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# Semimartingales and Shrinkage of Filtration

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## Abstract

We consider a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , which is endowed with two filtrations,  $\mathbb{G}$  and  $\mathbb{F}$ , assumed to satisfy the usual conditions and such that  $\mathbb{F} \subset \mathbb{G}$ . On this probability space we consider a real valued special  $\mathbb{G}$ -semimartingale  $X$ . The results can be generalized to the case of  $\mathbb{R}^n$  valued special semimartingales, in a straightforward manner. We fix a truncation function with respect to which the semimartingale characteristics are computed.

The purpose of this work is to study the following two problems:

- A. If  $X$  is  $\mathbb{F}$ -adapted, compute the  $\mathbb{F}$ -semimartingale characteristics of  $X$  in terms of the  $\mathbb{G}$ -semimartingale characteristics of  $X$ .

- B. If  $X$  is not  $\mathbb{F}$ -adapted, given that the  $\mathbb{F}$ -optional projection of  $X$  is a special semimartingale, compute the  $\mathbb{F}$ -semimartingale characteristics of  $\mathbb{F}$ -optional projection of  $X$  in terms of the  $\mathbb{G}$ -canonical decomposition and  $\mathbb{G}$ -semimartingale characteristics of  $X$ .

An important motivation behind the study originated in this paper is coming from the theory of stochastic structures that has been under works in recent years. One of the problems arising in this theory can be summarized as follows: Suppose that  $S = (S^1, \dots, S^n)$  is a multivariate semimartingale. Suppose that  $(B^i, C^i, \nu^i)$  are the semimartingale characteristics of the semimartingale  $S^i$  in the natural filtration of  $S$ . The problem is to find the semimartingale characteristics of  $S^i$  in the filtration of a subgroup of coordinates  $S^{i_1}, \dots, S^{i_k}$ ,  $i_1, \dots, i_k \in \{1, \dots, n\}$ , of  $S$ , in terms of  $(B^i, C^i, \nu^i)$ .

**Keywords:** Special semimartingale, filtration shrinkage, optional projection, semimartingale characteristics

**Mathematics Subjects Classification (2010):** 60G99, 60H99

## 1 Introduction

This paper is meant to initiate a systematic study of the change of properties of semimartingales under shrinkage of filtrations and, when appropriate, under respective projections. The paper does not aim at a complete and comprehensive study of the topic. Rather, we analyze in some special settings a selection of relevant research problems. Our study contributes, we believe, to understanding and solution of these problems.

We consider a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , which is endowed with two filtrations,  $\mathbb{G}$  and  $\mathbb{F}$ , assumed to satisfy the usual conditions and such that  $\mathbb{F} \subset \mathbb{G}$ . On this probability space we consider a real valued special  $\mathbb{G}$ -semimartingale  $X$ . The results can be generalized to the case of  $\mathbb{R}^n$  valued special semimartingales, in a straightforward manner. We fix a truncation function with respect to which the semimartingale characteristics are computed.

The purpose of this work is to study the following two problems:

- A. If  $X$  is  $\mathbb{F}$ -adapted, compute the  $\mathbb{F}$ -semimartingale characteristics of  $X$  in terms of the  $\mathbb{G}$ -semimartingale characteristics of  $X$ .
- B. If  $X$  is not  $\mathbb{F}$ -adapted, given that the  $\mathbb{F}$ -optional projection of  $X$  is a special semimartingale, compute the  $\mathbb{F}$ -semimartingale characteristics of  $\mathbb{F}$ -optional projection of  $X$  in terms of the  $\mathbb{G}$ -canonical decomposition and  $\mathbb{G}$ -semimartingale characteristics of  $X$ .

So, in a sense, we study problems, which are complementary to problems that arise when one studies what happens to a semimartingale under enlargement of filtration. The literature regarding enlargement of filtrations is quite abundant (see, e.g., the recent monograph [AJ17] and the references therein). On the contrary, the literature regarding the shrinkage of filtration and its effect on the properties of a semimartingale is essentially non-existent.

Two notable exceptions are Section II§6 in [LS89] and Section IX.2 in [Jac79], that feature partial versions of some of our results. Related study is also done in [BY78] where, however, a different, from our special semimartingales, class of processes was investigated (called semimartingales there); interestingly enough, our formula (4.22) can be obtained from Proposition 4 in [BY78]. Special cases of our Lemma 3.3 are present in the literature in the context of the filtering theory. So, to the great extent our work provides an original contribution to the systematic study of the change of properties of semimartingales under shrinkage of filtrations.

Also, contrary to the theory of the enlargement of the filtrations, where only initial and progressive enlargements are studied, here we do not make any specific restrictions regarding relation between the filtrations  $\mathbb{G}$  and  $\mathbb{F}$ , except for the inclusion condition  $\mathbb{F} \subset \mathbb{G}$ , and, additionally the immersion condition in Section 4.

An important motivation behind the study originated in this paper is coming from the theory of stochastic structures that has been under works in recent years (cf. [BJNng]). One of the problems arising in this theory can be summarized as follows: Suppose that  $S = (S^1, \dots, S^n)$  is a multivariate semimartingale. Suppose that  $(B^i, C^i, \nu^i)$  are the semimartingale characteristics of the semimartingale  $S^i$  in the natural filtration of  $S$ . The problem is to find the semimartingale characteristics of  $S^i$  in the filtration of a sub-group of coordinates  $S^{i_1}, \dots, S^{i_k}$ ,  $i_1, \dots, i_k \in \{1, \dots, n\}$ , of  $S$ , in terms of  $(B^i, C^i, \nu^i)$ .

