Adaptive estimation in the linear random coefficients model when regressors have limited variation

Christophe Gaillac, Eric Gautier

ADAPTIVE ESTIMATION IN THE LINEAR RANDOM COEFFICIENTS MODEL WHEN REGRESSORS HAVE LIMITED VARIATION

CHRISTOPHE GAILLAC[(1),(2)] AND ERIC GAUTIER[(1)]

Abstract. We consider a linear model where the coefficients - intercept and slopes - are random and independent from regressors which support is a proper subset. When the density has finite weighted L^2 norm, for well chosen weights, the joint density of the random coefficients is identified. Lower bounds on the supremum risk for the estimation of the density are derived for this model and a related white noise model. We present an estimator, its rates of convergence, and a data-driven rule which delivers adaptive estimators. An R package RandomCoefficients that implements our estimator is available on CRAN.R.

1. Introduction

For a random variable α and random vectors $\boldsymbol{X}$ and $\boldsymbol{\beta}$ of dimension $p \times 1$, the linear random coefficients model is

$$Y = \alpha + \boldsymbol{\beta}^\top \boldsymbol{X}, \tag{1}$$

$$(\alpha, \boldsymbol{\beta}^\top) \text{ and } \boldsymbol{X} \text{ are independent}, \tag{2}$$

from which the researcher has at her disposal n observations $(Y_i, \boldsymbol{X}_i^\top)_{i=1}^n$ of $(Y, \boldsymbol{X}^\top)$ but does not observe the realizations $(\alpha_i, \boldsymbol{\beta}_i^\top)$ of $(\alpha, \boldsymbol{\beta}^\top)$ for $i = 1, \ldots, n$. The coefficient α subsumes the usual intercept and error term and the vector of slope coefficients $\boldsymbol{\beta}$ is heterogeneous (namely, it varies across $i = 1, \ldots, n$). For example, a researcher interested in the effect of class size

(1) Toulouse School of Economics, Toulouse Capitole University, 21 allée de Brienne, 31000 Toulouse, France

(2) CREST, 5 avenue Henry Le Chatelier, 91764 Palaiseau, France

E-mail addresses: christophe.gaillac@ensae.fr, eric.gautier@tse-fr.eu.

Date: This version: May 15, 2019.

Keywords: Adaptation, Ill-posed Inverse Problem, Minimax, Random Coefficients.

AMS 2010 Subject Classification: Primary 62P20 ; secondary 42A99, 62C20, 62G07, 62G08, 62G20.

Eric Gautier acknowledges financial support from the grants ERC POEMH 337665.

on pupils' achievements can find it important to allow some pupils to be more sensitive than others to a decrease in the size and to estimate the density of the effect.

The coefficients $(\alpha, \boldsymbol{\beta}^\top)$ correspond to multidimensional unobserved heterogeneity and $\boldsymbol{X}$ to observed heterogeneity. Restricting unobserved heterogeneity to a scalar, as when one only assumes that α is random, can have undesirable implications on the implied law of the observed data such as monotonicity in the literature on public policy evaluation (see [22] for a solution involving random coefficients). Parametric assumptions on unobserved heterogeneity are often made by convenience and can drive the results (see [27]). For this reason, this paper considers a nonparametric law of the coefficients. Model (1) is also a type of linear model with homegeneous slopes and heteroscedasticity, hence the averages of the coefficients are easy to obtain by least squares. However, the law of coefficients or quantiles can also be of interest. For example, [3] considers prediction intervals for Y for $\boldsymbol{X} = \boldsymbol{x}$.

Estimation of the density of random coefficients $f_{\alpha,\boldsymbol{\beta}}$ when the support of $\boldsymbol{X}$ is $\mathbb{R}^p$ and $\boldsymbol{X}$ has heavy enough tails has been studied in [4, 28]. These papers notice that using a renormalization, the inverse problem is related to a tomography problem (see, *e.g.*, [10, 11]) involving the Radon transform. Assuming that the support of $\boldsymbol{X}$ is $\mathbb{R}^p$ amounts to assuming that the law of angles has full support, moreover a lower bound on the density of $\boldsymbol{X}$ is assumed so that the law of the angles is nondegenerate. When $p = 1$ this is implied by densities of $\boldsymbol{X}$ which follow a Cauchy distribution. The corresponding tomography problem has a nonuniform and estimable density of angles and the dimension can be larger than in tomography due to more than one regressor.

This paper considers the case where the support of $\boldsymbol{X}$ is a proper (*i.e.*, strict) subset. This is a much more useful and realistic framework for the random coefficients model. When $p = 1$, this is related to limited angle tomography (see, *e.g.*, [18, 29]). There, one has measurements over a subset of angles and the unknown density has support in the unit disk. This is too restrictive for a density of random coefficients and implies that α has compact support, ruling out usual parametric assumptions on error terms. Due to (2), the conditional characteristic function of Y given $\boldsymbol{X} = \boldsymbol{x}$ at t is the Fourier transform of $f_{\alpha,\boldsymbol{\beta}}$ at $(t, t\boldsymbol{x}^\top)^\top$. Hence, the family of conditional characteristic functions indexed by $\boldsymbol{x}$ in the support of $\boldsymbol{X}$ gives access to the Fourier transform of $f_{\alpha,\boldsymbol{\beta}}$ on a double cone of axis $(1, 0, \dots, 0) \in \mathbb{R}^{p+1}$ and apex 0. When $\alpha = 0$, $\mathbb{S}_{\boldsymbol{\beta}}$ is compact, and $\mathcal{X} \subseteq \mathbb{S}_{\boldsymbol{X}}$ is an arbitrary compact set of nonempty interior, this is the problem of out-of-band extrapolation or super-resolution (see, *e.g.*, [5] sections 11.4 and 11.5). Because we allow α to be nonzero, we generalize this approach. Estimation of

$f_{\alpha,\boldsymbol{\beta}}$ is a statistical inverse problem for which the deterministic problem is the inversion of a truncated Fourier transform (see, *e.g.*, [2] and the references therein). The companion paper [21] presents conditions on the law of $\left(\alpha, \boldsymbol{\beta}^\top\right)^\top$ and the support of $\boldsymbol{X}$ that imply nonparametric identification. It considers weak conditions on α which could have infinite absolute moments and the marginals of $\boldsymbol{\beta}$ could have heavy tails. In this paper, we obtain rates of convergence when the marginals of $\boldsymbol{\beta}$ do not have heavy tails but can have noncompact support.

A somewhat related approach is extrapolation. It is used in [39] to perform deconvolution of compactly supported densities while allowing the Fourier transform of the error density to vanish on a set of positive measure. Rather, in this paper, the operator to be inverted is viewed as a composition of two operators based on partial Fourier transforms. One involves inverting a truncated Fourier transform defined on a weighted space of square-integrable functions and we make use of properties of its singular value decomposition for inversion rather than extrapolation of the Fourier transform.

Similar to [23, 24] for the random coefficients binary choice, we study optimality in the minimax sense. We obtain lower bounds on the supremum risk for the estimation of the density under weak to strong integrability in the first argument of $f_{\alpha,\boldsymbol{\beta}}$ for this model and a related white noise model. We present an estimator which involves: series based estimation of the partial Fourier transform of the density with respect to the first variable, interpolation around zero, and inversion of the partial Fourier transform. We give rates of convergence and use a Goldenshluger-Lepski type method (see [26] and the references therein) to obtain data-driven estimators. We consider estimation of $f_{\boldsymbol{\beta}}$ in Appendix C. Finally, note that an R package RandomCoefficients that implements our estimator in the case where the random slope has compact support is available on CRAN.R.

2. Notations

The notation $\mathbb{Z}$ stands for the integers, $\mathbb{N}$ and $\mathbb{N}_0$ for the positive and nonnegative integers, $(\cdot)_+$ for $\max(\cdot, 0)$, $a \wedge b$ (resp. $a \vee b$) for the minimum (resp. maximum) between a and b, and $\mathbb{1}\{\cdot\}$ for the indicator function. Bold letters are used for vectors. For a real number r, $\underline{\boldsymbol{r}}$ is the vector, which dimension will be clear from the text, where each entry is r. The iterated logarithms are defined by $\ln_0(t) = t$ and, for $j \geq 1$, by $\ln_j(t) = \ln(\ln_{j-1}(t))$ provided t is large enough for the quantity to be defined. The notation $|\cdot|_q$ for $q \in [1, \infty]$ stands for the ℓ_q norm of a vector. For $\boldsymbol{\beta} \in \mathbb{C}^d$, $(f_m)_{m \in \mathbb{N}_0}$ functions with values in $\mathbb{C}$, and $\boldsymbol{m} \in \mathbb{N}_0^d$,

denote by $\boldsymbol{\beta}^{\boldsymbol{m}} = \prod_{k=1}^d \boldsymbol{\beta}_k^{\boldsymbol{m}_k}$, $|\boldsymbol{\beta}|^{\boldsymbol{m}} = \prod_{k=1}^d |\boldsymbol{\beta}_k|^{\boldsymbol{m}_k}$, and $f_{\boldsymbol{m}} = \prod_{k=1}^d f_{\boldsymbol{m}_k}$. For a differentiable function f of real variables, $f^{(\boldsymbol{m})}$ denotes $\prod_{j=1}^d \frac{\partial^{\boldsymbol{m}_j}}{\partial x_j^{\boldsymbol{m}_j}} f$ and $\mathrm{supp}(f)$ its support. $C^\infty\left(\mathbb{R}^d\right)$ is the space of infinitely differentiable functions. The inverse of a mapping f, when it exists, is denoted by f^I. We denote the interior of a subset $\mathcal{S}$ of $\mathbb{R}^d$ by $\overset{\circ}{\mathcal{S}}$ and its closure by $\overline{\mathcal{S}}$. When $\mathcal{S}$ is a measurable subset of $\mathbb{R}^d$ and μ is a function from $\mathcal{S}$ to $[0,\infty]$, $L^2(\mu)$ is the space of complex-valued square integrable functions equipped with the hermitian product $\langle f, g\rangle_{L^2(\mu)} = \int_{\mathcal{S}} f(\boldsymbol{x})\overline{g}(\boldsymbol{x})\mu(\boldsymbol{x})d\boldsymbol{x}$. This is denoted by $L^2(\mathcal{S})$ when $\mu = 1$. For $W_{\mathcal{S}} = \mathbb{1}\{\mathcal{S}\} + \infty\, \mathbb{1}\{\mathcal{S}^c\}$, we have $L^2(W_{\mathcal{S}}) = \left\{f \in L^2\left(\mathbb{R}^d\right) :\ \mathrm{supp}(f) \subseteq \mathcal{S}\right\}$ and $\langle f, g\rangle_{L^2(W_{\mathcal{S}})} = \int_{\mathcal{S}} f(\boldsymbol{x})\overline{g}(\boldsymbol{x})d\boldsymbol{x}$. Denote by $\mathcal{D}$ the set of densities, by $\Pi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ such that $\Pi f(\boldsymbol{x}) = f(-\boldsymbol{x})$, and by $\otimes$ the product of weighting functions (*e.g.*, $W^{\otimes d}(\boldsymbol{b}) = \prod_{j=1}^d W(\boldsymbol{b}_j)$) or the product of measures. The Fourier transform of f in $L^1\left(\mathbb{R}^d\right)$ is $\mathcal{F}[f](\boldsymbol{x}) = \int_{\mathbb{R}^d} e^{i\boldsymbol{b}^\top \boldsymbol{x}} f(\boldsymbol{b})d\boldsymbol{b}$ and we also use the notation $\mathcal{F}[f]$ for its Fourier transform in $L^2\left(\mathbb{R}^d\right)$. For $c > 0$, denote the Paley-Wiener space of bandlimited functions by $PW(c) = \left\{f \in L^2(\mathbb{R}) : \mathrm{supp}\left(\mathcal{F}[f]\right) \subseteq [-c, c]\right\}$, by $\mathcal{P}_c$ the projector from $L^2(\mathbb{R})$ to $PW(c)$ ($\mathcal{P}_c[f] = \mathcal{F}^I\left[\mathbb{1}\{[-c,c]\}\mathcal{F}[f]\right]$), and, for $c \neq 0$, by

$$
\begin{array}{rccccccc}
\mathcal{F}_c : & L^2\left(W^{\otimes d}\right) & \to & L^2\left([-1,1]^d\right) & \text{and} & \mathcal{C}_c : & L^2\left(\mathbb{R}^d\right) & \to & L^2\left(\mathbb{R}^d\right) \\
& f & \to & \mathcal{F}[f](c\,\cdot) & & & f & \to & |c|^d f(c\,\cdot).
\end{array} \tag{3}
$$

With a slight abuse of notations, for $f \in L^2\left(W^{\otimes d}\right)$, we sometimes use $\mathcal{F}_c[f]$ for both the function in $L^2\left([-1,1]^d\right)$ and in $L^2(\mathbb{R})$. Denote by $\mathcal{E}xt$ the operator which extends a function in $L^2\left([-1,1]^d\right)$ to $L^2\left(\mathbb{R}^d\right)$ by assigning the value 0 outside $[-1,1]^d$, by $\mathcal{F}_{1\mathrm{st}}[f](t,\cdot)$ the partial Fourier transform of f with respect to the first variable. For a random vector $\boldsymbol{X}$, $\mathbb{P}_{\boldsymbol{X}}$ is its law, $f_{\boldsymbol{X}}$ its density and $f_{\boldsymbol{X}|\mathcal{X}}$ the truncated density of $\boldsymbol{X}$ given $\boldsymbol{X} \in \mathcal{X}$ when they exist, and $\mathbb{S}_{\boldsymbol{X}}$ its support. Denote by $f_{Y|\boldsymbol{X}=\boldsymbol{x}}$ for $\boldsymbol{x} \in \mathbb{S}_{\boldsymbol{X}}$ the conditional density. For a sequence of random variables $(X_{n_0,n})_{(n_0,n)\in\mathbb{N}_0^2}$, $X_{n_0,n} = \underset{\mathcal{U}}{O_p}(1)$ means that, for all $\epsilon > 0$, there exists M such that $\mathbb{P}(|X_{n_0,n}| \geq M) \leq \epsilon$ for all $(n_0, n) \in \mathbb{N}_0^2$ such that $\mathcal{U}$ holds. When there is no constraint, we drop $\mathcal{U}$ from the notation and with a single index the $O_p(1)$ notation also requires a bound holding for all value of the index (this is equivalent to the usual notation if all random variables are bounded in probability).

3. Preliminaries

Assumption 1. (H1.1) $f_{\boldsymbol{X}}$ and $f_{\alpha,\boldsymbol{\beta}}$ exist;

(H1.2) $f_{\alpha,\boldsymbol{\beta}} \in L^2\left(w \otimes W^{\otimes p}\right)$, where $w \geq 1$ and W is even, nondecreasing on $[0,\infty)$, such that $W(0) > 0$ and $\lim_{x\to\infty} W(x) = \infty$, $\sum_{k\in\mathbb{N}} M_k^{-1/k} = \infty$ with $M_k = \left(\int_{\mathbb{R}} b^{2k} W^{-1}(b) db\right)^{1/2}$;

(H1.3) For $x_0 \in (0,\infty)$ and $\mathcal{X} = [-x_0, x_0]^p \subseteq \mathbb{S}_{\boldsymbol{X}}$, we have at our disposal an i.i.d sample $(Y_i, \boldsymbol{X}_i)_{i=1}^n$ and an estimator $\widehat{f}_{\boldsymbol{X}|\mathcal{X}}$ based on $\mathcal{G}_{n_0} = (\boldsymbol{X}_i)_{i=-n_0+1}^0$ independent of $(Y_i, \boldsymbol{X}_i)_{i=1}^n$;

(H1.4) $\mathcal{E}$ is a set of smooth densities on $\mathcal{X}$ such that, for $c_{\boldsymbol{X}}, C_{\boldsymbol{X}} \in (0,\infty)$, for all $f \in \mathcal{E}$, $\|f\|_{L^\infty(\mathcal{X})} \leq C_{\boldsymbol{X}}$ and $\|1/f\|_{L^\infty(\mathcal{X})} \leq c_{\boldsymbol{X}}$, and, for $(v(n_0,\mathcal{E}))_{n_0\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ which tends to 0 when n_0 tends to infinity, we have

$$\frac{1}{v(n_0,\mathcal{E})} \sup_{f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}} \left\|\widehat{f}_{\boldsymbol{X}|\mathcal{X}} - f_{\boldsymbol{X}|\mathcal{X}}\right\|_{L^\infty(\mathcal{X})}^2 = O_p(1).$$

When $w = 1$ in (H1.2), the moments $\mathbb{E}\left[\alpha^k\right]$, for $k \in \mathbb{N}$ might not exist. Due to Theorem 3.14 in [17], if there exist $R > 0$, $(a_j)_{j\in\mathbb{N}_0} \in (0,\infty)^{\mathbb{N}_0}$, and $(p_j)_{j\in\mathbb{N}_0} \in (-\infty,1]^{\mathbb{N}_0}$ equal to 0 for j large enough, such that, for all $x \in \mathbb{R}$,

$$W(x) \geq \exp\left(\frac{x^2}{\prod_{j=0}^\infty \log_j^{p_j}(a_j|x|)}\right) \mathbb{1}\{|x| \geq R\} \quad \left(e.g.,\ W(x) = \exp\left(\frac{|x|}{a_0 \log(a_1|x|)}\right)\mathbb{1}\{|x| \geq R\}\right)$$

then $\sum_{m\in\mathbb{N}} 1/\left\|x \mapsto x^{2m}/W(x)\right\|_{L^\infty(\mathbb{R})}^{1/(2m)} = \infty$ which implies (H1.2). Each marginal can have an infinite moment generating function hence be heavy-tailed and their Fourier transforms belong to a quasi-analytic class but might not be analytic. For simplicity, we use $W_{[-R,R]}$ and $\cosh(\cdot/R)$ in the rest of the paper. This rules out heavy tails and nonanalytic Fourier transforms. When $W = W_{[-R,R]}$ with $R > 0$, integrability in $\boldsymbol{b}$ amounts to $\mathbb{S}_{\boldsymbol{\beta}} \subseteq [-R,R]^p$, but other W allow for non compact $\mathbb{S}_{\boldsymbol{\beta}}$. The condition $\mathcal{X} = [-x_0,x_0]^p \subseteq \mathbb{S}_{\boldsymbol{X}}$ in (H1.4) is not restrictive because, for all $\underline{\boldsymbol{x}} \in \mathring{\mathbb{S}}_{\boldsymbol{X}}$, we can rewrite (1) as $Y = \alpha + \boldsymbol{\beta}^\top \underline{\boldsymbol{x}} + \boldsymbol{\beta}^\top(\boldsymbol{X} - \underline{\boldsymbol{x}})$, take $\underline{\boldsymbol{x}} \in \mathbb{R}^p$ and x_0 such that $\mathcal{X} \subseteq \mathbb{S}_{\boldsymbol{X}-\underline{\boldsymbol{x}}}$, and there is a one-to-one mapping between $f_{\alpha+\boldsymbol{\beta}^\top\underline{\boldsymbol{x}},\boldsymbol{\beta}}$ and $f_{\alpha,\boldsymbol{\beta}}$. We assume (H1.4) because the estimator considered involves estimators of $f_{\boldsymbol{X}}$ in denominators (more precisely of $f_{\boldsymbol{X}|\mathcal{X}}$). Alternative solutions exist in the specific case when $p = 1$ (see, *e.g.*, [32]) only. Assuming the availability of an estimator of $f_{\boldsymbol{X}|\mathcal{X}}$ using the preliminary sample $\mathcal{G}_{n_0}$ is common in the deconvolution literature (see, *e.g.*, [14]). By using estimators of $f_{\boldsymbol{X}|\mathcal{X}}$ for a well chosen $\mathcal{X}$ rather than of $f_{\boldsymbol{X}}$, the assumption that $\left\|f_{\boldsymbol{X}|\mathcal{X}}\right\|_{L^\infty(\mathcal{X})} \leq C_{\boldsymbol{X}}$ and $\left\|1/f_{\boldsymbol{X}|\mathcal{X}}\right\|_{L^\infty(\mathcal{X})} \leq c_{\boldsymbol{X}}$ in (H1.4) becomes very mild. This is feasible because of (2).

3.1. **Inverse problem in Hilbert spaces.** Estimation of $f_{\alpha,\boldsymbol{\beta}}$ is a statistical ill-posed inverse problem. The operator depends on the choice of w and W such that $f_{\alpha,\boldsymbol{\beta}} \in L^2\left(W^{\otimes p}\right)$. Now

on, the functions w and W are those of (H1.2). We have, for all $t \in \mathbb{R}$ and $\boldsymbol{u} \in [-1,1]^p$, $\mathcal{K} f_{\alpha,\boldsymbol{\beta}}(t,\boldsymbol{u}) = \mathcal{F}\left[f_{Y|\boldsymbol{X}=x_0\boldsymbol{u}}\right](t)\,|tx_0|^{p/2}$, where

$$\begin{array}{rccc} \mathcal{K}: & L^2\left(w \otimes W^{\otimes p}\right) & \to & L^2(\mathbb{R}\times[-1,1]^p) \\ & f & \to & (t,\boldsymbol{u}) \mapsto \mathcal{F}\left[f\right](t, tx_0\boldsymbol{u})\,|tx_0|^{p/2}\,. \end{array} \tag{4}$$

Proposition 1. $L^2\left(w \otimes W^{\otimes p}\right)$ is continuously embedded into $L^2(\mathbb{R}^{p+1})$. Moreover, $\mathcal{K}$ is injective and continuous, and not compact if $w = 1$.

The case $w = 1$ corresponds to mild integrability assumptions in the first variable of $f_{\alpha,\boldsymbol{\beta}}$. In this situation, the SVD of $\mathcal{K}$ does not exist which makes it difficult to prove rates of convergence even for estimators which do not rely explicitly on the SVD such as the Tikhonov and Landweber method (Gerchberg algorithm in out-of-band extrapolation, see, *e.g.*, [5]). In order to make mild integrability assumptions in the first variable of $f_{\alpha,\boldsymbol{\beta}}$, we do not work with $\mathcal{K}$ directly but use that $\mathcal{K}$ is the composition of operators which are easier to analyze

$$\text{for } t \in \mathbb{R},\ \mathcal{K}[f](t,\star) = \mathcal{F}_{tx_0}\left[\mathcal{F}_{1\text{st}}\left[f\right](t,\cdot)\right](\star)\,|tx_0|^{p/2} \text{ in } L^2([-1,1]^p). \tag{5}$$

For $f \in L^2\left(w \otimes W^{\otimes p}\right)$, W either $W_{[-R,R]}$ or $\cosh(\cdot/R)$, and $t \in \mathbb{R}$, $\mathcal{F}_{1\text{st}}\left[f\right](t,\cdot)$ belongs to $L^2(W^{\otimes p})$ and, for $c \neq 0$, $\mathcal{F}_c:\ L^2(W^{\otimes p}) \to L^2([-1,1]^p)$ admits a SVD, where both orthonormal systems are complete. This is a tensor product of the SVD when $p = 1$ that we denote by $\left(\sigma_m^{W,c}, \varphi_m^{W,c}, g_m^{W,c}\right)_{m\in\mathbb{N}_0}$, where $\left(\sigma_m^{W,c}\right)_{m\in\mathbb{N}_0} \in (0,\infty)^{\mathbb{N}_0}$ is in decreasing order repeated according to multiplicity, $\left(\varphi_m^{W,c}\right)_{m\in\mathbb{N}_0}$ and $\left(g_m^{W,c}\right)_{m\in\mathbb{N}_0}$ are orthonormal systems of, respectively, $L^2(W)$ and $L^2([-1,1])$. This holds for the following reason. Because $\mathcal{F}_c = \mathcal{F}\mathcal{C}_{c^{-1}} = |c|^{-1}\mathcal{C}_c\mathcal{F}$, $\Pi\mathcal{F}_c = \mathcal{F}_c\Pi$, $\mathcal{F}_c^* = W^{-1}\Pi\mathcal{F}_c\mathcal{E}xt$, and W is even, we obtain $\mathcal{F}_c^* = \Pi\left(W^{-1}\mathcal{F}_c\mathcal{E}xt\right)$ and

$$\begin{aligned} \mathcal{F}_c\mathcal{F}_c^* &= \Pi\mathcal{F}_c\left(W^{-1}\mathcal{F}_c\mathcal{E}xt\right) \\ &= \frac{2\pi}{|c|}\mathcal{F}^I\left(\mathcal{C}_{c^{-1}}\left(W^{-1}\mathcal{C}_c\mathcal{F}\mathcal{E}xt\right)\right) \\ &= 2\pi\mathcal{F}^I\left(\mathcal{C}_{c^{-1}}\left(W^{-1}\right)\mathcal{F}\mathcal{E}xt\right). \end{aligned}$$

The operator $\mathcal{Q}_c^W = (|c|/(2\pi))\mathcal{F}_c\mathcal{F}_c^*$ is a compact positive definite hermitian operator (see [42] and [47] for the two choices of W). Its eigenvalues in decreasing order repeated according to multiplicity are denoted by $\left(\rho_m^{W,c}\right)_{m\in\mathbb{N}_0}$ and a basis of eigenfunctions by $\left(g_m^{W,c}\right)_{m\in\mathbb{N}_0}$. The other elements of the SVD are $\sigma_m^{W,c} = \sqrt{2\pi\rho_m^{W,c}/|c|}$ and $\varphi_m^{W,c} = \mathcal{F}_c^* g_m^{W,c}/\sigma_m^{W,c}$.

Proposition 2. For $c \neq 0$, $\left(\varphi_m^{W,c}\right)_{m\in\mathbb{N}_0}$ is a basis of $L^2(W)$.

The singular vectors $\left(g_m^{W_{[-1,1]},c}\right)_{m\in\mathbb{N}_0}$ are the Prolate Spheroidal Wave Functions (hereafter PSWF, see, *e.g.*, [42]). They can be extended as entire functions in $L^2(\mathbb{R})$ and form a complete orthogonal system of $PW(c)$ for which we use the same notation. They are useful to carry interpolation and extrapolation (see, *e.g.*, [38]) with Hilbertian techniques. In this paper, for all $t \neq 0$, $\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot)$ plays the role of the Fourier transform in the definition of $PW(c)$. The weight $\cosh(\cdot/R)$ allows for larger classes than $PW(c)$ and noncompact $\mathbb{S}_{\boldsymbol{\beta}}$. This is useful even if $\mathbb{S}_{\boldsymbol{\beta}}$ is compact when the researcher does not know a superset containing $\mathbb{S}_{\boldsymbol{\beta}}$. The useful results on the corresponding SVD and a numerical algorithm to compute it are given in [20].

3.2. **Sets of smooth and integrable functions.** Define, for $(\phi(t))_{t\geq 0}$ and $(\omega_m)_{m\in\mathbb{N}_0}$ increasing, $\phi(0) = \omega_0 = 1$, $l, M > 0$, $q \in \{1, \infty\}$, $t \in \mathbb{R}$, $\boldsymbol{m} \in \mathbb{N}_0^p$, $k \in \mathbb{N}_0$, and $c(t) = tx_0$,

$$\mathcal{H}_{w,W}^{q,\phi,\omega}(l,M) = \left\{ f : \sum_{k\in\mathbb{N}_0} \int_{\mathbb{R}} \phi^2(|t|)\theta_{q,k}^2(t)dt \bigvee \sum_{k\in\mathbb{N}_0} \omega_k^2 \|\theta_{q,k}\|_{L^2(\mathbb{R})}^2 \leq 2\pi l^2,\ \|f\|_{L^2(w\otimes W^{\otimes p})} \leq M \right\}$$

and $\mathcal{H}_{w,W}^{q,\phi,\omega}(l)$ when we replace $\|f\|_{L^2(w\otimes W^{\otimes p})} \leq M$ by $\|f\|_{L^2(w\otimes W^{\otimes p})} < \infty$, where

$$(6) \qquad b_{\boldsymbol{m}}(t) = \left\langle \mathcal{F}_{1\text{st}}\left[f\right](t,\cdot), \varphi_{\boldsymbol{m}}^{W,c(t)}(\cdot) \right\rangle_{L^2(W^{\otimes p})},\ \theta_{q,k}(t) = \left(\sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_q = k} |b_{\boldsymbol{m}}(t)|^2 \right)^{1/2}.$$

The first inequality in the definition of $\mathcal{H}_{w,W}^{q,\phi,\omega}(l,M)$ defines the notion of smoothness for functions in $L^2\left(1\otimes W^{\otimes p}\right)$ analyzed in this paper. It involves a maximum of two terms, thus two inequalities: the first corresponds to smoothness in the first variable and the second to smoothness in the other variables. The additional inequality imposes integrability in the first variable. The asymmetry in the treatment of the first and remaining variables is due to the fact that, in the statistical problem, only the random slopes are multiplied by regressors which have limited variation and we make integrability assumptions in the first variable which are as mild as possible. The use of the Fourier transform to express smoothness in the first variable is classical. For the remaining variables, we choose a framework that allows for both functions with compact and noncompact support and work with the bases $\left(\varphi_{\boldsymbol{m}}^{W,c(t)}\right)_{\boldsymbol{m}\in\mathbb{N}_0^p}$ for $t \neq 0$. For functions with compact support, it is possible to use Fourier series and we make a comparison in Section A.3. The use of different bases for different values of t is motivated by (5). Though the spaces are

chosen for mathematical convenience, we analyze all types of smoothness. The smoothness being unknown anyway, we provide an adaptive estimator. We analyze two values of q and show that the choice of the ℓ_q norm matters for the rates of convergence for supersmooth functions.

Remark 1. The next model is related to (1) under Assumption 1 when $f_{\boldsymbol{X}}$ is known:

$$dZ(t) = \mathcal{K}[f](t,\cdot)dt + \frac{\sigma}{\sqrt{n}}dG(t), \quad t \in \mathbb{R}, \tag{7}$$

where $f \in L^2(w \otimes W^{\otimes p})$ plays the role of $f_{\alpha,\boldsymbol{\beta}}$, $\sigma > 0$ is known, and $(G(t))_{t\in\mathbb{R}}$ is a complex two-sided cylindrical Gaussian process on $L^2([-1,1]^p)$. This means that, for Φ Hilbert-Schmidt from $L^2([-1,1]^p)$ to a separable Hilbert space H, $(\Phi G(t))_{t\in\mathbb{R}}$ is a Gaussian process in H of covariance $\Phi\Phi^*$ (see [16]). Taking $\Phi G(t) = \sum_{\boldsymbol{m}\in\mathbb{N}_0^p} \Phi\left[g_{\boldsymbol{m}}^{W,c(t)}\right] B_{\boldsymbol{m}}(t)$, where $B_{\boldsymbol{m}}(t) = B_{\boldsymbol{m}}^{\mathfrak{R}}(t) + iB_{\boldsymbol{m}}^{\mathfrak{I}}(t)$, $(B_{\boldsymbol{m}}^{\mathfrak{R}}(t))_{t\in\mathbb{R}}$ and $\left(B_{\boldsymbol{m}}^{\mathfrak{I}}(t)\right)_{t\in\mathbb{R}}$ are independent two-sided Brownian motions, the system of independent equations, for $Z_{\boldsymbol{m}}(t) = \left\langle Z(t), g_{\boldsymbol{m}}^{W,c(t)}\right\rangle_{L^2([-1,1]^p)}$ and $\boldsymbol{m} \in \mathbb{N}_0^p$,

$$Z_{\boldsymbol{m}}(t) = \int_0^t \sigma_{\boldsymbol{m}}^{W,c(s)} b_{\boldsymbol{m}}(s)ds + \frac{\sigma}{\sqrt{n}}B_{\boldsymbol{m}}(t), \quad t \in \mathbb{R} \tag{8}$$

is equivalent to (7). The coefficient $\sigma_{\boldsymbol{m}}^{W,c(s)}$ is small when $|\boldsymbol{m}|_q$ is large or s is small (see Lemma A.1). This is why the estimator of Section 4.1 truncates large values of $|\boldsymbol{m}|_q$ and does not rely on small values of s in absolute value but uses interpolation.

Remark 2. [29] considers the Gaussian sequence model corresponding to (7), when $\mathcal{K}$ is the Radon transform, $p = 1$, G is a two-sided cylindrical Wiener process, and $L^2(W^{\otimes p})$ is instead a weighted L^2 space of functions with support in the disk of $\mathbb{R}^2$ for which $\mathcal{K}$ has a SVD with a known rate of decay of the singular values.

3.3. **Interpolation.** We present the interpolation procedure with Hilbertian techniques and the main results which can be of independent interest. Define, for $\underline{a}, \epsilon > 0$, the operator

$$\mathcal{I}_{\underline{a},\epsilon}[f](\cdot) = \sum_{m\in\mathbb{N}_0} \frac{\rho_m^{W_{[-1,1]},\underline{a}\epsilon}}{\left(1-\rho_m^{W_{[-1,1]},\underline{a}\epsilon}\right)\epsilon} \left\langle f(\star), g_m^{W_{[-1,1]},\underline{a}\epsilon}\left(\frac{\star}{\epsilon}\right)\right\rangle_{L^2(\mathbb{R}\setminus(-\epsilon,\epsilon))} g_m^{W_{[-1,1]},\underline{a}\epsilon}\left(\frac{\cdot}{\epsilon}\right) \tag{9}$$

on $L^2(\mathbb{R})$ with domain $PW(\underline{a})$. For an arbitrary function $f \in L^2(\mathbb{R})$, $\mathcal{I}_{\underline{a},\epsilon}[f]$ is a distribution (*i.e.*, a generalized function). The following proposition clarifies this statement.

Proposition 3. For $\underline{a}, \epsilon > 0$, $\mathcal{I}_{\underline{a},\epsilon}\left(L^2(\mathbb{R})\right) \subseteq L^2([-\epsilon,\epsilon])$. Moreover, we have, for $g \in PW(\underline{a})$ $\mathcal{I}_{\underline{a},\epsilon}[g] = g$ in $L^2(\mathbb{R})$ and, for $C(\underline{a},\epsilon) = 4\underline{a}\epsilon/\left(\pi\left(1-\rho_0^{W_{[-1,1]},\underline{a}\epsilon}\right)^2\right)$ and all $f, h \in L^2(\mathbb{R})$,

$$\left\|f - \mathcal{I}_{\underline{a},\epsilon}[h]\right\|^2_{L^2([-\epsilon,\epsilon])} \leq 2(1 \vee C(\underline{a},\epsilon))\left\|f - \mathcal{P}_{\underline{a}}[f]\right\|^2_{L^2(\mathbb{R})} + 2C(\underline{a},\epsilon)\left\|f-h\right\|^2_{L^2(\mathbb{R}\setminus(-\epsilon,\epsilon))}. \tag{10}$$

If $f \in PW(\underline{a})$, $\mathcal{I}_{\underline{a},\epsilon}[f]$ only relies on $f\mathbb{1}\{\mathbb{R}\setminus(-\epsilon,\epsilon)\}$ and $\mathcal{I}_{\underline{a},\epsilon}[f] = f$ on $\mathbb{R}\setminus(-\epsilon,\epsilon)$, so (9) provides an analytic formula to carry interpolation on $[-\epsilon,\epsilon]$ of functions in $PW(\underline{a})$. Else, (10) provides an upper bound on the error made by approximating f by $\mathcal{I}_{\underline{a},\epsilon}[h]$ on $[-\epsilon,\epsilon]$ when h approximates f outside $[-\epsilon,\epsilon]$. We use interpolation when the variance of an initial estimator $\widehat{f}^0$ of f is large due to its values near 0 but $\left\|f-\widehat{f}^0\right\|^2_{L^2(\mathbb{R}\setminus(-\epsilon,\epsilon))}$ is small and work with

$$\forall t \in \mathbb{R},\ \widehat{f}(t) = \widehat{f}^0(t)\mathbb{1}\{|t| \geq \epsilon\} + \mathcal{I}_{\underline{a},\epsilon}\left[\widehat{f}^0\right](t)\mathbb{1}\{|t| < \epsilon\},$$

in which case, (10) yields

$$\left\|f-\widehat{f}\right\|^2_{L^2(\mathbb{R})} \leq (1+2C(\underline{a},\epsilon))\left\|f-\widehat{f}^0\right\|^2_{L^2(\mathbb{R}\setminus(-\epsilon,\epsilon))} + 2(1\vee C(\underline{a},\epsilon))\left\|f-\mathcal{P}_{\underline{a}}[f]\right\|^2_{L^2(\mathbb{R})}. \tag{11}$$

When $\mathrm{supp}\left(\mathcal{F}[f]\right)$ is compact, we take $\underline{a}$ such that $\mathrm{supp}\left(\mathcal{F}[f]\right) \subseteq [-\underline{a},\underline{a}]$. Else, we take $\underline{a}$ going to infinity so the second term in (11) goes to 0. We take ϵ such that $\underline{a}\epsilon$ is constant because, due to (3.87) in [42], $\lim_{\underline{a}\epsilon\to\infty} C(\underline{a},\epsilon) = \infty$ and (10) and (11) become useless. Then $C(\underline{a},\epsilon)$ is constant and we set $C = 2\left(1 \vee (2C(\underline{a},\epsilon))\right)$. For $\underline{a}\epsilon = 1$, we get $\rho_0^{W_{[-1,1]},\underline{a}\epsilon} \approx 0.3019$ and $C = 2$.

3.4. **Risk.** The risk of an estimator $\widehat{f}_{\alpha,\boldsymbol{\beta}}$ is the mean integrated squared error (MISE)

$$\mathcal{R}^W_{n_0}\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) = \mathbb{E}\left[\left\|\widehat{f}_{\alpha,\boldsymbol{\beta}} - f_{\alpha,\boldsymbol{\beta}}\right\|^2_{L^2(1\otimes W^{\otimes p})}\middle|\mathcal{G}_{n_0}\right].$$

When $W = W_{[-R,R]}$ and $\mathrm{supp}\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}\right) \subseteq \mathbb{R}\times[-R,R]^p$, the risk is $\mathbb{E}\left[\left\|\widehat{f}_{\alpha,\boldsymbol{\beta}} - f_{\alpha,\boldsymbol{\beta}}\right\|^2_{L^2(\mathbb{R}^{p+1})}\middle|\mathcal{G}_{n_0}\right]$, else,

$$\mathbb{E}\left[\left\|\widehat{f}_{\alpha,\boldsymbol{\beta}} - f_{\alpha,\boldsymbol{\beta}}\right\|^2_{L^2(\mathbb{R}^{p+1})}\middle|\mathcal{G}_{n_0}\right] \leq \left\|W^{-1}\right\|^p_{L^\infty(\mathbb{R})}\mathcal{R}^W_{n_0}\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right). \tag{12}$$

We consider a risk conditional on $\mathcal{G}_{n_0}$ for simplicity of the treatment of the random regressors with unknown law. We adopt the minimax approach and consider the supremum risk.

4. Estimation

The sets of densities in the supremum risk and of estimators in this section depend on $q \in \{1,\infty\}$. The rates of convergence depend on q via $k_q = 1 + (p-1)\mathbb{1}\{q=\infty\}$.

4.1. **Estimator considered.** For $q \in \{1, \infty\}$, $0 < \epsilon < 1 < T$ and $N : \mathbb{R} \to \mathbb{N}_0$ such that $N(t) = N(\epsilon)$ for $t \in [-\epsilon, \epsilon]$ and $N(t) = N(T)$ for $|t| > T$, a regularized inverse is obtained by:

(S.1) for all $t \neq 0$, obtain a preliminary approximation of $F_1(t, \cdot) = \mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t, \cdot)$

$$F_1^{q,N,T,0}(t, \cdot) = \mathbb{1}\{|t| \leq T\} \sum_{|\boldsymbol{m}|_q \leq N(t)} \frac{c_{\boldsymbol{m}}(t)}{\sigma_{\boldsymbol{m}}^{W,tx_0}} \varphi_{\boldsymbol{m}}^{W,tx_0}(\cdot), \tag{13}$$

where $c_{\boldsymbol{m}}(t) = \left\langle \mathcal{F}\left[f_{Y|\mathcal{X},\boldsymbol{X}=x_0\cdot}\right](t), g_{\boldsymbol{m}}^{W,tx_0}(\cdot)\right\rangle_{L^2([-1,1]^p)}$,

(S.2) refine for $t \in [-\epsilon, \epsilon]$

$$F_1^{q,N,T,\epsilon}(t, \cdot) = F_1^{q,N,T,0}(t, \cdot)\mathbb{1}\{|t| \geq \epsilon\} + \mathcal{I}_{\underline{a},\epsilon}\left[F_1^{q,N,T,0}(\star, \cdot)\right](t)\mathbb{1}\{|t| < \epsilon\},$$

(S.3) take $f_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}(\cdot_1, \cdot_2) = \mathcal{F}_{1\text{st}}^I\left[F_1^{q,N,T,\epsilon}(\star, \cdot_2)\right](\cdot_1)$.

To deal with the statistical problem, we carry (S.1)-(S.3) replacing $c_{\boldsymbol{m}}$ by the estimator

$$\widehat{c}_{\boldsymbol{m}}(t) = \frac{1}{n}\sum_{j=1}^{n} \frac{e^{itY_j}}{x_0^p \widehat{f}_{\boldsymbol{X}|\mathcal{X}}^{\delta}(\boldsymbol{X}_j)} \overline{g_{\boldsymbol{m}}^{W,tx_0}}\left(\frac{\boldsymbol{X}_j}{x_0}\right) \mathbb{1}\left\{\boldsymbol{X}_j \in \mathcal{X}\right\}, \tag{14}$$

where $\widehat{f}_{\boldsymbol{X}|\mathcal{X}}^{\delta}(\boldsymbol{X}_j) = \widehat{f}_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{X}_j) \vee \sqrt{\delta(n_0)}$ and $\delta(n_0)$ is a trimming factor converging to zero with n_0. This yields the estimators $\widehat{F}_1^{q,N,T,0}$, $\widehat{F}_1^{q,N,T,\epsilon}$, and $\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}$. We use $\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}\right)_+$ as a final estimator of $f_{\alpha,\boldsymbol{\beta}}$ which always has a smaller risk than $\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}$ (see [23, 46]).

4.2. **Effective sample size.** We use $n_e = n \wedge (\delta(n_0)/v(n_0, \mathcal{E}))$ for the sample size required for an ideal estimator when $f_{\boldsymbol{X}|\mathcal{X}}$ is known to achieve the same rate as the estimator in this paper.

4.3. **Logarithmic rates when ω is a power.** The first result below involves, for $t, u > 0$ and $R, x_0 > 0$, the inverse $Q_{t,u}$ of $x \in (0, \infty) \mapsto x\ln(1 \vee (7\pi(2x+1)/(Rx_0t))) + u\ln(2x+1)$ which is such that, for all $x, u \in (0, \infty)$, $t \in (0, \infty) \mapsto Q_{t,u}(x)$ is increasing.

Theorem 1. Make Assumption 1. Take $\phi(\cdot) = 1 \vee |\cdot|^s$, $(\omega_k)_{k\in\mathbb{N}_0} = (k^\sigma)_{k\in\mathbb{N}_0}$, $q \in \{1, \infty\}$, $w(\cdot) = 1 \vee |\cdot|$, $l, M, s, R > 0$, $\sigma > 1/2$, $\mathbb{S}_{\boldsymbol{\beta}} \subseteq [-R, R]^p$, $N(t) = \lceil \overline{N}(t) \rceil$ for $\epsilon \leq |t| \leq T$, and $\underline{a} = 1/\epsilon$.

(T1.1) When $W = W_{[-R,R]}$, $\overline{N}(t) = Q_{|t|,(2\sigma+p-1)/4}\left(\ln(n_e)/(8k_q)\right)$, $T = n_e^{1/(2(p+1))}$, and $\epsilon = \left(\ln\left(n_e\right)/\ln_2\left(n_e\right)\right)^{-2\sigma}$, we have

$$\sup_{\substack{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}_{w,W}^{q,\phi,\omega}(l,M) \cap \mathcal{D} \\ f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}}} \mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right)\left(\frac{\ln\left(n_e\right)}{\ln_2\left(n_e\right)}\right)^{2\sigma} = O_p(1), \tag{15}$$

(T1.2) When $W(\cdot)=\cosh(\cdot/R)$,

$$\overline{N}(t)=\frac{\ln(n_e)}{2}\left(\frac{\mathbb{1}\left\{|t|>\frac{\pi}{4Rx_0}\right\}}{2\sigma+p-k_q+\pi k_q\frac{(2+p\mathbb{1}\{q=1\})}{2Rx_0|t|}}+\frac{\mathbb{1}\left\{|t|\leq\frac{\pi}{4Rx_0}\right\}}{2\sigma+p-k_q+2k_q(2-\ln(Rx_0\,|t|))}\right),$$

$T=n_e^{1/(2(p+2))}$, and $\epsilon=\ln(n_e)^{-2\sigma}$, then (15) also holds.

Theorem 2. Take $q\in\{1,\infty\}$, $\phi(\cdot)=1\vee|\cdot|^s$, $(\omega_k)_{k\in\mathbb{N}_0}=(k^\sigma)_{k\in\mathbb{N}_0}$, $w(\cdot)=1\vee|\cdot|$, and $0<l,s,R<\infty$.

(1) Let $W=W_{[-R,R]}$, $\sigma>2+k_q/2$, assume that $f_{\boldsymbol{X}}$ is known, $\mathbb{S}_{\boldsymbol{X}}=\mathcal{X}$, and $\|f_{\boldsymbol{X}}\|_{L^\infty(\mathcal{X})}<\infty$, there exists $\nu>0$ such that

$$\underline{\lim}_{n\to\infty}\inf_{\widehat{f}_{\alpha,\boldsymbol{\beta}}}\sup_{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l)\cap\mathcal{D}}\mathbb{E}\left[\left\|\widehat{f}_{\alpha,\boldsymbol{\beta}}-f_{\alpha,\boldsymbol{\beta}}\right\|^2_{L^2(\mathbb{R}^{p+1})}\right]\left(\frac{\ln(n)}{\ln_2(n)}\right)^{2\sigma}\geq\nu.$$

(2) In model (8) with $W=\cosh(\cdot/R)$ and $\sigma>1/2$, there exists $\nu>0$ such that

$$\underline{\lim}_{n\to\infty}\inf_{\widehat{f}}\sup_{f\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l)}\mathbb{E}\left[\left\|\widehat{f}-f\right\|^2_{L^2(\mathbb{R}^{p+1})}\right]\ln\left(\frac{n}{\ln(n)}\right)^{2\sigma}\geq\nu.$$

Theorem 2 shows that the rate in Theorem 1 (T1.1) is minimax optimal in model (1) where $f_{\boldsymbol{X}}$ is known and $\mathbb{S}_{\boldsymbol{X}}=\mathcal{X}$. This is the same rate as in [39] for deconvolution on the line with a known characteristic function of the noise on a bounded interval when the density of the signal has compact support, though for different smoothness assumptions. The rate in (T1.2) is for densities where we do not assume $\mathbb{S}_{\boldsymbol{\beta}}$ is compact but $\int\int f_{\alpha,\boldsymbol{\beta}}(a,\boldsymbol{b})^2w(a)W^{\otimes p}(\boldsymbol{b})dad\boldsymbol{b}<\infty$. In a similar spirit, the discussion after Theorem 2 in [39] considers classes of densities with non compact support but with a pointwise bound on the density outside $[-1,1]$. Using (12) and Theorem 2-(2), we obtain a lower bound for the risk $\mathcal{R}^W_{n_0}\left(\widehat{f}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right)$ for a class of densities with unbounded support. This lower bound differs slightly from the bound in Theorem 1 (T1.2), hence the minimax rate is somewhere between both rates. The rates in Theorem 1 are independent of p as is common for severely ill-posed problems (see [13, 20]).

4.4. **Polynomial and nearly parametric rates when ω is exponential.** Here $Q_{t,u}$ is the inverse of the increasing function $x\in(0,\infty)\mapsto x\ln\left(1\vee(7\pi(x+1)/(Rx_0t))\right)+ux\ln(x+1)$.

Theorem 3. Make Assumption 1 and take $s,\kappa,\gamma,l,M,R,\underline{a}>0$, $r,\rho\geq1$ and $q\in\{1,\infty\}$.

(T3.1) When $W = W_{[-R,R]}$, $\mathbb{S}_{\boldsymbol{\beta}} \subseteq [-R,R]^p$, $\phi(\cdot) = 1 \vee |\cdot|^s$, $(\omega_k)_{k\in\mathbb{N}_0} = \left(e^{\kappa(k\ln(1+k))^r}\right)_{k\in\mathbb{N}_0}$, $\kappa > k_q$, $N(t) = \left\lfloor \overline{N}(t) \right\rfloor$ for $\epsilon \leq |t| \leq T$, $\overline{N}(t) = Q_{|t|,\kappa/k_q}(\ln(n_e)/(2k_q))$, and $T = 7\pi e^{\kappa N(\epsilon)\ln(N(\epsilon)+1)/s}/(Rx_0)$, we have

(T3.1.1) for $r = \rho = 1$, $w(\cdot) = e^{\gamma(|\cdot|\ln(1\vee|\cdot|))^\rho}$, $\gamma > 1/(4k_q)$, $s > \kappa/(2k_q)$, $\epsilon\underline{a} = 7\pi/(Rx_0)$, and $\epsilon = 7\pi\ln_2(n)/(\ln(n)Rx_0)$,

$$\sup_{\substack{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}_{w,W}^{q,\phi,\omega}(l,M)\cap\mathcal{D}\\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}} \mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right) \frac{n_e^{\kappa/(\kappa+2k_q)}}{\ln(n_e)^{2\kappa+p+1+\mathbb{1}\{q=\infty\}}} = O_p(1), \tag{16}$$

(T3.1.2) for $r = 1$, $w = W_{[-\underline{a},\underline{a}]}$, $\mathbb{S}_\alpha \subseteq [-\underline{a},\underline{a}]$, $s > \kappa/k_q$, and $\epsilon = 7\pi/(Rx_0)$,

$$\sup_{\substack{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}_{w,W}^{q,\phi,\omega}(l)\cap\mathcal{D}\\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}} \mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right) \frac{n_e^{\kappa/(\kappa+k_q)}}{\ln(n_e)^{2\kappa+p+1+\mathbb{1}\{q=\infty\}}} = O_p(1). \tag{17}$$

(T3.2) When $W = W_{[-R,R]}$, $\mathbb{S}_{\boldsymbol{\beta}} \subseteq [-R,R]^p$, for $r > 1$, $w(\cdot) = e^{\gamma(|\cdot|\ln(1\vee|\cdot|))^\rho}$, $\rho \geq r$, $\gamma > \kappa$, $\phi(\cdot) = e^{\nu|\cdot|}$, $\underline{a} = 1/\epsilon$, $\nu > 0$ large enough (to satisfy (A.118)), $N(t) = \lceil \overline{N} \rceil$ for $\epsilon \leq |t| \leq T$, $\epsilon = 14\pi/\left(Rx_0\left(1+\overline{N}\right)\right)$, $T = \exp\left(2\overline{N}\ln\left(1+\overline{N}\right)\right)$,

$$\overline{N} = Q_{\infty,1}\left(\left(\frac{1}{2\kappa}\left(\ln(n_e) - \sum_{i=0}^{k} d_i \ln(n_e)^{(i+1)/r-i}\right)\right)^{1/r}\right),$$

where $k/(k+1) \leq 1/r \leq (k+1)/(k+2)$, $d_0 = 4/(2\kappa)^{1/r}$ and $(d_i)_{i=1}^k$ are such that

$$d_i = \frac{4(-1)^k}{(2\kappa)^{1/r}} \sum_{j=1}^{i} \frac{(1/r)\ldots(1/r-j+1)}{j!} \sum_{p_1+\cdots+p_j=i} d_{p_1-1}\ldots d_{p_j-1},$$

and $\varphi(\cdot) = \exp\left(\sum_{i=0}^k d_i \ln(\cdot)^{(i+1)/r-i}\right)/\ln(\cdot)^{4p/r}$, we have

$$\sup_{\substack{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}_{w,W}^{q,\phi,\omega}(l,M)\cap\mathcal{D}\\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}} \mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right) \varphi(n_e)\, n_e = O_p(1).$$

(T3.3) When $W(\cdot) = \cosh(\cdot/R)$, $\phi(\cdot) = 1\vee|\cdot|^s$, $(\omega_k)_{k\in\mathbb{N}_0} = (e^{\kappa k})_{k\in\mathbb{N}_0}$, $\kappa > k_q(\pi(s/(p+1)+1)/4-1)$ and $Rx_0 > \pi\kappa/(2(p+1))$, $w = W_{[-\underline{a},\underline{a}]}$, $\mathbb{S}_\alpha \subseteq [-\underline{a},\underline{a}]$, $\epsilon = \pi/(4Rx_0)$, $N(t) = \left\lfloor \overline{N}(t) \right\rfloor$

for $\epsilon \leq |t| \leq T$, $\overline{N}(t) = \ln(n_e)/(2\kappa + \pi k_q/(2Rx_0|t|))$, and $T = e^{\kappa \overline{N}(\epsilon)/s}/(Rx_0)$, we have

$$\sup_{\substack{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}^{q,\phi,\omega}_{w,W}(l) \cap \mathcal{D} \\ f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}}} \mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) \frac{n_e^{\kappa/(\kappa+k_q)}}{\ln(n_e)^{2(p+1)-p\mathbf{1}\{q=\infty\}}} = O_p(1).$$

We relax the assumption that $\mathbb{S}_{\boldsymbol{\beta}}$ is compact maintained in (T3.1) in (T3.3). Similarly, we can extend (T3.2) to noncompact $\mathbb{S}_{\boldsymbol{\beta}}$. The results of Theorem 3 are related to those for the "2exp-severely ill-posed problems" (see [12] for a survey and [45] which obtains the same polynomial rates up to logarithmic factor as in (T3.1.2) when $1/v(n_0, \mathcal{E}) \geq n$ and $p = 1$). When $1/v(n_0, \mathcal{E}) \geq n$, the rate in (T3.1.2) matches that of the lower bound in model (8) given below.

Theorem 4. Consider model (8) with $\phi(\cdot) = 1 \vee |\cdot|^s$, $w(\cdot) = 1 \vee |\cdot|$, $s, \kappa, l, R > 0$.

