq\frac{1}{2\pi}.$$

Using $\mathcal{K}\left[f_k\right](\cdot,\cdot_2)=\sigma_{\underline{\boldsymbol{0}}}^{W,x_0\cdot}v_k(\cdot)g_{\underline{\boldsymbol{0}}}^{W,x_0\cdot}(\cdot_2)\left|x_0\cdot\right|^{p/2}$, $c\in(0,\infty)\mapsto\rho_0^{W,c}$ is nondecreasing (by Lemma 1 in [23] which holds for all W satisfying (H1.2)), and using for all $j\in\mathbb{N}_0$, $\left\|v_j\right\|_{L^2(\mathbb{R})}=1$,

we obtain, for all $(j,k) \in \mathbb{N}_0^2$, $j < k$,

$$\begin{aligned}\|\mathcal{K}[f_j] - \mathcal{K}[f_k]\|^2_{L^2(\mathbb{R}\times[-1,1]^p)} &\geq \rho_{\underline{\mathbf{0}}}^{W,2^j x_0}(2\pi)^p \int_{\mathbb{R}} \left(v_j(t)^2 + v_k(t)^2\right) dt \\ &\geq 2(2\pi)^p \rho_{\underline{\mathbf{0}}}^{W,x_0} > 0,\end{aligned}$$

so $\mathcal{K}$ is not compact. □

Proof of Proposition 2. This holds by Theorem 15.16 in [38] and the injectivity of $\mathcal{F}_c$. □

Proof of Proposition 3. Take $f \in L^2(\mathbb{R})$ and start by showing that $\mathcal{I}_{\underline{a},\epsilon}[f] \in L^2([-\epsilon,\epsilon])$. The terms $1 - \rho_m^{W_{[-1,1]},\underline{a}\epsilon}$ in the denominator of (9) are nonzero because $\left(\rho_m^{W_{[-1,1]},\underline{a}\epsilon}\right)_{m\in\mathbb{N}_0}$ is nonincreasing and $\rho_0^{W_{[-1,1]},\underline{a}\epsilon} < 1$ (see (3.49) in [46]). Using that $\left(g_m^{W_{[-1,1]},\underline{a}\epsilon}(\cdot/\epsilon)/\sqrt{\epsilon}\right)_{m\in\mathbb{N}_0}$ is a basis of $L^2([-\epsilon,\epsilon])$, that $\left(\rho_m^{W_{[-1,1]},\underline{a}\epsilon}\right)_{m\in\mathbb{N}_0}$ is nonincreasing, and the Cauchy-Schwarz inequality for the first display, using that $\sum_{m\in\mathbb{N}_0} \rho_m^{W_{[-1,1]},\underline{a}\epsilon} = 2\underline{a}\epsilon/\pi$ (see (3.55) in [46]) and $\left\|g_m^{W_{[-1,1]},\underline{a}\epsilon}\right\|^2_{L^2(\mathbb{R})} = 1/\rho_m^{W_{[-1,1]},\underline{a}\epsilon}$ (see (3) in [7]) for the second inequality, we obtain

$$\begin{aligned}&\sum_{m\in\mathbb{N}_0}\left(\frac{\rho_m^{W_{[-1,1]},\underline{a}\epsilon}}{\left(1-\rho_m^{W_{[-1,1]},\underline{a}\epsilon}\right)\epsilon}\right)^2 \left|\left\langle f, g_m^{W_{[-1,1]},\underline{a}\epsilon}\left(\frac{\star}{\epsilon}\right)\right\rangle_{L^2(\mathbb{R}\setminus[-\epsilon,\epsilon])}\right|^2 \left\|g_m^{W_{[-1,1]},\underline{a}}\left(\frac{\cdot}{\epsilon}\right)\right\|^2_{L^2([-\epsilon,\epsilon])} \\ &\leq \frac{\|f\|^2_{L^2(\mathbb{R}\setminus[-\epsilon,\epsilon])}}{\left(1-\rho_0^{W_{[-1,1]},\underline{a}\epsilon}\right)^2} \sum_{m\in\mathbb{N}_0}\left(\rho_m^{W_{[-1,1]},\underline{a}\epsilon}\right)^2 \left\|g_m^{W_{[-1,1]},\underline{a}\epsilon}\right\|^2_{L^2(\mathbb{R})} \leq \frac{2\underline{a}\epsilon\,\|f\|^2_{L^2(\mathbb{R}\setminus[-\epsilon,\epsilon])}}{\pi\left(1-\rho_0^{W_{[-1,1]},\underline{a}\epsilon}\right)^2}.\end{aligned} \tag{A.3}$$

Let us now show the second statement. Take $\epsilon > 0$ and $g \in PW(\underline{a})$. Let $(\alpha_m)_{m\in\mathbb{N}}$ be the sequence of coefficients of $g(\epsilon\cdot) \in PW(\underline{a}\epsilon)$ on the complete orthogonal system $\left(g_m^{W_{[-1,1]},\underline{a}\epsilon}\right)_{m\in\mathbb{N}_0}$. Because $\left(g_m^{W_{[-1,1]},\underline{a}\epsilon}\right)_{m\in\mathbb{N}_0}$ is a basis of $L^2([-1,1])$, we have $\sum_{m\in\mathbb{N}_0} \alpha_m g_m^{W_{[-1,1]},\underline{a}\epsilon} = g(\epsilon\cdot)\mathbb{1}\{|\cdot| \geq 1\} + \sum_{m\in\mathbb{N}_0} \alpha_m g_m^{W_{[-1,1]},\underline{a}\epsilon}\mathbb{1}\{|\cdot| \leq 1\}$. We identify the coefficients by taking the Hermitian product in $L^2(\mathbb{R})$ with $g_m^{W_{[-1,1]},\underline{a}\epsilon}$ and obtain $\mathcal{I}_{\underline{a},\epsilon}[g] = g$ in $L^2(\mathbb{R})$ and, for all $f, h \in L^2(\mathbb{R})$,

$$\left\|f - \mathcal{I}_{\underline{a},\epsilon}[h]\right\|^2_{L^2([-\epsilon,\epsilon])} \leq 2\left(\left\|f - \mathcal{P}_{\underline{a}}[f]\right\|^2_{L^2([-\epsilon,\epsilon])} + \left\|\mathcal{I}_{\underline{a},\epsilon}\left[\mathcal{P}_{\underline{a}}[f] - h\right]\right\|^2_{L^2([-\epsilon,\epsilon])}\right). \tag{A.4}$$

Replacing f by $\mathcal{P}_{\underline{a}}[f] - h$ in (A.3) yields

$$\left\|\mathcal{I}_{\underline{a},\epsilon}\left[\mathcal{P}_{\underline{a}}[f] - h\right]\right\|^2_{L^2([-\epsilon,\epsilon])} \leq \frac{C_0(\underline{a}\epsilon)}{2}\left\|\mathcal{P}_{\underline{a}}[f] - h\right\|^2_{L^2(\mathbb{R}\setminus[-\epsilon,\epsilon])}. \tag{A.5}$$

Using (A.4) and (A.5) for the first display, $\mathcal{P}_{\underline{a}}[f] - h = \left(\mathcal{P}_{\underline{a}}[f] - f\right) + (f - h)$ and the Jensen inequality for the second display, we obtain

$$\left\|f - \mathcal{I}_{\underline{a},\epsilon}[h]\right\|^2_{L^2([-\epsilon,\epsilon])} \leq 2\left\|f - \mathcal{P}_{\underline{a}}[f]\right\|^2_{L^2([-\epsilon,\epsilon])} + C_0(\underline{a}\epsilon)\left\|\mathcal{P}_{\underline{a}}[f] - h\right\|^2_{L^2(\mathbb{R}\setminus[-\epsilon,\epsilon])}$$

$$\leq 2(1+C_0(\underline{a}\epsilon))\left\|f-\mathcal{P}_{\underline{a}}[f]\right\|^2_{L^2(\mathbb{R})}+2C_0(\underline{a}\epsilon)\left\|f-h\right\|^2_{L^2(\mathbb{R}\setminus[-\epsilon,\epsilon])}. \quad \square$$

A.2. **Lower bounds.** We denote by $\mathbb{P}_j$ the law of density $f_{j,n}$ and by $\mathbb{P}_{j,n}$ the law of an iid sample of size n, and use

$$\inf_{\widehat{f}}\ \sup_{f\in\mathcal{H}}\mathbb{E}\left[\left\|\widehat{f}-f\right\|^2_{L^2(\mathbb{R}^{p+1})}\right]\geq \inf_{\widehat{f}}\ \max_{f_{j,n}\in\mathcal{H},\ j\in\{1,2\}}\mathbb{E}\left[\left\|\widehat{f}-f_{j,n}\right\|^2_{L^2(\mathbb{R}^{p+1})}\right]$$

and the next lemma (see Theorem 2.2, (2.5), and (2.9) in [50]).

Lemma A.1. If there exists $\xi<\sqrt{2}$ such that

(i) $\forall j\in\{1,2\},\ f_{j,n}\in\mathcal{H}$,
(ii) $\|f_{1,n}-f_{2,n}\|^2_{L^2(\mathbb{R}^{p+1})}\geq 4r(n)^2>0$,
(iii) $\chi_2(\mathbb{P}_{2,n},\mathbb{P}_{1,n})\leq\xi^2$ or $K(\mathbb{P}_{2,n},\mathbb{P}_{1,n})\leq\xi^2$,

then we have

$$\frac{1}{r(n)^2}\inf_{\widehat{f}}\ \max_{f_{j,n}\in\mathcal{H},\ j\in\{1,2\}}\mathbb{E}\left[\left\|\widehat{f}-f_{j,n}\right\|^2_{L^2(\mathbb{R}^{p+1})}\right]\geq\frac{1}{2}\left(\frac{e^{-\xi^2}}{2}\bigvee\left(1-\frac{\xi}{\sqrt{2}}\right)\right).$$

Proof of (T1.1a). For $j=1,2$, $f_{j,n}$ is a possible $f_{\alpha,\boldsymbol{\beta}}$, $\left(b^j_{\boldsymbol{m}}\right)_{\boldsymbol{m}\in\mathbb{N}_0^p}$ the sequence of its coefficients (see (6)), $\mathbb{P}_{j,n}$ have marginal $f_{\boldsymbol{X}|\mathcal{X}}$, and $f^j_{Y|\boldsymbol{X}}(y|\boldsymbol{x})$ are the implied conditional densities. Steps 1-3 give conditions under which (i)-(iii) in Lemma A.1 are satisfied when

$$f_{1,n}:=f_0 \text{ and } f_{2,n}:=f_0+F,\ f_0(a,\boldsymbol{b}):=\frac{1}{\pi\tau\left(1+(a/\tau)^2\right)}\frac{\mathbb{1}\{|\boldsymbol{b}|_\infty\leq R\}}{(2R)^{p/2}}, \tag{A.6}$$

$$\forall(a,\boldsymbol{b})\in\mathbb{R}^{p+1},\ F(a,\boldsymbol{b}):=\gamma\mathcal{F}^I_{1\text{st}}\left[\left(\frac{c(|\star|)}{2\pi}\right)^{p/2}\lambda(\star)\psi^{Rc(\star)}_{\widetilde{\boldsymbol{N}}(q)}\left(\frac{\boldsymbol{b}}{R}\right)\right](a)\mathbb{1}\{|\boldsymbol{b}|_\infty\leq R\}, \tag{A.7}$$

$$\forall U/2\leq|t|\leq U,\ \lambda(t):=\exp\left(1-\frac{1}{1-\left(4\left|t\right|-3U\right)^2/U^2}\right),\ \text{else } \lambda(t):=0, \tag{A.8}$$

$$\widetilde{\boldsymbol{N}}(1):=\left(N,\underline{\boldsymbol{N}(\boldsymbol{R}x_0\boldsymbol{U})}^\top\right)^\top,\ \widetilde{\boldsymbol{N}}(\infty):=\underline{\boldsymbol{N}}\in\mathbb{N}^p,\ N(Rx_0U):=\lceil H(Rx_0U)\rceil, \tag{A.9}$$

for H from Section B.1.2, n large enough, N (odd), γ, $\tau\geq1$, and U from Step 4 and such that $N\geq N(Rx_0U)$, hence $N\geq Rx_0U\vee2$ by the discussion before Lemma B.6. Note $\|\lambda\|_{L^\infty(\mathbb{R})}\leq1$.

Step 1.1. We prove that $f_{1,n}$ and $f_{2,n}$ are nonnegative when

(A.10)
$$\gamma UN^{k_q/2}\left(\frac{1}{1+p/2}\left(\frac{Rx_0U}{\pi}\right)^{\frac{p}{2}}\left(\frac{5}{4}\right)^{\frac{k_q}{2}}\left(\frac{5}{4}N(Rx_0U)\right)^{\frac{p-1}{2q}}\right)\bigvee\left(2^{\frac{p}{2}-1}C_8(Rx_0U,p,U)N^2\right)\leq\frac{1}{\tau+1/\tau},$$

where $C_8(Rx_0U,p,U)$ is defined in Lemma B.10. Let $(a,\boldsymbol{b})\in\mathbb{R}\times[-R,R]^p$. We show that (A.10) yield $f_0(a,\boldsymbol{b})\geq|F(a,\boldsymbol{b})|$ which ensures that $f_{2,n}$ is nonnegative. The first bound in

(A.10) yields the result when $|a|<1$ because, by the third assertion in Lemma B.7,

$$|F(a,\boldsymbol{b})| \leq \frac{\gamma}{2\pi}\left(\frac{x_0}{2\pi}\right)^{p/2}\left(N+\frac{1}{2}\right)^{k_q/2}(N(Rx_0U)+1/2)^{(p-1)/(2q)}\int_{\mathbb{R}}|t|^{p/2}\lambda(t)dt$$

$$\leq \frac{\gamma U}{\pi(1+p/2)}\left(\frac{Ux_0}{2\pi}\right)^{p/2}\left(\frac{5}{4}N\right)^{k_q/2}\left(\frac{5}{4}N(Rx_0U)\right)^{(p-1)/(2q)}. \tag{A.11}$$

Because $t\mapsto \psi^{Rx_0t}_{\widetilde{\boldsymbol{N}}(q)}(\boldsymbol{b}/R)$ is analytic (see [22] page 320), $t\mapsto (x_0|t|/(2\pi))^{p/2}\lambda(t)\psi^{Rx_0t}_{\widetilde{\boldsymbol{N}}(q)}(\boldsymbol{b}/R)\in C^\infty(\mathbb{R})$ and its derivatives have compact support. By integration by parts, we obtain

$$\forall a\neq 0,\ |F(a,\boldsymbol{b})| \leq \frac{\gamma}{\pi a^2R^{p/2}}\int_{U/2}^{U}\left|\frac{\partial^2}{\partial t^2}\left(\left(\frac{Rx_0t}{2\pi}\right)^{p/2}\lambda(t)\psi^{Rx_0t}_{\widetilde{\boldsymbol{N}}(q)}\left(\frac{\boldsymbol{b}}{R}\right)\mathbb{1}\{|\boldsymbol{b}|_\infty\leq R\}\right)\right|dt.$$

The result when $|a|\geq 1$ is obtained by $1+(a/\tau)^2\leq(1+1/\tau^2)a^2$, so by the second bound in (A.10), $\gamma UC_8(Rx_0U,p,U)N^{2+k_q/2}/(2a^2)\leq 1/\left(2^{p/2}\tau(1+(a/\tau)^2)\right)$, and by Lemma B.10

$$\forall(a,\boldsymbol{b})\in\mathbb{R}^{p+1}:\ |a|\geq 1,\ |F(a,\boldsymbol{b})|\leq\frac{\gamma UC_8(Rx_0U,p,U)}{2\pi a^2R^{p/2}}N^{2+k_q/2}\mathbb{1}\{|\boldsymbol{b}|_\infty\leq R\}. \tag{A.12}$$

$f_{1,n}=f_0$ has integral 1 and so has $f_{2,n}$ by Fubini's theorem and that ψ^c_N is odd when N is odd.

Step 1.2. We give conditions for $f_{1,n},f_{2,n}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l)$. By (A.6)-(A.7), and because, by Step 1.1, for all $(a,\boldsymbol{b})\in\mathbb{R}^{p+1}$, $f_{2,n}(a,\boldsymbol{b})^2\leq 4f_{1,n}(a,\boldsymbol{b})^2$, $f_{1,n}$ and $f_{2,n}$ belong to $L^2\left(w\otimes W^{\otimes p}_{[-R,R]}\right)$. Let us show that $f_{2,n}$, hence $f_{1,n}$ ($f_{2,n}$ with $\gamma=0$), satisfy the first condition in $\mathcal{H}^{q,\phi,\omega}_{w,W}(l)$ if

$$2\left(\int_0^\infty\phi(t)^2e^{-2\tau t}dt+\gamma^2\left(\frac{Rx_0U}{2\pi}\right)^p\frac{\phi(U)^2U}{p+1}\right)\leq\pi l^2 \tag{A.13}$$

$$\frac{C_{12}(\sigma,p)}{\tau p^{2\sigma/q}}+\gamma^2\frac{2Up^{2\sigma/q}N^{2\sigma}}{p+1}\left(\frac{Rx_0U}{2\pi}\right)^p\leq\pi l^2. \tag{A.14}$$

Let $\boldsymbol{m}\in\mathbb{N}_0^p$ and $c^P_{\boldsymbol{m}}(t):=\left\langle 1/2^{p/2},\psi^{Rx_0t}_{\boldsymbol{m}}\right\rangle_{L^2([-1,1]^p)}$. By Proposition B.1 (iii), change of variables, and for all $t\in\mathbb{R}$, $\mathcal{F}_{1\text{st}}\left[f_0(\cdot,\star)\right](t)=e^{-|t|\tau}\mathbb{1}\{|\star|_\infty\leq R\}/(2R)^{p/2}$, we have

$$b^2_{\boldsymbol{m}}(t)=i^{-|\boldsymbol{m}|_1}\left(e^{-\tau|t|}c^P_{\boldsymbol{m}}(t)+\gamma\mathbb{1}\{\boldsymbol{m}=\widetilde{\boldsymbol{N}}(q)\}\left(\frac{Rx_0|t|}{2\pi}\right)^{p/2}\lambda(t)\right). \tag{A.15}$$

Because $\left(\psi^{Rx_0t}_{\boldsymbol{m}}\right)_{\boldsymbol{m}\in\mathbb{N}_0^p}$ is an orthonormal basis, we have

$$\forall t\neq 0,\ \sum_{\boldsymbol{m}\in\mathbb{N}_0^p}\left|b^2_{\boldsymbol{m}}(t)\right|^2\leq 2\left(e^{-2\tau|t|}+\gamma^2\left(\frac{Rx_0|t|}{2\pi}\right)^p\lambda(t)^2\right). \tag{A.16}$$

The first part of the first condition in $\mathcal{H}^{q,\phi,\omega}_{w,W}(l)$ holds by (A.13) and because, by (A.16),

$$\sum_{\boldsymbol{m}\in\mathbb{N}_0^p}\int_{\mathbb{R}}\phi(t)^2\left|b^2_{\boldsymbol{m}}(t)\right|^2dt\leq 4\left(\int_0^\infty\phi(t)^2e^{-2\tau t}dt+\gamma^2\left(\frac{Rx_0}{2\pi}\right)^p\int_{U/2}^U\phi(t)^2t^p\lambda^2(t)dt\right).$$

The second part of the first condition holds by (A.14) and because, by (A.15) and Lemma B.11, for all $\tau \geq \left(3e^{1/2}Rx_0/8\right) \vee (1/2)$ and $N \geq N(Rx_0U)$,

$$\sum_{\boldsymbol{m}\in\mathbb{N}_0^p} |\boldsymbol{m}|_q^{2\sigma} \int_{\mathbb{R}} \left|b_{\boldsymbol{m}}^2(t)\right|^2 dt \leq 2\left(\int_{\mathbb{R}} e^{-2\tau|t|} \sum_{\boldsymbol{m}\in\mathbb{N}_0^p} |\boldsymbol{m}|_q^{2\sigma} \left(c_{\boldsymbol{m}}^P(t)\right)^2 dt + \gamma^2 \left(\frac{Rx_0}{2\pi}\right)^p \left|\widetilde{\boldsymbol{N}}(q)\right|_q^{2\sigma} \int_{\mathbb{R}} \lambda^2(t)\,|t|^p\, dt\right)$$
$$\leq 2\left(\frac{C_{12}(\sigma,p)}{\tau p^{2\sigma/q}} + \gamma^2 \frac{2Up^{2\sigma/q}N^{2\sigma}}{p+1}\left(\frac{Rx_0U}{2\pi}\right)^p\right).$$

Step 2. (ii) holds with $4r(n)^2 = \gamma^2 \left(Rx_0/(2\pi)\right)^p \int_{U/2}^U t^p \lambda(t)^2 dt/\pi$.

Step 3. By (ii) page 97 in [50], we have $\chi_2(\mathbb{P}_{2,n}, \mathbb{P}_{1,n}) = \left(1+\chi_2\left(\mathbb{P}_2,\mathbb{P}_1\right)\right)^n - 1$ so

$$\chi_2(\mathbb{P}_{2,n},\mathbb{P}_{1,n}) = n\int_0^{\chi_2(\mathbb{P}_2,\mathbb{P}_1)} (1+u)^{n-1} du \leq n\chi_2\left(\mathbb{P}_2,\mathbb{P}_1\right) \exp\left((n-1)\chi_2\left(\mathbb{P}_2,\mathbb{P}_1\right)\right).$$

Thus, if $\chi_2\left(\mathbb{P}_2,\mathbb{P}_1\right) \leq 1/n$, we have $\chi_2(\mathbb{P}_{2,n},\mathbb{P}_{1,n}) \leq en\chi_2\left(\mathbb{P}_2,\mathbb{P}_1\right)$. Moreover, we have

$$\chi_2\left(\mathbb{P}_2,\mathbb{P}_1\right) = \int_{\mathbb{S}_{\boldsymbol{X}}} \int_{\mathbb{R}} \frac{f_{\boldsymbol{X}}(\boldsymbol{x})\left(f^1_{Y|\boldsymbol{X}}(y|\boldsymbol{x}) - f^2_{Y|\boldsymbol{X}}(y|\boldsymbol{x})\right)^2}{f^1_{Y|\boldsymbol{X}}(y|\boldsymbol{x})} d\boldsymbol{x}dy$$

and, for all $(y,\boldsymbol{x}) \in \mathbb{R}\times\mathbb{S}_{\boldsymbol{X}}$ such that $\boldsymbol{x}^{\underline{1}} \neq 0$,

$$f^1_{Y|\boldsymbol{X}}(y|\boldsymbol{x}) = \frac{1}{\pi\tau(2R)^{p/2}\left|\boldsymbol{x}^{\underline{1}}\right|} \int_{\mathbb{R}^p} \frac{\prod_{k=1}^p \mathbb{1}\{|\boldsymbol{u}_k| \leq |\boldsymbol{x}_k|\, R\}}{\left(\left(y - \sum_{k=1}^p \boldsymbol{u}_k\right)/\tau\right)^2 + 1} d\boldsymbol{u} \geq \frac{(2R)^{p/2}}{\pi\tau} \inf_{|u|\leq|\boldsymbol{x}|_1 R} \frac{1}{((y-u)/\tau)^2+1}.$$

This yields, using $\mathbb{S}_{\boldsymbol{X}} = [-x_0,x_0]^p$ and Parseval's identity,

$$\chi_2\left(\mathbb{P}_2,\mathbb{P}_1\right) \leq \frac{\pi\tau C_{\boldsymbol{X}}}{(2R)^{p/2}} \int_{[-x_0,x_0]^p} \int_{\mathbb{R}} \left(\frac{2y^2}{\tau^2} + \frac{2(|\boldsymbol{x}|_1 R)^2}{\tau^2} + 1\right)\left(f^1_{Y|\boldsymbol{X}}(y|\boldsymbol{x}) - f^2_{Y|\boldsymbol{X}}(y|\boldsymbol{x})\right)^2 d\boldsymbol{x}dy$$
$$= \frac{C_{\boldsymbol{X}} x_0^p \gamma^2}{\tau(2R)^{p/2}} \int_{[-1,1]^p} \int_{\mathbb{R}} \left(\left|\partial_t \mathcal{F}\left[F\right](t, x_0 t\boldsymbol{x})\right|^2 + \left((x_0 pR)^2 + \frac{\tau^2}{2}\right)\left|\mathcal{F}\left[F\right](t,x_0 t\boldsymbol{x})\right|^2\right) d\boldsymbol{x}dt.$$

By lemmas B.12 and B.4, we have $\chi_2\left(\mathbb{P}_2,\mathbb{P}_1\right) \leq C_{18}(U,x_0,R,\tau)\gamma^2 N^2 \exp\left(-2k_q N \ln\left(4N/(eRx_0U)\right)\right)$,

$$C_{18}(U,x_0,R,\tau) := \frac{C_{\boldsymbol{X}}}{\tau}\left(\frac{R^{3/2}x_0Ue^3}{9\sqrt{2}}\right)^p \left(C_{17}(Rx_0U,p,U) + U\frac{2(x_0pR)^2+\tau^2}{2N(Rx_0U)^2}\right)\left(\frac{eRx_0U}{4N(Rx_0U)}\right)^{\frac{2(p-1)N(Rx_0U)}{q}}.$$

As a result, (iii) is satisfied if

$$n\gamma^2 N^2 \exp\left(-2k_q N \ln\left(\frac{4N}{eRx_0U}\right)\right) \leq \frac{\xi^2}{eC_{18}(U,x_0,R,\tau)}. \tag{A.17}$$

Step 4. We take $U = 4/(eRx_0)$, $N = 2\lceil \underline{N} \rceil + 1$ for $\underline{N}$ going to infinity with n, and τ such that

$$\int_0^\infty \phi(t)^2 e^{-2\tau t} dt \bigvee \frac{C_{12}(\sigma,p)}{2\tau p^{2\sigma/q}} \leq \frac{\pi l^2}{4}.$$

Thus $N(Rx_0U)$ is universal and $\underline{N} \geq N(Rx_0U)$ and $N \leq (9/2)\underline{N}$ for n large enough. (A.10), (A.13)-(A.14) (by the pigeonhole principle), and (A.17) hold for n large enough if

$$\gamma \underline{N}^{2+k_q/2} \leq \frac{1}{(\tau + 1/\tau)\,(9/2)^{1+(p+k_q)/2} C_8(4/e, p, U) U}, \tag{A.18}$$

$$\gamma \underline{N}^{\sigma} \leq \frac{l}{2p^{\sigma/q}(9/2)2^{\sigma}} \sqrt{\frac{\pi(p+1)}{U}} \left(\frac{e\pi}{2}\right)^{p/2}, \quad \gamma \leq \frac{l}{\phi(U)} \sqrt{\frac{\pi(p+1)}{U}} \left(\frac{e\pi}{2}\right)^{p/2} \tag{A.19}$$

$$n\gamma^2 \underline{N}^2 \exp\left(-4k_q \underline{N} \ln\left(\underline{N}\right)\right) \leq \frac{\xi^2}{(9/2)^2 e C_{18}(U, x_0, R, \tau)}, \tag{A.20}$$

and γ goes to 0 with n. By taking $\gamma = C_\gamma \underline{N}^{-(2+k_q/2)\vee\sigma} / \left((C_8(4/e, p, U)U) \wedge \sqrt{U}\right)$ for a small enough C_γ depending on l, ϕ, σ, p, and q, (A.18)-(A.19) hold because Rx_0, hence U is fixed. Then, with $\underline{N} = 3\ln(n)/\left(8k_q \ln_2(n)\right)$, (A.20) becomes

$$\frac{C_\gamma^2}{\sqrt{n}} \exp\left(\frac{3\ln(n)\ln(8k_q\ln_2(n)/3)}{4\ln_2(n)} - \left(\left(2 + \frac{k_q}{2}\right) \vee \sigma - 1\right) \ln\left(\frac{3\ln(n)}{8k_q \ln_2(n)}\right)\right) \leq \frac{\xi^2 \left(C_8(4/e, p, U)U\right)^2 \wedge U}{8eC_{18}(U, x_0, R, \tau)}$$

which holds for n large enough and $r(n)^2 = \underline{N}^{-2((2+k_q/2)\vee\sigma)} C_\gamma^2 \left(Rx_0/(2\pi)\right)^p \int_{U/2}^{U} t^p \lambda(t)^2 dt/(4\pi)$.□

All other steps 2 are the same as for (T1.1a).