The paper is organized as follows. In Section 2 we formulate the mathematical set-up for our study and we recall some useful concepts and results. In Section 3 we study problem A. In Section 4 we study problem B. In Section 5 we provide several examples illustrating and complementing our theoretical developments. The complexity of the examples varies. But all of them are meant to illustrate our theoretical developments, even though results presented in some of the examples might possibly be obtained directly.

Finally, in Section 6 we formulate some non-trivial open problems, solution of which will require more in-depth understanding of subject matters discussed in this paper.

## 2 Preliminaries

We assume that the semimartingale  $X$  has jumps with absolute value bounded from above by some constant  $a > 0$ . Without loss of generality we take  $a = 1$ .

The (special)  $\mathbb{G}$ -semimartingale  $X$  admits the unique canonical decomposition of the form

$$X_t = X_0 + M_t^{\mathbb{G}} + B_t^{\mathbb{G}}, \quad t \geq 0,$$

where  $M^{\mathbb{G}}$  is a  $\mathbb{G}$ -local martingale such that  $M_0^{\mathbb{G}} = 0$ , and  $B^{\mathbb{G}}$  is a  $\mathbb{G}$ -predictable process with finite variation and  $B_0^{\mathbb{G}} = 0$ . Under our set-up, the process  $X$  can be written as

$$X_t = X_0 + X_t^{c, \mathbb{G}} + \int_0^t \int_{\mathbb{R}} x(\mu(ds, dx) - \nu^{\mathbb{G}}(ds, dx)) + B_t^{\mathbb{G}}, \quad t \geq 0,$$

where  $X^{c, \mathbb{G}}$  is the continuous  $\mathbb{G}$ -martingale part,  $\mu$  is the jump measure of  $X$  as defined in Proposition II.1.16 in [JS03], and  $\nu^{\mathbb{G}}$  is the  $\mathbb{G}$ -compensator of  $\mu$ . The above representation is the consequence of the Corollary II 2.38 in [JS03].

In what follows, we will use the standard truncation function  $\chi(x) = x\mathbb{1}_{|x|\leq 1}$ , and all the semimartingale characteristics will be considered with respect to this function.

For  $X$  as above, i.e., with jumps bounded by 1, the  $\mathbb{G}$ -semimartingale characteristics are  $(B^{\mathbb{G}}, C^{\mathbb{G}}, \nu^{\mathbb{G}})$ , where  $C^{\mathbb{G}}$  is the  $\mathbb{G}$ -predictable bracket of  $X^{c, \mathbb{G}}$  (cf. [JS03] Proposition II.2.29).

In view of Proposition II.2.9 in [JS03], there exists a  $\mathbb{G}$ -predictable, locally integrable increasing process, say  $A^{\mathbb{G}}$ , such that

$$B^{\mathbb{G}} = b^{\mathbb{G}} \cdot A^{\mathbb{G}}, \quad C^{\mathbb{G}} = c^{\mathbb{G}} \cdot A^{\mathbb{G}}, \quad \nu^{\mathbb{G}}(dt, dx) = K_t^{\mathbb{G}}(dx) dA_t^{\mathbb{G}},$$

where

- i.  $b^{\mathbb{G}}$  is an  $\mathbb{R}$ -valued and  $\mathbb{G}$ -predictable process,
- ii.  $c^{\mathbb{G}}$  is an  $\mathbb{R}_+$ -valued and  $\mathbb{G}$ -predictable process,
- iii.  $K_t^{\mathbb{G}}(\omega, dx)$  is a transition kernel from  $(\Omega \times \mathbb{R}_+, \mathcal{P}_{\mathbb{G}})$  to  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ , satisfying condition analogous to condition II.2.11 in [JS03], and where  $\mathcal{P}_{\mathbb{G}}$  is the  $\mathbb{G}$ -predictable  $\sigma$ -field on  $\Omega \times \mathbb{R}_+$ ,

and where  $\cdot$  denotes the stochastic or Stieltjes integral, wherever appropriate.

We will assume that

$$A_t^{\mathbb{G}} = \int_0^t a_u^{\mathbb{G}} du, \tag{2.1}$$

where  $a^{\mathbb{G}}$  is a  $\mathbb{G}$ -progressively measurable process. This assumption will be satisfied in examples studied in Section 5.