(1) When $W = W_{[-R,R]}$, and $(\omega_k)_{k\in\mathbb{N}_0} = \left(e^{\kappa k \ln(1+k)}\right)_{k\in\mathbb{N}_0}$, there exists $\nu > 0$ such that

$$\underline{\lim}_{n\to\infty} \inf_{\widehat{f}} \sup_{f \in \mathcal{H}^{q,\phi,\omega}_{w,W}(l)} \mathbb{E}\left[\left\|\widehat{f} - f\right\|^2_{L^2(\mathbb{R}^{p+1})}\right] n^{\kappa/(\kappa+k_q)} \geq \nu.$$

(2) When $W = \cosh(\cdot/R)$, and $(\omega_k)_{k\in\mathbb{N}_0} = \left(e^{\kappa k}\right)_{k\in\mathbb{N}_0}$, there exists $\nu > 0$ such that

$$\underline{\lim}_{n\to\infty} \inf_{\widehat{f}} \sup_{f \in \mathcal{H}^{q,\phi,\omega}_{w,W}(l)} \mathbb{E}\left[\left\|\widehat{f} - f\right\|^2_{L^2(\mathbb{R}^{p+1})}\right] n^{\kappa/(\kappa+k_q)} \geq \nu.$$

4.5. **Data-driven estimator.** We present a type of Goldenshluger-Lepski method (see [26] and the references therein) following a variant proposed by [36] to determine data-driven smoothing parameters. Let $\epsilon > 0$, $K_{\max} = \lfloor \zeta_0 \log(n)/\log(2) \rfloor$, where $\zeta_0 > 0$ depends on p and W and which value is given in Theorem 5, and $T_{\max} = 2^{K_{\max}}$. Theorems 1 and 3 are obtained by optimizing over functions $N \in \mathbb{N}_0^{\mathbb{R}}$. We consider the class

$$\mathcal{N}_n = \left\{N \in \mathbb{N}_0^{\mathbb{R}} : \ \forall t \in \mathbb{R} \setminus (-\epsilon, \epsilon), \ N(t) \in \{0, \ldots, N_{\max,q}(W,t)\}\right\}.$$

$\widehat{N}$ is obtained by solving univariate minimisation problems

$$\forall t \in \mathbb{R} \setminus (-\epsilon, \epsilon), \quad \widehat{N}(t) \in \operatorname*{argmin}_{0 \leq N \leq N_{\max,q}(W,t)} \left(B_1(t,N) + c_1 \Sigma(t,N)\right), \tag{18}$$

where $c_1 \geq 31/30$ is greater than 1 to handle the estimation of $f_{\boldsymbol{X}|\mathcal{X}}$ and

$$B_1(t,N) = \max_{N_{\max,q}(W,t) \geq N' \geq N} \left(\sum_{N \leq |\boldsymbol{m}|_q \leq N'} \frac{|\widehat{c}_{\boldsymbol{m}}(t)|^2}{\left(\sigma^{W,tx_0}_{\boldsymbol{m}}\right)^2} - \Sigma\left(t, N'\right) \right)_+,$$

$$\Sigma(t,N) = \frac{84(1+2((2\log(n))\vee 3))c_{\boldsymbol{X}}}{n}\left(\frac{|t|}{2\pi}\right)^p \nu_q(W,N,tx_0).$$

Define $\widehat{T}$ as

$$\widehat{T} \in \underset{T\in\mathcal{T}_n}{\operatorname{argmin}} \left(B_2\left(T,\widehat{N}\right) + \int_{\epsilon\le|t|\le T} \Sigma\left(t,\widehat{N}(t)\right) dt \right), \tag{19}$$

where

$$B_2\left(T,\widehat{N}\right) = \max_{T'\in\mathcal{T}_n, T'\ge T} \left(\int_{T\le|t|\le T'} \sum_{|\boldsymbol{m}|_q\le\widehat{N}(t)} \frac{|\widehat{c}_{\boldsymbol{m}}(t)|^2}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2} - \Sigma\left(t,\widehat{N}(t)\right) dt \right)_+,$$

$$\mathcal{T}_n = \left\{2^k :\ k = 1,\dots,K_{\max}\right\}.$$

The parameters $\nu_q(W,N,tx_0)$ and $N_{\max,q}$ are defined, for $t\neq 0$ and $q\in\{1,\infty\}$, by

(N.1) when $W(\cdot)=\cosh(\cdot/R)$,

$$\begin{aligned}
\nu_1(W,N,t) &= \frac{2^{p+1}(1\vee(N+p-1))^{p-1}}{(p-1)!}\left(1\vee\frac{\sqrt{2}R|t|}{\pi}\right)\exp\left(\frac{\pi(N+p)}{2R|t|}\right)\mathbb{1}\left\{|t|>\frac{\pi}{4R}\right\}\\
&\quad+\frac{(1\vee(N+p-1))^{p-1}}{(p-1)!}\left(\frac{\pi^2}{4}\right)^p\exp\left(2\left(2-\ln\left(R|t|\right)\right)N\right)\mathbb{1}\left\{|t|\le\frac{\pi}{4R}\right\},\\
\nu_\infty(W,N,t) &= 4^p\left(1\vee\frac{\sqrt{2}R|t|}{\pi}\right)^p\exp\left(\frac{\pi p(N+1)}{2R|t|}\right)\mathbb{1}\left\{|t|>\frac{\pi}{4R}\right\}\\
&\quad+\left(\frac{\pi^2}{4}\right)^p\exp\left(2p\left(2-\ln\left(R|t|\right)\right)N\right)\mathbb{1}\left\{|t|\le\frac{\pi}{4R}\right\},\\
N_{\max,q}(W,t) &= \left\lfloor\frac{2R|t|\ln(n)}{\pi k_q}-p\right\rfloor\mathbb{1}\left\{|t|>\frac{\pi}{4R}\right\}+\left\lfloor\frac{\ln(n)}{2k_q\left(2-\ln\left(R|t|\right)\right)}\right\rfloor\mathbb{1}\left\{|t|\le\frac{\pi}{4R}\right\}.
\end{aligned}$$

(N.2) when $W=W_{[-R,R]}$ and $\mathcal{W}$ the inverse of $x\in[0,\infty)\mapsto xe^x$,

$$\begin{aligned}
\nu_1(W,N,t) &= \frac{(1\vee(N+p-1))^{p-1}}{((R|t|)\vee 1)^p(p-1)!(1-\exp(-2))}\exp\left(2N\ln\left(\left(\frac{7\pi(N+1)}{R|t|}\right)\vee 1\right)\right),\\
\nu_\infty(W,N,t) &= \left(\frac{2(1\vee N)}{(R|t|)\vee 1}\right)^p\exp\left(2Np\ln\left(\left(\frac{7\pi(N+1)}{R|t|}\right)\vee 1\right)\right),\\
N_{\max,q}(W,t) &= \left\lfloor\frac{\ln(n)}{2k_q}\left(\mathcal{W}\left(\frac{7\pi}{R|t|}\frac{\ln(n)}{2k_q}\right)\vee 1\right)^{-1}\right\rfloor.
\end{aligned}$$

4.5.1. *Heuristic behind* (18)-(19). The method selects $\widehat{N}$ and $\widehat{T}$ as solutions of minimisation problems that mimic the upper bound on the bias-variance trade-off which allows to obtain theorems 1 and 3. Assume, for simplicity, that $f_{\boldsymbol{X}|\mathcal{X}}$ is known and remove the index n_0 from the risk. Let $N \in \mathbb{N}_0^{\mathbb{R}}$ and $T \in \mathbb{N}_0$. Denote by $\widetilde{c}_{\boldsymbol{m}}(t)$ and $\widetilde{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}$ the ideal estimators $\widehat{c}_{\boldsymbol{m}}(t)$ and $\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}$ with $\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}} = f_{\boldsymbol{X}|\mathcal{X}}$ and $\widetilde{\Delta}_{\boldsymbol{m}}(t) = \widetilde{c}_{\boldsymbol{m}}(t) - c_{\boldsymbol{m}}(t)$. Proposition 3 and Lemma A.10 yield

$$\begin{aligned}\frac{2\pi}{C}\mathcal{R}^W\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) &\leq \mathbb{E}\left[\left\|\mathbb{1}\{|\star| \geq \epsilon\}\left(\widehat{F}_1^{q,N,T,0} - \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(\star,\cdot)\right\|^2_{L^2(1\otimes W^{\otimes p})}\right] + \frac{2\pi M^2}{w(\underline{a})} \\ &= \int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \mathbb{E}\left[\left\|\widehat{F}_1^{q,N,T,0}(t,\cdot) - \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot)\right\|^2_{L^2(W^{\otimes p})}\right] dt + \frac{2\pi M^2}{w(\underline{a})}.\end{aligned}$$

Using (A.69) and denoting by $N \vee N' : t \mapsto N(t) \vee N'(t)$, the first term can be written as

$$\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \left\|\left(F_1^{q,N,T,0} - \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})} + \mathbb{E}\left[\left\|\left(\widehat{F}_1^{q,N,T,0} - F_1^{q,N,T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}\right] dt$$

(20)

$$\leq \int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \left(\left\|\left(F_1^{q,N,T,0} - \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})} dt + \Sigma(t,N)\mathbb{1}\{|t| \leq T\}\right) dt$$

(21)

$$= \int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \left(\sup_{N'}\left\|\left(F_1^{q,N\vee N',T,0} - F_1^{q,N,T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})} + \Sigma(t,N)\mathbb{1}\{|t| \leq T\}\right) dt.$$

For all $t \in \mathbb{R}\setminus(-\epsilon,\epsilon)$, $\widehat{N}(t)$ minimizes an estimator of the integrand in (21) obtained by replacing

$$\sup_{N'}\left\|\left(F_1^{q,N\vee N',T,0} - F_1^{q,N,T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}$$

by

$$\sup_{N'}\left(\left\|\left(\widehat{F}_1^{q,N\vee N',T,0} - \widehat{F}_1^{q,N,T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})} - \Sigma\left(t,N'\right)\right)_+ \tag{22}$$

and, using Proposition 2, by replacing the supremum in (22) by the maximum $B_1(t,N)$ over the grid $\{0,\ldots,N_{\max,q}(W,t)\}$. Indeed, we have

$$\begin{aligned}&\left\|\left(F_1^{q,N\vee N',T,0} - F_1^{q,N,T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})} \\ &= \mathbb{E}\left[\left\|\left(\widehat{F}_1^{q,N\vee N',T,0} - \widehat{F}_1^{q,N,T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}\right] - \mathbb{E}\left[\sum_{N\leq|\boldsymbol{m}|_q\leq N\vee N'} \frac{|\widehat{c}_{\boldsymbol{m}}(t)|^2 - |c_{\boldsymbol{m}}(t)|^2}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2}\right]\end{aligned}$$

and, using Lemma (A.12), $\mathbb{E}\left[|\widehat{c}_{\boldsymbol{m}}(t)|^2 - |c_{\boldsymbol{m}}(t)|^2\right] = \mathbb{E}\left[|\widehat{c}_{\boldsymbol{m}}(t) - c_{\boldsymbol{m}}(t)|^2\right]$, and (A.69), we can show that the second term is larger than $-\Sigma(t, N')$.
$\widehat{T}$ is obtained by (19) and minimises in T an estimator of (20) given $\widehat{N}$. Because the first term in (20) is equal to

$$\sup_{T'} \int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \left\| \left(F_1^{q,N,T\vee T',0} - F_1^{q,\infty,T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})} dt, \tag{23}$$

and using

$$\begin{aligned}
&\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \left\| \left(F_1^{q,\widehat{N},T\vee T',0} - F_1^{q,\widehat{N},T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})} dt \\
&= \mathbb{E}\left[\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \left\| \left(\widehat{F}_1^{q,\widehat{N},T\vee T',0} - \widehat{F}_1^{q,\widehat{N},T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})} dt\right] - \int_{T\le|t|\le T\vee T'} \mathbb{E}\left[\sum_{|\boldsymbol{m}|_q\le\widehat{N}} \frac{|\widehat{c}_{\boldsymbol{m}}(t)|^2 - |c_{\boldsymbol{m}}(t)|^2}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2}\right] dt
\end{aligned}$$

and we can show that the second term above is larger than $-\int_{T\le|t|\le T\vee T'} \Sigma\left(t,\widehat{N}\right) dt$ (see (A.69)), we replace (23) in (20) by a maximum $B_2(T,N)$ over a grid.

4.5.2. *Adaptation.*

Theorem 5. Make Assumption 1. Take $0 < l, M, s, R, \underline{a} < \infty$, $H \in \mathbb{N}$, $q \in \{1,\infty\}$, $\zeta > 1/12$, $\phi(\cdot) = 1 \vee |\cdot|^s$.

(T5.1) When $(\omega_k)_{k\in\mathbb{N}_0} = (k^\sigma)_{k\in\mathbb{N}_0}$, $\sigma > p/2$, for all σ_0 such that $\sigma_0 > \sigma$, $s > 1$, $w(\cdot) = 1 \vee |\cdot|$, and $\epsilon = (\ln_2(n)/\ln(n))^{-2\sigma_0}$, we have

(a) for $W = W_{[-R,R]}$, $\mathbb{S}_{\boldsymbol{\beta}} \subseteq [-R,R]^p$, and $\zeta_0 = 1/(6p)$,

$$\sup_{\substack{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)\cap\mathcal{D}\\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}} \mathcal{R}^W_{n_0}\left(\widehat{f}^{q,\widehat{N},\widehat{T},\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right)\left(\frac{\ln(n)}{\ln_2(n)}\right)^{2\sigma} = \underset{\substack{v(n_0,\mathcal{E})/\delta(n_0)\le n^{-(2+\zeta)}\\ n_e\ge e}}{O_p}(1), \tag{24}$$

(b) for $W(\cdot) = \cosh(\cdot/R)$, $\epsilon = (\ln_2(n))^{-2\sigma_0}$, and $\zeta_0 = 1/(14p)$ then (24) holds.

(T5.2) When $W = W_{[-R,R]}$, $\mathbb{S}_{\boldsymbol{\beta}} \subseteq [-R,R]^p$, $(\omega_k)_{k\in\mathbb{N}_0} = (e^{\kappa k \ln(1+k)})_{k\in\mathbb{N}_0}$, $w = W_{[-\underline{a},\underline{a}]}$, $\mathbb{S}_\alpha \subseteq [-\underline{a},\underline{a}]$, $\epsilon = 7\pi/(Rx_0)$, $\kappa > k_q$, $s > 3p$, and $\zeta_0 = 1/(6p)$, we have

$$\sup_{\substack{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l)\cap\mathcal{D}\\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}} \mathcal{R}^W_{n_0}\left(\widehat{f}^{q,\widehat{N},\widehat{T},\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right)\frac{n^{\kappa/(\kappa+k_q)}}{\ln(n)^{2\kappa+p+2+\mathbb{1}\{q=\infty\}}} = \underset{\substack{v(n_0,\mathcal{E})/\delta(n_0)\le n^{-(2+\zeta)}\\ n_e\ge e}}{O_p}(1). \tag{25}$$

(T5.3) When $W(\cdot)=\cosh(\cdot/R)$, $(\omega_k)_{k\in\mathbb{N}_0}=(e^{\kappa k})_{k\in\mathbb{N}_0}$, $\kappa>k_q(\pi(s/(p+1)+1)/4-1)$, $Rx_0>\pi\kappa/(2(p+1))$, $w=W_{[-\underline{a},\underline{a}]}$, $\mathbb{S}_\alpha\subseteq[-\underline{a},\underline{a}]$, $\epsilon=\pi/(4Rx_0)$, $s>7p\kappa/(\kappa+k_q)$, and $\zeta_0=1/(14p)$, we have

$$\sup_{\substack{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l)\cap\mathcal{D}\\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}}\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,\widehat{N},\widehat{T},\epsilon}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right)\frac{n^{\kappa/(\kappa+k_q)}}{\ln(n)^{2p+3-p\mathbb{1}\{q=\infty\}}}=\underset{\substack{v(n_0,\mathcal{E})/\delta(n_0)\leq n^{-(2+\zeta)}\\ n_e\geq e}}{O_p}(1). \tag{26}$$

The results in Theorem 5 are for $v(n_0,\mathcal{E})/\delta(n_0)\leq n^{-(2+\zeta)}$ with $\zeta>1/12$, in which case $n_e=n$. Theorem 2 and (T5.1) (a) show that $\widehat{f}^{q,\widehat{N},\widehat{T},\epsilon}_{\alpha,\boldsymbol{\beta}}$ is adaptive. The rate in (T5.2) matches, up to a logarithmic factor, the lower bound in Theorem 4 (1) for model (8). For the other cases, the risk is different for the lower bounds and the upper bounds in Theorem 5, but using (12) we obtain the same rate up to logarithmic factors for the risk involving the weight W.

5. Simulations

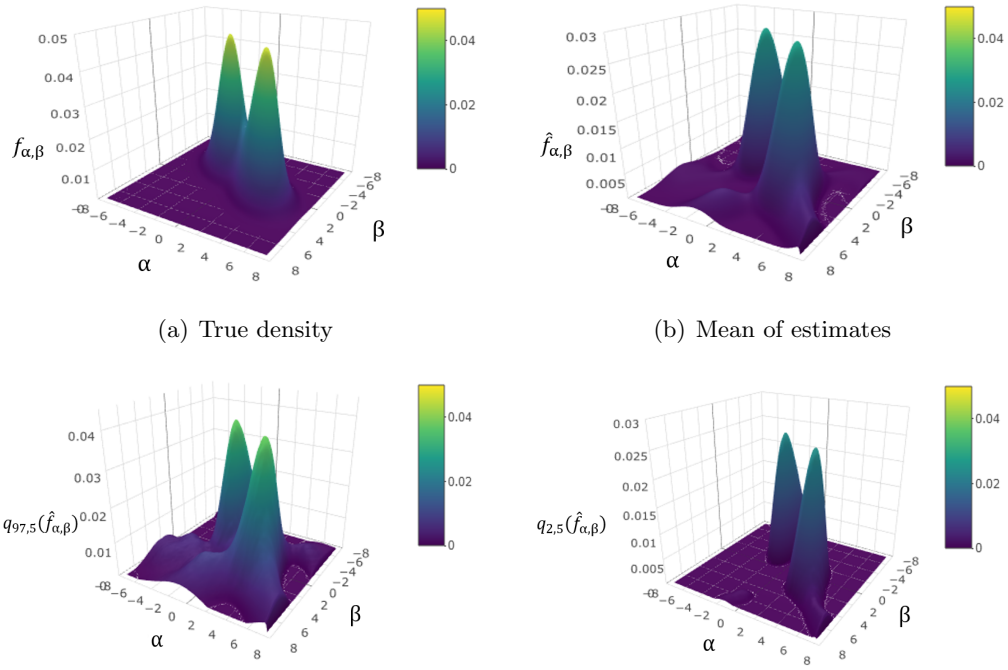
We take $p=1$, $q=\infty$, and $(\alpha,\beta)^\top=\xi_1 D+\xi_2(1-D)$ with $\mathbb{P}(D=1)=\mathbb{P}(D=0)=0.5$. The law of X is a truncated normal based on a normal of mean 0 and variance 2.5 and truncated to $\mathcal{X}$ with $x_0=1.5$. The laws of ξ_1 and ξ_2 are either: (Case 1) truncated normals based on normals with means $\mu_1=\begin{pmatrix}-2\\3\end{pmatrix}$ and $\mu_2=\begin{pmatrix}3\\0\end{pmatrix}$, same covariance $\begin{pmatrix}2&1\\1&2\end{pmatrix}$, and truncated to $[-6,6]^{p+1}$ or (Case 2) not truncated. Figure 1 (resp. Figure 2) displays summaries of the law of the estimator based on $\left(g^{W,c}_{\boldsymbol{m}}\right)_{\boldsymbol{m}\in\mathbb{N}_0^p}$ with $W=W_{[-R,R]}$ (resp. $W=\cosh(\cdot/R)$) with $R=7.5$ in Case 1 (resp. in Case 2). Table 1 compares $\mathbb{E}\left[\left\|\widehat{f}^{\infty,\widehat{N},\widehat{T},\epsilon}_{\alpha,\beta}-f_{\alpha,\beta}\right\|^2_{L^2([-7.5,7.5]^2)}\right]$ and the risk of the oracle $\min_{\substack{N\in\mathcal{N}_{n,H}\\ T\in\mathcal{T}_n}}\mathbb{E}\left[\left\|\widehat{f}^{\infty,N,T,\epsilon}_{\alpha,\beta}-f_{\alpha,\beta}\right\|^2_{L^2([-7.5,7.5]^2)}\right]$ for cases 1 and 2. $\widehat{f}_{X|X\in\mathcal{X}}$ is obtained with the same data and the risk is not conditional on $\mathcal{G}_{n_0}$.

The estimator requires the SVD of $\mathcal{F}_c$ for $c\neq 0$. By Proposition A.1 (i), we have $g^{W(\cdot/R),c}_m=g^{W,Rc}_m$ for all $m\in\mathbb{N}_0$. When $W=W_{[-1,1]}$, the first coefficients of the decomposition of the eigenfunctions on the Legendre polynomials can be obtained by solving for the eigenvectors of two tridiagonal symmetric Toeplitz matrices (for even and odd values of m, see Section 2.6 in [42]). When $W=\cosh$, we refer to Section 7 in [20]. We use that $\mathcal{F}^*_c\left(g^{W,Rc}_m\right)=\sigma^{W,Rc}_m\varphi^{W,Rc}_m$ and that $\varphi^{W,Rc}_m$ has norm 1 to obtain the remaining of the SVD. Note that $\mathcal{F}^*_c\left(g^{W,Rc}_m\right)$ is easy to compute when $W=W_{[-1,1]}$ using that $\mathcal{F}^*_c=\mathbb{1}\{[-1,1]\}\mathcal{F}_{-c}\mathcal{E}xt$ applied to the Legendre

polynomials has a closed form involving the Bessel functions of the first kind (see (18.17.19) in [41]). The Fourier inverse is computed using the fast Fourier transform (FFT).

	$W = W_{[-R,R]}$, Case 1, $R = 7.5$			$W(\cdot) = \cosh(\cdot/R)$, Case 2, $R = 7.5$		
	$n = 300$	$n = 500$	$n = 1000$	$n = 300$	$n = 500$	$n = 1000$
MISE (data-driven)	0.092	0.086	0.083	0.089	0.087	0.085
MISE (oracle)	0.091	0.086	0.082	0.088	0.087	0.085

TABLE 1. Risk of the data-driven estimator and of the oracle with 1000 simulations (to guarantee the use of three digits with a 95% confidence level).



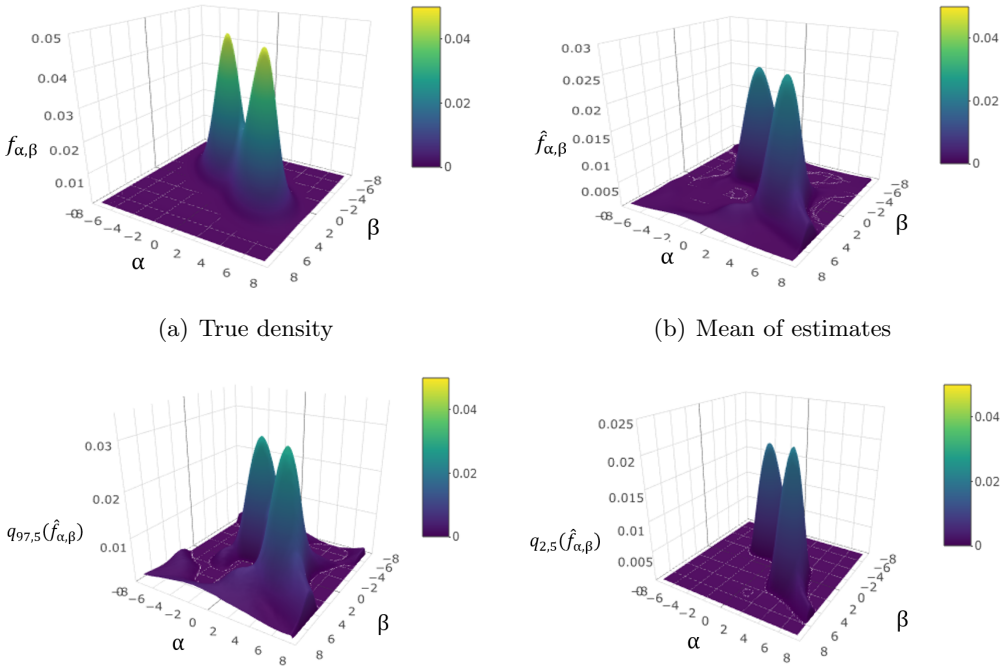
(a) True density

(b) Mean of estimates

(c) 97.5% quantile of estimates

(d) 2.5% quantile of estimates

FIGURE 1. Truth and data-driven estimator for case 1 and estimator based on $g_{\boldsymbol{m}}^{W,c}$ with $W = W_{[-R,R]}$, 1000 simulations, $n = 1000$.



(a) True density (b) Mean of estimates

(c) 97.5% quantile of estimates (d) 2.5% quantile of estimates

FIGURE 2. Truth and data-driven estimator for case 2 and estimator based on $g_{\boldsymbol{m}}^{W,c}$ with $W(\cdot) = \cosh(\cdot/6)$, 1000 simulations, $n = 1000$.

APPENDIX

APPENDIX A. USEFUL RESULTS FROM HARMONIC ANALYSIS

A.1. **Preliminaries.** In this appendix, we denote, for $m \in \mathbb{N}_0$, by ψ_m^c the function $g_m^{W_{[-1,1]},c}$ and use $\mu_m^c = i^m \sigma_m^{W_{[-1,1]},c}$. It is notable that $\psi_m^c = \mathcal{F}_c(\mathcal{E}xt[\psi_m^c])/\mu_m^c$ in $L^2([-1,1])$. This is the reason why ψ_m^c can be extended as an entire function which we denote with the same notation. Also, using the injectivity of $\mathcal{F}_c$ (see the proof of Proposition 1 below), we have $\varphi_m^{W_{[-1,1]},c} = i^{-m}\mathcal{E}xt[\psi_m^c]$. Also, P_m is the m^{th} normalized Legendre polynomial, normalized so that $\|P_m\|_{L^2([-1,1])} = 1$. We make use of the following proposition.

Proposition A.1. For all weighting function W, $c \in \mathbb{R}$, $R > 0$, $m \in \mathbb{N}_0$, we have

(i) $g_m^{W(\cdot/R),c} = g_m^{W,Rc}$ in $L^2([-1,1])$,
(ii) $\sigma_m^{W(\cdot/R),c} = \sigma_m^{W,Rc}\sqrt{R}$,

(iii) $\varphi_m^{W(\cdot/R),c}(\star) = \varphi_m^{W,Rc}(\star/R)/\sqrt{R}$ *a.e.*

Proof of Proposition A.1. (i) follows from $\mathcal{Q}_c^{W(\cdot/R)} = \mathcal{Q}_{Rc}^W$.
(ii) follows from

$$\begin{aligned}\sigma_m^{W(\cdot/R),c} &= \sqrt{\frac{2\pi\rho_m^{W(\cdot/R),c}}{|c|}} \\ &= \sqrt{\frac{2\pi\rho_m^{W,Rc}}{|c|}} \quad \text{(by the argument yielding (i))}.\end{aligned}$$

Now, for almost every $t \in \mathbb{R}$, we have

$$\begin{aligned}\sigma_m^{W,Rc}\varphi_m^{W,Rc}\left(\frac{t}{R}\right) &= \mathcal{F}_{Rc}^*\left[g_m^{W,Rc}\right]\left(\frac{t}{R}\right) \quad \left(\text{where } \mathcal{F}_{Rc}^*:\ L^2([-1,1]) \to L^2(W)\right) \\ &= \mathcal{F}_{Rc}^*\left[g_m^{W(\cdot/R),c}\right]\left(\frac{t}{R}\right) \quad \text{(by (i))} \\ &= \mathcal{F}_c^*\left[g_m^{W(\cdot/R),c}\right](t) \quad \left(\text{where } \mathcal{F}_c^*:\ L^2([-1,1]) \to L^2(W(\cdot/R))\right) \\ &= \sigma_m^{W(\cdot/R),c}\varphi_m^{W(\cdot/R),c}(t) \\ &= \sigma_m^{W,Rc}\sqrt{R}\varphi_m^{W(\cdot/R),c}(t) \quad \text{(by (ii))},\end{aligned}$$

hence (iii) when we divide by $\sigma_m^{W,Rc}$ which is nonzero. □

A.2. **Properties of the PSWF and eigenvalues.**

Lemma A.1. For all $m \in \mathbb{N}_0$ and $c \neq 0$, we have

$$|\mu_m^c| \leq \frac{\sqrt{2\pi}e^{3/2}}{3}\left(\frac{e\,|c|}{4(m+1/2)}\right)^m.$$

Proof of Lemma A.1. This follows from the following inequalities, for $m \in \mathbb{N}_0$ and $c \neq 0$,

$$\begin{aligned}|\mu_m^c| &\leq \frac{\sqrt{\pi}\,|c|^m\,(m!)^2}{(2m)!\Gamma(m+3/2)} \\ &\leq \frac{\pi\,|c|^m}{4^m\Gamma\left(m+3/2\right)}\frac{\Gamma(m+1)}{\Gamma\left(m+1/2\right)} \\ &\leq \frac{\pi\,|c|^m}{4^m\Gamma(m+3/2)}(m+1)^{1/2} \\ &\leq \frac{\sqrt{\pi e^3}(e\,|c|)^m(m+1)^{1/2}}{4^m\sqrt{2}(m+3/2)^{m+1}}\end{aligned}$$

$$\leq \frac{\sqrt{2\pi}e^{3/2}}{3}\left(\frac{e\,|c|}{4(m+1/2)}\right)^m,$$

using respectively (69) in [43], 6.1.18 in [1], (7) in [25], (1.3) in [40], and $\sup_{x\geq 0}(x+1)^{1/2}(x+1/2)^x/(x+3/2)^{x+1} \leq 2/3$. □

Lemma A.2. The following lower bound on $\rho_m^{W_{[-1,1],c}}$ holds for $c \neq 0$

$$\rho_m^{W_{[-1,1],c}} \geq \begin{cases} \frac{1}{2} & \text{if } 0 \leq m \leq \max\left(\lfloor 2\,|c|/\pi \rfloor - 1, 0\right) \\ \frac{(1\vee c)e^3}{9}\left(\frac{c}{7\pi(m+1)}\right)^{2m} & \text{if } m \geq \lfloor 2\,|c|/\pi \rfloor \vee 1 \end{cases}. \tag{A.1}$$

Proof of Lemma A.2. This is a consequence of Proposition 2.1 in [7] and $\rho_{\lfloor 2|c|/\pi \rfloor - 1}^{W_{[-1,1],c}} > 1/2$ (see [37]). □

Lemma A.3. For all $c \neq 0$ and $m \geq 2$, we have

$$\left|\frac{\mu_m^c}{\mu_{m-2}^c}\right| \leq \frac{\Pi(c)}{m^2}, \quad \text{where} \quad \Pi(c) = \frac{3c^2}{16}\exp\left(\frac{2c^2}{\sqrt{3}}\right).$$

Proof of Lemma A.3. Consider the case $c > 0$. From Theorem 8.1 in [42] we have that, for all $m \in \mathbb{N}_0$ and $c > 0$,

$$|\mu_m^c| = \frac{\sqrt{\pi}c^m(m!)^2}{(2m)!\Gamma(m+3/2)}e^{F_m(c)} \tag{A.2}$$

where

$$F_m(c) = \int_0^c \left(\frac{2\left(\psi_m^t(1)\right)^2 - 1}{2t} - \frac{m}{t}\right)dt. \tag{A.3}$$

Moreover, using (65) in [9], we have for all $m \in \mathbb{N}_0$ and $t > 0$,

$$\left(\sqrt{m+\frac{1}{2}} - \frac{t^2}{\sqrt{3}\sqrt{m+1/2}}\right)^2 \leq \left(\psi_m^t(1)\right)^2 \leq \left(\sqrt{m+\frac{1}{2}} + \frac{t^2}{\sqrt{3}\sqrt{m+1/2}}\right)^2$$

which yields, for $m \geq 2$,

$$\begin{aligned}
&\left(\psi_m^t(1)\right)^2 - \left(\psi_{m-2}^t(1)\right)^2 \\
&\leq \left(\sqrt{m+\frac{1}{2}} + \frac{t^2}{\sqrt{3}\sqrt{m+1/2}}\right)^2 - \left(\sqrt{(m-2)+\frac{1}{2}} - \frac{t^2}{\sqrt{3}\sqrt{(m-2)+1/2}}\right)^2 \\
&\leq 2 + \frac{4t^2}{\sqrt{3}} + \frac{t^4}{3}\left(\frac{1}{m+1/2} - \frac{1}{m-3/2}\right)
\end{aligned}$$

$$\leq 2 + \frac{4t^2}{\sqrt{3}}. \tag{A.4}$$

Using $\sup_{x\geq 2} x^3(x-1)/\left((x^2-1/4)(x-1/2)(x-3/2)\right) \leq 3$ and (A.3)-(A.4), we obtain for $m \geq 2$,

$$\begin{aligned}
\left|\frac{\mu_m^c}{\mu_{m-2}^c}\right| &\leq \frac{c^2}{16}\frac{m(m-1)}{(m^2-1/4)(m-1/2)(m-3/2)} \exp\left(F_m(c) - F_{m-2}(c)\right) \\
&\leq \frac{3c^2}{16m^2} \exp\left(\int_0^c \left(\frac{\left(\psi_m^t(1)\right)^2 - \left(\psi_{m-2}^t(1)\right)^2}{t} - \frac{2}{t}\right) dt\right) \\
&\leq \frac{3c^2}{16m^2} \exp\left(\frac{2c^2}{\sqrt{3}}\right),
\end{aligned}$$

which yields the result for $c > 0$. For all $c < 0$, we obtain the same results using $\mu_m^c = \overline{\mu_m^{-c}}$ and $\mathcal{E}xt[\psi_m^{-c}] = \mathcal{E}xt[\psi_m^c]$. □

Lemma A.4. For all $c \neq 0$ and $k \in \mathbb{N}$, we have $\left(\psi_k^c(1)\right)^2 \leq (k+1/2)\left(1 + 2c^2/3^{3/2}\right)^2$ and $\|\psi_k^c\|_{L^\infty([-1,1])}^2 \leq (k+1/2)\left(1 + 4c^2/3^{3/2}\right)^2$. For all $c \neq 0$ and $k \geq c$, we have $\|\psi_k^c\|_{L^\infty([-1,1])}^2 \leq k + 1/2$. We also have $\|\psi_0^c\|_{L^\infty([-1,1])}^2 \leq 2|c|/\pi$.

Proof of Lemma A.4. For all $c > 0$ and $k \in \mathbb{N}$, we obtain the first assertion using (65) in [9]. For all $c > 0$ and $k \in \mathbb{N}$, we obtain using (66) in [9] for the first display, and 22.14.7 and 22.2.10 in [1] which yield $\|P_k\|_{L^\infty([-1,1])} \leq \sqrt{k+1/2}$ for the second display,

$$\begin{aligned}
\|\psi_k^c\|_{L^\infty([-1,1])} &\leq \|P_k\|_{L^\infty([-1,1])} + \frac{c^2}{\sqrt{3(k+1/2)}}\left(1 + \frac{\sqrt{3/2}}{\sqrt{k+1/2}}\right) \\
&\leq \sqrt{k+1/2}\left(1 + \frac{c^2}{\sqrt{3}(k+1/2)}\left(1 + \frac{\sqrt{3/2}}{\sqrt{k+1/2}}\right)\right) \\
&\leq \sqrt{k+1/2}\left(1 + \frac{4c^2}{3^{3/2}}\right) \quad (\text{for } k \in \mathbb{N}).
\end{aligned}$$

This yields the second assertion for all $c > 0$ and $k \in \mathbb{N}$. We obtain the third one with (3.4) and (3.125) in [42]. We obtain the last one using the proof of Proposition 1 in [31] which yields $\|\psi_0^c\|_{L^\infty([-1,1])}^2 \leq 2/(\mu_0^c)^2$ and we conclude using Lemma A.2. For all $c < 0$, we obtain the same results using $\mu_m^c = \overline{\mu_m^{-c}}$ and $\mathcal{E}xt[\psi_m^{-c}] = \mathcal{E}xt[\psi_m^c]$. □

Lemma A.5. For all $c \neq 0$ and $N \geq H(c)$, where $H(c) = \sqrt{2\Pi(c)} \vee 2$ and Π is defined in Lemma A.3, using

$$C_1(c) = \frac{1}{H(c)}\left(1 + \frac{9}{2H(c)}\right),$$
$$C_2(c) = 2\left(\frac{2\,|c|}{\pi(H(c)-1)^2} + \frac{1}{2}\left(1 + \frac{4c^2}{3^{3/2}}\right)\left(1 + \frac{2c^2}{3^{3/2}}\right)\right),$$
$$C_3(c) = 4c_f\sqrt{1 + \frac{1}{2H(c)}},$$

and $c_f = 4/3$, we have

$$\left\|\frac{\partial \psi_N^c}{\partial c}\right\|_{L^\infty([-1,1])} \leq (4C_1(H,c) + C_2(H,c))\, C_3(H,c)\,|c|\,\sqrt{N}.$$

Proof of Lemma A.5. Take $N \geq H(c)$ and $w \in [-1,1]$. Theorem 7.11 in [42] yields

$$\frac{\partial \psi_N^c}{\partial c}(w) = \frac{2\psi_N^c(1)}{|c|} \sum_{\substack{k \neq N \\ k \equiv N\ [2]}} \frac{\mu_N^c \mu_k^c}{(\mu_N^c)^2 - (\mu_k^c)^2} \psi_k^c(1)\psi_k^c(w). \tag{A.5}$$

Using that $\mu_k^c/\mu_N^c \in \mathbb{R}$ if $k \equiv N\ [2]$, Lemma A.4, and denoting by $f:\ x \in (-1,1) \mapsto |x|/(1-x^2)$ and by $r(c) = \left(1 + 4c^2/3^{3/2}\right)\left(1 + 2c^2/3^{3/2}\right)$, we obtain

$$\left|\frac{\partial \psi_N^c}{\partial c}(w)\right|$$
$$\leq \frac{\sqrt{4N+2}}{|c|}\left(f\left(\frac{\mu_N^c}{\mu_0^c}\right)\frac{2|c|\mathbb{1}\{N \equiv 0[2]\}}{\pi} + \sum_{\substack{0<k<N \\ k\equiv N\ [2]}} f\left(\frac{\mu_N^c}{\mu_k^c}\right) r(c)\left(k + \frac{1}{2}\right) + \sum_{\substack{k>N \\ k\equiv N\ [2]}} f\left(\frac{\mu_k^c}{\mu_N^c}\right)\left(k + \frac{1}{2}\right)\right).$$

Lemma A.3 yields, if $k < N$ and $k \equiv N\ [2]$,

$$\left|\frac{\mu_N^c}{\mu_k^c}\right| \leq \left|\frac{\mu_N^c}{\mu_{N-2}^c}\right| \leq \frac{\Pi(c)}{N^2} \quad \left(\leq \frac{1}{2}\right) \tag{A.6}$$

while, if $k > N$ and $k \equiv N\ [2]$,

$$\left|\frac{\mu_k^c}{\mu_N^c}\right| \leq \frac{\Pi(c)}{(N+2)^{k-N}} \quad \left(\leq \frac{1}{2}\right). \tag{A.7}$$

Using that, for all $x \in [-1/2, 1/2]$, $f(x) \leq c_f\,|x|$, where $c_f = 4/3$, (A.6), (A.7), and that

$$2\sum_{\substack{0<k<N \\ k\equiv N\ [2]}} (2k+1) \leq \sum_{0 \leq k \leq N-2} (2k+1) = (N-1)^2,$$

we obtain

$$\left|\frac{\partial \psi_N^c}{\partial c}(w)\right| \tag{A.8}$$

$$\leq \frac{c_f\sqrt{4N+2}}{|c|}\left(\left(\frac{2\,|c|}{\pi}+\frac{r(c)(N-1)^2}{2}\right)\left(\frac{2c}{N}\right)^2+\sum_{\substack{k>N\\ k\equiv N\ [2]}}\frac{k+1/2}{2^{(k-N)/2}}\left(\frac{2^{3/2}c}{N}\right)^{k-N}\right)$$

$$\leq \frac{c_f\sqrt{4N+2}}{|c|}\left((2c)^2\left(\frac{2\,|c|}{\pi(H(c)-1)^2}+\frac{r(c)}{2}\right)+\frac{8c^2}{N^2}\sum_{\substack{l\geq 2\\ l\equiv 0\ [2]}}\left(l+N+\frac{1}{2}\right)2^{-l/2}\right) \tag{A.9}$$

$$\leq \frac{c_f\sqrt{4N+2}}{|c|}\left(2c^2C_2(c)+\frac{8c^2}{N^2}\left(N+\frac{9}{2}\right)\right)$$

$$\leq 4c_f\sqrt{1+\frac{1}{2H(c)}}\left(C_2(c)+\frac{4}{H(c)}\left(1+\frac{9}{2H(c)}\right)\right)|c|\sqrt{N}. \tag{A.10}$$

□

Lemma A.6. For all $c\neq 0$ and $N\geq H(c)$, where H is defined in Lemma A.5, using $C_1(c)$, $C_2(c)$, and $C_3(c)$ defined in Lemma A.5, we have

$$\left\|\frac{\partial^2\psi_N^c}{\partial c^2}\right\|_{L^\infty([-1,1])}\leq 4\sqrt{1+\frac{1}{2H(c)}}\left(C_4(c)N^{5/2}+C_5(c)N^{3/2}+C_6(c)\sqrt{N}+C_7(c)\right),$$

where $C_4(c)=4\left(C_2(c)-4C_1(c)\right)$,

$$C_5(c)=4\left(\left(1+\frac{c_f}{3}\right)C_2(c)+16\left(\frac{2c_f}{3}-1\right)C_1(c)\right)+\frac{c^2}{2}\left(\left(4C_1(c)+C_2(c)\right)C_3(c)\right)^2\left(1+\frac{4c^2}{3^{3/2}}\right),$$

$$C_6(c)=\frac{4}{3}\left(4C_1(c)+C_2(c)\right)\left(c^2+\frac{4}{9}\right)+\frac{32}{3}(4C_1(c)+C_2(c))(C_2(c)+8C_1(c))c^2,$$

$$C_7(c)=\frac{32}{\sqrt{H(c)}}\left(14+\frac{123}{H(c)}\right).$$

Proof of Lemma A.6. Consider first the case where $c>0$. Using that, for $x\in[-1,1]$, $c\in(0,\infty)\mapsto\psi_N^c(x)$ is analytic (see [19] page 320) and (7.99) in [42], we differentiate with respect to c and x, using the Lebesgue dominated convergence theorem, the equality

$$\mu_N^c\psi_N^c(x)=\int_{-1}^{1}e^{icxt}\psi_N^c(t)dt. \tag{A.11}$$

We obtain

(A.12)
$$\left(\frac{\partial^2\mu_N^c}{\partial c^2}\psi_N^c + 2\frac{\partial\mu_N^c}{\partial c}\frac{\partial\psi_N^c}{\partial c} + \mu_N^c\frac{\partial^2\psi_N^c}{\partial c^2}\right)(x) = \int_{-1}^{1} e^{icxt}\left(\frac{\partial^2\psi_N^c}{\partial c^2}(t) + 2ixt\frac{\partial\psi_N^c}{\partial c}(t) - (xt)^2\psi_N^c(t)\right)dt;$$

(A.13)
$$\mu_N^c\frac{\partial\psi_N^c}{\partial x}(x) = \int_{-1}^{1} icte^{icxt}\psi_N^c(t)dt;$$

(A.14)
$$\mu_N^c\frac{\partial^2\psi_N^c}{\partial x^2}(x) = -\int_{-1}^{1}(ct)^2e^{icxt}\psi_N^c(t)dt.$$

For $k \neq N$, multiplying both sides of (A.12) by $\psi_k^c(x)$, integrating with respect to x over $[-1,1]$, and using successively (A.11), (A.13), and (A.14), we obtain

$$2\frac{\partial\mu_N^c}{\partial c}\int_{-1}^{1}\frac{\partial\psi_N^c}{\partial c}(x)\psi_k^c(x)dx + \mu_N^c\int_{-1}^{1}\frac{\partial^2\psi_N^c}{\partial c^2}(x)\psi_k^c(x)dx$$
$$= \mu_k^c\int_{-1}^{1}\frac{\partial^2\psi_N^c}{\partial c^2}(x)\psi_k^c(x)dx + 2\frac{\mu_k^c}{c}\int_{-1}^{1}x\frac{\partial\psi_N^c}{\partial c}(x)\frac{\partial\psi_k^c}{\partial x}(x)dx + \frac{\mu_k^c}{c^2}\int_{-1}^{1}x^2\psi_N^c(x)\frac{\partial^2\psi_k^c}{\partial x^2}(x)dx.$$

Recombining and using that, for $k \neq N$, $\mu_k^c \neq \mu_N^c$ (see (3.45) in [42]), we obtain

$$\int_{-1}^{1}\frac{\partial^2\psi_N^c}{\partial c^2}(x)\psi_k^c(x)dx = \frac{1}{\mu_N^c - \mu_k^c}\left(2\frac{\mu_k^c}{c}\int_{-1}^{1}x\frac{\partial\psi_N^c}{\partial c}(x)\frac{\partial\psi_k^c}{\partial x}(x)dx\right.$$
$$\left. + \frac{\mu_k^c}{c^2}\int_{-1}^{1}x^2\psi_N^c(x)\frac{\partial^2\psi_k^c}{\partial x^2}(x)dx - 2\frac{\partial\mu_N^c}{\partial c}\int_{-1}^{1}\frac{\partial\psi_N^c}{\partial c}(x)\psi_k^c(x)dx\right).$$

This yields, for $k \not\equiv N[2]$, using (A.5), (7.69)-(7.70), and Theorem 7.11 in [42],

$$\int_{-1}^{1}\frac{\partial^2\psi_N^c}{\partial c^2}(x)\psi_k^c(x)dx = 0,$$

while, for $k \equiv N$ [2], using (7.69)-(7.70) and Theorem 7.11 and (7.99) in [42],

$$\int_{-1}^{1}\frac{\partial^2\psi_N^c}{\partial c^2}(x)\psi_k^c(x)dx = \frac{2}{c}\frac{\mu_k^c}{\mu_N^c - \mu_k^c}\int_{-1}^{1}x\frac{\partial\psi_N^c}{\partial c}(x)\frac{\partial\psi_k^c}{\partial x}(x)dx + \Xi_{N,k},$$

$$\Xi_{N,k} = \left(\frac{\mu_N^c\mu_k^c(\chi_k^c - \chi_N^c)}{\left(\mu_N^c - \mu_k^c\right)^2} - 2\frac{\mu_N^c\mu_k^c}{\left(\mu_N^c\right)^2 - \left(\mu_k^c\right)^2}\left(2 + \frac{\left(2\psi_N^c(1)^2 - 1\right)\mu_N^c}{\mu_N^c - \mu_k^c}\right)\right)\frac{\psi_N^c(1)\psi_k^c(1)}{c^2},$$

and $(\chi_N^c)_{N\in\mathbb{N}_0}$ are the eigenvalues of $\mathcal{L}_c^{W_{[-1,1]}}$. Also, differentiating (7.114) in [42] with respect to c yields

$$\int_{-1}^{1}\frac{\partial^2\psi_N^c}{\partial c^2}(x)\psi_N^c(x)dx = -\int_{-1}^{1}\left(\frac{\partial\psi_N^c}{\partial c}(x)\right)^2dx.$$

We obtain, using Lemma A.4 for the first term and for all $w \in [-1,1]$,

$$\left\| \frac{\partial^2 \psi_N^c}{\partial c^2} \right\|_{L^\infty([-1,1])} \leq \sqrt{N+\frac{1}{2}} \left(1+\frac{4c^2}{3^{3/2}}\right) \int_{-1}^{1} \left(\frac{\partial \psi_N^c}{\partial c}(x)\right)^2 dx + \sum_{\substack{k\neq N \\ k\equiv N \ [2]}} |\Xi_{N,k}| \, \|\psi_k^c\|_{L^\infty([-1,1])}$$

$$+ \sum_{\substack{k\neq N \\ k\equiv N \ [2]}} \frac{2\,|\mu_k^c|}{c\,|\mu_N^c - \mu_k^c|} \left| \int_{-1}^{1} x \frac{\partial \psi_N^c}{\partial c}(x) \frac{\partial \psi_k^c}{\partial x}(x) dx \right| \|\psi_k^c\|_{L^\infty([-1,1])}. \tag{A.15}$$

For the first term on the right-hand side of (A.15), using Lemma A.5, we obtain

$$\sqrt{N+\frac{1}{2}} \int_{-1}^{1} \left(\frac{\partial \psi_N^c}{\partial c}(x)\right)^2 dx \leq 2\left((4C_1(c)+C_2(c))\,C_3(c)\right)^2 \sqrt{1+\frac{1}{2H(c)}} N^{3/2} c^2.$$

For the second term in (A.15), using that for $k \equiv N\ [2]$, $\mu_N^c/\mu_k^c \in \mathbb{R}$, (A.6), (A.7), and denoting by $f:\ x \in (-1,1) \mapsto |x|/(1-x^2)$ and by $g:\ x \in (-1,1) \mapsto |x|/(1-x)^2$, we obtain, for $k < N$,

$$|\Xi_{N,k}| \leq \frac{|\psi_N^c(1)|\,|\psi_k^c(1)|}{c^2} \left(g\left(\frac{\mu_N^c}{\mu_k^c}\right)(\chi_k^c - \chi_N^c) + 2\left(2 + \frac{\left|2\psi_N^c(1)^2-1\right|\,|\mu_N^c|}{|\mu_N^c - \mu_k^c|}\right) f\left(\frac{\mu_N^c}{\mu_k^c}\right)\right),$$

and, for $k > N$,

$$|\Xi_{N,k}| \leq \frac{|\psi_N^c(1)|\,|\psi_k^c(1)|}{c^2} \left(g\left(\frac{\mu_k^c}{\mu_N^c}\right)(\chi_k^c - \chi_N^c) + 2\left(2 + \frac{\left|2\psi_N^c(1)^2-1\right|\,|\mu_N^c|}{|\mu_N^c - \mu_k^c|}\right) f\left(\frac{\mu_k^c}{\mu_N^c}\right)\right).$$

Then, using that for $k \equiv N\ [2]$, if $k < N$, $|\mu_N^c|/|\mu_N^c - \mu_k^c| \leq 1$ (using (A.6)), if $k > N$, $|\mu_N^c|/|\mu_N^c - \mu_k^c| \leq 2$ (using (A.7)), that $|\chi_N^c - \chi_k^c| \leq |N-k|\,(k+N+1) + c^2$ (see (13) in [8]), that $\left|2\psi_N^c(1)^2 - 1\right| \leq 2N$ (using Lemma A.4), Remark A.4, that for all $x \in [-1/2, 1/2]$ $f(x) \leq c_f\,|x|$ where $c_f = 4/3$ and $g(x) \leq c_g\,|x|$ where $c_g = 4$, using (A.9) for the second inequality, and denoting by $r(c) = \left(1+4c^2/3^{3/2}\right)\left(1+2c^2/3^{3/2}\right)$ we have

$$\begin{aligned}
&\sum_{\substack{k\neq N \\ k\equiv N \ [2]}} |\Xi_{N,k}| \, \|\psi_k^c\|_{L^\infty([-1,1])} \\
&\leq \frac{c_g\sqrt{4N+2}\mathbb{1}\{N\equiv 0[2]\}}{|c|\,\pi} \left|\frac{\mu_N^c}{\mu_0^c}\right| \left(N(N+1) + c^2 + \frac{4c_f}{3c_g}(N+1)\right) \\
&+ \frac{c_g\sqrt{4N+2}}{c^2} \sum_{\substack{0<k<N \\ k\equiv N \ [2]}} \left(k+\frac{1}{2}\right) r(c) \left|\frac{\mu_N^c}{\mu_k^c}\right| \left((N-k)(k+N+1) + c^2 + \frac{4c_f}{3c_g}(N+1)\right) \\
&+ \frac{c_g\sqrt{4N+2}}{c^2} \sum_{\substack{k>N \\ k\equiv N \ [2]}} \left(k+\frac{1}{2}\right) \left|\frac{\mu_k^c}{\mu_N^c}\right| \left(|N-k|\,|k+N+1| + c^2 + \frac{4c_f}{3c_g}(2N+1)\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq 2c_g C_2(c)\sqrt{4N+2}\left(N(N+1)+c^2+\frac{4c_f}{3c_g}(N+1)\right)\\
&+\frac{c_g\sqrt{4N+2}}{c^2}\sum_{\substack{k>N\\ k\equiv N\ [2]}}\frac{(k+1/2)}{2^{(k-N)/2}}\left((k-N)(k+N+1)+c^2+\frac{4c_f}{3c_g}(2N+1)\right)\left(\frac{2^{3/2}c}{N}\right)^{k-N}\\
&\leq 2c_g\sqrt{4N+2}C_2(c)\left(N(N+1)+c^2+\frac{4c_f}{3c_g}(N+1)\right)\\
&+8c_g\sqrt{4N+2}C_1(c)\left(c^2+\frac{4c_f}{3c_g}(2N+1)-N-N^2\right)\\
&+\frac{16c_g\sqrt{1+1/(2H(c))}}{N^{3/2}}\sum_{\substack{k\geq 2\\ k\equiv 0\ [2]}}\left(k+N+\frac{1}{2}\right)k(k+1)2^{-k/2}\quad\text{(using (A.10))}\\
&\leq 4c_g\sqrt{1+\frac{1}{2H(c)}}N^{5/2}\left(C_2(c)-4C_1(c)\right)\\
&+4c_g\sqrt{1+\frac{1}{2H(c)}}N^{3/2}\left(\left(1+\frac{4c_f}{3c_g}\right)C_2(c)+4\left(\frac{8c_f}{3c_g}-1\right)C_1(c)\right)\\
&+4c_g\sqrt{1+\frac{1}{2H(c)}}\sqrt{N}\left(C_2(c)\left(c^2+\frac{4c_f}{3c_g}\right)+4C_1(c)\left(c^2+\frac{4c_f}{3c_g}\right)\right)\\
&+\frac{32c_g}{\sqrt{H(c)}}\sqrt{1+\frac{1}{2H(c)}}\left(14+\frac{123}{H(c)}\right)\quad\left(\text{using }\sum_{k=1}^{\infty}\frac{k^3}{2^k}=26,\ \sum_{k=1}^{\infty}\frac{k^2}{2^k}=6,\ \text{and}\sum_{k=1}^{\infty}\frac{k}{2^k}=2\right).
\end{aligned}
$$

For the third term in (A.15), using (A.5), the triangle inequality, and (7.74) in [42], we obtain

$$
\begin{aligned}
\left|\int_{-1}^{1}x\frac{\partial\psi_N^c}{\partial c}(x)\frac{\partial\psi_k^c}{\partial x}(x)dx\right| &\leq \frac{4\left|\psi_N^c(1)\right|\left|\psi_k^c(1)\right|}{c}\sum_{\substack{m\equiv N\ [2]\\ m\neq N}}\frac{\left|\mu_N^c\right|\left|\mu_m^c\right|\left|\psi_m^c(1)\right|^2}{\left|\left(\mu_m^c\right)^2-\left(\mu_N^c\right)^2\right|}\frac{\left|\mu_m^c\right|}{\left|\mu_m^c+\mu_k^c\right|}\\
&\leq 8c_f\left|\psi_N^c(1)\right|\left|\psi_k^c(1)\right|\left(4C_1(c)+C_2(c)\right)|c|\quad\text{(using (A.10)).}
\end{aligned}
\tag{A.16}
$$

Using (A.6) and (A.7), we have that for $k\equiv N\ [2]$ and $N\geq H(c)$, if $k<N$, $\left|\mu_k^c\right|/\left|\mu_N^c-\mu_k^c\right|\leq 1$ and if $k>N$, $\left|\mu_k^c\right|/\left|\mu_N^c-\mu_k^c\right|\leq 2\left|\mu_k^c/\mu_N^c\right|$, which yields,

$$
\sum_{\substack{k\equiv N\ [2]\\ k\neq N}}\frac{2\left|\mu_k^c\right|}{c\left|\mu_N^c-\mu_k^c\right|}\left|\int_{-1}^{1}x\frac{\partial\psi_N^c}{\partial c}(x)\frac{\partial\psi_k^c}{\partial x}(x)dx\right|
$$

$$\leq 8c_f\sqrt{4N+2}(4C_1(c)+C_2(c))\sum_{\substack{k\equiv N\ [2]\\ k\neq N}}\frac{|\mu_k^c|}{|\mu_N^c-\mu_k^c|}\,|\psi_k^c(1)|\,\|\psi_k^c\|^2_{L^\infty([-1,1])}$$

$$\leq 8c_f\sqrt{4N+2}(4C_1(c)+C_2(c))\left(2c^2C_2(c)+\frac{16c^2}{N^2}\left(N+\frac{9}{2}\right)\right)\quad\text{(using (A.9))}$$

$$\leq 32c_f\sqrt{1+\frac{1}{2H(c)}}(4C_1(c)+C_2(c))(C_2(c)+8C_1(c))c^2\sqrt{N}.$$

For $c<0$, we obtain the same result using $\mu_m^c=\overline{\mu_m^{-c}}$ and $\mathcal{E}xt[\psi_m^{-c}]=\mathcal{E}xt[\psi_m^c]$. □

Lemma A.7. For all $N\geq H(Rc(U))$ and $t\in\mathbb{R}$, where H is defined in Lemma A.5 and $U>0$, there exists $C_8(Rx_0,p,U)>0$ such that,

$$\sup_{\boldsymbol{b}\in[-R,R]^p}\left|\frac{\partial^2}{\partial t^2}\left(\left(\frac{Rc(t)}{2\pi}\right)^{p/2}\phi(t)\psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}\left(\frac{\boldsymbol{b}}{R}\right)\right)\right|\leq\mathbb{1}\{U/2\leq|t|\leq U\}C_8(Rx_0,p,U)N^{k_q/2+2},$$

where $H_1(U)=\lceil H(Rc(U))\rceil=\lceil\sqrt{2\Pi(Rc(U))}\vee 2\rceil$, $c:\ t\mapsto tx_0$, ϕ is defined in (A.31), $\widetilde{\boldsymbol{N}}(1)=(N,\underline{\boldsymbol{H_1(U)}})\in\mathbb{N}^p$, $\widetilde{\boldsymbol{N}}(\infty)=\underline{\boldsymbol{N}}\in\mathbb{N}^p$, and $k_q=1+(p-1)\mathbb{1}\{q=\infty\}$.

