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HOLOMORPHIC APPROXIMATION AND MIXED BOUNDARY VALUE PROBLEMS FOR $\bar{\partial}$

CHRISTINE LAURENT-THIÉBAUT AND MEI-CHI SHAW

ABSTRACT. In this paper, we study holomorphic approximation using boundary value problems for $\bar{\partial}$ on an annulus in the Hilbert space setting. The associated boundary conditions for $\bar{\partial}$ are the mixed boundary problems on an annulus. We characterize pseudoconvexity and Runge type property of the domain by the vanishing of related L^2 cohomology groups.

Holomorphic approximation theory plays an important role in function theory in one and several complex variables. In one complex variable, the classical Runge approximation theorem is related to solving the $\bar{\partial}$ equation with compact support (see e.g. Theorem 1.3.1 in Hörmander's book [8]). In several complex variables, it is shown in [15] that holomorphic approximation can also be formulated in terms of Dolbeault cohomology groups. We refer the reader to the recent paper [4] for a comprehensive and up-to-date account of this rich subject.

The purpose of this paper is to associate holomorphic approximation to a mixed boundary value problem for $\bar{\partial}$ on an annulus in the L^2 setting. Let Ω_1 and Ω_2 be two relatively compact domains in a complex hermitian manifold X of complex dimension n such that $\Omega_2 \subset\subset \Omega_1$. Consider the annulus $\Omega = \Omega_1 \setminus \bar{\Omega}_2$ between Ω_1 and Ω_2 . Let $\bar{\partial}$ and $\bar{\partial}_c$ be the (weak) maximal closure and the (strong) minimal closure of the differential operator $\bar{\partial}$. The two operators are naturally dual to each other (see [2]). The $\bar{\partial}$ -Neumann problem on a domain arises naturally and is of fundamental importance in several complex variables (see [8, 9], [5] or [1]).

The $\bar{\partial}$ -Neumann problem on an annulus between two pseudoconvex domains in \mathbb{C}^n has been studied earlier (see [18], [19], [10] and [2]). Recently, Li and Shaw [16] introduce the following mixed boundary problem for $\bar{\partial}$ on the annulus Ω . It was then extended by Chakrabarti and Harrington in [3] where, in particular, they weaken the regularity condition on the inner boundary of the annulus from the earlier work in [18] and [16]. In the L^2 setting, the $\bar{\partial}_{\text{mix}}$ operator on the annulus is the closed realization of $\bar{\partial}$ which satisfies the $\bar{\partial}$ -Neumann boundary condition on the outer boundary $b\Omega_1$ and the $\bar{\partial}$ -Cauchy problem on the inner boundary $b\Omega_2$. For $0 \leq p, q \leq n$ and $u \in L^2_{p,q}(\Omega)$, $u \in \text{Dom}(\bar{\partial}_{\text{mix}})$ if and only if there exists $v \in L^2_{p,q+1}(\Omega)$ and a sequence $(u_\nu)_{\nu \in \mathbb{N}} \subset L^2_{p,q}(\Omega)$ which vanish near $\partial\Omega_2$ such that $u_\nu \rightarrow u$ in $L^2_{p,q}(\Omega)$ and $\bar{\partial}u_\nu \rightarrow v$ in $L^2_{p,q+1}(\Omega)$. If $u \in \text{Dom}(\bar{\partial}_{\text{mix}})$, then we define $\bar{\partial}_{\text{mix}}u = v$. It is obvious that $\bar{\partial}_{\text{mix}}$ is a densely defined closed operator from

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one Hilbert space to another and

$$\bar{\partial}_c \subseteq \bar{\partial}_{\text{mix}} \subseteq \bar{\partial}.$$

Let D be a domain in X and $\mathcal{O}(D)$ denote the space of holomorphic functions in D and $W^1(D)$ be the Sobolev 1-space on D . The following theorem is proved in Theorems 2.2 and 2.4 in [16].

Theorem 0.1. *Assume X is Stein and both Ω_1 and Ω_2 are pseudoconvex with $C^{1,1}$ boundary then, for any $2 \leq q \leq n$ and $q = 0$, $H_{\bar{\partial}_{\text{mix}}}^{0,q}(\Omega) = 0$. When $q = 1$, we have*

$$(0.1) \quad H_{\bar{\partial}_{\text{mix}}}^{0,1}(\Omega) \cong \mathcal{O}(\Omega_2) \cap W^1(\Omega_2) / \mathcal{O}(\Omega_1) \cap L^2(\Omega_1).$$

Moreover, $H_{\bar{\partial}_{\text{mix}}}^{0,1}(\Omega)$ is infinite dimensional (see [16]). In fact, it is even non-Hausdorff (see section 5 in [3]). The non-Hausdorff property of the quotient group is equivalent to that the space $\mathcal{O}(\Omega_1) \cap L^2(\Omega_1)$ is not a closed subspace in $\mathcal{O}(\Omega_2) \cap W^1(\Omega_2)$ under the $W^1(\Omega_2)$ norm (see Proposition 4.5 in [22]).