Proof of (T1.2a). Denote by $E := L^2(\mathbb{R}) \times L^2(\mathbb{R})$. Equip E with $\langle \boldsymbol{g}, \boldsymbol{h}\rangle_E^2 = \langle \boldsymbol{g}_1, \boldsymbol{h}_1\rangle_{L^2(\mathbb{R})}^2 + \langle \boldsymbol{g}_2, \boldsymbol{h}_2\rangle_{L^2(\mathbb{R})}^2$. Denote by $\mathbb{P}_{j,n}^{\boldsymbol{m}}$ the law of $\left(\left(\Re\left(Z_{\boldsymbol{m}}^j(t)\right)\right)_{t\in\mathbb{R}}, \left(\Im\left(Z_{\boldsymbol{m}}^j(t)\right)\right)_{t\in\mathbb{R}}\right)$ in E and by $\mathbb{P}_{j,n}$ the law on $\ell_2(E)$ of the sequence indexed by $\boldsymbol{m} \in \mathbb{N}_0^p$. The latter can be defined as a function of $f_{j,n}$ or $\left(b_{\boldsymbol{m}}^j(t)\right)_{\boldsymbol{m}\in\mathbb{N}_0^p,\ t\in\mathbb{R}}$, for $j = 1, 2$. Take $f_{1,n} := 0$ and $f_{2,n}$ like (A.6) with $\widetilde{\boldsymbol{N}}(1) := (N, \underline{\boldsymbol{0}}^\top)^\top \in \mathbb{N}_0^p$. (A.15) yields, for all $\boldsymbol{m} \in \mathbb{N}_0^p$, $b_{\boldsymbol{m}}^2(t) = i^{-|\boldsymbol{m}|_1}\gamma \mathbb{1}\{\boldsymbol{m} = \widetilde{\boldsymbol{N}}(q)\} \left(Rx_0\,|t|\,/2\pi\right)^{p/2} \lambda(t)$. By independence, we have, for $j = 1, 2$, $\mathbb{P}_{j,n} = \bigotimes_{\boldsymbol{m}\in\mathbb{N}_0^p} \mathbb{P}_{j,n}^{\boldsymbol{m}}$.

Step 1. Using (A.11) and (A.12), we have $f_{2,n} \in L^2\left(w \otimes \cosh\left(\cdot/R\right)^{\otimes p}\right)$ and $f_{2,n} \in \mathcal{H}_{w,W}^{q,\phi,\omega}(l)$ if

$$\left(\frac{Rx_0U}{2\pi}\right)^p \frac{2U\gamma^2}{p+1} \left(\phi(U) \vee N^\sigma\right)^2 \leq \pi l^2. \tag{A.21}$$

Step 3. Let $\xi < \sqrt{2}$, $G_{\widetilde{\boldsymbol{N}}(q)}^W = \left(\Re\left(\sigma_{\widetilde{\boldsymbol{N}}(q)}^{W,x_0\cdot} b_{\widetilde{\boldsymbol{N}}(q)}^2(\cdot)\right), \Im\left(\sigma_{\widetilde{\boldsymbol{N}}(q)}^{W,x_0\cdot} b_{\widetilde{\boldsymbol{N}}(q)}^2(\cdot)\right)\right)^\top$, $\mathcal{Q}$ the covariance operator of $\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}$ on E, and, for all $\boldsymbol{h} \in E$, $\mathcal{L}[\boldsymbol{h}] := (\sigma/\sqrt{n})\left(\int_0^\cdot \boldsymbol{h}_1(s)ds, \int_0^\cdot \boldsymbol{h}_2(s)ds\right)^\top$. The reproducing kernel Hilbert space $H_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}}$ of $\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}$ on E is the image of $\mathcal{Q}^{1/2}$ with the scalar product of the image structure. By Corollary B.3 in [17] and $\mathcal{Q} = \mathcal{L}\mathcal{L}^*$, it is the image of $\mathcal{L}$ with the norm $\|\boldsymbol{f}\|_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}}^2 = (n/\sigma^2)\left(\|\boldsymbol{h}_1\|_2^2 + \|\boldsymbol{h}_2\|_2^2\right)$ for $\boldsymbol{f} = \mathcal{L}[\boldsymbol{h}]$ and derived scalar product. By (2.12) in [17], the scalar product is also defined when one function belongs to $H_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}}$ for

$\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}$ *a.e.* other function in E. By the Cameron-Martin formula (Proposition 2.26 in [17]),

$$\frac{\mathrm{d}\mathbb{P}_{2,n}^{\widetilde{\boldsymbol{N}}(q)}}{\mathrm{d}\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}}(y) = \exp\left(\left\langle y, \frac{\sqrt{n}}{\sigma}\mathcal{L}\left[G^W_{\widetilde{\boldsymbol{N}}(q)}\right]\right\rangle_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}} - \frac{1}{2}\left|\frac{\sqrt{n}}{\sigma}\mathcal{L}\left[G^W_{\widetilde{\boldsymbol{N}}(q)}\right]\right|^2_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}}\right) \quad \mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}\ a.s.,$$

and, because $K(\mathbb{P}_{2,n},\mathbb{P}_{1,n}) = \int_E \ln\left(\mathrm{d}\mathbb{P}_{2,n}^{\widetilde{\boldsymbol{N}}(q)}/\mathrm{d}\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}(y)\right) d\mathbb{P}_{2,n}^{\widetilde{\boldsymbol{N}}(q)}(y)$, we have

$$K(\mathbb{P}_{2,n},\mathbb{P}_{1,n}) = \mathbb{E}\left[\left\langle Z^2_{\widetilde{\boldsymbol{N}}(q)}, \frac{\sqrt{n}}{\sigma}\mathcal{L}\left[G^W_{\widetilde{\boldsymbol{N}}(q)}\right]\right\rangle_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}}\right] - \frac{n}{2\sigma^2}\int_{\mathbb{R}}\left|\sigma^{W,x_0 s}_{\widetilde{\boldsymbol{N}}(q)} b^2_{\widetilde{\boldsymbol{N}}(q)}(s)\right|^2 ds.$$

Because

$$\left\langle Z^2_{\widetilde{\boldsymbol{N}}(q)}, \frac{\sqrt{n}}{\sigma}\mathcal{L}\left[G^W_{\widetilde{\boldsymbol{N}}(q)}\right]\right\rangle_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}} = \left|\frac{\sqrt{n}}{\sigma}\mathcal{L}\left[G^W_{\widetilde{\boldsymbol{N}}(q)}\right]\right|^2_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}} + \left\langle \begin{pmatrix} B^{\Re}_{\widetilde{\boldsymbol{N}}(q)} \\ B^{\Im}_{\widetilde{\boldsymbol{N}}(q)} \end{pmatrix}, \mathcal{L}\left[G^W_{\widetilde{\boldsymbol{N}}(q)}\right]\right\rangle_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}},$$

and the second term in the right-hand side is a limit in quadratic mean of mean zero Gaussian random variables, hence has mean zero (see the arguments page 41 in [17]), we have

$$K(\mathbb{P}_{2,n},\mathbb{P}_{1,n}) = \frac{n}{2\sigma^2}\int_{\mathbb{R}}\left|\sigma^{W,x_0 t}_{\widetilde{\boldsymbol{N}}(q)} b^2_{\widetilde{\boldsymbol{N}}(q)}(t)\right|^2 dt. \tag{A.22}$$

By Proposition B.1 (ii), we have $K(\mathbb{P}_{2,n},\mathbb{P}_{1,n}) = \gamma^2 n R^p \int_{\mathbb{R}}\left(\sigma^{\cosh,Rx_0 t}_{\widetilde{\boldsymbol{N}}(q)}\right)^2 \left(Rx_0|t|/(2\pi)\right)^p \lambda(t)^2 dt/(2\sigma^2) = \gamma^2 n R^p \int_{\mathbb{R}} \rho^{\cosh,Rx_0 t}_{\widetilde{\boldsymbol{N}}(q)} \lambda(t)^2 dt/(2\sigma^2)$ and, by Theorem 3 in [23] (there is difference of normalisation for $\mathcal{Q}_t$ by a factor $1/(2\pi)$), for all $U/2 \le |t| \le U$ and $2/(Rx_0U) \ge 1$, $\rho^{\cosh,Rx_0 t}_{\widetilde{\boldsymbol{N}}(q)} \le \left(2Rx_0 Ue/\left(\pi(1-(Rx_0U/2)^2)\right)\right)^p \exp\left(2k_q N \ln\left(Rx_0U/2\right)\right)$. Thus Lemma A.1 (iii) holds if

$$n\gamma^2 \exp\left(-2k_q N \ln\left(\frac{2}{Rx_0U}\right)\right) \le \xi^2\left(\frac{\pi(1-(Rx_0U/2)^2)}{2R^2x_0Ue}\right)^p \frac{2\sigma^2}{U}. \tag{A.23}$$

Step 4. Let $U = 2/(eRx_0)$, $N = \lceil \underline{N} \rceil$, $\gamma = C_\gamma \xi\sigma\sqrt{eRx_0}\left(\pi\left(1-e^{-2}\right)/(4R)\right)^{p/2}/\underline{N}^{\overline{\sigma}\vee\sigma}$ for $\overline{\sigma} > 1/2$ and $C_\gamma = l\left(8eR/\left(1-e^{-2}\right)\right)^{p/2}\sqrt{(p+1)e\pi Rx_0}/\left(2^{\sigma+1}\xi\sigma\sqrt{eRx_0}\right)$, and $\underline{N} = \ln(n/\ln(n))/(2k_q)$. (A.23) holds if $nC_\gamma^2 \exp\left(-2k_q\underline{N} - 2\left(\overline{\sigma}\vee\sigma\right)\ln\left(\underline{N}\right)\right) = C_\gamma^2 (2k_q)^{2(\overline{\sigma}\vee\sigma)} \ln(n)^{1-2(\overline{\sigma}\vee\sigma)} \ln_2(n)^{2(\overline{\sigma}\vee\sigma)} \le 1$, so (A.21) and (A.23) hold for n large enough. □

The proof of theorems (T1.1b) and (T1.2b) are similar and postponed to Section B.2.

A.3. **Upper bounds.** We use, for all $\epsilon > 0$, $N \in \mathbb{N}_0^{\mathbb{R}}$, $N_0 \in \mathbb{N}_0$, $T > 0$, $\widetilde{F}_1^{q,N,T,0}$ and $\widetilde{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}$ which are defined like $\widehat{F}_1^{q,N,T,0}$ and $\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}$ replacing $\widehat{c}_{\boldsymbol{m}}$ by $\widetilde{c}_{\boldsymbol{m}}$ and $\widehat{F}_1^{q,N,T,0}$ by $\widetilde{F}_1^{q,N,T,0}$, $Z_{n_0} := \sup_{f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}\left\|\Delta_f f_{\boldsymbol{X}|\mathcal{X}}\right\|^2_{L^\infty(\mathcal{X})}$, $K_1 := \left\|\mathbb{1}\left\{\epsilon \le |\star|\right\}\left(\widehat{F}_1^{q,N,T,0} - \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(\star,\cdot_2)\right\|^2_{L^2(1\otimes W^{\otimes p})}$, $K_2 :=$

$$\left\| \mathbb{1}\{|\star| < \epsilon\} \left(\mathcal{I}_{\underline{a},\epsilon} \left[\widehat{F}_1^{q,N,T,0} \right] - \mathcal{F}_{1\mathrm{st}} \left[f_{\alpha,\boldsymbol{\beta}} \right] \right) (\star, \cdot_2) \right\|^2_{L^2(1 \otimes W^{\otimes p})}, \ \Delta_f := 1/\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}} - 1/f_{\boldsymbol{X}|\mathcal{X}},$$

$$\widetilde{c}_{\boldsymbol{m}} := \sum_{j=1}^{n} \frac{e^{i\star Y_j} \overline{g_{\boldsymbol{m}}^{W,x_0\star}} (\boldsymbol{X}_j/x_0) \, \mathbb{1}\{\boldsymbol{X}_j \in \mathcal{X}\}}{n x_0^p f_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{X}_j)}, \quad \widetilde{\omega}^{q,W,c}_{N(\star)} := \sup_{|\boldsymbol{m}|_q \le N(\star)} \frac{1}{\rho^{W,c}_{\boldsymbol{m}}}, L := (2\pi)^p \left\| \mathcal{F}_{1\mathrm{st}} \left[f_{\alpha,\boldsymbol{\beta}} \right] (\star, \cdot_2) \right\|^2_{L^2(\mathbb{R}^p)},$$

$$R_1(\star, \cdot_2) := \mathbb{1}\{\epsilon \le |\star|\} \left(\widetilde{F}_1^{q,N,T,0} - F_1^{q,N,T,0} \right) (\star, \cdot_2), \ R_2(\star, \cdot_2) := \mathbb{1}\{\epsilon \le |\star|\} \left(\widehat{F}_1^{q,N,T,0} - \widetilde{F}_1^{q,N,T,0} \right) (\star, \cdot_2),$$

$$R_3(\star, \cdot_2) := \mathbb{1}\{\epsilon \le |\star|\} \left(F_1^{q,N,T,0} - F_1^{q,\infty,T,0} \right) (\star, \cdot_2), \ R_4(\star, \cdot_2) := \mathbb{1}\{\epsilon \le |\star|\} \left(F_1^{q,\infty,T,0} - \mathcal{F}_{1\mathrm{st}} \left[f_{\alpha,\boldsymbol{\beta}} \right] \right) (\star, \cdot_2)$$

$$S_0^N(\star, \cdot_2) := \sum_{|\boldsymbol{m}|_q \le N(\star)} g_{\boldsymbol{m}}^{W,x_0\star}(\cdot_2) \Delta_{\boldsymbol{m}}(\star), \ Z_j^{\boldsymbol{m},\star} := \frac{e^{i\star Y_j}}{x_0^p} \Delta_f(\boldsymbol{X}_j) \overline{g_{\boldsymbol{m}}^{W,x_0\star} \left(\frac{\boldsymbol{X}_j}{x_0} \right)} \mathbb{1}\{\boldsymbol{X}_j \in \mathcal{X}\},$$

$$\Delta_{\boldsymbol{m}} := \frac{1}{n} \sum_{j=1}^{n} Z_j^{\boldsymbol{m},\star}, \ \mathcal{R}^W_{n_0,\sup} := \sup_{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}^{q,\phi,\omega}_{w,W}(l,M) \cap \mathcal{D}, \ f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}} \mathcal{R}^W_{n_0} \left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}} \right),$$

$$S_1^N(\star, \cdot_2) := \sum_{|\boldsymbol{m}|_q \le N(\star)} g_{\boldsymbol{m}}^{W,x_0\star}(\cdot_2) \mathbb{E}\left[\Delta_{\boldsymbol{m}}(\star) \right], \ S_2^N(\star, \cdot_2) := \sum_{|\boldsymbol{m}|_q \le N(\star)} g_{\boldsymbol{m}}^{W,x_0\star}(\cdot_2) \left(\Delta_{\boldsymbol{m}}(\star) - \mathbb{E}\left[\Delta_{\boldsymbol{m}}(\star) \right] \right),$$

$$\Delta_0^W(\star, N_0, n, z) := \frac{2}{\pi(2\pi)^p} \left(\frac{c_{\boldsymbol{X}} |\star|^p}{n} \nu_q^W(x_0\star, N_0) + z \left(L(\star) + \frac{c_{\boldsymbol{X}} (N_0+1)^p |\star|^p}{n} \right) \widetilde{\omega}^{q,W,x_0\star}_{N_0} \right).$$

Lemma A.2. For all $\boldsymbol{m} \in \mathbb{N}_0^p$, we have $\mathbb{E}\left[\widetilde{c}_{\boldsymbol{m}}(t) \right] = c_{\boldsymbol{m}}(t)$ and $\mathbb{E}\left[|\widetilde{c}_{\boldsymbol{m}}(t) - c_{\boldsymbol{m}}(t)|^2 \right] \le c_{\boldsymbol{X}}/(n x_0^p)$.

Proof. This comes from

$$\mathbb{E}\left[\widetilde{c}_{\boldsymbol{m}}(t) \right] = \frac{1}{x_0^p} \mathbb{E}\left[\frac{e^{itY}}{f_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{X})} \overline{g_{\boldsymbol{m}}^{W,x_0 t} \left(\frac{\boldsymbol{X}}{x_0} \right)} \mathbb{1}\{\boldsymbol{X} \in \mathcal{X}\} \right] = \frac{1}{x_0^p} \int_{\mathcal{X}} \mathbb{E}\left[e^{it\alpha + it\boldsymbol{\beta}^\top \boldsymbol{x}} \right] \overline{g_{\boldsymbol{m}}^{W,x_0 t} \left(\frac{\boldsymbol{x}}{x_0} \right)} \, d\boldsymbol{x},$$

$$\begin{aligned} \mathbb{E}\left[|\widetilde{c}_{\boldsymbol{m}}(t) - c_{\boldsymbol{m}}(t)|^2 \right] &\le \frac{1}{n x_0^{2p}} \mathbb{E}\left[\left| \frac{e^{itY}}{f_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{X})} \overline{g_{\boldsymbol{m}}^{W,x_0 t} \left(\frac{\boldsymbol{X}}{x_0} \right)} \right|^2 \middle| \boldsymbol{X} \in \mathcal{X} \right] \\ &\le \frac{1}{n x_0^{2p}} \int_{\mathcal{X}} \frac{1}{f_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{x})} \left| \overline{g_{\boldsymbol{m}}^{W,x_0 t} \left(\frac{\boldsymbol{x}}{x_0} \right)} \right|^2 d\boldsymbol{x} \le \frac{c_{\boldsymbol{X}}}{n x_0^p} \int_{[-1,1]^p} \left| \overline{g_{\boldsymbol{m}}^{W,x_0 t}}(\boldsymbol{u}) \right|^2 d\boldsymbol{u}. \quad \square \end{aligned}$$

Lemma A.3. If $\widehat{f}_{\boldsymbol{X}|\mathcal{X}}$ satisfies (H1.4) then $Z_{n_0} = O_p\left(v(n_0, \mathcal{E})/\delta(n_0) \right)$.

Proof. For all n_0 sufficiently large so that $\sqrt{\delta(n_0)} c_{\boldsymbol{X}} \le 1$, we have, for all $x \in \mathcal{X}$,

$$\begin{aligned} \left| \left(\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}} - f_{\boldsymbol{X}|\mathcal{X}} \right)(x) \right| &\le \left| \left(\widehat{f}_{\boldsymbol{X}|\mathcal{X}} - f_{\boldsymbol{X}|\mathcal{X}} \right)(x) \right| \mathbb{1}\left\{ \widehat{f}_{\boldsymbol{X}|\mathcal{X}}(x) \ge \sqrt{\delta(n_0)} \right\} \\ &\quad + \left| \sqrt{\delta(n_0)} - f_{\boldsymbol{X}|\mathcal{X}}(x) \right| \mathbb{1}\left\{ \widehat{f}_{\boldsymbol{X}|\mathcal{X}}(x) - f_{\boldsymbol{X}|\mathcal{X}}(x) < \sqrt{\delta(n_0)} - f_{\boldsymbol{X}|\mathcal{X}}(x) \right\} \\ &\le \left| \left(\widehat{f}_{\boldsymbol{X}|\mathcal{X}} - f_{\boldsymbol{X}|\mathcal{X}} \right)(x) \right|, \end{aligned}$$

hence $\delta(n_0) Z_{n_0} \le \sup_{f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}} \left\| \widehat{f}_{\boldsymbol{X}|\mathcal{X}} - f_{\boldsymbol{X}|\mathcal{X}} \right\|^2_{L^\infty(\mathcal{X})}$. We conclude by (H1.4). $\square$

In the remaining, $\mathcal{E}$ is a class of densities, $f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}$, $\eta, l, M > 0$, and $f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}^{q,\phi,\omega}_{w,W}(l,M) \cap \mathcal{D}$. By Lemma A.3, there exists $M_{\mathcal{E},\eta}$ such that, for all $n_0 \in \mathbb{N}$, $\mathbb{P}\left(E\left(\mathcal{G}_{n_0}, \mathcal{E}, \eta\right)\right) \geq 1-\eta$, where $E\left(\mathcal{G}_{n_0}, \mathcal{E}, \eta\right) := \{Z_{n_0} \leq M_{\mathcal{E},\eta} v(n_0, \mathcal{E})/\delta(n_0)\}$. We work on $E\left(\mathcal{G}_{n_0}, \mathcal{E}, \eta\right)$.

Proof of theorems 2 and 3. The Plancherel and Chasles identities yield $\left\| \widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}} - f_{\alpha,\boldsymbol{\beta}} \right\|^2_{L^2(1 \otimes W^{\otimes p})} \leq (K_1 + K_2)/(2\pi)$. By the Jensen inequality, we have $K_1 \leq 4 \sum_{j=1}^4 \|R_j\|^2_{L^2(1\otimes W^{\otimes p})}$ and, using (10) for the first display and Lemma B.1 for the second,

$$
\begin{aligned}
K_1 + K_2 \leq & K_1 + \int_{\mathbb{R}^p} 2(1 + C_0(\underline{a}\epsilon)) \left\| \mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](\star, \boldsymbol{b}) - \mathcal{P}_{\underline{a}}\left[\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](\cdot, \boldsymbol{b})\right](\star) \right\|^2_{L^2(\mathbb{R})} W^{\otimes p}(\boldsymbol{b}) d\boldsymbol{b} \\
& + \int_{\mathbb{R}^p} 2C_0(\underline{a}\epsilon) \left\| \mathbb{1}\{|\star| \geq \epsilon\} \left(\widehat{F}^{q,N,T,0}_1 - \mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right] \right)(\star, \boldsymbol{b}) \right\|^2_{L^2(\mathbb{R})} W^{\otimes p}(\boldsymbol{b}) d\boldsymbol{b} \\
\leq & K_1 + 4\pi(1 + C_0(\underline{a}\epsilon)) \widetilde{w}(\underline{a}) \int_{\mathbb{R}^p} \|f_{\alpha,\boldsymbol{\beta}}(\cdot, \boldsymbol{b})\|^2_{L^2(w)} W^{\otimes p}(\boldsymbol{b}) d\boldsymbol{b} + 2C_0(\underline{a}\epsilon) K_1 \\
\leq & (1 + 2C_0(\underline{a}\epsilon)) K_1 + 4\pi(1 + C_0(\underline{a}\epsilon)) M^2 \widetilde{w}(\underline{a}) \leq C\left(K_1 + 2\pi M^2 \widetilde{w}(\underline{a})\right). \qquad \text{(A.24)}
\end{aligned}
$$

Using successively Proposition 2 and lemmas A.2 and B.2, we have

$$
\begin{aligned}
\mathbb{E}\left[\|R_1\|^2_{L^2(1\otimes W^{\otimes p})} \right] &= \int_{\epsilon \leq |t| \leq T} \sum_{|\boldsymbol{m}|_q \leq N(t)} \frac{\mathbb{E}\left[|\widetilde{c}_{\boldsymbol{m}}(t) - c_{\boldsymbol{m}}(t)|^2 \right]}{\left(\sigma^{W,x_0 t}_{\boldsymbol{m}}\right)^2} dt \qquad \text{(A.25)} \\
&\leq \frac{c_{\boldsymbol{X}}}{(2\pi)^p n} \int_{\epsilon \leq |t| \leq T} |t|^p \nu^W_q(x_0 t, N(t)) dt, \qquad \text{(A.26)}
\end{aligned}
$$

also

$$
\begin{aligned}
\|R_2\|^2_{L^2(1\otimes W^{\otimes p})} &\leq \int_{\epsilon \leq |t| \leq T} \left(\frac{x_0|t|}{2\pi} \right)^p \widetilde{\omega}^{q,W,x_0 t}_{N(t)} \left\| S^N_0(t, \cdot_2) \right\|^2_{L^2([-1,1]^p)} dt, \\
\mathbb{E}\left[\left\| S^N_0(t,\cdot_2) \right\|^2_{L^2([-1,1]^p)} \right] &= \left\| S^N_1(t,\cdot_2) \right\|^2_{L^2([-1,1]^p)} + \mathbb{E}\left[\left\| S^N_2(t,\cdot_2) \right\|^2_{L^2([-1,1]^p)} \right], \\
\left\| S^N_1(t,\cdot_2) \right\|^2_{L^2([-1,1]^p)} &= \left\| \sum_{|\boldsymbol{m}|_q \leq N(t)} g^{W,x_0 t}_{\boldsymbol{m}} \left\langle \mathcal{F}\left[f_{Y|\boldsymbol{X}=x_0\cdot_2}\right](t) \left(\Delta_f f_{\boldsymbol{X}|\mathcal{X}}\right)(x_0\cdot_2), g^{W,x_0 t}_{\boldsymbol{m}} \right\rangle_{L^2([-1,1]^p)} \right\|^2_{L^2([-1,1]^p)} \\
&\leq \left\| \mathcal{F}\left[f_{Y|\boldsymbol{X}=x_0\cdot_2}\right](t) \left(\Delta_f f_{\boldsymbol{X}|\mathcal{X}}\right)(x_0\cdot_2) \right\|^2_{L^2([-1,1]^p)} \\
&\leq Z_{n_0} \left\| \mathcal{F}\left[f_{\alpha,\boldsymbol{\beta}}\right](t, x_0 t\cdot_2) \right\|^2_{L^2([-1,1]^p)} \leq Z_{n_0} \left(\frac{2\pi}{x_0|t|} \right)^p \left\| \mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot_2) \right\|^2_{L^2(\mathbb{R}^p)}, \qquad \text{(A.27)}
\end{aligned}
$$

and, by independence and $\sum_{|\boldsymbol{m}|_q \leq N} 1 = \binom{N+p}{p} \mathbb{1}\{q=1\} + (N+1)^p \mathbb{1}\{q=\infty\} \leq (N+1)^p$,

$$
\mathbb{E}\left[\left\| S^N_2(t,\cdot_2) \right\|^2_{L^2([-1,1]^p)} \right] = \sum_{|\boldsymbol{m}|_q \leq N(t)} \frac{1}{n} \mathbb{E}\left[\left| Z^{\boldsymbol{m},t}_j - \mathbb{E}\left[Z^{\boldsymbol{m},t}_j \right] \right|^2 \right]
$$

$$\leq \sum_{|\boldsymbol{m}|_q \leq N(t)} \frac{Z_{n_0}}{n x_0^{2p}} \int_{\mathcal{X}} \frac{1}{f_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{x})} \left| \overline{g_{\boldsymbol{m}}^{W,x_0 t}} \left(\frac{\boldsymbol{x}}{x_0} \right) \right|^2 d\boldsymbol{x} \leq \frac{(N(t)+1)^p c_{\boldsymbol{X}} Z_{n_0}}{n x_0^p}. \tag{A.28}$$

Collecting (A.27) and (A.28), we obtain

$$\mathbb{E}\left[\|R_2\|^2_{L^2(1 \otimes W^{\otimes p})} \right] \leq \frac{Z_{n_0}}{(2\pi)^p} \int_{\epsilon \leq |t| \leq T} \left(L(t) + \frac{c_{\boldsymbol{X}} (N(t)+1)^p |t|^p}{n} \right) \widetilde{\omega}_{N(t)}^{q,W,x_0 t} dt. \tag{A.29}$$

By Lemma B.3 and Proposition 2, we have

$$\|R_3\|^2_{L^2(1 \otimes W^{\otimes p})} \leq \int_{\mathbb{R}} \sum_{k > N(t)} \sum_{|\boldsymbol{m}|_q = k} |b_{\boldsymbol{m}}(t)|^2 \, dt \leq \int_{\mathbb{R}} \sum_{k > N(t)} \frac{\omega_k^2 \theta_{q,k}^2(t)}{\omega_{N(t)+1}^2} dt \leq \sup_{t \in \mathbb{R}} \frac{2\pi l^2}{\omega_{N(t)+1}^2} \tag{A.30}$$

and, by Proposition 2,

$$\|R_4\|^2_{L^2(1 \otimes W^{\otimes p})} \leq \sum_{k \in \mathbb{N}_0} \int_{|t| \geq T} \sum_{|\boldsymbol{m}|_q = k} |b_{\boldsymbol{m}}(t)|^2 \, dt \leq \sum_{k \in \mathbb{N}_0} \int_{\mathbb{R}} \frac{\phi^2(|t|)}{\phi^2(T)} \theta_{q,k}^2(t) dt \leq \frac{2\pi l^2}{\phi^2(T)}. \tag{A.31}$$

Thus, we have

$$\mathcal{R}_{n_0,\sup}^W \leq C \left(\int_{\epsilon \leq |t| \leq T} \Delta_0^W(t, N, n, Z_{n_0}) dt + 4l^2 \left(\sup_{t \in \mathbb{R}} \frac{1}{\omega_{N(t)+1}^2} + \frac{1}{\phi(T)^2} \right) + M^2 \widetilde{w}(\underline{a}) \right). \tag{A.32}$$

The remaining of the proof is in Section B.2. It particularises (A.32) to the different smoothness.