Proof of Lemma A.7. Consider the case $q=1$. Similar computations yield the result when $q=\infty$. Denote, for $(t,\boldsymbol{b})\in\mathbb{R}\times[-R,R]^p$, by

$$R(t,\boldsymbol{b})=\left|\frac{\partial^2}{\partial t^2}\left(\left(\frac{R\,|c(t)|}{2\pi}\right)^{p/2}\phi(t)\psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}\left(\frac{\boldsymbol{b}}{R}\right)\right)\right|.$$

We have

$$R(t,\boldsymbol{b})\leq\left(\frac{Rx_0}{2\pi}\right)^{p/2}|t|^{p/2}\left[\left(\frac{p\,|p-2|}{4\,|t|^2}\phi(t)+\frac{p}{|t|}\,|\phi'(t)|+|\phi''(t)|\right)\left|\psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}\left(\frac{\boldsymbol{b}}{R}\right)\right|\right.$$
$$\left.+(Rx_0)\left(\frac{p}{|t|}\phi(t)+2|\phi'(t)|\right)\left|\frac{\partial\psi^c_{\widetilde{\boldsymbol{N}}(q)}}{\partial c}\right|_{c=Rc(t)}\left(\frac{\boldsymbol{b}}{R}\right)\right|+(Rx_0)^2\,\phi(t)\left|\frac{\partial^2\psi^c_{\widetilde{\boldsymbol{N}}(q)}}{\partial c^2}\right|_{c=Rc(t)}\left(\frac{\boldsymbol{b}}{R}\right)\left|\right].$$

Using that, for $\boldsymbol{b}\in[-R,R]^p$ and $c\neq 0$,

$$\frac{\partial\psi^c_{\widetilde{\boldsymbol{N}}(q)}}{\partial c}\left(\frac{\boldsymbol{b}}{R}\right)=\sum_{j=2}^p\psi^c_N\left(\frac{\boldsymbol{b}_1}{R}\right)\left(\prod_{\substack{l=2\\ l\neq j}}^p\psi^c_{H_1(U)}\left(\frac{\boldsymbol{b}_l}{R}\right)\right)\frac{\partial\psi^c_{H_1(U)}}{\partial c}\left(\frac{\boldsymbol{b}_j}{R}\right)$$
$$+\left(\prod_{l=2}^p\psi^c_{H_1(U)}\left(\frac{\boldsymbol{b}_l}{R}\right)\right)\frac{\partial\psi^c_N}{\partial c}\left(\frac{\boldsymbol{b}_1}{R}\right),$$

$$\frac{\partial^2\psi^c_{\widetilde{\boldsymbol{N}}(q)}}{\partial c^2}\left(\frac{\boldsymbol{b}}{R}\right) = 2\sum_{j=2}^{p}\left(\prod_{\substack{l=2\\l\neq j}}^{p}\psi^c_{H_1(U)}\left(\frac{\boldsymbol{b}_l}{R}\right)\right)\frac{\partial\psi^c_{H_1(U)}}{\partial c}\left(\frac{\boldsymbol{b}_j}{R}\right)\frac{\partial\psi^c_N}{\partial c}\left(\frac{\boldsymbol{b}_1}{R}\right)$$
$$+\sum_{k=2}^{p}\sum_{\substack{j=2\\j\neq k}}^{p}\psi^c_N\left(\frac{\boldsymbol{b}_1}{R}\right)\left(\prod_{\substack{l=2\\l\neq j,l\neq k}}^{p}\psi^{c(t)}_{H_1(U)}\left(\frac{\boldsymbol{b}_l}{R}\right)\right)\frac{\partial\psi^c_{H_1(U)}}{\partial c}\left(\frac{\boldsymbol{b}_j}{R}\right)\frac{\partial\psi^c_{H_1(U)}}{\partial c}\left(\frac{\boldsymbol{b}_k}{R}\right)$$
$$+\left(\prod_{l=2}^{p}\psi^c_{H_1(U)}\left(\frac{\boldsymbol{b}_l}{R}\right)\right)\frac{\partial^2\psi^c_N}{\partial c^2}\left(\frac{\boldsymbol{b}_1}{R}\right)$$
$$+\sum_{j=2}^{p}\psi^c_N\left(\frac{\boldsymbol{b}_l}{R}\right)\left(\prod_{\substack{l=2\\l\neq j}}^{p}\psi^c_{H_1(U)}\left(\frac{\boldsymbol{b}_l}{R}\right)\right)\frac{\partial^2\psi^c_{H_1(U)}}{\partial c^2}\left(\frac{\boldsymbol{b}_j}{R}\right),$$

the third assertion of Lemma A.4, that $N+1/2\leq 3N/2$, and Lemma A.6, we obtain, for $(t,\boldsymbol{b})\in\{[-U,-U/2]\cup[U/2,U]\}\times[-R,R]^p$,

$$R(t,\boldsymbol{b})\leq\left(\frac{3URx_0}{4\pi}\right)^{p/2}N^{1/2}(H_1(U))^{(p-1)/2}\Bigg(\left(\frac{p|p-2|}{U^2}+C_9\frac{2p}{U}+C_{10}\right)$$
$$+(Rx_0)\left(\frac{2p}{U}+2C_9\right)p\left(4C_1(c(U))+C_2(c(U))\right)C_3(c(U))+(Rx_0)^2C_{11}(U)p(p-1)$$
$$+(Rx_0)^2\,4p\sqrt{1+\frac{1}{2H(c(U))}}\left(C_4(c(U))N^{5/2}+C_5(c(U))N^{3/2}+C_6(c(U))\sqrt{N}+C_7(c(U))\right)\Bigg),$$

where we use that $N\geq H_1(U)$, denote by $C_9=\sup_{U/2\leq|t|\leq U}|\phi'(t)|$, $C_{10}=\sup_{U/2\leq|t|\leq U}|\phi''(t)|$, and $C_{11}(U)=\left(4C_1(c(U))+C_2(c(U))\right)^2C_3(c(U))^2$.

Thus, because $\mathrm{supp}(\phi)\subseteq[-U,-U/2]\cup[U/2,U]$, there exists $C_8(Rx_0,p,U)$ which only depends on Rx_0, p, and U, such that, for all $t\in\mathbb{R}$, $\sup_{\boldsymbol{b}\in[-R,R]^p}R(t,\boldsymbol{b})\leq\mathbb{1}\{U/2\leq|t|\leq U\}C_8(Rx_0,p,U)N^{2+1/2}$. □

Lemma A.8. For $R,x_0>0$, $\sigma>k_q/2+2$, and $q\in\{1,\infty\}$, we have

$$\forall\tau\geq\frac{3}{8}e^{\sigma+p/2-1/4}Rx_0,\quad\int_{\mathbb{R}}e^{-2\tau|t|}\left(\sum_{\boldsymbol{m}\in\mathbb{N}_0^p}|\boldsymbol{m}|_q^{2\sigma}\left|c^P_{\boldsymbol{m}}(t)\right|^2\right)dt\leq\frac{C_{12}(Rx_0,\sigma,p)}{\tau k_q^{2\sigma}},$$

where

$$c^P_{\boldsymbol{m}}(t)=\left\langle\frac{1}{2^{p/2}},\psi^{c(t)}_{\boldsymbol{m}}\right\rangle_{L^2([-1,1]^p)},$$

$$C_{12}(Rx_0,\sigma,p)=\frac{2^{p-1}p\Gamma(2\sigma+p+1)}{(2\sigma+p)}\left(\frac{8}{3}\right)^{2\sigma+p}+\frac{\pi e^3 p 2^p\sqrt{3}}{9\tau}\frac{\Gamma(2\sigma+p+1/2)}{(2\sigma+p-1/2)^{2\sigma+p+1/2}}.$$

Proof of Lemma A.8. When $q=1$, we make use of $|\boldsymbol{m}|_1\leq p\,|\boldsymbol{m}|_\infty$. When $q=\infty$, using that $P_0(\cdot)=\mathbb{1}\{|\cdot|_\infty\leq 1\}/2^{p/2}$, that for all $m\in\mathbb{N}_0$, $\left|\langle P_0,\psi_m^c\rangle_{L^2([-1,1])}\right|\leq 1$, that for $m>|c|$, $\left|\langle P_0,\psi_m^c\rangle_{L^2([-1,1])}\right|\leq|\mu_m^c|/\sqrt{2}$ (see Proposition 3 in [8]), that for all $m\in\mathbb{N}_0$ $|\mu_m^c|\leq\sqrt{2\pi/c(t)}$ (see (3.45) in [42]), and using Lemma A.1, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} e^{-2\tau|t|}\left(\sum_{\boldsymbol{m}\in\mathbb{N}_0^p}|\boldsymbol{m}|_\infty^{2\sigma}\left|c_{\boldsymbol{m}}^P(t)\right|^2\right)dt\\
&\leq\int_{\mathbb{R}} e^{-2\tau|t|}\left(\sum_{|\boldsymbol{m}|_\infty\leq|c(t)|}|\boldsymbol{m}|_\infty^{2\sigma}\mathbb{1}\{|c(t)|\geq 1\}+\sum_{|\boldsymbol{m}|_\infty>|c(t)|}\frac{|\boldsymbol{m}|_\infty^{2\sigma}\left|\mu_{|\boldsymbol{m}|_\infty}^{c(t)}\right|^2}{2}\right)dt.\\
&\leq\int_{\mathbb{R}} e^{-2\tau|t|}\left(\sum_{|\boldsymbol{m}|_\infty\leq|c(t)|}|\boldsymbol{m}|_\infty^{2\sigma}\mathbb{1}\{|c(t)|\geq 1\}+\sum_{|\boldsymbol{m}|_\infty>|c(t)|}\frac{|\boldsymbol{m}|_\infty^{2\sigma}\pi e^3}{9}\left(\frac{ec(t)}{4|\boldsymbol{m}|_\infty}\right)^{2|\boldsymbol{m}|_\infty}\right)dt.
\end{aligned}$$

Then, using for the third inequality that (see (1.3) in [40])

$$\int_0^\infty e^{-2\tau t}t^{2m}dt=\frac{\Gamma(2m+1)}{(2\tau)^{2m+1}}\leq e^{-2m}e^{2m\ln(2m+1)}\frac{\sqrt{2m+1}}{(2\tau)^{2m+1}}$$

we have, for $\tau\geq 3e^{\sigma+p/2-1/4}Rx_0/8$,

$$\begin{aligned}
&\int_{\mathbb{R}} e^{-2\tau|t|}\left(\sum_{\boldsymbol{m}\in\mathbb{N}_0^p}|\boldsymbol{m}|_\infty^{2\sigma}\left|c_{\boldsymbol{m}}^P(t)\right|^2\right)dt\\
&\leq 2p\int_0^\infty e^{-2\tau t}\left(\sum_{m\leq c(t)}\frac{m^{2\sigma}}{(m+1)^{1-p}}\mathbb{1}\{|c(t)|\geq 1\}+\frac{\pi e^3}{9}\sum_{m>c(t)}\frac{m^{2\sigma}}{(m+1)^{1-p}}\left(\frac{ec(t)}{4m}\right)^{2m}\right)dt\\
&\leq 2^p p\int_0^\infty e^{-2\tau t}\int_1^{c(t)+1}u^{2\sigma+p-1}du\mathbb{1}\{|c(t)|\geq 1\}dt\\
&\quad+\frac{\pi e^3 p 2^{p+1}}{9}\sum_{m\geq 1}m^{2\sigma+p-1}\left(\frac{eRx_0}{4m}\right)^{2m}\int_0^\infty e^{-2\tau t}t^{2m}dt\\
&\leq\frac{2^{2(\sigma+p)}p}{2\sigma+p}\int_{1/(Rx_0)}^\infty c(t)^{2\sigma+p}e^{-2\tau t}dt+\frac{\pi e^3 p 2^p\sqrt{3}}{9\tau}\sum_{m\geq 1}m^{2\sigma+p-1/2}e^{2m\ln(3Rx_0/(8\tau))}.
\end{aligned}$$

Then, we have, for $\tau \geq 3e^{\sigma+p/2-1/4}Rx_0/8$,

$$\begin{aligned}
&\int_{\mathbb{R}} e^{-2\tau|t|}\left(\sum_{\boldsymbol{m}\in\mathbb{N}_0^p}|\boldsymbol{m}|_\infty^{2\sigma}\left|c_{\boldsymbol{m}}^P(t)\right|^2\right)dt \\
&\leq \frac{2^{p-1}p}{2\sigma+p}\frac{\Gamma(2\sigma+p+1)(Rx_0)^{2\sigma+p}}{\tau^{2\sigma+p+1}} + \frac{\pi e^3 p 2^p\sqrt{3}}{9\tau}\sum_{m\geq 1} m^{2\sigma+p-1/2}e^{-(2\sigma+p-1/2)m} \\
&\leq \frac{2^{p-1}p}{(2\sigma+p)\tau}\Gamma(2\sigma+p+1)\left(\frac{8}{3}\right)^{2\sigma+p} + \frac{\pi e^3 p 2^p\sqrt{3}}{9\tau}\int_0^\infty t^{2\sigma+p-1/2}e^{-t(2\sigma+p-1/2)}dt \\
&\leq \frac{C_{12}(Rx_0,\sigma,p)}{\tau k_q^{2\sigma}}.
\end{aligned}$$

□

A.3. **Relation between $\mathcal{H}^{q,\phi,\omega}_{1,W_{[-R,R]}}$ and Sobolev ellipsoids.** Define, for $q \in \{1,\infty\}$, the Sobolev ellipsoid

$$H^{q,\sigma,s}(l) = \left\{ f : \int_{\mathbb{R}}\sum_{\boldsymbol{k}\in\mathbb{Z}^p}|\mathcal{F}[f](t,\boldsymbol{k})|^2\,|\boldsymbol{k}|_q^{2\sigma}\;(1\vee|t|^2)^s dt \leq 2\pi l^2\right\},$$

where

$$\mathcal{F}[f](t,\boldsymbol{k}) = \frac{1}{(2R)^{p/2}}\int_{\mathbb{R}} e^{ita}\int_{[-R,R]^p} e^{i\pi\boldsymbol{k}^\top\boldsymbol{b}/R}\; f(a,\boldsymbol{b})da d\boldsymbol{b}. \tag{A.17}$$

Lemma A.9. For all $\sigma,\delta,l,M,R>0$, $\sigma' = \sigma+1/2+\delta$, $(q',q)\in\{1,\infty\}^2$, $\phi=(1\vee|\cdot|)^s$, where $s>\sigma'+p/2$, and $(\omega_k)_{k\in\mathbb{N}_0} = (k^\sigma)_{k\in\mathbb{N}_0}$, there exists a finite constant A such that

$$H^{q',s,\sigma'}(l) \subseteq \mathcal{H}^{q,\phi,\omega}_{1,W_{[-R,R]}}\left(\sqrt{A}l\right),$$

where $k_q = 1+(p-1)\mathbb{1}\{q=\infty\}$.

In Lemma A.9, we do not explicit A as a function of $\sigma,\sigma',\delta,s,l,M,R$ for simplicity but it is given in (A.24) in the proof.

Proof of Lemma A.9. Take $f\in H^{q',s,\sigma'}(l)$. We have

$$\begin{aligned}
\int_{\mathbb{R}}\sum_{\boldsymbol{m}\in\mathbb{N}_0^p}|b_{\boldsymbol{m}}(t)|^2\left(1\vee|t|^2\right)^s dt &= \int_{\mathbb{R}}\|\mathcal{F}_1[f](t,\cdot)\|^2_{L^2([-R,R]^p)}\left(1\vee|t|^2\right)^s dt \\
&= \int_{\mathbb{R}}\sum_{\boldsymbol{k}\in\mathbb{Z}^p}|\mathcal{F}[f](t,\boldsymbol{k})|^2\left(1\vee|t|^2\right)^s dt \\
&\leq 2\pi l^2 \quad \left(\text{because } f\in H^{q',s,\sigma'}(l)\right).
\end{aligned}$$

Thus, f satisfies the first the first inequality in the definition of $\mathcal{H}^{q,\phi,\omega}_{1,W_{[-R,R]}}(\sqrt{A}l)$.
Let us now check the second inequality in the definition of $\mathcal{H}^{q,\phi,\omega}_{1,W_{[-R,R]}}(\sqrt{A}l)$, *i.e.*

$$\int_{\mathbb{R}} \sum_{\boldsymbol{m}\in\mathbb{N}_0^p} |\boldsymbol{m}|_q^{2\sigma} \left|b_{\boldsymbol{m}}(t)\right|^2 dt \leq 2\pi A l^2. \tag{A.18}$$

To show (A.18), we show that, there exists a finite $A' > 1$ such that for all $N \in \mathbb{N}$,

$$\int_{\mathbb{R}} \sum_{|\boldsymbol{m}|_\infty \geq N} \left|b_{\boldsymbol{m}}(t)\right|^2 dt \leq \frac{2\pi A' l^2}{N^{2\sigma'}}, \tag{A.19}$$

and then denote by $A = A' \left(k_\infty/k_q\right)^{2\sigma} \left(1 + 1/(2\delta)\right)$, which yields

$$\begin{aligned}
\int_{\mathbb{R}} \sum_{\boldsymbol{m}\in\mathbb{N}_0^p} |\boldsymbol{m}|_\infty^{2\sigma} \left|b_{\boldsymbol{m}}(t)\right|^2 dt &= \int_{\mathbb{R}} \sum_{\boldsymbol{m}\in\mathbb{N}_0^p, \boldsymbol{m}\neq\underline{\boldsymbol{0}}} \frac{|\boldsymbol{m}|_\infty^{2\sigma'}}{|\boldsymbol{m}|_\infty^{1+2\delta}} \left|b_{\boldsymbol{m}}(t)\right|^2 dt \\
&= \int_{\mathbb{R}} \sum_{k=1}^{\infty} \sum_{\boldsymbol{m}\in\mathbb{N}_0^p, |\boldsymbol{m}|_\infty = k} \frac{1}{k^{1+2\delta}} \left|b_{\boldsymbol{m}}(t)\right|^2 k^{2\sigma'} dt \\
&\leq \sum_{k=1}^{\infty} \frac{1}{k^{1+2\delta}} \left(\int_{\mathbb{R}} \sum_{|\boldsymbol{m}|_\infty \geq k} \left|b_{\boldsymbol{m}}(t)\right|^2 k^{2\sigma'} dt \right) \quad \text{(using Fubini-Tonelli's theorem)} \\
&\leq 2\pi A' l^2 \left(\sum_{k=1}^{\infty} \frac{1}{k^{1+2\delta}} \right) \quad \text{(using (A.19))} \\
&\leq 2\pi A' l^2 \left(1 + \frac{1}{2\delta} \right) \quad \text{(using integral test for convergence).} \qquad \text{(A.20)}
\end{aligned}$$

Thus, when $q = 1$, using $|\boldsymbol{m}|_1 \leq p\,|\boldsymbol{m}|_\infty$ we have

$$\begin{aligned}
\int_{\mathbb{R}} \sum_{\boldsymbol{m}\in\mathbb{N}_0^p} |\boldsymbol{m}|_1^{2\sigma} \left|b_{\boldsymbol{m}}(t)\right|^2 dt &\leq p^{2\sigma} \int_{\mathbb{R}} \sum_{\boldsymbol{m}\in\mathbb{N}_0^p} |\boldsymbol{m}|_\infty^{2\sigma} \left|b_{\boldsymbol{m}}(t)\right|^2 dt \\
&\leq \left(\frac{k_\infty}{k_q}\right)^{2\sigma} 2\pi A' l^2 \left(1 + \frac{1}{2\delta} \right) \quad \text{(using (A.20))} \qquad \text{(A.21)} \\
&\leq 2\pi A l^2
\end{aligned}$$

hence (A.18). When $q = \infty$, (A.21) yields (A.18).
Note that (A.19) is true for $N = 1$, hence we now show (A.19) for $N \geq 2$. For $\varphi \in L^2\left(W_{[-R,R]}^{\otimes p}\right)$,

we denote the projection of φ on the vector space spanned by $\left(\psi_{\boldsymbol{m}}^{c(t)}\left(\cdot/R\right)\right)_{|\boldsymbol{m}|_\infty \leq N}$ by

$$P_{c(t)}^N \varphi(\cdot) = \sum_{\substack{|\boldsymbol{m}|_\infty < N \\ \boldsymbol{m}\in\mathbb{N}_0^p}} \left\langle \varphi(\cdot), \frac{1}{(2R)^{p/2}} \psi_{\boldsymbol{m}}^{c(t)}\left(\frac{\cdot}{R}\right)\right\rangle_{L^2([-R,R]^p)} \frac{1}{(2R)^{p/2}} \psi_{\boldsymbol{m}}^{c(t)}\left(\frac{\cdot}{R}\right).$$

Then, using that $\left\|\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\cdot) - P_{c(t)}^N \mathcal{F}_{1\mathrm{st}}\left[f\right](t,\cdot)\right\|_{L^2([-R,R]^p)}^2 = \sum_{|\boldsymbol{m}|_\infty \geq N} |b_{\boldsymbol{m}}(t)|^2$, (A.19) can be rewritten as

$$\int_{\mathbb{R}} \left\|\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\cdot) - P_{c(t)}^N \mathcal{F}_{1\mathrm{st}}\left[f\right](t,\cdot)\right\|_{L^2([-R,R]^p)}^2 dt \leq \frac{2\pi A' l^2}{N^{2\sigma'}}. \tag{A.22}$$

Proof of (A.22). Define $\left(\phi_{\boldsymbol{m}}\left(\cdot/R\right)\right)_{\boldsymbol{m}\in\mathbb{N}_0^p} = \left(e^{i\pi \boldsymbol{m}^\top \cdot/R}/(2R)^{p/2}\right)_{\boldsymbol{m}\in\mathbb{N}_0^p}$ which is an orthonormal basis of $L^2([-R,R]^p)$ and denote, for $\boldsymbol{m}\in\mathbb{N}_0^p$, by $\mathcal{F}[f](\boldsymbol{m}) = \int_{[-R,R]^p} e^{i\pi \boldsymbol{m}^\top \boldsymbol{b}/R}\, f(\boldsymbol{b}) d\boldsymbol{b}/(2R)^{p/2}$ the decomposition on $\left(\phi_{\boldsymbol{m}}\left(\cdot/R\right)\right)_{\boldsymbol{m}\in\mathbb{N}_0^p}$ by

$$\mathcal{E}^N \varphi(\cdot) = \sum_{\substack{|\boldsymbol{m}|_\infty < N \\ \boldsymbol{m}\in\mathbb{N}_0^p}} \frac{\mathcal{F}\left[\varphi\right](\boldsymbol{m})}{(2R)^{p/2}} e^{i\pi \boldsymbol{m}^\top \cdot/R}.$$

For $t\neq 0$, consider the function $\varphi \in L^2\left(W_{[-R,R]^p}\right)$ defined by $\varphi(\cdot) = \mathcal{F}_{1\mathrm{st}}\left[f\right](t,\cdot)$ and denote by $\overline{N} = \lfloor \theta N \rfloor$ with $\theta > 0$. This yields

$$\begin{aligned}
&\left\|\varphi(\cdot) - P_{c(t)}^N \varphi(\cdot)\right\|_{L^2([-R,R]^p)}^2 \\
&\leq 2\left[\left\|\left(\varphi - \mathcal{E}^{\overline{N}}\varphi\right) - P_{c(t)}^N\left(\varphi - \mathcal{E}^{\overline{N}}\varphi\right)\right\|_{L^2([-R,R]^p)}^2 + \left\|\mathcal{E}^{\overline{N}}\varphi - P_{c(t)}^N \mathcal{E}^{\overline{N}}\varphi\right\|_{L^2([-R,R]^p)}^2\right] \\
&\leq 2\left[\left\|\varphi - \mathcal{E}^{\overline{N}}\varphi\right\|_{L^2([-R,R]^p)}^2 + \left\|\mathcal{E}^{\overline{N}}\varphi - P_{c(t)}^N \mathcal{E}^{\overline{N}}\varphi\right\|_{L^2([-R,R]^p)}^2\right].
\end{aligned}$$

The second term, which we denote by $K_R = (2R)^p \left\|\mathcal{E}^{\overline{N}}\varphi - P_{c(t)}^N \mathcal{E}^{\overline{N}}\varphi\right\|_{L^2([-R,R]^p)}^2$, satisfies

$$\begin{aligned}
K_R = \Bigg\| &\sum_{\substack{|\boldsymbol{k}|_\infty < \overline{N} \\ \boldsymbol{k}\in\mathbb{Z}^p}} \left\langle \varphi(\cdot), \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right)\right\rangle_{L^2([-R,R]^p)} \phi_{\boldsymbol{k}}\left(\frac{\star}{R}\right) \\
&- \sum_{\substack{|\boldsymbol{m}|_\infty \geq N \\ \boldsymbol{m}\in\mathbb{N}_0^p}} \sum_{\substack{|\boldsymbol{k}|_\infty < \overline{N} \\ \boldsymbol{k}\in\mathbb{Z}^p}} \left\langle \varphi(\cdot), \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right)\right\rangle_{L^2([-R,R]^p)} \left\langle \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right), \psi_{\boldsymbol{m}}^{c(t)}\left(\frac{\cdot}{R}\right)\right\rangle_{L^2([-R,R]^p)} \psi_{\boldsymbol{m}}^{c(t)}\left(\frac{\star}{R}\right)\Bigg\|_{L^2([-R,R]^p)}^2.
\end{aligned}$$

This yields, using the Cauchy-Schwarz inequality for the third display,

$$
\begin{aligned}
K_R &= \left\| \sum_{\substack{|\boldsymbol{k}|_\infty < \overline{N} \\ \boldsymbol{k} \in \mathbb{Z}^p}} \left\langle \varphi(\cdot), \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right)\right\rangle \left(\sum_{\boldsymbol{m} \in \mathbb{N}_0^p :\ |\boldsymbol{m}|_\infty \geq N} \left\langle \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right), \psi_{\boldsymbol{m}}^{c(t)}\left(\frac{\cdot}{R}\right)\right\rangle \psi_{\boldsymbol{m}}^{c(t)}\left(\frac{\star}{R}\right)\right) \right\|^2_{L^2([-R,R]^p)} \\
&= \sum_{\boldsymbol{m} \in \mathbb{N}_0^p :\ |\boldsymbol{m}|_\infty \geq N} \left| \sum_{\substack{|\boldsymbol{k}|_\infty < \overline{N} \\ \boldsymbol{k} \in \mathbb{Z}^p}} \left\langle \varphi(\cdot), \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right)\right\rangle_{L^2([-R,R]^p)} \left\langle \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right), \psi_{\boldsymbol{m}}^{c(t)}\left(\frac{\cdot}{R}\right)\right\rangle_{L^2([-R,R]^p)} \right|^2 \\
&\leq \sum_{\boldsymbol{m} \in \mathbb{N}_0^p :\ |\boldsymbol{m}|_\infty \geq N} \left(\sum_{\substack{|\boldsymbol{k}|_\infty < \overline{N} \\ \boldsymbol{k} \in \mathbb{Z}^p}} \left| \left\langle \varphi(\cdot), \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right)\right\rangle \right|^2 \right) \left(\sum_{\substack{|\boldsymbol{k}|_\infty < \overline{N} \\ \boldsymbol{k} \in \mathbb{Z}^p}} \left| \left\langle \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right), \psi_{\boldsymbol{m}}^{c(t)}\left(\frac{\cdot}{R}\right)\right\rangle \right|^2 \right) \\
&\leq \left(\sum_{\substack{|\boldsymbol{k}|_\infty < \overline{N} \\ \boldsymbol{k} \in \mathbb{Z}^p}} \left| \left\langle \varphi(\cdot), \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right)\right\rangle \right|^2 \right) \left(\sum_{\substack{|\boldsymbol{m}|_\infty \geq N \\ \boldsymbol{m} \in \mathbb{N}_0^p}} \sum_{\substack{|\boldsymbol{k}|_\infty < \overline{N} \\ \boldsymbol{k} \in \mathbb{Z}^p}} \left| \left\langle \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right), \psi_{\boldsymbol{m}}^{c(t)}\left(\frac{\cdot}{R}\right)\right\rangle \right|^2 \right) \\
&\leq \left(\sum_{\boldsymbol{k} \in \mathbb{Z}^p} \left| \mathcal{F}\left[\varphi\right](\boldsymbol{k}) \right|^2 \right) I_{N,\overline{N}}(t),
\end{aligned}
$$

where

$$
I_{N,\overline{N}}(t) = \sum_{\boldsymbol{m} \in \mathbb{N}_0^p :\ |\boldsymbol{m}|_\infty \geq N} \ \sum_{\boldsymbol{k} \in \mathbb{Z}^p :\ |\boldsymbol{k}|_\infty < \overline{N}} \left| \left\langle \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right), \psi_{\boldsymbol{m}}^{c(t)}\left(\frac{\cdot}{R}\right)\right\rangle_{L^2([-R,R]^p)} \right|^2.
$$

We now compute the coefficients $\left| \left\langle \phi_{\boldsymbol{k}}\left(\cdot/R\right), \psi_{\boldsymbol{m}}^{c(t)}\left(\cdot/R\right)\right\rangle_{L^2([-R,R]^p)} \right|^2$ as the product of coefficients in the case $p = 1$. Denote, for $(n, m) \in \mathbb{N}_0^2$, by $\beta_n^m = \left\langle \psi_m^{c(t)}, P_n \right\rangle_{L^2([-1,1])}$. We have

$$
\begin{aligned}
&\left\langle \phi_k\left(\frac{\cdot}{R}\right), \psi_m^{c(t)}\left(\frac{\cdot}{R}\right)\right\rangle_{L^2([-R,R])} \\
&= \left(\frac{R}{2}\right)^{1/2} \int_{-1}^{1} e^{i\pi k x} \psi_m^{c(t)}(x) dx \\
&= \sum_{n=0}^{\infty} \left(\frac{R}{2}\right)^{1/2} \beta_n^m \left\langle e^{i\pi k x}, P_n \right\rangle_{L^2([-1,1])}
\end{aligned}
$$

$$= \left(\frac{R}{2}\right)^{1/2} \left(\sum_{n=0}^{\lfloor m/M \rfloor - 1} \beta_n^m \left\langle e^{i\pi kx}, P_n \right\rangle_{L^2([-1,1])} + \sum_{n \geq \lfloor m/M \rfloor} \beta_n^m \sqrt{\frac{2}{k}} \sqrt{n + \frac{1}{2}} J_{n+1/2}(k\pi) \right)$$

$$\leq (2R)^{1/2} \left(I_{m,k} \vee J_{m,k} \right),$$

where

$$I_{m,k} = \sum_{n=0}^{\lfloor m/M \rfloor - 1} \beta_n^m \left\langle e^{i\pi kx}, P_n \right\rangle$$

$$J_{m,k} = \sum_{n \geq \lfloor m/M \rfloor} \beta_n^m \sqrt{\frac{2}{k}} \sqrt{n + 1/2} J_{n+1/2}(k\pi).$$

Using that for all $k \in \mathbb{N}_0$ $\left|\left\langle e^{i\pi kx}, P_n \right\rangle\right| \leq 1$, Proposition 3 in [8] for the second inequality, we obtain, for $m \geq M \vee (e\,|c(t)|)$ and $M = 8/5 > 1.4$,

$$\begin{aligned}
|I_{m,k}| &\leq \sum_{n=0}^{\lfloor m/M \rfloor - 1} |\beta_n^m| \\
&\leq \sqrt{\frac{5}{4\pi}} \left|\mu_m^{c(t)}\right| \sum_{n=0}^{\lfloor m/M \rfloor - 1} \left(\frac{2\sqrt{\chi_m^{c(t)}}}{|c(t)|} \right)^n \\
&\leq \sqrt{\frac{5}{4\pi}} \left|\mu_m^{c(t)}\right| \int_0^{\lfloor m/M \rfloor} \left(\frac{2\sqrt{\chi_m^{c(t)}}}{|c(t)|} \right)^x dx \quad \left(\text{because } 2\sqrt{\chi_m^{c(t)}} > |c(t)| \right) \\
&\leq \frac{1}{\ln\left(2\sqrt{\chi_m^{c(t)}} / |c(t)| \right)} \sqrt{\frac{5}{4\pi}} \left|\mu_m^{c(t)}\right| \left(\frac{2\sqrt{\chi_m^{c(t)}}}{|c(t)|} \right)^{\lfloor m/M \rfloor}.
\end{aligned}$$

Thus, for $m \geq M \vee (e^2\,|c(t)|))$, using Lemma A.1 and (3.4) page 34 in [42] which yields $\ln\left(2\sqrt{\chi_m^{c(t)}} / |c(t)| \right) \geq \ln(2) + 2$ for the first inequality, and decomposing the exponent as $m = m/M + (1 - 1/M)m$ for the third inequality, we obtain

$$\begin{aligned}
&|I_{m,k}| \\
&\leq \sqrt{\frac{5e^3}{18}} \frac{1}{\ln(2) + 2} \left(\frac{2\sqrt{m(m+1) + c(t)^2}}{|c(t)|} \right)^{m/M} \left(\frac{e\,|c(t)|}{4(m + 1/2)} \right)^m \\
&\leq \sqrt{\frac{5e^3}{18}} \frac{1}{\ln(2) + 2} \left(\frac{2\sqrt{1 + 1/e^2}(m+1)}{|c(t)|} \right)^{m/M} \left(\frac{e\,|c(t)|}{4m} \right)^m
\end{aligned}$$

$$
\begin{aligned}
&\leq \sqrt{\frac{5e^3}{18}} \frac{1}{\ln(2)+2} \left(\frac{\sqrt{e^2+1}(m+1)}{(2m+1)} \right)^{m/M} \left(\frac{1}{4} \right)^{(1-1/M)m} \exp\left(- \left(1 - \frac{1}{M} \right) m \ln\left(\frac{m}{e\,|c(t)|} \right) \right) \\
&\leq \sqrt{\frac{5e^3}{18}} \frac{\sqrt{e^2+1}(M+1)}{(\ln(2)+2)4^{M-1}(2M+1)} \exp\left(- \left(1 - \frac{1}{M} \right) m \ln\left(\frac{m}{e\,|c(t)|} \right) \right) \\
&\leq \alpha_M \exp\left(- \left(1 - \frac{1}{M} \right) m \ln\left(\frac{m}{e\,|c(t)|} \right) \right),
\end{aligned}
$$

where

$$
\alpha_M = \sqrt{\frac{5e^3}{18}} \frac{\sqrt{e^2+1}(M+1)}{(\ln(2)+2)4^{M-1}(2M+1)}.
$$

Using that, for $\alpha > -1/2$ and $x \in \mathbb{R}$, $|J_\alpha(x)| \leq |x|^\alpha / \left(2^\alpha \Gamma(\alpha+1)\right)$ (see 9.1.20 in [1]) and that $|\beta_n^m| \leq 1$ for the first display, then that $m > M$ for the third display, we obtain, for all $k \in \mathbb{N}_0$,

$$
\begin{aligned}
|J_{m,k}| \leq & \sum_{n \geq \lfloor m/M \rfloor} \frac{1}{n!} \left(\frac{k\pi}{2} \right)^n \\
\leq & \frac{1}{\lfloor m/M \rfloor !} \left(\frac{k\pi}{2} \right)^{\lfloor m/M \rfloor} \exp\left(\frac{k\pi}{2} \right) \\
\leq & \sqrt{\frac{m}{2\pi M} \theta_{M/m}} \frac{1}{e} \exp\left(- \frac{m}{M} \ln\left(\frac{2m}{k\pi e M} \theta_{M/m} \right) \right) \exp\left(\frac{k\pi}{2} \right) \quad \left(\text{because } n! \geq \left(\frac{n}{e} \right)^n \sqrt{2\pi n} \right),
\end{aligned}
$$

where $\theta_{M/m} = 1 - M/m$. We have, taking $\overline{N} = \lfloor \theta N \rfloor$, where $\theta > 0$, for all t such that $N \geq M \vee (e^2\,|c(t)|)$,

$$
I_{N,\overline{N}}(t) \leq (2R)^p \left(I_1(t) \vee J_1 \right),
$$

where

$$
\begin{aligned}
I_1(t) &= \sum_{\substack{|\boldsymbol{k}|_\infty < \overline{N} \\ \boldsymbol{k} \in \mathbb{Z}^p}} \sum_{\substack{m \geq N \\ m \in \mathbb{N}_0}} m^p \alpha_M^p \exp\left(-2p \left(1 - \frac{1}{M} \right) m \ln\left(\frac{m}{e\,|c(t)|} \right) \right) \\
J_1 &= \sum_{\substack{|\boldsymbol{k}|_\infty < \overline{N} \\ \boldsymbol{k} \in \mathbb{Z}^p}} \sum_{\substack{m \geq N \\ m \in \mathbb{N}_0}} m^p \prod_{j=1}^p \left(\frac{m\theta_{M/m}}{2\pi M e^2} \exp\left(- \frac{2m}{M} \ln\left(\frac{2m}{k_j \pi e M} \theta_{M/m} \right) \right) \exp\left(k_j \pi \right) \right).
\end{aligned}
$$

Then, denoting by

$$
\gamma_M = \frac{1}{M} \ln\left(\frac{2}{\theta \pi e M} \right)
$$

and using that $\overline{N}-1 \leq \theta N$ and that $\sup_{t\geq 1} m^{2p} e^{-\gamma_M m} = e((2p-1)/(p\gamma_M e))^{2p}$ for the second display, taking $\theta = 2/(\pi e^2 M)$ for the last display, and $\gamma = \gamma_M - \theta\pi > 3/4$, we obtain

$$\begin{aligned}
J_1 &\leq \sum_{\substack{|\boldsymbol{k}|_\infty<\overline{N}\\ \boldsymbol{k}\in\mathbb{Z}^p}} \sum_{\substack{m\geq N\\ m\in\mathbb{N}_0}} \left(\frac{m^2\theta_{M/m}}{2\pi M e^2} \exp\left(-\frac{2m}{M} \ln\left(\frac{2m}{\overline{N}\pi e M}\right)\right) \exp\left(\overline{N}\pi\right)\right)^p \\
&\leq \sum_{\substack{|\boldsymbol{k}|_\infty<\overline{N}\\ \boldsymbol{k}\in\mathbb{Z}^p}} \left(\frac{\theta_{M/m}\exp\left(\theta N\pi\right)}{2\pi M e^2}\right)^p \left(\frac{2}{e\gamma_M}\right)^{2p} \int_N^\infty \exp\left(-pm\gamma_M\right) dm \\
&\leq \sum_{\substack{|\boldsymbol{k}|_\infty<\overline{N}\\ \boldsymbol{k}\in\mathbb{Z}^p}} \frac{1}{p\gamma_M} \left(\frac{2\theta_{M/m}}{\pi M\gamma_M^2 e^4}\right)^p \exp\left(-p\gamma N\right) \\
&\leq (2\theta N+1)^p\, \alpha_{M,J} \exp\left(-p\gamma N\right)
\end{aligned}$$

where

$$\alpha_{M,J} = \frac{1}{p\gamma_M}\left(\frac{2\theta_{M/m}}{\pi M\gamma_M^2 e^4}\right)^p.$$

Then, denoting by

$$\kappa_M(t) = \left(1-\frac{1}{M}\right)\ln\left(\frac{N}{e\left|c(t)\right|}\right)$$

and using for the second display that $\sup_{m\geq 1} m^p e^{-p\kappa_M(t)m} = (\kappa_M(t)e)^{-p}$, yields, for $N > M \vee (e^2\left|c(t)\right|)$,

$$\begin{aligned}
I_1(t) &\leq (2\overline{N}+1)^p \alpha_M^p \sum_{\substack{m\geq N\\ m\in\mathbb{N}_0}} m^p \exp\left(-2p\left(1-\frac{1}{M}\right) m\ln\left(\frac{N}{e\left|c(t)\right|}\right)\right) \\
&\leq \left(\frac{(2\overline{N}+1)\alpha_M}{\kappa_M(t)e}\right)^p \int_N^\infty \exp\left(-p\kappa_M(t)m\right) dm \\
&\leq \left(\frac{(2\theta N+1)\alpha_M}{\kappa_M(t)e}\right)^p \frac{1}{p\kappa_M(t)} \exp\left(-p\kappa_M(t)N\right).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
I_{N,\overline{N}}(t) \leq& (2R(2\theta N+1))^p \left(\alpha_{M,J} e^{-p\gamma N} \bigvee \left(\left(\frac{\alpha_M}{\kappa_M(t)e}\right)^p \frac{1}{p\kappa_M(t)} e^{-p\kappa_M(t)N}\right)\right) \\
\leq& N^p\,(2R(2\theta+1))^p \left(\alpha_{M,J} \bigvee \left(\left(\frac{\alpha_M}{\kappa_M(t)e}\right)^p \frac{1}{p\kappa_M(t)}\right)\right) e^{-p(\gamma\wedge\kappa_M(t))N}
\end{aligned}$$

hence

(A.23)

$$\sup_{t:\ |t|\le N/(e^2x_0)} I_{N,\overline{N}}(t) \le N^p \left(2R(2\theta+1)\right)^p \left(\alpha_{M,J} \bigvee \left(\left(\frac{\alpha_M}{(1-1/M)e}\right)^p \frac{1}{p(1-1/M)}\right)\right) e^{-p(\gamma\wedge(1-1/M))N}.$$

Denote by

$$\tilde{\theta} = (2\theta+1)^p \left(\alpha_{M,J} \bigvee \left(\left(\frac{\alpha_M}{(1-1/M)e}\right)^p \frac{1}{p(1-1/M)}\right)\right)$$
$$\tilde{\gamma} = p\gamma \wedge (1-1/M).$$

We conclude, using that for $|t| > N/(e^2x_0)$,

$$\begin{aligned} I_{N,\overline{N}}(t) &\le R^{2p} \sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\overline{N}} \|\phi_{\boldsymbol{k}}(\cdot)\|^2_{L^2([-1,1]^p)} \\ &\le R^{2p}(2\overline{N}+1)^p, \end{aligned}$$

and for the last display that $\sup_{t\ge 1} t^a/e^{bt} = (a/(be))^a$, where $a, b > 0$, that we have

$$\begin{aligned} &\int_{\mathbb{R}} \left\| \mathcal{F}_{1\mathrm{st}}[f](t,\cdot) - P^N_{c(t)} \mathcal{F}_{1\mathrm{st}}[f](t,\cdot) \right\|^2_{L^2([-R,R]^p)} dt \\ &\le \frac{2}{(2R)^p} \sup_{t:\ |t|\le N/(e^2x_0)} I_{N,\overline{N}}(t) \int_{-N/(e^2x_0)}^{N/(e^2x_0)} \sum_{\boldsymbol{k}\in\mathbb{Z}^p} |\mathcal{F}_{1\mathrm{st}}[f](t,\boldsymbol{k})|^2 dt \\ &\quad + \frac{2(2\theta+1)^p N^p R^p}{2^p} \frac{1}{(1\vee(N/(e^2x_0))^2)^s} \int_{|t|>-N/(e^2x_0)} \sum_{\boldsymbol{k}\in\mathbb{Z}^p} |\mathcal{F}_{1\mathrm{st}}[f](t,\boldsymbol{k})|^2 (1\vee|t|^2)^s dt \\ &\quad + 2\int_{\mathbb{R}} \left\| \mathcal{F}_{1\mathrm{st}}[f](t,\cdot) - \mathcal{E}^{\overline{N}} \mathcal{F}_{1\mathrm{st}}[f](t,\cdot) \right\|^2_{L^2([-R,R]^p)} dt \\ &\le 4\tilde{\theta} N^p e^{-\tilde{\gamma}pN} \pi l^2 + \frac{4(2\theta+1)^p R^p \pi l^2}{2^p} \frac{N^p}{(1\vee(N/(e^2x_0))^{2\sigma'+p})} \\ &\quad + \frac{4\pi l^2 \theta^{2\sigma'}}{(2R)^p N^{2\sigma'}} \quad \left(\text{using } f\in H^{q',s,\sigma'}(l),\ s>\sigma'+p/2, \text{ and } |\boldsymbol{m}|_1 \ge |\boldsymbol{m}|_\infty\right) \end{aligned}$$

(A.24)

$$\le \left(2\tilde{\theta}\left(\frac{p+2\sigma'}{\tilde{\gamma}pe}\right)^{p+2\sigma'} + \frac{2(2\theta+1)^p R^p}{2^p} \frac{N^{2\sigma'+p}}{(1\vee(N/(e^2x_0))^{2\sigma'+p})} + \frac{2\theta^{2\sigma'}}{(2R)^p}\right) \frac{2\pi l^2}{N^{2\sigma'}}.$$

This yields (A.22) hence (A.19) then (A.18), hence the result. □

A.4. **Properties of the SVD for $W(\cdot) = \cosh(\cdot/R)$.** The following proposition is from [20].

Proposition A.2. For $\boldsymbol{m} \in \mathbb{N}_0^p$, $R > 0$, and $t \neq 0$, we have

$$\left\| g_{\boldsymbol{m}}^{W,tx_0} \right\|_{L^\infty([-1,1]^p)} \leq H_W(t) \prod_{j=1}^{p} \sqrt{m_j + 1/2},$$

where

(1) when $W = W_{[-R,R]}$, using (66) in [9], $H_W(t) = H_0^p \left(1 + (|t|x_0)^2\right)^p$, where $H_0 = 2(1 + 1/\sqrt{3})$,
(2) when $W(\cdot) = \cosh(\cdot/R)$, $H_W(t) = H_1^p((|t|x_0)^6 \vee (1/\sqrt{|t|x_0}))^p$ from Corollary 1 in [20], where $H_1 > 0$.

Proposition A.2 is used to prove the adaptive selection of the parameters.

Appendix B. Proofs

A.1. **Proofs of Proposition 1, 2 and 3.**

Proof of Proposition 1. The first assertion comes from the fact that W is nondecreasing on $[0, \infty)$ and $W(0) > 0$. For the rest, we use that, for every $h \in L^2(W^{\otimes p})$, if we do not restrict the argument in the definition of $\mathcal{F}_c[h]$ to $[-1,1]^p$, $\mathcal{F}_c[h]$ can be defined as a function in $L^2(\mathbb{R}^p)$. In what follows, for simplicity, we use $\mathcal{F}_c[h]$ for both the function in $L^2([-1,1]^p)$ and in $L^2(\mathbb{R}^p)$. Let us now show that, for all $c \neq 0$, $\mathcal{F}_c$ defined in (3) is injective. Take $h \in L^2\left(W^{\otimes p}\right) \subseteq L^2(\mathbb{R}^p)$ such that $\mathcal{F}_c[h] = 0$ in $L^2\left([-1,1]^p\right)$. When W^{-1} vanishes at one point, this implies that h is compactly supported, thus, by the Paley-Wiener theorem (see, *e.g.*, Theorem 7.23 (a) in [44]), its Fourier transform can be extended as an entire function which restriction to $\mathbb{R}^p$ belongs to $L^2\left(\mathbb{R}^p\right)$. Because the Fourier transform vanishes on a subset with nonempty interior, then $\mathcal{F}[h] = 0$ on $\mathbb{R}^p$, thus $h = 0$ in $L^2\left(\mathbb{R}^p\right)$. Now, consider the case where $W^{-1}(x) > 0$ for all $x \in \mathbb{R}$. $\mathcal{F}_c[h]$ belongs to $C^\infty(\mathbb{R}^p)$ by the Lebesgue dominated convergence theorem because, for all $(\boldsymbol{k}, \boldsymbol{u}) \in \mathbb{N}_0^p \times \mathbb{R}^p$,

$$\int_{\mathbb{R}^p} \left| c^{|\boldsymbol{k}|_1} \boldsymbol{b}^{\boldsymbol{k}} e^{ic\boldsymbol{b}^T \boldsymbol{u}} h(\boldsymbol{b}) \right| d\boldsymbol{b} \leq c^{|\boldsymbol{k}|_1} \left\| h \right\|_{L^2(W^{\otimes p})} \prod_{j=1}^{p} M_{\boldsymbol{k}_j}.$$

We obtain, for all $(\boldsymbol{k}, \boldsymbol{u}) \in \mathbb{N}_0^p \times \mathbb{R}^p$,

$$\left| \mathcal{F}_c[h]^{(\boldsymbol{k})}(\boldsymbol{u}) \right| \leq c^{|\boldsymbol{k}|_1} \left\| h \right\|_{L^2(W^{\otimes p})} \prod_{j=1}^{p} M_{\boldsymbol{k}_j}.$$

Theorem B.1 in [17] and the fact that, by the Cauchy-Schwarz inequality, for all $j \in \{1, \dots, p\}$, $\boldsymbol{k} \in \mathbb{N}_0^p$, $M_{\boldsymbol{k}_j} \leq M_{\boldsymbol{k}_j - 1} M_{\boldsymbol{k}_j + 1}$ yield that $\mathcal{F}_c[h]$ is zero on $\mathbb{R}^p$. Thus, $\mathcal{F}[h]$ and h are zero almost everywhere on $\mathbb{R}^p$.

We now show that $\mathcal{K}$ is injective. Take $f \in L^2\left(w \otimes W^{\otimes p}\right)$ such that $\mathcal{K}[f] = 0$. By the Plancherel identity and the fact that $w \geq 1$, we have

$$\int_{\mathbb{R}^{p+1}} \left|\mathcal{F}_{1\mathrm{st}}\left[f\right](t, \boldsymbol{b})\right|^2 W^{\otimes p}(\boldsymbol{b}) dt d\boldsymbol{b} \leq 2\pi \int_{\mathbb{R}^{p+1}} \left|f(a, \boldsymbol{b})\right|^2 w(a) W^{\otimes p}(\boldsymbol{b}) da d\boldsymbol{b} < \infty$$

thus, there exists $\Omega_1 \subseteq \mathbb{R}$ of Lebesgue measure 1, such that, for all $t \in \Omega_1$, $\boldsymbol{b} \mapsto \mathcal{F}_{1\mathrm{st}}\left[f\right](t, \boldsymbol{b}) \in L^2\left(W^{\otimes p}\right)$. Hence, by the above, for all $t \in \Omega_1$ and $c \in \mathbb{R}$, $\boldsymbol{u} \mapsto \mathcal{F}_c\left[\mathcal{F}_{1\mathrm{st}}\left[f\right](t, \cdot)\right](\boldsymbol{u})$ is continuous. Also, because $\|\mathcal{K}[f]\|_{L^2(\mathbb{R} \times [-1,1]^p)} = 0$, there exists $\Omega_2 \subseteq \mathbb{R}$ of Lebesgue measure 1, such that, for all $t \in \Omega_2$, $\|\mathcal{K}[f](t, \cdot)\|_{L^2([-1,1]^p)} = 0$. As a result, using (5), we have that, for all $(t, \boldsymbol{u}) \in \Omega_1 \cap \Omega_2 \times [-1, 1]^p$, $\mathcal{K}[f](t, \boldsymbol{u}) = 0$. Using again (5) and the injectivity of $\mathcal{F}_c$ for $c \neq 0$, we obtain that for all $t \in (\Omega_1 \cap \Omega_2) \setminus \{0\}$, $\mathcal{F}_{1\mathrm{st}}\left[f\right](t, \cdot) = 0$ in $L^2\left(W^{\otimes p}\right)$, thus $\mathcal{F}_{1\mathrm{st}}\left[f\right](\star, \cdot) = 0$ in $L^2\left(1 \otimes W^{\otimes p}\right)$ and $f = 0$ in $L^2\left(1 \otimes W^{\otimes p}\right)$, hence in $L^2\left(w \otimes W^{\otimes p}\right)$.

We now show that $\mathcal{K}$ is continuous by showing it is continuous at 0. Indeed, for $f \in L^2\left(w \otimes W^{\otimes p}\right)$, by using successively the change of variables, the Plancherel identity, and the lower bounds on the weights, we have

$$\begin{aligned}
\|\mathcal{K}[f]\|^2_{L^2(\mathbb{R} \times [-1,1]^p)} &\leq \int_{\mathbb{R}^{p+1}} \left|\mathcal{F}[f](t, \boldsymbol{v})\right|^2 (t, \boldsymbol{v}) dt d\boldsymbol{v} \\
&= (2\pi)^p \int_{\mathbb{R}^{p+1}} \left|f(a, \boldsymbol{b})\right|^2 (a, \boldsymbol{b}) da d\boldsymbol{b} \\
&\leq \left(\frac{2\pi}{W(0)}\right)^p \int_{\mathbb{R}^{p+1}} \left|f(a, \boldsymbol{b})\right|^2 (a, \boldsymbol{b}) w(a) W^{\otimes p}(\boldsymbol{b}) da d\boldsymbol{b} \\
&= \left(\frac{2\pi}{W(0)}\right)^p \|f\|^2_{L^2(w \otimes W^{\otimes p})}.
\end{aligned}$$

Consider the particular case where $w = 1$. To show that $\mathcal{K}$ is not compact, we exhibit a bounded sequence $(f_k)_{k \in \mathbb{N}_0}$ in $L^2(1 \otimes W^{\otimes p})$ such that there does not exist a subsequence of $(\mathcal{K}\left[f_k\right])_{k \in \mathbb{N}_0}$ which converges in $L^2(\mathbb{R} \times [-1, 1]^p)$. Take $v_0 : \mathbb{R} \to \mathbb{R}$ with $\mathrm{supp}(v_0) \subset [1, 2]$ and $\|v_0\|_{L^2(\mathbb{R})} = 1$. For $k \in \mathbb{N}_0$, define $v_k(\cdot) = 2^{-k/2} v_0(2^{-k} \cdot)$ and, for $(a, \boldsymbol{b}^\top)^\top \in \mathbb{R}^{p+1}$, $f_k(a, \boldsymbol{b}) = \mathcal{F}^I\left[v_k(\cdot) \varphi_{\underline{\boldsymbol{0}}}^{W, x_0 \cdot}(\boldsymbol{b})\right](a)$. The sequence $(f_k)_{k \in \mathbb{N}_0}$ is bounded in $L^2\left(1 \otimes W^{\otimes p}\right)$ because of the Plancherel identity and the fact that, for $k \in \mathbb{N}_0$,

$$\|f_k\|^2_{L^2(1 \otimes W^{\otimes p})} = \frac{1}{2\pi} \int_{\mathbb{R}} v_k(t)^2 \int_{\mathbb{R}^p} \left|\varphi_{\underline{\boldsymbol{0}}}^{W, t x_0}(\boldsymbol{b})\right|^2 W^{\otimes p}(\boldsymbol{b}) dt d\boldsymbol{b}$$

$$\leq \frac{1}{2\pi}.$$

Moreover, using that $\mathcal{K}[f_k](\cdot,*) = \sigma_{\underline{0}}^{W,x_0\cdot} v_k(\cdot) g_{\underline{0}}^{W,x_0\cdot}(*)\,|x_0\cdot|^{p/2}$ for the first display, that for $m \in \mathbb{N}_0$, $c \in (0,\infty) \mapsto \rho_m^{W,c}$ is nondecreasing (see Lemma 1 in [20] for the case of $W = \cosh$, which can be directly extended to weight W that satisfies Assumption 1 (H1.2)) for the third display, and that, for all $j \in \mathbb{N}_0$, $\|v_j\|_{L^2(\mathbb{R})} = 1$ for the fourth display, we obtain, for all $(j,k) \in \mathbb{N}_0^2$, $j < k$,

$$\begin{aligned}
\|\mathcal{K}[f_j] - \mathcal{K}[f_k]\|^2_{L^2(\mathbb{R}\times[-1,1]^p)} &= \int_{\mathbb{R}} \frac{\left|\sigma_{\underline{0}}^{W,tx_0}\right|^2 |tx_0|^p}{(2\pi)^p} (2\pi)^p \left(v_j(t)^2 + v_k(t)^2\right) dt \\
&= \int_{\mathbb{R}} \rho_{\underline{0}}^{W,tx_0} (2\pi)^p \left(v_j(t)^2 + v_k(t)^2\right) dt \\
&\geq \rho_{\underline{0}}^{W,2^j x_0} (2\pi)^p \int_{\mathbb{R}} \left(v_j(t)^2 + v_k(t)^2\right) dt \\
&\geq 2(2\pi)^p \rho_{\underline{0}}^{W,x_0} > 0.
\end{aligned}$$

□

Proof of Proposition 2. This holds by Theorem 15.16 in [33] and the injectivity of $\mathcal{F}_c$ (see the proof above). □

Proof of Proposition 3. Take $f \in L^2(\mathbb{R})$ and start by showing that $\mathcal{I}_{a_0,\epsilon}[f] \in L^2([-\epsilon,\epsilon])$. First, note that the terms $1 - \rho_m^{W_{[-1,1]},a_0\epsilon}$ in the denominator of (9) are nonzero because $\left(\rho_m^{W_{[-1,1]},a_0\epsilon}\right)_{m\in\mathbb{N}_0}$ is nonincreasing and $\rho_0^{W_{[-1,1]},a_0\epsilon} < 1$ (see (3.49) in [42]). Now, using that $\left(g_m^{W_{[-1,1]},a_0\epsilon}(\cdot/\epsilon)/\sqrt{\epsilon}\right)_{m\in\mathbb{N}_0}$ is a basis of $L^2([-\epsilon,\epsilon])$ for the first display, that $\left(\rho_m^{W_{[-1,1]},a_0\epsilon}\right)_{m\in\mathbb{N}_0}$ is nonincreasing for the second display, using the Cauchy-Schwarz inequality for the third display, and that $\sum_{m\in\mathbb{N}_0} \rho_m^{W_{[-1,1]},a_0\epsilon} = 2a_0\epsilon/\pi$ (see (3.55) in [42]) and $\left\|g_m^{W_{[-1,1]},a_0\epsilon}\right\|^2_{L^2(\mathbb{R})} = 1/\rho_m^{W_{[-1,1]},a_0\epsilon}$ (see (3) in [8]) for the fourth display, we obtain

$$\begin{aligned}
&\sum_{m\in\mathbb{N}_0} \left(\frac{\rho_m^{W_{[-1,1]},a_0\epsilon}}{\left(1-\rho_m^{W_{[-1,1]},a_0\epsilon}\right)\epsilon}\right)^2 \left|\left\langle f(\star), g_m^{W_{[-1,1]},a_0\epsilon}\left(\frac{\star}{\epsilon}\right)\right\rangle_{L^2(\mathbb{R}\setminus[-\epsilon,\epsilon])}\right|^2 \left\|g_m^{W_{[-1,1]},a_0}\left(\frac{\cdot}{\epsilon}\right)\right\|^2_{L^2([-\epsilon,\epsilon])} \\
&= \sum_{m\in\mathbb{N}_0} \frac{\left(\rho_m^{W_{[-1,1]},a_0\epsilon}\right)^2}{\left(1-\rho_m^{W_{[-1,1]},a_0\epsilon}\right)^2} \left|\left\langle f(\star), \frac{g_m^{W_{[-1,1]},a_0\epsilon}(\star/\epsilon)}{\sqrt{\epsilon}}\right\rangle_{L^2(\mathbb{R}\setminus[-\epsilon,\epsilon])}\right|^2
\end{aligned}$$

$$\leq \frac{\|f\|^2_{L^2(\mathbb{R}\setminus[-\epsilon,\epsilon])}}{\left(1-\rho_0^{W_{[-1,1]},a_0\epsilon}\right)^2}\sum_{m\in\mathbb{N}_0}\left(\rho_m^{W_{[-1,1]},a_0\epsilon}\right)^2\left\|g_m^{W_{[-1,1]},a_0\epsilon}\right\|^2_{L^2(\mathbb{R})}$$

$$\leq \frac{2a_0\epsilon}{\pi\left(1-\rho_0^{W_{[-1,1]},a_0\epsilon}\right)^2}\|f\|^2_{L^2(\mathbb{R}\setminus[-\epsilon,\epsilon])}\,. \tag{A.25}$$

Let us now show the second statement in the proposition. Take $\epsilon>0$ and $g\in PW(a_0)$. Because $g(\epsilon\cdot)\in PW(a_0\epsilon)$ and $\left(g_m^{W_{[-1,1]},a_0\epsilon}\right)_{m\in\mathbb{N}_0}$ is a complete orthogonal system of $PW(a_0\epsilon)$, there exists a unique sequence $(\alpha_m)_{m\in\mathbb{N}}$ such that, in $L^2(\mathbb{R})$,

$$g(\epsilon\cdot)=\sum_{m\in\mathbb{N}_0}\alpha_m g_m^{W_{[-1,1]},a_0\epsilon}(\cdot)\,. \tag{A.26}$$

Hence, because $\left(g_m^{W_{[-1,1]},a_0\epsilon}\right)$ is a basis of $L^2([-1,1])$, we have $\alpha_m=\left\langle g(\epsilon\cdot),g_m^{W_{[-1,1]},a_0\epsilon}(\cdot)\right\rangle_{L^2([-1,1])}$. Moreover, we have

$$\sum_{m=0}^{\infty}\alpha_m g_m^{W_{[-1,1]},a_0\epsilon}(\cdot)=g(\epsilon\cdot)\mathbb{1}\{|\cdot|\geq 1\}+\sum_{m=0}^{\infty}\alpha_m g_m^{W_{[-1,1]},a_0\epsilon}(\cdot)\,\mathbb{1}\{|\cdot|\leq 1\}.$$

Taking the Hermitian product in $L^2(\mathbb{R})$ with $g_m^{W_{[-1,1]},a_0\epsilon}(\cdot)$ on both sides yields

$$\frac{\alpha_m}{\rho_m^{W_{[-1,1]},a_0\epsilon}}=\left\langle g(\epsilon\cdot)\mathbb{1}\{|\cdot|\geq\epsilon\},g_m^{W_{[-1,1]},a_0\epsilon}(\cdot)\right\rangle_{L^2(\mathbb{R})}+\alpha_m.$$

Hence, we obtain $\mathcal{I}_{a_0,\epsilon}[g]=g$ in $L^2(\mathbb{R})$.