In what follows we use the following notions and notation:

1. For a given process  $Z$ , we denote by  ${}^{o, \mathbb{F}}Z$  the optional projection of  $Z$  on  $\mathbb{F}$  defined in the sense of He et al. [HWY92], i.e., the unique  $\mathbb{F}$ -optional, finite valued process such that for every  $\mathbb{F}$ -stopping time  $\tau$  we have

$$\mathbb{E}(Z_{\tau} \mathbb{1}_{\tau < \infty} | \mathcal{F}_{\tau}) = {}^{o, \mathbb{F}}Z_{\tau} \mathbb{1}_{\tau < \infty}.$$

Note that by Theorem 5.1 in [HWY92] this optional projection exists if  $Z$  is a measurable process such that  $Z_{\tau} \mathbb{1}_{\tau < \infty}$  is  $\sigma$ -integrable with respect to  $\mathcal{F}_{\tau}$  for every  $\mathbb{F}$ -stopping time  $\tau$ . That is, there exists a sequence of sets  $(A_n)_{n=1}^{\infty}$  such that  $A_n \in \mathcal{F}_{\tau}$ ,  $A_n \uparrow \Omega$  and  $\mathbb{E}(Z_{\tau} \mathbb{1}_{\tau < \infty} \mathbb{1}_{A_n}) < \infty$  for  $n = 1, 2, \dots$

2. For a given process  $Z$ , we denote by  ${}^{p, \mathbb{F}}Z$  the predictable projection of  $Z$  on  $\mathbb{F}$  defined in the sense of He et al. [HWY92], i.e., the unique  $\mathbb{F}$ -predictable, finite valued process such that for every  $\mathbb{F}$ -stopping time  $\tau$  we have

$$\mathbb{E}(Z_{\tau} \mathbb{1}_{\tau < \infty} | \mathcal{F}_{\tau-}) = {}^{p, \mathbb{F}}Z_{\tau} \mathbb{1}_{\tau < \infty}.$$

Note that by Theorem 5.2 in [HWY92] this predictable projection exists if  $Z$  is a measurable process such that  $Z_{\tau} \mathbb{1}_{\tau < \infty}$  is  $\sigma$ -integrable with respect to  $\mathcal{F}_{\tau-}$  for every predictable  $\mathbb{F}$ -stopping time  $\tau$ . That is, there exists a sequence of sets  $(A_n)_{n=1}^{\infty}$  such that  $A_n \in \mathcal{F}_{\tau-}$ ,  $A_n \uparrow \Omega$  and  $\mathbb{E}(Z_{\tau} \mathbb{1}_{\tau < \infty} \mathbb{1}_{A_n}) < \infty$  for  $n = 1, 2, \dots$

3. We will also need a notion of  $\mathbb{F}$ -optional and  $\mathbb{F}$ -predictable projections for any function  $W : \tilde{\Omega} \rightarrow \mathbb{R}$ , which is measurable with respect to  $\tilde{\mathcal{F}}$ , where

$$\tilde{\Omega} := \Omega \times \mathbb{R}_+ \times \mathbb{R}, \quad \tilde{\mathcal{F}} := \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}).$$

The  $\mathbb{F}$ -optional projection of such a function  $W$  is defined as the function  ${}^{o,\mathbb{F}}W$  on  $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ , which is such that for all  $x \in \mathbb{R}$  the process  ${}^{o,\mathbb{F}}W(\cdot, x)$  is the optional projection on  $\mathbb{F}$  of the process  $W(\cdot, x)$ . Similarly, the  $\mathbb{F}$ -predictable projection of such a function  $W$  is defined as the function  ${}^{p,\mathbb{F}}W$  on  $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ , which is such that for all  $x \in \mathbb{R}$  the process  ${}^{p,\mathbb{F}}W(\cdot, x)$  is the predictable projection on  $\mathbb{F}$  of the process  $W(\cdot, x)$ .

4. We denote by  $\mathcal{O}_{\mathbb{F}}$  (resp.  $\mathcal{P}_{\mathbb{F}}$ ), the  $\mathbb{F}$ -optional (resp. the  $\mathbb{F}$ -predictable) sigma-field on  $\Omega \times \mathbb{R}_+$  generated by  $\mathbb{F}$ -adapted càdlàg (resp. continuous) processes. Analogously we introduce the sigma fields  $\tilde{\mathcal{O}}_{\mathbb{F}}$  and  $\tilde{\mathcal{P}}_{\mathbb{F}}$  on  $\tilde{\Omega}$  defined by

$$\tilde{\mathcal{O}}_{\mathbb{F}} := \mathcal{O}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}), \quad \tilde{\mathcal{P}}_{\mathbb{F}} := \mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}).$$

5. A random measure  $\pi$  on  $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$  is  $\mathbb{F}$ -optional (resp.  $\mathbb{F}$ -predictable) if for any  $\tilde{\mathcal{O}}_{\mathbb{F}}$ -measurable (resp.  $\tilde{\mathcal{P}}_{\mathbb{F}}$ -measurable) positive real function  $W$ , the real valued process

$$V(\omega, t) := \int_{[0, t] \times \mathbb{R}} W(\omega, s, x) \pi(\omega; ds, dx)$$

is  $\mathbb{F}$ -optional (resp.  $\mathbb{F}$ -predictable); equivalently if for any positive real, measurable function  $W$  on  $\tilde{\Omega}$

$$\mathbb{E} \left( \int_{\mathbb{R}_+ \times \mathbb{R}} W(s, x) \pi(ds, dx) \right) = \mathbb{E} \left( \int_{\mathbb{R}_+ \times \mathbb{R}} {}^{q,\mathbb{F}}W(s, x) \pi(ds, dx) \right),$$

where  $q = o$  (resp.  $q = p$ ).