A.4. **Data-driven choice of the parameters.** Let $\mathcal{N}_n$ be the set of functions $N \in \mathbb{N}_0^{\mathbb{R}}$ such that, for all $t \in \mathbb{R} \setminus (-\epsilon, \epsilon)$, $N(t) \in \{0, \ldots, N_{\max,q}^W\}$, $\Delta_{\boldsymbol{m}} := \widehat{c}_{\boldsymbol{m}} - \widetilde{c}_{\boldsymbol{m}}$, and $\widetilde{\Delta}_{\boldsymbol{m}} := \widetilde{c}_{\boldsymbol{m}} - c_{\boldsymbol{m}}$,

$$\Xi(t, N) := \sum_{|\boldsymbol{m}|_q > N} \left| \frac{c_{\boldsymbol{m}}(t)}{\sigma_{\boldsymbol{m}}^{W,x_0 t}} \right|^2, \; S_1(t, N) := \sum_{|\boldsymbol{m}|_q \leq N} \left| \frac{\mathbb{E}\left[\Delta_{\boldsymbol{m}}(t)\right]}{\sigma_{\boldsymbol{m}}^{W,x_0 t}} \right|^2,$$

$$S_2(t, N) := \sum_{|\boldsymbol{m}|_q \leq N} \left| \frac{\Delta_{\boldsymbol{m}}(t) - \mathbb{E}\left[\Delta_{\boldsymbol{m}}(t)\right]}{\sigma_{\boldsymbol{m}}^{W,x_0 t}} \right|^2, \; S_3(t, N) := \sum_{|\boldsymbol{m}|_q \leq N} \left| \frac{\widetilde{\Delta}_{\boldsymbol{m}}(t)}{\sigma_{\boldsymbol{m}}^{W,x_0 t}} \right|^2,$$

$$K_n(t) := H_W(t) \left(N_{\max,q}^W + \frac{1}{2} \right)^p, \; L := \frac{1}{42} \sqrt{\frac{2x_0^p}{c_{\boldsymbol{X}}}},$$

$$\Psi_{0,n}(t) := \exp\left(-\frac{p_n}{6} \right) + \frac{294 c_{\boldsymbol{X}} K_n^2(t)}{x_0^p n} \exp\left(-\frac{L\sqrt{p_n n}}{K_n(t)} \right),$$

$$\widetilde{B}\left(\widehat{N}\right) := \mathbb{E}\left[\sup_{T' \in \mathcal{T}_n} \int_{\epsilon \leq |t| \leq T'} \left(S_3\left(t, \widehat{N}(t)\right) - \frac{\Sigma\left(t, \widehat{N}(t)\right)}{2(2+\sqrt{5})} \right)_+ dt \right],$$

$$\Pi(n, Z_{n_0}, T_{\max}, N_{\max,q}^W) := Z_{n_0} \int_{\epsilon \leq |t| \leq T_{\max}} \Psi_n(t) dt + \Pi_1\left(n, T_{\max}, N_{\max,q}^W\right),$$

$$\Pi_1(n, T_{\max}, N^W_{\max,q}) := \frac{96\left(1+2\sqrt{5}\right) c_{\boldsymbol{X}} K_{\max}}{(2\pi)^p n} \int_\epsilon^{T_{\max}} \left(N^W_{\max,q}+1\right) t^p \nu_q^W\left(x_0 t, N^W_{\max,q}\right) \Psi_{0,n}(t) dt,$$

$$\Psi_n := \left(2+\frac{1}{\sqrt{5}}\right)\left(\left(\frac{2\pi}{x_0\,|\star|}\right)^p \widetilde{\omega}^{q,W,x_0 t}_{N^W_{\max,q}} \left\|\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](\star,\cdot_2)\right\|^2_{L^2(\mathbb{R}^p)} + \left(\frac{|\star|}{2\pi}\right)^p \frac{c_{\boldsymbol{X}} \nu_q^W\left(x_0\star, N^W_{\max,q}\right)}{n}\right),$$

$$\widetilde{\Delta}_0^W(\star, N, n, z) := \frac{c_{\boldsymbol{X}}|\star|^p}{n}\left(1+2(1+2p_n)(1+c_1)\right)\nu_q^W(x_0\star, N) + z\left(L(\star) + \frac{c_{\boldsymbol{X}}(N+1)^p\,|\star|^p}{n}\right)\widetilde{\omega}_N^{q,W,x_0\star},$$

where $H_W(t)$ is defined in Proposition B.2. For all $t \in [-T,T]\setminus[-\epsilon,\epsilon]$ and $N \in \mathbb{N}_0$, using (A.2) with $c=\sqrt{5}$, we have

$$\mathcal{L}_q^W(t,N) \le \Xi(t,N) + \left(1+\frac{2}{\sqrt{5}}\right)\left(S_1(t,N)+S_2(t,N)\right) + (1+2\sqrt{5})S_3(t,N). \tag{A.33}$$

Lemma A.4. For all $q \in \{1,\infty\}$, $0<\epsilon<1<T<T_{\max}=2^{K_{\max}}$, $t\in[-T,T]\setminus(-\epsilon,\epsilon)$, and $N \in \{0,\dots,N^W_{\max,q}\}$, we have

$$\mathbb{E}\left[S_1\left(t,\widehat{N}(t)\right)\right] \le Z_{n_0}\left(\frac{2\pi}{x_0|t|}\right)^p \widetilde{\omega}^{q,W,x_0 t}_{N^W_{\max,q}} \left\|\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot_2)\right\|^2_{L^2(\mathbb{R}^p)}, \tag{A.34}$$

$$\mathbb{E}\left[S_2\left(t,\widehat{N}(t)\right)\right] \le Z_{n_0}\frac{c_{\boldsymbol{X}}}{n}\left(\frac{|t|}{2\pi}\right)^p \nu_q^W(x_0 t, N^W_{\max,q}), \tag{A.35}$$

$$\mathbb{E}\left[\left(S_3(t,N) - \frac{\Sigma(t,N)}{2(2+\sqrt{5})}\right)_+\right] \le 48\frac{c_{\boldsymbol{X}}}{n}\left(\frac{|t|}{2\pi}\right)^p \nu_q^W(x_0 t, N)\Psi_{0,n}(t). \tag{A.36}$$

Proof. Let the parameters in the for all statement be given. (A.34) follows from

$$\begin{aligned} S_1\left(t,\widehat{N}(t)\right) &\le \widetilde{\omega}^{q,W,x_0 t}_{\widehat{N}(t)} \left\|\mathcal{F}\left[f_{Y|\boldsymbol{X}=x_0\cdot}\right](t)\left(\Delta_f f_{\boldsymbol{X}|\mathcal{X}}\right)(x_0\cdot)\right\|^2_{L^2([-1,1]^p)} \\ &\le Z_{n_0}\left(\frac{2\pi}{x_0|t|}\right)^p \widetilde{\omega}^{q,W,x_0 t}_{N^W_{\max,q}} \left\|\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot_2)\right\|^2_{L^2(\mathbb{R}^p)} \quad \text{(by (A.27))}. \end{aligned}$$

By Lemma B.2, $\sum_{|\boldsymbol{m}|_q \le N}\left(\sigma^{W,x_0 t}_{\boldsymbol{m}}\right)^{-2} \le |x_0 t|^p \nu_q^W(x_0 t, N)/(2\pi)^p$, so we obtain (A.35) by the following sequence of inequalities, which uses (A.28) for the second display,

$$\mathbb{E}\left[S_2\left(t,\widehat{N}(t)\right)\right] \le \sum_{|\boldsymbol{m}|_q \le N^W_{\max,q}} \frac{\mathbb{E}\left[\left|\Delta_{\boldsymbol{m}}(t) - \mathbb{E}\left[\Delta_{\boldsymbol{m}}(t)\right]\right|^2\right]}{\left(\sigma^{W,x_0 t}_{\boldsymbol{m}}\right)^2} \le \frac{c_{\boldsymbol{X}} Z_{n_0}|t|^p \nu_q^W\left(x_0 t, N^W_{\max,q}\right)}{(2\pi)^p n}.$$

To prove (A.36), we use

$$S_3(t,N) = \int_{\mathbb{R}^p}\left|\widetilde{F}_1^{q,N,T,0}(t,\boldsymbol{b}) - F_1^{q,N,T,0}(t,\boldsymbol{b})\right|^2 W^{\otimes p}(\boldsymbol{b})d\boldsymbol{b} = \sup_{u\in\mathcal{U}}\left|\nu_n^t(u)\right|^2,$$

$$\nu_n^t(u) := \left\langle \widetilde{F}_1^{q,N,T,0}(t,\cdot_2) - F_1^{q,N,T,0}(t,\cdot_2), u(\cdot)\right\rangle_{L^2(W^{\otimes p})} = \frac{1}{n}\sum_{j=1}^n\left(f_u^t(Y_j,X_j) - \mathbb{E}\left[f_u^t(Y_j,X_j)\right]\right),$$

$$f_u^t(\star,\cdot) := \mathbb{1}\{\cdot \in \mathcal{X}\} \frac{e^{it\star}}{x_0^p f_{\boldsymbol{X}|\mathcal{X}}(\cdot)} \int_{\mathbb{R}^p} \sum_{|\boldsymbol{m}|_q \le N} \overline{g_{\boldsymbol{m}}^{W,x_0 t}}\left(\frac{\cdot}{x_0}\right) \frac{1}{\sigma_{\boldsymbol{m}}^{W,x_0 t}} \varphi_{\boldsymbol{m}}^{W,x_0 t}(\boldsymbol{b}) \overline{u}(\boldsymbol{b}) W^{\otimes p}(\boldsymbol{b}) d\boldsymbol{b},$$

and $\mathcal{U}$ is a countable dense set of measurable functions of $\left\{u : \|u\|_{L^2(W^{\otimes p})} = 1\right\}$ and check the conditions of the Talagrand inequality given in Lemma B.15 with $\eta = p_n$ and $\Lambda(p_n) = 1$. For all $u \in \mathcal{U}$, the Cauchy-Schwarz inequality yield

$$\begin{aligned}\left\|f_u^t\right\|_{L^\infty(\mathbb{R}\times\mathcal{X})} &\le c_{\boldsymbol{X}} \left(\frac{|t|}{2\pi x_0}\right)^{p/2} \left\| \left(\sum_{|\boldsymbol{m}|_q \le N} \frac{\left|\overline{g_{\boldsymbol{m}}^{W,x_0 t}}(\cdot/x_0)\right|^2}{\rho_{\boldsymbol{m}}^{W,x_0 t}} \int_{\mathbb{R}^p} \left|\varphi_{\boldsymbol{m}}^{W,x_0 t}(\boldsymbol{b})\right|^2 W^{\otimes p}(\boldsymbol{b}) d\boldsymbol{b} \right)^{1/2} \right\|_{L^\infty(\mathcal{X})} \\ &\le c_{\boldsymbol{X}} K_n(t) \left(\frac{|t|}{2\pi x_0}\right)^{p/2} \sqrt{\nu_q^W(x_0 t, N)}.\end{aligned}$$

By the Cauchy-Schwarz inequality and the computation leading to (A.26), we have

$$\begin{aligned}\mathbb{E}\left[\sup_{u\in\mathcal{U}} \left|\nu_n^t(u)\right|\right]^2 \le \mathbb{E}\left[\sup_{u\in\mathcal{U}} \left|\nu_n^t(u)\right|^2\right] &\le \mathbb{E}\left[\left\|\widetilde{F}_1^{q,N,T,0}(t,\cdot_2) - F_1^{q,N,T,0}(t,\cdot_2)\right\|_{L^2(W^{\otimes p})}^2\right] \\ &\le \frac{c_{\boldsymbol{X}}}{n}\left(\frac{|t|}{2\pi}\right)^p \nu_q^W(x_0 t, N) = \frac{\Sigma(t,N)}{8(2+\sqrt{5})(1+2p_n)}.\end{aligned}$$

Finally, by the Cauchy-Schwarz inequality and Proposition B.2 for the second display and Lemma B.2 for the third display, we have

$$\begin{aligned}\operatorname{Var}\left(\Re(f_u^t(Y_j,X_j))\right) \vee \operatorname{Var}\left(\Im(f_u^t(Y_j,X_j))\right) &\le \int_{\mathbb{R}\times\mathcal{X}} \left|f_u^t(y,\boldsymbol{x})\right|^2 f_{Y,\boldsymbol{X}}(y,\boldsymbol{x}) dy d\boldsymbol{x} \\ &\le c_{\boldsymbol{X}} \left(\frac{|t|}{2\pi}\right)^p \nu_q^W(x_0 t, N). \quad \square\end{aligned}$$

Lemma A.5. For all $\epsilon > 0$, $q \in \{1,\infty\}$, and $T \in \mathcal{T}_n$, we have

$$\begin{aligned}\mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,\widehat{N},\widehat{T},\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right) \le& \frac{C\left(2+\sqrt{5}\right)^2}{2\pi} \int_{\epsilon\le|t|\le T} \mathbb{E}\left[\mathcal{L}_q^W\left(t,\widehat{N}(t),T\right)\right] + \frac{1}{2+\sqrt{5}} \mathbb{E}\left[\Sigma\left(t,\widehat{N}(t)\right)\right] dt \\ &+ \frac{C2(2+\sqrt{5})^2}{\pi} \Pi(n, Z_{n_0}, T_{\max}, N_{\max,q}^W) + CM^2 \widetilde{w}(\underline{a}).\end{aligned}$$

Proof. Let $\epsilon > 0$, $q \in \{1,\infty\}$, and $T \in \mathcal{T}_n$. Start from (19). Using, for all $T_1, T_2 \ge \epsilon$, $R_{T_1}^{T_2} := \mathbb{1}\{\epsilon \le |\star|\}\left(\widehat{F}_1^{q,\widehat{N},T_1,0} - \widehat{F}_1^{q,\widehat{N},T_2\vee T_1,0}\right)(\star,\cdot_2)$ and $R^{T_1} := \mathbb{1}\{\epsilon \le |\star|\}\left(\widehat{F}_1^{q,\widehat{N},T_1,0} - \mathcal{F}_{1\text{st}}[f_{\alpha,\boldsymbol{\beta}}]\right)(\star,\cdot_2)$, we have $R^{\widehat{T}} = R_{\widehat{T}}^T - R_T^{\widehat{T}} + R^T$ and $\left\|R^{\widehat{T}}\right\|_{L^2(W^{\otimes p})}^2 = \mathbb{1}\{\epsilon \le |\star|\}\mathcal{L}_q^W\left(\star,\widehat{N}(\star),\widehat{T}\right)$. Because

$$B_2\left(T_1,\widehat{N}\right) = \max_{T'\in\mathcal{T}_n}\left(\int_{T_1\le|t|\le T_1\vee T'} \left\|R_{T_1}^{T'}(t,\cdot_2)\right\|_{L^2(W^{\otimes p})}^2 - \Sigma\left(t,\widehat{N}(t)\right) dt\right)_+, \tag{A.37}$$

we have $\mathbb{E}\left[\left\|R_{T_1}^{T_2}\right\|^2_{L^2(1\otimes W^{\otimes p})}\right] \leq \mathbb{E}\left[B_2\left(T_1,\widehat{N}\right)\right] + \mathbb{E}\left[\Sigma_2\left(T_2,\widehat{N}\right)\right]$ for possibly random T_1 and T_2. By (A.2) with $c=\sqrt{5}$ and (17), we have

$$\mathbb{E}\left[\left\|R^{\widehat{T}}\right\|^2_{L^2(1\otimes W^{\otimes p})}\right] \leq 2(2+\sqrt{5})\left(\mathbb{E}\left[B_2\left(T,\widehat{N}\right)\right] + \mathbb{E}\left[\Sigma_2\left(T,\widehat{N}\right)\right]\right) + \left(1+\frac{2}{\sqrt{5}}\right)\mathbb{E}\left[\left\|R^T\right\|^2_{L^2(1\otimes W^{\otimes p})}\right].$$

Using, for all $T'\in\mathcal{T}_n$, $R_{T,1}^{T'} := \widehat{F}_1^{q,\widehat{N},T\vee T',0} - F_1^{q,\widehat{N},T\vee T',0}$, $R_{T,2}^{T'} := F_1^{q,\widehat{N},T,0} - \widehat{F}_1^{q,\widehat{N},T,0}$, and $R_{T,3}^{T'} := F_1^{q,\widehat{N},T\vee T',0} - F_1^{q,\widehat{N},T,0}$, by (A.2), the objective function in (A.37) is smaller than

$$\int_{T\leq|t|\leq T\vee T'}\left((2+\sqrt{5})\sum_{j=1}^{2}\left\|R_{T,j}^{T'}(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})} + \left(1+\frac{2}{\sqrt{5}}\right)\left\|R_{T,3}^{T'}(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})} - \Sigma\left(t,\widehat{N}(t)\right)\right)_+ dt.$$

Using that $F_1^{q,\infty,\infty,0} = \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\beta}\right]$, we have, for all $t\in\mathbb{R}\setminus(-\epsilon,\epsilon)$,

$$\left\|R_{T,3}^{T'}(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})} = \mathbb{1}\{T\leq|t|\leq T\vee T'\}\sum_{0\leq|\boldsymbol{m}|_q\leq\widehat{N}}\left|\frac{c_{\boldsymbol{m}}(t)}{\sigma_{\boldsymbol{m}}^{W,x_0t}}\right|^2 \leq \left\|\left(F_1^{q,\widehat{N},T,0} - \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\beta}\right]\right)(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})},$$

hence

$$\begin{aligned}B_2\left(T,\widehat{N}\right) \leq \max_{T'\in\mathcal{T}_n}&\int_{T\leq|t|\leq T'}\left(2(2+\sqrt{5})\left\|\left(\widehat{F}_1^{q,\widehat{N},T',0} - F_1^{q,\widehat{N},T',0}\right)(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})} - \Sigma\left(t,\widehat{N}(t)\right)\right)_+ dt\\ &+\left(1+\frac{2}{\sqrt{5}}\right)\int_{\epsilon\leq|t|}\left\|R^T(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})}dt.\end{aligned}$$

Finally, we have

$$\begin{aligned}&\mathbb{E}\left[\left\|R^{\widehat{T}}\right\|^2_{L^2(1\otimes W^{\otimes p})}\right] \leq 2(2+\sqrt{5})\mathbb{E}\left[\Sigma_2\left(T,\widehat{N}\right)\right] + (5+2\sqrt{5})\left(1+\frac{2}{\sqrt{5}}\right)\mathbb{E}\left[\left\|R^T\right\|^2_{L^2(1\otimes W^{\otimes p})}\right]\\ &+4(2+\sqrt{5})^2\mathbb{E}\left[\max_{T'\in\mathcal{T}_n}\int_{T\leq|t|\leq T'}\left(\left\|\left(\widehat{F}_1^{q,\widehat{N},T',0} - F_1^{q,\widehat{N},T',0}\right)(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})} - \frac{\Sigma\left(t,\widehat{N}(t)\right)}{2(2+\sqrt{5})}\right)_+\right].\end{aligned}$$

Using (A.33) and Lemma A.4, we have

$$\begin{aligned}&\mathbb{E}\left[\max_{T'\in\mathcal{T}_n}\int_{T\leq|t|\leq T'}\left(\left\|\left(\widehat{F}_1^{q,\widehat{N},T',0} - F_1^{q,\widehat{N},T',0}\right)(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})} - \frac{\Sigma\left(t,\widehat{N}(t)\right)}{2(2+\sqrt{5})}\right)_+ dt\right]\\ &\leq \mathbb{E}\left[\max_{T'\in\mathcal{T}_n}\int_{\epsilon\leq|t|\leq T'}\left(\sum_{|\boldsymbol{m}|_q\leq\widehat{N}(t)}\left(\frac{|\widehat{c}_{\boldsymbol{m}}(t)-c_{\boldsymbol{m}}(t)|}{\sigma_{\boldsymbol{m}}^{W,x_0t}}\right)^2 - \frac{\Sigma\left(t,\widehat{N}(t)\right)}{2(2+\sqrt{5})}\right)_+ dt\right]\\ &\leq \left(1+2\sqrt{5}\right)\widetilde{B}\left(\widehat{N}\right) + Z_{n_0}\int_{\epsilon\leq|t|\leq T_{\max}}\Psi_n(t)dt.\end{aligned}$$

Considering the first term of the last inequality and using (A.36) for the second display yields

$$\begin{aligned}\widetilde{B}\left(\widehat{N}\right) &\leq \mathbb{E}\left[\sum_{T'\in\mathcal{T}_n}\int_{\epsilon\leq|t|\leq T'}\left(S_3\left(t,\widehat{N}(t)\right)-\frac{\Sigma\left(t,\widehat{N}(t)\right)}{2(2+\sqrt{5})}\right)_+ dt\right]\\ &\leq \sum_{T'\in\mathcal{T}_n}\int_{\epsilon}^{T'}\sum_{0\leq N\leq N^W_{\max,q}} 96\frac{c_{\boldsymbol{X}}}{n}\left(\frac{t}{2\pi}\right)^p \nu_q^W(x_0 t,N)\Psi_{0,n}(t)dt\\ &\leq \frac{96 c_{\boldsymbol{X}} K_{\max}}{(2\pi)^p n}\int_{\epsilon}^{T_{\max}}\left(N^W_{\max,q}+1\right)t^p\nu_q^W\left(x_0 t,N^W_{\max,q}\right)\Psi_{0,n}(t)dt. \qquad \square\end{aligned}$$

Lemma A.6. For all $\epsilon>0$, $q\in\{1,\infty\}$, and $(T,N)\in\mathcal{T}_n\times\mathcal{N}_n$,

$$\begin{aligned}&\int_{\epsilon\leq|t|\leq T}\mathbb{E}\left[\mathcal{L}_q^W\left(t,\widehat{N}(t),T\right)\right]+\frac{1}{2+\sqrt{5}}\mathbb{E}\left[\Sigma\left(T,\widehat{N}(t)\right)\right]dt\\ &\leq\left(2+\sqrt{5}\right)^2\left(\int_{\epsilon\leq|t|\leq T}\mathbb{E}\left[\mathcal{L}_q^W\left(t,N(t),T\right)\right]+\frac{1+c_1}{2+\sqrt{5}}\mathbb{E}\left[\Sigma(t,N(t))\right]dt+4\Pi(n,Z_{n_0},T_{\max},N^W_{\max,q})\right).\end{aligned}$$

The proof of Lemma A.6 is similar to that of Lemma A.5 and postponed to Section B.2.

Proof of Theorem 4. Let n,n_0 such that $v(n_0,\mathcal{E})/\delta(n_0)\leq n^{-2}\ln(n)^{-p}$, $T\in\mathcal{T}_n$, and $N\in\mathcal{N}_n$. By Lemma A.5 and Lemma A.6, we have

$$\begin{aligned}\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,\widehat{N},\widehat{T},\epsilon}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right)\leq&\frac{C\left(2+\sqrt{5}\right)^4}{2\pi}\int_{\epsilon\leq|t|\leq T}\mathbb{E}\left[\mathcal{L}_q^W\left(t,N(t),T\right)\right]+\frac{1+c_1}{2+\sqrt{5}}\mathbb{E}\left[\Sigma(t,N(t))\right]dt\\ &+\frac{2(2+\sqrt{5})^2C\left(1+\left(2+\sqrt{5}\right)^2\right)}{\pi}\Pi(n,Z_{n_0},T_{\max},N_{\max,q})+CM^2\widetilde{w}(\underline{a}).\end{aligned}$$

The definition of Σ, (A.26), (A.29), (A.30), (A.31), and Lemma B.14 yield

$$\begin{aligned}\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,\widehat{N},\widehat{T},\epsilon}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right)\leq&\frac{C2\left(2+\sqrt{5}\right)^4}{\pi(2\pi)^p}\int_{\epsilon\leq|t|\leq T}\widetilde{\Delta}_0^W(t,N(t),n,Z_{n_0})dt+\sup_{t\in[-T,T]}\frac{8\pi l^2}{\omega^2_{N(t)+1}}+\frac{8\pi l^2}{\phi(T)^2}\\ &+CM^2\widetilde{w}(\underline{a})+\frac{2(2+\sqrt{5})^2C\left(1+\left(2+\sqrt{5}\right)^2\right)(A_0+A_1)}{\pi n}.\end{aligned}\tag{A.38}$$

The rest of the proof considers the different smoothness and is given in Section B.2.

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SUPPLEMENTAL APPENDIX

Appendix B.1. Harmonic analysis

B.1.1. **Preliminaries.** P_m is the Legendre polynomial of degree m with $\|P_m\|_{L^2([-1,1])} = 1$.

Lemma B.1. For all $f \in L^2_w(\mathbb{R})$, w even, nondecreasing on $[0,\infty)$, and $w(0), R > 0$, we have $\left\|\mathcal{P}_R\left[\mathcal{F}\left[f\right]\right] - \mathcal{F}\left[f\right]\right\|^2_{L^2(\mathbb{R})} \leq \left(2\pi/w(R)\right)\|f\|^2_{L^2(w)}$.

Proof. The result uses the Plancherel identity and

$$\left\|\mathcal{P}_R\left[\mathcal{F}\left[f\right]\right] - \mathcal{F}\left[f\right]\right\|^2_{L^2(\mathbb{R})} = 2\pi \int_{\mathbb{R}} \mathbb{1}\{|a| > R\}\left|f\left(a\right)\right|^2 da \leq \frac{2\pi}{w(R)}\int_{\mathbb{R}}\left|f\left(a\right)\right|^2 w(a)da. \quad \square$$

Proposition B.1. For all weighting function W, $c \in \mathbb{R}$, $R > 0$, and $m \in \mathbb{N}_0$, we have

(i) $g_m^{W(\cdot/R),c} = g_m^{W,Rc}$ in $L^2([-1,1])$,
(ii) $\sigma_m^{W(\cdot/R),c} = \sigma_m^{W,Rc}\sqrt{R}$,
(iii) $\varphi_m^{W(\cdot/R),c} = \varphi_m^{W,Rc}\left(\star/R\right)/\sqrt{R}$ *a.e.*

Proof. (i) follows from $\mathcal{Q}_c^{W(\cdot/R)} = \mathcal{Q}_{Rc}^{W}$ and (ii) from $\sigma_m^{W(\cdot/R),c} = \sqrt{2\pi\rho_m^{W(\cdot/R),c}/\left|c\right|} = \sqrt{2\pi\rho_m^{W,Rc}/\left|c\right|}$ (by the argument yielding (i)). Now, using (i) in the first display and (ii) in the last display, we have, for a.e. $t \in \mathbb{R}$,

$$\begin{aligned}
\sigma_m^{W,Rc}\varphi_m^{W,Rc}\left(\frac{t}{R}\right) &= \mathcal{F}_{Rc}^*\left[g_m^{W(\cdot/R),c}\right]\left(\frac{t}{R}\right) \quad \left(\text{where } \mathcal{F}_{Rc}^*\colon\ L^2([-1,1]) \to L^2(W)\right) \\
&= \mathcal{F}_c^*\left[g_m^{W(\cdot/R),c}\right](t) \quad \left(\text{where } \mathcal{F}_c^*\colon\ L^2([-1,1]) \to L^2(W(\cdot/R))\right) \\
&= \sigma_m^{W(\cdot/R),c}\varphi_m^{W(\cdot/R),c}(t) = \sigma_m^{W,Rc}\sqrt{R}\varphi_m^{W(\cdot/R),c}(t),
\end{aligned}$$

hence (iii) when we divide by $\sigma_m^{W,Rc}$ which is nonzero. $\square$

Proposition B.2. For all $\boldsymbol{m} \in \mathbb{N}_0^p$, $R > 0$, $W = W_{[-R,R]}$ or $W = \cosh(\cdot/R)$, $t \neq 0$, we have $\left\|g_{\boldsymbol{m}}^{W,x_0 t}\right\|_{L^\infty([-1,1]^p)} \leq H_W(t)\prod_{j=1}^p\sqrt{m_j + 1/2}$, where $H_{W_{[-R,R]}}(t) = H_0^p\left(1 + (x_0|t|)^2\right)^p$, $H_0 = 2(1 + 1/\sqrt{3})$, $H_{\cosh(\cdot/R)}(t) = H_1^p(1 \vee (x_0|t|)^4)^p$, $H_1 > 0$.

Proof. When $W = W_{[-R,R]}$, this is (66) in [8] else this is Corollary 1 in [23]. $\square$

Lemma B.2. For all $q \in \{1,\infty\}$, $t \neq 0$, $R > 0$, $N \in \mathbb{N}_0$, in cases (N.1) and (N.2) of Section 5.3, we have $\sum_{|\boldsymbol{m}|_q \leq N} 1/\rho_{\boldsymbol{m}}^{W,t} \leq \nu_q^W(t,N)$.