Now, for f and h in $L^2(\mathbb{R})$, using $\mathcal{I}_{a_0,\epsilon}[\mathcal{P}_{a_0}[f]]=\mathcal{P}_{a_0}[f]$ in the second display, we have

$$\begin{aligned}\|f-\mathcal{I}_{a_0,\epsilon}[h]\|^2_{L^2([-\epsilon,\epsilon])}&\leq 2\left(\|f-\mathcal{P}_{a_0}[f]\|^2_{L^2([-\epsilon,\epsilon])}+\|\mathcal{P}_{a_0}[f]-\mathcal{I}_{a_0,\epsilon}[h]\|^2_{L^2([-\epsilon,\epsilon])}\right)\\&\leq 2\left(\|f-\mathcal{P}_{a_0}[f]\|^2_{L^2([-\epsilon,\epsilon])}+\|\mathcal{I}_{a_0,\epsilon}[\mathcal{P}_{a_0}[f]-h]\|^2_{L^2([-\epsilon,\epsilon])}\right).\end{aligned} \tag{A.27}$$

Replacing f by $\mathcal{P}_{a_0}[f]-h$ in (A.25) yields

$$\|\mathcal{I}_{a_0,\epsilon}[\mathcal{P}_{a_0}[f]-h]\|^2_{L^2([-\epsilon,\epsilon])}\leq\frac{C(a_0,\epsilon)}{2}\|\mathcal{P}_{a_0}[f]-h\|^2_{L^2(\mathbb{R}\setminus[-\epsilon,\epsilon])}\,. \tag{A.28}$$

Using (A.27) and (A.28) for the first display, $\mathcal{P}_{a_0}[f]-h=(\mathcal{P}_{a_0}[f]-f)+(f-h)$ and the triangle inequality for the second display, we obtain

$$\begin{aligned}\|f-\mathcal{I}_{a_0,\epsilon}[h]\|^2_{L^2([-\epsilon,\epsilon])}&\leq 2\|f-\mathcal{P}_{a_0}[f]\|^2_{L^2([-\epsilon,\epsilon])}+C(a_0,\epsilon)\|\mathcal{P}_{a_0}[f]-h\|^2_{L^2(\mathbb{R}\setminus[-\epsilon,\epsilon])}\\&\leq 2(1\vee C(a_0,\epsilon))\|f-\mathcal{P}_{a_0}[f]\|^2_{L^2(\mathbb{R})}+2C(a_0,\epsilon)\|f-h\|^2_{L^2(\mathbb{R}\setminus[-\epsilon,\epsilon])}\,.\end{aligned}$$

□

Lemma A.10. For $f \in L^2_w(\mathbb{R})$, where w is even, nondecreasing on $[0,\infty)$, such that $w(0) > 0$, and with $R > 0$, we have

$$\left\|\mathcal{P}_R\left[\mathcal{F}\left[f\right]\right] - \mathcal{F}\left[f\right]\right\|^2_{L^2(\mathbb{R})} \leq \frac{2\pi}{w(R)}\|f\|^2_{L^2(w)}.$$

Proof of Lemma A.10. The result follows by the Plancherel identity

$$\begin{aligned}\left\|\mathcal{P}_R\left[\mathcal{F}\left[f\right]\right] - \mathcal{F}\left[f\right]\right\|^2_{L^2(\mathbb{R})} &= 2\pi \int_{\mathbb{R}} \mathbb{1}\{|a| > R\}\left|f\left(a\right)\right|^2 da \\ &\leq \frac{2\pi}{w(R)}\int_{\mathbb{R}} \left|f\left(a\right)\right|^2 w(a) da.\end{aligned}$$

□

A.2. **Lower bounds.** We consider the problem of nonparametric estimation of a function f in a class $\mathcal{H} \subseteq L^2(\mathbb{R}^{p+1})$. We use

$$\inf_{\widehat{f}_n}\sup_{f\in\mathcal{H}}\mathbb{E}\left[\left\|\widehat{f}_n - f\right\|^2_{L^2(\mathbb{R}^{p+1})}\right] \geq \inf_{\widehat{f}_n}\max_{f_{j,n}\in\mathcal{H},\ j\in\{1,2\}}\mathbb{E}\left[\left\|\widehat{f}_n - f_{j,n}\right\|^2_{L^2(\mathbb{R}^{p+1})}\right],$$

where $f_{j,n}$, for $j = 1, 2$, are two well chosen functions. We denote by $\mathbb{P}_{j,n}$ the law of the data implied by a function $f_{j,n}$, for $j = 1, 2$. We use the following lemma (see Theorem 2.2, (2.5), and (2.9) in [46]), where χ_2 and K denote respectively the χ_2 and Kullback-Leibler divergences.

Lemma A.11. If, for $\xi < \sqrt{2}$, we have

(i) $\forall m \in \{1, 2\},\ f_{m,n} \in \mathcal{H}$,
(ii) $\|f_{1,n} - f_{2,n}\|^2_{L^2(\mathbb{R}^{p+1})} \geq 4h_n^2 > 0$,
(iii) $\chi_2(\mathbb{P}_{2,n}, \mathbb{P}_{1,n}) \leq \xi^2$ or $K(\mathbb{P}_{2,n}, \mathbb{P}_{1,n}) \leq \xi^2$,

then

$$\inf_{\widehat{f}_n}\max_{f_{j,n}\in\mathcal{H},\ j\in\{1,2\}}\mathbb{E}\left[\left\|\widehat{f}_n - f_{j,n}\right\|^2_{L^2(\mathbb{R}^{p+1})}\right] \geq e^{-\xi^2}\bigvee\left(2\left(1 - \frac{\xi}{\sqrt{2}}\right)\right).$$

Proof of Theorem 2 (1). Here, for $j = 1, 2$, $\mathbb{P}_{j,n}$ is the law of an i.i.d sample of $(Y_i, \boldsymbol{X}_i)_{i=1}^n$ when $f_{\alpha,\boldsymbol{\beta}} = f_{j,n}$ and for the same marginal $f_{\boldsymbol{X}|\mathcal{X}}$. We use the notation $C_{\boldsymbol{X}} = \left\|f_{\boldsymbol{X}|\mathcal{X}}\right\|_{L^\infty(\mathcal{X})}$. We show that conditions (i)-(iii) in Lemma A.11 are satisfied for n large enough using

$$f_{1,n} = f_0 \text{ and } f_{2,n} = f_0 + \gamma H_N, \tag{A.29}$$

where N is odd and γ are chosen later as a function of n, for $(a, \boldsymbol{b}) \in \mathbb{R}^{p+1}$,

$$H_N(a,\boldsymbol{b}) = \mathcal{F}^I_{1\text{st}}\left[\left(\frac{c(|t|)}{2\pi}\right)^{p/2} i^{-|\widetilde{\boldsymbol{N}}(q)|_1}\phi(t)\psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}\left(\frac{\boldsymbol{b}}{R}\right)\right](a)\mathbb{1}\{|\boldsymbol{b}|_\infty \le R\}, \tag{A.30}$$

$\widetilde{\boldsymbol{N}}(1) = \left(N, \underline{\boldsymbol{H_1(U)}}^\top\right)^\top \in \mathbb{N}^p$, where $H_1(U) = \lceil H(Rc(U))\rceil$ where $U > 0$ is chosen later independently of n, $H(Rc(U)) \ge 2$ is defined in Lemma A.5, $\widetilde{\boldsymbol{N}}(\infty) = \underline{\boldsymbol{N}} \in \mathbb{N}^p$, $R > 0$, $c : t \in \mathbb{R} \mapsto tx_0$, $\phi \in C^\infty(\mathbb{R})$ with $\text{supp}(\phi) \subset [-U, U] \setminus [-U/2, U/2]$ and

$$\forall t \in [-U,U]\setminus[-U/2,U/2], \quad \phi(t) = \exp\left(1 - \frac{1}{1 - 16\left(|t| - 3U/4\right)^2/U^2}\right), \tag{A.31}$$

and, for $\tau \in (0,\infty)$ chosen later independently of n,

$$f_0(a,\boldsymbol{b}) = \frac{1}{\pi\tau\left(1+(a/\tau)^2\right)}\frac{\mathbb{1}\{|\boldsymbol{b}|_\infty \le R\}}{(2R)^{p/2}}.$$

Clearly $\mathbb{P}_{2,n}$ is absolutely continuous with respect to $\mathbb{P}_{1,n}$. We denote by $b^1_{\boldsymbol{m}}$ and $b^2_{\boldsymbol{m}}$ the coefficients $(b_{\boldsymbol{m}}(\cdot))_{\boldsymbol{m}\in\mathbb{N}_0^p}$ of $f_{1,n}$ and $f_{2,n}$.

Step 1. We start by showing that $f_{1,n}$ and $f_{2,n}$ satisfy (i) in Lemma A.11.

Step 1.1. We prove that $f_{1,n}$ and $f_{2,n}$ are densities when $N \ge H_1(U)$ and γ satisfies

$$\frac{\gamma U}{(1+p/2)}\left(\frac{Ux_0}{2\pi}\right)^{p/2}\left(N+\frac{1}{2}\right)^{k_q/2}\Xi_q(U) \le \frac{1}{(2R)^{p/2}\tau(1+(1/\tau)^2)} \qquad \text{when } |a|<1, \tag{A.32}$$

$$\gamma\frac{UC_8(Rx_0,p,U)}{2a^2}N^{k_q/2+2} \le \frac{1}{2^{p/2}\tau(1+(a/\tau)^2)} \qquad \text{when } |a|\ge 1, \tag{A.33}$$

where $\Xi_q(U) = (H_1(U)+1/2)^{(p-1)/2} + \left(1-(H_1(U)+1/2)^{(p-1)/2}\right)\mathbb{1}\{q=\infty\}$ and $C_8(Rx_0,p,U)$ is defined in Lemma A.7. We show that (A.32) and (A.33) imply that, for all $(a,\boldsymbol{b}) \in \mathbb{R}\times[-R,R]^p$, $f_0(a,\boldsymbol{b}) \ge |\gamma H_N(a,\boldsymbol{b})|$ which ensures that $f_{2,n}(\cdot)$ is nonnegative.

Using the third assertion in Lemma A.4, we obtain that, for $N \ge H_1(U)$ and all $(a,\boldsymbol{b}) \in \mathbb{R}\times[-R,R]^p$,

$$\begin{aligned}
|\gamma H_N(a,\boldsymbol{b})| &\le \frac{\gamma}{2\pi}\left(\frac{x_0}{2\pi}\right)^{p/2}\left(N+\frac{1}{2}\right)^{k_q/2}\Xi_q(U)\int_{\mathbb{R}}|t|^{p/2}\,|\phi(t)|\,dt \\
&\le \frac{\gamma U}{\pi(1+p/2)}\left(\frac{Ux_0}{2\pi}\right)^{p/2}\left(N+\frac{1}{2}\right)^{k_q/2}\Xi_q(U)\ \left(\text{because } \|\phi\|_{L^\infty(\mathbb{R})} \le 1\right).
\end{aligned}$$

This yields the condition (A.32) in the case $|a| < 1$. Additionally, because for all $\boldsymbol{b} \in \mathbb{R}^p$, $t \mapsto \psi^{tRx_0}_{\widetilde{\boldsymbol{N}}(q)}(\boldsymbol{b}/R)$ is analytic (see [19] page 320),

$$t \mapsto \left(\frac{c(|t|)}{2\pi}\right)^{p/2} i^{-|\widetilde{\boldsymbol{N}}(q)|_1} \phi(t) \psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}\left(\frac{\boldsymbol{b}}{R}\right)$$

belongs to $C^\infty(\mathbb{R})$ and its derivatives are square integrable because their support is compact. Thus, by integration by parts, we obtain, for $|a| \geq 1$ and $\boldsymbol{b} \in \mathbb{R}$,

$$\begin{aligned}|H_N(a,\boldsymbol{b})| &= \frac{1}{2\pi a^2 R^{p/2}} \left| \int_{\mathbb{R}} e^{-ita} \frac{\partial^2}{\partial t^2} \left(\left(\frac{Rc(|t|)}{2\pi}\right)^{p/2} \phi(t) \psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}\left(\frac{\boldsymbol{b}}{R}\right) \mathbb{1}\{|\boldsymbol{b}|_\infty \leq R\} \right) dt \right| \\ &\leq \frac{1}{\pi a^2 R^{p/2}} \int_{U/2}^{U} \left| \frac{\partial^2}{\partial t^2} \left(\left(\frac{Rc(|t|)}{2\pi}\right)^{p/2} \phi(t) \psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}\left(\frac{\boldsymbol{b}}{R}\right) \mathbb{1}\{|\boldsymbol{b}|_\infty \leq R\} \right) \right| dt.\end{aligned}$$

Using Lemma A.7, for $N \geq H_1(U)$ we have

$$\forall |a| \geq 1,\ \forall \boldsymbol{b} \in \mathbb{R}^p,\ |H_N(a,\boldsymbol{b})| \leq \frac{UC_8(Rx_0,p,U)}{2\pi a^2 R^{p/2}} N^{k_q/2+2}. \tag{A.34}$$

Thus, (A.34) yields the condition (A.33). Due to (A.32) and (A.33), $f_{1,n}$ and $f_{2,n}$ are nonnegative.

Step 1.2. $f_{1,n}$ integrates to one because $\int_{\mathbb{R}^{p+1}} f_0(a,\boldsymbol{b}) da d\boldsymbol{b} = 1$ and $f_{2,n}$ as well by the Fubini theorem and because for odd values of N the functions $b \in [-1,1] \mapsto \psi^c_N(b)$ are odd.

Step 1.3: We prove that $f_{1,n}$ and $f_{2,n}$ belong to in $\mathcal{H}^{q,\phi,\omega}_{w,W}(l)$.

Clearly, we have that $\int_{\mathbb{R}^p} \int_{\mathbb{R}} f_{1,n}(a,\boldsymbol{b})^2 (1 \vee |a|) W^{\otimes p}_{[-R,R]}(\boldsymbol{b}) d\boldsymbol{b} da < \infty$. The same inequality holds for $f_{2,n}$ when (A.32) and (A.33) hold. Indeed, we then have for all $(a,\boldsymbol{b}) \in \mathbb{R}^{p+1}$ $f_{2,n}(a,\boldsymbol{b})^2 \leq 4 f_{1,n}(a,\boldsymbol{b})^2$. Thus, $f_{1,n}$ and $f_{2,n}$ belong to $L^2\left(w \otimes W^{\otimes p}_{[-R,R]}\right)$.

Let us show that if

$$2\left(\frac{\Gamma(2s+1)}{(2\tau)^{2s+1}} + \frac{1}{2\tau} + \gamma^2 \left(\frac{Rx_0}{2\pi}\right)^p \frac{\left(1 \vee U^{2s}\right) U^{p+1}}{p+1}\right) \leq \pi l^2 \tag{A.35}$$

$$\frac{C_{12}(Rx_0,\sigma,p)}{\tau k_q^{2\sigma}} + \frac{2U\left(\mathbb{1}\{q=\infty\} + p^{2\sigma}\mathbb{1}\{q=1\}\right)\gamma^2 N^{2\sigma}}{p+1} \left(\frac{URx_0}{2\pi}\right)^p \leq \pi l^2. \tag{A.36}$$

then because $\operatorname{supp}\left(\boldsymbol{b} \mapsto \int_{\mathbb{R}} f_{2,n}(a,\boldsymbol{b}) da\right) \subseteq [-R,R]^p$, $f_{2,n}$ belongs to $\mathcal{H}^{q,\phi,\omega}_{w,W}(l)$ and because $f_{1,n}$ corresponds to $f_{2,n}$ when $\gamma = 0$, this holds for both $f_{1,n}$ and $f_{2,n}$. Using the definition of $b^2_{\boldsymbol{m}}(t)$ for the first equality, then Proposition A.1-(iii) for the second equality, a change of variables,

and that for $t \in \mathbb{R}$, $\mathcal{F}_{1\mathrm{st}}\left[f_0(\cdot,\star)\right](t) = e^{-|t|\tau}\mathbb{1}\{|\star|_\infty \leq R\}/(2R)^{p/2}$, we have for $\boldsymbol{m} \in \mathbb{N}_0^p$,

$$
\begin{aligned}
b^2_{\boldsymbol{m}}(t) &= \left\langle \mathcal{F}_{1\mathrm{st}}\left[f_0(\cdot,\star)\right](t) + \gamma \mathcal{F}_{1\mathrm{st}}\left[H_N(\cdot,\star)\right](t), \varphi^{W,c(t)}_{\boldsymbol{m}}(\star) \right\rangle_{L^2\left(W^{\otimes p}_{[-R,R]}\right)} \\
&= \left\langle \mathcal{F}_{1\mathrm{st}}\left[f_0(\cdot,\star)\right](t) + \gamma \mathcal{F}_{1\mathrm{st}}\left[H_N(\cdot,\star)\right](t), \frac{i^{-|\boldsymbol{m}|_1}}{R^{p/2}} \mathcal{E}\mathrm{xt}\left[\psi^{Rc(t)}_{\boldsymbol{m}}\right]\left(\frac{\star}{R}\right) \right\rangle_{L^2\left(W^{\otimes p}_{[-R,R]}\right)} \\
&= e^{-\tau|t|} i^{-|\boldsymbol{m}|_1} \left\langle \frac{1}{2^{p/2}}, \psi^{Rc(t)}_{\boldsymbol{m}}(\star) \right\rangle_{L^2([-1,1]^p)} + \gamma \mathbb{1}\{\boldsymbol{m} = \widetilde{\boldsymbol{N}}(q)\}(-1)^{|\widetilde{\boldsymbol{N}}(q)|_1} \left(\frac{Rc(|t|)}{2\pi}\right)^{p/2} \phi(t).
\end{aligned} \tag{A.37}
$$

Thus, using that $\left(\psi^{Rc(t)}_{\boldsymbol{m}}\right)_{\boldsymbol{m}\in\mathbb{N}_0^p}$ is a basis of $L^2([-1,1]^p)$ and that $\left\|1/2^{p/2}\right\|_{L^2([-1,1]^p)} = 1$, We obtain that, for all $t \neq 0$,

$$
\sum_{\boldsymbol{m}\in\mathbb{N}_0^p} \left|b^2_{\boldsymbol{m}}(t)\right|^2 \leq 2\left(e^{-2\tau|t|} + \gamma^2 \left(\frac{Rc(|t|)}{2\pi}\right)^p \phi(t)^2\right).
$$

Thus, using the Fubini-Tonelli theorem for the first equality we obtain

$$
\begin{aligned}
&\sum_{\boldsymbol{m}\in\mathbb{N}_0^p} \int_{\mathbb{R}} \left(1 \vee |t|^{2s}\right) \left|b^2_{\boldsymbol{m}}(t)\right|^2 dt \\
&= \int_{\mathbb{R}} \left(1 \vee |t|^{2s}\right) \sum_{\boldsymbol{m}\in\mathbb{N}_0^p} \left|b^2_{\boldsymbol{m}}(t)\right|^2 dt \\
&\leq 2\left(\int_{\mathbb{R}} e^{-2\tau|t|}\left(1 + |t|^{2s}\right) dt + \gamma^2 \left(\frac{Rx_0}{2\pi}\right)^p \int_{\mathbb{R}} \left(1 \vee |t|^{2s}\right)|t|^p \phi^2(t) dt\right) \\
&\leq 4\left(\frac{\Gamma(2s+1)}{(2\tau)^{2s+1}} + \frac{1}{2\tau} + \gamma^2 \left(\frac{Rx_0}{2\pi}\right)^p \frac{\left(1 \vee U^{2s}\right) U^{p+1}}{p+1}\right)
\end{aligned}
$$

which yields (A.35). Moreover, using the Fubini-Tonelli theorem, (A.37), denoting by

$$
C_{13,q}(\tau, Rx_0, \sigma, p) = \int_{\mathbb{R}} e^{-2\tau|t|} \sum_{\boldsymbol{m}\in\mathbb{N}_0^p} |\boldsymbol{m}|_q^{2\sigma} \left| \left\langle \frac{1}{2^{p/2}}, \psi^{Rc(t)}_{\boldsymbol{m}} \right\rangle_{L^2([-1,1]^p)} \right|^2 dt,
$$

and using Lemma A.8, there exists $C_{12}(Rx_0, \sigma, p) > 0$ such that, for $\tau \geq 3e^{\sigma+p/2-1/4}Rx_0/8$ and $N \geq H_1(U)$,

$$
\sum_{\boldsymbol{m}\in\mathbb{N}_0^p} |\boldsymbol{m}|_q^{2\sigma} \int_{\mathbb{R}} \left|b^2_{\boldsymbol{m}}(t)\right|^2 dt
$$

$$\leq 2\left(C_{13,q}(\tau, Rx_0, \sigma, p) + \gamma^2\left(\frac{Rx_0}{2\pi}\right)^p \left|\widetilde{\boldsymbol{N}}(q)\right|_q^{2\sigma} \int_{\mathbb{R}} \phi^2(t)\,|t|^p\,dt\right)$$
$$\leq 2\left(\frac{C_{12}(Rx_0, \sigma, p)}{\tau k_q^{2\sigma}} + \frac{2U\left(\mathbb{1}\{q=\infty\} + p^{2\sigma}\,\mathbb{1}\{q=1\}\right)\gamma^2 N^{2\sigma}}{p+1}\left(\frac{URx_0}{2\pi}\right)^p\right).$$

This yields (A.36).

Step 2. $f_{1,n}$ and $f_{2,n}$ satisfy (ii) in Lemma A.11 with $4h_n^2 = \gamma^2 I_\phi(U)/(2\pi)\,(Rx_0/(2\pi))^p$ where $I_\phi(U) = 2\int_{U/2}^{U} |t|^p\,\phi(t)^2 dt$ because

$$\|f_{1,n} - f_{2,n}\|^2_{L^2(\mathbb{R}^{p+1})} = \frac{\gamma^2 I_\phi(U)}{2\pi}\left(\frac{Rx_0}{2\pi}\right)^p. \tag{A.38}$$

Step 3. Let us now show that the functions $f_{1,n}$ and $f_{2,n}$ satisfy (iii) in Lemma A.11. Let $\xi < \sqrt{2}$. The χ^2 divergence between $\mathbb{P}_{1,n} = (\mathbb{P}_1)^{\otimes n}$ and $\mathbb{P}_{2,n} = (\mathbb{P}_2)^{\otimes n}$ is (see (ii) page 97 in [46])

$$\begin{aligned}\chi_2(\mathbb{P}_{2,n}, \mathbb{P}_{1,n}) &= (1 + \chi_2(\mathbb{P}_2, \mathbb{P}_1))^n - 1\\ &= n\int_0^{\chi_2(\mathbb{P}_2,\mathbb{P}_1)} (1+u)^{n-1} du\\ &\leq n\chi_2(\mathbb{P}_2, \mathbb{P}_1)\exp\left((n-1)\chi_2(\mathbb{P}_2, \mathbb{P}_1)\right).\end{aligned}$$

Thus, if $\chi_2(\mathbb{P}_{2,n}, \mathbb{P}_1) \leq 1/n$, we have $\chi_2(\mathbb{P}_{2,n}, \mathbb{P}_{1,n}) \leq en\chi_2(\mathbb{P}_2, \mathbb{P}_1)$. Then, the following inequalities hold

$$\begin{aligned}\chi_2(\mathbb{P}_2, \mathbb{P}_1) &= \int_{\mathbb{S}_{\boldsymbol{X}}}\int_{\mathbb{R}} \frac{\left(f^1_{Y,\boldsymbol{X}}(y,\boldsymbol{x}) - f^2_{Y,\boldsymbol{X}}(y,\boldsymbol{x})\right)^2}{f^1_{Y,\boldsymbol{X}}(y,\boldsymbol{x})} d\boldsymbol{x}dy\\ &\leq \int_{\mathbb{S}_{\boldsymbol{X}}}\int_{\mathbb{R}} f_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{x})\frac{\left(f^1_{Y|\boldsymbol{X}}(y|\boldsymbol{x}) - f^2_{Y|\boldsymbol{X}}(y|\boldsymbol{x})\right)^2}{f^1_{Y|\boldsymbol{X}}(y|\boldsymbol{x})} d\boldsymbol{x}dy.\end{aligned}$$

Using (1)-(2) and that under f_0 the components of $\left(\alpha, \boldsymbol{\beta}^\top\right)$ are independent we obtain, for all $y \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^p$ such that $\boldsymbol{x} \in \mathbb{S}_{\boldsymbol{X}}$,

$$\begin{aligned}f^1_{Y|\boldsymbol{X}}(y|\boldsymbol{x}) &= \int_{\mathbb{R}} f^1_\alpha(y-v)\, f^1_{\boldsymbol{\beta}^\top\boldsymbol{x}}(v)\,dv\\ &= \left(\prod_{k=1}^p \frac{1}{|\boldsymbol{x}_k|}\right)\int_{\mathbb{R}^p} f^1_\alpha\left(y - \sum_{k=1}^p \boldsymbol{u}_k\right)\prod_{k=1}^p f^1_{\boldsymbol{\beta}_k}\left(\frac{\boldsymbol{u}_k}{\boldsymbol{x}_k}\right) d\boldsymbol{u}\end{aligned}$$

$$= \left(\prod_{j=1}^{p} \frac{1}{|\boldsymbol{x}_j|}\right) \frac{1}{\pi\tau(2R)^p} \int_{\mathbb{R}^p} \frac{1}{\left|\left(y - \sum_{k=1}^{p} \boldsymbol{u}_k\right)/\tau\right|^2 + 1} \prod_{k=1}^{p} \mathbb{1}\{|\boldsymbol{u}_k| \leq |\boldsymbol{x}_k| R\} d\boldsymbol{u}$$
$$\geq \frac{1}{\pi\tau} \inf_{|u| \leq |\boldsymbol{x}|_1 R} \frac{1}{(|y-u|/\tau)^2 + 1}.$$

This yields, using that $\mathbb{S}_{\boldsymbol{X}} = [-x_0, x_0]^p$ for the third inequality,

$$\begin{aligned}
\chi_2(\mathbb{P}_2, \mathbb{P}_1) &\leq \pi\tau C_{\boldsymbol{X}} \int_{\mathbb{S}_{\boldsymbol{X}}} \int_{\mathbb{R}} \max_{|u| \leq |\boldsymbol{x}|_1 R} \left((|y-u|/\tau)^2 + 1\right) \left(f^1_{Y|\boldsymbol{X}}(y|\boldsymbol{x}) - f^2_{Y|\boldsymbol{X}}(y|\boldsymbol{x})\right)^2 d\boldsymbol{x} dy \\
&\leq \pi\tau C_{\boldsymbol{X}} \int_{[-x_0,x_0]^p} \int_{\mathbb{R}} \left(\frac{2y^2}{\tau^2} + \frac{2(|\boldsymbol{x}|_1 R)^2}{\tau^2} + 1\right) \left(f^1_{Y|\boldsymbol{X}}(y|\boldsymbol{x}) - f^2_{Y|\boldsymbol{X}}(y|\boldsymbol{x})\right)^2 d\boldsymbol{x} dy \\
&\leq \frac{2\pi C_{\boldsymbol{X}}}{\tau} \int_{[-x_0,x_0]^p} \left(\int_{\mathbb{R}} y^2 \left| \left(f^2_{Y|\boldsymbol{X}} - f^1_{Y|\boldsymbol{X}}\right)(y|\boldsymbol{x}) \right|^2 dy \right. \\
&\qquad \left. + \left((x_0 pR)^2 + \frac{\tau^2}{2}\right) \int_{\mathbb{R}} \left| \left(f^2_{Y|\boldsymbol{X}} - f^1_{Y|\boldsymbol{X}}\right)(y|\boldsymbol{x}) \right|^2 dy \right) d\boldsymbol{x} \\
&\leq \frac{(2\pi)^2 C_{\boldsymbol{X}}}{\tau} \left(\int_{[-x_0,x_0]^p} \int_{\mathbb{R}} \left|\partial_t \left(\mathcal{F}[f_{2,n}] - \mathcal{F}[f_{1,n}]\right)(t, t\boldsymbol{x})\right|^2 d\boldsymbol{x} dt \right. \\
&\qquad \left. + \left((x_0 pR)^2 + \frac{\tau^2}{2}\right) \int_{[-x_0,x_0]^p} \int_{\mathbb{R}} \left|\left(\mathcal{F}[f_{2,n}] - \mathcal{F}[f_{1,n}]\right)(t, t\boldsymbol{x})\right|^2 d\boldsymbol{x} dt \right) \\
&\leq \frac{(2\pi)^2 C_{\boldsymbol{X}} x_0^p \gamma^2}{\tau} \left(\int_{[-1,1]^p} \int_{\mathbb{R}} \left|\partial_t \mathcal{F}[H_N](t, t x_0 \boldsymbol{x})\right|^2 d\boldsymbol{x} dt \right. \\
&\qquad \left. + \left((x_0 pR)^2 + \frac{\tau^2}{2}\right) \int_{[-1,1]^p} \int_{\mathbb{R}} \left|\mathcal{F}[H_N](t, t x_0 \boldsymbol{x})\right|^2 d\boldsymbol{x} dt \right) \\
&\leq \frac{(2\pi)^2 C_{\boldsymbol{X}} x_0^p \gamma^2}{\tau} \left(I_1 + \left((x_0 pR)^2 + \frac{\tau^2}{2}\right) I_2 \right).
\end{aligned}$$

The term I_1 is bounded using that, for $\boldsymbol{x} \in [-1,1]^p$,

$$\begin{aligned}
\mathcal{F}[H_N](t, t x_0 \boldsymbol{x}) &= \left(\frac{c(|t|)}{2\pi}\right)^{p/2} i^{-|\widetilde{\boldsymbol{N}}(q)|_1} \phi(t) \mathcal{F}\left[\psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}\left(\frac{\cdot}{R}\right) \mathbb{1}\{|\cdot|_\infty \leq R\}\right](t x_0 \boldsymbol{x}) \\
&= \left(\frac{c(|t|)}{2\pi}\right)^{p/2} i^{-|\widetilde{\boldsymbol{N}}(q)|_1} \phi(t) \mathcal{F}_{Rc(t)}\left[\psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}\right](\boldsymbol{x}) \\
&= R^{p/2} \phi(t) \sqrt{\rho^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}} \psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}(\boldsymbol{x}) \quad \left(\text{using } \mu^{Rc(t)}_m = i^m \left(\frac{2\pi}{Rc(|t|)}\right)^{1/2} \sqrt{\rho^{Rc(t)}_m}\right), \qquad \text{(A.39)}
\end{aligned}$$

which yields

$$\frac{I_1}{R^p} \leq \int_{\mathbb{R}} \int_{[-1,1]^p} \left(\left(\frac{\mathrm{d}\sqrt{\rho^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}}}{\mathrm{d}t} \phi(t) + \phi'(t) \sqrt{\rho^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}} \right) \psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}(\boldsymbol{x}) + \sqrt{\rho^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}} \phi(t) \frac{\partial \psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}(\boldsymbol{x})}{\partial t} \right)^2 dt d\boldsymbol{x}.$$

Using (7.114) in [42] and that cross-products terms of the last inequality are zero, we obtain

$$\begin{aligned} \frac{I_1}{R^p} \leq & \int_{\mathbb{R}} \phi(t)^2 \left(\frac{\mathrm{d}\sqrt{\rho^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}}}{\mathrm{d}t} \right)^2 dt + \int_{\mathbb{R}} \left(\left(\phi'(t)\right)^2 \rho^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)} + 2\phi(t)|\phi'(t)| \sqrt{\rho^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}} \frac{\mathrm{d}\sqrt{\rho^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}}}{\mathrm{d}t} \right) dt \\ & + \int_{\mathbb{R}} \left(\int_{[-1,1]^p} \left(\frac{\psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}(\boldsymbol{x})}{\partial t} \right)^2 d\boldsymbol{x} \right) \phi(t)^2 \rho^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)} dt. \end{aligned} \tag{A.40}$$

Then, using (7.100) in [42] for the second equality yields, for $|t| \geq \epsilon$,

$$\begin{aligned} \frac{\mathrm{d}\sqrt{\rho^{Rc(t)}_N}}{\mathrm{d}t} &= \frac{x_0 R}{2\sqrt{\rho^{Rc(t)}_N}} \left. \frac{\mathrm{d}\rho^c_N}{\mathrm{d}c} \right|_{c=Rc(t)} \\ &= \frac{\sqrt{\rho^{Rc(t)}_N}}{|t|} \left(\psi^{Rc(t)}_N(1) \right)^2, \end{aligned}$$

hence, using the first and the last assertions of Lemma A.4, for $U/2 \leq |t| \leq U$,

$$\frac{\mathrm{d}\sqrt{\rho^{Rc(t)}_N}}{\mathrm{d}t} \leq \frac{3N\sqrt{\rho^{Rc(t)}_N}}{2\,|t|} \left(1 + \frac{2(Rc(U))^2}{3^{3/2}} \right)^2 \mathbb{1}\{N > 0\} + \sqrt{\rho^{Rc(t)}_0} \left(\frac{2x_0 R}{\pi} \right) \mathbb{1}\{N = 0\}. \tag{A.41}$$

For $q = 1$ and $N \geq H_1(U)$, we have

$$\begin{aligned} \frac{\mathrm{d}\sqrt{\rho^{Rc(t)}_N}}{\mathrm{d}t} &= (p-1) \left(\sqrt{\rho^{Rc(t)}_{H_1(U)}} \right)^{p-2} \sqrt{\rho^{Rc(t)}_N} \left(\frac{\mathrm{d}\sqrt{\rho^{Rc(t)}_{H_1(U)}}}{\mathrm{d}t} \right) + \left(\sqrt{\rho^{c(t)}_{H_1(U)}} \right)^{p-1} \left(\frac{\mathrm{d}\sqrt{\rho^{Rc(t)}_{H_1(U)}}}{\mathrm{d}t} \right) \\ &\leq \frac{3pN}{2\,|t|} \left(1 + \frac{2(Rc(U))^2}{3^{3/2}} \right)^2 \sqrt{\rho^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}}. \end{aligned}$$

Because $N \geq H_1(U)$, the same inequality holds for $q = \infty$. Hence, denoting by $C_{15}(U) = \sup_{t \in [U/2, U]} |\phi'(t)|$, using (A.41) and that for $m \in \mathbb{N}_0$, $c \in (0, \infty) \mapsto \rho^{W_{[-1,1]}, c}_m$ is nondecreasing

(using a direct adaptation of Lemma 1 in [20]), we have

$$2\int_{U/2}^{U}\phi(t)^2\left(\frac{\mathrm{d}\sqrt{\rho^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}}}{\mathrm{d}t}\right)^2+\left(\phi'(t)\right)^2\rho^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}+2\left|\phi'(t)\right|\sqrt{\rho^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}}\frac{\mathrm{d}\sqrt{\rho^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}}}{\mathrm{d}t}dt\leq C_{16}(H,U)N^2\rho^{Rc(U)}_{\widetilde{\boldsymbol{N}}(q)},$$

where

$$C_{16}(H,U)=2p\left(1+\frac{2(Rc(U))^2}{3^{3/2}}\right)^2\left(\frac{9p}{2U}\left(1+\frac{2(Rc(U))^2}{3^{3/2}}\right)^2+\frac{3C_{15}(U)\ln(2)}{H}\right)+\frac{2UC_{15}(U)^2}{H^2}.$$

Using Lemma A.5 and (7.114) in [42] we have, for $q=1$, $N\geq H_1(U)$, and $U/2\leq|t|\leq U$,

$$\begin{aligned}&\int_{[-1,1]^p}\left(\frac{\partial\psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}(\boldsymbol{z})}{\partial t}\right)^2d\boldsymbol{z}\\&=(Rx_0)^2\left((p-1)\int_{[-1,1]}\left(\frac{\partial\psi^{c}_{H_1(U)}(z)}{\partial c}\bigg|_{c=Rc(t)}\right)^2dz+\int_{[-1,1]}\left(\frac{\partial\psi^{c}_{N}(z)}{\partial c}\bigg|_{c=Rc(t)}\right)^2dz\right)\\&\leq NC_{17}(U).\end{aligned}$$

where $C_{17}(H,U)=2p\left(4C_1(H)+C_2(H,URx_0)\right)^2C_3(H)^2U^2(Rx_0)^4$. Because $N\geq H_1(U)$, the same inequality holds for $q=\infty$. Thus, denoting by $C_{19}(H,U)=R^p(C_{16}(H,U)+UC_{17}(H,U)/H_1(U))$ and using (A.40), we obtain for $N\geq H_1(U)$

$$I_1\leq C_{19}(H,U)N^2\rho^{Rc(U)}_{\widetilde{\boldsymbol{N}}(q)}. \tag{A.42}$$

By (A.39),

$$\begin{aligned}I_2=&\int_{\mathbb{R}}\int_{[-1,1]^p}R^p\rho^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}\phi(t)^2\left|\psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}(\boldsymbol{z})\right|^2dtd\boldsymbol{z}\\\leq&\int_{\mathbb{R}}R^p\phi(t)^2dt\rho^{Rc(U)}_{\widetilde{\boldsymbol{N}}(q)}\quad\left(\rho^{c}_{\widetilde{\boldsymbol{N}}(q)}\text{ is nonincreasing in }c,\text{ see Lemma 1 in [20]}\right)\\\leq&R^pU\rho^{Rc(U)}_{\widetilde{\boldsymbol{N}}(q)}.\end{aligned}$$

Using the upper bounds on I_1 and I_2 and Lemma A.1, we have, for U such that $4/(eRc(U))\geq1$ and $N\geq H_1(U)$,

$$\chi_2\left(\mathbb{P}_2^{Y,\boldsymbol{Z}},\mathbb{P}_1^{Y,\boldsymbol{Z}}\right)\leq C_{20}(H,U,x_0,R,\tau)\gamma^2N^2\exp\left(-2Nk_q\ln\left(\frac{4N}{eRc(U)}\right)\right), \tag{A.43}$$

where

$$C_{20}(H,U,x_0,R,\tau)=\frac{(2\pi)^2(x_0Rc(U))^pC_{\boldsymbol{X}}e^{3p}\left(C_{19}(H,U)+R^p(U(x_0pR)^2+\tau^2/2)/H_1(U)^2\right)}{9^p\tau(\exp\left(2(p-1)H_1(U)\ln\left(4H_1(U)/(eRc(U))\right)\right)\mathbb{1}\{q=1\}+\mathbb{1}\{q=\infty\})}.$$

As a result, (iii) is satisfied if

$$C_{20}(H,U,x_0,R,\tau)e\gamma^2N^2\exp\left(-2Nk_q\ln\left(\frac{4N}{eRc(U)}\right)\right)n\leq\xi^2. \tag{A.44}$$

Step 4. We tune the parameters as follows. We take $U=4/(Rx_0e)$, $\tau\geq 1$ such that

$$\tau\geq\left(\frac{2}{\pi l^2}\left(\frac{\Gamma(2s+1)}{2^{2s}}+1\right)\right)^{1/(2s+1)}\bigvee\left(\frac{2C_{12}(Rx_0,\sigma,p)}{\pi l^2k_q^{2\sigma}}\right)\bigvee\left(\frac{3Rx_0e^{\sigma+p/2-1/4}}{8}\right),$$

$N=\lceil\overline{N}\rceil$, where $\overline{N}=3(\ln(n)/\ln_2(n))/(4k_q)$, and $\gamma=C_\gamma/\overline{N}^\sigma$, where $C_\gamma=M_1\wedge\sqrt{M_2}\wedge\sqrt{M_3}$, where

$$M_1=\frac{1}{\tau(1+(1/\tau)^2)2^{p/2}U}\left(\left(\left(\frac{Ux_0}{2\pi}\right)^{-p/2}\frac{Q_1(1+p/2)}{\Xi_q(U)R^{p/2}}\right)\bigwedge\left(\frac{2Q_2}{C_8(Rx_0,p,U)}\right)\right),$$

$$M_2=\frac{\pi l^2(p+1)U}{4}\left(\frac{URx_0}{2\pi}\right)^{-p}\left(\frac{1}{(1\vee U^{2s})}\bigwedge\frac{1}{\mathbb{1}\{q=\infty\}+p^{2\sigma}\mathbb{1}\{q=1\}}\right),$$

$$M_3=\frac{\xi^2}{C_{20}(H,U,x_0,R,\tau)e},$$

where, using that $\sigma>2+k_q/2$, Q_1 and Q_2 are such that $\overline{N}^\sigma/(N+1/2)^{k_q/2}\leq Q_1$ and $\overline{N}^\sigma/N^{2+k_q/2}\leq Q_2$. Hence, (A.32)-(A.33) and (A.35)-(A.36) are satisfied. Moreover, this yields that, for $\overline{N}\geq H_1(c(U))$ (this is satisfied for n large enough),

$$\begin{aligned}n\gamma^2N^2\exp\left(-2k_qN\ln\left(N\right)\right)&\leq\frac{2C_\gamma^2n\overline{N}^2}{\overline{N}^{2\sigma}}\exp\left(-2k_q\overline{N}\ln(\overline{N})\right)\\&\leq\frac{2C_\gamma^2}{\sqrt{n}}\exp\left(\frac{3}{2}\frac{\ln(n)\ln(4k_q\ln_2(n)/3)}{\ln_2(n)}-2(\sigma-1)\ln\left(\frac{3\ln(n)}{4k_q\ln_2(n)}\right)\right),\end{aligned}$$

which yields

$$\lim_{n\to\infty}n\gamma^2\exp\left(-2k_qN\ln\left(N\right)\right)N^2=0.$$

Thus, we have the required condition (A.44) for n large enough.

For this choice of γ, we have $h_n^2=\overline{N}^{-2\sigma}C_\gamma^2I_\phi(U)/(8\pi)\left((Rx_0/(2\pi))^p\right)$. Using Lemma A.11, there exists $\nu>0$ independent of n such that, for n large enough,

$$\inf_{\widehat{f}_n}\sup_{f_{\alpha,\boldsymbol{\beta}}\in H_{w,W}^{q,\phi,\omega}(l)\cap\mathcal{D}}\mathbb{E}\left[\int_{L^2(\mathbb{R}^{p+1})}\left(\widehat{f}_n-f_{\alpha,\boldsymbol{\beta}}\right)^2(a,\boldsymbol{b})\,dad\boldsymbol{b}\right]\geq\nu h_n^2.$$

□

Proof of Theorem 2 (2). We underline the differences with the proof of Theorem 2 (1). Equip $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ with $\langle \boldsymbol{g}, \boldsymbol{h} \rangle^2_{L^2(\mathbb{R})\times L^2(\mathbb{R})} = \langle \boldsymbol{g}_1, \boldsymbol{h}_1 \rangle^2_{L^2(\mathbb{R})} + \langle \boldsymbol{g}_2, \boldsymbol{h}_2 \rangle^2_{L^2(\mathbb{R})}$. It is a separable Hilbert space. Denote by $\mathbb{P}^{\boldsymbol{m}}_{j,n}$ the law of $\left(\mathfrak{Re}\left(Z^j_{\boldsymbol{m}}(t)\right), \mathfrak{Im}\left(Z^j_{\boldsymbol{m}}(t)\right)\right)_{t\in\mathbb{R}}$ in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ and by $\mathbb{P}_{j,n}$ the law on the space $\ell^2\left(L^2(\mathbb{R}) \times L^2(\mathbb{R})\right)$ of square summable sequences with values in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ of $\left(Z^j_{\boldsymbol{m}}(t)\right)_{\boldsymbol{m}\in\mathbb{N}_0^p,\ t\in\mathbb{R}} = \left(\mathfrak{Re}\left(Z^j_{\boldsymbol{m}}(t)\right), \mathfrak{Im}\left(Z^j_{\boldsymbol{m}}(t)\right)\right)_{\boldsymbol{m}\in\mathbb{N}_0^p,\ t\in\mathbb{R}}$ defined using $f_{j,n}$, hence $\left(b^j_{\boldsymbol{m}}(t)\right)_{\boldsymbol{m}\in\mathbb{N}_0^p,\ t\in\mathbb{R}}$, for $j = 1, 2$.

Take $f_{1,n} = 0$ and $f_{2,n}$ like (A.29) replacing $\widetilde{\boldsymbol{N}}(1)$ by $\widetilde{\boldsymbol{N}}(1) = (N, \underline{\boldsymbol{0}}^\top)^\top \in \mathbb{N}_0^p$, where N is odd and N and γ are chosen as a function of n. Using (A.37), this yields, for all $\boldsymbol{m} \in \mathbb{N}_0^p$, $b^2_{\boldsymbol{m}}(t) = \gamma \mathbb{1}\{\boldsymbol{m} = \widetilde{\boldsymbol{N}}(q)\}(-1)^{\left|\widetilde{\boldsymbol{N}}(q)\right|_1} \left(Rc(|t|)/2\pi\right)^{p/2} \phi(t)$.

By independence, we have, for $j = 1, 2$, $\mathbb{P}_{j,n} = \bigotimes_{\boldsymbol{m}\in\mathbb{N}_0^p} \mathbb{P}^{\boldsymbol{m}}_{j,n}$, hence

$$\frac{\mathrm{d}\mathbb{P}_{2,n}}{\mathrm{d}\mathbb{P}_{1,n}} = \frac{\mathrm{d}\mathbb{P}^{\widetilde{\boldsymbol{N}}(q)}_{2,n}}{\mathrm{d}\mathbb{P}^{\widetilde{\boldsymbol{N}}(q)}_{1,n}}.$$

Because $\mathbb{P}_{2,n}$ is absolutely continuous with respect to $\mathbb{P}_{1,n}$, we have

$$\begin{aligned} K(\mathbb{P}_{2,n}, \mathbb{P}_{1,n}) &= \int_{\ell^2(L^2(\mathbb{R})\times L^2(\mathbb{R}))} \ln\left(\frac{\mathrm{d}\mathbb{P}_{2,n}}{\mathrm{d}\mathbb{P}_{1,n}}(y)\right) d\mathbb{P}_{2,n}(y) \\ &= \int_{L^2(\mathbb{R})\times L^2(\mathbb{R})} \ln\left(\frac{\mathrm{d}\mathbb{P}^{\widetilde{\boldsymbol{N}}(q)}_{2,n}}{\mathrm{d}\mathbb{P}^{\widetilde{\boldsymbol{N}}(q)}_{1,n}}(y)\right) d\mathbb{P}^{\widetilde{\boldsymbol{N}}(q)}_{2,n}(y). \end{aligned}$$

Step 1. We derive conditions so that $f_{1,n}$ and $f_{2,n}$ satisfy (i) in Lemma A.11. Unlike in the proof of Theorem 2, we do not have to ensure that $f_{1,n}$ and $f_{2,n}$ are densities but simply that $f_{1,n}$ and $f_{2,n}$ belong to $\mathcal{H}^{q,\phi,\omega}_{w,W}(l)$. Using (A.34), we have that $\int_{\mathbb{R}} \int_{\mathbb{R}^p} f_{2,n}(a, \boldsymbol{b})^2 \left(1 \vee |a|\right) W^{\otimes p}_{\cosh(\cdot/R)}(\boldsymbol{b}) da d\boldsymbol{b} < \infty$. Like in (A.35)-(A.36), we obtain the following conditions

$$2\gamma^2 \left(\frac{Rx_0}{2\pi}\right)^p \frac{\left(1 \vee U^{2s}\right) U^{p+1}}{p+1} \leq \pi l^2, \tag{A.45}$$

$$\frac{2U\left(\mathbb{1}\{q = \infty\} + p^{2\sigma}\mathbb{1}\{q = 1\}\right)\gamma^2 N^{2\sigma}}{p+1} \left(\frac{URx_0}{2\pi}\right)^p \leq \pi l^2. \tag{A.46}$$

Step 2. This is the same as for Theorem 2.

Step 3. Let $\xi < \sqrt{2}$. We give conditions which imply that $f_{1,n}$ and $f_{2,n}$ satisfy (iii).

We start by proving that, for $y \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ and $\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}$ *a.s.*,

$$\frac{\mathrm{d}\mathbb{P}_{2,n}^{\widetilde{\boldsymbol{N}}(q)}}{\mathrm{d}\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}}(y)$$
$$= \exp\left(\left\langle y, \begin{pmatrix} \int_0^\cdot \mathfrak{Re}\left(\sigma_{\widetilde{\boldsymbol{N}}(q)}^{W,c(s)} b_{\widetilde{\boldsymbol{N}}(q)}^2(s)\right) ds \\ \int_0^\cdot \mathfrak{Im}\left(\sigma_{\widetilde{\boldsymbol{N}}(q)}^{W,c(s)} b_{\widetilde{\boldsymbol{N}}(q)}^2(s)\right) ds \end{pmatrix} \right\rangle_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}} - \frac{1}{2} \left| \begin{pmatrix} \int_0^\cdot \mathfrak{Re}\left(\sigma_{\widetilde{\boldsymbol{N}}(q)}^{W,c(s)} b_{\widetilde{\boldsymbol{N}}(q)}^2(s)\right) ds \\ \int_0^\cdot \mathfrak{Im}\left(\sigma_{\widetilde{\boldsymbol{N}}(q)}^{W,c(s)} b_{\widetilde{\boldsymbol{N}}(q)}^2(s)\right) ds \end{pmatrix} \right|^2_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}} \right),$$

where $\langle \cdot, \star \rangle_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}}$ is the scalar product on

$$H_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}} = \left\{ \frac{\sigma}{\sqrt{n}} \begin{pmatrix} \int_0^\cdot \boldsymbol{h}_1(s) ds \\ \int_0^\cdot \boldsymbol{h}_2(s) ds \end{pmatrix} : \quad \boldsymbol{h} \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \right\}$$

defined, for $\boldsymbol{f_j} = \left(\int_0^\cdot \boldsymbol{h_j}_1(s) ds, \int_0^\cdot \boldsymbol{h_j}_2(s) ds\right)^\top$, $\boldsymbol{h_j} \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$, $j \in \{1, 2\}$ (the functions $\boldsymbol{h_j}$ are unique *a.e.* because they are the derivatives of $\boldsymbol{f_j}$ in the sense of distributions), as

$$\langle \boldsymbol{f_1}, \boldsymbol{f_2} \rangle^2_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}} = \frac{n}{\sigma^2} \left(\langle \boldsymbol{h_1}_1, \boldsymbol{h_2}_1 \rangle^2_{L^2(\mathbb{R})} + \langle \boldsymbol{h_1}_2, \boldsymbol{h_2}_2 \rangle^2_{L^2(\mathbb{R})} \right)$$

and using (2.12) page 41 in [16] when one function belongs to $H_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}}$ and for $\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}$ *a.e.* other function in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. Indeed, the reproducing kernel Hilbert space $H_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}}$ of $\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}$ on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ is the image of $\mathcal{Q}^{1/2}$ with the scalar product of the image structure and where $\mathcal{Q}$ is its covariance operator. Using Corollary B.3 in [16] and that $\mathcal{Q} = \mathcal{L}\mathcal{L}^*$, where

$$\mathcal{L} : \begin{array}{ccc} L^2(\mathbb{R}) \times L^2(\mathbb{R}) & \to & L^2(\mathbb{R}) \times L^2(\mathbb{R}) \\ \boldsymbol{h} & \mapsto & \frac{\sigma}{\sqrt{n}} \begin{pmatrix} \int_0^\cdot \boldsymbol{h}_1(s) ds \\ \int_0^\cdot \boldsymbol{h}_2(s) ds \end{pmatrix} \end{array},$$

$H_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}}$ is also the image of $\mathcal{L}$ with the norm of the image structure (*i.e.*, $\|\boldsymbol{f}\|^2_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}} = \|\boldsymbol{h}_1\|_2^2 + \|\boldsymbol{h}_2\|_2^2$, where $\boldsymbol{f} = \mathcal{L}[\boldsymbol{h}]$). Finally, we use the Cameron-Martin formula (Proposition 2.26 in [16]).