6. We say that a random measure  $\pi$  on  $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$  is  $\mathbb{F}$ -optionally (resp.  $\mathbb{F}$ -predictably),  $\sigma$ -integrable if the measure  $M_{\pi}$  on  $\tilde{\mathcal{F}}$  defined by

$$M_{\pi}(\tilde{B}) := \mathbb{E} \left( \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\tilde{B}}(\omega, t, x) \pi(\omega, dt, dx) \right), \quad \tilde{B} \in \tilde{\mathcal{F}}, \quad (2.2)$$

restricted to  $\tilde{\mathcal{O}}_{\mathbb{F}}$  (resp. to  $\tilde{\mathcal{P}}_{\mathbb{F}}$ ), is a  $\sigma$ -finite measure. In other words  $\pi$  is  $\mathbb{F}$ -optionally, resp.  $\mathbb{F}$ -predictably,  $\sigma$ -integrable if there exist a sequence of sets  $\tilde{A}_k \in \tilde{\mathcal{O}}_{\mathbb{F}}$ , resp.  $\tilde{A}_k \in \tilde{\mathcal{P}}_{\mathbb{F}}$ , such that  $\tilde{A}_k \uparrow \tilde{\Omega}$  and  $M_{\pi}(\tilde{A}_k) < \infty$  for each  $k$ .

7. For a random measure  $\pi$  on  $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$  we denote by  $\pi^{o,\mathbb{F}}$  the  $\mathbb{F}$ -dual optional projection of  $\pi$  on  $\mathbb{F}$ , i.e., the unique  $\mathbb{F}$ -optional measure on  $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$  such that it is  $\mathbb{F}$ -optionally  $\sigma$ -integrable and for every positive  $\tilde{\mathcal{O}}_{\mathbb{F}}$ -measurable function  $W$  on  $\tilde{\Omega}$ , we have

$$\mathbb{E} \left( \int_{\mathbb{R}_+ \times \mathbb{R}} W(t, x) \pi(dt, dx) \right) = \mathbb{E} \left( \int_{\mathbb{R}_+ \times \mathbb{R}} W(t, x) \pi^{o,\mathbb{F}}(dt, dx) \right).$$

The  $\mathbb{F}$ -dual predictable projection of  $\pi$  on  $\mathbb{F}$ , denoted by  $\pi^{p,\mathbb{F}}$ , is defined analogously, as the unique  $\mathbb{F}$ -predictable measure on  $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$  such that it is  $\mathbb{F}$ -predictably  $\sigma$ -integrable and for every positive  $\tilde{\mathcal{P}}_{\mathbb{F}}$ -measurable function  $W$  on  $\tilde{\Omega}$ , we have

$$\mathbb{E}\left(\int_{\mathbb{R}_+ \times \mathbb{R}} W(t, x) \pi(dt, dx)\right) = \mathbb{E}\left(\int_{\mathbb{R}_+ \times \mathbb{R}} W(t, x) \pi^{p,\mathbb{F}}(dt, dx)\right).$$

We note that existence and uniqueness of  $\pi^{o,\mathbb{F}}$  (resp.  $\pi^{p,\mathbb{F}}(dt, dx)$ ) holds under assumption that  $\pi$  is  $\mathbb{F}$ -optionally (resp.  $\mathbb{F}$ -predictably),  $\sigma$ -integrable (see e.g. [HWY92, Theorem 11.8]).

8. For any process  $A$  and any (stopping) time  $T$ , we denote by  $A^T$  the process  $A$  stopped at  $T$ .

In the rest of the paper we shall study the  $\mathbb{F}$ -characteristics of  $X$  in the case when  $X$  is  $\mathbb{F}$ -adapted, and the  $\mathbb{F}$ -characteristics of the optional projection of  $X$  on  $\mathbb{F}$  in the case when  $X$  is not  $\mathbb{F}$ -adapted, assuming that such optional projection exists.

### 3 Study of Problem A: The Case of $X$ adapted to $\mathbb{F}$

In this section, we consider the case where  $X$  is  $\mathbb{F}$ -adapted. Then, according to [Str77]  $X$  is a special  $\mathbb{F}$ -semimartingale and it can be decomposed as

$$X_t = X_0 + M_t^{\mathbb{F}} + B_t^{\mathbb{F}} = X_0 + X_t^{c,\mathbb{F}} + \int_0^t \int_E x(\mu(ds, dx) - \nu^{\mathbb{F}}(ds, dx)) + B_t^{\mathbb{F}}.$$

As before, we have that in view of Proposition II.2.9 in [JS03], there exists an  $\mathbb{F}$ -predictable, locally integrable increasing process, say  $A^{\mathbb{F}}$ , such that

$$B^{\mathbb{F}} = b^{\mathbb{F}} \cdot A^{\mathbb{F}}, \quad C^{\mathbb{F}} = c^{\mathbb{F}} \cdot A^{\mathbb{F}}, \quad \nu^{\mathbb{F}}(dt, dx) = K_t^{\mathbb{F}}(dx) dA_t^{\mathbb{F}}.$$

We make the following assumptions:

- A1. For every  $t \geq 0$  we have

$$\mathbb{E} \int_0^t |b_u^{\mathbb{G}} a_u^{\mathbb{G}}| du < \infty,$$

where  $a^{\mathbb{G}}$  is defined in 2.1.