Instead of considering the non-Hausdorff cohomology group $H_{\bar{\partial}_{\text{mix}}}^{0,1}(\Omega)$, we consider the associated Hausdorff cohomology group $\sigma H_{\bar{\partial}_{\text{mix}}}^{0,1}(\Omega)$ defined by

$$(0.2) \quad \sigma H_{\bar{\partial}_{\text{mix}}}^{0,1}(\Omega) \cong \mathcal{O}(\Omega_2) \cap W^1(\Omega_2) / \overline{\mathcal{O}(\Omega_1) \cap L^2(\Omega_1)},$$

where $\overline{\mathcal{O}(\Omega_1) \cap L^2(\Omega_1)}$ is the closure of the space $\mathcal{O}(\Omega_1) \cap L^2(\Omega_1)$ under the $W^1(\Omega_2)$ -norm. Then the space defined on the right-hand side of (0.2) is Hausdorff.

It is easy to see that the space $\mathcal{O}(\Omega_1) \cap L^2(\Omega_1)$ is dense in $\mathcal{O}(\Omega_2) \cap W^1(\Omega_2)$ for the W^1 topology on Ω_2 if and only if $\sigma H_{\bar{\partial}_{\text{mix}}}^{0,1}(\Omega) = 0$. Thus the associated Hausdorff cohomology group $\sigma H_{\bar{\partial}_{\text{mix}}}^{0,1}(\Omega)$ is directly related to holomorphic approximation. This simple observation motivates the present paper. However, the L^2 condition on the holomorphic functions near the boundary of Ω_1 is of no interest in holomorphic approximation. We avoid the growth condition and reformulate another $\bar{\partial}$ problem with mixed boundary conditions which is more suitable for holomorphic approximation.

We consider the more general situation: let D be a relatively compact domain in a complex hermitian manifold X , For $0 \leq p, q \leq n$, we define a new operator $\bar{\partial}_{\text{Mix}}$ on $(L_{\text{loc}}^2)^{p,q}(X \setminus \bar{D})$, whose domain is the set of all $u \in (L_{\text{loc}}^2)^{p,q}(X)$ such that u is vanishing on D and $\bar{\partial}u \in (L_{\text{loc}}^2)^{p,q+1}(X)$, where $\bar{\partial}u$ is taken in the sense of currents. Then we set $\bar{\partial}_{\text{Mix}}f = \bar{\partial}f$ in the sense of currents. Compare to the $\bar{\partial}_{\text{mix}}$ operator, we do not assume any growth condition at infinity of X .

The plan of the paper is as follows: In the first section, we formulate a new mixed boundary conditions of $\bar{\partial}$, denoted by $\bar{\partial}_{\text{Mix}}$, which is associated naturally with holomorphic approximation. We prove a theorem (see Theorem 1.2) analogous to Theorem 0.1.

In the second section, we introduce the transposed operator ${}^t\bar{\partial}_{\text{Mix}}$ to $\bar{\partial}_{\text{Mix}}$ defined on $(L_{\text{loc}}^2)^{n-p,n-q-1}(X \setminus \bar{D})$, whose domain is the $u \in L_{n-p,n-q-1}^2(X \setminus \bar{D})$ and u is vanishing outside a compact subset of X such that $\bar{\partial}u \in L_{n-p,n-q}^2(X \setminus \bar{D})$, where $\bar{\partial}u$ is taken in the sense of currents,. We prove the following characterization of approximation of $\bar{\partial}$ -closed forms using a version of the Serre duality.

Theorem 0.2. *Let X be a Stein manifold of complex dimension $n \geq 2$, $D \subset\subset X$ a relatively compact pseudoconvex domain in X with Lipschitz boundary. Let q be a fixed*

integer such that $0 \leq q \leq n - 1$. Then, for any $0 \leq p \leq n$, the following assertions are equivalent.

- (1) The space of W_{loc}^1 $\bar{\partial}$ -closed (p, q) -forms on X is dense in the space W^1 $\bar{\partial}$ -closed (p, q) -forms on D for the W^1 topology on D ;
- (2) The natural map $H_{\bar{D}, W^{-1}}^{n-p, n-q}(X) \rightarrow H_c^{n-p, n-q}(X)$ is injective;
- (3) $H_{\bar{\partial}_{\text{Mix}}}^{n-p, n-q-1}(X \setminus \bar{D}) = 0$.

Finally, we obtain the following characterization of a pseudoconvex domain satisfying some Runge type property (see Corollary 2.5).