This yields

$$K(\mathbb{P}_{2,n}, \mathbb{P}_{1,n}) = \mathbb{E}\left[\left\langle Z^2_{\widetilde{\boldsymbol{N}}(q)}, \begin{pmatrix} \int_0^\cdot \mathfrak{Re}\left(\sigma_{\widetilde{\boldsymbol{N}}(q)}^{W,c(s)} b_{\widetilde{\boldsymbol{N}}(q)}^2(s)\right) ds \\ \int_0^\cdot \mathfrak{Im}\left(\sigma_{\widetilde{\boldsymbol{N}}(q)}^{W,c(s)} b_{\widetilde{\boldsymbol{N}}(q)}^2(s)\right) ds \end{pmatrix} \right\rangle_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}} \right] - \frac{n}{2\sigma^2} \int_{\mathbb{R}} \left| \sigma_{\widetilde{\boldsymbol{N}}(q)}^{W,c(s)} b_{\widetilde{\boldsymbol{N}}(q)}^2(s) \right|^2 ds.$$

Using that

$$\left\langle Z^2_{\widetilde{\boldsymbol{N}}(q)}, \begin{pmatrix} \int_0^\cdot \mathfrak{Re}\left(\sigma^{W,c(s)}_{\widetilde{\boldsymbol{N}}(q)} b^2_{\widetilde{\boldsymbol{N}}(q)}(s)\right) ds \\ \int_0^\cdot \mathfrak{Im}\left(\sigma^{W,c(s)}_{\widetilde{\boldsymbol{N}}(q)} b^2_{\widetilde{\boldsymbol{N}}(q)}(s)\right) ds \end{pmatrix} \right\rangle_{\mathbb{P}^{\widetilde{\boldsymbol{N}}(q)}_{1,n}}$$

$$= \left| \begin{pmatrix} \int_0^\cdot \mathfrak{Re}\left(\sigma^{W,c(s)}_{\widetilde{\boldsymbol{N}}(q)} b^2_{\widetilde{\boldsymbol{N}}(q)}(s)\right) ds \\ \int_0^\cdot \mathfrak{Im}\left(\sigma^{W,c(s)}_{\widetilde{\boldsymbol{N}}(q)} b^2_{\widetilde{\boldsymbol{N}}(q)}(s)\right) ds \end{pmatrix} \right|^2_{\mathbb{P}^{\widetilde{\boldsymbol{N}}(q)}_{1,n}} + \frac{\sigma}{\sqrt{n}} \left\langle \begin{pmatrix} B^{\mathfrak{R}}_{\widetilde{\boldsymbol{N}}(q)} \\ B^{\mathfrak{I}}_{\widetilde{\boldsymbol{N}}(q)} \end{pmatrix}, \begin{pmatrix} \int_0^\cdot \mathfrak{Re}\left(\sigma^{W,c(s)}_{\widetilde{\boldsymbol{N}}(q)} b^2_{\widetilde{\boldsymbol{N}}(q)}(s)\right) ds \\ \int_0^\cdot \mathfrak{Im}\left(\sigma^{W,c(s)}_{\widetilde{\boldsymbol{N}}(q)} b^2_{\widetilde{\boldsymbol{N}}(q)}(s)\right) ds \end{pmatrix} \right\rangle_{\mathbb{P}^{\widetilde{\boldsymbol{N}}(q)}_{1,n}}$$

and that the second term in the right-hand side is a limit in quadratic mean of mean zero Gaussian random variables, hence has mean zero (see the arguments page 41 in [16]), we have

$$K(\mathbb{P}_{2,n}, \mathbb{P}_{1,n}) = \frac{n}{2\sigma^2} \int_{\mathbb{R}} \left| \sigma^{W,c(t)}_{\widetilde{\boldsymbol{N}}(q)} b^2_{\widetilde{\boldsymbol{N}}(q)}(t) \right|^2 dt. \tag{A.47}$$

Using Proposition A.1 (ii) we obtain

$$\begin{aligned} K(\mathbb{P}_{2,n}, \mathbb{P}_{1,n}) =& \frac{\gamma^2 n R^p}{2\sigma^2} \int_{\mathbb{R}} \left(\sigma^{W_{\cosh(\cdot)}, Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}\right)^2 \left(\frac{Rc(|t|)}{2\pi}\right)^p \phi(t)^2 dt \\ =& \frac{\gamma^2 n R^p}{2\sigma^2} \int_{\mathbb{R}} \rho^{W_{\cosh(\cdot)}, Rc(t)}_{\widetilde{\boldsymbol{N}}(q)} \phi(t)^2 dt. \end{aligned}$$

Using Theorem 3 in [20], we have that, for all $U/2 \leq |t| \leq U$ and $U = 2/(Rx_0 e)$,

$$\begin{aligned} \rho^{W_{\cosh(\cdot)}, Rc(t)}_{\widetilde{\boldsymbol{N}}(q)} &\leq \left(\frac{Rx_0 Ue}{\pi(1-(Rx_0U/2)^2)}\right)^p \exp\left(-2k_q N \log\left(\frac{2}{Rx_0U}\right)\right) \\ &\leq \left(\frac{2}{\pi(1-e^{-2})}\right)^p \exp\left(-2k_q N\right). \end{aligned}$$

Hence, we have

$$K(\mathbb{P}_{2,n}, \mathbb{P}_{1,n}) \leq \left(\frac{2R}{\pi(1-e^{-2})}\right)^p \frac{U\gamma^2 n}{2\sigma^2} \exp\left(-2k_q N\right).$$

As a result, (iii) is satisfied if

$$\left(\frac{2R}{\pi(1-e^{-2})}\right)^p \frac{U\gamma^2 n}{2\sigma^2} \exp\left(-2k_q N\right) \leq \xi^2. \tag{A.48}$$

Step 4. We tune the parameters as follows. We take $U = 1/(Rx_0 e)$, $N = \lceil \overline{N} \rceil$, where $\overline{N} = \ln(n/\ln(n))/2k_q$, $\gamma = C_{\gamma,1}/\overline{N}^\sigma$ and

$$\begin{aligned} & C^2_{\gamma,1} \\ &= \left(\left(\frac{p+1}{2U(URx_0)^p}\right)\left[\left(\frac{\pi\,(2\pi)^p\, l^2}{1 \vee U^{2s}}\right) \bigwedge \left(\frac{\pi\,(2\pi)^p\, l^2}{\mathbb{1}\{q=\infty\} + p^{2\sigma}\,\mathbb{1}\{q=1\}}\right)\right]\right) \bigwedge \left(\frac{2(\sigma\xi)^2}{U}\left(\frac{\pi(1-e^{-2})}{R^2 x_0 Ue}\right)^p\right) \end{aligned}$$

which guarantee that (A.45)-(A.46) are satisfied and $f_{1,n}$ and $f_{2,n}$ belong to $\mathcal{H}_W^{\omega,\phi}(l)$. Moreover, (A.51) is also satisfied because

$$\begin{aligned}\gamma^2 n \exp\left(-2k_q N\right) &\leq C_{\gamma,1}^2 n \exp(-\ln(n) + \ln_2(n) - 2\sigma \ln_2(n) + 2\sigma \ln_3(n)) \\ &\leq C_{\gamma,1}^2 \quad (\text{using } \sigma > 1/2).\end{aligned}$$

For this choice of γ, we have $h_n^2 = C_{\gamma,1}^2 \ln(n/\ln(n))^{2\sigma} I_\phi(U)/(8\pi) \left(Rx_0/(2\pi)\right)^p$, hence the result. □

Proof of Theorem 4 (1). We underline the differences with the proof of Theorem 2 (1).

Step 1. We derive conditions so that $f_{1,n}$ and $f_{2,n}$ satisfy (i) in Lemma A.11. Unlike in the proof of Theorem 2, we do not have to ensure that $f_{1,n}$ and $f_{2,n}$ are densities but simply that $f_{1,n}$ and $f_{2,n}$ belong to $\mathcal{H}_{w,W}^{q,\phi,\omega}(l)$. Using (A.34), we have that $\int_{\mathbb{R}} \int_{\mathbb{R}^p} f_{2,n}(a,\boldsymbol{b})^2 \left(1 \vee |a|\right) W_{[-R,R]}^{\otimes p}(\boldsymbol{b}) da d\boldsymbol{b} < \infty$. Like for (A.35)-(A.36), we obtain the following conditions

$$2\gamma^2 \left(\frac{Rx_0}{2\pi}\right)^p \frac{\left(1 \vee U^{2s}\right) U^{p+1}}{p+1} \leq \pi l^2, \tag{A.49}$$

$$\frac{2U\gamma^2 e^{2\kappa N \ln(N)}}{p+1} \left(\frac{URx_0}{2\pi}\right)^p \leq \pi l^2. \tag{A.50}$$

Step 2. This is the same as for Theorem 2.

Step 3. Let $\xi < \sqrt{2}$. We give conditions which imply that $f_{1,n}$ and $f_{2,n}$ satisfy (iii). Using (A.47) then Proposition A.1 (ii) we have

$$K(\mathbb{P}_{2,n}, \mathbb{P}_{1,n}) = \frac{\gamma^2 n R^p}{2\sigma^2} \int_{\mathbb{R}} \left(\sigma_{\widetilde{\boldsymbol{N}}(q)}^{W_{[-1,1]}, Rc(t)}\right)^2 \left(\frac{Rc(|t|)}{2\pi}\right)^p \phi(t)^2 dt.$$

Using Lemma A.1, we have, for all $U/2 \leq |t| \leq U$ such that $4/(eRc(U)) \geq 1$,

$$\left(\sigma_{\widetilde{\boldsymbol{N}}(q)}^{W_{[-1,1]}, Rc(t)}\right)^2 \leq \left(\frac{2\pi e^3}{9}\right)^p \exp\left(-2k_q N \ln\left(\frac{4N}{eRc(U)}\right)\right),$$

hence

$$K(\mathbb{P}_{2,n}, \mathbb{P}_{1,n}) \leq \left(\frac{R^2 x_0 e^3}{9}\right)^p \frac{U^{p+1}}{(p+1)\sigma^2} \gamma^2 n \exp\left(-2k_q N \ln\left(N\right)\right).$$

As a result, (iii) is satisfied if

$$\left(\frac{R^2 x_0 e^3}{9}\right)^p \frac{U^{p+1}}{(p+1)\sigma^2} \gamma^2 n \exp\left(-2k_q N \ln\left(N\right)\right) \leq \xi^2. \tag{A.51}$$

Step 4. We tune the parameters as follows. We take $U = 4/(Rx_0e)$, $N = \lceil \overline{N} \rceil$, where $\overline{N}\ln(\overline{N}) = \ln(n)/(2(\kappa + k_q))$, $\gamma = C_{\gamma,2}\exp(-\kappa \overline{N}\ln(\overline{N}))$ and

$$C_{\gamma,2}^2 = \left(\frac{p+1}{U(URx_0)^p}\right)\left[\left(\frac{\pi\,(2\pi)^p\,l^2}{2\,(1\vee U^{2s})}\right)\bigwedge\left(\frac{\pi\,(2\pi)^p\,l^2}{2}\right)\bigwedge\left((\sigma\xi)^2\left(\frac{9}{Re^3}\right)^p\right)\right]$$

which guarantee that (A.49)-(A.50) are satisfied and $f_{1,n}$ and $f_{2,n}$ belong to $\mathcal{H}_W^{\omega,\phi}(l)$. Moreover, (A.51) is also satisfied because

$$\begin{aligned}\gamma^2 n\exp\left(-2k_qN\ln\left(N\right)\right) &\leq C_{\gamma,2}^2 n\exp(-2(\kappa+k_q)\overline{N}\ln(\overline{N}))\\ &\leq C_{\gamma,2}^2.\end{aligned}$$

For this choice of γ, we have $h_n^2 = C_{\gamma,2}^2 n^{-\kappa/(\kappa+k_q)} I_\phi(U)/(8\pi)\,(Rx_0/(2\pi))^p$, hence the result. □

Proof of Theorem 4 (2). We underline the differences with the proof of Theorem 2 (1).

Step 1. We derive conditions so that $f_{1,n}$ and $f_{2,n}$ satisfy (i) in Lemma A.11. Unlike in the proof of Theorem 2, we do not have to ensure that $f_{1,n}$ and $f_{2,n}$ are densities but simply that $f_{1,n}$ and $f_{2,n}$ belong to $\mathcal{H}_{w,W}^{q,\phi,\omega}(l)$. Using (A.34), we have that $\int_{\mathbb{R}}\int_{\mathbb{R}^p} f_{2,n}(a,\boldsymbol{b})^2\,(1\vee|a|)\,W_{\cosh(\cdot/R)}^{\otimes p}(\boldsymbol{b})dad\boldsymbol{b} < \infty$. Like for (A.35)-(A.36), we obtain the following conditions

$$2\gamma^2\left(\frac{Rx_0}{2\pi}\right)^p\frac{\left(1\vee U^{2s}\right)U^{p+1}}{p+1} \leq \pi l^2, \tag{A.52}$$

$$\frac{2U\gamma^2 e^{2\kappa N}}{p+1}\left(\frac{URx_0}{2\pi}\right)^p \leq \pi l^2. \tag{A.53}$$

Step 2. This is the same as for Theorem 2.

Step 3. Let $\xi < \sqrt{2}$. We give conditions which imply that $f_{1,n}$ and $f_{2,n}$ satisfy (iii).
This is the same reasoning as for Theorem 2 (2), hence (iii) is satisfied if (A.48) holds.

Step 4. We tune the parameters as follows. We take $U = 2/(Rx_0e)$, $N = \lceil \overline{N} \rceil$, where $\overline{N}\ln(\overline{N}) = \ln(n)/(2(\kappa + k_q))$, $\gamma = C_{\gamma,3}\exp(-\kappa\overline{N})$ and

$$C_{\gamma,3}^2 = \left(\left(\frac{p+1}{2U(URx_0)^p}\right)\left[\left(\frac{\pi^{p+1}l^2}{2\,(1\vee U^{2s})}\right)\bigwedge\left(\pi\,(2\pi)^p\,l^2\right)\right]\right)\bigwedge\left(\frac{2(\sigma\xi)^2}{U}\left(\frac{\pi(1-e^{-2})}{R^2x_0Ue}\right)^p\right)$$

which guarantee that (A.52)-(A.53) are satisfied and $f_{1,n}$ and $f_{2,n}$ belong to $\mathcal{H}_W^{\omega,\phi}(l)$. Moreover, (A.48) is also satisfied because

$$\begin{aligned}\gamma^2 n\exp\left(-2k_qN\right) &\leq C_{\gamma,3}^2 n\exp(-2(\kappa+k_q)\overline{N})\\ &\leq C_{\gamma,3}^2.\end{aligned}$$

For this choice of γ, we have $h_n^2 = C_{\gamma,3}^2 n^{-\kappa/(\kappa+k_q)} I_\phi(U)/(8\pi)\,(Rx_0/(2\pi))^p$, hence the result. □

A.3. **Upper bounds.** Throughout this section we use the notation

$$Z_{n_0} = \sup_{f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}} \left\| 1 - \frac{f_{\boldsymbol{X}|\mathcal{X}}}{\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}}} \right\|^2_{L^\infty(\mathcal{X})}.$$

Lemma A.12. For $\boldsymbol{m} \in \mathbb{N}_0^p$, $\widetilde{c}_{\boldsymbol{m}}(t) = n^{-1} \sum_{j=1}^n e^{itY_j} \overline{g_{\boldsymbol{m}}^{W,tx_0}}\,(\boldsymbol{X}_j/x_0)\,\mathbb{1}\{\boldsymbol{X}_j \in \mathcal{X}\}/(x_0^p f_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{X}_j))$ satisfies

$$\mathbb{E}\left[\widetilde{c}_{\boldsymbol{m}}(t)\right] = c_{\boldsymbol{m}}(t) \quad \text{and} \quad \mathbb{E}\left[\left|\widetilde{c}_{\boldsymbol{m}}(t) - c_{\boldsymbol{m}}(t)\right|^2\right] \le \frac{c_{\boldsymbol{X}}}{n x_0^p}. \tag{A.54}$$

Proof of Lemma A.12. The first assertion comes from

$$\begin{aligned}
\mathbb{E}\left[\widetilde{c}_{\boldsymbol{m}}(t)\right] &= \frac{1}{x_0^p}\mathbb{E}\left[\frac{e^{itY}}{f_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{X})}\overline{g_{\boldsymbol{m}}^{W,tx_0}}\left(\frac{\boldsymbol{X}}{x_0}\right)\middle|\, \boldsymbol{X} \in \mathcal{X}\right] \quad \text{(the observations are i.i.d.)} \\
&= \frac{1}{x_0^p}\int_{\mathcal{X}} \mathbb{E}\left[e^{it\alpha + it\boldsymbol{\beta}^\top \boldsymbol{x}}\right]\overline{g_{\boldsymbol{m}}^{W,tx_0}}\left(\frac{\boldsymbol{x}}{x_0}\right) d\boldsymbol{x} \quad \text{(by the law of iterated expectations)} \\
&= \int_{[-1,1]^p} \mathcal{F}\left[f_{Y|\mathcal{X},\boldsymbol{X}=x_0\boldsymbol{u}}\right](t)\overline{g_{\boldsymbol{m}}^{W,tx_0}}\,(\boldsymbol{u})\, d\boldsymbol{u}.
\end{aligned}$$

Similarly, the second assertion follows from

$$\begin{aligned}
\mathbb{E}\left[\left|\widetilde{c}_{\boldsymbol{m}}(t) - c_{\boldsymbol{m}}(t)\right|^2\right] &\le \frac{1}{n x_0^{2p}}\mathbb{E}\left[\left|\frac{e^{itY}}{f_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{X})}\overline{g_{\boldsymbol{m}}^{W,tx_0}}\left(\frac{\boldsymbol{X}}{x_0}\right)\right|^2\middle|\, \boldsymbol{X} \in \mathcal{X}\right] \\
&\le \frac{1}{n x_0^{2p}}\int_{\mathcal{X}} \frac{1}{f_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{x})}\left|\overline{g_{\boldsymbol{m}}^{W,tx_0}}\left(\frac{\boldsymbol{x}}{x_0}\right)\right|^2 d\boldsymbol{x} \\
&\le \frac{c_{\boldsymbol{X}}}{n x_0^p}\int_{[-1,1]^p}\left|\overline{g_{\boldsymbol{m}}^{W,tx_0}}\,(\boldsymbol{u})\right|^2 d\boldsymbol{u}.
\end{aligned}$$

□

Lemma A.13. For $q \in \{1, \infty\}$, we have, for $c \neq 0$ and $N \in \mathbb{N}_0$

$$\sum_{\substack{|\boldsymbol{m}|_q \le N \\ \boldsymbol{m} \in \mathbb{N}_0^p}} \frac{1}{\rho_{\boldsymbol{m}}^{W,c}} \le \nu_q(W, N, c), \tag{A.55}$$

where

(1) $\nu_q(W, N, c)$ is defined in (N.1) in Section 4.5 when $W(\cdot) = \cosh(\cdot/R)$ for $R > 0$,
(2) $\nu_q(W, N, c)$ is defined in (N.2) in Section 4.5 when $W = W_{[-R,R]}$.

Proof of Lemma A.13. Let us start by proving (A.55) in case (1).

Case $|c| > \pi/4$. Let $m \in \mathbb{N}_0$ and $c_m = \pi(m+1)/2$. Using Equation (8) in [20] (note the difference between normalisations for $\mathcal{Q}_c$, with a factor $1/(2\pi)$), we have, for all $m \in \mathbb{N}_0$,

$$\rho_m^{\cosh(\cdot),c} \geq \frac{1}{2}\exp\left(-\frac{\pi(m+1)}{2\,|c|}\right). \tag{A.56}$$

Consider the case $q = 1$. Using Theorem 5.2 in [6] for the second display, $(k+p-1)!/k! \leq (k+p-1)^{p-1}$ for the third display, and that

$$\begin{aligned}
\sum_{k\leq N}\left(\exp\left(-\frac{\pi}{2\,|c|}\right)\right)^k &= \frac{1-\exp\left(-\pi(N+1)/(2\,|c|)\right)}{1-\exp\left(-\pi/(2\,|c|)\right)} \\
&\leq 2\mathbb{1}\left\{|c| \leq \frac{\pi}{2\ln(2)}\right\} + \frac{\exp\left(\pi/(4\,|c|)\right)}{2\sinh\left(\pi/(4c)\right)}\mathbb{1}\left\{|c| \geq \frac{\pi}{2\ln(2)}\right\} \\
&\leq 2\mathbb{1}\left\{|c| \leq \frac{\pi}{2\ln(2)}\right\} + \frac{2\sqrt{2}\,|c|}{\pi}\mathbb{1}\left\{|c| \geq \frac{\pi}{2\ln(2)}\right\} \text{ (using } \sinh(|x|) \geq |x|) \qquad \text{(A.57)}
\end{aligned}$$

for the fifth display, we obtain,

$$\begin{aligned}
\sum_{|\boldsymbol{m}|_1\leq N}\frac{1}{\rho_{\boldsymbol{m}}^{\cosh(\cdot),c}} &\leq 2^p\sum_{k\leq N}\sum_{|\boldsymbol{m}|_1=k}\exp\left(\frac{\pi(|\boldsymbol{m}|_1+p)}{2\,|c|}\right) \\
&\leq 2^p\sum_{k\leq N}\binom{k+p-1}{p-1}\exp\left(\frac{\pi(k+p)}{2\,|c|}\right) \\
&\leq \frac{2^p}{(p-1)!}\sum_{k\leq N}(k+p-1)^{p-1}\exp\left(\frac{\pi(k+p)}{2\,|c|}\right) \\
&\leq \frac{2^p(N+p-1)^{p-1}}{(p-1)!}\exp\left(\frac{\pi(N+p)}{2\,|c|}\right)\sum_{k\leq N}\left(\exp\left(-\frac{\pi}{2\,|c|}\right)\right)^k \\
&\leq \frac{2^{p+1}(N+p-1)^{p-1}}{(p-1)!}\left(1\bigvee\frac{\sqrt{2}\,|c|}{\pi}\right)\exp\left(\frac{\pi(N+p)}{2\,|c|}\right), \qquad \text{(A.58)}
\end{aligned}$$

which yields the result using $\rho_m^{\cosh(\cdot),Rc} = \rho_m^{\cosh(\cdot/R),c}$, for all $m \in \mathbb{N}_0$.

Consider now the case $q = \infty$. Using (A.58) with $p = 1$, we have

$$\sum_{m=0}^{N}\frac{1}{\rho_m^{\cosh(\cdot),c}} \leq 4\left(1\bigvee\frac{\sqrt{2}\,|c|}{\pi}\right)\exp\left(\frac{\pi(N+1)}{2\,|c|}\right) \text{ (using } \sinh(x) \geq x \text{ for } x \geq 0). \tag{A.59}$$

Then, we have

$$\sum_{\substack{|\boldsymbol{m}|_\infty \le N \\ \boldsymbol{m}\in\mathbb{N}_0^p}} \frac{1}{\rho_{\boldsymbol{m}}^{W,c}} \le \prod_{j=1}^{p} \left(\sum_{\boldsymbol{m}_j=0}^{N} \frac{1}{\rho_{\boldsymbol{m}_j}^{W,c}} \right) \tag{A.60}$$

which, using (A.59), yields the result.

Case $|c| \le \pi/4$. Consider now the case $q = 1$. Then, using Theorem 1 in [20] we have, for all $m \in \mathbb{N}_0$,

$$\rho_m^{W,c} \ge \frac{4}{\pi^2} \exp\left(-2\left(1 + \ln\left(\frac{e}{|c|}\right)\right) m \right). \tag{A.61}$$

Hence, we obtain

(A.62)

$$\begin{aligned}
&\sum_{|\boldsymbol{m}|_1 \le N} \frac{1}{\rho_{\boldsymbol{m}}^{W,c}} \\
&\le \left(\frac{\pi^2}{4}\right)^p \sum_{k\le N} \sum_{|\boldsymbol{m}|_1 = k} \exp\left(2\left(1 + \ln\left(\frac{e}{|c|}\right)\right)|\boldsymbol{m}|_1\right) \\
&\le \left(\frac{\pi^2}{4}\right)^p \sum_{k\le N} \binom{k+p-1}{p-1} \exp\left(2\left(1 + \ln\left(\frac{e}{|c|}\right)\right)k\right) \\
&\le \frac{1}{(p-1)!}\left(\frac{\pi^2}{4}\right)^p \sum_{k\le N} (k+p-1)^{p-1} \exp\left(2\left(1 + \ln\left(\frac{e}{|c|}\right)\right)k\right) \\
&\le \frac{(N+p-1)^{p-1}}{(p-1)!}\left(\frac{\pi^2}{4}\right)^p \exp\left(2\left(1 + \ln\left(\frac{e}{|c|}\right)\right)N\right) \sum_{k\le N} \left(2\exp\left(-1 - \ln\left(\frac{e}{|c|}\right)\right)\right)^k \\
&\le \frac{(N+p-1)^{p-1}}{(p-1)!}\left(\frac{\pi^2}{4}\right)^p \exp\left(2\left(1 + \ln\left(\frac{e}{|c|}\right)\right)N\right),
\end{aligned}$$

yields the result. Consider now the case $q = \infty$. Using (A.58) with $p = 1$, we have

$$\sum_{m=0}^{N} \frac{1}{\rho_m^{\cosh(\cdot),c}} \le \frac{\pi^2}{4} \exp\left(2\left(1 + \ln\left(\frac{e}{|c|}\right)\right)N\right) \quad (\text{using } \sinh(x) \ge x \text{ for } x \ge 0). \tag{A.63}$$

Using again (A.60) and using $\rho_m^{\cosh(\cdot),Rc} = \rho_m^{\cosh(\cdot/R),c}$, for all $m \in \mathbb{N}_0$, yields the result.

Let us now prove (A.55) in case (2). We use Lemma A.2 which yields that for all $m \in \mathbb{N}_0$,

$c \neq 0$,

$$\rho_m^{W_{[-1,1]},c} \geq \frac{|c| \vee 1}{2} \left(\left(\frac{|c|}{7\pi(m+1)} \right) \vee 1 \right)^{2m}. \tag{A.64}$$

Consider the case $q = 1$. Using Theorem 5.2. in [6] for the third inequality and $(k+p-1)!/k! \leq (k+p-1)^{p-1}$ for the fourth inequality, we obtain

$$\begin{aligned}
\sum_{|\boldsymbol{m}|_1 \leq N} \frac{1}{\rho_m^{W_{[-1,1]},c}} \leq & \frac{2^p}{(|c| \vee 1)^p} \sum_{k \leq N} \sum_{|\boldsymbol{m}|_1 = k} \prod_{j=1}^{p} \exp\left(2\boldsymbol{m}_j \ln\left(\left(\frac{7\pi(\boldsymbol{m}_j+1)}{|c|} \right) \bigvee 1 \right) \right) \\
\leq & \frac{2^p}{(|c| \vee 1)^p} \sum_{k \leq N} \sum_{|\boldsymbol{m}|_1 = k} \exp\left(2k \ln\left(\left(\frac{7\pi(N+1)}{|c|} \right) \bigvee 1 \right) \right) \\
\leq & \frac{2^p}{(|c| \vee 1)^p} \sum_{k \leq N} \begin{pmatrix} k+p-1 \\ p-1 \end{pmatrix} \exp\left(2k \ln\left(\left(\frac{7\pi(N+1)}{|c|} \right) \bigvee 1 \right) \right) \\
\leq & \frac{2^p (N+p-1)^{p-1}}{(|c| \vee 1)^p (p-1)!} \sum_{k \leq N} \exp\left(2k \ln\left(\left(\frac{7\pi(N+1)}{|c|} \right) \bigvee 1 \right) \right) \\
\leq & \frac{2^p (N+p-1)^{p-1}}{(|c| \vee 1)^p (p-1)! (1 - \exp(-2))} \exp\left(2N \ln\left(\left(\frac{7\pi(N+1)}{|c|} \right) \bigvee 1 \right) \right)
\end{aligned}$$

which yields the result.

Now, consider the case $q = \infty$. We use Lemma A.2 and (A.60). If $N > \lfloor 2|c|/\pi \rfloor \vee 1$, we use

$$\begin{aligned}
\sum_{m=\lfloor 2|c|/\pi \rfloor \vee 1}^{N} \frac{1}{\rho_m^{W_{[-1,1]},c}} \leq & \frac{2}{|c| \vee 1} \exp\left(2N \ln\left(\frac{7\pi(N+1)}{|c|} \right) \right) \sum_{m=\lfloor 2|c|/\pi \rfloor \vee 1}^{N} 1 \\
\leq & \frac{2(N+1)}{|c| \vee 1} \left(\frac{7\pi(N+1)}{|c|} \right)^{2N}
\end{aligned}$$

and conclude using $\rho_m^{W_{[-1,1]},Rc} = \rho_m^{W_{[-R,R]},c}$ for all $m \in \mathbb{N}_0$. □

Lemma A.14. If $\widehat{f}_{\boldsymbol{X}|\mathcal{X}}$ satisfies (H1.3) and (H1.4) then $Z_{n_0} = O_p\left(v(n_0, \mathcal{E})/\delta(n_0)\right)$.

Proof of Lemma A.14. We use

$$\begin{aligned}
& \left\| 1 - \frac{f_{\boldsymbol{X}|\mathcal{X}}}{\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}}} \right\|^2_{L^\infty(\mathcal{X})} \\
& \leq \frac{\left\| \widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}} - f_{\boldsymbol{X}|\mathcal{X}} \right\|^2_{L^\infty(\mathcal{X})}}{\delta(n_0)}
\end{aligned}$$

$$\leq 2\frac{\left\|\widehat{f}_{\boldsymbol{X}|\mathcal{X}}-f_{\boldsymbol{X}|\mathcal{X}}\right\|^2_{L^\infty(\mathcal{X})}}{\delta(n_0)}+2\mathbb{1}\left\{\left\|\widehat{f}_{\boldsymbol{X}|\mathcal{X}}\right\|_{L^\infty(\mathcal{X})}<\sqrt{\delta(n_0)}\right\}$$
$$\leq 2\frac{\left\|\widehat{f}_{\boldsymbol{X}|\mathcal{X}}-f_{\boldsymbol{X}|\mathcal{X}}\right\|^2_{L^\infty(\mathcal{X})}}{\delta(n_0)}+2\mathbb{1}\left\{\left\|f_{\boldsymbol{X}|\mathcal{X}}\right\|_{L^\infty(\mathcal{X})}-\left\|\widehat{f}_{\boldsymbol{X}|\mathcal{X}}-f_{\boldsymbol{X}|\mathcal{X}}\right\|_{L^\infty(\mathcal{X})}<\sqrt{\delta(n_0)}\right\}$$
$$\leq 2\frac{\left\|\widehat{f}_{\boldsymbol{X}|\mathcal{X}}-f_{\boldsymbol{X}|\mathcal{X}}\right\|^2_{L^\infty(\mathcal{X})}}{\delta(n_0)}+2\mathbb{1}\left\{\left\|\widehat{f}_{\boldsymbol{X}|\mathcal{X}}-f_{\boldsymbol{X}|\mathcal{X}}\right\|_{L^\infty(\mathcal{X})}\geq c_{\boldsymbol{X}}-\sqrt{\delta(n_0)}\right\}.$$

Hence, we have, for n_0 large enough,

$$\left\|1-\frac{f_{\boldsymbol{X}|\mathcal{X}}}{\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}}}\right\|^2_{L^\infty(\mathcal{X})}\leq\frac{2}{\delta(n_0)}\left\|\widehat{f}_{\boldsymbol{X}|\mathcal{X}}-f_{\boldsymbol{X}|\mathcal{X}}\right\|^2_{L^\infty(\mathcal{X})}\left(1+\frac{\delta(n_0)}{\left(c_{\boldsymbol{X}}-\sqrt{\delta(n_0)}\right)^2}\right)$$
$$\leq\frac{4}{\delta(n_0)}\left\|\widehat{f}_{\boldsymbol{X}|\mathcal{X}}-f_{\boldsymbol{X}|\mathcal{X}}\right\|^2_{L^\infty(\mathcal{X})}$$

else

$$\left\|1-\frac{f_{\boldsymbol{X}|\mathcal{X}}}{\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}}}\right\|^2_{L^\infty(\mathcal{X})}\leq 2\frac{\left\|\widehat{f}_{\boldsymbol{X}|\mathcal{X}}-f_{\boldsymbol{X}|\mathcal{X}}\right\|^2_{L^\infty(\mathcal{X})}}{\delta(n_0)}+2.$$

We conclude using (H1.4). □

Proof of Theorem 1. We take $f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)$. Using the Plancherel and Chasles identities, we obtain $\left\|\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}-f_{\alpha,\boldsymbol{\beta}}\right\|^2_{L^2(1\otimes W^{\otimes p})}\leq(K_1+K_2)/(2\pi)$, where

$$K_1=\left\|\mathbb{1}\{|\cdot|\geq\epsilon\}\left(\widehat{F}^{q,N,T,0}_1-\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(\cdot,\star)\right\|^2_{L^2(1\otimes W^{\otimes p})} \tag{A.65}$$

$$K_2=\left\|\mathbb{1}\{|\cdot|<\epsilon\}\left(\mathcal{I}_{a_0,\epsilon}\left[\widehat{F}^{q,N,T,0}_1\right]-\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(\cdot,\star)\right\|^2_{L^2(1\otimes W^{\otimes p})}. \tag{A.66}$$

Let us consider K_2. Because for almost every $\boldsymbol{b}\in\mathbb{R}^p$, $a\mapsto f_{\alpha,\boldsymbol{\beta}}(a,\boldsymbol{b})\in L^2_w(\mathbb{R})$, for those $\boldsymbol{b}$ we have $\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](\cdot,\boldsymbol{b})\in L^2(\mathbb{R})$. Using (10) in Proposition 3 for the first display and Lemma A.10 for the second display, we obtain

$$K_2\leq\int_{\mathbb{R}^p}2(1\vee C(\underline{a},\epsilon))\left\|\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](\cdot,\boldsymbol{b})-\mathcal{P}_{\underline{a}}\left[\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](\star,\boldsymbol{b})\right](\cdot)\right\|^2_{L^2(\mathbb{R})}W^{\otimes p}(\boldsymbol{b})d\boldsymbol{b}$$
$$+\int_{\mathbb{R}^p}2C(\underline{a},\epsilon)\left\|\mathbb{1}\{|\cdot|\geq\epsilon\}\left(\widehat{F}^{q,N,T,0}_1-\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(\cdot,\boldsymbol{b})\right\|^2_{L^2(\mathbb{R})}W^{\otimes p}(\boldsymbol{b})d\boldsymbol{b}$$

$$\leq \frac{4\pi(1 \vee C(\underline{a},\epsilon))}{w(\underline{a})} \int_{\mathbb{R}^p} \|f_{\alpha,\boldsymbol{\beta}}(\cdot,\boldsymbol{b})\|^2_{L^2(w)} W^{\otimes p}(\boldsymbol{b}) d\boldsymbol{b} + 2C(\underline{a},\epsilon)K_1$$

$$\leq (1 \vee C(\underline{a},\epsilon)) \frac{4\pi M^2}{w(\underline{a})} + 2C(\underline{a},\epsilon)K_1, \tag{A.67}$$

hence

$$K_1 + K_2 \leq (1 + 2C(\underline{a},\epsilon))K_1 + (1 \vee C(\underline{a},\epsilon)) \frac{4\pi M^2}{w(\underline{a})}. \tag{A.68}$$

Then, by the Jensen inequality, we have

$$K_1 \leq 4 \sum_{j=1}^{4} \|R_j\|^2_{L^2(1 \otimes W^{\otimes p})},$$

where

$$R_1 : (t,\boldsymbol{b}) \mapsto \mathbb{1}\{\epsilon \leq |t|\} \left(\widetilde{F}^{q,N,T,0} - F^{q,N,T,0}\right)(t,\boldsymbol{b}),$$
$$R_2 : (t,\boldsymbol{b}) \mapsto \mathbb{1}\{\epsilon \leq |t|\} \left(\widehat{F}^{q,N,T,0} - \widetilde{F}^{q,N,T,0}\right)(t,\boldsymbol{b}),$$
$$R_3 : (t,\boldsymbol{b}) \mapsto \mathbb{1}\{\epsilon \leq |t|\} \left(F^{q,N,T,0} - F^{q,\infty,T,0}\right)(t,\boldsymbol{b}),$$
$$R_4 : (t,\boldsymbol{b}) \mapsto \mathbb{1}\{\epsilon \leq |t|\} \left(F^{q,\infty,T,0} - \mathcal{F}_{\text{1st}}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(t,\boldsymbol{b}),$$

$\widetilde{F}_1^{q,N,T,0}$ is defined like $\widehat{F}_1^{q,N,T,0}$ replacing $\widehat{c}_{\boldsymbol{m}}(t)$ by $\widetilde{c}_{\boldsymbol{m}}(t)$ (c.f. Lemma A.12), $\widetilde{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}$ is defined as $\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}$ replacing $\widehat{F}_1^{q,N,T,0}$ by $\widetilde{F}_1^{q,N,T,0}$.

Term R_1. Using Proposition 2 for the second display and Lemma A.12 for the third display, we obtain

$$\mathbb{E}\left[\|R_1\|^2_{L^2(1 \otimes W^{\otimes p})}\right] = \int_{\mathbb{R}} \mathbb{1}\{\epsilon \leq |t| \leq T\} \int_{\mathbb{R}^p} \mathbb{E}\left[\left|\widetilde{F}_1^{q,N,T,0}(t,\boldsymbol{b}) - F_1^{q,N,T,0}(t,\boldsymbol{b})\right|^2\right] W^{\otimes p}(\boldsymbol{b}) dt d\boldsymbol{b}$$
$$\leq \int_{\mathbb{R}} \mathbb{1}\{\epsilon \leq |t| \leq T\} \sum_{|\boldsymbol{m}|_q \leq N(t)} \frac{\mathbb{E}\left[|\widetilde{c}_{\boldsymbol{m}}(t) - c_{\boldsymbol{m}}(t)|^2\right]}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2} dt$$
$$\leq \frac{c_{\boldsymbol{X}}}{(2\pi)^p n} \int_{\mathbb{R}} \mathbb{1}\{\epsilon \leq |t| \leq T\} \sum_{|\boldsymbol{m}|_q \leq N(t)} \frac{|t|^p}{\rho_{\boldsymbol{m}}^{W,tx_0}} dt.$$

Using Lemma A.13, we have

$$\mathbb{E}\left[\|R_1\|^2_{L^2(1 \otimes W^{\otimes p})}\right] \leq \frac{c_{\boldsymbol{X}}}{(2\pi)^p n} \int_{\mathbb{R}} \mathbb{1}\{\epsilon \leq |t| \leq T\} |t|^p \nu_q(W, N(t), tx_0) dt. \tag{A.69}$$

Term R_2. Denoting by

$$\left(\widetilde{\omega}^{q,W,c}_{N(t)}\right)^2 = \sup_{|\boldsymbol{m}|_q \le N(t)} \frac{1}{\rho^{W,c}_{\boldsymbol{m}}}, \tag{A.70}$$

we have, denoting by $\Delta_f : x \mapsto \left(1/\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}} - 1/f_{\boldsymbol{X}|\mathcal{X}}\right)(x)$,

$$\begin{aligned}
&\|R_2\|^2_{L^2(1\otimes W^{\otimes p})} \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^p} \mathbb{1}\{\epsilon \le |t| \le T\} \left|\widehat{F}^{q,N,T,0}_1(t,\boldsymbol{b}) - \widetilde{F}^{q,N,T,0}_1(t,\boldsymbol{b})\right|^2 W^{\otimes p}(\boldsymbol{b})dtd\boldsymbol{b} \\
&= \int_{\epsilon\le|t|\le T}\int_{\mathbb{R}^p} \left| \sum_{|\boldsymbol{m}|_q \le N(t)} \frac{\varphi^{W,tx_0}_{\boldsymbol{m}}(\boldsymbol{b})}{\sigma^{W,tx_0}_{\boldsymbol{m}}} \sum_{j=1}^n \frac{e^{itY_j}}{nx_0^p} \Delta_f(\boldsymbol{X}_j) \overline{g^{W,tx_0}_{\boldsymbol{m}}\left(\frac{\boldsymbol{X}_j}{x_0}\right)} \mathbb{1}\{\boldsymbol{X}_j \in \mathcal{X}\} \right|^2 W^{\otimes p}(\boldsymbol{b})dtd\boldsymbol{b} \\
&= \int_{\epsilon\le|t|\le T} \sum_{k\le N} \sum_{|\boldsymbol{m}|_q = k} \left| \frac{1}{\sigma^{W,tx_0}_{\boldsymbol{m}}} \sum_{j=1}^n \frac{e^{itY_j}}{nx_0^p} \Delta_f(\boldsymbol{X}_j) \overline{g^{W,tx_0}_{\boldsymbol{m}}\left(\frac{\boldsymbol{X}_j}{x_0}\right)} \mathbb{1}\{\boldsymbol{X}_j \in \mathcal{X}\} \right|^2 dt \\
&\le \int_{\epsilon\le|t|\le T} \frac{|tx_0|^p \left(\widetilde{\omega}^{q,W,tx_0}_{N(t)}\right)^2}{(2\pi)^p} \sum_{|\boldsymbol{m}|_q \le N(t)} \left| \sum_{j=1}^n \frac{e^{itY_j}}{nx_0^p} \Delta_f(\boldsymbol{X}_j) \overline{g^{W,tx_0}_{\boldsymbol{m}}\left(\frac{\boldsymbol{X}_j}{x_0}\right)} \mathbb{1}\{\boldsymbol{X}_j \in \mathcal{X}\} \right|^2 dt \\
&= \int_{\epsilon\le|t|\le T} \frac{|tx_0|^p \left(\widetilde{\omega}^{q,W,tx_0}_{N(t)}\right)^2}{(2\pi)^p} \left(\int_{[-1,1]^p} \left| \sum_{|\boldsymbol{m}|_q \le N(t)} \Delta_{\boldsymbol{m}}(t) g^{W,tx_0}_{\boldsymbol{m}}(\boldsymbol{x}) \right|^2 d\boldsymbol{x} \right) dt \\
&\le \int_{\epsilon\le|t|\le T} \frac{|tx_0|^p \left(\widetilde{\omega}^{q,W,tx_0}_{N(t)}\right)^2}{(2\pi)^p} \left\|S^N_0(\cdot,t)\right\|^2_{L^2([-1,1]^p)} dt,
\end{aligned}$$

where $S^N_0(\cdot,t) = \sum_{\boldsymbol{m}\in\mathbb{N}^p_0:\ |\boldsymbol{m}|_q \le N} g^{W,tx_0}_{\boldsymbol{m}}(\cdot)\Delta_{\boldsymbol{m}}(t)$ and

$$\Delta_{\boldsymbol{m}}(t) = \sum_{j=1}^n \frac{e^{itY_j}}{nx_0^p} \left(\frac{1}{\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}}} - \frac{1}{f_{\boldsymbol{X}|\mathcal{X}}} \right)(\boldsymbol{X}_j) \overline{g^{W,tx_0}_{\boldsymbol{m}}\left(\frac{\boldsymbol{X}_j}{x_0}\right)} \mathbb{1}\{\boldsymbol{X}_j \in \mathcal{X}\}.$$

Then, we have $\mathbb{E}\left[\left\|S^N_0(\cdot,t)\right\|^2_{L^2([-1,1]^p)}\right] = \mathbb{E}\left[\left\|S^N_1(\cdot,t)\right\|^2_{L^2([-1,1]^p)}\right] + \mathbb{E}\left[\left\|S^N_2(\cdot,t)\right\|^2_{L^2([-1,1]^p)}\right]$, where

$$\begin{aligned}
S^N_1(\cdot,t) &= \sum_{|\boldsymbol{m}|_q \le N(t)} g^{W,tx_0}_{\boldsymbol{m}}(\cdot)\mathbb{E}\left[\Delta_{\boldsymbol{m}}(t)|\mathcal{G}_{n_0}\right], \\
S^N_2(\cdot,t) &= \sum_{|\boldsymbol{m}|_q \le N(t)} g^{W,tx_0}_{\boldsymbol{m}}(\cdot)\left(\Delta_{\boldsymbol{m}}(t) - \mathbb{E}\left[\Delta_{\boldsymbol{m}}(t)|\mathcal{G}_{n_0}\right]\right).
\end{aligned}$$

First, considering $S_1^N(\cdot,t)$, we obtain

$$
\begin{aligned}
&\left\|S_1^N(\cdot,t)\right\|^2_{L^2([-1,1]^p)} \\
&= \left\| \sum_{|\boldsymbol{m}|_q \leq N(t)} g_{\boldsymbol{m}}^{W,tx_0}(\cdot) \int_{[-1,1]^p} \mathcal{F}\left[f_{Y|\mathcal{X},\boldsymbol{X}=x_0\boldsymbol{u}}\right](t) \left(\frac{f_{\boldsymbol{X}|\mathcal{X}}}{\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}}}(x_0\boldsymbol{u}) - 1 \right) \overline{g_{\boldsymbol{m}}^{W,tx_0}(\boldsymbol{u})} d\boldsymbol{u} \right\|^2_{L^2([-1,1]^p)} \\
&\leq \left\| \mathcal{F}\left[f_{Y|\mathcal{X},\boldsymbol{X}=x_0\cdot}\right](t) \left(\frac{f_{\boldsymbol{X}|\mathcal{X}}}{\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}}}(x_0\cdot) - 1 \right) \right\|^2_{L^2([-1,1]^p)} \\
&\leq Z_{n_0} \left\| \mathcal{F}\left[f_{\alpha,\boldsymbol{\beta}}\right](t, tx_0\cdot) \right\|^2_{L^2([-1,1]^p)} \\
&\leq Z_{n_0} \left(\frac{2\pi}{|t|x_0} \right)^p \left\| \mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot) \right\|^2_{L^2(\mathbb{R}^p)}.
\end{aligned} \tag{A.71}
$$

Second, denoting by

$$
Z_j^{\boldsymbol{m},t} = \frac{e^{itY_j}}{x_0^p} \left(\frac{1}{\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}}} - \frac{1}{f_{\boldsymbol{X}|\mathcal{X}}} \right) (\boldsymbol{X}_j) \overline{g_{\boldsymbol{m}}^{W,tx_0}\left(\frac{\boldsymbol{X}_j}{x_0}\right)} \mathbb{1}\{\boldsymbol{X}_j \in \mathcal{X}\},
$$

and using independence of the $Z_j^{\boldsymbol{m},t}$ for $j = 1, \ldots, n$, we have

$$
\begin{aligned}
\mathbb{E}\left[\left\|S_2^N(\cdot,t)\right\|^2_{L^2([-1,1]^p)} \middle| \mathcal{G}_{n_0} \right] &= \sum_{|\boldsymbol{m}|_q \leq N(t)} \mathbb{E}\left[\left| \Delta_{\boldsymbol{m}}(t) - \mathbb{E}\left[\Delta_{\boldsymbol{m}}(t) | \mathcal{G}_{n_0}\right] \right|^2 \middle| \mathcal{G}_{n_0} \right] \\
&\leq \sum_{|\boldsymbol{m}|_q \leq N(t)} \frac{1}{n} \mathbb{E}\left[\left| Z_j^{\boldsymbol{m},t} - \mathbb{E}\left[Z_j^{\boldsymbol{m},t} \middle| \mathcal{G}_{n_0} \right] \right|^2 \middle| \mathcal{G}_{n_0} \right] \\
&\leq \sum_{|\boldsymbol{m}|_q \leq N(t)} \frac{Z_{n_0}}{n x_0^{2p}} \int_{\mathcal{X}} \frac{1}{f_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{x})} \left| \overline{g_{\boldsymbol{m}}^{W,tx_0}}\left(\frac{\boldsymbol{x}}{x_0}\right) \right|^2 d\boldsymbol{x} \\
&\leq \sum_{|\boldsymbol{m}|_q \leq N(t)} \frac{c_{\boldsymbol{X}} Z_{n_0}}{n x_0^p} \int_{[-1,1]^p} \left| \overline{g_{\boldsymbol{m}}^{W,tx_0}}(\boldsymbol{u}) \right|^2 d\boldsymbol{u} \\
&\leq \sum_{|\boldsymbol{m}|_q \leq N(t)} \frac{c_{\boldsymbol{X}} Z_{n_0}}{n x_0^p} \\
&\leq \frac{(N(t)+1)^p c_{\boldsymbol{X}} Z_{n_0}}{n x_0^p},
\end{aligned} \tag{A.72}
$$

where $\sum_{|\boldsymbol{m}|_q \leq N} 1 = \binom{N+p}{p} \mathbb{1}\{q=1\} + (N+1)^p \mathbb{1}\{q=\infty\} \leq (N+1)^p$.
Collecting (A.71) and (A.72) and denoting by $L^2(t) = (2\pi)^p \left\| \mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot)\right\|^2_{L^2(\mathbb{R}^p)}$, we obtain

$$\text{(A.73)}\quad \mathbb{E}\left[\|R_2\|^2_{L^2(1\otimes W^{\otimes p})}\middle|\mathcal{G}_{n_0}\right] \leq \frac{Z_{n_0}}{(2\pi)^p}\int_{\epsilon\leq|t|\leq T}\left(L^2(t) + \frac{c_{\boldsymbol{X}}(N(t)+1)^p|t|^p}{n}\right)\left(\widetilde{\omega}^{q,W,tx_0}_{N(t)}\right)^2 dt.$$

Term R_3. We have, for $\boldsymbol{m}\in\mathbb{N}_0^p$ and $t\neq 0$, when $W(\cdot)=\cosh(\cdot/R)$

$$\begin{aligned}
c_{\boldsymbol{m}}(t) &= \int_{[-1,1]^p}\int_{\mathbb{R}^p} e^{itx_0\boldsymbol{b}^\top\boldsymbol{u}}\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\boldsymbol{b})\overline{g^{W,tx_0}_{\boldsymbol{m}}(\boldsymbol{u})}d\boldsymbol{u}d\boldsymbol{b}\\
&= \int_{\mathbb{R}^p}\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\boldsymbol{b})\overline{\left((W^{\otimes p})^{-1}(\boldsymbol{b})\int_{[-1,1]^p} e^{-itx_0\boldsymbol{b}^\top\boldsymbol{u}}g^{W,tx_0}_{\boldsymbol{m}}(\boldsymbol{u})\,d\boldsymbol{u}\right)}W^{\otimes p}(\boldsymbol{b})d\boldsymbol{b}\\
&= \int_{\mathbb{R}^p}\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\boldsymbol{b})\overline{\mathcal{F}^*_{tx_0}\left[g^{W,tx_0}_{\boldsymbol{m}}\right](\boldsymbol{b})}W^{\otimes p}(\boldsymbol{b})\,d\boldsymbol{b}\\
&= \sigma^{W,tx_0}_{\boldsymbol{m}}\int_{\mathbb{R}^p}\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\boldsymbol{b})\overline{\varphi^{W,tx_0}_{\boldsymbol{m}}}(\boldsymbol{b})\,W^{\otimes p}(\boldsymbol{b})\,d\boldsymbol{b}\\
&= \sigma^{W,tx_0}_{\boldsymbol{m}}b_{\boldsymbol{m}}(t),
\end{aligned}$$

while when $W = W_{[-R,R]}$

$$\begin{aligned}
c_{\boldsymbol{m}}(t) =& \int_{[-1,1]^p}\int_{\mathbb{R}^p} e^{itx_0\boldsymbol{b}^\top\boldsymbol{u}}\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\boldsymbol{b})\overline{g^{W_{[-R,R]},tx_0}_{\boldsymbol{m}}}(\boldsymbol{u})\;\;d\boldsymbol{u}d\boldsymbol{b}\\
=& \int_{\mathbb{R}^p}\mathbb{1}\left\{|\boldsymbol{b}|_\infty\leq R\right\}\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\boldsymbol{b})\overline{\mathcal{F}^*_{tx_0}\left[g^{W_{[-R,R]},tx_0}_{\boldsymbol{m}}\right](\boldsymbol{b})}d\boldsymbol{b}\quad(\text{because }\mathbb{S}_{\boldsymbol{\beta}}\subseteq[-R,R]^p)\\
=&\sigma^{W_{[-R,R]},tx_0}_{\boldsymbol{m}}\int_{[-R,R]^p}\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\boldsymbol{b})\varphi^{W_{[-R,R]},tx_0}_{\boldsymbol{m}}(\boldsymbol{b})\,d\boldsymbol{b}\\
=&\sigma^{W_{[-R,R]},tx_0}_{\boldsymbol{m}}b_{\boldsymbol{m}}(t).
\end{aligned}$$

Thus, in both cases, using that $\left(\varphi^{W,tx_0}_{\boldsymbol{m}}\right)_{\boldsymbol{m}\in\mathbb{N}_0^p}$ is a basis of $L^2_{W^{\otimes p}}(\mathbb{R}^p)$, we obtain

$$\begin{aligned}
\|R_3\|^2_{L^2(1\otimes W^{\otimes p})} &= \int_{\mathbb{R}}\int_{\mathbb{R}^p}\mathbb{1}\{\epsilon\leq|t|\leq T\}\left|F_1^{q,N,T,0}(t,\boldsymbol{b}) - \mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\boldsymbol{b})\right|^2 W^{\otimes p}(\boldsymbol{b})dtd\boldsymbol{b}\\
&= \int_{\epsilon\leq|t|\leq T}\sum_{|\boldsymbol{m}|_q>N(t)}|b_{\boldsymbol{m}}(t)|^2\,dt,
\end{aligned}$$

hence

$$\|R_3\|^2_{L^2(1\otimes W^{\otimes p})} \leq \int_{\mathbb{R}}\sum_{k>N(t)}\sum_{|\boldsymbol{m}|_q=k}|b_{\boldsymbol{m}}(t)|^2\,dt$$

$$\leq \int_{\mathbb{R}} \frac{1}{\omega_{N(t)}^2} \sum_{k>N(t)} \omega_k^2 \theta_{q,k}^2(t)$$

$$\leq 2\pi l^2 \sup_{t\in\mathbb{R}} \frac{1}{\omega_{N(t)}^2} \quad \left(\text{because } f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}_{w,W}^{q,\phi,\omega}(l,M)\right). \tag{A.74}$$

Term R_4. We obtain

$$\begin{aligned}
\|R_4\|_{L^2(1\otimes W^{\otimes p})}^2 &= \int_{\mathbb{R}} \mathbb{1}\{\epsilon \leq |t|\} \int_{\mathbb{R}^p} \left|\mathbb{1}\{|t|\leq T\}\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\boldsymbol{b}) - \mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\boldsymbol{b})\right|^2 W^{\otimes p}(\boldsymbol{b})dtd\boldsymbol{b} \\
&\leq \int_{\mathbb{R}} \mathbb{1}\{|t|\geq T\} \sum_{\boldsymbol{m}\in\mathbb{N}_0^p} |b_{\boldsymbol{m}}(t)|^2 dt \quad \left(\text{because } (\varphi_{\boldsymbol{m}}^{W,tx_0})_{\boldsymbol{m}\in\mathbb{N}_0^p} \text{ is a basis of } L^2\left(W^{\otimes p}\right)\right) \\
&= \sum_{k\in\mathbb{N}_0} \int_{\mathbb{R}} \mathbb{1}\{|t|\geq T\} \sum_{|\boldsymbol{m}|_q=k} |b_{\boldsymbol{m}}(t)|^2 dt \\
&\leq \frac{1}{\phi^2(T)} \sum_{k\in\mathbb{N}_0} \int_{\mathbb{R}} \phi^2(|t|)\theta_{q,k}^2(t)dt \\
&\leq \frac{2\pi l^2}{\phi^2(T)}.
\end{aligned} \tag{A.75}$$

Thus we have

$$\begin{aligned}
&\sup_{\substack{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}_{w,W}^{q,\phi,\omega}(l,M)\cap\mathcal{D} \\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}} \mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right) \\
&\leq C\int_{\epsilon\leq|t|\leq T} \frac{2}{\pi(2\pi)^p}\left(\frac{c_{\boldsymbol{X}}|t|^p}{n}\nu_q(W,N(t),tx_0) + Z_{n_0}\left(L^2(t) + \frac{c_{\boldsymbol{X}}(1+N(t))^p|t|^p}{n}\right)\left(\widetilde{\omega}_{N(t)}^{q,W,tx_0}\right)^2\right)dt \\
&\quad + C\left(4l^2\left(\sup_{t\in\mathbb{R}}\frac{1}{\omega_{N(t)}^2} + \frac{1}{\phi(T)^2}\right) + \frac{M^2}{w(\underline{a})}\right).
\end{aligned} \tag{A.76}$$

Proof of (T1.2). We now consider the above equation in each different smoothness case.

Case $q = 1$. Using (A.56) and (A.61), we obtain the upper bound on $\left(\widetilde{\omega}_{N(t)}^{1,W,tx_0}\right)^2$, which is defined in (A.70) and enters (A.73),

$$\begin{aligned}
\left(\widetilde{\omega}_{N(t)}^{1,W,tx_0}\right)^2 \leq \left(\frac{\pi^2}{4}\right)^p \exp\left(2\left(1+\ln\left(\frac{e}{Rx_0|t|}\right)\right)N(t)\right)\mathbb{1}\left\{|t|\leq\frac{\pi}{4Rx_0}\right\} \\
+ 2^p\exp\left(\frac{\pi(N(t)+p)}{2Rx_0|t|}\right)\mathbb{1}\left\{|t|>\frac{\pi}{4Rx_0}\right\}.
\end{aligned}$$

In order to collect the upper bounds on $\mathbb{E}\left[\|R_1\|_{L^2_{1\otimes W^{\otimes p}}}\middle|\mathcal{G}_{n_0}\right]$ and $\mathbb{E}\left[\|R_2\|_{L^2_{1\otimes W^{\otimes p}}}\middle|\mathcal{G}_{n_0}\right]$, we denote, for $z>0$, $N\in\mathbb{N}$, and *a.e.* $t\neq 0$, for $|t|>\pi/(4Rx_0)$ by

$$\Delta_1(t,N,n,z) = \frac{4c_{\boldsymbol{X}}(N+p-1)^{p-1}\,|t|^p}{\pi^{p+1}n(p-1)!}\left(1\bigvee\left(\frac{\sqrt{2}Rx_0\,|t|}{\pi}\right)\right)\exp\left(\frac{\pi(N+p)}{2\,|t|\,Rx_0}\right)$$
$$+\frac{2z}{\pi^{p+1}}\left(L^2(t)+\frac{c_{\boldsymbol{X}}(N+1)^p\,|t|^p}{n}\right)\exp\left(\frac{\pi(N+p)}{2\,|t|\,Rx_0}\right),$$

and, for $|t|\leq\pi/(4Rx_0)$, by

$$\Delta_1(t,N,n,z) = \frac{c_{\boldsymbol{X}}(N+p-1)^{p-1}\,|t|^p}{n(p-1)!}\frac{2\pi^{p-1}}{8^p}\exp\left(2\left(1+\ln\left(\frac{e}{Rx_0|t|}\right)\right)N\right)$$
$$+z\left(L^2(t)+\frac{c_{\boldsymbol{X}}(N+1)^p\,|t|^p}{n}\right)\frac{2\pi^{p-1}}{8^p}\exp\left(2\left(1+\ln\left(\frac{e}{Rx_0|t|}\right)\right)N\right).$$

Using (A.69), (A.73)-(A.75), and the definition of C (*i.e* $C=2(1\vee(2C(\underline{a},\epsilon)))$), we have

(A.77)

$$\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right)\leq C\left(\int_{\epsilon\leq|t|\leq T}\Delta_1(t,N(t),n,Z_{n_0})dt+4l^2\sup_{t\in\mathbb{R}}N(t)^{-2\sigma}+\frac{4l^2}{(1\vee T)^{2s}}+M^2\epsilon\right).$$

Take $\mathcal{E}$ and $\eta>0$. Using Lemma A.14, there exists $M_{1,\mathcal{E},\eta}\in\mathbb{R}$, such that, for all n_0, $\mathbb{P}\left(E\left(\mathcal{G}_{n_0},\mathcal{E},\eta\right)\right)\geq 1-\eta$, where $E\left(\mathcal{G}_{n_0},\mathcal{E},\eta\right)=\{Z_{n_0}\leq M_{1,\mathcal{E},\eta}v(n_0,\mathcal{E})/\delta(n_0)\}$.