- A2. The process  $b^{\mathbb{G}} a^{\mathbb{G}}$  admits an  $\mathbb{F}$ -optional projection.

- A3. The process  $M^{\mathbb{G}}$  is a true  $\mathbb{G}$ -martingale.

*Remark 3.1.* In view of assumption A3 process  $M^{\mathbb{G}}$  is a true  $\mathbb{F}$ -martingale as well. If the process  $M^{\mathbb{G}}$  were a  $\mathbb{G}$ -local-martingale but not a true  $\mathbb{G}$ -martingale, then it might not necessarily be an  $\mathbb{F}$ -local-martingale.

We will need the following two technical results.

**Lemma 3.2.** *Suppose that  $A$  is an  $\mathbb{F}$ -adapted process with prelocally integrable variation,  $H$  is a process admitting an  $\mathbb{F}$ -optional projection, and such that  $H \cdot A$  has an integrable variation. Then*

$$M = {}^{o,\mathbb{F}}(H \cdot A) - ({}^{o,\mathbb{F}}H) \cdot A$$

*is a uniformly integrable  $\mathbb{F}$ -martingale.*

*Proof.* Applying [HWY92, Corollary 5.31.(2)] to the process  $H \cdot A$ , we conclude that the process

$$M = {}^{o,\mathbb{F}}(H \cdot A) - (H \cdot A)^{o,\mathbb{F}}$$

is a uniformly integrable martingale. Now, by [HWY92, Theorem 5.25] and the remark following this theorem, we have

$$(H \cdot A)^{o,\mathbb{F}} = ({}^{o,\mathbb{F}}H) \cdot A,$$

which finishes the proof.  $\square$

**Lemma 3.3.** *Let A1 and A2 be satisfied. Then,  ${}^{o,\mathbb{F}}B^{\mathbb{G}}$  and  $\int_0^\cdot {}^{o,\mathbb{F}}(b^{\mathbb{G}}a^{\mathbb{G}})_u du$  exist and the process  $M^B$  given as*

$$M_t^B = {}^{o,\mathbb{F}}B_t^{\mathbb{G}} - \int_0^t {}^{o,\mathbb{F}}(b^{\mathbb{G}}a^{\mathbb{G}})_u du, \quad t \geq 0, \quad (3.1)$$

*is an  $\mathbb{F}$ -martingale. In particular, if  $\mathbb{F}$  is immersed in  $\mathbb{G}$  then  $M^B$  is a null process.*

*Proof.* Since, by assumption A1, the process  $B_t^{\mathbb{G}} = \int_0^t H_s ds$ , where  $H_s = a_s^{\mathbb{G}} b_s^{\mathbb{G}}$ , is prelocally integrable and, by assumption A2,  $H$  has an optional projection, we may apply [HWY92, Theorem 5.25] and conclude that  ${}^{o,\mathbb{F}}B^{\mathbb{G}}$  and  $\int_0^\cdot {}^{o,\mathbb{F}}(b^{\mathbb{G}}a^{\mathbb{G}})_u du$  exist.

Now, fix  $T > 0$  and let

$$L_t = a_t^{\mathbb{G}} b_t^{\mathbb{G}} \mathbb{1}_{\{t \leq T\}}, \quad t \geq 0.$$

Then, invoking A1, A2 and applying Lemma 3.2 with  $A_t = t$  we conclude that  ${}^{o,\mathbb{F}}(L \cdot A)$  and  ${}^{o,\mathbb{F}}L \cdot A$  exist and the process

$$N_t := {}^{o,\mathbb{F}}(L \cdot A)_t - ({}^{o,\mathbb{F}}L \cdot A)_t, \quad t \geq 0, \quad (3.2)$$

is a uniformly integrable  $\mathbb{F}$ -martingale. Now note that  $L \cdot A = (L \cdot A)^T = (H \cdot A)^T$  so, by [HWY92, Theorem 5.7], we have for  $t \in [0, T]$

$${}^{o,\mathbb{F}}(L \cdot A)_t = {}^{o,\mathbb{F}}((H \cdot A)^T)_t = {}^{o,\mathbb{F}}(H \cdot A)_t. \quad (3.3)$$

Using the definition of  $L$  and applying again [HWY92, Theorem 5.7] we have

$${}^{o,\mathbb{F}}L = {}^{o,\mathbb{F}}(L \mathbb{1}_{[0,T]}) = \mathbb{1}_{[0,T]} {}^{o,\mathbb{F}}L = \mathbb{1}_{[0,T]} {}^{o,\mathbb{F}}(L^T) = \mathbb{1}_{[0,T]} {}^{o,\mathbb{F}}(H^T) = \mathbb{1}_{[0,T]} {}^{o,\mathbb{F}}H,$$

which implies that

$${}^{o,\mathbb{F}}L \cdot A = (\mathbb{1}_{[0,T]} {}^{o,\mathbb{F}}H) \cdot A = ({}^{o,\mathbb{F}}H \cdot A)^T. \quad (3.4)$$



Using (3.3) and (3.4) we see that the martingale  $N$  defined by (3.2) can be written on  $[0, T]$  as

$$N_t = {}^{o,\mathbb{F}}(H \cdot A)_t - ({}^{o,\mathbb{F}}H \cdot A)_t = {}^{o,\mathbb{F}}B_t^{\mathbb{G}} - \int_0^t {}^{o,\mathbb{F}}(b^{\mathbb{G}}a^{\mathbb{G}})_u du, \quad t \in [0, T].$$

Since  $T$  was arbitrary, this proves that the process given by (3.1) is an  $\mathbb{F}$ -martingale.