Theorem 0.3. *Let X be a Stein manifold of complex dimension $n \geq 2$ and $D \subset\subset X$ a relatively compact domain in X with $\mathcal{C}^{1,1}$ boundary such that $X \setminus D$ is connected. Then the following assertions are equivalent:*

- (1) the domain D is pseudoconvex and the space $\mathcal{O}(X)$ is dense in the space $\mathcal{O}(D) \cap W^1(D)$ for the W^1 topology on D ;
- (2) $H_{\bar{D}, W^{-1}}^{n, r}(X) = 0$, for $2 \leq r \leq n - 1$, and the natural map $H_{\bar{D}, W^{-1}}^{n, n}(X) \rightarrow H_c^{n, n}(X)$ is injective;
- (3) $H_{\bar{\partial}_{\text{Mix}}}^{n, q}(X \setminus \bar{D}) = 0$, for all $1 \leq q \leq n - 1$.

From (1) and (3) in Theorem 0.3, we see that the vanishing of the cohomology groups $H_{\bar{\partial}_{\text{Mix}}}^{n, q}(X \setminus \bar{D})$ for all $1 \leq q \leq n - 1$ characterizes pseudoconvexity and a Runge type property of D . This is in contrast to earlier results using cohomology groups on $X \setminus \bar{D}$ to characterize holomorphic convexity (see Trapani [21]). It is proved in [21] that the vanishing of the Dolbeault cohomology groups $H^{n, q}(X \setminus \bar{D})$ for $1 \leq q \leq n - 2$ and the Hausdorff property for $q = n - 1$ characterizes the holomorphic convexity of \bar{D} . More recently, it is proved in Fu-Laurent-Shaw [6] that the vanishing of the L^2 Dolbeault cohomology groups $H_{L^2}^{n, q}(X \setminus \bar{D})$ for $1 \leq q \leq n - 2$ and the Hausdorff property for $q = n - 1$ characterizes pseudoconvexity of D (see [6]). Thus different cohomology groups characterize different holomorphic property of the domain D . Our results show that $\bar{\partial}_{\text{Mix}}$ and its transpose ${}^t\bar{\partial}_{\text{Mix}}$ are naturally associated with holomorphic approximation.

1. W^1 -MERGELYAN DOMAINS AND L^2 THEORY FOR $\bar{\partial}$ WITH MIXED BOUNDARY CONDITIONS

Let X be a complex hermitian manifold X of complex dimension n , where $n \geq 2$.

Definition 1.1. A relatively compact domain D with Lipschitz boundary in a complex manifold X is called W^1 -Mergelyan in X if and only if, for any $0 \leq p \leq n$, the space $H^{p, 0}(X)$ of holomorphic $(p, 0)$ -forms in X is dense in the space $H_{W^1}^{p, 0}(D)$ of W^1 holomorphic $(p, 0)$ -forms in D for the W^1 topology on D .

We would like to characterize domains which are W^1 -Mergelyan in X by means of some adapted mixed boundary value problem for the $\bar{\partial}$ -operator. Let $L_{loc}^2(X)$ be the space of L_{loc}^2 functions in X endowed with the Fréchet topology of L^2 convergence on compact subsets, and $L_c^2(X)$ the space of L^2 functions with compact support in X with the inductive limit topology. These two spaces are dual of each other (see [17] or [13]). We use $(L_c^2)^{p, q}(X)$ to denote the space of (p, q) -forms with $L_c^2(X)$ coefficients. For $0 \leq p, q \leq n$, we define the densely defined operator $\bar{\partial}_K$ from $(L_c^2)^{p, q}(X)$ into $(L_c^2)^{p, q+1}(X)$, whose domain is the

space of all $f \in (L^2_c)^{p,q}(X)$ with $\bar{\partial}f \in (L^2_c)^{p,q+1}(X)$, such that for any $f \in \text{Dom}(\bar{\partial}_K)$, $\bar{\partial}_K f = \bar{\partial}f$ in the sense of currents. We denote by $\bar{\partial}_{loc}$ the densely defined transposed operator of $\bar{\partial}_K$, then $\bar{\partial}_{loc}$ maps $(L^2_{loc})^{n-p,n-q-1}(X)$ into $(L^2_{loc})^{n-p,n-q}(X)$ and the domain of $\bar{\partial}_{loc}$ is the space of all $f \in (L^2_{loc})^{n-p,n-q-1}(X)$ such that $\bar{\partial}f \in (L^2_{loc})^{n-p,n-q}(X)$.

Let D be a relatively compact domain with Lipschitz boundary in a complex manifold X . We are interested in the study in the L^2 setting of some operators $\bar{\partial}_{\text{Mix}}$ on $X \setminus \bar{D}$ such that $\bar{\partial}_K \subseteq \bar{\partial}_{\text{Mix}} \subseteq \bar{\partial}_{loc}$, where $\bar{\partial}_K$ and $\bar{\partial}_{loc}$ are the previously defined operators. The domain of $\bar{\partial}_{\text{Mix}}$ is defined as follows:

For $0 \leq p, q \leq n$ and $u \in (L^2_{loc})^{p,q}(X \setminus \bar{D})$, $u \in \text{Dom}(\bar{\partial}_{\text{Mix}})$ if and only if $u \in (L^2_{loc})^{p,q}(X)$, u is vanishing on D and $\bar{\partial}u \in (L^2_{loc})^{p,q+1}(X)$, where $\bar{\partial}u$ is taken in the sense of currents. Then we set $\bar{\partial}_{\text{Mix}}f = \bar{\partial}f$ in the sense of currents. The transposed operator ${}^t\bar{\partial}_{\text{Mix}}$ is then an operator whose domain is given by the set of all $u \in (L^2_{loc})^{n-p,n-q-1}(X \setminus \bar{D})$, $u \in L^2_{n-p,n-q-1}(X \setminus \bar{D})$, $\bar{\partial}u \in L^2_{n-p,n-q}(X \setminus \bar{D})$, where $\bar{\partial}u$ is taken in the sense of currents, and u is vanishing outside a compact subset of X .