Step 1. We show that, on $E\left(\mathcal{G}_{n_0},\mathcal{E},\eta\right)$, $\int_{\epsilon\leq|t|\leq T}\Delta_1(t,N(t),n,Z_{n_0})N(t)^{2\sigma}dt\leq M_{2,\mathcal{E},\eta,1}$, where

$$M_{2,\mathcal{E},\eta,1} = \frac{4c_{\boldsymbol{X}}p^{p-1}}{\pi(p+1)(p-1)!}\left(\frac{\tau_1'\pi^p}{8^p}\bigvee\frac{2\tau_1}{\pi^p}\left(1\bigvee\frac{\sqrt{2}Rx_0}{\pi}\right)\right)+2^{p+2}\pi^pM_{1,\mathcal{E},\eta}\left(\frac{\tau_1'\pi^p}{8^p}\bigvee\frac{\tau_1}{\pi^p}\right)l^2$$
$$+\left(\frac{\tau_1'\pi^p}{8^p}\bigvee\frac{\tau_1}{\pi^p}\right)\frac{4M_{1,\mathcal{E},\eta}c_{\boldsymbol{X}}}{\pi(p+1)}\left(\frac{p}{e}+2\right)^p,$$

$\tau_1=\exp\left(2(p+1)\right)+1$, and $\tau_1'=e^4\left(8\sigma/e\right)^{4\sigma}+1$, from which we deduce

(A.78)

$$\sup_{\substack{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)\cap\mathcal{D}\\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}}\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right)\leq C\left(\left(\sup_{\epsilon\leq|t|\leq T}N(t)^{-2\sigma}\right)\left(M_{2,\mathcal{E},\eta,1}+4l^2\right)+\frac{4l^2}{(1\vee T)^{2s}}+M^2\epsilon\right).$$

Step 1.1. Inequalities for $|t|>\pi/(4Rx_0)$. Indeed, on $E\left(\mathcal{G}_{n_0},\mathcal{E},\eta\right)$, we have using $(N(t)+p-1)^{p-1}\leq(pN(t))^{p-1}$, for *a.e.* t such that $|t|>\pi/(4Rx_0)$,

(A.79) $$\frac{\pi}{2}\Delta_1(t,N(t),n,Z_{n_0})N(t)^{2\sigma}$$

$$\leq \left(\frac{2c_{\boldsymbol{X}} n_e p^{p-1}}{\pi^p (p-1)! n}\right)\left(1 \bigvee \frac{\sqrt{2} R x_0 |t|}{\pi}\right) \frac{|t|^p N(t)^{2\sigma+p-1}}{n_e} \exp\left(\frac{\pi(N(t)+p)}{2|t| R x_0}\right)$$
$$+ \left(\frac{M_{1,\mathcal{E},\eta} v(n_0,\mathcal{E}) n_e}{\pi^p}\right) L^2(t) \frac{N(t)^{2\sigma}}{n_e} \exp\left(\frac{\pi(N(t)+p)}{2|t| R x_0}\right)$$
$$+ \left(\frac{c_{\boldsymbol{X}} (N(t)+1)^p M_{1,\mathcal{E},\eta} n_e v(n_0,\mathcal{E})}{n\pi^p}\right) \frac{|t|^p N(t)^{2\sigma}}{n_e} \exp\left(\frac{\pi(N(t)+p)}{2|t| R x_0}\right).$$

Using that for $N(t) > 1$, $N(t) + p \leq \overline{N}(t)(p+2)$ and $N(t)^{2\sigma+p-1} \leq \left(2\overline{N}(t)\right)^{2\sigma+p-1}$ for the first display and that, for all $t > 0$, $\ln(t) \leq t - 1$ for the second display, we obtain, for $|t| \geq (\epsilon \vee (\pi/(4Rx_0)))$,

$$\frac{N(t)^{2\sigma+p-1}}{n_e} \exp\left(\frac{\pi(N(t)+p)}{2|t| R x_0}\right)$$
$$\leq \frac{\mathbb{1}\{N(t)=1\}}{n_e} \exp\left(\frac{\pi(p+1)}{2|t| R x_0}\right) + \frac{\mathbb{1}\{N(t)>1\} 2^{2\sigma+p-1}}{n_e} \exp\left((2\sigma+p-1)\ln\left(\overline{N}(t)\right) + \frac{\pi(p+2)}{2|t| R x_0}\overline{N}(t)\right)$$
$$\leq \frac{\mathbb{1}\{N(t)=1\}}{n_e} \exp\left(2(p+1)\right) + \frac{\mathbb{1}\{N(t)>1\}}{n_e} \exp\left(\left(2\sigma+p-1+\frac{\pi(p+2)}{2|t| R x_0}\right)\overline{N}(t)\right)$$
$$\leq \frac{\tau_1}{n_e^{1/2}} \quad \left(\text{because } n_e \geq 1 \text{ and by definition of } \overline{N}\right).$$

For the remaining terms we use

$$\frac{N(t)^{2\sigma}}{n_e} \exp\left(\frac{\pi(N(t)+p)}{2|t| R x_0}\right) \leq \frac{N(t)^{2\sigma+p-1}}{n_e} \exp\left(\frac{\pi(N(t)+p)}{2|t| R x_0}\right)$$

which holds for $N(t) = 0$ and when $N(t) \geq 1$.

Then, using that $n_e/n \leq 1$, $n_e v(n_0,\mathcal{E})/\delta(n_0) \leq 1$, we obtain

$$\frac{\pi}{2} \int_{(\epsilon\vee(\pi/(4Rx_0)))\leq|t|\leq T} \Delta_1(t, N(t), n, Z_{n_0}) N(t)^{2\sigma} dt$$
$$\leq \frac{\tau_1 4 c_{\boldsymbol{X}} p^{p-1}}{\pi^p (p-1)!} \left(1 \bigvee \frac{\sqrt{2} R x_0}{\pi}\right) \frac{1}{n_e^{1/2}} \int_{(\epsilon\vee(\pi/(4Rx_0)))}^{T} |t|^p (1 \vee |t|) dt$$

(A.80)

$$+ \frac{\tau_1}{\pi^p} M_{1,\mathcal{E},\eta} \int_{(\epsilon\vee(\pi/(4Rx_0)))\leq|t|\leq T} L^2(t) dt + \frac{\tau_1 2 c_{\boldsymbol{X}} M_{1,\mathcal{E},\eta}}{\pi^p n_e^{1/2}} \int_{(\epsilon\vee(\pi/(4Rx_0)))}^{T} \frac{|t|^p (1+N(t))^p}{n} dt.$$

Step 1.2. Inequalities for $|t| \leq \pi/(4Rx_0)$. On $E(\mathcal{G}_{n_0}, \mathcal{E}, \eta)$, we have using $(N(t)+p-1)^{p-1} \leq (pN(t))^{p-1}$, for *a.e.* t such that $|t| \leq \pi/(4Rx_0)$,

(A.81) $$\frac{\pi}{2} \Delta_1(t, N(t), n, Z_{n_0}) N(t)^{2\sigma}$$

$$\leq \frac{c_{\boldsymbol{X}} p^{p-1} n_e \left|t\right|^p \pi^p}{n(p-1)! 8^p} \frac{N(t)^{2\sigma+p+1}}{n_e} \exp\left(2\left(1+\ln\left(\frac{e}{Rx_0|t|}\right)\right)N(t)\right)$$
$$+ \frac{M_{1,\mathcal{E},\eta} v(n_0,\mathcal{E}) n_e L^2(t) \pi^p}{8^p} \frac{N(t)^{2\sigma}}{n_e} \exp\left(2\left(1+\ln\left(\frac{e}{Rx_0|t|}\right)\right)N(t)\right)$$
$$+ \frac{c_{\boldsymbol{X}} M_{1,\mathcal{E},\eta} n_e v(n_0,\mathcal{E}) (N(t)+1)^p (\left|t\right|\pi)^p}{8^p n} \frac{N(t)^{2\sigma}}{n_e} \exp\left(2\left(1+\ln\left(\frac{e}{Rx_0|t|}\right)\right)N(t)\right).$$

Using that for $N(t) > 1$, $N(t)^{2\sigma+p-1} \leq (2\overline{N}(t))^{2\sigma+p-1}$ and, for all $t > 0$, $\ln(t) \leq t - 1$ for the first display, we obtain, for $\epsilon \leq |t| \leq (\epsilon \vee (\pi/(4Rx_0)))$,

$$\frac{N(t)^{2\sigma+p-1}}{n_e} \exp\left(2\left(1+\ln\left(\frac{e}{Rx_0|t|}\right)\right)N(t)\right)$$
$$\leq \frac{\mathbb{1}\{N(t)=1\} e^4}{n_e}\left(\frac{1}{Rx_0\epsilon}\right)^2 + \frac{\mathbb{1}\{N(t)>1\}}{n_e} \exp\left(\left(2\sigma+p+1+2\ln\left(\frac{e}{Rx_0|t|}\right)\right)\overline{N}(t)\right)$$
$$\leq \frac{\mathbb{1}\{N(t)=1\} e^4 \ln(n_e)^{4\sigma}}{(Rx_0)^2 n_e} + \frac{\mathbb{1}\{N(t)>1\}}{n_e^{1/2}} \quad \left(\text{because } n_e \geq 1 \text{ and by definition of } \overline{N}\right)$$
$$\leq \frac{\tau_1'}{n_e^{1/2}} \quad \left(\text{because } \sup_{t\geq 1} \frac{\ln(t)^a}{t^b} = \left(\frac{a}{eb}\right)^a,\ a,b>0\right).$$

For the remaining terms we use

$$\frac{N(t)^{2\sigma}}{n_e} \exp\left(2\left(1+\ln\left(\frac{e}{Rx_0|t|}\right)\right)N(t)\right) \leq \frac{N(t)^{2\sigma+p-1}}{n_e} \exp\left(2\left(1+\ln\left(\frac{e}{Rx_0|t|}\right)\right)N(t)\right)$$

which holds for $N(t) = 0$ and when $N(t) \geq 1$.

Then, using that $n_e/n \leq 1$, $n_e v(n_0,\mathcal{E})/\delta(n_0) \leq 1$, we obtain

$$\frac{\pi}{2} \int_{\epsilon\leq|t|\leq(\epsilon\vee(\pi/(4Rx_0)))} \Delta_1(t, N(t), n, Z_{n_0}) N(t)^{2\sigma} dt$$
$$\leq \frac{2\tau_1' c_{\boldsymbol{X}} p^{p-1} \pi^p}{(p-1)! 8^p} \frac{1}{n_e^{1/2}} \int_{\epsilon\leq t\leq(\epsilon\vee(\pi/(4Rx_0)))} \left|t\right|^p dt + M_{1,\mathcal{E},\eta} \frac{\tau_1' \pi^p}{8^p} \int_{\epsilon\leq|t|\leq(\epsilon\vee(\pi/(4Rx_0)))} L^2(t) dt$$
$$+ \frac{2\tau_1' M_{1,\mathcal{E},\eta} c_{\boldsymbol{X}} \pi^p}{n_e^{1/2} 8^p} \int_{\epsilon\leq t\leq(\epsilon\vee(\pi/(4Rx_0)))} \frac{\left|t\right|^p (1+N(t))^p}{n} dt. \tag{A.82}$$

Step 1.3. Conclusion. We have

$$\int_\epsilon^T \frac{\left|t\right|^p (1+N(t))^p}{n} dt \leq \frac{T^{p+1}(\ln(n_e)+2)^p}{(p+1)n} \quad \left(\text{because } N(t)+1 \leq \overline{N}(t)+2 \leq \ln(n_e)+2\right)$$
$$\leq \frac{T^{p+1}}{p+1}\left(\frac{p}{e}+2\right)^p \quad \left(\text{because } n \geq n_e \text{ and } \sup_{t\geq 1} \frac{\ln(t)^a}{t^b} = \left(\frac{a}{eb}\right)^a,\ a,b>0\right), \tag{A.83}$$

hence with (A.80) and (A.82) we obtain

$$\begin{aligned}
&\frac{\pi}{2}\int_{\epsilon\le|t|\le(\epsilon\vee(4/(\pi Rx_0)))}\Delta_1(t,N(t),n,Z_{n_0})N(t)^{2\sigma}dt\\
&\le\frac{2c_{\boldsymbol{X}}p^{p-1}}{(p-1)!(p+2)}\left(\frac{\tau_1'\pi^p}{8^p}\bigvee\frac{2\tau_1}{\pi^p}\left(1\bigvee\frac{\sqrt{2}Rx_0}{\pi}\right)\right)\frac{1}{n_e^{1/2}}\left(T^{p+2}+\frac{1}{p+1}\right)\\
&\quad+M_{1,\mathcal{E},\eta}\left(\frac{\tau_1'\pi^p}{8^p}\bigvee\frac{\tau_1}{\pi^p}\right)\int_{\epsilon\le|t|\le T}L^2(t)dt+\left(\frac{\tau_1'\pi^p}{8^p}\bigvee\frac{\tau_1}{\pi^p}\right)\frac{2M_{1,\mathcal{E},\eta}c_{\boldsymbol{X}}T^{p+1}}{n_e^{1/2}(p+1)}\left(\frac{p}{e}+2\right)^p.
\end{aligned}$$

Then, using that $\int_{\epsilon\le|t|\le T}L^2(t)dt\le(2\pi)^{p+1}l^2$ (because $f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{1,\phi,\omega}_{w,W}(l,M)$) and $\phi\ge1$) and that $T^{p+2}=n_e^{1/2}$, we have $\int_{\epsilon\le|t|\le T}\Delta_1(t,N(t),n,Z_{n_0})N(t)^{2\sigma}dt\le M_{2,\mathcal{E},\eta}$ hence (A.78).

Step 2. Using $\sup_{|t|\in[\epsilon,T]}N(t)^{-2\sigma}=N(\epsilon)^{-2\sigma}\le\overline{N}(\epsilon)^{-2\sigma}$ and (A.78), we obtain on $E(\mathcal{G}_{n_0},\mathcal{E},\eta)$,

(A.84)
$$\sup_{\substack{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)\cap\mathcal{D}\\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}}\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right)\le\frac{C}{\overline{N}(\epsilon)^{2\sigma}}\left(M_{2,\mathcal{E},\eta,1}+4l^2+\frac{4l^2\overline{N}(\epsilon)^{2\sigma}}{(1\vee T)^{2s}}+M^2\epsilon\overline{N}(\epsilon)^{2\sigma}\right).$$

Because, $\epsilon=1/\ln(n_e)^{2\sigma}$, then for $n_e\ge n_1$, where $n_1=\exp\left((Rx_0/e^{2\sigma+p+2})^{1/2\sigma}\right)\vee\exp(\exp(1))$, we have $2\sigma+p+1+\ln\left(\ln(n_e)^{2\sigma}e/(Rx_0)\right)\ge1$ which yields

$$\begin{aligned}
\epsilon\overline{N}(\epsilon)^{2\sigma}&=(\ln(n_e))^{-2\sigma}\left(\frac{\ln(n_e)}{2\left(2\sigma+p+1+\ln\left(\ln(n_e)^{2\sigma}e/(Rx_0)\right)\right)}\right)^{2\sigma}\\
&\le1.
\end{aligned}$$

We also obtain, for $n_e\ge n_1$,

$$\begin{aligned}
\overline{N}(\epsilon)^{-2\sigma}&=\left(\frac{\ln(n_e)}{2\left(2\sigma+p+1+4\sigma\ln_2(n_e)+2\ln(e/(Rx_0))\right)}\right)^{-2\sigma}\\
&\le\left(\frac{\ln(n_e)}{\ln_2(n_e)}\right)^{-2\sigma}\left(1\vee\left(2\left(6\sigma+p+1+2\ln\left(\frac{e}{Rx_0}\right)\right)\right)^{2\sigma}\right)
\end{aligned}$$

Thus, using, the definition of $\overline{N}(\epsilon)$, $T^{p+2}=n_e^{1/2}$, and $n_e\ge n_1$, we have

$$\begin{aligned}
\frac{\overline{N}(\epsilon)^{2\sigma}}{(1\vee T)^{2s}}&\le\frac{1}{2^{2\sigma}}\frac{\ln(n_e)^{2\sigma}}{n_e^{s/(p+2)}}\\
&\le\left(\frac{\sigma(p+2)}{se}\right)^{2\sigma}\quad\left(\text{because }\sup_{t\ge1}\frac{\ln(t)^a}{t^b}=\left(\frac{a}{eb}\right)^a\text{ for }a,b>0\right).
\end{aligned}$$

Finally using (A.84) we obtain, for all n_0, on $E(\mathcal{G}_{n_0}, \mathcal{E}, \eta)$, if $n_e \geq n_1$,

$$\left(\frac{\ln(n_e)}{\ln_2(n_e)}\right)^{2\sigma} \sup_{\substack{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}_{w,W}^{q,\phi,\omega}(l,M)\cap\mathcal{D} \\ f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}}} \mathcal{R}_{n_0}^{W}\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right) \leq M_{3,\mathcal{E},\eta,1}, \tag{A.85}$$

where

$$M_{3,\mathcal{E},\eta,1} = C\left(1 \vee \left(2\left(6\sigma + p + 1 + 2\ln\left(\frac{e}{Rx_0}\right)\right)\right)^{2\sigma}\right)\left(M_{2,\mathcal{E},\eta} + 4l^2 + 4l^2\left(\frac{\sigma(p+2)}{se}\right)^{2\sigma} + M^2\right).$$

Case $q = \infty$. Similarly, using that

$$\left(\widetilde{\omega}_N^{\infty,W,tx_0}\right)^2 \leq \left(\frac{\pi}{8}\right)^p \exp\left(2p\left(1 + 1\ln\left(\frac{e}{Rx_0|t|}\right)\right)N\right) \mathbb{1}\left\{|t| \leq \frac{\pi}{4Rx_0}\right\} + 2^p \exp\left(\frac{\pi p(N+1)}{2Rx_0\,|t|}\right) \mathbb{1}\left\{|t| > \frac{\pi}{4Rx_0}\right\},$$

we obtain (A.85) in the case $q = \infty$ with

$$M_{3,\mathcal{E},\eta,\infty} = C\left(1 \vee \left(2\left(6\sigma + 2p\left(1 + \ln\left(\frac{e}{Rx_0}\right)\right)\right)\right)\right)^{2\sigma}\left(M_{2,\mathcal{E},\eta} + 4l^2 + 4l^2\left(\frac{\sigma(p+2)}{\sigma se}\right)^{2\sigma} + M^2\right).$$

Proof of (T1.1). Case $q = 1$.

Using (A.64), for all $t \neq 0$, we obtain

$$\left(\widetilde{\omega}_N^{1,W,tx_0}\right)^2 \leq \left(\frac{2}{1 \vee (Rx_0\,|t|)}\right)^p \left(1 \bigvee \left(\frac{7\pi(N+1)}{Rx_0\,|t|}\right)\right)^{2N}. \tag{A.86}$$

and, using (A.73), we have

$$\mathbb{E}\left[\|R_2\|^2_{L^2(1\otimes W^{\otimes p})}\middle|\mathcal{G}_{n_0}\right] \tag{A.87}$$
$$\leq \frac{Z_{n_0}}{\pi^p}\int_{\epsilon\leq|t|\leq T}\left(L(t)^2 + \frac{c_{\boldsymbol{X}}(N+1)^p\,|t|^p}{n}\right)\left(\frac{1}{1 \vee |Rx_0 t|}\right)^p\left(1 \bigvee \left(\frac{7\pi(N+1)}{Rx_0|t|}\right)\right)^{2N} dt.$$

In order to collect the upper bounds on $\mathbb{E}\left[\|R_1\|_{L^2(1\otimes W^{\otimes p})}\middle|\mathcal{G}_{n_0}\right]$ and $\mathbb{E}\left[\|R_2\|_{L^2(1\otimes W^{\otimes p})}\middle|\mathcal{G}_{n_0}\right]$, we denote, for $z > 0$, $N \in \mathbb{N}$, and *a.e.* $t \neq 0$, by

$$\Delta_{2,q}(t,N,n,z) = \Delta_{3,q}(t,N)\frac{c_{\boldsymbol{X}}(N(t)p)^{p-1}}{(2\pi)^p(p-1)!(1-e^{-2})n}\left(\frac{|t|}{1 \vee (Rx_0\,|t|)}\right)^p + z\Delta_{3,q}(t,N)\left(\frac{1}{\pi(1 \vee (Rx_0\,|t|))}\right)^p\left(L(t)^2 + \frac{c_{\boldsymbol{X}}(N(t)+1)^p\,|t|^p}{n}\right), \tag{A.88}$$

and by

$$\Delta_{3,q}(t,N) = \frac{2}{\pi}\left(1 \bigvee \left(\frac{7\pi(N+1)}{Rx_0\,|t|}\right)\right)^{2N}. \tag{A.89}$$

Using Lemma A.13 (2), (A.76), and $(N(t)+p-1)^{p-1} \leq (pN(t))^{p-1}$, we have, for all $N \in \mathbb{N}_0$,

(A.90)

$$\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) \leq C\left(\int_{\epsilon\leq|t|\leq T} \Delta_{2,q}(t,N(t),n,Z_{n_0})dt + 4l^2 \sup_{t\in\mathbb{R}} N(t)^{-2\sigma} + \frac{4l^2}{(1\vee T)^{2s}} + M^2\epsilon\right).$$

Step 1. We show that, on $E\left(\mathcal{G}_{n_0},\mathcal{E},\eta\right) = \{Z_{n_0} \leq M_{1,\mathcal{E},\eta} v(n_0,\mathcal{E})/\delta(n_0)\}$ and for all $n_e \geq \exp(\exp(1))$, we have $\int_{\epsilon\leq|t|\leq T} \Delta_{2,q}(t,N(t),n,Z_{n_0})N(t)^{2\sigma}dt \leq M_{4,\mathcal{E},\eta}$, where

$$M_{4,\mathcal{E},\eta,1} = \frac{4\tau_2}{\pi^{p+1}(Rx_0)^p}\left(\frac{c_{\boldsymbol{X}}p^{p-1}}{2^p(p-1)!(1-e^{-2})} + M_{1,\mathcal{E},\eta}c_{\boldsymbol{X}}\left(\frac{p}{e}+1\right)^p\right) + 2^{p+2}\tau_2 M_{1,\mathcal{E},\eta}l^2$$

and $\tau_2 = 1 + (8\sigma/e)^{4\sigma}(1\vee(14\pi/(Rx_0))^2)$, from which we deduce that

$$\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) \leq C\left(\sup_{t\in\mathbb{R}} N(t)^{-2\sigma}\left(M_{4,\mathcal{E},\eta}+4l^2\right) + \frac{4l^2}{(1\vee T)^{2s}} + M^2\epsilon\right). \tag{A.91}$$

Using that $n_e/n \leq 1$ and $n_e v(n_0,\mathcal{E})/\delta(n_0) \leq 1$, we obtain for all $t>0$,

$$\begin{aligned}
&\left(1\vee\left(\frac{7\pi(N(t)+1)}{Rx_0\,|t|}\right)\right)^{2N(t)} N(t)^{2\sigma+p-1}\\
&\leq \mathbb{1}\{N(t)=1\}\left(1\vee\left(\frac{14\pi}{Rx_0\epsilon}\right)\right)^2\\
&\quad + \mathbb{1}\{N(t)>1\}\exp\left(4\left(\overline{N}(t)\ln\left(1\vee\left(\frac{7\pi(2\overline{N}(t)+1)}{Rx_0\,|t|}\right)\right) + \frac{2\sigma+p+1}{4}\ln\left(2\overline{N}(t)+1\right)\right)\right)\\
&\leq \mathbb{1}\{N(t)=1\}\left(1\vee\left(\frac{14\pi}{Rx_0}\right)\right)^2 n_e^{1/2}\left(\frac{\ln(n_e)^{4\sigma}}{n_e^{1/2}}\right) + \mathbb{1}\{N(t)>1\}n_e^{1/2} \quad \left(\text{by definition of } \overline{N}(t)\right),\\
&\leq \tau_2 n_e^{1/2}\left(\text{because} \quad \sup_{t\geq 1}\frac{\ln(t)^a}{t^b} = \left(\frac{a}{eb}\right)^a,\ a,b>0\right).
\end{aligned}$$

This yields, on $E\left(\mathcal{G}_{n_0},\mathcal{E},\eta\right)$ and for *a.e.* $t\neq 0$,

$$\begin{aligned}
\Delta_{2,q}(t,N(t),n,Z_{n_0})N(t)^{2\sigma} \leq &\frac{2}{\pi}\frac{\tau_2}{n_e^{1/2}}\frac{c_{\boldsymbol{X}}p^{p-1}}{(2\pi)^p(p-1)!(1-e^{-2})}\left(\frac{|t|}{1\vee(Rx_0\,|t|)}\right)^p\\
&+ \frac{2}{\pi}\frac{\tau_2}{n_e^{1/2}}\frac{M_{1,\mathcal{E},\eta}}{(\pi(1\vee(|t|\,Rx_0)))^p}\left(L(t)^2 + \frac{c_{\boldsymbol{X}}(N(t)+1)^p\,|t|^p}{n}\right).
\end{aligned}$$

Then, using that $n_e/n \leq 1$, $n_e v(n_0,\mathcal{E})/\delta(n_0) \leq 1$, and that

$$\int_\epsilon^T \frac{1}{(1\vee(tRx_0))^p}\frac{t^p(1+N(t))^p}{n}dt \leq \frac{T}{(Rx_0)^p}\frac{(\ln(n_e)+1)^p}{n} \left(\text{using } N(t)+1 \leq \overline{N}(t)+1 \leq \ln(n_e)+1\right)$$

$$\leq \frac{T}{(Rx_0)^p}\left(\frac{p}{e}+1\right)^p \left(\sup_{t\geq 1}\frac{\ln(t)^a}{t^b} = \left(\frac{a}{eb}\right)^a,\ a,b>0\right), \tag{A.92}$$

we obtain

$$\frac{\pi}{2}\int_{\epsilon\leq|t|\leq T}\Delta_2(t,N(t),n,Z_{n_0})N(t)^{2\sigma}dt$$

$$\leq \frac{2\tau_2 c_{\boldsymbol{X}} p^{p-1}T}{n_e^{1/2}(2\pi)^p(p-1)!(1-e^{-2})(Rx_0)^p} + \frac{\tau_2 M_{1,\mathcal{E},\eta}}{n_e^{1/2}\pi^p}\int_{\epsilon\leq|t|\leq T}L(t)^2dt + \frac{2c_{\boldsymbol{X}}\tau_2 M_{1,\mathcal{E},\eta}}{n_e^{1/2}\pi^p}\frac{T}{(Rx_0)^p}\left(\frac{p}{e}+1\right)^p.$$

Using that $\int_{\epsilon\leq|t|\leq T}L^2(t)dt \leq (2\pi)^{p+1}l^2$ and using that $T^{p+1} = n_e^{1/2}$ we obtain, for $t\neq 0$,

$$\begin{aligned}
&\int_{\epsilon\leq|t|\leq T}\Delta_{2,q}(t,N(t),n,Z_{n_0})N(t)^{2\sigma}dt\\
&\leq \frac{4\tau_2}{\pi^{p+1}}\left(\frac{c_{\boldsymbol{X}}p^{p-1}}{2^p(p-1)!(1-e^{-2})} + M_{1,\mathcal{E},\eta}c_{\boldsymbol{X}}\left(\frac{p}{e}+1\right)^p\right)\frac{1}{(Rx_0)^p} + 2^{p+2}\tau_2 M_{1,\mathcal{E},\eta}l^2\\
&= M_{4,\mathcal{E},\eta,1},
\end{aligned}$$

which yields (A.91).

Step 2. Using $\sup_{|t|\in[\epsilon,T]}N(t)^{-2\sigma} = N(\epsilon)^{-2\sigma} \leq \overline{N}(\epsilon)^{-2\sigma}$ and (A.91), we obtain on $E(\mathcal{G}_{n_0},\mathcal{E},\eta)$,

$$\sup_{\substack{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}_{w,W}^{q,\phi,\omega}(l,M)\cap\mathcal{D}\\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}}\mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon},f_{\alpha,\boldsymbol{\beta}}\right) \leq C\overline{N}(\epsilon)^{-2\sigma}\left(M_{4,\mathcal{E},\eta,1}+4l^2+\frac{4l^2\overline{N}(\epsilon)^{2\sigma}}{(1\vee T)^{2s}}+M^2\epsilon\overline{N}(\epsilon)^{2\sigma}\right).$$

Because $\epsilon = (\ln(n_e)/\ln_2(n_e))^{-2\sigma}$, then

$$\begin{aligned}
&\overline{N}(\epsilon)\ln\left(\frac{7\pi(2\overline{N}(\epsilon)+1)}{Rx_0\epsilon}\right)+\frac{2\sigma+p-1}{4}\ln\left(2\overline{N}(\epsilon)+1\right)\\
&\leq \overline{N}(\epsilon)\ln\left(\frac{7\pi}{Rx_0\epsilon}\right)+\left(\frac{2\sigma+p-1}{4}+\overline{N}(\epsilon)\right)\ln\left(2\overline{N}(\epsilon)+1\right)\\
&\leq 2\left(\left(2\sigma\overline{N}(\epsilon)\ln\left(\left(\frac{7\pi}{Rx_0}\right)^{1/2\sigma}\frac{\ln(n_e)}{\ln_2(n_e)}\right)\right)\bigvee\left(1+\frac{2\sigma+p-1}{4}\right)(2\overline{N}(\epsilon)+1)\ln\left(2\overline{N}(\epsilon)+1\right)\right)
\end{aligned}$$

which yields $\overline{N}(\epsilon) \geq \widetilde{N}\wedge\widetilde{N}_1$ where

$$2\left(1+\frac{2\sigma+p-1}{4}\right)\left(2\widetilde{N}+1\right)\ln\left(2\widetilde{N}+1\right) = \frac{\ln(n_e)}{8k_q}$$

and

$$4\sigma\widetilde{N}_1 \ln\left(\left(\frac{7\pi}{Rx_0}\right)^{1/2\sigma}\frac{\ln(n_e)}{\ln_2(n_e)}\right) = \frac{\ln(n_e)}{8k_q}.$$

Denoting by $\tau_{2,q} = 4k_q(3+2\sigma+p)$, by $\mathcal{W}$ the W-Lambert function (satisfying $e^{W(z)}W(z) = z$ for $z \in [0,\infty)$), and using that $\mathcal{W}(x) \leq \ln(x+1)$ for $x > 0$ (see Theorem 2.3 in [30]), we have

$$\begin{aligned} 2\widetilde{N}+1 \geq & \frac{\ln(n_e)}{\tau_{2,1}\mathcal{W}\left(\frac{\ln(n_e)}{\tau_{2,1}}\right)} \\ \geq & \frac{\ln(n_e)}{\tau_{2,1}\ln\left(\ln(n_e)+\tau_{2,1}\right)} \\ \geq & \frac{\ln(n_e)}{\tau_{2,1}\left(1+\ln(1+\tau_{2,1})\right)\ln_2(n_e)} \quad (\text{for } n_e \geq \exp(\exp(1))). \end{aligned} \tag{A.93}$$

Moreover, denoting by $\tau_{3,q} = 4\sigma k_q\left(1+\left(\ln\left((7\pi/(Rx_0))^{1/2\sigma}\right)\vee 1\right)\right) > 0$, we have

$$\begin{aligned} \widetilde{N}_1 \geq & \frac{\ln(n_e)}{\ln_2(n_e)}\frac{1}{4\sigma k_q\left(1+\left(\ln\left((7\pi/(Rx_0))^{1/2\sigma}/\ln_2(n_e)\right)\vee 1\right)/\ln_2(n_e)\right)} \\ \geq & \frac{\ln(n_e)}{\tau_{3,1}\ln_2(n_e)}. \end{aligned}$$

Thus, using (A.93) and denoting by $\tau_{4,q} = \tau_{2,q}(1+\ln(1+\tau_{2,q}))$, we have

$$\left(1+\overline{N}(\epsilon)\right)^{-2\sigma} \leq \left(\frac{\ln(n_e)}{\tau_{4,1}\ln_2(n_e)} \bigwedge \frac{\ln(n_e)}{\tau_{3,1}\ln_2(n_e)}\right)^{-2\sigma}.$$

Then, using $\overline{N}(\epsilon) \leq N(\epsilon)$, hence $2^{-2\sigma}(1\vee N(\epsilon))^{-2\sigma} \leq \left(1+\overline{N}(\epsilon)\right)^{-2\sigma}$, and $N(\epsilon) \geq 1$ we obtain

$$N(\epsilon)^{-2\sigma} \leq \left(\frac{1}{2}\left(\frac{1}{\tau_{4,1}} \bigwedge \frac{1}{\tau_{3,1}}\right)\frac{\ln(n_e)}{\ln_2(n_e)}\right)^{-2\sigma}. \tag{A.94}$$

For $n_e \geq 1$, we have

$$\begin{aligned} \frac{N(\epsilon)^{2\sigma}}{(1\vee T)^{2s}} \leq & \frac{\ln(n_e)^{2\sigma}}{n_e^{s/(p+1)}} \\ \leq & \left(\frac{2\sigma(p+1)}{se}\right)^{2\sigma} \left(\text{because } \sup_{t\geq 1}\frac{\ln(t)^a}{t^b} = \left(\frac{a}{eb}\right)^a \text{ for } a,b>0\right). \end{aligned}$$

Moreover, denoting by $\theta = 7\pi/(Rx_0)$ and using that

$$\overline{N}(\epsilon)\ln\left(\frac{\theta\left(2\overline{N}(\epsilon)+1\right)}{\epsilon}\right) + \frac{2\sigma+p-1}{4}\ln\left(2\overline{N}(\epsilon)+1\right) \geq \overline{N}(\epsilon)\ln\left(\frac{2\theta\overline{N}(\epsilon)}{\epsilon}\right),$$

we have, similarly to (A.93) and using that for $x \geq e$, $\mathcal{W}(x) \geq \ln(x) - \ln_2(x)$ for $x > 0$ (see Corollary 2.4 in [30]), for all n_e such that $n_e \geq \exp(2ek_1(\ln(n_e)/\ln_2(n_e))^{-2\sigma}/\theta) \vee e^e$

$$\begin{aligned}\overline{N}(\epsilon) &\leq \frac{\ln(n_e)}{4k_1\mathcal{W}\left(\ln(n_e)\theta/(2k_1\epsilon)\right)} \\ &\leq \frac{\ln(n_e)}{4k_1\ln_2(n_e)} \frac{1}{(1+\ln(\theta/(2k_1\epsilon))/\ln_2(n_e) - \ln_2(\ln(n_e)\theta/(2k_1\epsilon)/\ln_2(n_e))},\end{aligned}$$

hence there exists $\tau_{5,1} > 0$ such that for n_e large enough we have $\epsilon\overline{N}(\epsilon)^{2\sigma} \leq 1/(4k_1\tau_{5,1})^{2\sigma}$. Thus, we obtain on $E\left(\mathcal{G}_{n_0}, \mathcal{E}, \eta\right)$

$$\left(\frac{\ln(n_e)}{\ln_2(n_e)}\right)^{2\sigma} \sup_{\substack{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}_{w,W}^{q,\phi,\omega}(l,M)\cap\mathcal{D} \\ f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}}} \mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right) \leq M_{5,\mathcal{E},\eta,q}, \tag{A.95}$$

where

$$M_{5,\mathcal{E},\eta,1} = C\left(\frac{1}{2\tau_{4,1}} \bigwedge \frac{1}{2\tau_{3,1}}\right)^{-2\sigma}\left(4l^2\left(1+\left(\frac{2\sigma}{se}\right)^{2\sigma}\right) + M_{4,\mathcal{E},\eta} + \frac{M^2}{(4k_1\tau_{5,1})^{2\sigma}}\right) \tag{A.96}$$

which yields the result. Similar computations in the case $q = \infty$ using

$$\left(\widetilde{\omega}_N^{\infty,W,tx_0}\right)^2 \leq \left(\frac{2}{1\vee(|t|\,Rx_0)}\right)^p \left(1 \bigvee \left(\frac{7\pi(N+1)}{|t|\,Rx_0}\right)\right)^{2pN}.$$

also yield (A.95). □

Proof of Theorem 3. Take $\mathcal{E}$ and $\eta > 0$.

Proof of (T3.1.1). Case $q = 1$. Using (A.76), (A.88), and (A.89) we obtain

$$\begin{aligned}&\mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right) \\ &\leq C\left(\int_{\epsilon\leq|t|\leq T} \Delta_{2,q}(t, N(t), n, Z_{n_0})dt + 4l^2 \sup_{t\in\mathbb{R}} e^{-2\kappa(N(t)\ln(1+N(t)))^r} + \frac{4l^2}{(1\vee T)^{2s}} + \frac{M^2}{w(\underline{a})}\right).\end{aligned} \tag{A.97}$$

Hereafter, we denote by $\theta = 7\pi/(Rx_0)$. Using Lemma A.14, there exists $M_{1,\mathcal{E},\eta} \in \mathbb{R}$, such that, for all $n_0 \in \mathbb{N}$, $\mathbb{P}\left(E\left(\mathcal{G}_{n_0}, \mathcal{E}, \eta\right)\right) \geq 1-\eta$, where $E\left(\mathcal{G}_{n_0}, \mathcal{E}, \eta\right) = \{Z_{n_0} \leq M_{1,\mathcal{E},\eta} v(n_0, \mathcal{E})/\delta(n_0)\}$. Using (A.88), $n_e/n \leq 1$, $n_e v(n_0, \mathcal{E}))/\delta(n_0) \leq 1$, and that, for $t \neq 0$,

$$\begin{aligned}&\left(1 \vee \left(\frac{7\pi(1+N(t))}{Rx_0\,|t|}\right)\right)^{2N(t)} e^{2\kappa N(t)\ln(1+N(t))} \\ &\leq \exp\left(2\overline{N}(t)\ln\left(1\vee\left(\frac{7\pi(1+\overline{N}(t))}{Rx_0\,|t|}\right)\right) + 2\kappa\overline{N}(t)\ln\left(1+\overline{N}(t)\right)\right)\end{aligned}$$

$$\leq n_e \text{ (by definition of } \overline{N}(t)),$$

we have, on $E(\mathcal{G}_{n_0}, \mathcal{E}, \eta)$,

$$\frac{\Delta_{2,q}(t, N(t), n, Z_{n_0}) e^{2\kappa N(t)\ln(1+N(t))}}{(1 \vee N(t))^{p-1}(((1+\overline{N}(T))\theta) \vee |t|)^2}$$
$$\leq \frac{2}{\pi(\theta \vee |t|)^2 (1 \vee (|t| \, Rx_0))^p} \left(\frac{c_{\boldsymbol{X}} p^{p-1} |t|^p}{(2\pi)^p (p-1)!(1-e^{-2})} + \frac{M_{1,\mathcal{E},\eta}}{\pi^p} \left(L(t)^2 + \frac{c_{\boldsymbol{X}} (N(t)+1)^p \, |t|^p}{n} \right) \right).$$

Hence, on $E(\mathcal{G}_{n_0}, \mathcal{E}, \eta)$ and using that $(N(t)+1)^p/n \leq (\ln(n_e)+1)^p/n \leq (p/e+1)^p$, we have

$$\int_\epsilon^T \frac{(N(t)+1)^p t^p}{(\theta \vee t)^2 (1 \vee (tRx_0))^p n} dt \leq \frac{(\ln(n_e)+1)^p}{n(Rx_0)^p} \int_\epsilon^T \frac{1}{(\theta \vee t)^2} dt$$
$$\leq \frac{2}{\theta (Rx_0)^p} \left(\frac{p}{e} + 1 \right)^p \left(\text{using } \sup_{t \geq 1} \frac{\ln(t)^a}{t^b} = \left(\frac{a}{eb} \right)^a, \; a, b > 0 \right).$$

Then, we obtain using that $\int_{\epsilon \leq |t| \leq T} L^2(t) dt \leq (2\pi)^{p+1} l^2$,

$$\int_{\epsilon \leq |t| \leq T} \frac{\Delta_{2,q}(t, N(t), n, Z_{n_0}) e^{2\kappa N(t)\ln(1+N(t))}}{(1 \vee N(t))^{p-1} \left(\left(1 + \overline{N}(T)\right) \theta \right) \vee |t|)^2} dt \leq M_{6,\mathcal{E},\eta,q},$$

where

$$M_{6,\mathcal{E},\eta,1} = \frac{8}{\theta \pi (Rx_0)^p} \frac{c_{\boldsymbol{X}} p^{p-1}}{(2\pi)^p (p-1)!(1-e^{-2})}$$
$$+ 4 M_{1,\mathcal{E},\eta} \left(2^p l^2 + \frac{c_{\boldsymbol{X}}}{\pi^{p+1} (Rx_0)^p} \frac{2}{\theta} \left(\frac{p}{e} + 1 \right)^p \right).$$

Thus, we obtain in (A.97)

$$\mathcal{R}_{n_0}^W \left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}, f_{\alpha,\boldsymbol{\beta}} \right)$$

(A.98)

$$\leq C \left(\sup_{|t| \in [\epsilon, T]} \frac{(1 \vee N(t))^{p-1} \left(\left(\left(1 + \overline{N}(T)\right) \theta \right) \vee |t| \right)^2}{e^{2\kappa (N(t) \ln(1+N(t)))}} \left(M_{6,\mathcal{E},\eta} + 4l^2 \right) + \frac{4l^2}{(1 \vee T)^{2s}} + \frac{M^2}{w(\underline{a})} \right).$$

Using that for $t \neq 0$, $\overline{N}(t) - 1 \leq N(t) \leq \overline{N}(t)$, we obtain, for $n_e \in \mathbb{N}$,

$$\frac{\exp\left(2\kappa N(t) \ln(1+N(t))\right)}{(1+N(t))^{p-1}} \geq \frac{\exp\left(2\kappa \overline{N}(t) \ln(1+N(t)) - 2\kappa \ln(1+\overline{N}(t))\right)}{(1+N(t))^{p-1}}$$
$$\geq \frac{\exp\left(2\kappa \overline{N}(t) \ln\left(1+\overline{N}(t)\right)\right)}{(1+\overline{N}(t))^{2\kappa+p-1}} \exp\left(-2\kappa \overline{N}(t) \ln\left(1 + \frac{1}{\overline{N}(t)} \right) \right)$$
$$\geq \frac{\exp\left(2\kappa \overline{N}(t) \ln\left(1+\overline{N}(t)\right)\right)}{e^{2\kappa} (1+\overline{N}(t))^{2\kappa+p-1}} \text{ (using } \ln(1+x) \leq x)$$

$$\geq \frac{\exp\left(2\kappa\overline{N}(t)\ln\left(1+\overline{N}(t)\right)\right)}{\sup_{t\in[\epsilon,T]} e^{2\kappa}(1+\overline{N}(t))^{2\kappa+p-1}}.$$

Thus, denoting by

$$g : t \in [\epsilon, T] \mapsto \frac{\left(\left(\left(1+\overline{N}(T)\right)\theta\right)\vee|t|\right)^2}{e^{2\kappa(\overline{N}(t)\ln(1+\overline{N}(t)))}},$$

we have

$$\sup_{|t|\in[\epsilon,T]} \frac{(1\vee N(t))^{p-1}\left(\left(\left(1+\overline{N}(T)\right)\theta\right)\vee|t|\right)^2}{e^{2\kappa(N(t)\ln(1+N(t)))}} \leq \left(\sup_{t\in[\epsilon,T]} e^{2\kappa}(1+\overline{N}(t))^{2\kappa+p-1}\right)\sup_{|t|\in[\epsilon,T]} g(t)$$

$$\leq \left(\sup_{t\in[\epsilon,T]} e^{2\kappa}(1+\overline{N}(t))^{2\kappa+p-1}\right) g(\epsilon), \tag{A.99}$$

because we show below that $\|g\|_{L^\infty([\epsilon,T])} = g(\epsilon)$. Indeed, for $x \in (0,\infty)$, differentiating $Q^{-1}_{t,u}(Q_{t,u}(x)) = x$ with respect to t yields

$$\frac{\partial Q^{-1}_{t,u}}{\partial t}(Q_{t,u}(x)) + \frac{\partial Q^{-1}_{t,u}}{\partial x}(Q_{t,u}(x))\frac{\partial Q_{t,u}}{\partial t}(x) = 0$$

thus

$$\frac{\partial Q_{t,u}}{\partial t}(x) = \frac{-\frac{\partial Q^{-1}_{t,u}}{\partial t}(Q_{t,u}(x))}{\frac{\partial Q^{-1}_{t,u}}{\partial x}(Q_{t,u}(x))}.$$

Hence, for $t \neq 0$ such that $|t| < (1+\overline{N}(t))\theta$, we obtain

$$\overline{N}'(t) = \frac{\overline{N}(t)}{|t|\left((1+\kappa/k_q)\overline{N}(t)/(1+\overline{N}(t)) + \ln((1+\overline{N}(t))^2\theta/|t|)\right)}, \tag{A.100}$$

which is positive, while for $|t| \geq (1+\overline{N}(t))\theta$, $\overline{N}'(t) = 0$. Thus, we have, for $|t| \leq \left(1+\overline{N}(T)\right)\theta$,

$$g'(t) = -2\kappa\overline{N}'(t)\left(\ln(\overline{N}(t)+1) + \frac{\overline{N}(t)}{\overline{N}(t)+1}\right)\left(\left(1+\overline{N}(T)\right)\theta\right)^2 e^{-2\kappa\overline{N}(t)\ln(\overline{N}(t)+1)},$$

which yields that g is decreasing for $|t| \leq (1+\overline{N}(T))\theta$, and for $|t| \geq (1+\overline{N}(T))\theta$, $\overline{N}(\cdot)$ being independent of $|t|$, g is increasing. This yields, using that g is positive,

$$\|g\|_{L^\infty([\epsilon,T])} = g(\epsilon)\vee g(T). \tag{A.101}$$

Moreover, we have

$$k_q\overline{N}(\epsilon)\ln\left(\frac{7\pi\left(\overline{N}(\epsilon)+1\right)}{Rx_0\epsilon}\right) + \kappa\overline{N}(\epsilon)\ln\left(\overline{N}(\epsilon)+1\right) \geq (k_q+\kappa)\,\overline{N}(\epsilon)\ln\left(\overline{N}(\epsilon)+1\right) \tag{A.102}$$

which yields that $\overline{N}(\epsilon) \leq N_1$, where

$$N_1 \ln(N_1+1) = \frac{\ln(n_e)}{2k_q(k_q+\kappa)}. \tag{A.103}$$

Then, using that $T \geq \left(1+\overline{N}(T)\right)\theta$ (see (A.109)-(A.110)) and the definition of $\overline{N}(T)$, we have

$$\kappa \overline{N}(T) \ln\left(\overline{N}(T)+1\right) = \frac{\ln(n_e)}{2} \tag{A.104}$$

and with (A.102) and the definition of $\overline{N}(\epsilon)$, this yields

$$\frac{\ln(n_e)}{2} \geq (k_q+\kappa)\,\overline{N}(\epsilon)\ln\left(\overline{N}(\epsilon)+1\right) \tag{A.105}$$

hence using (A.104)-(A.105), we obtain

$$\begin{aligned} -\frac{1}{s} + \frac{\overline{N}(T)\ln\left(\overline{N}(T)+1\right)}{\overline{N}(\epsilon)\ln\left(\overline{N}(\epsilon)+1\right)} &\geq -\frac{1}{s} + \frac{k_q+\kappa}{\kappa} \\ &\geq 1 \quad \left(\text{using } s > \frac{\kappa}{k_q}\right). \end{aligned} \tag{A.106}$$

Thus we have

$$\begin{aligned} g(T) &= \exp\left(-2\kappa\overline{N}(\epsilon)\ln(\overline{N}(\epsilon)+1)\left[-\frac{1}{s}+\frac{\overline{N}(T)\ln(\overline{N}(T)+1)}{\overline{N}(\epsilon)\ln(\overline{N}(\epsilon)+1)}\right]\right) \\ &\leq \exp\left(-2\kappa\overline{N}(\epsilon)\ln(\overline{N}(\epsilon)+1)\right), \\ &\leq g(\epsilon), \end{aligned} \tag{A.107}$$

hence $\|g\|_{L^\infty([\epsilon,T])} = g(\epsilon)$.
Now, by definition of $\overline{N}$, for all $t \in [\epsilon, T]$, $\overline{N}(t) \leq \ln(n_e)/(2\kappa)$, hence we have

$$\sup_{t\in[\epsilon,T]} (1+\overline{N}(t))^{2\kappa+p-1} \leq \left(1 \bigvee \left(\frac{\ln(n_e)}{2\kappa}\right)\right)^{2\kappa+p-1},$$

which yields using (A.99)

$$\sup_{|t|\in[\epsilon,T]} \frac{(1\vee N(t))^{p-1}\left(\left(\left(\overline{N}(T)+1\right)\theta\right)\vee|t|\right)^2}{4e^{2\kappa N(t)\ln(1+N(t))}} \leq \frac{e^{2\kappa}\theta^2(1\vee(\ln(n_e)/(2\kappa)))^{2\kappa+p+1}}{4e^{2\kappa\overline{N}(\epsilon)\ln(\overline{N}(\epsilon)+1)}}. \tag{A.108}$$

Then, distinguishing the case $\ln(n_e)/\ln_2(n_e) \leq 1+N(\epsilon)$ from $\ln(n_e)/\ln_2(n_e) > 1+N(\epsilon)$ we have

$$\begin{aligned} &k_q\overline{N}(\epsilon)\ln\left(\frac{7\pi(\overline{N}(\epsilon)+1)}{Rx_0\epsilon}\right) + \kappa\overline{N}(\epsilon)\ln\left(\overline{N}(\epsilon)+1\right) \\ &\leq \left((\kappa+2k_q)\overline{N}(\epsilon)\ln\left(\frac{\ln(n_e)}{\ln_2(n_e)}\right)\right) \bigvee \left((\kappa+2k_q)\,\overline{N}(\epsilon)\ln\left(\overline{N}(\epsilon)+1\right)\right) \end{aligned}$$

which yields $\overline{N}(\epsilon) \geq \widetilde{N}_1 \wedge \widetilde{N}$, where

$$(\kappa + 2k_q)\widetilde{N}_1 \ln\left(\frac{\ln(n_e)}{\ln_2(n_e)}\right) = \frac{\ln(n_e)}{2} \tag{A.109}$$

and

$$(\kappa + 2k_q)\widetilde{N} \ln\left(\widetilde{N}+1\right) = \frac{\ln(n_e)}{2}. \tag{A.110}$$

This yields

$$e^{2\kappa\overline{N}(\epsilon)\ln\left(1+\overline{N}(\epsilon)\right)} \geq n_e^{\kappa/(\kappa+2k_q)} \left(1 \bigwedge \left(\exp\left(1 + \frac{\ln\left(\ln_2(n_e)/(2(\kappa+2k_q)\ln(\ln(n_e)/\ln_2(n_e)))\right)}{\ln(\ln(n_e)/\ln_2(n_e))}\right)\right)$$

hence for large enough n_e, using (A.108) on $E(n_0, \mathcal{E})$, we obtain

$$\begin{aligned}
&\frac{n_e^{\kappa/(\kappa+2k_q)}}{\ln(n_e)^{2\kappa+p+1}} \sup_{\substack{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}_{w,W}^{q,\phi,\omega}(l,M)\cap\mathcal{D} \\ f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}}} \mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right) \\
&\leq C\left(\frac{e^{2\kappa}\theta^2}{(2\kappa)^{2\kappa+p+1}}\left(4l^2 + M_{6,\mathcal{E},\eta,q}\right) + \frac{e^{2\kappa\overline{N}(\epsilon)\ln(1+\overline{N}(\epsilon))}}{\ln(n_e)^{2\kappa+p+1}}\left(\frac{4l^2}{(1\vee T)^{2s}} + \frac{M^2}{e^{2\gamma\underline{a}\ln(1+\underline{a})}}\right)\right).
\end{aligned}$$

Then, using $\underline{a}\epsilon = 7\pi/Rx_0$, the definitions of ϵ and $\overline{N}(\epsilon)$ for the first display, and that $\gamma > 1/(4k_q)$ for the second display, there exists $\widetilde{M}_{6,\mathcal{E},\eta}$ such that

$$\begin{aligned}
\exp\left(-2\gamma\underline{a}\ln(1+\underline{a}) + 2\kappa\overline{N}(\epsilon)\ln\left(1+\overline{N}(\epsilon)\right)\right) &\leq \exp\left(-2\gamma\frac{7\pi}{Rx_0\epsilon}\ln\left(\frac{7\pi}{Rx_0\epsilon}\right) + \frac{\ln(n_e)}{2k_q}\right) \\
&\leq \exp\left(\left(\frac{1}{2k_q} - 2\gamma\right)\ln(n_e) + 2\gamma\frac{\ln(n_e)\ln_3(n_e)}{\ln_2(n_e)}\right) \\
&\leq \widetilde{M}_{6,\mathcal{E},\eta}.
\end{aligned}$$

Denoting by $M_{7,\mathcal{E},\eta,1} = C\left(4l^2 + e^{2\kappa}\theta^2(2\kappa)^{-(2\kappa+p+1)}\left(4l^2 + M_{6,\mathcal{E},\eta,q}\right) + M^2\widetilde{M}_{6,\mathcal{E},\eta}\right)$, on the event $E(n_0, \mathcal{E})$ we obtain

$$\frac{n_e^{\kappa/(\kappa+2)}}{\ln(n_e)^{2\kappa+p+1}} \sup_{\substack{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}_{w,W}^{1,\phi,\omega}(l,M)\cap\mathcal{D} \\ f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}}} \mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{1,N,T\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right) \leq M_{7,\mathcal{E},\eta,1}.$$

Similar computations in the case $q = \infty$ yield

$$\frac{n_e^{\kappa/(\kappa+2p)}}{\ln(n_e)^{2\kappa+p+2}} \sup_{\substack{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}_{w,W}^{\infty,\phi,\omega}(l,M)\cap\mathcal{D} \\ f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}}} \mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{\infty,N,T\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right) \leq M_{7,\mathcal{E},\eta,\infty}.$$

Proof of (T3.1.2). Consider the case $q = 1$ and denote by $\theta = 7\pi/(Rx_0)$. Proceeding like in the proof of (16) in (T3.1) to obtain (A.98) and using that $w = W_{\mathcal{A}}$, we have, on the event $E(n_0, \mathcal{E})$,

$$\sup_{\substack{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}^{q,\phi,\omega}_{w,W}(l,M) \cap \mathcal{D} \\ f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}}} \mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right)$$

(A.111)

$$\leq 2(1 \vee (2C(\underline{a}, \epsilon))) \left(\sup_{\epsilon \leq |t| \leq T} \frac{(1 \vee N(t))^{p-1} \left(\left(\left(\overline{N}(T) + 1 \right) \theta \right) \vee |t| \right)^2}{e^{2\kappa (N(t) \ln(1+N(t)))}} \left(M_{6,\mathcal{E},\eta,q} + 4l^2 \right) + \frac{4l^2}{(1 \vee T)^{2s}} \right).$$

Then, using (A.101-A.107), that $\overline{N}(T) \ln(1 + \overline{N}(T)) = \ln(n_e)/(2\kappa)$, that

$$\overline{N}(\epsilon) \ln\left(1 + \overline{N}(\epsilon)\right) = \frac{\ln(n_e)}{2 + 2\kappa} - \frac{\overline{N}(\epsilon) \ln(\theta/\epsilon)}{1 + 2\kappa},$$