Proof. Let $R > 0$. We use repeatedly, for all $x > 0$ and $N \in \mathbb{N}_0$,

$$\sum_{k \leq N}\exp\left(kx\right) \leq \frac{\exp\left((N+1/2)x\right)}{2\sinh\left(x/2\right)} \leq \frac{\exp\left((N+1/2)x\right)}{x} \quad \left(\text{because } \sinh(|x|) \geq |x|\right), \tag{B.1}$$

$$\leq \frac{\exp\left(Nx\right)}{1 - \exp(-x)}, \tag{B.2}$$

the cardinal of $\{\boldsymbol{m} \in \mathbb{N}_0^p : \ |\boldsymbol{m}|_1 = k\}$ is $\binom{k+p-1}{p-1}$, and $(k+p-1)!/k! \leq (k+p-1)^{p-1}$, and for all $m \in \mathbb{N}_0$, $\rho_m^{\cosh,Rt} = \rho_m^{\cosh(\cdot/R),t}$ and $\rho_m^{W_{[-1,1]},Rt} = \rho_m^{W_{[-R,R]},t}$.
Start by case (N.2). Let $|t| > \pi/4$ and $q = 1$. By (8) in [23] (there $\mathcal{Q}_t$ differs by a factor $1/(2\pi)$), we have, for all $m \in \mathbb{N}_0$,

$$\rho_m^{\cosh,t} \geq \frac{1}{2} \exp\left(-\frac{\pi(m+1)}{2\,|t|}\right). \tag{B.3}$$

The result is obtained from the above with (B.1) and

$$\begin{aligned}\sum_{|\boldsymbol{m}|_1 \leq N} \frac{1}{\rho_{\boldsymbol{m}}^{\cosh,t}} \leq & 2^p \sum_{k \leq N} \sum_{|\boldsymbol{m}|_1 = k} \exp\left(\frac{\pi(|\boldsymbol{m}|_1 + p)}{2\,|t|}\right) \\ \leq & \frac{2^{p+1}(N+p-1)^{p-1} e\,|t|}{\pi(p-1)!} \exp\left(\frac{\pi(N+p)}{2\,|t|}\right).\end{aligned} \tag{B.4}$$

Let $|t| \leq \pi/4$ and $q = 1$. By Theorem 1 in [23], we have, for all $m \in \mathbb{N}_0$,

$$\rho_m^{\cosh,t} \geq \left(\frac{2}{e\pi}\right)^2 \exp\left(-2 \ln\left(\frac{7e^2}{4|t|}\right) m\right). \tag{B.5}$$

The result is obtained from the above with (B.2) and

$$\begin{aligned}\sum_{|\boldsymbol{m}|_1 \leq N} \frac{1}{\rho_{\boldsymbol{m}}^{\cosh,t}} \leq & \left(\frac{e\pi}{2}\right)^{2p} \sum_{k \leq N} \sum_{|\boldsymbol{m}|_1 = k} \exp\left(2 \ln\left(\frac{7e^2}{4|t|}\right) |\boldsymbol{m}|_1\right) \\ \leq & \left(\frac{e\pi}{2}\right)^{2p} \frac{(N+p-1)^{p-1}}{(p-1)!} \exp\left(2 \ln\left(\frac{7e^2}{4|t|}\right) N\right) \frac{1}{1 - (\pi/(14e^2))^2}.\end{aligned} \tag{B.6}$$

The results for $q = \infty$ are obtained using (B.4) and (B.6) with $p = 1$ and

$$\sum_{|\boldsymbol{m}|_\infty \leq N} \frac{1}{\rho_{\boldsymbol{m}}^{\cosh,t}} \leq \prod_{j=1}^{p} \left(\sum_{\boldsymbol{m}_j = 0}^{N} \frac{1}{\rho_{\boldsymbol{m}_j}^{\cosh,t}}\right). \tag{B.7}$$

Consider case (N.1). Let $t \neq 0$. Because $7e/\pi \geq 1$ and by Lemma B.5, we have, for all $m \in \mathbb{N}_0$,

$$\rho_m^{W_{[-1,1]},t} \geq \frac{1}{2} \left(\frac{2\,|t|}{7e(m+1)} \bigwedge 1\right)^{2m}. \tag{B.8}$$

When $q = 1$, the result follows from the following sequence of inequalities

$$\begin{aligned}\sum_{|\boldsymbol{m}|_1 \leq N} \frac{1}{\rho_{\boldsymbol{m}}^{W_{[-1,1]},t}} \leq & 2^p \sum_{k \leq N} \sum_{|\boldsymbol{m}|_1 = k} \prod_{j=1}^{p} \exp\left(2\boldsymbol{m}_j \ln\left(\frac{7e(\boldsymbol{m}_j+1)}{2\,|t|} \bigvee 1\right)\right) \\ \leq & \frac{2^p (N+p-1)^{p-1}(N+1)}{(p-1)!} \exp\left(2N \ln\left(\frac{7e(N+1)}{2\,|t|} \bigvee 1\right)\right).\end{aligned}$$

When $q = \infty$, we obtain the result using the above with $p = 1$ and (B.7). □

The proof of the next lemma is straightforward.

Lemma B.3. Let $f_{\alpha,\boldsymbol{\beta}} \in L^2(w \otimes W^{\otimes p})$. For all $\boldsymbol{m} \in \mathbb{N}_0^p$, $t \neq 0$, we have $c_{\boldsymbol{m}} = \sigma_{\boldsymbol{m}}^{W,x_0 t} b_{\boldsymbol{m}}$ *a.e.*

B.1.2. **Properties of the PSWF and eigenvalues.**

Lemma B.4. For all $c \neq 0$ and $m \in \mathbb{N}_0$, we have $|\mu_m^c| \leq \sqrt{2\pi} e^{3/2} \left(e|c| / (4(m+3/2))\right)^m / 3$.

Proof. Let $c \neq 0$ and $m \in \mathbb{N}_0$. By (69) in [48], 6.1.18 in [1], (7) in [28], (1.3) in [43], we obtain

$$\begin{aligned}
|\mu_m^c| &\leq \frac{\sqrt{\pi}\,|c|^m\,(m!)^2}{(2m)!\Gamma(m+3/2)} \\
&\leq \frac{\pi\,|c|^m}{4^m\Gamma(m+3/2)} \frac{\Gamma(m+1)}{\Gamma(m+1/2)} \\
&\leq \frac{\pi\,|c|^m}{4^m\Gamma(m+3/2)}(m+1)^{1/2} \leq \frac{\sqrt{\pi e^3}(e|c|)^m(m+1)^{1/2}}{4^m\sqrt{2}(m+3/2)^{m+1}}
\end{aligned}$$

and conclude using $\sup_{x\geq 0}(x+1)^{1/2}/(x+3/2) \leq 2/3$. □

Lemma B.5. For all $c \neq 0$ and $m \in \mathbb{N}_0$, we have

$$\rho_m^{W_{[-1,1]},c} \geq \frac{1}{2}\left(\mathbb{1}\left\{m \leq \frac{2|c|}{\pi} - 1\right\} + \frac{2c}{7e(m+1)}\mathbb{1}\left\{m > \frac{2|c|}{\pi} - 1\right\}\right)^{2m}.$$

Proof. When $m \geq 2|c|/\pi - 1$, the result follows from the fact that, by Proposition 5.1 in [6] and the Turán-Nazarov inequality (see [44] page 240), $\rho_m^{W_{[-1,1]},c} \geq \left(2c/\left(7e(m+1)\right)\right)^{2m}/2$. For all $m \leq 2|c|/\pi - 1$, the result follows from Remark 5.2 in [6] and that, for all $m \in \mathbb{N}_0$, $c \in (0,\infty) \mapsto \rho_m^c$ is nondecreasing (by the arguments in the proof of Lemma 1 in [23]). □

We now use $\Pi(c) := 3c^2 \exp\left(2c^2/\sqrt{3}\right)/16$, $H(c) := \sqrt{2\Pi(c)} \vee 2$, $l(c) := \left(1 + 4c^2/3^{3/2}\right)\left(1 + 2c^2 3^{3/2}\right)$, if $N \geq H(c)$ then $N \geq c$ (for all $c \geq 2$, $N \geq c\sqrt{3\exp(8/\sqrt{3})/16} > c$ else $N \geq H(c) \geq 2 > c$), $f(x) := |x|/(1-x^2)$, $g(x) := |x|/(1-x)^2$, $h(x) := |x|/(1-|x|)$, $c_f := 4/3$, $c_g := 4$, $c_h := 2$,

$$\forall x \in [-1/2, 1/2],\ f(x) \leq c_f|x|,\ g(x) \leq c_g|x|,\ h(x) \leq c_h|x|; \tag{B.9}$$

$$2 \sum_{k \equiv N[2],\ 0<k<N} 2k+1 = N(N-1). \tag{B.10}$$

(B.10) is obtained because for all N even the sum is $2\sum_{p=1}^{N/2-1} 4p+1$ and else $2\sum_{p=0}^{(N-1)/2-1} 4p+3$.

Lemma B.6. For all $c \neq 0$ and $m \geq 2$, we have $\left|\mu_m^c/\mu_{m-2}^c\right| \leq \Pi(c)/m^2$.

Proof. Let $c > 0$ and $m \in \mathbb{N}_0$ (for $c < 0$, we use $\mu_m^c = \overline{\mu_m^{-c}}$). By Theorem 8.1 in [46], we have

$$|\mu_m^c| = \frac{\sqrt{\pi}c^m(m!)^2}{(2m)!\Gamma(m+3/2)} e^{F_m(c)},\ F_m(c) = \int_0^c \left(\frac{2\left(\psi_m^t(1)\right)^2 - 1}{2t} - \frac{m}{t}\right) dt.$$

Moreover, by (65) in [8], for all $t > 0$,

$$\left(\sqrt{m+\frac{1}{2}} - \frac{t^2}{\sqrt{3}\sqrt{m+1/2}}\right)^2 \leq \left(\psi_m^t(1)\right)^2 \leq \left(\sqrt{m+\frac{1}{2}} + \frac{t^2}{\sqrt{3}\sqrt{m+1/2}}\right)^2$$

which yields, if $m \geq 2$,

$$\left(\psi_m^t(1)\right)^2 - \left(\psi_{m-2}^t(1)\right)^2 \leq \left(\sqrt{m+\frac{1}{2}} + \frac{t^2}{\sqrt{3}\sqrt{m+1/2}}\right)^2 - \left(\sqrt{(m-2)+\frac{1}{2}} - \frac{t^2}{\sqrt{3}\sqrt{(m-2)+1/2}}\right)^2$$

$$= 2 + \frac{4t^2}{\sqrt{3}} + \frac{t^4}{3}\left(\frac{1}{m+1/2} - \frac{1}{m-3/2}\right) \leq 2 + \frac{4t^2}{\sqrt{3}}. \tag{B.11}$$

Using $\sup_{x\geq 2} x^3(x-1)/\left((x^2-1/4)(x-1/2)(x-3/2)\right) \leq 3$ and (B.11), for all $m \geq 2$,

$$\left|\frac{\mu_m^c}{\mu_{m-2}^c}\right| = \frac{c^2}{16}\frac{m(m-1)}{(m^2-1/4)(m-1/2)(m-3/2)}\exp\left(F_m(c) - F_{m-2}(c)\right)$$

$$\leq \frac{3c^2}{16m^2}\exp\left(\int_0^c \left(\frac{\left(\psi_m^t(1)\right)^2 - \left(\psi_{m-2}^t(1)\right)^2}{t} - \frac{2}{t}\right)dt\right) \leq \frac{3c^2}{16m^2}\exp\left(\frac{2c^2}{\sqrt{3}}\right). \quad \square$$

Lemma B.7. For all $c \neq 0$ and $k \in \mathbb{N}$, we have $\left(\psi_k^c(1)\right)^2 \leq (k+1/2)\left(1 + 2c^2/3^{3/2}\right)^2$ and $\|\psi_k^c\|_{L^\infty([-1,1])}^2 \leq (k+1/2)\left(1+4c^2/3^{3/2}\right)^2$. For all $c \neq 0$ and $k \geq c$, we have $\|\psi_k^c\|_{L^\infty([-1,1])}^2 \leq k + 1/2$. We also have $\|\psi_0^c\|_{L^\infty([-1,1])}^2 \leq 2|c|/\pi$.

Proof. The first assertion follows from (65) in [8]. For the second, we use (66) in [8] in the first display, 22.14.7 and 22.2.10 in [1], hence $\|P_k\|_{L^\infty([-1,1])} \leq \sqrt{k+1/2}$, in the second inequality,

$$\|\psi_k^c\|_{L^\infty([-1,1])} \leq \|P_k\|_{L^\infty([-1,1])} + \frac{c^2}{\sqrt{3(k+1/2)}}\left(1 + \frac{\sqrt{3/2}}{\sqrt{k+1/2}}\right)$$

$$\leq \sqrt{k+1/2}\left(1 + \frac{c^2}{\sqrt{3}(k+1/2)}\left(1 + \frac{\sqrt{3/2}}{\sqrt{k+1/2}}\right)\right) \leq \sqrt{k+1/2}\left(1 + \frac{4c^2}{3^{3/2}}\right).$$

The third uses (3.4) and (3.125) in [46]. We obtain the last by the proof of Proposition 1 in [36] which yields $\|\psi_0^c\|_{L^\infty([-1,1])}^2 \leq 2/(\mu_0^c)^2$ and Lemma B.5. For all $c < 0$, we use $\psi_m^{-c} = \psi_m^c$. $\square$

Lemma B.8. For all $c \neq 0$ and $N \geq H(c)$, we have

$$\left\|\frac{\partial \psi_N^c}{\partial c}\right\|_{L^\infty([-1,1])} \leq \frac{2c_f\left(C_1(c) + C_2(c)\right)C_3(c)\Pi(c)}{|c|}\sqrt{N},$$

$$C_1(c) := \frac{2H(c)+9}{(H(c)+2)^2},\ C_2(c) := \frac{2\,|c|}{\pi H(c)(H(c)-1)} + \frac{l(c)}{4},\ C_3(c) := \sqrt{1 + \frac{1}{2H(c)}}.$$

Proof. Take $c \neq 0$, $N \geq H(c)$, and $w \in [-1,1]$. Theorem 7.11 in [46] yields

$$\frac{\partial \psi_N^c}{\partial c}(w) = \frac{2\psi_N^c(1)}{|c|}\sum_{k\equiv N[2],\ k\neq N}\frac{\mu_N^c\mu_k^c}{\left(\mu_N^c\right)^2 - \left(\mu_k^c\right)^2}\psi_k^c(1)\psi_k^c(w). \tag{B.12}$$

Using $\mu_k^c/\mu_N^c \in \mathbb{R}$ if $k \equiv N[2]$ and Lemma B.7, we obtain

$$\left|\frac{\partial \psi_N^c}{\partial c}(w)\right| \leq \frac{\sqrt{4N+2}}{|c|}\mathcal{C}(f,N,c),$$

$$\mathcal{C}(f,N,c) := f\left(\frac{\mu_N^c}{\mu_0^c}\right)\frac{2|c|\mathbb{1}\{N\equiv 0[2]\}}{\pi} + \sum_{\substack{0<k<N\\ k\equiv N[2]}} f\left(\frac{\mu_N^c}{\mu_k^c}\right) l(c)\left(k+\frac{1}{2}\right) + \sum_{\substack{k>N\\ k\equiv N[2]}} f\left(\frac{\mu_k^c}{\mu_N^c}\right)\left(k+\frac{1}{2}\right).$$

Lemma B.6 yields, if $k \equiv N[2]$,

$$\left|\frac{\mu_N^c}{\mu_k^c}\right| \leq \left|\frac{\mu_N^c}{\mu_{N-2}^c}\right| \leq \frac{\Pi(c)}{N^2} \leq \frac{1}{2} \text{ if } k<N \text{ and } \left|\frac{\mu_k^c}{\mu_N^c}\right| \leq \left(\frac{\sqrt{\Pi(c)}}{N+2}\right)^{k-N} \leq \frac{1}{2} \text{ if } k>N. \tag{B.13}$$

Using (B.10), (B.9), (B.13), and $\sum_{k\in\mathbb{N}} k2^{-k} = 2$ in the third display, the result follows from

$$\begin{aligned}
\mathcal{C}(f,N,c) &\leq c_f\left(\left(\frac{2\,|c|}{\pi}+\frac{l(c)N(N-1)}{4}\right)\frac{\Pi(c)}{N^2} + \sum_{k\equiv N[2],\ k>N}\frac{k+1/2}{2^{(k-N)/2}}\left(\frac{\sqrt{2\Pi(c)}}{N+2}\right)^{k-N}\right)\\
&\leq c_f\Pi(c)\left(\frac{2\,|c|}{\pi H(c)(H(c)-1)}+\frac{l(c)}{4}+\frac{2}{(N+2)^2}\sum_{l\equiv 0[2],\ l\geq 2}\left(l+N+\frac{1}{2}\right)\frac{1}{2^{l/2}}\right)\\
&\leq c_f\Pi(c)\left(C_2(c)+\frac{2}{(N+2)^2}\left(N+\frac{9}{2}\right)\right) \leq c_f\Pi(c)\left(C_1(c)+C_2(c)\right). \qquad \text{(B.14)}
\end{aligned}$$

□

Lemma B.9. For all $c \neq 0$ and $N \geq H(c)$, we have

$$\left\|\frac{\partial^2\psi_N^c}{\partial c^2}\right\|_{L^\infty([-1,1])} \leq \frac{\Pi(c)C_3(c)}{c^2}\left(C_4(c)N^{5/2}+C_5(c)N^{3/2}+C_6(c)\sqrt{N}+C_7(c)\right),$$

$$C_4(c) := c_g\left(C_2(c)-C_1(c)\right),\ C_7(c) := \frac{c_g}{(H(c)+2)^{1/2}}\left(85+\frac{246}{H(c)+2}\right),$$

$$C_5(c) := 8\left(c_f\left(C_1(c)+C_2(c)\right)C_3(c)\right)^2\Pi(c) + \left(c_g+4c_f\right)C_2(c) + \left(8c_f-c_g\right)C_1(c) + 2c_g,$$

$$C_6(c) := 8c_hc_f(C_1(c)+C_2(c))^2\Pi(c) + \left(C_1(c)+C_2(c)\right)\left(c^2c_g+4c_f\right) + 19c_g.$$

Proof. For all $c<0$, $\mu_m^c = \overline{\mu_m^{-c}}$ and $\psi_m^{-c} = \psi_m^c$, hence we only consider $c>0$. Using $c \in (0,\infty) \mapsto \psi_N^c(x)$ is analytic (see [22] page 320) and (7.99) in [46], we have by differentiating

$$\mu_N^c\psi_N^c(x) = \int_{-1}^1 e^{icxt}\psi_N^c(t)dt\,: \tag{B.15}$$

$$\mu_N^c\frac{\partial\psi_N^c}{\partial x}(x) = \int_{-1}^1 icte^{icxt}\psi_N^c(t)dt, \tag{B.16}$$

$$\mu_N^c\frac{\partial^2\psi_N^c}{\partial x^2}(x) = -\int_{-1}^1 (ct)^2e^{icxt}\psi_N^c(t)dt, \tag{B.17}$$

(B.18)
$$\left(\frac{\partial^2\mu_N^c}{\partial c^2}\psi_N^c + 2\frac{\partial\mu_N^c}{\partial c}\frac{\partial\psi_N^c}{\partial c} + \mu_N^c\frac{\partial^2\psi_N^c}{\partial c^2}\right)(x) = \int_{-1}^{1} e^{icxt}\left(\frac{\partial^2\psi_N^c}{\partial c^2}(t) + 2ixt\frac{\partial\psi_N^c}{\partial c}(t) - (xt)^2\psi_N^c(t)\right)dt.$$

Multiplying (B.18) by $\psi_k^c(x)$, integrating, and using (B.15)-(B.17), we obtain, for all $k \neq N$,

$$\begin{aligned}&2\frac{\partial\mu_N^c}{\partial c}\int_{-1}^{1}\frac{\partial\psi_N^c}{\partial c}(x)\psi_k^c(x)dx + \mu_N^c\int_{-1}^{1}\frac{\partial^2\psi_N^c}{\partial c^2}(x)\psi_k^c(x)dx\\ &= \mu_k^c\int_{-1}^{1}\frac{\partial^2\psi_N^c}{\partial c^2}(x)\psi_k^c(x)dx + 2\frac{\mu_k^c}{c}\int_{-1}^{1}x\frac{\partial\psi_N^c}{\partial c}(x)\frac{\partial\psi_k^c}{\partial x}(x)dx + \frac{\mu_k^c}{c^2}\int_{-1}^{1}x^2\psi_N^c(x)\frac{\partial^2\psi_k^c}{\partial x^2}(x)dx.\end{aligned}$$

Recombining and using that, for all $k \neq N$, $\mu_k^c \neq \mu_N^c$ (see (3.45) in [46]), we obtain

$$\begin{aligned}&(\mu_N^c - \mu_k^c)\int_{-1}^{1}\frac{\partial^2\psi_N^c}{\partial c^2}(x)\psi_k^c(x)dx\\ &= 2\frac{\mu_k^c}{c}\int_{-1}^{1}x\frac{\partial\psi_N^c}{\partial c}(x)\frac{\partial\psi_k^c}{\partial x}(x)dx + \frac{\mu_k^c}{c^2}\int_{-1}^{1}x^2\psi_N^c(x)\frac{\partial^2\psi_k^c}{\partial x^2}(x)dx - 2\frac{\partial\mu_N^c}{\partial c}\int_{-1}^{1}\frac{\partial\psi_N^c}{\partial c}(x)\psi_k^c(x)dx.\end{aligned}$$

This yields, for all $k \not\equiv N[2]$, using (B.12), (7.69)-(7.70), and Theorem 7.11 in [46], $\int_{-1}^{1}\frac{\partial^2\psi_N^c}{\partial c^2}(x)\psi_k^c(x)dx = 0$, while, for all $k \equiv N[2]$ and $k \neq N$, using (7.69)-(7.70), Theorem 7.11, (7.99) and the eigenvalues $(\chi_N^c)_{N\in\mathbb{N}_0}$ of the differential operator in (1.1) in [46],

$$\int_{-1}^{1}\frac{\partial^2\psi_N^c}{\partial c^2}(x)\psi_k^c(x)dx = \frac{2}{c}\frac{\mu_k^c}{\mu_N^c - \mu_k^c}\int_{-1}^{1}x\frac{\partial\psi_N^c}{\partial c}(x)\frac{\partial\psi_k^c}{\partial x}(x)dx + \Xi_{N,k},$$

$$\Xi_{N,k} := \frac{\psi_N^c(1)\psi_k^c(1)}{c^2}\left(\frac{\mu_N^c\mu_k^c(\chi_k^c - \chi_N^c)}{\left(\mu_N^c - \mu_k^c\right)^2} - 2\frac{\mu_N^c\mu_k^c}{\left(\mu_N^c\right)^2 - \left(\mu_k^c\right)^2}\left(2 + \frac{\mu_N^c\left(2\psi_N^c(1)^2 - 1\right)}{\mu_N^c - \mu_k^c}\right)\right).$$

Differentiating (7.114) in [46] in c yields $\int_{-1}^{1}\frac{\partial^2\psi_N^c}{\partial c^2}(x)\psi_N^c(x)dx = -\int_{-1}^{1}\left(\frac{\partial\psi_N^c}{\partial c}(x)\right)^2 dx$. Also, by (B.13), for all $k \equiv N[2]$,

$$\frac{|\mu_N^c|}{|\mu_N^c - \mu_k^c|} \leq 1 \text{ if } k < N \text{ and else } \frac{|\mu_N^c|}{|\mu_N^c - \mu_k^c|} \leq 2. \tag{B.19}$$

We obtain, using Lemma B.7 and $N \geq c$ for the first term,

$$\begin{aligned}\left\|\frac{\partial^2\psi_N^c}{\partial c^2}\right\|_{L^\infty([-1,1])} \leq &\sqrt{N+\frac{1}{2}}\int_{-1}^{1}\left(\frac{\partial\psi_N^c}{\partial c}(x)\right)^2 dx + \sum_{k\equiv N[2],\ k\neq N}|\Xi_{N,k}|\,\|\psi_k^c\|_{L^\infty([-1,1])}\\ &+ \sum_{k\equiv N[2],\ k\neq N}\frac{2\,|\mu_k^c|}{c\,|\mu_N^c - \mu_k^c|}\left|\int_{-1}^{1}x\frac{\partial\psi_N^c}{\partial c}(x)\frac{\partial\psi_k^c}{\partial x}(x)dx\right|\|\psi_k^c\|_{L^\infty([-1,1])}.\end{aligned} \tag{B.20}$$

For the first term on the right-hand side of (B.20), using Lemma B.8, we obtain

$$\sqrt{N+\frac{1}{2}}\int_{-1}^{1}\left(\frac{\partial\psi_N^c}{\partial c}(x)\right)^2 dx \leq 8\left(c_f\left(C_1(c) + C_2(c)\right)C_3(c)\right)^2 C_3(c)\left(\frac{\Pi(c)}{c}\right)^2 N^{3/2}.$$

For the second term in (B.20), using that for all $k \equiv N[2]$, $\mu_N^c/\mu_k^c \in \mathbb{R}$ and (B.13) we obtain

$$|\Xi_{N,k}| \leq \frac{|\psi_N^c(1)|\,|\psi_k^c(1)|}{c^2}\left(g\left(\rho_k\right)\left(\chi_k^c-\chi_N^c\right)+2\left(2+\frac{\left|2\psi_N^c(1)^2-1\right|\,|\mu_N^c|}{\left|\mu_N^c-\mu_k^c\right|}\right)f\left(\rho_k\right)\right),$$

where $\rho_k = \mu_N^c/\mu_k^c$ when $k < N$ and $\rho_k = \mu_k^c/\mu_N^c$ when $k > N$. Using $N \geq c$, (B.19), $|\chi_N^c - \chi_k^c| \leq |N-k|\,(k+N+1) + c^2$ (see (13) in [7]), (B.9), and $\left|2\psi_N^c(1)^2-1\right| \leq 2N$ (by Lemma B.7) for the first inequality, $(N-k)(k+N+1) \leq N(N+1)$ for all $0 < k < N$, (B.13), and (B.10) for the second, $(k-N)(k+N+1) = k(k+1) - N^2 - N$ for the third, the computations in (B.14), $\sum_{k=1}^{\infty} k^2 2^{-k} = 6$ and $\sum_{k=1}^{\infty} k^3 2^{-k} = 26$, and Euclidean division for the fourth, yield

$$\begin{aligned}
&\sum_{k\equiv N[2],\ k\neq N}|\Xi_{N,k}|\,\|\psi_k^c\|_{L^\infty([-1,1])} \leq \frac{c_g\sqrt{4N+2}\mathbb{1}\{N\equiv 0[2]\}}{|c|\,\pi}\left|\frac{\mu_N^c}{\mu_0^c}\right|\left(N(N+1)+c^2+\frac{4c_f}{c_g}\left(N+1\right)\right)\\
&\quad+\frac{c_g\sqrt{4N+2}}{2c^2}\sum_{k\equiv N[2],\ 0<k<N}\left(k+\frac{1}{2}\right)l(c)\left|\frac{\mu_N^c}{\mu_k^c}\right|\left((N-k)(k+N+1)+c^2+\frac{4c_f}{c_g}\left(N+1\right)\right)\\
&\quad+\frac{c_g\sqrt{4N+2}}{2c^2}\sum_{k\equiv N[2],\ k>N}\left(k+\frac{1}{2}\right)\left|\frac{\mu_k^c}{\mu_N^c}\right|\left(|N-k|\,(k+N+1)+c^2+\frac{4c_f}{c_g}(2N+1)\right)\\
&\leq \frac{c_g\sqrt{4N+2}}{2c^2}\left(N(N+1)+c^2+\frac{4c_f}{c_g}\left(N+1\right)\right)\left(\frac{2\,|c|}{\pi}+\frac{l(c)N(N-1)}{4}\right)\frac{\Pi(c)}{N^2}\\
&\quad+\frac{c_g\sqrt{4N+2}}{2c^2}\sum_{k\equiv N[2],\ k>N}\frac{k+1/2}{2^{(k-N)/2}}\left(\frac{\sqrt{2\Pi(c)}}{N+2}\right)^{k-N}\left((k-N)(k+N+1)+c^2+\frac{4c_f}{c_g}(2N+1)\right)\\
&\leq \frac{c_g\sqrt{4N+2}\Pi(c)}{2c^2}\left(N(N+1)+c^2+\frac{4c_f}{c_g}(N+1)\right)\left(\frac{2\,|c|}{\pi H(c)(H(c)-1)}+\frac{l(c)}{4}\right)\\
&\quad+\frac{c_g\sqrt{4N+2}}{2c^2}\frac{2\Pi(c)}{(N+2)^2}\sum_{l\equiv 0[2],\ l\geq 2}\frac{l+N+1/2}{2^{l/2}}\left(c^2+\frac{4c_f}{c_g}(2N+1)-N-N^2\right)\\
&\quad+\frac{c_g\sqrt{4N+2}}{2c^2}\frac{2\Pi(c)}{(N+2)^2}\sum_{l\equiv 0[2],\ l\geq 2}\left(l+N+\frac{1}{2}\right)(l+N)(l+N+1)\frac{1}{2^{l/2}}\\
&\leq \frac{c_g\sqrt{4N+2}\Pi(c)}{2c^2}\left[C_2(c)\left(N(N+1)+c^2+\frac{4c_f}{c_g}(N+1)\right)+C_1(c)\left(c^2+\frac{4c_f}{c_g}(2N+1)-N-N^2\right)\right.\\
&\qquad\left.+2N+19+\frac{85}{N+2}+\frac{246}{(N+2)^2}\right]\\
&\leq \frac{c_g\Pi(c)}{c^2}C_3(c)\left[N^{5/2}\left(C_2(c)-C_1(c)\right)+N^{3/2}\left(\left(1+\frac{4c_f}{c_g}\right)C_2(c)+\left(\frac{8c_f}{c_g}-1\right)C_1(c)+2\right)\right.
\end{aligned}$$

$$+\sqrt{N}\left((C_1(c)+C_2(c))\left(c^2+\frac{4c_f}{c_g}\right)+19\right)+\frac{85}{(H(c)+2)^{1/2}}+\frac{246}{(H(c)+2)^{3/2}}\Bigg].$$