For any $0 \leq p \leq n$, we get two new differential complexes $((L^2_{loc})^{p,\bullet}(X \setminus \bar{D}), \bar{\partial}_{\text{Mix}})$ and $((L^2_{loc})^{n-p,\bullet}(X \setminus \bar{D}), {}^t\bar{\partial}_{\text{Mix}})$, which are dual complexes since the boundary of D is Lipschitz (see [14]). We denote by $H_{\bar{\partial}_{\text{Mix}}}^{p,q}(X \setminus \bar{D})$ and $H_{{}^t\bar{\partial}_{\text{Mix}}}^{p,q}(X \setminus \bar{D})$, $0 \leq q \leq n$, the cohomology groups of the complexes $(L^2_{p,\bullet}(X \setminus \bar{D}), \bar{\partial}_{\text{Mix}})$ and $(L^2_{n-p,\bullet}(X \setminus \bar{D}), {}^t\bar{\partial}_{\text{Mix}})$ respectively. We endow the cohomology groups with quotient topology. Then it follows from Serre duality [17] that $H_{\bar{\partial}_{\text{Mix}}}^{p,q}(X \setminus \bar{D})$ is Hausdorff if and only if $H_{{}^t\bar{\partial}_{\text{Mix}}}^{n-q+1}(X \setminus \bar{D})$ is Hausdorff. Moreover, if $H_{{}^t\bar{\partial}_{\text{Mix}}}^{p,q}(X \setminus \bar{D})$ is Hausdorff, then $H_{\bar{\partial}_{\text{Mix}}}^{p,q}(X \setminus \bar{D})$ is the dual space of $\bar{H}_{\bar{\partial}_{\text{Mix}}}^{n-p,n-q}(X \setminus \bar{D})$ the Hausdorff group associated to $H_{\bar{\partial}_{\text{Mix}}}^{n-p,n-q}(X \setminus \bar{D})$.

Theorem 1.2. *Let X be a Stein manifold of complex dimension $n \geq 2$ with a hermitian metric and D a relatively compact pseudoconvex domain with $C^{1,1}$ boundary in X . Then, for any $0 \leq p \leq n$, we have*

- (1) $H_{\bar{\partial}_{\text{Mix}}}^{p,q}(X \setminus \bar{D}) = 0$, if $2 \leq q \leq n$ or $q = 0$.
- (2) $H_{\bar{\partial}_{\text{Mix}}}^{p,1}(X \setminus \bar{D})$ is topologically isomorphic to $H_{W^1}^{p,0}(D)/H^{p,0}(X)$, endowed with the quotient topology.

Proof. The proof is similar to the proof of Theorems 2.2 and 2.4 in [16]. If $q = 0$, $H_{\bar{\partial}_{\text{Mix}}}^{p,0}(X \setminus \bar{D})$ is the space of holomorphic $(p, 0)$ -forms in X , which vanish identically on D . Since X is Stein, hence connected, by analytic continuation we get $H_{\bar{\partial}_{\text{Mix}}}^{p,0}(X \setminus \bar{D}) = 0$.

We now assume that $2 \leq q \leq n$. Let $f \in \ker(\bar{\partial}_{\text{Mix}}) \cap \text{Dom}(\bar{\partial}_{\text{Mix}})$. Then $f \in (L^2_{loc})^{p,q}(X)$, $f = 0$ in D and $\bar{\partial}f = 0$ in X . Since X is Stein, $H^{p,q}(X) = 0$ and by the Dolbeault isomorphism and the interior regularity of the $\bar{\partial}$, we get $H_{L^2_{loc}}^{p,q}(X) = 0$. More precisely there exists $v \in (W^1_{loc})^{p,q-1}(X)$ such that $\bar{\partial}v = f$. Moreover we have $\bar{\partial}v = 0$ on D .

Since $q > 1$ and D is a relatively compact pseudoconvex domain with $C^{1,1}$ boundary, it follows from [7] or Theorem 2.2 in [3] (see also [11] for smooth boundary) that there exists $w \in W^1_{p,q-2}(D)$ such that $\bar{\partial}w = v$ in D . Let \tilde{w} be a W^1_{loc} extension of w to X . We set $u = v - \bar{\partial}\tilde{w}$. Then u is in $(L^2_{loc})^{p,q-1}(X)$, u vanishes on D and satisfies $\bar{\partial}u = f$. This proves (1).