Finally, we will now prove that if  $\mathbb{F}$  is immersed in  $\mathbb{G}$ , then the martingale  $M^B$  is a null process. Indeed, for any  $t \geq 0$ , we have

$$\begin{aligned} {}^{o,\mathbb{F}}B_t^{\mathbb{G}} &= \mathbb{E} \left( \int_0^t b_u^{\mathbb{G}} a_u^{\mathbb{G}} du \middle| \mathcal{F}_t \right) = \int_0^t \mathbb{E} (b_u^{\mathbb{G}} a_u^{\mathbb{G}} | \mathcal{F}_t) du \\ &= \int_0^t \mathbb{E} (b_u^{\mathbb{G}} a_u^{\mathbb{G}} | \mathcal{F}_u) du = \int_0^t {}^{o,\mathbb{F}}(b^{\mathbb{G}}a^{\mathbb{G}})_u du, \end{aligned}$$

where the third equality is a consequence of immersion of  $\mathbb{F}$  in  $\mathbb{G}$ .

The proof of the lemma is complete.  $\square$

*Remark 3.4.* Special versions of the above lemma are known in the filtering theory. See for example Lemma 8.4 in [LS01], or the proof of Theorem 8.11 in [RW00].

The next theorem is the main result in this section.

**Theorem 3.5.** *Assume A1-A3. Then, the  $\mathbb{F}$ -characteristic triple of  $X$  is given as*

$$B^{\mathbb{F}} = \int_0^\cdot {}^{o,\mathbb{F}}(b^{\mathbb{G}}a^{\mathbb{G}})_s ds, \quad C^{\mathbb{F}} = C^{\mathbb{G}}, \quad \nu^{\mathbb{F}}(dt, dx) = (K_t^{\mathbb{G}}(dx)a_t^{\mathbb{G}} dt)^{p,\mathbb{F}}.$$

*Proof.* Given our assumptions A1 and A2 we see from Lemma 3.3 that  ${}^{o,\mathbb{F}}B^{\mathbb{G}}$  exists. In view of A3 process  $M^{\mathbb{G}}$  is a martingale and thus the optional projection  ${}^{o,\mathbb{F}}M^{\mathbb{G}}$  exists as well (see [HWY92, Remark 5.3 and Theorem 5.8]). From this and from the linearity of the optional projection we thus conclude that the optional projection  ${}^{o,\mathbb{F}}X$  exists, and is given as

$${}^{o,\mathbb{F}}X = X_0 + {}^{o,\mathbb{F}}M^{\mathbb{G}} + {}^{o,\mathbb{F}}B^{\mathbb{G}}. \quad (3.5)$$

Since  $X$  is  $\mathbb{F}$ -adapted we have

$${}^{o,\mathbb{F}}X = X. \quad (3.6)$$

Combining (3.5) and (3.6) we obtain

$$X = X_0 + {}^{o,\mathbb{F}}M^{\mathbb{G}} + {}^{o,\mathbb{F}}B^{\mathbb{G}}.$$

Thus, since the process  ${}^{o,\mathbb{F}}B_t^{\mathbb{G}} - \int_0^t {}^{o,\mathbb{F}}(b^{\mathbb{G}}a^{\mathbb{G}})_u du$  is an  $\mathbb{F}$ -martingale (by Lemma 3.3 again), and since, in view of A3, the process  ${}^{o,\mathbb{F}}M^{\mathbb{G}}$  is an  $\mathbb{F}$ -martingale, we see that

$$X_t = X_0 + M_t^{\mathbb{F}} + \int_0^t {}^{o,\mathbb{F}}(b^{\mathbb{G}}a^{\mathbb{G}})_s ds,$$

where  $M_t^{\mathbb{F}} = {}^{o,\mathbb{F}}M_t^{\mathbb{G}} + {}^{o,\mathbb{F}}B_t^{\mathbb{G}} - \int_0^t {}^{o,\mathbb{F}}(b^{\mathbb{G}}a^{\mathbb{G}})_u du$ . Thus, by uniqueness of the decomposition of the special  $\mathbb{F}$ -semimartingale  $X$ , we conclude that

$$B_t^{\mathbb{F}} = \int_0^t {}^{o,\mathbb{F}}(b^{\mathbb{G}}a^{\mathbb{G}})_s ds.$$

The second formula,  $C^{\mathbb{F}} = C^{\mathbb{G}}$ , follows from [Jac79, Remark 9.20, p.288].