$\theta/\epsilon = 1$, and that $s > \kappa/k_q$, we obtain $\|g\|_{L^\infty([\epsilon,T])} = g(\epsilon)$. Thus, we have (A.108). Then, for $q \in \{1, \infty\}$, denoting by $M_{8,\mathcal{E},\eta,q} = 2(1 + C(\underline{a}, 1/\theta)) \left(4l^2 + e^{2\kappa}\theta^2 (2\kappa)^{-(2\kappa+p+1)} \left(M_{6,\mathcal{E},\eta,q} + 4l^2\right)\right)$, by definition of $\overline{N}(\epsilon)$, and using that (A.111) we have, for all n_e and on $E(\mathcal{G}_{n_0}, \mathcal{E})$,

$$\frac{n_e^{\kappa/(\kappa+1)}}{\ln(n_e)^{2\kappa+p+1}} \sup_{\substack{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}^{1,\phi,\omega}_{w,W}(l) \cap \mathcal{D} \\ f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}}} \mathcal{R}^W_{n_0,n}\left(\widehat{f}^{1,N,T\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) \leq M_{8,\mathcal{E},\eta,1}.$$

Proceeding similarly for the case $q = \infty$ yields

$$\frac{n_e^{\kappa/(\kappa+p)}}{\ln(n_e)^{2\kappa+p+2}} \sup_{\substack{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}^{\infty,\phi,\omega}_{w,W}(M) \cap \mathcal{D} \\ f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}}} \mathcal{R}^W_{n_0}\left(\widehat{f}^{\infty,N,T\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) \leq M_{8,\mathcal{E},\eta,\infty}.$$

Proof of (T3.2). Consider the case $q = 1$ and denote by $\theta = 7\pi/(Rx_0)$ and by $\widetilde{n}_e = n_e/\ln(n_e)^{p/r}$. We follow the arguments in [34]. Using (A.97), and denoting by

$$\Phi_{n_e} = \sup_{t \in \mathbb{R}:\ |t| \in [\epsilon,T]} \exp\left(2N \ln\left(1 \vee \left(\frac{(1+N)\theta}{|t|}\right)\right)\right) (1 \vee (Rx_0\,|t|))^2,$$

we have

$$\begin{aligned}
&\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) \\
&\leq C \left(\int_{\epsilon \leq |t| \leq T} \Delta_{2,q}(t, N, n, Z_{n_0}) dt + 4l^2 e^{-2\kappa (N \ln(1+N))^r} + \frac{4l^2}{\phi(T)} + \frac{M^2}{w(\underline{a})} \right)
\end{aligned}$$

$$\leq C\left(\frac{\Phi_{n_e}}{\widetilde{n}_e}\frac{4l^2\widetilde{n}_e}{\Phi_{n_e}e^{2\kappa(N\ln(1+N))^r}}+\frac{4l^2}{e^{\nu T}}\right.$$
$$\left.+\frac{\Phi_{n_e}}{\widetilde{n}_e}\int_{\epsilon\leq|t|\leq T}\frac{\Delta_{2,q}(t,N,n,Z_{n_0})\widetilde{n}_e}{(1\vee(Rx_0\,|t|))^2e^{2N\ln((1+N)\theta/|t|)}}dt+\frac{M^2}{e^{2\gamma(\underline{a}\ln(1+\underline{a}))^\rho}}\right)$$

Now, we study Φ_{n_e}. We use that

$$g_1: t\in(0,\infty)\mapsto\exp\left(2N\ln\left(1\vee\left(\frac{(1+N)\,\theta}{|t|}\right)\right)\right)(1\vee(Rx_0\,|t|))^2$$

is nonincreasing on $\left[\epsilon,(1+\overline{N})\theta\right]$ and increasing on $\left[(1+\overline{N})\theta,\infty\right)$, hence we obtain $\|g_1\|_{L^\infty([\epsilon,T])}=g_1(\epsilon)\vee g_1(T)$. Then, using that $N\leq\overline{N}+1$ and $1+N\leq 2\left(1+\overline{N}\right)$ for the first display and that $\theta/|t|\leq\theta/\epsilon=(1+\overline{N})/2$ for the second display, we have for $t\neq 0$,

$$\exp\left(2N\ln\left(1\vee\left(\frac{(1+N)\theta}{|t|}\right)\right)\right)\leq\exp\left(2\overline{N}\ln\left(1\vee\left(\frac{2\left(1+\overline{N}\right)\theta}{|t|}\right)\right)\right)\left(1\bigvee\left(\frac{2\theta\left(1+\overline{N}\right)}{|t|}\right)\right)^2$$
$$\leq\exp\left(2\overline{N}\ln\left(1\vee\left(\frac{2\left(1+\overline{N}\right)\theta}{|t|}\right)\right)\right)\left(1+\overline{N}\right)^3.$$

Thus, using $2\theta/\epsilon=\left(1+\overline{N}\right)$, $\overline{N}\leq(\ln(n_e)/(2\kappa))^{1/r}$, and $T^2=\exp\left(4\overline{N}\ln\left(\overline{N}+1\right)\right)$, we have

$$\Phi_{n_e}\leq\exp\left(4\overline{N}\ln\left(\overline{N}+1\right)\right)2\left(1\bigvee\left(\frac{\ln(n_e)^{3p/r}}{(2\kappa)^{3p/r}}\right)\right)(1\vee(Rx_0))^2. \tag{A.112}$$

Now, we show that there exists $\widetilde{D}<\infty$ such that

$$\exp\left(2N\ln\left(\frac{(N+1)\theta}{\epsilon}\right)\right)\exp\left(2\kappa(N\ln((N+1)))^r\right)\geq\widetilde{D}n_e. \tag{A.113}$$

Define d_i recursively by the equations $D_0=D_1=\cdots=D_k=0$, where $D_0=-d_0+4/(2\kappa)^{1/r}$, and for $1\leq i\leq k+1$,

$$D_i=-d_i+\frac{4(-1)^k}{(2\kappa)^{1/r}}\sum_{j=1}^{i}\frac{1/r\ldots(1/r-j+1)}{j!}\sum_{p_1+\cdots+p_j=i}d_{p_1-1}\ldots d_{p_j-1} \tag{A.114}$$

and $d_{k+1}=0$. Denote by $u_{n_e}=\sum_{i=0}^k d_i\ln\left(n_e\right)^{(i+1)/r-(i+1)}$ and, for $t\neq 0$, by

$$g\left(n_e,d,r,t\right)=\ln(n_e)-\sum_{i=0}^{k}d_i\ln\left(n_e\right)^{(i+1)/r-i}.$$

Using $\theta/\epsilon=\overline{N}+1$, that $N\geq\overline{N}$, and using the definition of $\overline{N}$, we obtain

$$\exp\left(2\kappa(N\ln((N+1))^r+2N\ln\left((N+1)\,\theta/\epsilon\right)\right)\geq\exp\left(2\kappa(\overline{N}\ln(\overline{N}+1))^r+4\overline{N}\ln\left(\overline{N}+1\right)\right)$$

$$\geq h(n_e, d, r, t),$$

where

$$h(n_e,d,r) = \exp\left(g\left(n_e,d,r\right) + \frac{4}{(2\kappa)^{1/r}} g\left(n_e,d,r\right)^{1/r}\right) \tag{A.115}$$
$$= n_e \exp\left(-\sum_{i=0}^{k} d_i \ln\left(n_e\right)^{(i+1)/r-i}\right)$$
$$\times \exp\left(\frac{4}{(2\kappa)^{1/r}} (\ln\left(n_e\right))^{1/r} \left(1-\nu(n_0,\mathcal{E},n)\right)\right),$$

with

$$\nu(n_0,\mathcal{E},n) = \frac{1}{r}u_{n_e} + \cdots + (-1)^k \frac{1/r\ldots(1/r-k)}{(k+1)!} u_{n_e}^{k+1} + o\left(u_{n_e}^{k+1}\right).$$

Thus, denoting by $\tau_4 = (k+1)r-(k+2)$, we have

$$h(n_e,d,r) = n_e \exp\left(1 + \sum_{i=1}^{k+1} D_i \ln\left(n_e\right))^{(i+1)r-i} + o\left((\ln\left(n_e\right))^{\tau_4}\right)\right) \tag{A.116}$$

which yields (A.113) with $\widetilde{D}$ depending only on $(D_i)_{i=1}^{k+1}$ thus on r, κ, and p. Then, we have on $E\left(\mathcal{G}_{n_0},\mathcal{E},\eta\right)$,

$$\frac{\Delta_{2,q}(t,N(t),n,Z_{n_0})\widetilde{n}_e}{(1\vee|Rx_0t|)^2 e^{2N\ln(1\vee((N+1)\theta/|t|))}}$$
$$\leq \frac{2}{\pi(1\vee(Rx_0\,|t|))^{p+2}}\left(\frac{c_{\boldsymbol{X}} p^{p-1}|t|^p}{(2\pi)^p(p-1)!(1-e^{-2})} + \frac{M_{1,\mathcal{E},\eta}}{\pi^p}\left(L(t)^2 + \frac{c_{\boldsymbol{X}}(N(t)+1)^p\,|t|^p}{n}\right)\right).$$

Hence, we obtain, on $E\left(\mathcal{G}_{n_0},\mathcal{E},\eta\right)$,

$$\int_{\epsilon\leq|t|\leq T} \frac{\Delta_{2,q}(t,N(t),n,Z_{n_0})\widetilde{n}_e}{(1\vee|Rx_0t|)^2 e^{2N\ln(1\vee((N+1)\theta/|t|))}} dt \leq M_{9,\mathcal{E},\eta,q},$$

where

$$M_{9,\mathcal{E},\eta,1} = \frac{4}{(2\pi)^p\pi(Rx_0)^{p+2}}\left(1+\frac{1}{p+1}\right)\left(\frac{c_{\boldsymbol{X}}p^{p-1}}{(p-1)!(1-e^{-2})} + 2^p M_{1,\mathcal{E},\eta}\left(1+\frac{p}{e}\right)^p c_{\boldsymbol{X}}\right)$$
$$+ 2^{p+2} M_{1,\mathcal{E},\eta} l^2.$$

Thus, we obtain

$$\mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right)$$
$$\leq \frac{Ce^{4\overline{N}\ln(1+\overline{N})}}{\tilde{n}_e \ln(n_e)^{-3p/r}}\left(2\left(1\vee\frac{1}{(2\kappa)^{3p/r}}\right)(1\vee(Rx_0))^2\left(4l^2 + M_{9,\mathcal{E},\eta,1} + M^2\right) + \frac{4l^2\ln(n_e)^{-3p/r}\tilde{n}_e}{e^{\nu T}\exp\left(4\overline{N}\ln(1+\overline{N})\right)}\right).$$

Then, using the definition of N for the first equality, then (A.115)-(A.116) for the second inequality yields that there exists $M_{10,\mathcal{E},\eta}$ such that

$$\begin{aligned}\frac{\widetilde{n}_e}{\exp\left(4\overline{N}\ln(1+\overline{N})\right)} &= \widetilde{n}_e \exp\left(-\frac{4}{(2\kappa)^{1/r}} g\left(n_e,d,r,t\right)^{1/r}\right)\\ &\geq \frac{M_{10,\mathcal{E},\eta}}{\ln(n_e)^{p/r}} \exp\left(g\left(n_e,d,r,t\right)\right)\\ &\geq \frac{M_{10,\mathcal{E},\eta} n_e}{\ln(n_e)^{p/r}} \exp\left(-\Sigma(n_e)\right), \end{aligned} \tag{A.117}$$

where

$$\Sigma(n_e) = \sum_{i=0}^{k} d_i \ln\left(n_e\right)^{(i+1)/r-i}.$$

Then, using that,

$$\begin{aligned}\frac{\widetilde{n}_e}{\exp\left(4\overline{N}\ln(1+\overline{N})\right)} &\leq \frac{\exp\left((2\kappa(\overline{N}\ln(1+\overline{N}))^r\right)}{\widetilde{D}\ln(n_e)^{p/r}} \quad \text{(using (A.113))}\\ &\leq \frac{n_e \exp\left(-\Sigma(n_e)\right)}{\widetilde{D}\ln(n_e)^{p/r}} \text{ (by definition of } \overline{N}),\end{aligned}$$

and using that there exists $\overline{D}$ depending only on $\{d_i\}_{i=1}^k$ such that $\exp\left(\Sigma(n_e)\right) \geq \exp\left(\overline{D}\ln(n_e)^{1/r}\right)$, we obtain

$$\begin{aligned}\frac{\widetilde{n}_e}{e^{\nu T}\exp\left(4\overline{N}\ln(1+\overline{N})\right)}\frac{1}{\ln(n_e)^{3p/r}} &\leq \frac{n_e\exp\left(-\Sigma(n_e)\right)}{\widetilde{D}\exp\left(\nu T\right)}\frac{1}{\ln(n_e)^{4p/r}}\\ &\leq \frac{n_e\exp\left(-\Sigma(n_e)\right)}{\widetilde{D}\exp\left(\nu\widetilde{D}\exp\left(\Sigma(n_e)\right)\right)}\frac{1}{\ln(n_e)^{4p/r}}\\ &\leq \frac{n_e\exp\left(-\Sigma(n_e)\right)}{\widetilde{D}\exp\left(\nu\widetilde{D}\exp\left(\overline{D}\ln(n_e)^{1/r}\right)\right)}\\ &\leq \frac{\exp\left(-\Sigma(n_e)\right)}{\widetilde{D}\ln(n_e)^{4p/r}} \leq \frac{1}{\widetilde{D}},\end{aligned}$$

for ν such that

$$\ln(\nu) \geq r\left(\ln\left(\frac{r}{\overline{D}}\right)-1\right) - \ln\left(\widetilde{D}\right), \tag{A.118}$$

where we use that $\sup_{n\geq e} \ln_2(n) - \overline{D}\ln(n)^{1/r} = r\left(\ln\left(r/\overline{D}\right) - 1\right)$. Thus, using (A.117), $\rho \geq r$, $\gamma > \kappa$, and the aforementioned choice of $\underline{a}$, we have, for all $n_e \geq 1$ and on $E(\mathcal{G}_{n_0}, \mathcal{E})$,

$$\frac{n_e \mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right)}{\ln(n_e)^{4p/r}\exp\left(\Sigma(n_e)\right)} \leq \frac{C}{M_{10,\mathcal{E},\eta}}\left(2\left(1 \vee \frac{1}{(2\kappa)^{3p/r}}\right)(1 \vee (Rx_0))^2\left(4l^2 + M_{9,\mathcal{E},\eta,1} + M^2\right) + \frac{4l^2}{\widetilde{D}}\right),$$

hence the result. Similar computations yield the result in the case $q = \infty$.

Proof of (T3.3). Denote by $\theta_1 = \pi/(4Rx_0)$. We start from equation (A.77), where, using that $w = W_{\mathcal{A}}$, we have, for all $N \in \mathbb{N}_0$,

$$\begin{aligned}(\text{A.119}) \quad & \mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) \\ & \leq 2(1 \vee (2C(\underline{a},\epsilon)))\left(\int_{\epsilon\leq|t|\leq T} \Delta_q(t, N(t), n, Z_{n_0})dt + 4l^2 \sup_{t\in\mathbb{R}} e^{-2\kappa N(t)} + \frac{4l^2}{(1\vee T)^{2s}}\right).\end{aligned}$$

Consider the case $q = 1$. Then, using (A.79), then for $|t| \neq 0$,

$$\begin{aligned}\frac{1}{n_e}\exp\left(\frac{\pi(N(t)+p)}{2|t|Rx_0} + 2\kappa N(t)\right) &\leq \frac{\exp\left(\pi p/(2Rx_0\epsilon)\right)}{n_e}\exp\left(2\left(\frac{\pi}{4|t|Rx_0} + \kappa\right)\overline{N}(t)\right) \\ &\leq \exp(2\pi p) \quad \left(\text{using the definition of } \overline{N}(t) \text{ and } \epsilon = \frac{\pi}{4Rx_0}\right),\end{aligned}$$

and using that $N(t) \leq \ln(n_e)/(2\kappa)$, we obtain, for $|t| > \epsilon$,

$$\begin{aligned}(\text{A.120}) \quad \frac{\Delta_1(t, N(t), n, Z_{n_0})e^{2\kappa N(t)}}{e^{2\pi p}((\ln(n_e)/(2\kappa)) \vee |t|)^{2p}} \leq & \left(\frac{8c_{\boldsymbol{X}}p^{p-1}\kappa}{\pi^{p+1}(p-1)!}\right)\left(1 \bigvee \frac{\sqrt{2}Rx_0}{\pi}\right) + \frac{2(2\kappa)^{2p}M_{1,\mathcal{E},\eta}}{\pi^{p+1}}\ L^2(t) \\ & + \frac{2(2\kappa)^p p^p c_{\boldsymbol{X}} M_{1,\mathcal{E},\eta}}{\pi^{p+1}n}.\end{aligned}$$

Thus, using (A.120) and , we obtain

$$(\text{A.121}) \qquad \int_{\epsilon\leq|t|\leq T} \frac{\Delta_1(t, N(t), n, Z_{n_0})e^{2\kappa N(t)}}{(\ln(n_e)/(2\kappa) \vee |t|)^{2(p+1)}}dt \leq M_{11,\mathcal{E},\eta,q},$$

where

$$M_{11,\mathcal{E},\eta,1} = e^{2\pi p}\left(\frac{8c_{\boldsymbol{X}}p^{p-1}\kappa}{\pi^{p+1}(p-1)!}\right)\left(1 \bigvee \frac{\sqrt{2}Rx_0}{\pi} + 2^{p+2}l^2(2\kappa)^{2p}M_{1,\mathcal{E},\eta} + \frac{2(2\kappa)^p p^p c_{\boldsymbol{X}} M_{1,\mathcal{E},\eta}}{\pi^{p+1}n}\right).$$

Then, we have

$$\begin{aligned}&\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) \\ &\leq 2(1 \vee (2C(\underline{a},\epsilon)))\left(\sup_{t\in[\epsilon,T]}\left(e^{-2\kappa N(t)}\left(\left(\frac{\ln(n_e)}{2\kappa}\right) \bigvee |t|\right)^{2(p+1)}\right)\left(4e^{2\kappa}l^2 + M_{11,\mathcal{E},\eta}\right) + \frac{4l^2}{(1\vee T)^{2s}}\right).\end{aligned}$$

The next step consists in showing that, denoting by $g : t \mapsto e^{-2\kappa\overline{N}(t)}(\ln(n_e)/(2\kappa) \vee |t|)^{2(p+1)}$, we have $\|g\|_{L^\infty([\epsilon,T])} = e^{-2\kappa\overline{N}(\epsilon)} \left(\ln(n_e)/(2\kappa)\right)^{2(p+1)}$. Indeed, using that, for $|t| \geq \epsilon$,

$$\overline{N}'(t) = \frac{\theta_1 k_q \ln(n_e)}{2(\kappa |t| + k_q\theta_1)^2},$$

where $\theta_1 = \pi/(4Rx_0)$, we have, for $t \in \mathbb{R}$,

$$g'(t) = \frac{1}{|t|}\left(2(p+1)\mathbb{1}\left\{|t| \geq \frac{\ln(n_e)}{2\kappa}\right\} - \frac{\kappa\theta_1 \ln(n_e)\,|t|}{(\kappa\,|t| + \theta_1)^2}\right) g(t).$$

Using that $t \in [0,\infty) \mapsto at/(bt+c)^2$ for $a, b, c > 0$ is decreasing for $t > c/b$ and increasing on $(0, c/b)$, for $t \geq \ln(n_e)/(2\kappa)$, we have

(1) if $\ln(n_e)/(2\kappa) \geq \theta_1/\kappa$, then using $(p+1) > 2\kappa\theta_1$

$$g'(t) \geq \frac{2}{|t|}\left((p+1) - \frac{4\kappa\theta_1}{(1+2\theta_1)^2}\right) g(t) > 0$$

(2) if $\ln(n_e)/(2\kappa) < \theta_1/\kappa$, then using $(p+1) > \kappa\theta_1/4$

$$g'(t) \geq \frac{1}{|t|}\left(2(p+1) - \frac{\kappa\theta_1}{2}\right) g(t) > 0$$

hence g' is positive for $t \geq \ln(n_e)/(2\kappa)$. Thus, g is positive and also decreasing on $[\epsilon, \ln(n_e)/(2\kappa)]$ and increasing on $[\ln(n_e)/(2\kappa), \infty)$ which yields

$$\|g\|_{L^\infty([\epsilon,T])} = \left(e^{-2\kappa\overline{N}(\epsilon)}\left(\frac{\ln(n_e)}{2\kappa}\right)^{2(p+1)}\right) \bigvee \left(e^{-2\kappa\overline{N}(T)} T^{2(p+1)}\right).$$

Using that $T^{2s} = e^{2\kappa\overline{N}(\epsilon)}/(Rx_0)^{2s}$, we have

$$e^{-2\kappa\overline{N}(T)} T^{2(p+1)} = \frac{1}{(Rx_0)^{2(p+1)}} e^{-2\kappa\overline{N}(\epsilon)} e^{-2\kappa N(T) + 2(p+1)\kappa N(\epsilon)/s + 2\kappa\overline{N}(\epsilon)}$$

and using $\overline{N}(t) - 1 \leq N(t) \leq \overline{N}(t)$, $\overline{N}(T) = \ln(n_e)/(2(\kappa + k_q\theta_1/T))$, $\overline{N}(\epsilon) = \ln(n_e)/(2(\kappa+k_q))$, and $\epsilon = \theta_1$ we have

$$\begin{aligned}
-2\kappa N(T) + 2(p+1)\kappa N(\epsilon)/s + 2\kappa\overline{N}(\epsilon) &\leq 2\kappa - 2\kappa\overline{N}(T) + 2(p+1)\kappa\overline{N}(\epsilon)/s + 2\kappa\overline{N}(\epsilon) \\
&\leq 2\kappa + \frac{\kappa \ln(n_e)}{\kappa + k_q}\left(\frac{p+1}{s} + 1\right) \\
&\leq 2\kappa + \frac{\kappa \ln(n_e)}{\kappa + k_q}\left(\frac{p+1}{s} + 1 - \frac{\kappa/k_q + 1}{\kappa/k_q + \pi/4}\right) \\
&\leq 2\kappa \quad \left(\text{using} \quad \frac{\kappa}{k_q} > \frac{\pi}{4}\left(1 + \frac{s}{p+1}\right) - 1\right).
\end{aligned}$$

This yields $\|g\|_{L^\infty([\epsilon,T])} = e^{-2\kappa\overline{N}(\epsilon)} \left(\ln(n_e)/(2\kappa)\right)^{2(p+1)}$.
Using $\overline{N}(t) - 1 \leq N(t) \leq \overline{N}(t)$, this yields

$$\frac{n_e^{\kappa/(\kappa+1)}}{\ln(n_e)^{2(p+1)}} \sup_{\substack{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}_{w,W}^{q,\phi,\omega}(l) \cap \mathcal{D} \\ f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}}} \mathcal{R}_{n_0}^W \left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{1,N,T,\epsilon}, f_{\alpha,\boldsymbol{\beta}} \right) \leq M_{12,\mathcal{E},\eta,q},$$

where $M_{12,\mathcal{E},\eta,1} = 2(1 \vee (2C(\underline{a},\epsilon)))e^{2\kappa}(2\kappa)^{-2(p+1)} \left(4(e^{2\kappa}+1)l^2 + M_{11,\mathcal{E},\eta,1}\right)$. Similar computations, considering

$$\int_{\epsilon \leq |t| \leq T} \frac{\Delta_\infty(t, N(t), n, Z_{n_0}) e^{2\kappa N(t)}}{((\ln(n_e)/(2\kappa) \vee |t|)^{p+2}} dt \leq M_{11,\mathcal{E},\eta,\infty}, \tag{A.122}$$

instead of (A.121) yield the result for $q = \infty$.

□

A.4. **Data-driven choice of the parameters.** For $t \in \mathbb{R} \setminus (-\epsilon, \epsilon)$ and arbitrary $T \in \mathcal{T}_n$ and $N \in \mathbb{N}_0$, we use in this section an abuse of notations and denote by $\widehat{F}_1^{q,\widehat{N},T,0}(t,\cdot)$ (resp. $\widehat{F}_1^{q,N,T,0}(t,\cdot)$ and $F_1^{q,N,T,0}(t,\cdot)$) the function

$$\widehat{F}_1^{q,\widehat{N}(t),T,0}(t,\cdot) = \mathbb{1}\{|t| \leq T\} \sum_{|\boldsymbol{m}|_q \leq \widehat{N}(t)} \frac{\widehat{c}_{\boldsymbol{m}}(t)}{\sigma_{\boldsymbol{m}}^{W,tx_0}} \varphi_{\boldsymbol{m}}^{W,tx_0}(\cdot)$$

(resp. $\mathbb{1}\{|t| \leq T\} \sum_{|\boldsymbol{m}|_q \leq N(t)} \widehat{c}_{\boldsymbol{m}}(t) \varphi_{\boldsymbol{m}}^{W,tx_0}(\cdot)/\sigma_{\boldsymbol{m}}^{W,tx_0}$ and $\mathbb{1}\{|t| \leq T\} \sum_{|\boldsymbol{m}|_q \leq N(t)} c_{\boldsymbol{m}}(t) \varphi_{\boldsymbol{m}}^{W,tx_0}(\cdot)/\sigma_{\boldsymbol{m}}^{W,tx_0}$), where the difference between the two notations will be clear from the text. Denote, for $t \in \mathbb{R}$ and $N \in \mathbb{N}_0$, by

$$\mathcal{R}_{0,q}(\mathcal{G}_{n_0}, N, t) = \mathbb{E}\left[\left\| \left(\widehat{F}_1^{q,N,T,0} - \mathcal{F}_{1st}\left[f_{\alpha,\boldsymbol{\beta}}\right] \right)(t,\cdot) \right\|_{L^2(W^{\otimes p})}^2 \middle| \mathcal{G}_{n_0} \right].$$

The upper bounds that we derive depend on the parameters of the class $\mathcal{H}_{w,W}^{q,\phi,\omega}(l,M)$. For $t \in [-T,T] \setminus [-\epsilon,\epsilon]$ and $N \in \mathbb{N}_0$, using the convexity of $x \mapsto x^2$, we have

$$\begin{aligned} &\mathcal{R}_{0,q}(\mathcal{G}_{n_0}, N, t) \\ &\leq \mathbb{E}\left[\Xi(t,N)|\mathcal{G}_{n_0}\right] + 3\left(\mathbb{E}\left[S_1(t,N)|\mathcal{G}_{n_0}\right] + \mathbb{E}\left[S_2(t,N)|\mathcal{G}_{n_0}\right] + \mathbb{E}\left[S_3(t,N)|\mathcal{G}_{n_0}\right]\right), \end{aligned} \tag{A.123}$$

where

$$\Xi : (t,N) \mapsto \sum_{\boldsymbol{m} \in \mathbb{N}_0^p :\ |\boldsymbol{m}|_q > N} \left| \frac{c_{\boldsymbol{m}}(t)}{\sigma_{\boldsymbol{m}}^{W,tx_0}} \right|^2,$$

$$S_1 : (t, N) \mapsto \sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_q\le N} \left| \frac{\mathbb{E}\left[\Delta_{\boldsymbol{m}}(t)|\mathcal{G}_{n_0}\right]}{\sigma_{\boldsymbol{m}}^{W,tx_0}} \right|^2,\ \Delta_{\boldsymbol{m}} : t \mapsto \widehat{c}_{\boldsymbol{m}}(t) - \widetilde{c}_{\boldsymbol{m}}(t),$$

$$S_2 : (t, N) \mapsto \sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_q\le N} \left| \frac{\Delta_{\boldsymbol{m}}(t) - \mathbb{E}\left[\Delta_{\boldsymbol{m}}(t)|\mathcal{G}_{n_0}\right]}{\sigma_{\boldsymbol{m}}^{W,tx_0}} \right|^2,$$

$$S_3 : (t, N) \mapsto \int_{\mathbb{R}^p} \left| \widetilde{F}_1^{q,N,T,0}(t, \boldsymbol{b}) - F_1^{q,N,T,0}(t, \boldsymbol{b}) \right|^2 W^{\otimes p}(\boldsymbol{b}) d\boldsymbol{b}$$

$$= \sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_q\le N} \left| \frac{\widetilde{\Delta}_{\boldsymbol{m}}(t)}{\sigma_{\boldsymbol{m}}^{W,tx_0}} \right|^2,\ \widetilde{\Delta}_{\boldsymbol{m}} : t \mapsto \widetilde{c}_{\boldsymbol{m}}(t) - c_{\boldsymbol{m}}(t).$$

Lemma A.15. Let $q \in \{1, \infty\}$, for $t \in [-T, T] \setminus (-\epsilon, \epsilon)$ with $0 < \epsilon < 1 < T < T_{\max} = 2^{K_{\max}}$, $\boldsymbol{m} \in \mathbb{N}_0^p$, $\widehat{N}(t)$ chosen from (18), and $N \in \{0, \dots, N_{\max,q}(W, t)\}$, the following inequalities hold

$$\mathbb{E}\left[S_1\left(t, \widehat{N}(t)\right)\middle|\mathcal{G}_{n_0}\right] \le Z_{n_0} \left\|\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t, \cdot)\right\|_{L^2(\mathbb{R}^p)}^2 |tx_0|^p \mathbb{E}\left[\nu_q\left(W, \widehat{N}(t), tx_0\right)\middle|\mathcal{G}_{n_0}\right], \tag{A.124}$$

$$\mathbb{E}\left[S_2\left(t, \widehat{N}(t)\right)\middle|\mathcal{G}_{n_0}\right] \le Z_{n_0} \left(\frac{|tx_0|}{2\pi}\right)^p \frac{c_{\boldsymbol{X}} \nu_q(W, N_{\max,q}(W, t), tx_0)}{n x_0^p}, \tag{A.125}$$

$$\mathbb{E}\left[\left(S_3(t, N) - \frac{\Sigma(t, N)}{2(2 + c_0)}\right)_+ \middle|\mathcal{G}_{n_0}\right] \le \frac{48 c_{\boldsymbol{X}} |t|^p \nu_q(W, N, tx_0)}{(2\pi)^p n} \Psi_{0,n}(t), \tag{A.126}$$

where

$$\Psi_{0,n} : t \mapsto x_0^p \exp(-p_n) + \frac{294 c_{\boldsymbol{X}}^2 K_n^2(t)}{x_0^p n} \exp\left(-\frac{K_1\sqrt{p_n n}}{K_n(t)}\right)$$

$$K_n : t \mapsto H_W(t) \left(N_{\max,q}(W, t) + \frac{1}{2}\right)^p,$$

$p_n = (2\ln(n)) \vee 3$, $K_1 = \sqrt{2x_0^p/3}/(42\sqrt{c_{\boldsymbol{X}}})$, and $H_W(t)$ is defined in Proposition A.2.

Proof of Lemma A.15.

Proof of (A.124). For $\boldsymbol{m} \in \mathbb{N}_0^p$, we have, like in (A.71),

$$\left|\mathbb{E}\left[\Delta_{\boldsymbol{m}}(t)|\mathcal{G}_{n_0}\right]\right|^2 = \left| \int_{[-1,1]^p} \mathcal{F}\left[f_{Y|\mathcal{X},\boldsymbol{X}=x_0\boldsymbol{u}}\right](t) \left(\frac{f_{\boldsymbol{X}|\mathcal{X}}}{\widehat{f}_{\boldsymbol{X}|\mathcal{X}}^{\delta}}(x_0\boldsymbol{u}) - 1\right) g_{\boldsymbol{m}}^{W,tx_0}(\boldsymbol{u}) d\boldsymbol{u} \right|^2$$

$$\le \left\| \mathcal{F}\left[f_{Y|\mathcal{X},\boldsymbol{X}=x_0\cdot}\right](t) \left(\frac{f_{\boldsymbol{X}|\mathcal{X}}}{\widehat{f}_{\boldsymbol{X}|\mathcal{X}}^{\delta}}(x_0\cdot) - 1\right) \right\|_{L^2([-1,1]^p)}^2$$

$$\le Z_{n_0} (2\pi)^p \left\|\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t, \cdot)\right\|_{L^2(\mathbb{R}^p)}^2. \tag{A.127}$$

Then, using Lemma A.13-(2), we obtain

$$\sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_q\le N}\frac{1}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2}\le\frac{|tx_0|^p}{(2\pi)^p}\nu_q(W,N,tx_0), \tag{A.128}$$

hence the result.

Proof of (A.125)**.** Using (A.128) and (A.72) for the second display, we have

$$\begin{aligned}\mathbb{E}\left[S_2\left(t,\widehat{N}(t)\right)\middle|\mathcal{G}_{n_0}\right]&\le\sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_q\le N_{\max,q}(W,t)}\frac{\mathbb{E}\left[\left|\Delta_{\boldsymbol{m}}(t)-\mathbb{E}\left[\Delta_{\boldsymbol{m}}(t)|\mathcal{G}_{n_0}\right]\right|^2\middle|\mathcal{G}_{n_0}\right]}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2}\\&\le\frac{c_{\boldsymbol{X}}Z_{n_0}}{nx_0^p}\frac{|tx_0|^p}{(2\pi)^p}\nu_q(W,N_{\max,q}(W,t),tx_0).\end{aligned}$$

Proof of (A.126)**.** For $N\in\{0,\dots,N_{\max,q}(W,t)\}$ and $t\in[-T,T]\setminus[-\epsilon,\epsilon]$, we have

$$\begin{aligned}\sum_{|\boldsymbol{m}|_q\le N}\frac{\left|\widetilde{\Delta}_{\boldsymbol{m}}(t)\right|^2}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2}&=\int_{\mathbb{R}^p}\left|\widetilde{F}_1^{q,N,T,0}(t,\boldsymbol{b})-F_1^{q,N,T,0}(t,\boldsymbol{b})\right|^2W^{\otimes p}(\boldsymbol{b})d\boldsymbol{b}\\&=\sup_{u\in\mathcal{U}}\left|\nu_n^t(u)\right|^2,\end{aligned} \tag{A.129}$$

where

$$\begin{aligned}\nu_n^t(u)&=\left\langle\widetilde{F}_1^{q,N,T,0}(t,\cdot)-F_1^{q,N,T,0}(t,\cdot),u(\cdot)\right\rangle_{L^2(W^{\otimes p})}\\&=\frac{1}{n}\sum_{j=1}^n\left(f_u^t(Y_j,X_j)-\mathbb{E}\left[f_u^t(Y_j,X_j)\right]\right),\\f_u^t(\star,\cdot)&=\int_{\mathbb{R}^p}\sum_{|\boldsymbol{m}|_q\le N(t)}\frac{e^{it\star}}{x_0^pf_{\boldsymbol{X}|\mathcal{X}}(\cdot)}\overline{g_{\boldsymbol{m}}^{W,tx_0}}\left(\frac{\cdot}{x_0}\right)\frac{\varphi_{\boldsymbol{m}}^{W,tx_0}(\boldsymbol{b})}{\sigma_{\boldsymbol{m}}^{W,tx_0}}\overline{u}(\boldsymbol{b})W^{\otimes p}(\boldsymbol{b})d\boldsymbol{b}\end{aligned}$$

and $\mathcal{U}$ is countable dense class of functions of $\left\{u:\ \|u\|_{L^2(W^{\otimes p})}=1\right\}$. Then, we obtain

$$\mathbb{E}\left[\left(S_3(t,N)-\frac{\Sigma(t,N)}{2(2+c_0)}\right)_+\middle|\mathcal{G}_{n_0}\right]=\mathbb{E}\left[\left(\sup_{u\in\mathcal{U}}\left|\nu_n^t(u)\right|^2-\frac{\Sigma(t,N)}{2(2+c_0)}\right)_+\middle|\mathcal{G}_{n_0}\right].$$

We now verify the conditions of the Talagrand inequality given in Lemma A.16.

Condition (A.130). For $u\in\mathcal{U}$, using Proposition A.2 and the Cauchy-Schwarz inequality, we have

$$\left\|f_u^t\right\|_{L^\infty(\mathbb{R}\times\mathcal{X})}$$

$$\leq \left\| \left(\int_{\mathbb{R}^p} \sum_{|\boldsymbol{m}|_q \leq N(t)} \frac{|t|^p}{(2\pi)^p f^2_{\boldsymbol{X}|\mathcal{X}}(\cdot)\rho^{W,tx_0}_{\boldsymbol{m}} x_0^p} \left| \overline{g^{W,tx_0}_{\boldsymbol{m}}} \left(\frac{\cdot}{x_0} \right) \right|^2 \left| \varphi^{W,tx_0}_{\boldsymbol{m}}(\boldsymbol{b}) \right|^2 W^{\otimes p}(\boldsymbol{b}) d\boldsymbol{b} \right)^{1/2} \right\|_{L^\infty(\mathcal{X})} \|u\|_{L^2(W^{\otimes p})}$$

$$\leq \frac{H_W(t)(N_{\max,q}(W,t)+1/2)^p \left|t\right|^{p/2} c_{\boldsymbol{X}}}{(2\pi x_0)^{p/2}} \left(\sum_{|\boldsymbol{m}|_q \leq N} \frac{1}{\rho^{W,tx_0}_{\boldsymbol{m}}} \right)^{1/2}$$

$$\leq M(t,N),$$

where $M(t,N) = K_n(t)\left|t\right|^{p/2} c_{\boldsymbol{X}} \left(\nu_q(W,N,tx_0)\right)^{1/2} / (2\pi x_0)^{p/2}$.

Condition (A.131). Using (A.69) we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{u\in\mathcal{U}} \left|\nu^t_n(u)\right|^2 \right] &\leq \mathbb{E}\left[\sup_{u\in\mathcal{U}} \left\| \widetilde{F}^{q,N,T,0}_1(t,\cdot) - F^{q,N,T,0}_1(t,\cdot) \right\|^2_{L^2(W^{\otimes p})} \|u\|^2_{L^2(W^{\otimes p})} \right] \\
&\leq \frac{\Sigma(N,t)}{8(2+c_0)p_n} \\
&= \frac{6c_{\boldsymbol{X}}\left|tx_0\right|^p}{n(2\pi)^p} \nu_q(W,N,tx_0) \\
&= H^2.
\end{aligned}$$

Condition (A.132). We have, using the Cauchy-Schwarz inequality and that $\left(\varphi^{W,tx_0}_{\boldsymbol{m}}\right)_{\boldsymbol{m}\in\mathbb{N}^p_0}$ is an orthonormal basis of $L^2\left(W^{\otimes p}\right)$ for the second display and Lemma A.13 for the third display

$$\begin{aligned}
\operatorname{Var}\left(\mathfrak{Re}(f^t_u(Y_j,X_j))\right) \vee \operatorname{Var}\left(\mathfrak{Im}(f^t_u(Y_j,X_j))\right) &\leq \int_{\mathbb{R}\times\mathcal{X}} \left|f^t_u(y,\boldsymbol{x})\right|^2 f_{Y,\boldsymbol{X}}(y,\boldsymbol{x}) dy d\boldsymbol{x} \\
&\leq c_{\boldsymbol{X}} \sum_{|\boldsymbol{m}|_q \leq N} \frac{\left|tx_0\right|^p}{(2\pi)^p \rho^{W,tx_0}_{\boldsymbol{m}}} \int_{[-1,1]^p} \left|g^{W,tx_0}_{\boldsymbol{m}}(\boldsymbol{x})\right|^2 d\boldsymbol{x} \\
&\leq \frac{c_{\boldsymbol{X}}\left|tx_0\right|^p}{(2\pi)^p} \nu_q(W,N,tx_0).
\end{aligned}$$

Then, using Lemma A.16 yields, with $\eta = p_n$,

$$\begin{aligned}
&\mathbb{E}\left[\left(\sup_{u\in\mathcal{U}} \left|\nu^t_n(u)\right|^2 - \frac{\Sigma(t,N)}{2(2+c_0)} \right)_+ \right] \\
&\leq \frac{42\nu_q(W,N,tx_0)c_{\boldsymbol{X}}\left|t\right|^p}{(2\pi)^p n} \left(x_0^p \exp\left(-p_n\right) + \frac{294 c^2_{\boldsymbol{X}} K_n(t)^2}{n\Lambda(p_n)^2 x_0^p} \exp\left(-\frac{K_1\sqrt{np_n}}{K_n(t)} \right) \right),
\end{aligned}$$

which yields the result with $\Lambda(p_n) \geq 1$. □

Lemma A.16 is the Talagrand inequality, as formulated in Lemma 7.1 in [15] (see also Lemma 5 in [35]), for complex functions.

Lemma A.16. Let $n \in \mathbb{N}$, $\eta > 0$, $X_1, \dots, X_n$ be independent random variables and $\nu_n(u) = \sum_{i=1}^n (u(X_i) - \mathbb{E}[u(X_i)])/n$, for u in a countable class $\mathcal{U}$ of complex measurable functions, and if there exist $M_1, H, v > 0$ such that

$$\sup_{u \in \mathcal{U}} \|u\|_{L^\infty(\mathbb{R}^p)} \leq M_1, \tag{A.130}$$

$$\mathbb{E}\left[\sup_{u \in \mathcal{U}} |\nu_n(u)|\right] \leq H, \tag{A.131}$$

$$\sup_{u \in \mathcal{U}} \frac{1}{n} \sum_{i=1}^n \max\left(\mathrm{Var}\left(\mathfrak{Re}(u(X_i))\right), \mathrm{Var}\left(\mathfrak{Im}(u(X_i))\right)\right) \leq v, \tag{A.132}$$

then

$$\begin{aligned}
&\mathbb{E}\left[\left(\sup_{u \in \mathcal{U}} |\nu_n(u)|^2 - 4(1+2\eta)H^2\right)_+\right] \\
\leq &\frac{8}{K_1}\left(\frac{v}{n}\exp\left(-K_2\eta\frac{nH^2}{v}\right) + \frac{49M_1^2}{K_2\Lambda(\eta)^2 n^2}\exp\left(-\frac{\sqrt{2}K_1\Lambda(\eta)\sqrt{\eta}}{7}\frac{nH}{M_1}\right)\right),
\end{aligned}$$

where $K_2 = 1/6$ and $\Lambda(\eta) = (\sqrt{1+\eta} - 1) \wedge 1$.

Proof of Lemma A.16. We use that

$$\begin{aligned}
&\mathbb{E}\left[\left(\sup_{u \in \mathcal{U}} |\nu_n(u)|^2 - 4(1+2\eta)H^2\right)_+\right] \\
&\leq \mathbb{E}\left[\left(\sup_{u \in \mathcal{U}} \mathfrak{Re}(\nu_n(u))^2 + \sup_{u \in \mathcal{U}} \mathfrak{Im}(\nu_n(u))^2 - 4(1+2\eta)H^2\right)_+\right] \\
&\leq \mathbb{E}\left[\left(\sup_{u \in \mathcal{U}} \mathfrak{Re}(\nu_n(u))^2 - 2(1+2\eta)H^2\right)_+\right] + \mathbb{E}\left[\left(\sup_{u \in \mathcal{U}} \mathfrak{Im}(\nu_n(u))^2 - 2(1+2\eta)H^2\right)_+\right]
\end{aligned}$$

and then apply Lemma 7.1 in [15] to both terms. □

Lemma A.17. Assume that $\|f\|_{L^2(1\otimes W^{\otimes p})} \leq M$ and $M, \epsilon > 0$. For weights W given in Section 4.5, $q \in \{1, \infty\}$, $\widehat{N}$ chosen from (18), and all $T \in \mathcal{T}_n$, we have

$$\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,\widehat{N},\widehat{T},\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) \leq \frac{C(5+2c_0)}{2\pi}\left(1+\frac{2}{c_0}\right)\mathbb{E}\left[\left\|\widehat{F}^{q,\widehat{N},T,0}_1 - \mathcal{F}_{1st}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right\|^2_{L^2(1\otimes W^{\otimes p})}\middle|\mathcal{G}_{n_0}\right] + \frac{CM^2}{w(\underline{a})}$$
$$+ C_{24}\Pi(n, Z_{n_0}, T_{\max}, N_{\max,q}) + \frac{C(2+c_0)}{\pi}\int_{\epsilon\leq|t|\leq T}\mathbb{E}\left[\Sigma\left(t, \widehat{N}(t)\right)\middle|\mathcal{G}_{n_0}\right]dt,$$

where $C_{24} = 2(2+c_0)^2C/\pi$, $\Pi(n, Z_{n_0}, T_{\max}, N_{\max,q}) = \Pi_1(n, Z_{n_0}, T_{\max}, N_{\max,q}) + \Pi_2(n, T_{\max}, N_{\max,q})$,

$$\Pi_1(n, Z_{n_0}, T_{\max}, N_{\max,q}) = Z_{n_0}\int_{\epsilon\leq|t|\leq T_{\max}}\Psi_n(t)dt,$$

$$\Pi_2(n, T_{\max}, N_{\max,q}) = \frac{(1+2c_0)48c_{\boldsymbol{X}}K_{\max}}{(2\pi)^p n}\int_{\epsilon\leq|t|\leq T_{\max}} N_{\max,q}(W,t)\,|t|^p\,\nu_q(W, N_{\max,q}(W,t), tx_0)\Psi_{0,n}(t)dt,$$

$$\Psi_n(\star) = \left(2+\frac{1}{c_0}\right)\left(\frac{|\star x_0|}{2\pi}\right)^p\frac{c_{\boldsymbol{X}}\nu_q(W, N_{\max,q}(W,\star), \star x_0)}{nx_0^p}$$
$$+\left(2+\frac{1}{c_0}\right)\|\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](\star,\cdot)\|^2_{L^2(\mathbb{R}^p)}\,|\star x_0|^p\,\mathbb{E}\left[\nu_q\left(W, \widehat{N}(\star), \star x_0\right)\middle|\mathcal{G}_{n_0}\right].$$

Proof of Lemma A.17. Using Proposition 3 and Lemma A.10, $\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,\widehat{N},\widehat{T},\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right)$ is smaller than

$$\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,\widehat{N},\widehat{T},\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right)$$
$$\leq \frac{C}{2\pi}\mathbb{E}\left[\left\|\mathbb{1}\{|\cdot|\geq\epsilon\}\left(\widehat{F}^{q,\widehat{N},\widehat{T},0}_1 - \mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(\cdot,\star)\right\|^2_{L^2(1\otimes W^{\otimes p})}\middle|\mathcal{G}_{n_0}\right] + \frac{CM^2}{w(\underline{a})}.$$

Then, for arbitrary $T \in \mathcal{T}_n$, consider the term $\widehat{F}^{q,\widehat{N},\widehat{T},0}_1 - \mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right] = \sum_{j=1}^3 R_j$, where $R_1 = \widehat{F}^{q,\widehat{N},\widehat{T},0}_1 - \widehat{F}^{q,\widehat{N},T\vee\widehat{T},0}_1$, $R_2 = \widehat{F}^{q,\widehat{N},T\vee\widehat{T},0}_1 - \widehat{F}^{q,\widehat{N},T,0}_1$, and $R_3 = \widehat{F}^{q,\widehat{N},T,0}_1 - \mathcal{F}_{1st}\left[f_{\alpha,\boldsymbol{\beta}}\right]$. For arbitrary $T \in \mathcal{T}_n$, using the Young inequality for products with $c_0 > 0$, we have

$$\mathbb{E}\left[\left\|\mathbb{1}\{|\cdot|\geq\epsilon\}\left(\widehat{F}^{q,\widehat{N},\widehat{T},0}_1 - \mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(\cdot,\star)\right\|^2_{L^2(1\otimes W^{\otimes p})}\middle|\mathcal{G}_{n_0}\right]$$
$$\leq (2+c_0)\sum_{j=1}^2\mathbb{E}\left[\|R_j\|^2_{L^2(1\otimes W^{\otimes p})}\middle|\mathcal{G}_{n_0}\right] + \left(1+\frac{2}{c_0}\right)\mathbb{E}\left[\|R_3\|^2_{L^2(1\otimes W^{\otimes p})}\middle|\mathcal{G}_{n_0}\right].$$

For $T \in \mathcal{T}_n$ and $N \in \mathcal{N}_n$, denote by

$$\Sigma_2(T, N) = \int_{\epsilon\leq|t|\leq T}\Sigma(t, N(t))dt.$$

For arbitrary $T \in \mathcal{T}_n$ and using fro the second display that by definition of $B_2\left(T, \widehat{N}\right)$,

$$B_2\left(T, \widehat{N}\right) = \max_{T' \in \mathcal{T}_n} \left(\int_{\epsilon \leq |t|} \left\| \left(\widehat{F}_1^{q, \widehat{N}, T \vee T', 0} - \widehat{F}_1^{q, \widehat{N}, T, 0} \right)(t, \cdot) \right\|^2_{L^2(W^{\otimes p})} - \Sigma\left(t, \widehat{N}(t)\right) dt \right)_+,$$

we have

$$\begin{aligned}
&\mathbb{E}\left[\|R_1\|^2_{L^2(1 \otimes W^{\otimes p})} \middle| \mathcal{G}_{n_0} \right] \\
&\leq \mathbb{E}\left[\max_{T' \in \mathcal{T}_n} \int_{\epsilon \leq |t|} \left(\left\| \left(\widehat{F}_1^{q, \widehat{N}, \widehat{T}, 0} - \widehat{F}_1^{q, \widehat{N}, T' \vee \widehat{T}, 0} \right)(t, \cdot) \right\|^2_{L^2(W^{\otimes p})} - \mathbb{1}\{|t| \leq T'\} \Sigma\left(t, \widehat{N}(t)\right) \right)_+ dt \middle| \mathcal{G}_{n_0} \right] \\
&\quad + \mathbb{E}\left[\int_{\epsilon \leq |t| \leq T} \Sigma\left(t, \widehat{N}(t)\right) dt \middle| \mathcal{G}_{n_0} \right] \\
&\leq \quad \mathbb{E}\left[B_2\left(\widehat{T}, \widehat{N}\right) \middle| \mathcal{G}_{n_0} \right] + \mathbb{E}\left[\Sigma_2\left(T, \widehat{N}\right) \middle| \mathcal{G}_{n_0} \right]
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\|R_2\|^2_{L^2(W^{\otimes p})} \middle| \mathcal{G}_{n_0} \right] \\
&\leq \mathbb{E}\left[\sup_{T' \in \mathcal{T}_n} \int_{\epsilon \leq |t|} \left(\left\| \left(\widehat{F}_1^{q, \widehat{N}, T, 0} - \widehat{F}_1^{q, \widehat{N}, T \vee T', 0} \right)(t, \cdot) \right\|^2_{L^2(1 \otimes W^{\otimes p})} - \mathbb{1}\{|t| \leq T'\} \Sigma\left(t, \widehat{N}(t)\right) \right)_+ dt \middle| \mathcal{G}_{n_0} \right] \\
&\quad + \mathbb{E}\left[\Sigma_2\left(\widehat{T}, \widehat{N}\right) \middle| \mathcal{G}_{n_0} \right] \\
&\leq \quad \mathbb{E}\left[B_2\left(T, \widehat{N}\right) \middle| \mathcal{G}_{n_0} \right] + \mathbb{E}\left[\Sigma_2\left(\widehat{T}, \widehat{N}\right) \middle| \mathcal{G}_{n_0} \right].
\end{aligned}$$

Thus, using the definition of $\widehat{T}$ for the second inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\left\| \mathbb{1}\{|\cdot| \geq \epsilon\} \left(\widehat{F}_1^{q, \widehat{N}, \widehat{T}, 0} - \mathcal{F}_{1\mathrm{st}}[f_{\alpha, \beta}] \right)(\cdot, \star) \right\|^2_{L^2(1 \otimes W^{\otimes p})} \middle| \mathcal{G}_{n_0} \right] \\
&\leq (2 + c_0) \left(\mathbb{E}\left[B_2\left(\widehat{T}, \widehat{N}\right) \middle| \mathcal{G}_{n_0} \right] + \mathbb{E}\left[\Sigma_2\left(T, \widehat{N}\right) \middle| \mathcal{G}_{n_0} \right] + \mathbb{E}\left[B_2\left(T, \widehat{N}\right) \middle| \mathcal{G}_{n_0} \right] + \mathbb{E}\left[\Sigma_2\left(\widehat{T}, \widehat{N}\right) \middle| \mathcal{G}_{n_0} \right] \right) \\
&\quad + \left(1 + \frac{2}{c_0}\right) \mathbb{E}\left[\|R_3\|^2_{L^2(1 \otimes W^{\otimes p})} \right] \\
&\leq 2(2 + c_0) \left(\mathbb{E}\left[B_2\left(T, \widehat{N}\right) \middle| \mathcal{G}_{n_0} \right] + \mathbb{E}\left[\Sigma_2\left(T, \widehat{N}\right) \middle| \mathcal{G}_{n_0} \right] \right) + \left(1 + \frac{2}{c_0}\right) \mathbb{E}\left[\|R_3\|^2_{L^2(1 \otimes W^{\otimes p})} \middle| \mathcal{G}_{n_0} \right].
\end{aligned} \tag{A.133}$$

Let us consider $\mathbb{E}\left[B_2\left(T,\widehat{N}\right)\middle|\mathcal{G}_{n_0}\right]$. Denoting, for all $T'\in\mathcal{T}_n$, by $K_1=\widehat{F}_1^{q,\widehat{N},T\vee T',0}-F_1^{q,\widehat{N},T\vee T',0}$, $K_2=F_1^{q,\widehat{N},T,0}-\widehat{F}_1^{q,\widehat{N},T,0}$, and by $K_3=F_1^{q,\widehat{N},T\vee T',0}-F_1^{q,\widehat{N},T,0}$, we obtain, for all $T'\in\mathcal{T}_n$,

$$
\begin{aligned}
&B_2\left(T,\widehat{N}\right)\\
&\leq\int_{\epsilon\leq|t|}\left((2+c_0)\sum_{j=1}^{2}\|K_j(t,\cdot)\|^2_{L^2(W^{\otimes p})}+\left(1+\frac{2}{c_0}\right)\|K_3(t,\cdot)\|^2_{L^2(W^{\otimes p})}-\mathbb{1}\{|t|\leq T'\}\Sigma\left(t,\widehat{N}(t)\right)\right)_+dt.
\end{aligned}
$$

Using that $F_1^{q,\infty,\infty,0}=\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\beta}\right]$ we have, for $t\in\mathbb{R}\setminus(-\epsilon,\epsilon)$,

$$
\begin{aligned}
\|K_3(t,\cdot)\|^2_{L^2(W^{\otimes p})}&=\mathbb{1}\{T\leq|t|\leq T\vee T'\}\sum_{0\leq|\boldsymbol{m}|_q\leq\widehat{N}}\left|\frac{c_{\boldsymbol{m}}(t)}{\sigma_{\boldsymbol{m}}^{W,tx_0}}\right|^2\\
&\leq\left\|\left(F_1^{q,\widehat{N},T,0}-\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\beta}\right]\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}
\end{aligned}
$$

hence we obtain

$$
\begin{aligned}
&B_2\left(T,\widehat{N}\right)\\
&\leq\sup_{T'\in\mathcal{T}_n}\int_{\epsilon\leq|t|}\left(2(2+c_0)\left\|\left(F_1^{q,\widehat{N},T',0}-\widehat{F}_1^{q,\widehat{N},T',0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}-\mathbb{1}\{|t|\leq T'\}\Sigma\left(t,\widehat{N}(t)\right)\right)_+dt\\
&\quad+\left(1+\frac{2}{c_0}\right)\sup_{T'\in\mathcal{T}_n}\int_{\epsilon\leq|t|}\left\|\left(F_1^{q,\widehat{N},T\vee T',0}-F_1^{q,\widehat{N},T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}dt\\
&\leq\sup_{T'\in\mathcal{T}_n}\int_{\epsilon\leq|t|}\left(2(2+c_0)\left\|\left(F_1^{q,\widehat{N},T',0}-\widehat{F}_1^{q,\widehat{N},T',0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}-\mathbb{1}\{|t|\leq T'\}\Sigma\left(t,\widehat{N}(t)\right)\right)_+dt\\
&\quad+\left(1+\frac{2}{c_0}\right)\int_{\epsilon\leq|t|}\left\|\left(F_1^{q,\widehat{N},T,0}-\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\beta}\right]\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}dt.
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
&\mathbb{E}\left[\left\|\mathbb{1}\left\{|\cdot|\geq\epsilon\right\}\left(\widehat{F}_1^{q,\widehat{N},\widehat{T},0}-\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(\cdot,\star)\right\|^2_{L^2(1\otimes W^{\otimes p})}\middle|\mathcal{G}_{n_0}\right]\\
&\leq 4(2+c_0)^2\mathbb{E}\left[\sup_{T'\in\mathcal{T}_n}\int_{\epsilon\leq|t|}\left(\left\|\left(\widehat{F}_1^{q,\widehat{N},T',0}-F_1^{q,\widehat{N},T',0}\right)(t,\star)\right\|^2_{L^2(W^{\otimes p})}-\mathbb{1}\{|t|\leq T'\}\frac{\Sigma\left(t,\widehat{N}(t)\right)}{2(2+c_0)}\right)_+\middle|\mathcal{G}_{n_0}\right]\\
&\quad+2(2+c_0)\mathbb{E}\left[\Sigma_2\left(T,\widehat{N}\right)\middle|\mathcal{G}_{n_0}\right]+\left(1+\frac{2}{c_0}\right)(5+2c_0)\mathbb{E}\left[\|R_3\|^2_{L^2(1\otimes W^{\otimes p})}\middle|\mathcal{G}_{n_0}\right].
\end{aligned}
$$

Using Young inequality for products for the second display, andLemma A.15 for the third display, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{T'\in\mathcal{T}_n}\int_{\epsilon\le|t|}\left(\left\|\left(\widehat{F}_1^{q,\widehat{N},T',0}-F_1^{q,\widehat{N},T',0}\right)(t,\star)\right\|^2_{L^2(W^{\otimes p})}-\mathbb{1}\{|t|\le T'\}\frac{\Sigma\left(t,\widehat{N}(t)\right)}{2(2+c_0)}\right)_+dt\middle|\mathcal{G}_{n_0}\right]\\
&\le\mathbb{E}\left[\sup_{T'\in\mathcal{T}_n}\int_{\epsilon\le|t|\le T'}\left(\sum_{|\boldsymbol{m}|_q\le\widehat{N}(t)}\frac{|\widehat{c}_{\boldsymbol{m}}(t)-c_{\boldsymbol{m}}(t)|^2}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2}-\frac{\Sigma\left(t,\widehat{N}(t)\right)}{2(2+c_0)}\right)_+dt\middle|\mathcal{G}_{n_0}\right]\\
&\le(1+2c_0)\mathbb{E}\left[\sup_{T'\in\mathcal{T}_n}\int_{\epsilon\le|t|\le T'}\left(\sum_{|\boldsymbol{m}|_q\le\widehat{N}(t)}\frac{\left|\widetilde{\Delta}_{\boldsymbol{m}}(t)\right|^2}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2}-\frac{\Sigma\left(t,\widehat{N}(t)\right)}{2(2+c_0)}\right)_+dt\middle|\mathcal{G}_{n_0}\right]\\
&+\left(2+\frac{1}{c_0}\right)\mathbb{E}\left[\sup_{T'\in\mathcal{T}_n}\int_{\epsilon\le|t|\le T'}S_1\left(\widehat{N}(t),t\right)+S_2\left(\widehat{N}(t),t\right)dt\middle|\mathcal{G}_{n_0}\right]\\
&\le(1+2c_0)\mathbb{E}\left[\sup_{T'\in\mathcal{T}_n}\int_{\epsilon\le|t|\le T'}\left(\sum_{|\boldsymbol{m}|_q\le\widehat{N}(t)}\frac{\left|\widetilde{\Delta}_{\boldsymbol{m}}(t)\right|^2}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2}-\frac{\Sigma\left(t,\widehat{N}(t)\right)}{2(2+c_0)}\right)_+dt\middle|\mathcal{G}_{n_0}\right]\\
&\quad+Z_{n_0}\int_{\epsilon\le|t|\le T_{\max}}\Psi_n(t)dt.
\end{aligned}$$

We now focus on the first term of the last inequality. Using (A.126) for the third display, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{T'\in\mathcal{T}_n}\int_{\epsilon\le|t|\le T'}\left(\sum_{|\boldsymbol{m}|_q\le\widehat{N}(t)}\frac{\left|\widetilde{\Delta}_{\boldsymbol{m}}(t)\right|^2}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2}-\frac{\Sigma\left(t,\widehat{N}(t)\right)}{2(2+c_0)}\right)_+dt\middle|\mathcal{G}_{n_0}\right]\\
&\le\mathbb{E}\left[\sum_{T'\in\mathcal{T}_n}\int_{\epsilon\le|t|\le T'}\left(\sum_{|\boldsymbol{m}|_q\le\widehat{N}(t)}\frac{\left|\widetilde{\Delta}_{\boldsymbol{m}}(t)\right|^2}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2}-\frac{\Sigma\left(t,\widehat{N}(t)\right)}{2(2+c_0)}\right)_+dt\middle|\mathcal{G}_{n_0}\right]\\
&\le\sum_{T'\in\mathcal{T}_n}\int_{\epsilon\le|t|\le T'}\mathbb{E}\left[\left(\sum_{|\boldsymbol{m}|_q\le\widehat{N}(t)}\frac{\left|\widetilde{\Delta}_{\boldsymbol{m}}(t)\right|^2}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2}-\frac{\Sigma\left(t,\widehat{N}(t)\right)}{2(2+c_0)}\right)_+\middle|\mathcal{G}_{n_0}\right]dt\\
&\le\sum_{T'\in\mathcal{T}_n}\int_{\epsilon\le|t|\le T'}\sum_{0\le N\le N_{\max,q}(W,t)}\mathbb{E}\left[\left(\sum_{|\boldsymbol{m}|_q\le N(t)}\frac{\left|\widetilde{\Delta}_{\boldsymbol{m}}(t)\right|^2}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2}-\frac{\Sigma\left(t,N\right)}{2(2+c_0)}\right)_+\middle|\mathcal{G}_{n_0}\right]dt
\end{aligned}$$

$$\leq \sum_{T' \in \mathcal{T}_n} \int_{\epsilon \leq |t| \leq T'} \sum_{0 \leq N \leq N_{\max,q}(W,t)} \frac{48 c_{\boldsymbol{X}} \, |t|^p \, \nu_q(W, N, t x_0)}{(2\pi)^p n} \Psi_{0,n}(t) dt$$
$$\leq \frac{48 c_{\boldsymbol{X}} K_{\max}}{(2\pi)^p n} \int_{\epsilon \leq |t| \leq T_{\max}} N_{\max,q}(W,t) \, |t|^p \, \nu_q(W, N_{\max,q}(W,t), t x_0) \Psi_{0,n}(t) dt.$$

This yields the result. □

Lemma A.18. For the two weights W given in Section 4.5, $q \in \{1, \infty\}$, and for all $T \in \mathcal{T}_n$ and $N \in \mathcal{N}_n$ we have

$$\int_{\epsilon \leq |t| \leq T} \mathcal{R}_{0,q}\left(\mathcal{G}_{n_0}, \widehat{N}(t), t\right) dt + C_{c_0} \int_{\epsilon \leq |t| \leq T} \mathbb{E}\left[\Sigma\left(T, \widehat{N}(t)\right) \middle| \mathcal{G}_{n_0}\right] dt$$
$$\leq 4(2+c_0)^2 \Pi(n, Z_{n_0}, T_{\max}, N_{\max,q})$$
$$+ (5+2c_0)\left(1 + \frac{2}{c_0}\right) \int_{\epsilon \leq |t| \leq T} \left(\mathcal{R}_{0,q}\left(\mathcal{G}_{n_0}, N(t), t\right) + C_{c_0,1} \mathbb{E}\left[\Sigma(t, N(t)) | \mathcal{G}_{n_0}\right]\right) dt,$$

where $C_{c_0,1} = (2(2+c_0) + C_{c_0})/((5+2c_0)\,(1+2/c_0))$ and $C_{c_0} = 2(2+c_0)/[(5+2c_0)(1+2/c_0)]$.