We now consider the case when $q = 1$. For any $f \in H_{W^1}^{p,0}(D)$, we extend f as a W_{loc}^1 $(p, 0)$ -form $\tilde{f} = E(f)$ on X , where E is a continuous extension operator from $W_{p,0}^1(D)$ into $(W_{loc}^1)^{p,0}(X)$. This is possible since the boundary of D is $C^{1,1}$. Then $\bar{\partial}\tilde{f} \in (L_{loc}^2)^{p,1}(X)$ and $\bar{\partial}\tilde{f} = 0$ on D . Thus $\bar{\partial}_{\text{Mix}}(\bar{\partial}\tilde{f}) = 0$ in $X \setminus \bar{D}$. We define a map

$$(1.1) \quad l : H_{W^1}^{p,0}(D) \rightarrow H_{\bar{\partial}_{\text{Mix}}}^{p,1}(X \setminus \bar{D})$$

by $l(f) = [\bar{\partial}\tilde{f}]$.

First, we show that l is well-defined. If \tilde{f}_1 is another W_{loc}^1 extension of f to X , then

$$\bar{\partial}\tilde{f} - \bar{\partial}\tilde{f}_1 = \bar{\partial}(\tilde{f} - \tilde{f}_1).$$

Since $\tilde{f} = \tilde{f}_1 = f$ on D , $\tilde{f} - \tilde{f}_1$ vanishes on D and $\bar{\partial}\tilde{f} - \bar{\partial}\tilde{f}_1 = \bar{\partial}_{\text{Mix}}(\tilde{f} - \tilde{f}_1)$, that is

$$[\bar{\partial}\tilde{f}] = [\bar{\partial}\tilde{f}_1] \quad \text{in} \quad H_{\bar{\partial}_{\text{Mix}}}^{p,1}(X \setminus \bar{D}).$$

Thus the map l is well-defined and it is continuous if $H_{\bar{\partial}_{\text{Mix}}}^{p,1}(X \setminus \bar{D})$ is endowed with the quotient topology.

We will show that the kernel of the map l is $H^{p,0}(X)$. Let $f \in H_{W^1}^{p,0}(D)$ such that $l(f) = [0]$. First we extend f as a W_{loc}^1 $(p, 0)$ -form on X . Thus we have that $\bar{\partial}\tilde{f}$ is a $\bar{\partial}_{\text{Mix}}$ -closed form and, since $l(f) = [0]$, it is $\bar{\partial}_{\text{Mix}}$ -exact. Therefore there exists $g \in (L_{loc}^2)^{p,0}(X)$ such that $g = 0$ on D and $\bar{\partial}_{\text{Mix}}g = \bar{\partial}\tilde{f}$. Let $F = \tilde{f} - g$. Then F is holomorphic in X and $F = f$ on D . Thus $l(f) = 0$ implies that f can be extended as a holomorphic $(p, 0)$ -form in X .

Next we prove that l is surjective. Let $f \in (L_{loc}^2)^{p,1}(X) \cap \ker(\bar{\partial}_{\text{Mix}})$, then $f = 0$ in D and $\bar{\partial}f = 0$ in X . Since X is a Stein manifold, using Dolbeault isomorphism and the interior regularity of the $\bar{\partial}$ operator, there exists a $(p, 0)$ -form $u \in (W_{loc}^1)^{p,0}(X)$ such that $\bar{\partial}u = f$ in X . Moreover $u|_D$ is a W^1 holomorphic $(p, 0)$ -form in D . Hence $l(u|_D) = [\bar{\partial}u] = [f]$.

Finally we get the topological isomorphism

$$H_{\bar{\partial}_{\text{Mix}}}^{p,1}(X \setminus \bar{D}) \cong H_{W^1}^{p,0}(D)/H^{p,0}(X)$$

if we endow the quotient space $H_{W^1}^{p,0}(D)/H^{p,0}(X)$ with the quotient topology. \square

Using the same arguments as in [16] and [3], one has that

Corollary 1.3. $H_{\bar{\partial}_{\text{Mix}}}^{0,1}(X \setminus \bar{D})$ is infinite dimensional and non-Hausdorff.

We note that the non-Hausdorff property of the quotient space $H_{\bar{\partial}_{\text{Mix}}}^{0,1}(X \setminus \bar{D})$ is equivalent to that the space $H^{p,0}(X)$ is not a closed subspace in $H_{W^1}^{p,0}(D)$ (see Proposition 4.5 in [22]).

Definition 1.4. We define the associated Hausdorff quotient

$$(1.2) \quad \sigma(H_{W^1}^{p,0}(D)/H^{p,0}(X)) = H_{W^1}^{p,0}(D)/\overline{H^{p,0}(X)}$$

where $\overline{H^{p,0}(X)}$ is the closure of the space $H^{p,0}(X)$ under the $W^1(D)$ -norm.