It remains to derive a formula for  $\nu^{\mathbb{F}}$ . Towards this end, we recall that  $\nu^{\mathbb{G}} = \mu^{p,\mathbb{G}}$  is a  $\tilde{\mathcal{P}}_{\mathbb{G}}$ -predictably  $\sigma$ -integrable random measure (i.e., using the notation (2.2),  $M_{\nu^{\mathbb{G}}}$  is  $\sigma$ -finite on  $\tilde{\mathcal{P}}_{\mathbb{G}}$ , ) such that

$$M_{\nu^{\mathbb{G}}}|_{\tilde{\mathcal{P}}_{\mathbb{G}}} = M_{\mu}|_{\tilde{\mathcal{P}}_{\mathbb{G}}}.$$

Thus since  $\tilde{\mathcal{P}}_{\mathbb{F}} \subset \tilde{\mathcal{P}}_{\mathbb{G}}$  we have

$$M_{\nu^{\mathbb{G}}}|_{\tilde{\mathcal{P}}_{\mathbb{F}}} = M_{\mu}|_{\tilde{\mathcal{P}}_{\mathbb{F}}}. \quad (3.7)$$

Since  $M_{\mu}$  is  $\sigma$ -finite on  $\tilde{\mathcal{P}}_{\mathbb{F}}$  (see the proof of Theorem 11.15 [HWY92], with  $\tilde{\mathcal{P}}$  there replaced by  $\tilde{\mathcal{P}}_{\mathbb{F}}$ ), the above implies that  $M_{\nu^{\mathbb{G}}}$  is also  $\sigma$ -finite on  $\tilde{\mathcal{P}}_{\mathbb{F}}$ . So  $\nu^{\mathbb{G}}$  is  $\mathbb{F}$ -predictably  $\sigma$ -integrable. Thus it has the  $\mathbb{F}$ -dual predictable projection  $(\nu^{\mathbb{G}})^{p,\mathbb{F}}$  which is characterized by

$$M_{(\nu^{\mathbb{G}})^{p,\mathbb{F}}}|_{\tilde{\mathcal{P}}_{\mathbb{F}}} = M_{\nu^{\mathbb{G}}}|_{\tilde{\mathcal{P}}_{\mathbb{F}}}. \quad (3.8)$$

This and (3.7) implies that

$$M_{(\nu^{\mathbb{G}})^{p,\mathbb{F}}}|_{\tilde{\mathcal{P}}_{\mathbb{F}}} = M_{\mu}|_{\tilde{\mathcal{P}}_{\mathbb{F}}}.$$

So, by the uniqueness of dual predictable projections we have  $(\nu^{\mathbb{G}})^{p,\mathbb{F}} = \nu^{\mathbb{F}}$ . The proof is complete.  $\square$

*Remark 3.6.* Let us note that we also have

$$(\nu^{\mathbb{G}})^{p,\mathbb{F}} = ((\nu^{\mathbb{G}})^{o,\mathbb{F}})^{p,\mathbb{F}}.$$

Indeed, by analogous reasoning as in the proof of [HWY92, Theorem 11.8] we can prove that the random measure  $\nu^{\mathbb{G}}$  admits an  $\mathbb{F}$ -dual optional projection if and only if it is  $\mathbb{F}$ -optionally  $\sigma$ -integrable. Now, recall that  $M_{\nu^{\mathbb{G}}}$  is  $\sigma$ -finite on  $\tilde{\mathcal{P}}_{\mathbb{F}}$ . This and the fact that  $\tilde{\mathcal{P}}_{\mathbb{F}} \subset \tilde{\mathcal{O}}_{\mathbb{F}}$  implies that  $M_{\nu^{\mathbb{G}}}$  is also  $\sigma$ -finite on  $\tilde{\mathcal{O}}_{\mathbb{F}}$ , so  $\nu^{\mathbb{G}}$  is  $\mathbb{F}$ -optionally  $\sigma$ -integrable. Thus there exists  $(\nu^{\mathbb{G}})^{o,\mathbb{F}}$  – the  $\mathbb{F}$ -dual optional projection of  $\nu^{\mathbb{G}}$ , i.e., the unique  $\mathbb{F}$ -optional measure which is  $\mathbb{F}$ -optionally  $\sigma$ -integrable such that

$$M_{\nu^{\mathbb{G}}}|_{\tilde{\mathcal{O}}_{\mathbb{F}}} = M_{(\nu^{\mathbb{G}})^{o,\mathbb{F}}}|_{\tilde{\mathcal{O}}_{\mathbb{F}}}.$$

Hence we have

$$M_{\nu^{\mathbb{G}}}|_{\tilde{\mathcal{P}}_{\mathbb{F}}} = M_{(\nu^{\mathbb{G}})^{o,\mathbb{F}}}|_{\tilde{\mathcal{P}}_{\mathbb{F}}}. \quad (3.9)$$

Since  $M_{\nu^{\mathbb{G}}}$  is  $\sigma$ -finite on  $\tilde{\mathcal{P}}_{\mathbb{F}}$ , so is  $M_{(\nu^{\mathbb{G}})^{o,\mathbb{F}}}$ . Therefore, invoking again [HWY92, Theorem 11.8], we conclude that there exists the  $\mathbb{F}$ -predictable projection of  $(\nu^{\mathbb{G}})^{o,\mathbb{F}}$ , i.e.  $((\nu^{\mathbb{G}})^{o,\mathbb{F}})^{p,\mathbb{F}}$ , for which we have

$$M_{((\nu^{\mathbb{G}})^{o,\mathbb{F}})^{p,\mathbb{F}}}|_{\tilde{\mathcal{P}}_{\mathbb{F}}} = M_{(\nu^{\mathbb{G}})^{o,\mathbb{F}}}|_{\tilde{\mathcal{P}}_{\mathbb{F}}}.$$