For the third term in (B.20), using (B.12), the triangle inequality, and (7.74) in [46] for the first inequality and using $|\mu_m^c|/|\mu_m^c+\mu_k^c|\leq 1$ for the second, we obtain

$$\begin{aligned}\left|\int_{-1}^1 x\frac{\partial\psi_N^c}{\partial c}(x)\frac{\partial\psi_k^c}{\partial x}(x)dx\right| &\leq \frac{4\,|\psi_N^c(1)|\,|\psi_k^c(1)|}{|c|}\sum_{m\neq N,\ m\equiv N[2]}\frac{|\mu_N^c|\,|\mu_m^c|\,|\psi_m^c(1)|^2}{\left|(\mu_m^c)^2-(\mu_N^c)^2\right|}\frac{|\mu_m^c|}{|\mu_m^c+\mu_k^c|}\\ &\leq \frac{4\,|\psi_N^c(1)|\,|\psi_k^c(1)|}{|c|}\mathcal{C}(f,N,c),\end{aligned}$$

hence, using (B.14) for the first inequality and (B.9) and (B.14) replacing c_f by c_h for the third,

$$\begin{aligned}&\sum_{k\equiv N[2],\ k\neq N}\frac{2\,|\mu_k^c|}{c\,|\mu_N^c-\mu_k^c|}\left|\int_{-1}^1 x\frac{\partial\psi_N^c}{\partial c}(x)\frac{\partial\psi_k^c}{\partial x}(x)dx\right|\|\psi_k^c\|_{L^\infty([-1,1])}\\ &\leq 4c_f\sqrt{4N+2}(C_1(c)+C_2(c))\frac{\Pi(c)}{c^2}\sum_{k\equiv N[2],\ k\neq N}\frac{|\mu_k^c|}{|\mu_N^c-\mu_k^c|}\,|\psi_k^c(1)|\,\|\psi_k^c\|_{L^\infty([-1,1])}\\ &\leq 4c_f\sqrt{4N+2}(C_1(c)+C_2(c))\frac{\Pi(c)}{c^2}\mathcal{C}(h,N,c)\\ &\leq 4c_hc_f\sqrt{4N+2}(C_1(c)+C_2(c))^2\frac{\Pi(c)^2}{c^2}\leq 8c_hc_fC_3(c)(C_1(c)+C_2(c))^2\frac{\Pi(c)^2}{c^2}\sqrt{N}.\quad\square\end{aligned}$$

Lemma B.10. For all $u,x_0,R>0$, $t\in\mathbb{R}$, $q\in\{1,\infty\}$, λ from (A.8) and $N(Rx_0U)$ and $\widetilde{\boldsymbol{N}}(q)$ from (A.9), for all $N\geq N(Rx_0U)$, we have

$$\sup_{\boldsymbol{b}\in[-R,R]^p}\left|\frac{\partial^2}{\partial t^2}\left(\left(\frac{Rx_0t}{2\pi}\right)^{p/2}\lambda(t)\psi_{\widetilde{\boldsymbol{N}}(q)}^{Rx_0t}\left(\frac{\boldsymbol{b}}{R}\right)\right)\right|\leq \mathbb{1}\{U/2\leq|t|\leq U\}C_8(Rx_0U,p,U)N^{2+k_q/2},$$

$$\begin{aligned}C_8(Rx_0U,p,U) &:= \left(\frac{Rx_0U}{\pi}\right)^{p/2}\frac{C_3(Rx_0U)^pN(Rx_0U)^{(p-1)/(2q)}}{N(Rx_0U)^2}\left(\frac{p|p-2|}{U^2}+C_9(U)\frac{2p}{U}+C_{10}(U)\right.\\ &\quad\left.+\left(\frac{2p}{U}+2C_9(U)\right)\frac{pC_{16}(Rx_0U)}{N(Rx_0U)^2}+\frac{p(p-1)C_{16}(Rx_0U)}{N(Rx_0U)^2}+pC_{11}(Rx_0U)\right),\end{aligned}$$

$$C_9(U):=\sup_{t\in[U/2,U]}\left|\lambda'(t)\right|,\ C_{10}(U):=\sup_{t\in[U/2,U]}\left|\lambda''(t)\right|,$$

$$C_{11}(Rx_0U):=\frac{(Rx_0)^2\,\Pi(Rx_0U)}{(Rx_0U)^2}\left(C_4(Rx_0U)+\frac{C_5(Rx_0U)}{N(Rx_0U)}+\frac{C_6(Rx_0U)}{N(Rx_0U)^2}+\frac{C_7(Rx_0U)}{N(Rx_0U)^{5/2}}\right),$$

$$C_{16}(Rx_0U):=2c_fRx_0\left(C_1(Rx_0U)+C_2(Rx_0U)\right)C_3(Rx_0U)\frac{\Pi(Rx_0U)}{Rx_0U}.$$

Proof. Let $q = 1$. By symmetry, we take $t \in [U/2, U]$, $\boldsymbol{b} \in [-R, R]^p$, and $c > 0$. We have

$$R(t,\boldsymbol{b}) := \left| \frac{\partial^2}{\partial t^2} \left(\left(\frac{Rx_0 t}{2\pi} \right)^{p/2} \lambda(t) \psi^{Rx_0 t}_{\widetilde{\boldsymbol{N}}(q)} \left(\frac{\boldsymbol{b}}{R} \right) \right) \right|$$

$$\leq \left(\frac{Rx_0}{2\pi} \right)^{p/2} t^{p/2} \Bigg[\left(\frac{p\,|p-2|}{4t^2} \lambda(t) + \frac{p}{t} \left| \lambda'(t) \right| + \left| \lambda''(t) \right| \right) \left| \psi^{Rx_0 t}_{\widetilde{\boldsymbol{N}}(q)} \left(\frac{\boldsymbol{b}}{R} \right) \right|$$

$$+ Rx_0 \left(\frac{p}{t} \lambda(t) + 2|\lambda'(t)| \right) \left| \frac{\partial \psi^c_{\widetilde{\boldsymbol{N}}(q)}}{\partial c} \right|_{c=Rx_0 t} \left(\frac{\boldsymbol{b}}{R} \right) \Bigg| + (Rx_0)^2 \lambda(t) \left| \frac{\partial^2 \psi^c_{\widetilde{\boldsymbol{N}}(q)}}{\partial c^2} \right|_{c=Rx_0 t} \left(\frac{\boldsymbol{b}}{R} \right) \Bigg| \Bigg],$$

$$\frac{\partial \psi^c_{\widetilde{\boldsymbol{N}}(q)}}{\partial c} \left(\frac{\boldsymbol{b}}{R} \right) = \sum_{j=2}^{p} \psi^c_N \left(\frac{\boldsymbol{b}_1}{R} \right) \frac{\partial \psi^c_{N(Rx_0U)}}{\partial c} \left(\frac{\boldsymbol{b}_j}{R} \right) \prod_{\substack{l=2 \\ l \neq j}}^{p} \psi^c_{N(Rx_0U)} \left(\frac{\boldsymbol{b}_l}{R} \right) + \frac{\partial \psi^c_N}{\partial c} \left(\frac{\boldsymbol{b}_1}{R} \right) \prod_{l=2}^{p} \psi^c_{N(Rx_0U)} \left(\frac{\boldsymbol{b}_l}{R} \right),$$

$$\frac{\partial^2 \psi^c_{\widetilde{\boldsymbol{N}}(q)}}{\partial c^2} \left(\frac{\boldsymbol{b}}{R} \right) = 2 \sum_{j=2}^{p} \frac{\partial \psi^c_{N(Rx_0U)}}{\partial c} \left(\frac{\boldsymbol{b}_j}{R} \right) \frac{\partial \psi^c_N}{\partial c} \left(\frac{\boldsymbol{b}_1}{R} \right) \prod_{\substack{l=2 \\ l \neq j}}^{p} \psi^c_{N(Rx_0U)} \left(\frac{\boldsymbol{b}_l}{R} \right)$$

$$+ \sum_{k=2}^{p} \sum_{\substack{j=2 \\ j \neq k}}^{p} \psi^c_N \left(\frac{\boldsymbol{b}_1}{R} \right) \frac{\partial \psi^c_{N(Rx_0U)}}{\partial c} \left(\frac{\boldsymbol{b}_j}{R} \right) \frac{\partial \psi^c_{N(Rx_0U)}}{\partial c} \left(\frac{\boldsymbol{b}_k}{R} \right) \prod_{\substack{l=2 \\ l \neq j, l \neq k}}^{p} \psi^{x_0 t}_{N(Rx_0U)} \left(\frac{\boldsymbol{b}_l}{R} \right)$$

$$+ \frac{\partial^2 \psi^c_N}{\partial c^2} \left(\frac{\boldsymbol{b}_1}{R} \right) \prod_{l=2}^{p} \psi^c_{N(Rx_0U)} \left(\frac{\boldsymbol{b}_l}{R} \right) + \sum_{j=2}^{p} \psi^c_N \left(\frac{\boldsymbol{b}_l}{R} \right) \frac{\partial^2 \psi^c_{N(Rx_0U)}}{\partial c^2} \left(\frac{\boldsymbol{b}_j}{R} \right) \prod_{\substack{l=2 \\ l \neq j}}^{p} \psi^c_{N(Rx_0U)} \left(\frac{\boldsymbol{b}_l}{R} \right).$$

We conclude using $N \geq Rx_0U$ (by the discussion before Lemma B.6), the third assertion of Lemma B.7, and Lemma B.9. The case $q = \infty$ is obtained with $N(Rx_0U) = N$. □

Lemma B.11. For all $R, x_0 > 0$, $2\sigma > k_q + 4$, $q \in \{1, \infty\}$, $2\tau \geq \left(3e^{1/2} Rx_0/4\right) \vee 1$, we have

$$\int_{\mathbb{R}} e^{-2\tau|t|} \sum_{\boldsymbol{m} \in \mathbb{N}_0^p} |\boldsymbol{m}|_q^{2\sigma} \left(c^P_{\boldsymbol{m}}(t) \right)^2 dt \leq \frac{C_{12}(\sigma, p)}{\tau p^{2\sigma/q}},$$

$$C_{12}(\sigma, p) := \Gamma(2\sigma + p + 1/2) \left(\frac{2^{p-1} p}{2\sigma + p} \left(\frac{8}{3e^{1/2}} \right)^{2\sigma+p} + \frac{\pi e^3 p 2^p \sqrt{3}}{9} \right).$$

Proof. When $q = 1$, we use $|\boldsymbol{m}|_1 \leq p\,|\boldsymbol{m}|_\infty$. Let $q = \infty$, R, x_0, σ, τ as in the lemma. Because $P_0 = \mathbb{1}\{|\cdot|_\infty \leq 1\}/2^{p/2}$, for all $m \in \mathbb{N}_0$, $\left| \langle P_0, \psi^c_m \rangle_{L^2([-1,1])} \right| \leq 1$, and, for all $m > |c|$, $\left| \langle P_0, \psi^c_m \rangle_{L^2([-1,1])} \right| \leq |\mu^c_m| / \sqrt{2}$ (see Proposition 3 and (13) in [7]) we obtain, for all $t \neq 0$,

$$\sum_{\boldsymbol{m} \in \mathbb{N}_0^p} |\boldsymbol{m}|_\infty^{2\sigma} \left(c^P_{\boldsymbol{m}}(t) \right)^2 \leq \sum_{|\boldsymbol{m}|_\infty \leq Rx_0|t|} |\boldsymbol{m}|_\infty^{2\sigma} \mathbb{1}\{Rx_0\,|t| \geq 1\} + \sum_{|\boldsymbol{m}|_\infty > Rx_0|t|} \frac{|\boldsymbol{m}|_\infty^{2\sigma} \left| \mu^{Rx_0 t}_{|\boldsymbol{m}|_\infty} \right|^2}{2}. \tag{B.21}$$

Using (B.21), Lemma B.4, and $\sum_{|\boldsymbol{m}|_\infty=k} 1 \leq p(k+1)^{p-1}$ for the first inequality, $m+1 \leq 2m$ when $m \geq 1$ for the second, and $2m+1 \leq 3m$, $(Rx_0t+1)^{2\sigma+p} \leq (2Rx_0t)^{2\sigma+p}$ when $m, Rx_0t \geq 1$, and (1.3) in [43] for the third, we have

$$\begin{aligned}
&\int_{\mathbb{R}} e^{-2\tau|t|} \sum_{\boldsymbol{m}\in\mathbb{N}_0^p} |\boldsymbol{m}|_\infty^{2\sigma} \left(c_{\boldsymbol{m}}^P(t)\right)^2 dt \\
&\leq \int_0^\infty 2pe^{-2\tau t} \left(\sum_{m\leq Rx_0t} (m+1)^{p-1} m^{2\sigma} \mathbb{1}\{Rx_0t \geq 1\} + \frac{\pi e^3}{9} \sum_{m>Rx_0t} (m+1)^{p-1} m^{2\sigma} \left(\frac{eRx_0t}{4m}\right)^{2m} \right) dt \\
&\leq \int_0^\infty 2^p pe^{-2\tau t} \int_1^{Rx_0t+1} u^{2\sigma+p-1} du \mathbb{1}\{Rx_0t \geq 1\} dt + \frac{\pi e^3 p 2^p}{9} \sum_{m\geq 1} m^{2\sigma+p-1} \left(\frac{eRx_0}{4m}\right)^{2m} \int_0^\infty e^{-2\tau t} t^{2m} dt \\
&\leq \frac{2^{2(\sigma+p)}p}{2\sigma+p} \int_{1/(Rx_0)}^\infty e^{-2\tau t} (Rx_0t)^{2\sigma+p} dt + \frac{\pi e^3 p 2^p \sqrt{3}}{9\tau} \sum_{m\geq 1} m^{2\sigma+p-1/2} e^{2m\ln(3Rx_0/(8\tau))} \\
&\leq \frac{2^{p-1}p\Gamma(2\sigma+p+1)}{(2\sigma+p)\tau} \left(\frac{8}{3e^{1/2}}\right)^{2\sigma+p} + \frac{\pi e^3 p 2^p \sqrt{3}}{9\tau} \int_0^\infty e^{-t} t^{2\sigma+p-1/2} dt \leq \frac{C_{12}(\sigma,p)}{\tau p^{2\sigma/q}}. \quad \square
\end{aligned}$$

Lemma B.12. For all $N \geq H(Rx_0U)$, $R, U > 0$, $q \in \{1, \infty\}$, and F from (A.7), we have

$$I_1 := \int_{[-1,1]^p} \int_{\mathbb{R}} \left|\partial_t \mathcal{F}[F](t, x_0t\boldsymbol{x})\right|^2 d\boldsymbol{x}dt \leq R^p C_{17}(Rx_0U, p, U) N^2 \rho_{\widetilde{\boldsymbol{N}}(q)}^{W_{[-1,1]}, Rx_0U} \tag{B.22}$$

$$C_{17}(Rx_0U, p, U) := C_{15}(Rx_0U, p, U) + \frac{2pUC_{16}(Rx_0U)^2}{N(Rx_0U)},$$

$$C_{15}(Rx_0U, p, U) := \frac{25p^2}{8U}\left(1 + \frac{2(Rx_0U)^2}{3^{3/2}}\right)^4 + \frac{UC_9(U)^2}{N(Rx_0U)^2} + \frac{5pC_9(U)\ln(2)}{2N(Rx_0U)}\left(1 + \frac{2(Rx_0U)^2}{3^{3/2}}\right)^2,$$

$$I_2 := \int_{[-1,1]^p} \int_{\mathbb{R}} \left|\mathcal{F}[F](t, x_0t\boldsymbol{x})\right|^2 d\boldsymbol{x}dt \leq R^p U \rho_{\widetilde{\boldsymbol{N}}(q)}^{W_{[-1,1]}, Rx_0U}. \tag{B.23}$$

Proof. Let $N \geq H(Rx_0U) \geq 2$. For simplicity of notations, we omit $W_{[-1,1]}$ from ρ. We have

$$\begin{aligned}
\mathcal{F}[F](t, x_0t\boldsymbol{x}) &= \left(\frac{Rx_0|t|}{2\pi}\right)^{p/2} R^{p/2} \lambda(t) \mathcal{F}_{Rx_0t}\left[\psi_{\widetilde{\boldsymbol{N}}(q)}^{Rx_0t}\right](\boldsymbol{x}) \\
&= R^{p/2} i^{|\widetilde{\boldsymbol{N}}(q)|_1} \lambda(t) \sqrt{\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rx_0t}} \psi_{\widetilde{\boldsymbol{N}}(q)}^{Rx_0t}(\boldsymbol{x}) \quad \left(\text{because } \mu_m^{Rx_0t} = i^m \left(\frac{2\pi}{Rx_0|t|}\right)^{1/2} \sqrt{\rho_m^{Rx_0t}}\right). \qquad (B.24)
\end{aligned}$$

This yields

$$I_1 \leq R^p \int_{\mathbb{R}} \int_{[-1,1]^p} \left(\left(\frac{\mathrm{d}\sqrt{\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rx_0t}}}{\mathrm{d}t} \lambda(t) + \lambda'(t) \sqrt{\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rx_0t}} \right) \psi_{\widetilde{\boldsymbol{N}}(q)}^{Rx_0t}(\boldsymbol{x}) + \sqrt{\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rx_0t}} \lambda(t) \frac{\partial \psi_{\widetilde{\boldsymbol{N}}(q)}^{Rx_0t}(\boldsymbol{x})}{\partial t} \right)^2 dt d\boldsymbol{x}.$$

Using (7.114) in [46], cross-products terms are zero and $I_1 \le R^p(I_{11}+I_{12})$, where

$$I_{11} = \int_{\mathbb{R}} \left(\lambda(t)^2 \left(\frac{\mathrm{d}\sqrt{\rho^{Rx_0t}_{\widetilde{\boldsymbol{N}}(q)}}}{\mathrm{d}t} \right)^2 + \lambda'(t)^2 \rho^{Rx_0t}_{\widetilde{\boldsymbol{N}}(q)} + 2\lambda(t)|\lambda'(t)| \sqrt{\rho^{Rx_0t}_{\widetilde{\boldsymbol{N}}(q)}} \frac{\mathrm{d}\sqrt{\rho^{Rx_0t}_{\widetilde{\boldsymbol{N}}(q)}}}{\mathrm{d}t} \right) dt,$$

$$I_{12} = \int_{\mathbb{R}} \lambda(t)^2 \rho^{Rx_0t}_{\widetilde{\boldsymbol{N}}(q)} \left(\int_{[-1,1]^p} \left(\frac{\partial \psi^{Rx_0t}_{\widetilde{\boldsymbol{N}}(q)}(\boldsymbol{x})}{\partial t} \right)^2 d\boldsymbol{x} \right) dt.$$

Then, using (7.100) in [46] for the second equality yields, for all $t \neq 0$,

$$\frac{\mathrm{d}\sqrt{\rho^{Rx_0t}_N}}{\mathrm{d}t} = \frac{x_0 R}{2\sqrt{\rho^{Rx_0t}_N}} \left. \frac{\mathrm{d}\rho^c_N}{\mathrm{d}c} \right|_{c=Rx_0t} = \frac{\sqrt{\rho^{Rx_0t}_N}}{|t|} \left(\psi^{Rx_0t}_N(1) \right)^2,$$

in particular $\rho^{Rx_0t}_N$ is increasing in t and, by the first assertion of Lemma B.7,

$$\text{(B.25)} \qquad \forall U/2 \le |t| \le U, \ \frac{\mathrm{d}\sqrt{\rho^{Rx_0t}_N}}{\mathrm{d}t} \le \frac{(N+1/2)\sqrt{\rho^{Rx_0t}_N}}{|t|} \left(1 + \frac{2(Rx_0U)^2}{3^{3/2}} \right)^2.$$

When $q=1$, using $N+1/2 \le 5N/4$ for all $N \ge 2$ and

$$\frac{\mathrm{d}\sqrt{\rho^{Rx_0t}_{\widetilde{\boldsymbol{N}}(q)}}}{\mathrm{d}t} = (p-1) \left(\sqrt{\rho^{Rx_0t}_{N(Rx_0U)}} \right)^{p-2} \sqrt{\rho^{Rx_0t}_N} \left(\frac{\mathrm{d}\sqrt{\rho^{Rx_0t}_{N(Rx_0U)}}}{\mathrm{d}t} \right) + \left(\sqrt{\rho^{Rx_0t}_{N(Rx_0U)}} \right)^{p-1} \left(\frac{\mathrm{d}\sqrt{\rho^{Rx_0t}_N}}{\mathrm{d}t} \right),$$

we have

$$\text{(B.26)} \qquad \frac{\mathrm{d}\sqrt{\rho^{Rx_0t}_{\widetilde{\boldsymbol{N}}(q)}}}{\mathrm{d}t} \le \frac{5pN}{4\,|t|} \left(1 + \frac{2(Rx_0U)^2}{3^{3/2}} \right)^2 \sqrt{\rho^{Rx_0t}_{\widetilde{\boldsymbol{N}}(q)}}.$$

The same inequality holds for $q=\infty$ (there $N = N(Rx_0U)$). This yields, for all $q \in \{1,\infty\}$,

$$\begin{aligned} I_{11} &\le \left(\frac{25p^2N^2}{8} \int_{U/2}^U \frac{dt}{t^2} \left(1 + \frac{2(Rx_0U)^2}{3^{3/2}} \right)^4 + UC_9(U)^2 + \frac{5pNC_9(U)}{2} \left(1 + \frac{2(Rx_0U)^2}{3^{3/2}} \right)^2 \int_{U/2}^U \frac{dt}{t} \right) \rho^{Rx_0U}_{\widetilde{\boldsymbol{N}}(q)} \\ &\le C_{15}(Rx_0U, p, U) N^2 \rho^{Rx_0U}_{\widetilde{\boldsymbol{N}}(q)}. \end{aligned}$$

Then, by (7.114) in [46] and Lemma B.8, we have, for all $U/2 \le |t| \le U$,

$$\begin{aligned} \int_{[-1,1]^p} \left(\frac{\partial \psi^{Rx_0t}_{\widetilde{\boldsymbol{N}}(q)}(\boldsymbol{x})}{\partial t} \right)^2 d\boldsymbol{x} &= (Rx_0)^2 \int_{[-1,1]} \left((p-1) \left(\left. \frac{\partial \psi^c_{N(Rx_0U)}(x)}{\partial c} \right|_{c=Rx_0t} \right)^2 + \left(\left. \frac{\partial \psi^c_N(x)}{\partial c} \right|_{c=Rx_0t} \right)^2 \right) dx \\ &\le 2p \left(C_{16}(Rx_0U) \right)^2 N \quad (\text{using } N \ge N(Rx_0U)). \end{aligned}$$

The same holds for $q=\infty$ (there $N = N(Rx_0U)$). This and $N \ge N(Rx_0U)$ yield (B.22).
(B.23) follows from (B.24) and the fact that $c \in (0,\infty) \mapsto \rho^c_m$ is nondecreasing. □

Appendix B.2. Complements on the proofs of the main results

Proof of (T1.1b). Step 1. By the proof of (T.1a), $f_{2,n} \in L^2\left(w \otimes W_{[-R,R]}^{\otimes p}\right)$ and $f_{2,n} \in \mathcal{H}_{w,W}^{q,\phi,\omega}(l)$ if

$$\left(\frac{Rx_0U}{2\pi}\right)^p \frac{2U\gamma^2}{p+1}\left(\phi(U)\bigvee \exp\left(\kappa N \ln\left(N+1\right)\right)\right)^2 \leq \pi l^2. \tag{B.27}$$

Step 3. Let $\xi < \sqrt{2}$ and $8/(eRx_0U) \geq 1$. By Lemma B.4, we have, for all $U/2 \leq |t| \leq U$, $\left(\sigma_{\widetilde{\boldsymbol{N}}(q)}^{W_{[-1,1]},Rx_0t}\right)^2 \leq \left(2\pi e^3/9\right)^p \exp\left(-2k_qN\ln\left(4(N+3/2)/(eRx_0U)\right)\right)$ and, by (A.22) and Proposition B.1 (ii), Lemma A.1 (iii) holds if

$$n\gamma^2 \exp\left(-2k_qN\ln\left(\frac{4(N+3/2)}{eRx_0U}\right)\right) \leq \frac{(p+1)\xi^2\sigma^2}{U}\left(\frac{9}{R^2Ux_0e^3}\right)^p. \tag{B.28}$$

Step 4. Let $U = 4/(eRx_0)$, $\gamma = \widetilde{C}_\gamma \exp\left(-\kappa N\ln\left(N+1\right)\right)$, $\widetilde{C}_\gamma = l(\pi e/2)^{p/2}\sqrt{(p+1)e\pi Rx_0/8}$, $N = \lceil \underline{N} \rceil$, $2(\kappa+k_q)\underline{N}\ln\left(\underline{N}+1\right) = \ln\left(C_\gamma^2 n\right)$, $C_\gamma = l(2\pi Re^3/9)^{p/2}\sqrt{\pi/2}/(\xi\sigma)$. Under such a choice, (B.27) and (B.28) hold for n large enough. Moreover, we have $r(n) = C_r\exp\left(-\kappa N\ln\left(N+1\right)\right)$, where $C_r = \widetilde{C}_\gamma\left(Rx_0/(2\pi)\right)^{p/2}\sqrt{\int_{U/2}^U |t|^p\,\lambda(t)^2dt/(4\pi)}$, $\underline{N}\ln\left(\underline{N}\right) \leq N\ln(N+1) \leq \left(\underline{N}+1\right)\ln\left(\underline{N}+2\right)$

$$\begin{aligned} N\ln(N+1) &\leq \underline{N}\ln\left(\underline{N}+1\right) + \ln\left(\underline{N}+1\right) + 1 + o(1) \\ &= \ln\left(\left(C_\gamma^2 n\right)^{1/(2\kappa+2k_q)}\right) + (1+o(1))\ln_2\left(\left(C_\gamma^2 n\right)^{1/(2\kappa+2k_q)}\right), \end{aligned}$$

indeed, using iteratively the definition of $\underline{N}$, $\ln\left(\underline{N}+1\right) = \ln\left(\underline{N}\right) + (1+o(1))/\underline{N}$ so $\ln\left(\underline{N}+1\right) = \ln\left(\underline{N}\right)\left(1+(1+o(1))/\ln\left(\left(C_\gamma^2 n\right)^{1/(2\kappa+2k_q)}\right)\right)$ and

$$\begin{aligned} \ln\left(\underline{N}\right) &= \ln_2\left(\left(C_\gamma^2 n\right)^{1/(2\kappa+2k_q)}\right) - \ln_2\left(\underline{N}+1\right) \\ &= \ln_2\left(\left(C_\gamma^2 n\right)^{1/(2\kappa+2k_q)}\right) - \ln_2\left(\underline{N}\right) + (1+o(1))/\ln\left(\left(C_\gamma^2 n\right)^{1/(2\kappa+2k_q)}\right) \\ &= \ln_2\left(\left(C_\gamma^2 n\right)^{1/(2\kappa+2k_q)}\right) - (1+o(1))\ln_3\left(\left(C_\gamma^2 n\right)^{1/(2\kappa+2k_q)}\right) \end{aligned}$$

so $1/\ln\left(\left(C_\gamma^2 n\right)^{(1+o(1))\kappa/(2\kappa+2k_q)}\right) \leq r(n)\left(C_\gamma^2 n\right)^{\kappa/(2\kappa+2k_q)}/C_r \leq 1$.