Proof of Lemma A.18. For $t \in [-T, T] \setminus (-\epsilon, \epsilon)$ and arbitrary $N \in \mathbb{N}_0$ and $T \in \mathcal{T}_n$, denote by

$$\left(\widehat{F}_1^{q,\widehat{N}(t),T,0} - F_1^{q,\infty,T,0}\right)(t, \cdot) = \sum_{j=1}^{3} R_j(t, \cdot),$$

where $R_1(t,\cdot) = \left(\widehat{F}_1^{q,\widehat{N}(t),T,0} - \widehat{F}_1^{q,\widehat{N}(t)\vee N,T,0}\right)(t,\cdot)$, $R_2(t,\cdot) = \left(\widehat{F}_1^{q,\widehat{N}(t)\vee N,T,0} - \widehat{F}_1^{q,N,T,0}\right)(t,\cdot)$, and $R_3(t,\cdot) = \left(\widehat{F}_1^{q,N,T,0} - F_1^{q,\infty,T,0}\right)(t,\cdot)$ (here, note that for t such that $|t| \in [\epsilon, T]$, we have $\mathcal{F}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot) = F_1^{q,\infty,T,0}(t,\cdot)$). For arbitrary $T \in \mathcal{T}_n$, $N \in \mathbb{N}_0$, using the Young inequality for products, we have

$$\mathcal{R}_{0,q}\left(\mathcal{G}_{n_0}, \widehat{N}(t), t\right) \leq (2+c_0) \sum_{j=1}^{2} \mathbb{E}\left[\|R_j(t,\cdot)\|^2_{L^2(W^{\otimes p})} \middle| \mathcal{G}_{n_0}\right] + \left(1 + \frac{2}{c_0}\right) \mathbb{E}\left[\|R_3(t,\cdot)\|^2_{L^2(W^{\otimes p})} \middle| \mathcal{G}_{n_0}\right].$$

We proceed similarly to (A.133). For arbitrary $N \in \{0, \ldots, N_{\max,q}(W,t)\}$ and $t \in [-T, T] \setminus [-\epsilon, \epsilon]$, using the definition of $B_1(t, N)$, which can be rewritten as

$$B_1(t, N) = \max_{N' :\ N' \leq N_{\max,q}(W,t)} \left(\sum_{N \leq |\boldsymbol{m}|_q \leq N' \vee N} \frac{|\widehat{c}_{\boldsymbol{m}}(t)|^2}{\left(\sigma_{\boldsymbol{m}}^{W,t x_0}\right)^2} - \Sigma\left(t, N'\right) \right)_+$$

and the definition of $\widehat{N}(t)$ (we take $c_0 = 1/6$ in the definition of (18), hence $1+C_{c_0}/(2(2+c_0)) = 31/30$), we obtain

$$
\begin{aligned}
&\mathcal{R}_{0,q}\left(\mathcal{G}_{n_0}, \widehat{N}(t), t\right) + C_{c_0}\mathbb{E}\left[\Sigma\left(t, \widehat{N}(t)\right)\right] \\
&\leq 2(2+c_0)\left(\mathbb{E}\left[B_1\left(t, \widehat{N}(t)\right)\middle|\mathcal{G}_{n_0}\right] + \mathbb{E}\left[\Sigma\left(t, \widehat{N}(t)\right)\middle|\mathcal{G}_{n_0}\right]\right) + \left(1+\frac{2}{c_0}\right)\mathbb{E}\left[\|R_3(t,\cdot)\|^2_{L^2(W^{\otimes p})}\middle|\mathcal{G}_{n_0}\right] \\
&\quad + C_{c_0}\mathbb{E}\left[\Sigma\left(t, \widehat{N}(t)\right)\right] \\
&\leq 2(2+c_0)\left(\mathbb{E}\left[B_1(t,N)|\mathcal{G}_{n_0}\right] + \left(1+\frac{C_{c_0}}{2(2+c_0)}\right)\mathbb{E}\left[\Sigma(t,N)|\mathcal{G}_{n_0}\right]\right) \\
&\quad + \left(1+\frac{2}{c_0}\right)\mathbb{E}\left[\|R_3(t,\cdot)\|^2_{L^2(W^{\otimes p})}\middle|\mathcal{G}_{n_0}\right].
\end{aligned}
$$

Consider $\mathbb{E}\left[B_1(t,N)|\mathcal{G}_{n_0}\right]$. For $t \in [-T,T]\setminus[-\epsilon,\epsilon]$, $N \in \mathbb{N}_0$, and denoting, for $N' \in \mathbb{N}_0$, by $K_1(t,\cdot) = \left(\widehat{F}_1^{q,N\vee N',T,0} - F_1^{q,N\vee N',T,0}\right)(t,\cdot)$, $K_2(t,\cdot) = \left(F_1^{q,N,T,0} - \widehat{F}_1^{q,N,T,0}\right)(t,\cdot)$, and $K_3(t,\cdot) = \left(F_1^{q,N\vee N',T,0} - F_1^{q,N,T,0}\right)(t,\cdot)$, we obtain

$$
\begin{aligned}
&B_1(t,N) \\
&\leq \sup_{0\leq N'\leq N_{\max,q}(W,t)}\left((2+c_0)\sum_{j=1}^{2}\|K_j(t,\cdot)\|^2_{L^2(W^{\otimes p})} + \left(1+\frac{2}{c_0}\right)\|K_3(t,\cdot)\|^2_{L^2(W^{\otimes p})} - \Sigma(t,N')\right)_+.
\end{aligned}
$$

Using $F_1^{q,\infty,T,0}(t,\cdot) = \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\beta}\right](t,\cdot)$, we have

$$
\begin{aligned}
\|K_3(t,\cdot)\|^2_{L^2(W^{\otimes p})} &= \sum_{N<|\boldsymbol{m}|_q\leq(N\vee N')}\left|\frac{c_{\boldsymbol{m}}(t)}{\sigma_{\boldsymbol{m}}^{W,tx_0}}\right|^2 \\
&\leq \left\|\left(F_1^{q,N,T,0} - \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\beta}\right]\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})},
\end{aligned}
$$

hence

$$
\begin{aligned}
B_1(t,N) &\leq \sup_{0\leq N'\leq N_{\max,q}(W,t)}\left(2(2+c_0)\left\|\left(F_1^{q,N',T,0} - \widehat{F}_1^{q,N',T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})} - \Sigma(t,N')\right)_+ \\
&\quad + \left(1+\frac{2}{c_0}\right)\sup_{0\leq N'\leq N_{\max,q}(W,t)}\left\|\left(F_1^{q,N'\vee N,T,0} - F_1^{q,N,T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})} \\
&\leq \sup_{0\leq N'\leq N_{\max,q}(W,t)}\left(2(2+c_0)\left\|\left(F_1^{q,N',T,0} - \widehat{F}_1^{q,N',T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})} - \Sigma(t,N')\right)_+ \\
&\quad + \left(1+\frac{2}{c_0}\right)\left\|\left(F_1^{q,N,T,0} - \mathcal{F}_{1st}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}.
\end{aligned}
$$

Finally, we have for arbitrary $N \in \{0, \ldots, N_{\max,q}(W,t)\}$,

$$
\begin{aligned}
&\mathcal{R}_{0,q}\left(\mathcal{G}_{n_0}, \widehat{N}(t), t\right) + C_{c_0}\mathbb{E}\left[\Sigma\left(t, \widehat{N}(t)\right)\right] \\
&\leq 4(2+c_0)^2\mathbb{E}\left[\sup_{0\leq N'\leq N_{\max,q}(W,t)}\left(\left\|\left(\widehat{F}_1^{q,N',T,0} - F_1^{q,N',T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})} - \frac{\Sigma_1(t,N')}{2(2+c_0)}\right)_+ \middle| \mathcal{G}_{n_0}\right] \\
&\quad + (2(2+c_0)+C_{c_0})\mathbb{E}\left[\Sigma\left(t, \widehat{N}(t)\right)\middle|\mathcal{G}_{n_0}\right] \\
&\quad + (5+2c_0)\left(1+\frac{2}{c_0}\right)\mathbb{E}\left[\left\|\left(\widehat{F}_1^{q,N,T,0} - \mathcal{F}_{1st}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}\middle|\mathcal{G}_{n_0}\right]
\end{aligned}
$$

Using Lemma A.15 for the third display, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq N'\leq N_{\max,q}(W,t)}\left\|\left(\widehat{F}_1^{q,N',T,0} - F_1^{q,N',T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})} - \frac{\Sigma(t,N')}{2(2+c_0)}\middle|\mathcal{G}_{n_0}\right] \\
\leq&\mathbb{E}\left[\sup_{0\leq N'\leq N_{\max,q}(W,t)}\left(\sum_{|\boldsymbol{m}|_q\leq N'}\frac{|\widehat{c}_{\boldsymbol{m}}(t) - c_{\boldsymbol{m}}(t)|^2}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2} - \frac{\Sigma(t,N')}{2(2+c_0)}\right)_+\middle|\mathcal{G}_{n_0}\right] \\
\leq&(1+2c_0)\mathbb{E}\left[\sup_{0\leq N'\leq N_{\max,q}(W,t)}\left(\sum_{|\boldsymbol{m}|_q\leq N'}\frac{\left|\widetilde{\Delta}_{\boldsymbol{m}}(t)\right|^2}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2} - \frac{\Sigma(t,N')}{2(2+c_0)}\right)_+\middle|\mathcal{G}_{n_0}\right] \\
&+\left(2+\frac{1}{c_0}\right)\mathbb{E}\left[\sup_{0\leq N'\leq N_{\max,q}(W,t)}\left(S_1(N',t)+S_2(N',t)\right)\middle|\mathcal{G}_{n_0}\right] \\
\leq&\frac{(1+2c_0)48N_{\max,q}(W,t)c_{\boldsymbol{X}}|t|^p\,\nu_q(W,N_{\max,q}(W,t),tx_0)}{(2\pi)^p n}\Psi_{0,n}(t) + Z_{n_0}\Psi_n(t).
\end{aligned}
$$

Thus, this yields that for all $N \in \mathcal{N}_n$,

$$
\begin{aligned}
&\int_{\epsilon\leq|t|\leq T}\mathcal{R}_{0,q}\left(\mathcal{G}_{n_0}, \widehat{N}(t), t\right)dt + C_{c_0}\int_{\epsilon\leq|t|\leq T}\mathbb{E}\left[\Sigma\left(t, \widehat{N}(t)\right)\middle|\mathcal{G}_{n_0}\right]dt \\
\leq&4(2+c_0)^2\int_{\epsilon\leq|t|\leq T}\left(\frac{(1+2c_0)48N_{\max,q}(W,t)c_{\boldsymbol{X}}|t|^p\,\nu_q(W,N_{\max,q}(W,t),tx_0)}{(2\pi)^p n}\Psi_{0,n}(t) + Z_{n_0}\Psi_n(t)\right)dt \\
&+(5+2c_0)\left(1+\frac{2}{c_0}\right) \\
&\times\int_{\epsilon\leq|t|\leq T}\left(\mathbb{E}\left[\left\|\left(\widehat{F}_1^{q,N(t),T,0} - \mathcal{F}_{1st}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}\middle|\mathcal{G}_{n_0}\right] + C_{c_0,1}\mathbb{E}\left[\Sigma(t,N(t))|\mathcal{G}_{n_0}\right]\right)dt,
\end{aligned}
$$

hence the result. □

Lemma A.19. Assume that $v(n_0, \mathcal{E}) \leq n^{-(2+\zeta)}$ with $\zeta > 0$. For the weights W given in Section 4.5 and $q \in \{1, \infty\}$, there exists $C_{25}(W, q)$ independent of n and t such that for $\epsilon \leq |t| \leq T_{\max}$,

$$\left(\frac{|t|\, x_0}{2\pi}\right)^p \nu_q\left(W, N_{\max,q}, tx_0\right) v(n_0, \mathcal{E}) \leq C_{25}(W, q) \ln(n)^{p-1} \left(\frac{\mathbb{1}\{W = W_{[-R,R]}\}}{n^{1+\zeta-p\zeta_0}} + \frac{\mathbb{1}\{W = \cosh(\cdot/R)\}}{n^{1+\zeta-(p+1)\zeta_0}}\right).$$

Proof of Lemma A.19. Consider the case $W = W_{[-R,R]}$ and $q = 1$. The choice of $N_{\max,q}$ ensures $\left((7\pi/(|t|\, Rx_0))\, N_{\max,q}(W,t)\right)^{2pN_{\max,q}(W,t)} \leq n$ and $N_{\max,q}(W,t) \leq 2\,|t|\, Rx_0 \ln(n)/(\pi k_q)$, thus, using the definition of $\nu\left(W, N, tx_0\right)$, we have, for $\epsilon \leq |t| \leq T_{\max} \leq n^{\zeta_0}$,

$$\begin{aligned}
&\left(\frac{|t|\, x_0}{2\pi}\right)^p \nu\left(W, N_{\max,q}, tx_0\right) \\
&\leq \left(\frac{x_0\,|t|}{2\pi(1 \vee |Rx_0 t|)}\right)^p \frac{(N_{\max,q}(W,t) + p - 1)^{p-1}}{(p-1)!(1-e^{-2})} \exp\left(2N_{\max,q}(W,t) \ln\left(\frac{7\pi(N_{\max,q}(W,t)+1)}{|t|} \vee 1\right)\right) \\
&\leq C_{25}(W,1) T_{\max}^p \ln(n)^{p-1} n, \quad \left(\text{using } (N_{\max,q}(W,t) + p - 1)^{p-1} \leq (pN_{\max,q}(W,t))^{p-1}\right) \\
&\leq C_{25}(W,1) \ln(n)^{p-1} n^{1+p\zeta_0},
\end{aligned} \tag{A.134}$$

where $C_{25}(W,1) = (2/(\pi k_q))^p p^{p-1} x_0^p/\pi^p (p-1)!(1-e^{-2})$, hence the result. Similar computations yield the result in case $q = \infty$.

Consider now the case $W = \cosh(\cdot/R)$ and $q = 1$. We have, for $\epsilon \leq |t| \leq T_{\max}$,

$$\begin{aligned}
\left(\frac{|t|\, x_0}{2\pi}\right)^p \nu\left(W, N_{\max,q}, tx_0\right) &\leq \left(\frac{x_0}{2\pi}\right)^p T_{\max}^{p+1} \left(1 \bigvee \frac{\sqrt{2}R}{\pi}\right) \frac{2x_0^p p^{p-1}}{\pi^p (p-1)!} n \ln(n)^{p-1} \\
&\leq C_{25}(W,1) \ln(n)^{p-1} n^{(p+1)\zeta_0},
\end{aligned}$$

where $C_{25}(W,1) = \left(1 \vee (\sqrt{2}R/\pi)\right) 2x_0^p p^{p-1}/(\pi^p (p-1)!)$, hence the result. Similar computations yields the result in the case $q = \infty$. □

Proof of Theorem 5. Use the notations of Theorem (T1.1). Take $\mathcal{E}$ and $\eta > 0$, using Lemma A.14, there exists $M_{1,\mathcal{E},\eta} \in \mathbb{R}$ such that, for all $n_0 \in \mathbb{N}$, $\mathbb{P}\left(E\left(\mathcal{G}_{n_0}, \mathcal{E}\right)\right) \geq 1 - \eta$, where $E\left(\mathcal{G}_{n_0}, \mathcal{E}\right) = \{Z_{n_0} \leq M_{1,\mathcal{E},\eta} v(n_0, \mathcal{E})/\delta(n_0)\}$.

Proof of (24). Consider the case $W = W_{[-R,R]}$ and $q = 1$, as the other cases can be treated similarly. Note that we take $c_0 = 1/6$ in the definition of (18). Denote by $C_{c_0,1} = (2(2 + c_0) + C_{c_0})/((5 + 2c_0)\,(1 + 2/c_0))$, $C_{c_0,2} = C(5 + 2c_0)\,(1 + 2/c_0)\,/(2\pi)$, and $C_{c_0,3} = C(2 + c_0)/\pi$. Using Lemma A.17 and (A.68) for the first display, $C_{c_0} = C_{c_0,3}/C_{c_0,2}$ for the second one, and Lemma

A.18 for the third one, we have for all $T \in \mathcal{T}_n$ and $N \in \mathcal{N}_n$,

$$
\begin{aligned}
&\mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,\widehat{N},\widehat{T},\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right) \\
&\leq C_{c_0,2}\mathbb{E}\left[\left\|\widehat{F}_1^{q,\widehat{N},T,0} - \mathcal{F}_{1st}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right\|^2_{L^2(1\otimes W^{\otimes p})}\middle|\mathcal{G}_{n_0}\right] + C_{c_0,3}\int_{\epsilon\leq|t|\leq T}\mathbb{E}\left[\Sigma\left(t,\widehat{N}(t)\right)\middle|\mathcal{G}_{n_0}\right]dt \\
&\quad + C_{24}\Pi(n, Z_{n_0}, T_{\max}, N_{\max,q}) + \frac{CM^2}{w(\underline{a})} \\
&\leq C_{c_0,2}\left(\int_{\epsilon\leq|t|\leq T}\mathcal{R}_{0,q}\left(\mathcal{G}_{n_0},\widehat{N}(t),t\right) + C_{c_0}\mathbb{E}\left[\Sigma\left(t,\widehat{N}(t)\right)\middle|\mathcal{G}_{n_0}\right]dt\right) \\
&\quad + C_{24}\Pi(n, Z_{n_0}, T_{\max}, N_{\max,q}) + \frac{CM^2}{w(\underline{a})} \\
&\leq \int_{\epsilon\leq|t|\leq T}\left(\frac{2\pi C_{c_0,2}^2}{C}\mathcal{R}_{0,q}\left(\mathcal{G}_{n_0},N(t),t\right) + C_{c_0,2}\left[2(2+c_0)+C_{c_0}\right]\mathbb{E}\left[\Sigma(t,N(t))|\mathcal{G}_{n_0}\right]\right)dt \\
&\quad + \left(C_{24} + 4(2+c_0)^2C_{c_0,2}\right)\left[\Pi_1(n, Z_{n_0}, T_{\max}, N_{\max,q}) + \Pi_2(T_{\max}, N_{\max,q})\right] + \frac{CM^2}{w(\underline{a})}.
\end{aligned}
$$

Step 1: upper bound on $\Pi_1(n, Z_{n_0}, T_{\max}, N_{\max,q})$.
The choice of $N_{\max,q}$ ensures that, for all $t \neq 0$,

$$
\left(\frac{7\pi N_{\max,q}(W,t)}{|t|\,Rx_0}\right)^{2N_{\max,q}(W,t)} \leq n.
$$

Then, on $E\left(\mathcal{G}_{n_0},\mathcal{E}\right)$ using Lemma A.19, we obtain

$$
\begin{aligned}
&Z_{n_0}\Psi_n(t) \\
&\leq M_{1,\mathcal{E},\eta}\left(2+\frac{1}{c_0}\right)\frac{c_{\boldsymbol{X}}}{nx_0^p}\left[\left(\frac{|tx_0|}{2\pi}\right)^p\nu_q(W, N_{\max,q}(W,t), tx_0)\frac{v(n_0,\mathcal{E})}{\delta(n_0)}\right] \\
&\quad + M_{1,\mathcal{E},\eta}\left(2+\frac{1}{c_0}\right)(2\pi)^p\left\|\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot)\right\|^2_{L^2(\mathbb{R}^p)}\mathbb{E}\left[\left(\frac{|tx_0|}{2\pi}\right)^p\nu_q\left(W,\widehat{N}(t),tx_0\right)v(n_0,\mathcal{E})\middle|\mathcal{G}_{n_0}\right] \\
&\leq \frac{C_{26,\mathcal{E},\eta}(t)\ln(n)^{p-1}}{n^{1+\zeta}},
\end{aligned}
$$

where $C_{26,\mathcal{E},\eta}(t) = M_{1,\mathcal{E},\eta}C_{25}(W,q)(2+1/c_0)\left(c_{\boldsymbol{X}}/x_0^p + (2\pi)^p\left\|\mathcal{F}_{1st}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot)\right\|^2_{L^2(\mathbb{R}^p)}\right)$. Thus, using $T_{\max} \leq n^{\zeta_0}$ and $\int_{\epsilon\leq|t|\leq T}\left\|\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot)\right\|^2_{L^2(\mathbb{R}^p)}dt \leq 2\pi l^2$, there exists $C_{27,\mathcal{E},\eta}$ such that

$$
\int_{\epsilon\leq|t|\leq T_{\max}} C_{26,\mathcal{E},\eta}(t)dt \leq n^{\zeta_0}(1\vee l^2)C_{27,\mathcal{E},\eta}.
$$

This yields, using that $\sup_{t\geq 1}\ln(t)^a/t^b = (a/(eb))^a$ for $a,b>0$,

$$\Pi_1\left(n, M_{1,\mathcal{E},\eta}\frac{v(n_0,\mathcal{E})}{\delta(n_0)}, T_{\max}, N_{\max,q}\right) \leq \frac{C_{27,\mathcal{E},\eta}\left((p-1)/(e(\zeta-\zeta_0))\right)^{p-1}(1\vee l^2)}{n}.$$

Step 2: upper bound on $\Pi_2(T_{\max}, N_{\max,q})$.
Using that $N_{\max,q}(t)\leq 2|t|\,Rx_0\ln(n)/(\pi k_q)$, that for all $\epsilon\leq|t|\leq T_{\max}$,

$$K_n(t)\leq T_{\max}^{3p}\left(\frac{4Rx_0H_0(1+x_0^2)}{\pi k_q}\right)^p,$$

and that $T_{\max}\leq n^{\zeta_0}$, for all $|t|\leq T_{\max}$, we obtain

$$\begin{aligned}\frac{K_1\sqrt{p_n n}}{K_n(t)} &\geq \frac{K_1(\pi k_q)^p\sqrt{2\log(n)n}}{H_0^p(4Rx_0)^p(1+x_0^2)^p\ln(n)^pT_{\max}^{3p}}\\ &\geq \frac{\sqrt{2}K_1(\pi k_q)^p n^{(1-6\zeta_0 p)/2}}{H_0^p(4Rx_0)^p(1+x_0^2)^p\ln(n)^{p-1/2}}\\ &\geq C_{28}\ln(n)^2, \qquad \text{(A.135)}\end{aligned}$$

where, using that $\sup_{t\geq 1}\ln(t)^a/t^b=(a/(eb))^a$ for $a,b>0$, we have

$$C_{28} = \frac{\sqrt{2}K_1(\pi k_q)^p(e(1-6p\zeta_0))^{p+3/2}}{H_0^p(4Rx_0)^p(1+x_0^2)^p\,(2p+3)^{p+3/2}}. \qquad \text{(A.136)}$$

Then, using (A.69) and (A.134), we obtain

$$\begin{aligned}&\Pi_2(T_{\max}, N_{\max,q})\\ &\leq \frac{(1+2c_0)8c_{\boldsymbol{X}}K_{\max}}{K_1(2\pi)^p n}\int_{\epsilon\leq|t|\leq T_{\max}} N_{\max,q}(W,t)\,|t|^p\,\nu_q(W,N_{\max,q}(W,t),tx_0)dt \sup_{t\in\mathbb{R}:\ |t|\in[\epsilon,T_{\max}]}\Psi_{0,n}(t)\\ &\leq \frac{16(1+2c_0)Rx_0c_{\boldsymbol{X}}\zeta_0\log(n)^2}{K_1(2\pi)^p\pi k_q\log(2)n}\int_{\epsilon\leq t\leq T_{\max}}|t|^{p+1}\,\nu_q(W,N_{\max,q}(W,t),tx_0)dt \sup_{t\in\mathbb{R}:\ |t|\in[\epsilon,T_{\max}]}\Psi_{0,n}(t)\\ &\leq \frac{16(1+2c_0)Rx_0c_{\boldsymbol{X}}\zeta_0C_{25}(W,q)}{K_1(2\pi)^p\pi k_q\log(2)}\log(n)^{p+1}n^{(p+2)\zeta_0}\sup_{t\in\mathbb{R}:\ |t|\in[\epsilon,T_{\max}]}\Psi_{0,n}(t)\\ &\leq \frac{96(1+2c_0)Rx_0c_{\boldsymbol{X}}\zeta_0C_{25}(W,q)}{(2\pi)^p\pi k_q\log(2)}\left(\frac{\log(n)^{p+1}}{n^{\zeta_0}}\right)\frac{n^2}{n^{2-(p+3)\zeta_0}}\sup_{t\in\mathbb{R}:\ |t|\in[\epsilon,T_{\max}]}\Psi_{0,n}(t),\end{aligned}$$

where $2-(p+3)\zeta_0>4/3$. Finally, using that $T_{\max}^{6p}\leq n$, (A.135) and $\sup_{n>0}\left(e^{-C_{28}\ln(n)^2}n^2\right)=e^{1/C_{28}}$, we obtain

$\Pi_2(T_{\max}, N_{\max,q})$

$$\leq \frac{96(1+2c_0)Rx_0c_{\boldsymbol{X}}\zeta_0 C_{25}(W,q)}{(2\pi)^p\pi k_q\log(2)n^{4/3}}\left(x_0^p+\frac{294c_{\boldsymbol{X}}^2\exp(1/C_{28})x_0^p}{(2\pi)^p}\left(\frac{4RH_0(1+x_0^2)}{\pi k_q}\right)^{2p}\right)\left(\frac{p+1}{e\zeta_0}\right)^{p+1}$$
$$\leq \frac{C_{29}}{n^{4/3}},$$

where

$$C_{29}=\frac{96(1+2c_0)Rx_0c_{\boldsymbol{X}}\zeta_0 C_{25}(W,q)x_0^p}{(2\pi)^p\pi k_q\log(2)}\left(1+\frac{294c_{\boldsymbol{X}}^2\exp(1/C_{28})}{(2\pi)^p}\left(\frac{4RH_0(1+x_0^2)}{\pi k_q}\right)^{p}\right)\left(\frac{p+1}{e\zeta_0}\right)^{p+1}.$$

Step 3: upper bound on $\mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,\widehat{N},\widehat{T},\epsilon},f_{\alpha,\boldsymbol{\beta}}\right)$ using Steps 1, Step 2, and the proof of Theorem 1.

Finally, using $v(n_0,\mathcal{E})/\delta(n_0)\leq n^{-(2+\zeta)}$, the definition of Σ, and (A.88-A.90), we obtain on $E(\mathcal{G}_{n_0},\mathcal{E})$, that for all $T\in\mathcal{T}_n$ and $N\in\mathcal{N}_n$,

$$\begin{aligned}\mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,\widehat{N},\widehat{T},\epsilon},f_{\alpha,\boldsymbol{\beta}}\right)\leq{}&\frac{CM^2}{w(\underline{a})}+\frac{C_{30,\mathcal{E},\eta}}{n}+\frac{2\pi C_{c_0,2}^2}{C}\left(4l^2\sup_{t\in\mathbb{R}}\frac{1}{N(t)^{2\sigma}}+\frac{4l^2}{(1\vee T)^{2s}}\right)\\&+\frac{2\pi C_{c_0,2}^2}{C}\int_{\epsilon\leq|t|\leq T}\widetilde{\Delta}_{2,q}\left(t,N(t),n,\frac{M_{1,\mathcal{E},\eta}v(n_0,\mathcal{E})}{\delta(n_0)}\right)dt,\end{aligned}\tag{A.137}$$

where

$$\begin{aligned}&\widetilde{\Delta}_{2,q}(t,N,n,z)\\&=\Delta_{3,q}(t,N)\left(1+\frac{60(1+2p_n)C(2(2+c_0)+C_{c_0})}{\pi C_{c_0,2}}\right)\frac{c_{\boldsymbol{X}}(N(t)p)^{p-1}}{(2\pi)^p(p-1)!(1-e^{-2})n}\left(\frac{|t|}{1\vee|Rx_0t|}\right)^p\\&+z\Delta_{3,q}(t,N)\left(\frac{1}{\pi(1\vee|Rx_0t|)}\right)^p\left(L(t)^2+\frac{c_{\boldsymbol{X}}(N(t)+1)^p|t|^p}{n}\right),\end{aligned}$$

$\Delta_{3,q}(t,N)$ is given in (A.89), and

$$C_{30,\mathcal{E},\eta}=\left(C_{24}+4(2+c_0)^2C_{c_0,2}\right)\left(C_{27,\mathcal{E},\eta}\left(\frac{p-1}{e(\zeta-\zeta_0)}\right)^{p-1}(1\vee l^2)+C_{29}\right).$$

Denote by $T^*=n_e^{1/(6s(p+1))}$ and $N^*(t)=\lfloor\overline{N}^*(t)\rfloor$ the choice of N made in Theorem 1 (T1.1) replacing $\overline{N}(t)$ by $\overline{N}^*(t)=Q_{|t|,(2\sigma+p)/2}(\ln(n_e)/4)$ (where $(2\sigma+p)/2$ replace $(2\sigma+p-1)/2$ to take into account the logarithmic penalty p_n). We have that $N^*(t)\leq N_{\max,q}(W,t)$ for $t\in\mathbb{R}\setminus(-\epsilon,\epsilon)$, thus $N^*\in\mathcal{N}_n$. We also have that $T^*\leq T_{\max}=n^{\zeta_0}$ for $s>1$. Then, we have

on $E\left(\mathcal{G}_{n_0},\mathcal{E}\right)$ and for all (n,n_0) such that $v(n_0,\mathcal{E})/\delta(n_0)\leq n^{-(2+\zeta)}$,

$$\begin{aligned}
&\sup_{\substack{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}_{w,W}^{q,\phi,\omega}(l,M)\cap\mathcal{D}\\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}} \mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,\widehat{N},\widehat{T},\epsilon},f_{\alpha,\boldsymbol{\beta}}\right)\\
&\leq \frac{CM^2}{\left(\ln(n)/\ln_2(n)\right)^{2\sigma_0}}+\frac{C_{30,\mathcal{E},\eta}}{n}+\frac{2\pi C_{c_0,2}^2}{C}\int_{\epsilon\leq|t|\leq T^*}\widetilde{\Delta}_{2,q}\left(t,N^*(t),n,\frac{M_{1,\mathcal{E},\eta}v(n_0,\mathcal{E})}{\delta(n_0)}\right)dt\\
&\quad+\frac{2\pi C_{c_0,2}^2}{C}\left(4l^2\sup_{t\in\mathbb{R}}\frac{1}{N^*(t)^{2\sigma}}+\frac{4l^2}{(1\vee T^*)^{2s}}\right).
\end{aligned}$$

Finally, we obtain an upper bound on the above infimum using the proof of Theorem 1 (T1.1), only with a minor adaptation of the constants to take into account the new value of the parameter T ($n_e^{1/(6s(p+1))}$ instead of $n_e^{1/(2(p+1))}$). Namely, there exists $M_{13,\mathcal{E},\eta,q}$ such that we have, on $E\left(\mathcal{G}_{n_0},\mathcal{E}\right)$ and for all (n,n_0) such that $v(n_0,\mathcal{E})/\delta(n_0)\leq n^{-(2+\zeta)}$,

$$\mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,\widehat{N},\widehat{T},\epsilon},f_{\alpha,\boldsymbol{\beta}}\right)\left(\frac{\ln(n)}{\ln_2(n)}\right)^{2\sigma}\leq M_{13,\mathcal{E},\eta,q},$$

where

$$M_{13,\mathcal{E},\eta,1}=2\pi C_{c_0,2}^2\left(\frac{1}{2\tau_{4,1}}\bigwedge\frac{1}{2\tau_{3,1}}\right)^{-2\sigma}\left(4l^2\left(1+\left(\frac{6\sigma(p+1)}{e}\right)^{2\sigma}\right)+M'_{4,\mathcal{E},\eta,1}+\frac{M^2}{(4k_1\tau_{5,1})^{2\sigma}}\right)$$

and

$$\begin{aligned}
M'_{4,\mathcal{E},\eta,1}&=\left(1+\frac{60(1+2p_n)C(2(2+c_0)+C_{c_0})}{\pi C_{c_0,2}}\right)\frac{4}{\pi}\frac{\tau_2 c_{\boldsymbol{X}}p^{p-1}}{(2\pi)^p(p-1)!(1-e^{-2})(Rx_0)^p}\\
&\quad+4M_{1,\mathcal{E},\eta}\left(2^pl^2+\frac{c_{\boldsymbol{X}}}{\pi^{p+1}(Rx_0)^p}\left(\frac{p}{e}+1\right)^p\right).
\end{aligned}$$

Proof of (25). Consider the case $q=1$, as the other cases can be treated similarly. Start from (A.137), where, because $w=W_{\mathcal{A}}$, the term $M^2/w(a_0)$ is zero and, using (A.97) this yields, on $E\left(\mathcal{G}_{n_0},\mathcal{E}\right)$, for all (n,n_0) such that $v(n_0,\mathcal{E})/\delta(n_0)\leq n^{-(2+\zeta)}$, and for all $N\in\mathcal{N}_n$ and $T\in\mathcal{T}_n$,

$$\begin{aligned}
&\sup_{\substack{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}_{w,W}^{q,\phi,\omega}(l,M)\cap\mathcal{D}\\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}} \mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,\widehat{N},\widehat{T},\epsilon},f_{\alpha,\boldsymbol{\beta}}\right)\\
&\leq\frac{C_{30,\mathcal{E},\eta}}{n}+\frac{2\pi C_{c_0,2}^2}{C}\sup_{t\in\mathbb{R}}\frac{4l^2}{e^{2\kappa(N(t)\ln(N(t)+1)}}\\
&\quad+\frac{2\pi C_{c_0,2}^2}{C}\left(\int_{\epsilon\leq|t|\leq T}\widetilde{\Delta}_{2,q}\left(t,N(t),n,\frac{M_{1,\mathcal{E},\eta}v(n_0,\mathcal{E})}{\delta(n_0)}\right)dt+\frac{4l^2}{(1\vee T)^{2s}}\right).
\end{aligned}$$

Denote by T^* and N^* the choices of T and N made in Theorem 3 (T3.2). This yields directly that $N^*(t) \leq N_{\max,q}(W,t)$ hence $N^* \in \mathcal{N}_n$ and that $T^* \leq n_e^{1/(2s)} \leq T_{\max} = n^{1/(6p)}$ because $s > 3p$. Thus, we obtain

$$\begin{aligned}
&\sup_{\substack{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}_{w,W}^{q,\phi,\omega}(l,M) \cap \mathcal{D} \\ f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}}} \mathcal{R}_{n_0}^W \left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,\widehat{N},\widehat{T},\epsilon}, f_{\alpha,\boldsymbol{\beta}} \right) \\
&\leq \frac{C_{30,\mathcal{E},\eta}}{n} + \frac{2\pi C_{c_0,2}^2}{C} \sup_{t \in \mathbb{R}} \frac{4l^2}{e^{2\kappa(N^*(t)\ln(N^*(t)+1)}} \\
&\quad + \frac{2\pi C_{c_0,2}^2}{C} \left(\int_{\epsilon \leq |t| \leq T^*} \widetilde{\Delta}_{2,q} \left(t, N^*(t), n, \frac{M_{1,\mathcal{E},\eta} v(n_0,\mathcal{E})}{\delta(n_0)} \right) dt + \frac{4l^2}{(1 \vee T^*)^{2s}} \right).
\end{aligned}$$

This upper bound yields the result, using (A.111) where we replace $(1 \vee N(t))^{p-1}$ by $(1 \vee N(t))^{p-1} p_n$ to take into account the logarithmic penalty p_n.

Proof of (26). The proof is similar to the proof of (25), using that with the choices of T^* and N^* as the choices of T and N made in Theorem 3 (T3.3), we have $T^* \leq n_e^{\kappa/(2s(\kappa+k_q))} \leq T_{\max} = n^{1/(14p)}$ because $s > 7p\kappa/(\kappa + k_q)$, where $\zeta_0 \leq 1/(14p)$ and replace C_{28} in (A.136) by

$$C_{31} = \frac{\sqrt{2} K_1 (\pi k_q)^p (e(1 - 14p\zeta_0))^{p+3/2}}{H_0^p (4Rx_0)^p \left(x_0^6 \vee (4R/\pi)^{1/2} \right)^p (2p+3)^{p+3/2}}.$$

□

Appendix C. Estimation of the marginal $f_{\boldsymbol{\beta}}$

For $q \in \{1, \infty\}$, let us work with the set of smooth functions,

$$\mathcal{H}_{w,W}^{q,\phi,\omega}(l,M) = \left\{ f : \|f\|_{L^2(w \otimes W^{\otimes p})} \leq M, \ \sum_{k \in \mathbb{N}_0} \omega_k \|\theta_{q,k}\|_{L^2(\mathbb{R})}^2 \leq 2\pi l^2 \right\},$$

where $w = 1$. Because $f_{\boldsymbol{\beta}}(\cdot) = \mathcal{F}_{1\text{st}}[f_{\alpha,\boldsymbol{\beta}}](0,\cdot)$, we consider the estimator

$$\widehat{f}_{\beta}^{q,N,\epsilon}(\cdot) = \widehat{F}_1^{q,N}(\epsilon,\cdot). \tag{A.138}$$

Proposition A.3. Take $q \in \{1,\infty\}$, $W = W_{[-R,R]}$, $\phi(\cdot) = 1 \vee |\cdot|^{2s}$, $(\omega_k)_{k \in \mathbb{N}_0} = \left(k^{2\sigma}\right)_{k \in \mathbb{N}_0}$, $w(\cdot) = 1 \vee |\cdot|^{3+\delta_1}$, $\sigma > 2$, $R, \delta_1, s > 0$, $0 < l, M < \infty$, $\epsilon = (\ln(n_e)/\ln_2(n_e))^{-\sigma}$, $n_e = n \wedge (\delta(n_0)/v(n_0,\mathcal{E}))$, $N = \lfloor \overline{N} \rfloor$, and $\overline{N} = \ln(n_e)/(4k_q(1+\sigma)\ln_2(n_e))$, we have

$$\sup_{\substack{f_{\boldsymbol{\beta}} \in \mathcal{H}_{w,W}^{q,\phi,\omega}(l,M) \cap D \\ f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}}} \mathbb{E}\left[\int_{\mathbb{R}^p} \left(\widehat{f}_{\boldsymbol{\beta}}^{q,N,\epsilon}(\boldsymbol{b}) - f_{\boldsymbol{\beta}}(\boldsymbol{b}) \right)^2 d\boldsymbol{b} \middle| \mathcal{G}_{n_0} \right] \left(\frac{\ln(n_e)}{\ln_2(n_e)} \right)^{2\sigma} = O_p(1).$$

Proof of Proposition A.3. We prove this result assuming that $f_{\boldsymbol{X}|\mathcal{X}}$ is known, the general case can be handled like in the proof of Theorem 1 (T3.1). Denote by $f^{\epsilon}_{\boldsymbol{\beta}} = F_1(\epsilon, \cdot)$ and $f^{q,\epsilon,N}_{\boldsymbol{\beta}} = \widetilde{F}^{q,N}_1(\epsilon, \cdot)$, where $\widetilde{F}^{q,N}_1$ (resp. $F^{q,N}_1$) is defined like $\widehat{F}^N_1$ with $\widetilde{c}_{\boldsymbol{m}}(t)$ (resp. $c_{\boldsymbol{m}}(t)$) defined in Lemma A.12 instead of $\widehat{c}_{\boldsymbol{m}}(t)$. Use the decomposition $\left\| \widehat{f}^{q,N,\epsilon}_{\boldsymbol{\beta}} - f_{\boldsymbol{\beta}} \right\|^2_{L^2(\mathbb{R}^p)} \leq 3 \sum_{j=1}^3 \|R_j\|^2_{L^2(\mathbb{R}^p)}$, where $R_1 = \widehat{f}_{\boldsymbol{\beta}} - f^{q,N,\epsilon}_{\boldsymbol{\beta}}$, $R_2 = f^{q,N,\epsilon}_{\boldsymbol{\beta}} - f^{\epsilon}_{\boldsymbol{\beta}}$, and $R_3 = f^{\epsilon}_{\boldsymbol{\beta}} - f_{\boldsymbol{\beta}}$.

Consider the case $q = 1$, as the case $q = \infty$ can be treated using similar computations. Using (A.69) and $(N + p - 1)^p \leq (pN)^{p-1}$ we have

$$\begin{aligned}
\mathbb{E}\left[\|R_1\|^2_{L^2(\mathbb{R}^p)} \right] \leq & \frac{c_{\boldsymbol{X}}}{(2\pi)^p} \frac{1}{n} \frac{(N+p-1)^p}{(Rx_0)^p (p-1)! (1-e^{-2})} \exp\left(2N \ln\left(\left(\frac{7\pi(N+1)}{Rx_0\epsilon} \right) \vee 1 \right) \right) \\
\leq & \frac{B}{n} N^{p-1} \exp\left(2N \ln\left(\left(\frac{7\pi(N+1)}{Rx_0\epsilon} \right) \vee 1 \right) \right). \qquad \text{(A.139)}
\end{aligned}$$

where

$$B = \frac{c_{\boldsymbol{X}} p^{p-1}}{(Rx_0)^p (2\pi)^p (p-1)! (1-e^{-2})}.$$

We directly have $\|R_2\|^2_{L^2(\mathbb{R}^{p+1})} \leq 2\pi l^2 / \omega_N$. We also have

$$\begin{aligned}
\|R_3\|^2_{L^2(\mathbb{R}^p)} = & \left\| \mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}} \right](\epsilon, \cdot) - \mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}} \right](0, \cdot) \right\|^2_{L^2(\mathbb{R}^p)} \\
\leq & \int_{[-R,R]^p} \left| \int_{\mathbb{R}} \left| e^{i\epsilon a} - 1 \right| f_{\alpha,\boldsymbol{\beta}}(a, \boldsymbol{b}) da \right|^2 d\boldsymbol{b} \\
\leq & \epsilon^2 \int_{[-R,R]^p} \left| \int_{\mathbb{R}} |a| \, f_{\alpha,\boldsymbol{\beta}}(a, \boldsymbol{b}) da \right|^2 d\boldsymbol{b} \\
\leq & \epsilon^2 M^2 \left(1 + \frac{1}{\delta_1} \right). \qquad \text{(A.140)}
\end{aligned}$$

Thus, using (A.139) and (A.140) we obtain

$$\begin{aligned}
& \sup_{\substack{f_{\boldsymbol{\beta}} \in \mathcal{H}^{q,\phi,\omega}_{w,W}(l,M) \cap D \\ f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}}} \mathbb{E}\left[\left\| \widehat{f}^{q,N,\epsilon}_{\boldsymbol{\beta}} - f_{\boldsymbol{\beta}} \right\|^2_{L^2(\mathbb{R}^p)} \right] \\
& \leq 3 \left(\frac{BN^{p-1}}{n} \left(\left(\frac{7\pi(N+1)}{Rx_0\epsilon} \right) \vee 1 \right)^{2N} + \frac{l^2}{N^{2\sigma}} + \epsilon^2 M^2 \left(1 + \frac{1}{\delta_1} \right) \right) \\
& \leq \frac{3}{N^{2\sigma}} \left(\frac{BN^{p-1}}{n} \left(\left(\frac{7\pi(N+1)^{1+2\sigma}}{Rx_0} \right) \vee 1 \right)^{2N} + l^2 + M^2 \left(1 + \frac{1}{\delta_1} \right) \right)
\end{aligned}$$

Thus, using (A.94), we have

$$\sup_{\substack{f_{\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)\cap D\\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}} \mathbb{E}\left[\left\|\widehat{f}^{q,N,\epsilon}_{\boldsymbol{\beta}} - f_{\boldsymbol{\beta}}\right\|^2_{L^2(\mathbb{R}^p)}\right] = O\left(\left(\frac{\ln(n)}{\ln_2(n)}\right)^{-2\sigma}\right).$$

□

References

[1] M. Abramowitz and I. Stegun. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*. Courier Corporation, 1965.

[2] N. Alibaud, P. Maréchal, and Y. Saesor. A variational approach to the inversion of truncated Fourier operators. *Inverse Probl.*, 25:045002, 2009.

[3] R. Beran. Prediction in random coefficient regression. *J. Stat. Plan. Infer.*, 43:205–213, 1995.

[4] R. Beran, A. Feuerverger, and P. Hall. On nonparametric estimation of intercept and slope distributions in random coefficient regression. *Ann. Stat.*, 24:2569–2592, 1996.

[5] M. Bertero and P. Boccacci. *Introduction to inverse problems in imaging*. CRC press, 1998.

[6] M. Bona. *A walk through combinatorics: an introduction to enumeration and graph theory*. World Scientific Publishing Co Inc, 2011.

[7] A. Bonami, P. Jaming, and A. Karoui. Further bounds on the eigenvalues of the sinc kernel operator. *Private communication*, 2016.

[8] A. Bonami and A. Karoui. Spectral decay of the sinc kernel operator and approximations by Prolate Spheroidal Wave Functions. Preprint arXiv:1012.3881.

[9] A. Bonami and A. Karoui. Uniform approximation and explicit estimates for the prolate spheroidal wave functions. *Constr. Approx.*, 43:15–45, 2016.

[10] L. Cavalier. Efficient estimation of a density in a problem of tomography. *Ann. Stat.*, 28:630–647, 2000.

[11] L. Cavalier. On the problem of local adaptive estimation in tomography. *Bernoulli*, 7:63–78, 2001.

[12] L. Cavalier, Y. Golubev, O. Lepski, and A. Tsybakov. Block thresholding and sharp adaptive estimation in severely ill-posed inverse problems. *Theory Probab. Appl.*, 48:426–446, 2004.

[13] X. Chen and M. Reiss. On rate optimality for ill-posed inverse problems in econometrics. *Economet. Theor.*, 27:497–521, 2011.

[14] F. Comte and C. Lacour. Deconvolution with estimated characteristic function of the errors. *J. R. Stat. Soc. Series B*, 73:601–627, 2011.

[15] F. Comte and T. Rebafka. Adaptive density estimation in the pile-up model involving measurement errors. *Electron. J. Stat.*, 6:2002–2037, 2012.

[16] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions, second ed.* Cambridge, 2014.

[17] M. de Jeu. Subspaces with equal closure. *Constr. Approx.*, 20:93–157, 2004.

[18] J. Frikel. Sparse regularization in limited angle tomography. *Appl. Comput. Harmon. Anal.*, 34(1):117–141, 2013.

[19] W. Fuchs. On the eigenvalues of an integral equation arising in the theory of band-limited signals. *J. Math. Anal. Appl.*, 9:317–330, 1964.

[20] C. Gaillac and E. Gautier. Estimates for the SVD of the truncated fourier transform on $L^2(\cosh(b\cdot))$ and stable analytic continuation. *Working paper*, 2019.

[21] C. Gaillac and E. Gautier. Identification in random coefficients linear and binary choice models when regressors have limited variation. *Working paper*, 2019.

[22] E. Gautier and S. Hoderlein. A triangular treatment effect model with random coefficients in the selection equation. Preprint arXiv:1109.0362v4.

[23] E. Gautier and Y. Kitamura. Nonparametric estimation in random coefficients binary choice models. *Econometrica*, 81:581–607, 2013.

[24] E. Gautier and E. Le Pennec. Adaptive estimation in the nonparametric random coefficients binary choice model by needlet thresholding. *Electron. J. Statist.*, 12:277–320, 2018.

[25] W. Gautschi. Some elementary inequalities relating to the Gamma and incomplete Gamma function. *J. Math. Phys.*, 38:77–81, 1959.

[26] A. Goldenshluger and O. Lepski. On adaptive minimax density estimation on r^d. *Probab. Theory Related Fields*, 159:479–543, 2014.

[27] J. Heckman and B. Singer. A method for minimizing the impact of distributional assumptions in econometric models for duration data. *Econometrica*, 52:271–320, 1984.

[28] S. Hoderlein, J. Klemelä, and E. Mammen. Analyzing the random coefficient model nonparametrically. *Economet. Theor.*, 26:804–837, 2010.

[29] D. Hohmann and H. Holzmann. Weighted angle radon transform: Convergence rates and efficient estimation. *Stat. Sinica*, 2015.

[30] A. Hoorfar and M. Hassani. Approximation of the Lambert W function and hyperpower function. *Research report*, 10, 2007.

[31] A. Karoui and T. Moumni. New efficient methods of computing the prolate spheroidal wave functions and their corresponding eigenvalues. *Appl. Comput. Harmon. Anal.*, 24:269–289, 2008.

[32] G. Kerkyacharian and D. Picard. Regression in random design and warped wavelets. *Bernoulli*, 10:1053–1105, 2004.

[33] R. Kress. *Linear integral equations*. Springer, 1999.

[34] C. Lacour. Rates of convergence for nonparametric deconvolution. *CR. Acad. Sci. I-Math.*, 342:877–882, 2006.

[35] C. Lacour. Adaptive estimation of the transition density of a particular hidden markov chain. *J. Multivariate Anal.*, 99(5):787–814, 2008.

[36] C. Lacour and P. Massart. Minimal penalty for Goldenshluger–Lepski method. *Stochastic Process. Appl.*, 126(12):3774–3789, 2016.

[37] H. Landau. On the density of phase-space expansions. *IEEE Trans. Inf. Theory*, 39:1152–1156, 1993.

[38] J. Lindberg. Mathematical concepts of optical superresolution. *J. Opt.*, 14:083001, 2012.

[39] A. Meister. Deconvolving compactly supported densities. *Math. Meth. Statist.*, 16:63–76, 2007.

[40] C. Mortici and C.-P. Chen. New sharp double inequalities for bounding the gamma and digamma function. *Analele Universitatii de Vest din Timisoara*, 49, 2011.

[41] F. Olver, D. Lozier, R. Boisvert, and C. Clark. *NIST Handbook of Mathematical Functions, first ed.* Cambridge University Press, 2010.

[42] A. Osipov, V. Rokhlin, and H. Xiao. *Prolate spheroidal wave functions of order zero*. Springer, 2013.

[43] V. Rokhlin and H. Xiao. Approximate formulae for certain prolate spheroidal wave functions valid for large values of both order and band-limit. *Appl. Comput. Harmon. Anal.*, 22:105–123, 2007.

[44] W. Rudin. *Functional analysis*. McGraw-Hill, 1987.

[45] A. Tsybakov. On the best rate of adaptive estimation in some inverse problems. *C. R. Acad. Sci. Paris Ser. I Math.*, 330:835–840, 2000.

[46] A. Tsybakov. *Introduction to nonparametric estimation*. Springer, 2008.

[47] H. Widom. Asymptotic behavior of the eigenvalues of certain integral equations. II. *Arch. Ration. Mech. Anal.*, 17:215–229, 1964.