Corollary 1.5. *Assume X is a Stein manifold of complex dimension $n \geq 2$ and D a relatively compact pseudoconvex domain with $C^{1,1}$ boundary in X .*

Suppose that D is W^1 -Mergelyan. Then for any $0 \leq p \leq n$, $H_{\bar{\partial}_{\text{Mix}}}^{n-p, n-1}(X \setminus \bar{D}) = 0$.

Proof. From (2) in Theorem 1.2 and (1.2), we have

$$\sigma H_{\bar{\partial}_{\text{Mix}}}^{p,1}(X \setminus \bar{D}) \cong H_{W^1}^{p,0}(D)/\overline{H^{p,0}(X)}.$$

Thus if D is W^1 -Mergelyan in X , $\sigma H_{\bar{\partial}_{\text{Mix}}}^{p,1}(X \setminus \bar{D}) = 0$. It follows from Serre duality and from Theorem 1.2 that $H_{\bar{\partial}_{\text{Mix}}}^{n-p, n-1}(X \setminus \bar{D})$ is Hausdorff, since $H_{\bar{\partial}_{\text{Mix}}}^{p,2}(X \setminus \bar{D}) = 0$. Using again Serre duality, we get $H_{\bar{\partial}_{\text{Mix}}}^{n-p, n-1}(X \setminus \bar{D}) = 0$. \square

2. THE W^1 q -MERGELYAN DENSITY PROPERTY

In this section we extend the approximation results to arbitrary (p, q) -forms.

Definition 2.1. A relatively compact domain D with Lipschitz boundary in X is W^1 q -Mergelyan, for $0 \leq q \leq n-1$, if and only if, for any $0 \leq p \leq n$, the space $Z_{W_{loc}^1}^{p,q}(X)$ of W_{loc}^1 $\bar{\partial}$ -closed (p, q) -forms in X is dense in the space $Z_{W^1}^{p,q}(D)$ of W^1 $\bar{\partial}$ -closed (p, q) -forms in D for the W^1 topology on D .

For $p = q = 0$, we will simply say that the domain is W^1 -Mergelyan in X .

If $D \subset\subset X$ is a relatively compact domain with Lipschitz boundary in X , we denote by $H_{\bar{D}, W^{-1}}^{r,s}(X)$ the Dolbeault cohomology groups of W^{-1} currents with prescribed support in \bar{D} and by $H_{\bar{\partial}_{\text{Mix}}}^{r,s}(X \setminus \bar{D})$ the Dolbeault cohomology groups of L^2 forms in $X \setminus \bar{D}$ vanishing outside a compact subset of X . We have that $W^s(D)$ is a reflexive Banach space, i.e. $(W_{\bar{D}, W^{-1}}^{-s}(X))' = W^s(D)$. This follows from the proof similar to [12]

Theorem 2.2. *Let X be a non compact complex manifold of complex dimension $n \geq 1$, $D \subset\subset X$ a relatively compact domain with Lipschitz boundary in X and q be a fixed integer such that $0 \leq q \leq n-1$. Assume that, for any $0 \leq p \leq n$, $H_c^{n-p, n-q}(X)$ and $H_{\bar{D}, W^{-1}}^{n-p, n-q}(X)$ are Hausdorff. Then D is a W^1 q -Mergelyan domain in X if and only if the natural map $H_{\bar{D}, W^{-1}}^{n-p, n-q}(X) \rightarrow H_c^{n-p, n-q}(X)$ is injective.*

Proof. Assume D is W^1 q -Mergelyan in X and let $T \in W_{n-p, n-q}^{-1}(X)$ with support contained in \bar{D} . We assume that the cohomological class $[T]$ of T vanishes in $H_c^{n-p, n-q}(X)$, which means that there exists $S \in W_{n-p, n-q-1}^{-1}(X)$ with compact support in X such that $T = \bar{\partial}S$. Since $H_{\bar{D}, W^{-1}}^{n-p, n-q}(X)$ is Hausdorff, then $[T] = 0$ in $H_{\bar{D}, W^{-1}}^{n-p, n-q}(X)$ if and only if, for any form $\varphi \in Z_{W^1}^{p,q}(D)$, we have $\langle T, \varphi \rangle = 0$. But, as D is W^1 q -Mergelyan in X , there exists a sequence $(\varphi_k)_{k \in \mathbb{N}}$ of W_{loc}^1 $\bar{\partial}$ -closed (p, q) -forms in X which converge to φ in $W^1(D)$. So

$$\langle T, \varphi \rangle = \lim_{k \rightarrow \infty} \langle T, \varphi_k \rangle = \lim_{k \rightarrow \infty} \langle \bar{\partial}S, \varphi_k \rangle = \pm \lim_{k \rightarrow \infty} \langle S, \bar{\partial}\varphi_k \rangle = 0.$$