Proof of (T.1.2b). Let $U = 2/(eRx_0)$, $\gamma = \widetilde{C}_\gamma\exp\left(-\kappa N\right)$, $\widetilde{C}_\gamma = l(\pi e)^{p/2}\sqrt{(p+1)e\pi Rx_0/2}$, $N = \lceil \underline{N} \rceil$, $\underline{N} = \ln(n)/(2\kappa+2k_q)$, $C_\gamma = l\left(4\pi e/\left(\pi\left(1-e^{-2}\right)\right)\right)^{p/2}\sqrt{\pi/2}/(\xi\sigma)$. Under such a choice, $4\left(Rx_0U/(2\pi)\right)^p\left(2U\gamma^2/(p+1)\right)\left(\phi(U)\bigvee\exp\left(\kappa N\right)\right)^2 \leq \pi l^2$ and (A.23) hold for n large enough, hence steps 1 and 3. By Step 2, we have $r(n) = C_r\exp\left(-\kappa N\right) \geq C_r\exp\left(-\kappa\underline{N}\right)/e$. □

End of proofs of theorems 2, 3. We consider all smoothness cases when $q \in \{1,\infty\}$. We $\theta := 7e/(2Rx_0)$, $\theta_0 := \pi/(4Rx_0)$, $\theta_1 := 7e^2/(4Rx_0)$, $Q_q := 2^{k_q}\left(\mathbb{1}\{q=\infty\} + (p/2)^p/(p!q)\right)$, for

all $k,l\geq 0$, $N\geq 1$, $f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}_{w,W}^{q,\phi,\omega}(l,M)$,

$$(N+l)^k\leq((l+1)N)^k,\ \int_{\epsilon\leq|t|\leq T}L(t)dt\leq(2\pi)^{p+1}l^2. \tag{B.29}$$

End of proof of Theorem 2. Let $t\neq 0$ and $z>0$. (B.8) yields

$$\widetilde{\omega}_N^{q,W,x_0t}\leq 2^p\left(1\bigvee\frac{\theta(N+1)}{|t|}\right)^{2k_qN}. \tag{B.30}$$

This yields, for all $N\geq 1$, $\Delta_0^{W_{[-R,R]}}(t,N,n,z)\leq\Delta^{W_{[-R,R]}}(t,N,n,z)$, where

$$\Delta^{W_{[-R,R]}}(\star,N,n,z):=\left(1\bigvee\frac{\theta(N+1)}{|\star|}\right)^{2k_qN}\frac{2}{\pi^{p+1}}\left(\frac{Q_qc_{\boldsymbol{X}}N^p\,|\star|^p}{n}+z\left(L(t)+\frac{c_{\boldsymbol{X}}(N+1)^p\,|\star|^p}{n}\right)\right).$$

Let n_e be large enough to ensure $\underline{N}\geq(p+1)/k_q$. Using $N\leq\underline{N}$, $\epsilon\leq\theta\leq\theta(\underline{N}+1)$,

$$\begin{aligned}\int_\epsilon^T\left(1\bigvee\frac{\theta(\underline{N}+1)}{t}\right)^{2k_q\underline{N}}t^pdt&=(\theta(\underline{N}+1))^{2k_q\underline{N}}\int_\epsilon^{\theta(\underline{N}+1)}t^{p-2k_q\underline{N}}dt+\mathbb{1}\{\theta(\underline{N}+1)\leq T\}\int_{\theta(\underline{N}+1)}^T t^pdt\\&\leq\frac{\epsilon^{p+1}}{2k_q\underline{N}-p-1}\left(\frac{\theta(\underline{N}+1)}{\epsilon}\right)^{2k_q\underline{N}}+\frac{T^{p+1}}{p+1},\end{aligned} \tag{B.31}$$

$$\int_{\epsilon\leq|t|\leq T}\left(1\bigvee\frac{\theta(\underline{N}+1)}{|t|}\right)^{2k_q\underline{N}}L(t)dt\leq(2\pi)^{p+1}l^2\left(\frac{\theta(\underline{N}+1)}{\epsilon}\right)^{2k_q\underline{N}}\quad\text{(by (B.29))}, \tag{B.32}$$

$n_e/n\leq 1$, and $n_ev(n_0,\mathcal{E})/\delta(n_0)\leq 1$, we have

$$\begin{aligned}&\int_{\epsilon\leq|t|\leq T}\Delta^{W_{[-R,R]}}(t,N,n,Z_{n_0})dt\\&\leq\frac{4c_{\boldsymbol{X}}\underline{N}^p}{\pi^{p+1}n_e}\left(\frac{\epsilon^{p+1}}{k_q\underline{N}}\left(\frac{\theta(\underline{N}+1)}{\epsilon}\right)^{2k_q\underline{N}}+\frac{T^{p+1}}{p+1}\right)\left(Q_q+\frac{M_{\mathcal{E},\eta}2^p}{n}\right)+\frac{M_{\mathcal{E},\eta}2^{p+2}l^2}{n_e}\left(\frac{\theta(\underline{N}+1)}{\epsilon}\right)^{2k_q\underline{N}}\\&\leq\frac{\tau_0\underline{N}^{p-1}}{e^{2k_q}n_e}\left(\frac{\theta(\underline{N}+1)}{\epsilon}\right)^{2k_q\underline{N}}+\frac{\tau_1\underline{N}^{p-1}T^{p+1}}{n_e},\end{aligned}$$

$$\tau_0:=\frac{e^{2k_q}4c_{\boldsymbol{X}}\theta^{p+1}}{\pi^{p+1}k_q}\left(Q_q+e^2M_{\mathcal{E},\eta}2^p\right)+M_{\mathcal{E},\eta}2^{p+2}l^2,\ \tau_1:=\frac{4c_{\boldsymbol{X}}}{\pi^{p+1}(p+1)}\left(Q_q+M_{\mathcal{E},\eta}2^p\right).$$

(A.32), $(\underline{N}+1)^{2k_q\underline{N}}\leq e^{2k_q}\underline{N}^{2k_q\underline{N}}$, $\theta/\epsilon=K_{\underline{a}}(1)$, $N+1\geq\underline{N}$, and the definition of $\underline{a}$, yield

$$\mathcal{R}_{n_0,\sup}^W\leq C\left(\tau_0\underline{N}^{p-1}\frac{\left(\underline{N}K_{\underline{a}}(1)\right)^{2k_q\underline{N}}}{n_e}+\tau_1\underline{N}^{p-1}\frac{T^{p+1}}{n_e}+\frac{8l^2+M^2\mathbb{1}\{w\neq W_{[-\underline{a},\underline{a}]}\}}{\omega_{\underline{N}}^2}\right). \tag{B.33}$$

The choices of $\underline{N}$ are such that the first and third terms have the same and largest order.

Proof of (T2.1). Let $n_e\geq e^e$ be large enough so that $(\ln(n_e)/\tau_2)^{\sigma((p+1)/s+2)+p-1}\leq n_e$, where

$\tau_2 := 4k_q(2\sigma/\mu+1)\mathcal{W}(1/(4k_q(2\sigma/\mu+1)))$. We have

$$2k_q\underline{N}\ln\left(\underline{N}w^I\left(\omega_{\underline{N}}^2\right)\right)+\ln\left(\omega_{\underline{N}}^2\right)+(p-1)\ln(\underline{N}) = 2k_q\left(\frac{2\sigma}{\mu}+1\right)\underline{N}\ln\left(\underline{N}\right)+2\sigma\ln\left(\underline{N}\right)+(p-1)\ln(\underline{N})$$
$$\geq 2k_q\left(\frac{2\sigma}{\mu}+1\right)\underline{N}\ln\left(\underline{N}\right)$$

and, for all $x \geq 1/e$, $\mathcal{W}\left(x\ln(x)\right) = \ln(x)$. Using as well the definition of $\mathcal{W}$, this yields

$$\underline{N} \leq \frac{\ln(n_e)}{4k_q(2\sigma/\mu+1)\mathcal{W}(\ln(n_e)/(4k_q(2\sigma/\mu+1)))} \leq \frac{\ln(n_e)}{\tau_2}. \tag{B.34}$$

Using (B.34), we have $T^{p+1}\underline{N}^{p-1}/n_e = \underline{N}^{\sigma(p+1)/s+p-1}/n_e \leq (\ln(n_e)/\tau_2)^{\sigma(p+1)/s+p-1}/n_e \leq (\ln(n_e)/\tau_2)^{-2\sigma} \leq \underline{N}^{-2\sigma} = \omega_{\underline{N}}^{-2}$. Using the definition of $\underline{N}$ and (B.33), we obtain

$$\mathcal{R}_{n_0,\sup}^W \leq \frac{C}{\underline{N}^{2\sigma}}\left(\tau_0+\tau_1+8l^2+M^2\right). \tag{B.35}$$

We also have

$$2k_q\underline{N}\ln\left(\underline{N}w^I\left(\omega_{\underline{N}}^2\right)\right)+\ln\left(\omega_{\underline{N}}^2\right)+(p-1)\ln(\underline{N}) \leq \left(2k_q\left(\frac{2\sigma}{\mu}+1\right)+\frac{2\sigma+p-1}{p+1}\right)\underline{N}\ln(\underline{N}),$$

hence $\underline{N}\ln(\underline{N}) \geq \ln(n_e)/\tau_3$, $\tau_3 := 2k_q\left(2\sigma/\mu+1\right)+(2\sigma+p-1)/(p+1)$. Similarly to (B.34) and using for the second inequality, for all $x>0$, $\mathcal{W}(x) \leq \ln(x+1)$ (see Theorem 2.3 in [35]), we have

$$\underline{N} \geq \frac{\ln(n_e)}{\tau_3\mathcal{W}\left(\ln(n_e)/\tau_3\right)} \geq \frac{\ln(n_e)}{\tau_3\ln\left(\ln(n_e)+\tau_3\right)} \geq \frac{\ln(n_e)}{\tau_3\ln_2(n_e)\left(1+\ln(1+\tau_3/e)\right)}. \tag{B.36}$$

This and (B.35) yield the result.

Proof of (T2.2a). We have

$$2k_q\underline{N}\ln\left(\underline{N}w^I\left(\omega_{\underline{N}}^2\right)\right)+\ln\left(\omega_{\underline{N}}^2\right) \geq 2\left(k_q(\nu+1)+\kappa\right)\underline{N}\ln\left(\underline{N}\right), \tag{B.37}$$

hence $\underline{N}\ln\left(\underline{N}\right) \leq \left(\ln(n_e)-(p-1)\ln(\underline{N})\right)/\left(2\left(k_q(\nu+1)+\kappa\right)\right)$ and, using $\tau_4 := \kappa(p+1)/\left(2s\left(k_q(\nu+1)+\kappa\right)\right)$,

$$\frac{T^{p+1}\underline{N}^{p-1}}{n_e} = \frac{e^{\kappa(p+1)\underline{N}\ln(\underline{N}+1)/s}\underline{N}^{p-1}}{n_e} \leq e^{\kappa(p+1)/s}n_e^{\tau_4-1}\underline{N}^{(p-1)(1-\tau_4)}.$$

Because $s \geq \kappa(p+1)/(2k_q(1+\nu))$, we have $\tau_4-1 \leq -\kappa/\left(\kappa+k_q(1+\nu)\right)$ and

$$\mathcal{R}_{n_0,\sup}^W \leq C\left(\frac{\tau_1 e^{\kappa(p+1)/s}}{n_e^{\kappa/(\kappa+k_q(1+\nu))}}\underline{N}^{(p-1)(1-\tau_4)}+\frac{\tau_0+8l^2+M^2\mathbb{1}\{w\neq W_{[-\underline{a},\underline{a}]}\}}{e^{2\kappa\underline{N}\ln(\underline{N}+1)}}\right). \tag{B.38}$$

Using again $\tau_4-1 \leq -\kappa/\left(\kappa+k_q(1+\nu)\right)$, $\tau_5 := 2(k_q(\nu+1)+\kappa)\ln(2)$, and

$$\ln\left(n_e\right)-(p-1)\ln\left(\underline{N}\right) = 2k_q\underline{N}\ln\left(\underline{N}w^I\left(\omega_{\underline{N}}^2\right)\right)+\ln\left(\omega_{\underline{N}}^2\right) \leq \frac{\tau_5}{\ln(2)}\underline{N}\ln\left(\underline{N}+1\right), \tag{B.39}$$

we obtain $e^{2\kappa\underline{N}\ln(\underline{N}+1)} \geq n_e^{\kappa/(\kappa+k_q(1+\nu))}/\underline{N}^{\kappa(p-1)/(k_q(\nu+1)+\kappa)} \geq n_e^{\kappa/(\kappa+k_q(1+\nu))}/\underline{N}^{(p-1)(1-\tau_4)}$. We conclude because, by (B.37), $\underline{N} \leq \ln(n_e)/\tau_5$.

Proof of (T2.2b). It is derived from (B.38) with $w = W_{[-\underline{a},\underline{a}]}$ and $\nu = 0$.

Proof of (T2.3). By $\ln(n_e) \geq 2k_q \underline{N} \ln\left(\underline{N} w^I\left(\omega_{\underline{N}}^2\right)\right) + \ln\left(\omega_{\underline{N}}^2\right) \geq \ln\left(\omega_{\underline{N}}^2\right)$, we have

$$(\underline{N}\ln(\underline{N}+1))^r \leq \frac{\ln(n_e)}{2\kappa}, \tag{B.40}$$

hence, using the value of T and $\underline{N} \leq \underline{N}\ln(\underline{N}+1)/\ln(p+2)$,

$$\frac{T^{p+1}\underline{N}^{p-1}}{n_e} = \frac{\kappa^{p+1}(\underline{N}\ln(\underline{N}+1))^{r(p+1)}\underline{N}^{p-1}}{\gamma^{p+1}n_e} \leq \frac{\ln(n_e)^{p+1+(p-1)/r}}{\kappa^{(p-1)/r}2^{p+1+(p-1)/r}\gamma^{p+1}\ln(p+2)^{p-1}n_e}.$$

Moreover, because $\ln(n_e)^{p+1+(p-1)/r}$ is smaller than $\varphi(n_e)$ by definition,

$$\mathcal{R}_{n_0,\sup}^W \leq C\left(\frac{\tau_1\varphi(n_e)}{\kappa^{(p-1)/r}2^{p+1+(p-1)/r}\gamma^{p+1}\ln(p+2)^{p-1}n_e} + \frac{\tau_0+8l^2+M^2}{e^{2\kappa(\underline{N}\ln(1+\underline{N}))^r}}\right). \tag{B.41}$$

We also have

$$2k_q\underline{N}\ln\left(\underline{N}w^I\left(\omega_{\underline{N}}^2\right)\right) + \ln\left(\omega_{\underline{N}}^2\right) + (p-1)\ln(\underline{N}) \leq 2\kappa\left(\underline{N}\ln(\underline{N}+1)\right)^r(1+h(\underline{N})), \tag{B.42}$$

where $h := (k_q(1+\nu)\cdot + p - 1)\ln(\cdot)/(\kappa(\cdot\ln(\cdot+1))^r)$. This yields, for n_e large enough,

$$\exp\left(2\kappa\left(\underline{N}\ln(\underline{N}+1)\right)^r\right) \geq \exp\left(\frac{\ln(n_e)}{1+h(\underline{N})}\right) = n_e\exp\left(\sum_{k=1}^{\infty}(-1)^k h(\underline{N})^k\ln(n_e)\right). \tag{B.43}$$

By (B.42), we have $\underline{N}\ln(\underline{N}+1) \geq \ln(n_e)^{1/r}/d_0^{1/r}$. We obtain, by (B.40) for the second inequality,

$$h(\underline{N}) \leq \frac{(k_q+1)p/(p+1)+k_q-1+k_q\nu}{\kappa(\underline{N}\ln(\underline{N}+1))^{r-1}} \leq \frac{((k_q+1)p/(p+1)+k_q-1+k_q\nu)d_0^{1-1/r}}{\kappa\ln(n_e)^{1-1/r}},$$

$$h(\underline{N}) \geq \frac{k_q(1+\nu)}{\kappa(1+1/((p+1)\ln(p+1)))^r(\underline{N}\ln(\underline{N}))^{r-1}} \geq \frac{k_q(1+\nu)(2\kappa)^{1-1/r}}{\kappa(1+1/((p+1)\ln(p+1)))^r\ln(n_e)^{1-1/r}},$$

and we conclude using that, for n_e large enough so that the remainder below is smaller in absolute value than a converging geometric series,

$$\exp\left(\sum_{k=1}^{\infty}(-1)^k h(\underline{N})^k\ln(n_e)\right) \geq \exp\left(\sum_{k=1}^{k_0}(-1)^k d_k\ln(n_e)^{(1/r-1)k+1} + O(1)\right). \quad \square$$

End of proof of Theorem 3. Let $t \neq 0$ and $z > 0$. By (B.3), (B.5), and Proposition B.1 (ii), we have, for $q \in 1, \infty$ and $|t| \leq \theta_0$

$$\begin{aligned}\widetilde{\omega}_N^{q,W,x_0t} &\leq \left(\frac{e\pi}{2}\right)^{2p}\exp\left(2k_q\ln\left(\frac{\theta_1}{|t|}\right)N\right)\mathbb{1}\{|t|\leq\theta_0\} \\ &\quad + 2^p\exp\left(\frac{2k_q\theta_0(N+k_q')}{|t|}\right)\mathbb{1}\{|t|>\theta_0\}.\end{aligned} \tag{B.44}$$

For all $N \geq 1$, we have $\Delta_0^{\cosh(\cdot/R)}(t,N,n,z) \leq \Delta^{\cosh(\cdot/R)}(t,N,n,z)$, where $\Delta^{\cosh(\cdot/R)}(\star,N,n,z)$

$$:= \frac{2}{\pi}\left(\frac{\pi e^2}{8}\right)^p \left(\frac{2^{p/q} Q_q c_{\boldsymbol{X}} N^{(p-1)/q} |\star|^p}{n} + z\left(L(\star) + \frac{c_{\boldsymbol{X}}(N+1)^p |\star|^p}{n}\right)\right)\left(\frac{\theta_1}{|\star|}\right)^{2k_q N} \mathbb{1}\{|\star| \le \theta_0\}$$
$$+ \frac{2}{\pi^{p+1}}\left(\frac{2^{p/q} Q_q c_{\boldsymbol{X}} N^{(p-1)/q} |\star|^{p+k_q}}{(4\theta_0/e)^{k_q} n} + z\left(L(\star) + \frac{c_{\boldsymbol{X}}(N+1)^p |\star|^p}{n}\right)\right)\exp\left(\frac{2k_q\theta_0(N+k_q')}{|\star|}\right)\mathbb{1}\{|\star| > \theta_0\}.$$

Let $n_e \ge e^e$ be large enough so $\underline{N} \ge (p+2)/(2k_q)$. We have

(B.45)
$$\int_{\theta_0}^T t^{p+k_q} e^{2k_q\theta_0(\underline{N}+k_q')/t} dt = \int_{1/T}^{1/\theta_0} \frac{e^{2k_q\theta_0(\underline{N}+k_q')u}}{u^{p+2+k_q}} du \le \frac{\theta_0^{p+1+k_q} e^{2k_q(\underline{N}+k_q')}}{2k_q(\underline{N}+k_q')} \le \frac{\theta_0^{p+1+k_q} e^{2k_q(\underline{N}+k_q')}}{p+2(1+k_q k_q')}.$$

Then, using $N \le \underline{N}$, $n_e/n \le 1$, $n_e v(n_0,\mathcal{E})/\delta(n_0) \le 1$, and $\epsilon \le \theta_0$, for the first display,

$$\int_\epsilon^{\theta_0} t^p \left(\frac{\theta_1}{t}\right)^{2k_q\underline{N}} dt \le \frac{\epsilon^{p+1}}{2k_q\underline{N} - p - 1}\left(\frac{\theta_1}{\epsilon}\right)^{2k_q\underline{N}}, \tag{B.46}$$

$$\int_{\epsilon\le|t|\le\theta_0} \left(\frac{\theta_1}{|t|}\right)^{2k_q\underline{N}} L(t)dt + \int_{\theta_0\le|t|\le T} e^{2k_q\theta_0(\underline{N}+k_q')/|t|} L(t)dt$$
$$\le (2\pi)^{p+1} l^2 \left(\left(\frac{\theta_1}{\epsilon}\right)^{2k_q\underline{N}} \mathbb{1}\{\epsilon < \theta_0\} \bigvee e^{2k_q(\underline{N}+k_q')}\right),$$

and (B.45) for the second display, we obtain

$$\int_{\epsilon\le|t|\le T} \Delta^{\cosh(\cdot/R)}(t,N,n,Z_{n_0})$$
$$\le \frac{2^{2+p/q} Q_q c_{\boldsymbol{X}} \underline{N}^{(p-1)/q}}{\pi n_e}\left(\frac{\pi e^2}{8}\right)^p \int_\epsilon^{\theta_0} t^p \left(\frac{\theta_1}{t}\right)^{2k_q\underline{N}} dt$$
$$+ \frac{M_{\mathcal{E},\eta}}{n_e}\left(\left(\frac{\pi e^2}{8}\right)^p \bigvee \frac{1}{\pi^p}\right) 2^{p+2}\pi^p l^2 \left(\left(\frac{\theta_1}{\epsilon}\right)^{2k_q\underline{N}} \mathbb{1}\{\epsilon < \theta_0\} \bigvee e^{2k_q(\underline{N}+k_q')}\right)$$
$$+ \frac{4c_{\boldsymbol{X}}}{\pi n_e}\left(\frac{\pi e^2}{8}\right)^p \frac{2^p M_{\mathcal{E},\eta}\underline{N}^p}{n} \int_\epsilon^{\theta_0} t^p \left(\frac{\theta_1}{t}\right)^{2k_q\underline{N}} dt + \frac{2^{p/q} Q_q c_{\boldsymbol{X}} \underline{N}^{(p-1)/q} e^{k_q}}{\pi^{p+1} 4^{k_q-1}\theta_0^{k_q} n_e} \int_{\theta_0}^T t^{p+k_q} e^{2k_q\theta_0(\underline{N}+k_q')/t} dt$$
$$+ \frac{4c_{\boldsymbol{X}}}{\pi^{p+1} n_e} \frac{2^p M_{\mathcal{E},\eta}\underline{N}^p}{n} \int_{\theta_0}^T t^p e^{2k_q\theta_0(\underline{N}+k_q')/t} dt$$
$$\le G_1\left(\frac{\underline{N}^{k_q}}{n}\right)\frac{\underline{N}^{(p-1)/q}}{n_e}\left(\frac{\theta_1}{\epsilon}\right)^{2k_q\underline{N}} \mathbb{1}\{\epsilon < \theta_0\} + G_2\left(\frac{\underline{N}^{k_q}}{n}\right)\frac{\underline{N}^{(p-1)/q}}{n_e} e^{2k_q(\underline{N}+k_q')}, \tag{B.47}$$

$$G := 4\left(\frac{2k_q}{p+2}\right)^{(p-1)/q} M_{\mathcal{E},\eta}\left(\frac{\pi e^2}{4}\right)^p \pi^p l^2,\ G_1 := \frac{4c_{\boldsymbol{X}}\theta_0^{p+1}}{\pi}\left(2^{p/q}Q_q + 2^p M_{\mathcal{E},\eta}\cdot\right)\left(\frac{\pi e^2}{8}\right)^p + G,$$

$$G_2 := \frac{4c_{\boldsymbol{X}}\theta_0^{p+1+k_q}}{(p+2(1+k_qk_q'))\pi^{p+1}}\left(2^{p/q}Q_q\left(\frac{e}{4\theta_0}\right)^{k_q}+2^pM_{\mathcal{E},\eta^\cdot}\right)+G.$$

Proof of (T3.1). We have, using $K_{\underline{a}}(e)=w^I\left(\omega_{\underline{N}}^2\right)$,

$$\ln(n_e)-\frac{p-1}{q}\ln(\underline{N})=2k_q\underline{N}\ln\left(w^I\left(\omega_{\underline{N}}^2\right)\right)+\ln\left(\omega_{\underline{N}}^2\right)=\frac{4k_q\sigma}{\mu}\underline{N}\ln\left(\underline{N}\right)+2\sigma\ln\left(\underline{N}\right)\geq\frac{4k_q\sigma}{\mu}\underline{N}\ln\left(\underline{N}\right),$$

hence, for n_e large enough so that $\ln(\underline{N})\geq\mu/(k_q\sigma)$,

$$\underline{N}\leq\frac{\ln(n_e)}{4k_q\sigma\ln(\underline{N})/\mu}\leq\frac{\ln(n_e)}{4k_q}. \tag{B.48}$$

Thus, using (A.1), we have $\underline{N}/n\leq 1/(4k_qe)$. This yields, using (B.47), (A.32), $\theta_1/\epsilon=w^I\left(\omega_{\underline{N}}^2\right)$, $N+1\geq\underline{N}$, and the definition of $\underline{a}$,

$$\mathcal{R}^W_{n_0,\sup}\leq C\left(G_1\left(\frac{k_q^{k_q-1}}{4e^{k_q}}\right)\underline{N}^{p-1}\frac{\left(w^I\left(\omega_{\underline{N}}^2\right)\right)^{2k_q\underline{N}}}{n_e}+G_2\left(\frac{k_q^{k_q-1}}{4e^{k_q}}\right)\underline{N}^{p-1}\frac{e^{2k_q(\underline{N}+k_q')}}{n_e}+\frac{8l^2+M^2}{\omega_{\underline{N}}^2}\right).$$

By (B.48), we obtain $\omega_{\underline{N}}^2\underline{N}^{(p-1)/q}e^{2k_q(\underline{N}+k_q')}/n_e\leq\ln(n_e)^{2\sigma+(p-1)/q}e^{2k_qk_q'}/(4^{2\sigma+p-1}\sqrt{n_e})$. Thus, using (A.1), we have $\underline{N}^{(p-1)/q}e^{2k_q(\underline{N}+k_q')}/n_e\leq e^{2k_qk_q'}((2\sigma+(p-1)/q)/(2e))^{2\sigma+(p-1)/q}/\omega_{\underline{N}}^2$ and using the definition of $\underline{N}$,

$$\mathcal{R}^W_{n_0,\sup}\leq\frac{C}{\underline{N}^{2\sigma}}\left(G_1\left(\frac{k_q^{k_q-1}}{4e^{k_q}}\right)+G_2\left(\frac{k_q^{k_q-1}}{4e^{k_q}}\right)e^{2k_qk_q'}\left(\frac{2\sigma+(p-1)/q}{2e}\right)^{2\sigma+(p-1)/q}+8l^2+M^2\right). \tag{B.49}$$

We also have

$$\ln(n_e)=2k_q\underline{N}\ln\left(w^I\left(\omega_{\underline{N}}^2\right)\right)+\ln\left(\omega_{\underline{N}}^2\right)+\frac{p-1}{q}\ln(\underline{N})\leq\left(2\left(\frac{2k_q}{\mu}+1\right)\sigma+\frac{2(p-1)}{q(p+2)}\right)\underline{N}\ln(\underline{N}),$$

hence $\underline{N}\ln(\underline{N})\geq\ln(n_e)/\tau_6$, $\tau_6:=2\left(2k_q/\mu+1\right)\sigma+2(p-1)/(q(p+2))$. Similarly to (B.36), we have $\underline{N}\geq\ln(n_e)/(\tau_6\left(1+\ln(1+\tau_6/e)\right)\ln_2(n_e))$, which yields the result with (B.49).

Proof of (T3.2). Because $K_{\underline{a}}(e)=1$ then $2(k_q+\kappa)\underline{N}+(p-1)\ln(\underline{N})\geq 2(k_q+\kappa)\underline{N}$, we obtain $\underline{N}\leq\ln(n_e)/(2(k_q+\kappa))$. Thus using $n\geq n_e$ and (A.1), we have $G_2(\underline{N}^{k_q}/n)\leq G_2(k_q^{k_q}/(2(k_q+\kappa)e)^{k_q})$. Using (A.32), $w=W_{\mathcal{A}}$, (B.47), $\epsilon=\theta_0$, yield

$$\mathcal{R}^W_{n_0,\sup}\leq C\left(G_2\left(\frac{k_q^{k_q}}{2(k_q+\kappa)e^{k_q}}\right)\underline{N}^{(p-1)/q}\frac{e^{2k_q(\underline{N}+k_q')}}{n_e}+\frac{8l^2}{\omega_{\underline{N}}^2}\right).$$

We conclude using the definition of $\underline{N}$, which yields $\underline{N}^{(p-1)/q}e^{2k_q\underline{N}}/n_e=n_e^{-\kappa/(k_q+\kappa)}\underline{N}^{\kappa(p-1)/(q(1+\kappa))}$ and $\omega_{\underline{N}}^{-2}=e^{-2\kappa\underline{N}}=n_e^{-\kappa/(k_q+\kappa)}\underline{N}^{\kappa(p-1)/(q(k_q+\kappa))}$. □

Proof of Lemma A.6. Let $t\in[-T,T]\setminus(-\epsilon,\epsilon)$, $N\in\{0,\dots,N^W_{\max,q}\}$, $T\in\mathcal{T}_n$. Using, for all $N_1,N_2\in\mathbb{N}_0$, $\widetilde{R}^{N_2}_{N_1}(t,\cdot_2):=\left(\widehat{F}_1^{q,N_1,T,0}-\widehat{F}_1^{q,N_2\vee N_1,T,0}\right)(t,\cdot_2)$, we have $\mathcal{L}_q^W\left(t,\widehat{N}(t),T\right)=$

$\widetilde{R}^N_{\widehat{N}(t)} - \widetilde{R}^{\widehat{N}(t)}_N + \mathcal{L}^W_q(t,N,T)$. Using (A.2) yields

$$\mathbb{E}\left[\mathcal{L}^W_q\left(t,\widehat{N}(t),T\right)\right] \leq (2+\sqrt{5})\left(\mathbb{E}\left[\left\|\widetilde{R}^N_{\widehat{N}(t)}(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})}\right] + \mathbb{E}\left[\left\|\widetilde{R}^{\widehat{N}(t)}_N(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})}\right]\right) + \left(1+\frac{2}{\sqrt{5}}\right)\mathbb{E}\left[\mathcal{L}^W_q(t,N,T)\right].$$

Because $B_1(t,N) = \max_{N'\in\mathbb{N}_0:\ N'\leq N^W_{\max,q}} \left(\sum_{N\leq|\boldsymbol{m}|_q\leq N'\vee N}\left(|\widehat{c}_{\boldsymbol{m}}(t)|/\sigma^{W,x_0t}_{\boldsymbol{m}}\right)^2 - \Sigma(t,N')\right)_+$, we have $\mathbb{E}\left[\left\|\widetilde{R}^{N_2}_{N_1}(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})}\right] \leq \mathbb{E}\left[B_1(t,N_1)\right] + \mathbb{E}\left[\Sigma(t,N_2)\right]$ for possibly random N_1 and N_2. Using $c_1 \geq 1 + 1/(2+\sqrt{5})^2$ and (16) yield

$$\begin{aligned}&\mathbb{E}\left[\mathcal{L}^W_q\left(t,\widehat{N}(t),T\right)\right] + \frac{1}{2+\sqrt{5}}\mathbb{E}\left[\Sigma\left(t,\widehat{N}(t)\right)\right]\\ &\leq (2+\sqrt{5})\left(2\mathbb{E}\left[B_1(t,N)\right] + (1+c_1)\mathbb{E}\left[\Sigma(t,N)\right]\right) + \left(1+\frac{2}{\sqrt{5}}\right)\mathbb{E}\left[\mathcal{L}^W_q(t,N,T)\right].\end{aligned}$$

By (A.2) and, for all $N'\in\mathcal{N}_n$, $\widetilde{R}^{N'}_{N,1}(t,\cdot_2) := \left(\widehat{F}^{q,N\vee N',T,0}_1 - F^{q,N\vee N',T,0}_1\right)(t,\cdot_2)$, $\widetilde{R}^{N'}_{N,2}(t,\cdot_2) := \left(F^{q,N,T,0}_1 - \widehat{F}^{q,N,T,0}_1\right)(t,\cdot_2)$, and $\widetilde{R}^{N'}_{N,3}(t,\cdot_2) := \left(F^{q,N\vee N',T,0}_1 - F^{q,N,T,0}_1\right)(t,\cdot_2)$, we have

$$B_1(t,N) \leq \max_{\substack{0\leq N'\leq N^W_{\max,q}\\ N'\in\mathbb{N}_0}}\left((2+\sqrt{5})\sum_{j=1}^2\left\|\widetilde{R}^{N'}_{N,j}(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})} + \left(1+\frac{2}{\sqrt{5}}\right)\left\|\widetilde{R}^{N'}_{N,3}(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})} - \Sigma(t,N')\right)_+.$$

Using $F^{q,\infty,T,0}_1(t,\cdot) = \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\beta}\right](t,\cdot)$, we have

$$\left\|\widetilde{R}^{N'}_{N,3}(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})} = \sum_{N<|\boldsymbol{m}|_q\leq N\vee N'}\left|\frac{c_{\boldsymbol{m}}(t)}{\sigma^{W,x_0t}_{\boldsymbol{m}}}\right|^2 \leq \left\|\left(F^{q,N,T,0}_1 - \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\beta}\right]\right)(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})},$$

$$\begin{aligned}B_1(t,N) \leq \max_{\substack{0\leq N'\leq N^W_{\max,q}\\ N'\in\mathbb{N}_0}}&\left(2(2+\sqrt{5})\left\|\left(F^{q,N',T,0}_1 - \widehat{F}^{q,N',T,0}_1\right)(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})} - \Sigma(t,N')\right)_+\\ &+\left(1+\frac{2}{\sqrt{5}}\right)\left\|\left(F^{q,N,T,0}_1 - \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})}.\end{aligned}$$

Finally, we have

$$\begin{aligned}&\mathbb{E}\left[\mathcal{L}^W_q\left(t,\widehat{N}(t),T\right)\right] + \frac{1}{2+\sqrt{5}}\mathbb{E}\left[\Sigma\left(t,\widehat{N}(t)\right)\right]\\ &\leq 4(2+\sqrt{5})^2\mathbb{E}\left[\max_{0\leq N'\leq N^W_{\max,q}}\left(\left\|\left(\widehat{F}^{q,N',T,0}_1 - F^{q,N',T,0}_1\right)(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})} - \frac{\Sigma(t,N')}{2(2+\sqrt{5})}\right)_+\right]\\ &+(2+\sqrt{5})(1+c_1)\mathbb{E}\left[\Sigma(t,N)\right] + \left(2+\sqrt{5}\right)^2\mathbb{E}\left[\mathcal{L}^W_q(t,N,T)\right].\end{aligned}$$

Using (A.33) for the second display and Lemma A.4 for the third, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\max_{0\le N'\le N^W_{\max,q}}\left\|\left(\widehat{F}_1^{q,N',T,0}-F_1^{q,N',T,0}\right)(t,\cdot_2)\right\|^2_{L^2(W^{\otimes p})}-\frac{\Sigma(t,N')}{2(2+\sqrt{5})}\right]\\
&=\mathbb{E}\left[\max_{0\le N'\le N^W_{\max,q}}\left(\sum_{|\boldsymbol{m}|_q\le N'}\left(\frac{|\widehat{c}_{\boldsymbol{m}}(t)-c_{\boldsymbol{m}}(t)|}{\sigma_{\boldsymbol{m}}^{W,x_0t}}\right)^2-\frac{\Sigma(t,N')}{2(2+\sqrt{5})}\right)_+\right]\\
&\le\left(1+2\sqrt{5}\right)\mathbb{E}\left[\max_{0\le N'\le N^W_{\max,q}}\left(S_3(N',t)-\frac{\Sigma(t,N')}{2(2+\sqrt{5})}\right)_+\right]\\
&\quad+\left(2+\frac{1}{\sqrt{5}}\right)\mathbb{E}\left[\max_{0\le N'\le N^W_{\max,q}}\left(S_1(N',t)+S_2(N',t)\right)\right]\\
&\le\left(N^W_{\max,q}+1\right)\left(1+2\sqrt{5}\right)48\frac{c_{\boldsymbol{X}}}{n}\left(\frac{|t|}{2\pi}\right)^p\nu_q^W\left(x_0t,N^W_{\max,q}\right)\Psi_{0,n}(t)+Z_{n_0}\Psi_n(t).
\end{aligned}$$

Hence the result. □

Hereafter, let $(n,n_0)\in\mathbb{N}^2$ such that $v(n_0,\mathcal{E})/\delta(n_0)\le n^{-2}\ln(n)^{-p}$, $n\ge e^{7e^2/(2\pi)}$ large enough so that $N^W_{\max,q}\ge(p+1)/k_q$. Using $\theta_1/\ln(n)\le 2\theta_0$, let $(\theta\mathbb{1}\{W=W_{[-R,R]}\}+\theta_1\mathbb{1}\{W=\cosh(\cdot/R)\})/\ln(n)\le\epsilon\le\theta\mathbb{1}\{W=W_{[-R,R]}\}+2\theta_0\mathbb{1}\{W=\cosh(\cdot/R)\}$. Using the definition of $N^W_{\max,q}$ yields $N^W_{\max,q}\le\underline{N^W_{\max,q}}$ and $\epsilon\le\theta$ yields $\underline{N^{W_{[-R,R]}}_{\max,q}}\ln\left(\underline{N^{W_{[-R,R]}}_{\max,q}}\right)\le\ln(n)/(2k_q)$. Then, using that, for all $x\ge 1/e$, $\mathcal{W}(x\ln(x))=\ln(x)$, and the definition of $\mathcal{W}$ for the bound on $N^{W_{[-R,R]}}_{\max,q}$ and else the definition of $N^{\cosh(\cdot/R)}_{\max,q}$, we have, for all $t\neq 0$,

$$N^W_{\max,q}\le\frac{\ln(n)}{\tau_7},\ \tau_7:=2k_q\mathcal{W}\left(\frac{7e^2}{4\pi k_q}\right)\mathbb{1}\{W=W_{[-R,R]}\}+2k_q\mathbb{1}\{W=\cosh(\cdot/R)\}. \tag{B.50}$$

Using, for all $|t|\ge\epsilon$ and $N\ge 1$, $2k_qN\ln(\theta(N+1)/|t|)\le 2k_qN\ln(\theta N/\epsilon)+2k_q$, we have $\left(\theta\left(N^{W_{[-R,R]}}_{\max,q}+1\right)/|t|\right)^{2k_qN^{W_{[-R,R]}}_{\max,q}}\le e^{2k_q}\left(\theta\underline{N^{W_{[-R,R]}}_{\max,q}}/\epsilon\right)^{2k_q\underline{N^{W_{[-R,R]}}_{\max,q}}}$. (B.30) and the definition of $N^{W_{[-R,R]}}_{\max,q}$ yield

$$\forall|t|\ge\epsilon,\ \widetilde{\omega}^{q,W_{[-R,R]},x_0t}_{N^W_{\max,q}}\le 2^pe^{2k_q}n. \tag{B.51}$$

Lemma B.13. For all $q\in\{1,\infty\}$, $\epsilon\le T_{\max}\le n^{\zeta_0}$, and $W\in\{W_{[-R,R]},\cosh(\cdot/R)\}$, we have

$$\begin{aligned}
&\int_\epsilon^{T_{\max}}t^p\nu_q^W\left(x_0t,N^W_{\max,q}\right)dt\le A_3^{W,q}\ln(n)^{a_0}n,\\
&a_0:=p\mathbb{1}\{W=W_{[-R,R]}\}+\frac{p-1}{q}\mathbb{1}\{W=\cosh(\cdot/R)\},\\
&A_3^{W_{[-R,R]},q}:=\frac{2^pQ_q}{\tau_7^p(p+1)}\left(\frac{\theta^{p+1}e^{2k_q}}{k_q}+e^{(1-\zeta_0(p+1))7e^2/(2\pi)}\right),
\end{aligned}$$

$$A_3^{\cosh(\cdot/R),q} := \frac{2^{1/q}Q_q}{k_q^{(p-1)/q}}\left(\left(\frac{e\pi}{2}\right)^{2p}\mathbb{1}\{\epsilon<\theta_0\}\frac{\theta_0^{p+1}}{p+1}+2^p\left(\frac{e}{4}\right)^{k_q}\frac{\theta_0^{p+1}e^{2k_qk_q'}}{p+2(1+k_qk_q')}\right).$$

Proof. Let $W = W_{[-R,R]}$. Using the definition of $\nu_q^{W_{[-R,R]}}$, $N_{\max,q}^W \geq 1$, and (B.29) for the first inequality, (B.31) and (B.50) for the second, and the arguments which yield (B.51) for the last inequality, the result follows from

$$\begin{aligned}\int_\epsilon^{T_{\max}} t^p\nu_q^W\left(x_0t,N_{\max,q}^W\right)dt &\leq 2^pQ_q\left(\underline{N_{\max,q}^W}\right)^p\int_\epsilon^{T_{\max}}t^p\left(1\bigvee\left(\frac{\theta\left(\underline{N_{\max,q}^W}+1\right)}{|t|}\right)\right)^{2k_q\underline{N_{\max,q}^W}}dt\\ &\leq \frac{2^pQ_q\ln(n)^p}{\tau_7^p}\left(\frac{\epsilon^{p+1}}{k_q\underline{N_{\max,q}^W}}\left(\frac{\theta(\underline{N_{\max,q}^W}+1)}{\epsilon}\right)^{2k_q\underline{N_{\max,q}^W}}+\frac{T_{\max}^{p+1}}{p+1}\right)\\ &\leq \frac{2^pQ_q\ln(n)^p}{\tau_7^p(p+1)}\left(\frac{\theta^{p+1}e^{2k_q}}{k_q}+\frac{1}{n^{(p+1)\zeta_0-1}}\right)n.\end{aligned}$$

Let now $W=\cosh(\cdot/R)$. Using the definition of ν_q^W, we have for all $N\geq 1$ and $t\neq 0$,

$$\begin{aligned}\nu_q^{\cosh(\cdot/R)}\left(x_0t,N\right) =&2^{p/q}Q_qN^{(p-1)/q}\left(\frac{e\pi}{2}\right)^{2p}\left(\frac{\theta_1}{|t|}\right)^{2k_qN}\mathbb{1}\{|t|\leq\theta_0\}\\ &+2^{p(1+1/q)}Q_qN^{(p-1)/q}\left(\frac{e|t|}{4\theta_0}\right)^{k_q}\exp\left(2\theta_0k_q(N+k_q')\right)\mathbb{1}\{|t|>\theta_0\}.\end{aligned}$$

Because of (B.50), we have, for $t\neq 0$, $e^{2k_q\underline{N_{\max,q}^{\cosh(\cdot/R)}}}\leq n$. By definition of $\underline{N_{\max,q}^{\cosh(\cdot/R)}}$, when $\epsilon<\theta_0$, we also have, for $|t|\leq\theta_0$, $(\theta_1/\epsilon)^{2k_q\underline{N_{\max,q}^{\cosh(\cdot/R)}}}\leq n$. Then, using (B.50) for the first display and using (B.45) and (B.46) for the second, the result follows from

$$\begin{aligned}\int_\epsilon^{T_{\max}} t^p\nu_q^W\left(x_0t,N_{\max,q}^W\right)dt \leq&2^{p/q}Q_q\left(\frac{\ln(n)}{2k_q}\right)^{(p-1)/q}\left(\frac{e\pi}{2}\right)^{2p}\mathbb{1}\{\epsilon<\theta_0\}\int_\epsilon^{\theta_0}t^p\left(\frac{\theta_1}{|t|}\right)^{2k_q\underline{N_{\max,q}^W}}dt\\ &+Q_q\left(\frac{\ln(n)}{2k_q}\right)^{(p-1)/q}\frac{2^{p(1+1/q)}e^{k_q}}{(4\theta_0)^{k_q}}\int_{\theta_0}^{T_{\max}}t^{p+k_q}e^{2\theta_0k_q\left(\underline{N_{\max,q}^W}+k_q'\right)}dt\\ \leq&2^{p/q}Q_q\left(\frac{\ln(n)}{2k_q}\right)^{(p-1)/q}\left(\frac{e\pi}{2}\right)^{2p}\mathbb{1}\{\epsilon<\theta_0\}\frac{\theta_0^{p+1}}{p+1}\left(\frac{\theta_1}{\epsilon}\right)^{2k_q\underline{N_{\max,q}^W}}\\ &+Q_q\left(\frac{\ln(n)}{2k_q}\right)^{(p-1)/q}\frac{2^{p(1+1/q)}e^{k_q}}{(4\theta_0)^{k_q}}\frac{\theta_0^{p+1+k_q}e^{2k_qk_q'}}{p+2(1+k_qk_q')}e^{2k_q\underline{N_{\max,q}^W}}\\ \leq&A_3^{\cosh(\cdot/R),q}\ln(n)^{(p-1)/q}n.\quad\square\end{aligned}$$

Lemma B.14. For all $q \in \{1, \infty\}$, $W \in \{W_{[-R,R]}, \cosh(\cdot/R)\}$, and $(\epsilon \vee 1) \leq T_{\max} \leq n^{\zeta_0}$, we have $\Pi(n, Z_{n_0}, T_{\max}, N_{\max,q}) \leq (A_0 + A_1)/n$, where

$$A_0 := \frac{M_{\mathcal{E},\eta}}{n}\left(2 + \frac{1}{\sqrt{5}}\right)\left(\left(\frac{4\pi^2}{\theta x_0}\right)^p b_0 (2\pi) l^2 + \frac{2c_{\boldsymbol{X}} A_3^{W,q}}{(2\pi)^p e^{7e^2/(2\pi)}}\right),$$

$$b_0 := 2^p e^{2k_q}\mathbb{1}\{W = W_{[-R,R]}\} + \left(\frac{e\pi}{2}\right)^{2p}\mathbb{1}\{W = \cosh(\cdot/R)\},$$

$$A_1 := \frac{96\left(1 + 2\sqrt{5}\right) c_{\boldsymbol{X}} \zeta_0 A_3^{W,q}}{(2\pi)^p k_q \ln(2)\left(1/\tau_7 + \pi/(7e^2)\right)^p}\left(\left(\frac{a_0 + 2}{e\zeta_0}\right)^{a_0+2} + \frac{e^{1/b_1} 294 c_{\boldsymbol{X}} a_1^2}{x_0^p}\left(\frac{a_0 + 2(p+1)}{e}\right)^{a_0+2(p+1)}\right),$$

$$a_1 := \left(\frac{1}{\tau_7} + \frac{\pi}{7e^2}\right)^p \left(H_0\mathbb{1}\{W = W_{[-R,R]}\} + H_1\mathbb{1}\{W = \cosh(\cdot/R)\}\right)^p \left(1 + x_0^2\right)^p,$$

$$b_1 := \frac{L\sqrt{p_2}}{a_1}\left(\frac{e(1 - 4(1 + \mathbb{1}\{W = \cosh(\cdot/R)\})p\zeta_0)}{2p + 3}\right)^{p+3/2}.$$

Proof. Let us show

$$Z_{n_0}\int_{\epsilon \leq |t| \leq T_{\max}} \Psi_n(t)dt \leq \frac{A_0}{n} \quad \text{and} \quad \Pi_1(n, T_{\max}, N_{\max,q}) \leq \frac{A_1}{n}. \tag{B.52}$$

Let $W = W_{[-R,R]}$. Using for the second display $v(n_0, \mathcal{E})/\delta(n_0) \leq n^{-2}\ln(n)^{-p}$, (B.51), $\epsilon \geq \theta/\ln(n)$, (B.29), and Lemma B.13, we obtain

$$\begin{aligned}
Z_{n_0}\int_{\epsilon \leq |t| \leq T_{\max}} \Psi_n(t)dt =& Z_{n_0}\left(2 + \frac{1}{\sqrt{5}}\right)\left(\frac{2\pi}{x_0}\right)^p \int_{\epsilon \leq |t| \leq T_{\max}} \widetilde{\omega}_{N_{\max,q}^W}^{q,W,x_0 t} |t|^{-p} \left\|\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t, \cdot_2)\right\|_{L^2(\mathbb{R}^p)}^2 dt \\
&+ Z_{n_0}\left(2 + \frac{1}{\sqrt{5}}\right)\frac{c_{\boldsymbol{X}}}{(2\pi)^p n}\int_{\epsilon \leq |t| \leq T_{\max}} |t|^p \nu_q^W(x_0 t, N_{\max,q}^W)dt \\
\leq& \frac{M_{\mathcal{E},\eta}}{n^2 \ln(n)^p}\left(2 + \frac{1}{\sqrt{5}}\right)\left(\left(\frac{4\pi}{\theta x_0}\right)^p e^{2k_q} n \ln(n)^p (2\pi)^{p+1} l^2 + \frac{2c_{\boldsymbol{X}} A_3^{W,q}\ln(n)^p}{(2\pi)^p}\right).
\end{aligned}$$

Using $n \geq e^{7e^2/(2\pi)}$ and (A.1) yield the first inequality in (B.52). Similarly, by definition of $N_{\max,q}^{\cosh(\cdot/R)}$ and (B.44), we have $\widetilde{\omega}_{N_{\max,q}^W}^{q,\cosh(\cdot/R),x_0 t} \leq (e\pi/2)^{2p} n$. This and (B.50) yield the first inequality in (B.52) for the other instances of W and q.
By (B.50), we have

$$K_n(t) \leq \left(\frac{\ln(n)}{\tau_7} + \frac{1}{2}\right)^p T_{\max}^{2p} H_0^p \left(\frac{1}{T_{\max}^2} + x_0^2\right)^p \leq a_1 \ln(n)^p T_{\max}^{2p}. \tag{B.53}$$

We obtain, using $T_{\max} \leq n^{\zeta_0}$ for the third inequality and (A.1) for the fourth,

$$\frac{L\sqrt{p_n n}}{K_n(t)} \geq \frac{L\sqrt{p_2 \ln(n) n}}{a_1 \ln(n)^p T_{\max}^{2p}} \geq \frac{L\sqrt{p_2} n^{(1-4\zeta_0 p)/2}}{a_1 \ln(n)^{p-1/2}} \geq b_1 \ln(n)^2.$$

Using (B.50) for the first inequality, Lemma B.13 for the second, and using the definition of p_n, $6(1+\zeta_0)\ln(n) > 3$, (B.53), and $T_{\max}^{4p} \le n^{4p\zeta_0}$, for the third, we have

$$\begin{aligned}
&\Pi_1(n, T_{\max}, N_{\max,q}) \\
&\le \frac{96\left(1+2\sqrt{5}\right) c_{\boldsymbol{X}} \zeta_0 \ln(n)^2}{(2\pi)^p k_q \ln(2) n} \left(\frac{1}{\tau_7} + \frac{\pi}{7e^2}\right)^p \int_\epsilon^{T_{\max}} t^p \nu_q^W\left(x_0 t, N_{\max,q}^W\right) dt \sup_{\epsilon \le t \le T_{\max}} \Psi_{0,n}(t) \\
&\le \frac{96\left(1+2\sqrt{5}\right) c_{\boldsymbol{X}} \zeta_0 A_3^{W,q}}{(2\pi)^p k_q \ln(2) n} \left(\frac{1}{\tau_7} + \frac{\pi}{7e^2}\right)^p \sup_{\epsilon \le t \le T_{\max}} \ln(n)^{p+2} n \Psi_{0,n}(t) \\
&\le \frac{96\left(1+2\sqrt{5}\right) c_{\boldsymbol{X}} \zeta_0 A_3^{W,q}}{(2\pi)^p k_q \ln(2) n} \left(\frac{1}{\tau_7} + \frac{\pi}{7e^2}\right)^p \left(\frac{\ln(n)^{p+2}}{n^{\zeta_0}} + \sup_{n>0}\left(e^{-b_1 \ln(n)^2} n^2\right) \frac{294 c_{\boldsymbol{X}} a_1^2}{x_0^p} \frac{\ln(n)^{3p+2}}{n^{2(1-2p\zeta_0)}}\right).
\end{aligned}$$

Thus, (A.1), $1-2p\zeta_0 > 1/2$, $\sup_{x>0}\left(e^{-b_1\ln(x)^2}x^2\right) = e^{1/b_1}$ yield the second inequality in (B.52). We obtain similarly the bounds for the other instances of W and q. □

End of proof of Theorem 4. We start from (A.38) and use $A_4 := 2(2+\sqrt{5})^2 C\left(1+\left(2+\sqrt{5}\right)^2\right)(A_0+A_1)/\pi$, $T^* := 2^{k^*}$, $k^* := \lfloor \ln(\phi^I(\omega_{\underline{N}^*}))/\ln(2) \rfloor$, $N^*(t) := \lfloor \underline{N}^* \rfloor$, where $\underline{N}^*$ is defined below, and

$$\mathcal{R}_{n_0,\sup}^{W,\mathrm{adp}} := \sup_{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}_{w,W}^{q,\phi,\omega}(l,M) \cap \mathcal{D},\ f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}} \mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,\widehat{N},\widehat{T},\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right).$$

We have, for all $|t| \ge \epsilon$, $W \in \{W_{[-R,R]}, \cosh(\cdot/R)\}$, and $N \ge 1$, $2\widetilde{\Delta}_0^W(t,N,n,Z_{n_0})/(\pi(2\pi)^p) \le \widetilde{\Delta}^W(t,N,n,Z_{n_0})$ where $\widetilde{\Delta}^W$ is defined like Δ^W replacing Q_q by $Q_{q,n} := Q_q\left(1+2(1+2p_n)(1+c_1)\right)$. Thus, by (A.38), we obtain, for all $W \in \{W_{[-R,R]}, \cosh(\cdot/R)\}$,

$$\begin{aligned}
\mathcal{R}_{n_0,\sup}^{W,\mathrm{adp}} \le& C\left(2+\sqrt{5}\right)^4 \int_{\epsilon \le |t| \le T} \widetilde{\Delta}^W\left(t, N(t), n, \frac{M_{\mathcal{E},\eta} v(n_0,\mathcal{E})}{\delta(n_0)}\right) dt \\
&+ C\left(2+\sqrt{5}\right)^4 \left(\sup_{t\in[-T,T]} \frac{4l^2}{\omega_{N(t)+1}^2} + \frac{4l^2}{\phi(T)^2}\right) + CM^2 \widetilde{w}(\underline{a}) + \frac{A_4}{n}. \qquad \text{(B.54)}
\end{aligned}$$

Proof of (T4.1a). Let $\underline{N}^*$ solution of

$$2k_q \underline{N}^* \ln\left(\underline{N}^* \ln(n)\right) + \ln\left(\omega_{\underline{N}^*}^2\right) + (p-1)\ln\left(\underline{N}^*\right) + \ln_2(n) = \ln(n), \qquad \text{(B.55)}$$

$n \ge e^{7e^2/(2\pi)}$ large enough so $N^* \ge 1$, and $(\ln(n)/\tau_2')^{\sigma/s} \le n^{\zeta_0}/2$, where $\tau_2' := 2k_q \mathcal{W}(e/(2k_q))$. By (B.55) and the definition of $\underline{N}_{\max,q}^W$, we have $N^* \le \underline{N}_{\max,q}^W$ for all $t \in \mathbb{R} \setminus (-\epsilon, \epsilon)$, hence $N^* \in \mathcal{N}_n$. Also $T^* \in \mathcal{T}_n$ because, by the arguments in the proof of (T2.1),

$$\underline{N}^* \le \frac{\ln(n)}{2k_q \mathcal{W}(\ln(n)/(2k_q))} \le \frac{\ln(n)}{\tau_2'},$$

so we have $T^* \leq (\ln(n)/\tau_2')^{\sigma/s} \leq n^{\zeta_0}/2 \leq T_{\max}$. (B.31), (B.32), and $p_n = 6\,(1+\zeta_0)\ln(n)$, yield

$$\mathcal{R}^{W,\mathrm{adp}}_{n_0,\sup} \leq C\left(2+\sqrt{5}\right)^4 \ln(n)(N^*)^{p-1}\left(\frac{\tau_0'}{n}\left(N^*\ln(n)\right)^{2k_qN^*} + \frac{\tau_1'}{n}(T^*)^{p+1}\right) + \frac{8C\left(2+\sqrt{5}\right)^4 l^2}{\omega^2_{N^*}} + \frac{\theta C M^2}{\ln(n)} + \frac{A_4}{n},$$

$$\tau_0' := \frac{e^{2k_q}4c_{\boldsymbol{X}}\theta^{p+1}}{\pi^{p+1}k_q}\left(Q_q\left(\frac{1}{e}+\left(\frac{1}{e}+6\,(1+\zeta_0)\right)2(1+c_1)\right) + eM_{\mathcal{E},\eta}2^p\right) + \frac{M_{\mathcal{E},\eta}2^{p+2}l^2}{e},$$

$$\tau_1' := \frac{4c_{\boldsymbol{X}}}{\pi^{p+1}(p+1)}\left(Q_q\left(\frac{1}{e}+\left(\frac{1}{e}+6\,(1+\zeta_0)\right)2(1+c_1)\right) + \frac{M_{\mathcal{E},\eta}2^p}{e}\right).$$

The computation below gives lower bounds on $N^*\ln(N^*)$ and N^*:

$$\begin{aligned}\ln(n) &= 2k_qN^*\ln\left(N^*\ln(n)\right) + (2\sigma+p-1)\ln\left(N^*\right) + \ln_2(n)\\ &\leq 2\left((2(k_q+\sigma)+p-1)N^*\ln(N^*)\bigvee(2k_qN^*+1)\ln_2(n)\right)\\ &\leq 2\left((2(k_q+\sigma)+p-1)N^*\ln(N^*)\bigvee(2k_q+1)N^*\ln_2(n)\right).\end{aligned}$$

Using both and $\mathcal{W}(x) \leq \ln(x+1)$ for all $x>0$, we obtain $N^* \geq \ln(n)/((\tau_8\bigvee(2k_q+1))\ln_2(n)(1+\ln(1+\tau_8/e))$, where $\tau_8 := 2(2(k_q+\sigma)+p-1)$. We conclude proceeding like for (B.35).