Conversely, by the Hahn-Banach theorem, it is sufficient to prove that, for any form $g \in Z_{W^1}^{p,q}(D)$ and any $(n-p, n-q)$ -current T in $W_{p,q}^{-1}(D)$ with compact support in \bar{D} such that $\langle T, f \rangle = 0$ for any form $f \in Z_{W_{loc}^1}^{p,q}(X)$, we have $\langle T, g \rangle = 0$. Since $H_c^{n-p, n-q}(X)$ is

Hausdorff, the hypothesis on T implies that there exists a $W^{-1}(n-p, n-q-1)$ -current S with compact support in X such that $T = \bar{\partial}S$. The injectivity of the natural map $H_{D, W^{-1}}^{n-p, n-q}(X) \rightarrow H_c^{n-p, n-q}(X)$ implies that there exists a $W^{-1}(n-p, n-q-1)$ -current U with compact support in \bar{D} such that $T = \bar{\partial}U$. Hence since the boundary of D is Lipschitz, for any $g \in Z_{W^1}^{p, q}(D)$, we get

$$\langle T, g \rangle = \langle \bar{\partial}U, g \rangle = \pm \langle U, \bar{\partial}g \rangle = 0.$$

□

Proposition 2.3. *Let X be a non compact complex manifold of complex dimension $n \geq 2$, $D \subset\subset X$ a relatively compact domain in X with Lipschitz boundary and q a fixed integer such that $0 \leq q \leq n-2$. Assume that, for some $0 \leq p \leq n$, $H_c^{n-p, n-q-1}(X) = 0$. Then $H_{\bar{\partial}_{\text{Mix}}}^{n-p, n-q-1}(X \setminus \bar{D}) = 0$ if and only if the natural map $H_{\bar{D}, W^{-1}}^{n-p, n-q}(X) \rightarrow H_c^{n-p, n-q}(X)$ is injective.*

Proof. We first consider the necessary condition. Let $T \in W_{n-p, n-q}^{-1}(X)$ be a $\bar{\partial}$ -closed current with support contained in \bar{D} such that the cohomological class $[T]$ of T vanishes in $H_c^{n-p, n-q}(X)$. By the interior regularity property of the $\bar{\partial}$ -operator and the Dolbeault isomorphism, there exists $g \in L_{n-p, n-q-1}^2(X)$ and compactly supported such that $T = \bar{\partial}g$. Since the support of T is contained in \bar{D} , we have $\bar{\partial}g = 0$ on $X \setminus \bar{D}$. Therefore the vanishing of the group $H_{\bar{\partial}_{\text{Mix}}}^{n-p, n-q-1}(X \setminus \bar{D})$ implies that there exists $u \in L_{n-p, n-q-2}^2(X \setminus \bar{D})$ vanishing outside a compact subset of X and such that $\bar{\partial}u = g$ on $X \setminus \bar{D}$. Since the boundary of D is Lipschitz there exists \tilde{u} a L^2 extension of u to X , we set $S = g - \bar{\partial}\tilde{u}$, then $S \in W^{-1}(X)$ satisfies $T = \bar{\partial}S$ and $\text{supp } S \subset \bar{D}$.

Conversely, let g be a $\bar{\partial}$ -closed $(n-p, n-q-1)$ -form in $L_{n-p, n-q-1}^2(X \setminus \bar{D})$ which vanishes outside a compact subset of X and \tilde{g} an L^2 extension of g to X , then \tilde{g} has compact support in X and $T = \bar{\partial}\tilde{g}$ is a current in $W_{n-p, n-q}^{-1}(X)$ with support in \bar{D} . By the injectivity of the natural map $H_{\bar{D}, W^{-1}}^{n-p, n-q}(X) \rightarrow H_c^{n-p, n-q}(X)$, there exists $S \in W_{n-p, n-q-1}^{-1}(X)$ with support contained in \bar{D} and such that $\bar{\partial}S = T$. We set $U = \tilde{g} - S$. Then U is a W^{-1} $\bar{\partial}$ -closed $(n-p, n-q-1)$ -current with compact support in X such that $U|_{X \setminus \bar{D}} = g$ in $X \setminus \bar{D}$. Since $H_c^{n-p, n-q-1}(X) = 0$, by the interior regularity property of the $\bar{\partial}$ -operator and the Dolbeault isomorphism, we have $U = \bar{\partial}w$ for some $w \in L_{n-p, n-q-2}^2(X)$ with compact support in X . Finally we get $g = U|_{X \setminus \bar{D}} = \bar{\partial}(w|_{X \setminus \bar{D}})$. □

Corollary 2.4. *Let X be a Stein hermitian manifold of complex dimension $n \geq 2$ and $D \subset\subset X$ a relatively compact pseudoconvex domain with $\mathcal{C}^{1,1}$ boundary in X . Then the following assertions are equivalent:*

- i) the domain D is W^1 -Mergelyan in X ,
- ii) the natural map $H_{\bar{D}, W^{-1}}^{n, n}(X) \rightarrow H_c^{n, n}(X)$ is injective,
- iii) $H_{\bar{\partial}_{\text{Mix}}}^{n, n-1}(X \setminus \bar{D}) = 0$.