Proof of (T4.2). Start from (B.54), where, because $w = W_{\mathcal{A}}$, the term $M^2\widetilde{w}(\underline{a})$ is zero. Let $\underline{N}^*$ solution of $2k_q\underline{N}^*\ln\left(\underline{N}^*\right) + \ln\left(\omega^2_{\underline{N}^*}\right) + (p-1)\ln\left(\underline{N}^*\right) + \ln_2(n) = \ln(n)$. By definition of $\underline{N^W_{\max,q}}$ this yields $N^* \leq N^W_{\max,q}$ hence $N^* \in \mathcal{N}_n$. Using arguments from the proof of (T2.2a) we have $T^* \leq n^{\kappa/(2(\kappa+k_q)s)}$ and, using $s > 2p+1/2$, for n large enough $n^{\kappa/(2(\kappa+k_q)s)} \leq n^{1/(4p+1)}/2 \leq T_{\max}$, hence $T^* \in \mathcal{T}_n$. Thus, we obtain

$$\mathcal{R}^{W,\mathrm{adp}}_{n_0,\sup} \leq C\left(2+\sqrt{5}\right)^4 \ln(n)(N^*)^{p-1}\left(\frac{\tau_0'}{n}\left(N^*\right)^{2k_qN^*} + \frac{\tau_1'}{n}(T^*)^{p+1}\right) + \frac{8C\left(2+\sqrt{5}\right)^4 l^2}{\omega^2_{N^*}} + \frac{A_4}{n}.$$

This yields the result following the proof of (T2.2a).

Proof of (T4.1b). Starting from (B.54), the proof is similar to that of (T4.1a) with elements from that of (T3.1), using $\underline{N}^*$ solution of $2k_q\underline{N}\ln_2(n) + (p-1)\ln(\underline{N})/q + \ln\left(\omega^2_{\underline{N}^*}\right) = \ln(n)$.

Proof of (T4.3). The proof is similar to that of (T4.2). Start from (B.54). Let $\underline{N}^*$ solution of $2(k_q+\kappa)\underline{N}^* + (p-1)\ln(\underline{N}^*)/q + \ln_2(n) = \ln(n)$. Then, using the definition of $\underline{N^W_{\max,q}}$, which satisfies $2k_q\underline{N^W_{\max,q}} = \ln(n)$, we have $N^* \in \mathcal{N}_n$. Using arguments from the proof of (T3.2), we have $T^* \leq n_e^{\kappa/(2s(\kappa+k_q))}$ and, using $s > 4p+1/2$, for n large enough $n_e^{\kappa/(2s(\kappa+k_q))} \leq n^{1/(8p+1)}/2 \leq T_{\max}$, hence $T^* \in \mathcal{T}_n$. This yields the result using the proof of (T3.2). □

Appendix B.3. Estimation of the marginal $f_{\boldsymbol{\beta}}$

For all $(\omega_m)_{m\in\mathbb{N}_0}$ increasing, $\omega_0=1$, $l,M>0$, $q\in\{1,\infty\}$, consider

$$\mathcal{H}^{q,\omega}_{w,W}(l,M):=\left\{f:\|f\|_{L^2(w\otimes W^{\otimes p})}\leq M,\ \sum_{k\in\mathbb{N}_0}\omega_k^2\left\|\theta_{q,k}\right\|^2_{L^2(\mathbb{R})}\leq 2\pi l^2\right\}.$$

For brevity, we present the slow rates and the estimator $\widehat{f}^{q,N,\epsilon}_{\beta}:=\sum_{|\boldsymbol{m}|_q\leq N(\epsilon)}\widehat{c}_{\boldsymbol{m}}(\epsilon)\varphi^{W,\epsilon x_0}_{\boldsymbol{m}}/\sigma^{W,\epsilon x_0}_{\boldsymbol{m}}$. It is based on $f_{\boldsymbol{\beta}}=\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](0,\cdot_2)$.

Proposition B.3. Let $W=W_{[-R,R]}$. For all $q\in\{1,\infty\}$, $l,M,R>0$, $\sigma>2$, $\mathbb{S}_\beta\subseteq[-R,R]^p$, $\underline{N}$ solution of $2(1+\sigma)k_q\underline{N}\ln(\underline{N})+p(1-\sigma)\ln(\underline{N})+\ln(\omega^2_{\underline{N}})=\ln(n_e)$, $\epsilon=\theta/\omega_{\underline{N}}$, $(\omega_k)_{k\in\mathbb{N}_0}=(k^\sigma)_{k\in\mathbb{N}_0}$, and w such that $\int_{\mathbb{R}}a^2/w(a)da<\infty$, we have

$$\left(\frac{\ln(n_e)}{\ln_2(n_e)}\right)^{2\sigma}\sup_{f_{\boldsymbol{\beta}}\in\mathcal{H}^{q,\omega}_{w,W}(l,M)\cap D,\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}\mathbb{E}\left[\left\|\widehat{f}^{q,N,\epsilon}_{\boldsymbol{\beta}}-f_{\boldsymbol{\beta}}\right\|^2_{L^2(\mathbb{R}^p)}\right]=O_p(1).$$

Proof. We assume $f_{\boldsymbol{X}|\mathcal{X}}$ is known. The general case can be handled like in the proof of (T2.1). Use $f^\epsilon_{\boldsymbol{\beta}}:=\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](\epsilon,\cdot)$ and define $f^{q,\epsilon,N}_{\boldsymbol{\beta}}$ like $\widehat{f}^{q,\epsilon,N}_{\boldsymbol{\beta}}$ with $\widetilde{c}_{\boldsymbol{m}}(t)$ (see Lemma A.2) instead of $\widehat{c}_{\boldsymbol{m}}(t)$. Use $\left\|\widehat{f}^{q,N,\epsilon}_{\boldsymbol{\beta}}-f_{\boldsymbol{\beta}}\right\|^2_{L^2(\mathbb{R}^p)}\leq 3\sum_{j=1}^3\|R_j\|^2_{L^2(\mathbb{R}^p)}$, where $R_1:=\widehat{f}_{\boldsymbol{\beta}}-f^{q,N,\epsilon}_{\boldsymbol{\beta}}$, $R_2:=f^{q,N,\epsilon}_{\boldsymbol{\beta}}-f^\epsilon_{\boldsymbol{\beta}}$, and $R_3:=f^\epsilon_{\boldsymbol{\beta}}-f_{\boldsymbol{\beta}}$. Let $n\geq e^e$ large enough so that $N\geq 1\bigvee((\sigma-1)p-\sigma)/(2k_q)$. By similar arguments from (A.26), (B.29), $N\leq\underline{N}$, and $(\underline{N}+1)^{2k_q\underline{N}}\leq e^{2k_q}\underline{N}^{2k_q\underline{N}}$, we have

$$\mathbb{E}\left[\|R_1\|^2_{L^2(\mathbb{R}^p)}\right]\leq\frac{Q_qc_{\boldsymbol{X}}e^{2k_q}}{\pi^pn}\epsilon^p\underline{N}^p\left(1\bigvee\frac{\theta\underline{N}}{\epsilon}\right)^{2k_q\underline{N}}=\frac{Q_qc_{\boldsymbol{X}}e^{2k_q}\theta^p}{\pi^pn}\underline{N}^{p(1-\sigma)+2(1+\sigma)k_q\underline{N}}.\tag{B.56}$$

We also obtain $\|R_2\|^2_{L^2(\mathbb{R}^p)}\leq 2\pi l^2/\omega^2_{\underline{N}}$ and

$$\begin{aligned}\|R_3\|^2_{L^2(\mathbb{R}^p)}&\leq\int_{[-R,R]^p}\left(\int_{\mathbb{R}}\left|e^{i\epsilon a}-1\right|f_{\alpha,\boldsymbol{\beta}}(a,\boldsymbol{b})da\right)^2d\boldsymbol{b}\\&\leq\epsilon^2\int_{[-R,R]^p}\left(\int_{\mathbb{R}}|a|\,f_{\alpha,\boldsymbol{\beta}}(a,\boldsymbol{b})da\right)^2d\boldsymbol{b}\leq\frac{\theta^2M^2}{\omega^2_{\underline{N}}}\int_{\mathbb{R}}\frac{a^2}{w(a)}da<\infty.\end{aligned}\tag{B.57}$$

Then, using $\ln(n)=2(1+\sigma)k_q\underline{N}\ln(\underline{N})+p(1-\sigma)\ln(\underline{N})+\ln(\omega^2_{\underline{N}})\geq 2\sigma k_q\underline{N}\ln(\underline{N})$ and $\mathcal{W}(x)\leq\ln(x+1)$, we have $\underline{N}\leq\ln(n)/(2\sigma k_q\ln_2(n_e)(1+\ln(1+2\sigma k_q/e)))$. The result follows from the definition of $\underline{N}$, (B.56), and (B.57). □

Similar ideas apply for the estimation of $f_{\boldsymbol{\beta}_j}$ for $j=1,\dots,p$.

Appendix B.4. Talagrand inequality for complex functions

Lemma B.15. Let $X_1,\dots,X_n$ n independent random vectors, $\Lambda:=(\sqrt{1+\cdot}-1)\wedge 1$, $\mathcal{U}$ a countable set of complex measurable functions, and, for all $u\in\mathcal{U}$, $\nu_n(u):=\sum_{i=1}^n\left(u(X_i)-\mathbb{E}\left[u(X_i)\right]\right)/n$.

If there exist $M, H, v > 0$ such that

$$\sup_{u\in\mathcal{U}} \|u\|_{L^\infty(\mathbb{R}^p)} \leq M,\ \mathbb{E}\left[\sup_{u\in\mathcal{U}} |\nu_n(u)|\right] \leq H,\ \sup_{u\in\mathcal{U}} \frac{1}{n}\sum_{i=1}^n \mathrm{Var}\left(\Re(u(X_i))\right) \bigvee \mathrm{Var}\left(\Im(u(X_i))\right) \leq v,$$

then, for all $\eta > 0$,

$$\mathbb{E}\left[\left(\sup_{u\in\mathcal{U}} |\nu_n(u)|^2 - 4(1+2\eta)H^2\right)_+\right] \leq 48\left(\frac{v}{n}e^{-\eta\frac{nH^2}{6v}} + \frac{294M^2}{\Lambda(\eta)^2 n^2}e^{-\frac{\sqrt{2}\Lambda(\eta)\sqrt{\eta}}{42}\frac{nH}{M}}\right).$$

Proof. The result follows from Theorem 7.3 in [14] and

$$\mathbb{E}\left[\left(\sup_{u\in\mathcal{U}} |\nu_n(u)|^2 - 4(1+2\eta)H^2\right)_+\right] \leq \mathbb{E}\left[\left(\sup_{u\in\mathcal{U}} \Re(\nu_n(u))^2 + \sup_{u\in\mathcal{U}} \Im(\nu_n(u))^2 - 4(1+2\eta)H^2\right)_+\right]$$
$$\leq \mathbb{E}\left[\left(\sup_{u\in\mathcal{U}} \Re(\nu_n(u))^2 - 2(1+2\eta)H^2\right)_+\right] + \mathbb{E}\left[\left(\sup_{u\in\mathcal{U}} \Im(\nu_n(u))^2 - 2(1+2\eta)H^2\right)_+\right]. \qquad \square$$

Appendix B.5. Approximation by PSWF in Sobolev ellipsoids.

For all $\sigma, s, l > 0$ and $q \in \{1, \infty\}$, denote by $(\phi_{\boldsymbol{m}}(\cdot/R))_{\boldsymbol{m}\in\mathbb{Z}^p} := \left(e^{i\pi\boldsymbol{m}^\top\cdot/R}/(2R)^{p/2}\right)_{\boldsymbol{m}\in\mathbb{Z}^p}$, $\mathcal{F}[f](\star, \boldsymbol{k}) := \int_{\mathbb{R}} e^{i\star a}\int_{[-R,R]^p} e^{i\pi\boldsymbol{k}^\top\boldsymbol{b}/R}\, f(a,\boldsymbol{b})da d\boldsymbol{b}/(2R)^{p/2}$, and

$$H^{q,s,\sigma}(l) := \left\{f : \int_{\mathbb{R}}\sum_{\boldsymbol{k}\in\mathbb{Z}^p} |\mathcal{F}[f](t,\boldsymbol{k})|^2 (1\vee t^{2s})dt \bigvee \int_{\mathbb{R}}\sum_{\boldsymbol{k}\in\mathbb{Z}^p} |\mathcal{F}[f](t,\boldsymbol{k})|^2 \left(1\vee|\boldsymbol{k}|_q^{2\sigma}\right)dt \leq 2\pi l^2\right\}.$$

Denote, for all $N \in \mathbb{N}$ and $c \neq 0$, by P_c^N (resp. $\mathcal{E}^N$) the projector in $L^2\left(W_{[-R,R]}^{\otimes p}\right)$ onto the vector space spanned by $\left(\psi_{\boldsymbol{m}}^c(\cdot/R)/R^{p/2}\right)_{|\boldsymbol{m}|_\infty<N}$ (resp. $(\phi_{\boldsymbol{m}}(\cdot/R))_{|\boldsymbol{m}|_\infty<N}$). For all $t \neq 0$ and $\left(n, m, N, \widetilde{N}\right) \in \mathbb{N}_0^4$, denote by $\varphi^t := \mathcal{F}_{1\mathrm{st}}[f](t,\cdot_2)$, $\beta_n^m(t) := \left\langle\psi_m^t, P_n\right\rangle_{L^2([-1,1])}$, J_j the Bessel function of the first kind and order $j > -1$, $K_t^{N,\widetilde{N}} := \left\|\mathcal{E}^{\widetilde{N}}\varphi^t - P_{x_0 t}^N\mathcal{E}^{\widetilde{N}}\varphi^t\right\|^2$, and $I_{N,\widetilde{N}} := \sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\widetilde{N}}\sum_{|\boldsymbol{m}|_\infty\geq N} |\langle\phi_{\boldsymbol{k}}(\cdot/R), \psi_{\boldsymbol{m}}^{x_0\star}(\cdot/R)\rangle|^2$.

Proposition B.4. For all $\sigma, l, M, R > 0$, $q \in \{1, \infty\}$, and $s \geq \sigma + p/2$, we have, for all $N \geq 10$,

$$\int_{\mathbb{R}} \left\|\mathcal{F}_{1\mathrm{st}}[f](t,\cdot_2) - P_{x_0 t}^N\mathcal{F}_{1\mathrm{st}}[f](t,\cdot_2)\right\|^2 dt \leq \frac{2\pi A l^2}{N^{2\sigma}}, \tag{B.58}$$

$$A := 2\left(\left(\frac{1}{1/(\pi e) - 1/10}\right)^{2\sigma} + c\left(\frac{p+2\sigma}{be}\right)^{p+2\sigma} + \left(\frac{2eRx_0}{\pi}\right)^p (e^2x_0)^{2\sigma}\right),$$
$$a := \frac{\sqrt{5e^3}(e^2+1/e^2)^{5/8}}{3(\ln(2)+2)2^{11/4}}, b := p\left(\frac{5}{8}\ln\left(\frac{21}{10}\right) - \frac{1}{e}\right),$$

$$c := \frac{p(4R^2/(\pi e))^p}{2R}\left(a^{2p}\frac{8^p(p-1)^{p-1}}{(3p)^p e^{p-1}} + \left(\frac{(2p-1)8}{5pe}\right)^{2p-1}\frac{5^{p-1}8}{p16^p\ln(21/10)}\right).$$

Proposition B.4 is an analogue of (A.30) with N constant and $(\omega_k)_{k\in\mathbb{N}_0} = (k^\sigma)_{k\in\mathbb{N}_0}$. It shows that the approximation error when we use a truncated series expansion in the PSWF basis is of order $N^{-2\sigma}$ whether we work on the class $H^{q,s,\sigma}(l)$ or $\mathcal{H}^{q,\phi,\omega}_{w,W_{[-R,R]}}(l,M)$ with $\phi = 1\vee|\cdot|^s$. (A.31) can be obtained using the first inequality in the definition of $H^{q,s,\sigma}(l)$ and, for all $t\neq 0$, $\sum_{\boldsymbol{k}\in\mathbb{Z}^p}|\mathcal{F}[f](t,\boldsymbol{k})|^2 = \|\mathcal{F}_{1\mathrm{st}}[f](t,\cdot_2)\|^2_{L^2\left(W^{\otimes p}_{[-R,R]}\right)} = \sum_{\boldsymbol{m}\in\mathbb{N}_0^p}|b_{\boldsymbol{m}}(t)|^2$. Thus (T2.1) also holds for functions in the intersection of $H^{q,s,\sigma}(l)\bigcap\left\{f:\ \|f\|_{L^2\left(w\otimes W^{\otimes p}_{[-R,R]}\right)}\leq M\right\}$. The proof below uses techniques from the proof of Lemma 11 in [7].

Proof. In this proof, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the scalar product and norm in $L^2([-R,R]^p)$. Take $f\in H^{q',s,\sigma'}(l)$. Let $N\geq 10$ and $\widetilde{N} := \lfloor\tau N\rfloor$, where $\tau := 1/(\pi e)$. We have

$$\begin{aligned}\left\|\varphi^t - P^N_{x_0t}\varphi^t\right\|^2 &\leq 2\left(\left\|\varphi^t - \mathcal{E}^{\widetilde{N}}\varphi^t - P^N_{x_0t}\left(\varphi^t - \mathcal{E}^{\widetilde{N}}\varphi^t\right)\right\|^2 + K_t^{N,\widetilde{N}}\right)\\ &\leq 2\left(\left\|\varphi^t - \mathcal{E}^{\widetilde{N}}\varphi^t\right\|^2 + K_t^{N,\widetilde{N}}\right). \qquad \text{(B.59)}\end{aligned}$$

Using that $\left(\psi^{x_0t}_{\boldsymbol{m}}(\cdot/R)/R^{p/2}\right)_{\boldsymbol{m}\in\mathbb{N}_0^p}$ are orthonormal in $L^2([-R,R]^p)$ and the Cauchy-Schwarz inequality in the second display yield

$$\begin{aligned}K_t^{N,\widetilde{N}} &= \left\|\sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\widetilde{N}}\left\langle\varphi^t,\phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right)\right\rangle\left(\sum_{|\boldsymbol{m}|_\infty\geq N}\left\langle\phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right),\psi^{x_0t}_{\boldsymbol{m}}\left(\frac{\cdot}{R}\right)\right\rangle\psi^{x_0t}_{\boldsymbol{m}}\left(\frac{\star}{R}\right)\frac{1}{R^p}\right)\right\|^2\\ &\leq \sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\widetilde{N}}\left|\left\langle\varphi^t,\phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right)\right\rangle\right|^2 I_{N,\widetilde{N}}(t) \leq \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^p}|\mathcal{F}[f](t,\boldsymbol{k})|^2\right)I_{N,\widetilde{N}}(t).\end{aligned}$$

We have, using (18.17.19) in [45] for the first equality and for all $k\in\mathbb{Z}$ and $m\in\mathbb{N}_0$,

$$\left|\left\langle\phi_k\left(\frac{\cdot}{R}\right),\psi^{x_0t}_m\left(\frac{\cdot}{R}\right)\right\rangle\right|^2 = \frac{R}{2}|I_{m,k}+O_{m,k}|^2 \leq R\left(|I_{m,k}|^2+|O_{m,k}|^2\right),$$

$$I_{m,k} := \sum_{n=0}^{\lfloor 5m/8\rfloor-1}\beta^m_n(x_0t)\left\langle e^{i\pi k\cdot},P_n\right\rangle_{L^2([-1,1])},\quad O_{m,k} := \sum_{n\geq\lfloor 5m/8\rfloor}\beta^m_n(x_0t)i^n\sqrt{\frac{2}{|k|}}\sqrt{n+\frac{1}{2}}J_{n+1/2}(|k|\pi).$$

Using, for all $k\in\mathbb{Z}$, $\left|\left\langle e^{i\pi k\cdot},P_n\right\rangle\right|\leq\sqrt{2}$, Proposition 3 in [7], and integral test for convergence (indeed, by (3.4) page 34 in [46], for all $m\geq 2\vee(e^2x_0|t|)$, $2\sqrt{\chi^{x_0t}_m}/(x_0|t|)\geq 2e^2>1$), we obtain, for all $m\geq 2\vee(e^2x_0|t|)$,

$$|I_{m,k}| \leq \sqrt{\frac{5}{2\pi}}\int_0^{\lfloor 5m/8\rfloor}\left(\frac{2\sqrt{\chi^{x_0t}_m}}{x_0|t|}\right)^x dx\,|\mu^{x_0t}_m| \leq \frac{\sqrt{5/(2\pi)}}{\ln\left(2\sqrt{\chi^{x_0t}_m}/(x_0|t|)\right)}\left(\frac{2\sqrt{\chi^{x_0t}_m}}{x_0|t|}\right)^{\lfloor 5m/8\rfloor}|\mu^{x_0t}_m|.$$

Let $m \geq 2 \vee (e^2 x_0 |t|)$. Using Lemma B.4 for the first inequality, we obtain

$$\begin{aligned}
|I_{m,k}| &\leq \frac{\sqrt{5e^3}}{3} \frac{1}{\ln(2)+2} \left(\frac{2\sqrt{m(m+1)+x_0^2 t^2}}{x_0 |t|} \right)^{5m/8} \left(\frac{e x_0 |t|}{4(m+3/2)} \right)^m \\
&\leq \frac{\sqrt{5e^3}}{3} \frac{1}{\ln(2)+2} \left(\frac{\sqrt{e^2+1/e^2}(m+1)}{2^{11/5}(m+3/2)} \right)^{5m/8} \exp\left(-\frac{3m}{8} \ln\left(\frac{m}{e x_0 |t|} \right) \right) \\
&\leq a \exp\left(-\frac{3m}{8} \ln\left(\frac{m}{e x_0 |t|} \right) \right).
\end{aligned}$$

Using, for all $j > -1/2$, $x \in \mathbb{R}$, and $n \in \mathbb{N}_0$, $|J_j(x)| \leq |x|^j / \left(2^j \Gamma(j+1)\right)$ (see 9.1.20 in [1]), $|\beta_n^m(x_0 t)| \leq 1$, and $\sqrt{n+1/2} < \Gamma(n+3/2)/n!$ (see (5.6.4) in [45]) for the first inequality and $m > 8/5$ and $n! \geq (n/e)^n \sqrt{2\pi n}$ for the third, we obtain, for all $k \in \mathbb{Z}$,

$$\begin{aligned}
|O_{m,k}| &\leq \sum_{n \geq \lfloor 5m/8 \rfloor} \frac{\sqrt{\pi}}{n!} \left(\frac{|k|\pi}{2} \right)^n \\
&\leq \frac{\sqrt{\pi}}{\lfloor 5m/8 \rfloor !} \left(\frac{|k|\pi}{2} \right)^{\lfloor 5m/8 \rfloor} \exp\left(\frac{|k|\pi}{2} \right) \leq \sqrt{\frac{5m}{16}} \left(\frac{|k|\pi e}{2(5m/8-1)} \right)^{5m/8} \exp\left(\frac{|k|\pi}{2} \right).
\end{aligned}$$

Using $\left| \left\langle \phi_k(\cdot/R), \psi_m^{x_0 t}(\cdot/R) \right\rangle \right|^2 \leq R$ for all $(k,m) \in \mathbb{N}_0^2$ for the first inequality, $\sum_{|\boldsymbol{m}|_\infty = j} 1 \leq p(j+1)^{p-1}$ for the second, (B.29) and the convexity of $x \mapsto x^p$ for the fourth inequality, we have, for all t such that $N \geq e^2 x_0 |t|$,

$$\begin{aligned}
I_{N,\widetilde{N}}(t) &\leq R^{p-1} \sum_{\boldsymbol{k} \in \mathbb{Z}^p:\ |\boldsymbol{k}|_\infty < \widetilde{N}} \sum_{j=N}^{\infty} \sum_{|\boldsymbol{m}|_\infty = j} \prod_{l=1}^{p} \left| \left\langle \phi_{\boldsymbol{k}_l}\left(\frac{\cdot}{R}\right), \psi_j^{x_0 t}\left(\frac{\cdot}{R}\right) \right\rangle \right|^2 \\
&\leq R^{2p-1} \sum_{\boldsymbol{k} \in \mathbb{Z}^p:\ |\boldsymbol{k}|_\infty < \widetilde{N}} \sum_{j=N}^{\infty} p(j+1)^{p-1} \prod_{l=1}^{p} \left(|I_{j,\boldsymbol{k}_l}|^2 + |O_{j,\boldsymbol{k}_l}|^2 \right) \\
&\leq p R^{2p-1} \sum_{\boldsymbol{k} \in \mathbb{Z}^p:\ |\boldsymbol{k}|_\infty < \widetilde{N}} \sum_{j=N}^{\infty} (j+1)^{p-1} \left(a^2 \left(\frac{e x_0 |t|}{j} \right)^{3j/4} + \frac{5 j e^{|\boldsymbol{k}_l|\pi}}{16} \left(\frac{|\boldsymbol{k}_l|\pi e}{2(5j/8-1)} \right)^{5j/4} \right) \\
&\leq \frac{p(4R^2 \tau N)^p}{2R} \sum_{j=N}^{\infty} j^{p-1} a^{2p} \left(\frac{e x_0 |t|}{j} \right)^{3pj/4} + \left(\frac{5 e^{\widetilde{N}\pi} j^2}{16} \right)^p \left(\frac{\widetilde{N}\pi e}{2(5j/8-1)} \right)^{5pj/4} \frac{1}{j}.
\end{aligned}$$

Using $\kappa(t) := -3 \ln\left(e x_0 |t| / N\right)/8$, $\kappa(t) \geq 3/8$ for $N \geq 2 \vee (e^2 x_0 |t|)$, and $\sup_{j \geq 1} j^{p-1} e^{-p\kappa(t) j} = (1-1/p)^{p-1}/(\kappa(t) e)^{p-1}$ for the second inequality, we obtain, for all $N \geq e^2 x_0 |t|$,

$$\sum_{j=N}^{\infty} j^{p-1} \left(\frac{e x_0 |t|}{j} \right)^{3pj/4} \leq \frac{(1-1/p)^{p-1}}{(\kappa(t) e)^{p-1}} \int_N^\infty e^{-p\kappa(t) j} dj \leq \frac{(p-1)^{p-1}}{(p\kappa(t))^p e^{p-1}} e^{-p\kappa(t) N}.$$

Using $1-8/(5N)\geq 1/5$, that for $j\geq N$, $\widetilde{N}\pi e/(2(5j/8-1))\leq 10\tau\pi e/21 = 10/21$, and $\sup_{j\geq 1} j^{2p-1}e^{-5pj/8} = ((2p-1)8/(5pe))^{2p-1}$ for the first inequality, we obtain

$$\begin{aligned}\sum_{j=N}^{\infty} j^{2p-1}\left(\frac{\widetilde{N}\pi e}{2(5j/8-1)}\right)^{5pj/4} &\leq \left(\frac{(2p-1)8}{5pe}\right)^{2p-1}\int_N^{\infty} e^{-5pj\ln(21/10)/8}dj\\ &\leq \left(\frac{(2p-1)8}{5pe}\right)^{2p-1}\frac{8}{5p\ln(21/10)}e^{-5pN\ln(21/10)/8}.\end{aligned}$$

Using (B.59) for the first display, using $\sup_{t:\ |t|\leq N/(e^2x_0)} I_{N,\widetilde{N}}(t)\leq cN^p e^{-bN}$ (because $5\ln(21/10)/8-\tau\pi < 3/8\leq\kappa(t)$), using $s\geq\sigma+p/2$, $f\in H^{q,s,\sigma}(l)$, and, for all $|t|>N/(e^2x_0)$, $I_{N,\widetilde{N}}(t)\leq R^p\sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\widetilde{N}}\|\phi_{\boldsymbol{k}}\|^2_{L^2([-1,1]^p)}\leq(2\tau NR)^p$ for the second, we have

$$\begin{aligned}&\int_{\mathbb{R}}\left\|\mathcal{F}_{1\mathrm{st}}[f](t,\cdot_2)-P^N_{x_0t}\mathcal{F}_{1\mathrm{st}}[f](t,\cdot_2)\right\|^2dt\\ &\leq 2\left\|\mathcal{F}_{1\mathrm{st}}[f]-\mathcal{E}^{\widetilde{N}}\mathcal{F}_{1\mathrm{st}}[f]\right\|^2_{L^2(\mathbb{R}\times[-R,R]^p)}+2\int_{-N/(e^2x_0)}^{N/(e^2x_0)}\sum_{\boldsymbol{k}\in\mathbb{Z}^p}\left|\mathcal{F}_{1\mathrm{st}}[f](t,\boldsymbol{k})\right|^2dt\sup_{t:\ |t|\leq N/(e^2x_0)}I_{N,\widetilde{N}}(t)\\ &\quad+\frac{2(2\tau NR)^p}{1\vee(N/(e^2x_0))^{2s}}\int_{|t|>N/(e^2x_0)}\sum_{\boldsymbol{k}\in\mathbb{Z}^p}\left|\mathcal{F}_{1\mathrm{st}}[f](t,\boldsymbol{k})\right|^2(1\vee t^{2s})dt\\ &\leq\frac{4\pi l^2}{(\tau N-1)^{2\sigma}}+\frac{4\pi l^2cN^p}{e^{bN}}+\frac{4\pi l^2\left(2\tau Re^2x_0\right)^p(e^2x_0)^{2\sigma}}{N^{2\sigma}}.\end{aligned}$$

Using $\tau-1/10>0$ and (A.1) yield the result. □