Proof. Since X is Stein, we have $H_c^{n, n-1}(X) = 0$ and $H_c^{n, n}(X)$ is Hausdorff. The domain D being relatively compact, pseudoconvex with $\mathcal{C}^{1,1}$ boundary in X , we have $H_{W^1}^{0, 1}(D) = 0$. Then Serre duality implies that $H_{\bar{D}, W^{-1}}^{n, n}(X)$ is Hausdorff. The corollary follows then from Theorem 2.2 and Proposition 2.3. □

Finally using the characterization of pseudoconvexity by mean of W^1 cohomology and Serre duality, we can prove the following corollary.

Corollary 2.5. *Let X be a Stein hermitian manifold of complex dimension $n \geq 2$ and $D \subset\subset X$ a relatively compact domain in X with $C^{1,1}$ boundary such that $X \setminus D$ is connected. Then the following assertions are equivalent:*

- (i) *the domain D is pseudoconvex and W^1 -Mergelyan in X ;*
- (ii) *$H_{\bar{D}, W^{-1}}^{n,r}(X) = 0$, for $2 \leq r \leq n-1$, and the natural map $H_{\bar{D}, W^{-1}}^{n,n}(X) \rightarrow H_c^{n,n}(X)$ is injective;*
- (iii) *$H_{t\bar{\partial}_{\text{Mix}}}^{n,q}(X \setminus \bar{D}) = 0$, for all $1 \leq q \leq n-1$.*

Proof. We first notice that a domain D with $C^{1,1}$ boundary is pseudoconvex if and only if $H_{W^1}^{0,q}(D) = 0$ for all $1 \leq q \leq n-1$. This follows from [7] or Theorem 2.2 in [3] for the necessary condition and Theorem 5.1 in [6] for the sufficient condition. Applying Serre duality, we get that D is pseudoconvex if and only if $H_{\bar{D}, W^{-1}}^{n,q}(X) = 0$ for all $2 \leq q \leq n-1$ and $H_{\bar{D}, W^{-1}}^{n,n}(X)$ is Hausdorff.

To get the equivalence between (i) and (ii), it remains to prove that the injectivity of the natural map $H_{\bar{D}, W^{-1}}^{n,n}(X) \rightarrow H_c^{n,n}(X)$ implies that $H_{\bar{D}, W^{-1}}^{n,n}(X)$ is Hausdorff and to apply Theorem 2.2.

Let T be a $W^{-1}(n, n)$ -current with support in \bar{D} such that $\langle T, \varphi \rangle = 0$ for any W^1 holomorphic function φ on D . In particular $\langle T, \varphi \rangle = 0$ for any holomorphic function φ on X and X being Stein, $H_c^{n,n}(X)$ is Hausdorff and therefore $T = \bar{\partial}S$ for some $W^{-1}(n, n-1)$ -current S with compact support in X , i.e. $[T] = 0$ in $H_c^{n,n}(X)$. By the injectivity of the map $H_{\bar{D}, W^{-1}}^{n,n}(X) \rightarrow H_c^{n,n}(X)$, we get that $T = \bar{\partial}U$ for some $W^{-1}(n, n-1)$ -current U with support in \bar{D} , which ends the proof.

We next prove the equivalence between (ii) and (iii). It follows from Theorem 4.8 in [6] that, $H_{W^1}^{0,q}(\bar{D}) = 0$ for all $1 \leq q \leq n-1$ if and only if $H_{L^2}^{n,q}(X \setminus \bar{D}) = 0$ for all $1 \leq q \leq n-2$ and $H_{L^2}^{n,n-1}(X \setminus \bar{D})$ is Hausdorff. But, by Serre duality, $H_{W^1}^{0,q}(\bar{D}) = 0$ for all $1 \leq q \leq n-1$ is equivalent to $H_{\bar{D}, W^{-1}}^{n,q}(X) = 0$ for all $2 \leq q \leq n-1$ and $H_{\bar{D}, W^{-1}}^{n,n}(X)$ is Hausdorff. It remains to prove that, for all $1 \leq q \leq n-2$, $H_{t\bar{\partial}_{\text{Mix}}}^{n,q}(X \setminus \bar{D}) = 0$ if and only if $H_{L^2}^{n,q}(X \setminus \bar{D}) = 0$ and that $H_{t\bar{\partial}_{\text{Mix}}}^{n,n-1}(X \setminus \bar{D}) = 0$, implies $H_{L^2}^{n-p, n-1}(X \setminus \bar{D})$ is Hausdorff. This can be done since X is Stein and both are equivalent to $H_{W^1}^{0,q}(\bar{D}) = 0$ for all $1 \leq q \leq n-1$ (see [14]). Then we apply Proposition 2.3 to get the result. The corollary is proved. \square

From Corollary 2.5, the vanishing of the cohomology groups $H_{t\bar{\partial}_{\text{Mix}}}^{n,q}(X \setminus \bar{D})$ characterizes pseudoconvexity and W^1 -Mergelyan property of D .

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