From the latter equality and from (3.7) and (3.9) we deduce that

$$M_{((\nu^{\mathbb{G}})^{o,\mathbb{F}})^{p,\mathbb{F}}}|_{\tilde{\mathcal{P}}_{\mathbb{F}}} = M_{\mu}|_{\tilde{\mathcal{P}}_{\mathbb{F}}}.$$

By uniqueness of the  $\mathbb{F}$ -dual predictable projection of  $\mu$  we finally obtain  $\nu^{\mathbb{F}} = \mu^{p,\mathbb{F}} = ((\nu^{\mathbb{G}})^{o,\mathbb{F}})^{p,\mathbb{F}}$ .  $\square$

### The case of immersion between $\mathbb{F}$ and $\mathbb{G}$ .

We briefly discuss here the case when  $\mathbb{F}$  is immersed in  $\mathbb{G}$ , i.e., any  $\mathbb{F}$ -local martingale is a  $\mathbb{G}$ -local martingale. In this case we may relax our assumptions regarding the semimartingale  $X$ .

In particular, we assume that  $X$  is a  $\mathbb{G}$ -semimartingale with  $\mathbb{G}$ -characteristic triple  $(B^{\mathbb{G}}, C^{\mathbb{G}}, \nu^{\mathbb{G}})$ , but we do not assume that its jumps are bounded by 1, neither that  $X$  is a special  $\mathbb{G}$ -semimartingale. As before,  $X$  being  $\mathbb{F}$ -adapted, it is an  $\mathbb{F}$ -semimartingale and we denote its  $\mathbb{F}$ -characteristic triple as  $(B^{\mathbb{F}}, C^{\mathbb{F}}, \nu^{\mathbb{F}})$ .

We will show that  $(B^{\mathbb{F}}, C^{\mathbb{F}}, \nu^{\mathbb{F}}) = (B^{\mathbb{G}}, C^{\mathbb{G}}, \nu^{\mathbb{G}})$ . Towards this end, let us consider the process  $\check{X}$  defined by

$$\check{X}_t = X_t - X_0 - \sum_{0 < u \leq t} \Delta X_u \mathbb{1}_{|\Delta X_u| > 1}.$$

Clearly, the process  $\check{X}$  has bounded jumps and is both  $\mathbb{G}$ -adapted and  $\mathbb{F}$ -adapted. Thus, it is a special semimartingale in both filtrations, and hence it has the canonical decompositions

$$\check{X} = M^{\mathbb{F}} + B^{\mathbb{F}} = M^{\mathbb{G}} + B^{\mathbb{G}}.$$

Since, by immersion,  $M^{\mathbb{F}}$  is a  $\mathbb{G}$  martingale and, obviously,  $B^{\mathbb{F}}$  is  $\mathbb{G}$ -predictable, one has that  $M^{\mathbb{G}} = M^{\mathbb{F}}$  and  $B^{\mathbb{G}} = B^{\mathbb{F}}$  (by uniqueness of canonical  $\mathbb{G}$ -decomposition of  $\check{X}$ ).

The fact that  $C^{\mathbb{F}} = C^{\mathbb{G}}$  follows, again, from [Jac79, Remark 9.20, p.288].

Finally, we verify that  $\nu^{\mathbb{G}} = \nu^{\mathbb{F}}$ . Note that, for any positive real measurable function  $g$ , the process  $g * \mu - g * \nu^{\mathbb{F}}$  is an  $\mathbb{F}$ -local martingale and hence, by immersion, a  $\mathbb{G}$ -local martingale. This implies, by uniqueness of the compensator and by the fact that  $\nu^{\mathbb{F}}$  is  $\mathbb{G}$ -predictable, that  $\nu^{\mathbb{F}} = \nu^{\mathbb{G}}$ .

In conclusion,

$$(B^{\mathbb{F}}, C^{\mathbb{F}}, \nu^{\mathbb{F}}) = (B^{\mathbb{G}}, C^{\mathbb{G}}, \nu^{\mathbb{G}}).$$

## 4 Study of Problem B: The Case of $X$ not adapted to $\mathbb{F}$

In this section we will work under the following additional standing assumptions:

B1. There exists a square integrable  $\mathbb{F}$ -martingale  $Z$  such that the predictable representation property holds for  $(\mathbb{F}, Z)$ : any square integrable  $\mathbb{F}$ -martingale  $M$  admits a representation  $M_t = M_0 + \int_0^t \psi_u dZ_u$ ,  $t \geq 0$ , with an  $\mathbb{F}$ -predictable process  $\psi$ .

B2. The  $\mathbb{F}$ -martingale  $Z$  is a  $\mathbb{G}$ -martingale.

B3.  $\mathcal{G}_0$  is trivial (so  $\mathcal{F}_0$  is also trivial).

Note that, under B1 and B2, the immersion property holds between  $\mathbb{F}$  and  $\mathbb{G}$ . Here we consider the case where  $X$  is a  $\mathbb{G}$ -special semimartingale, but it is not necessarily adapted to  $\mathbb{F}$ . We will additionally assume that  ${}^{\circ, \mathbb{F}}X$  exists and that it is an  $\mathbb{F}$ -special semimartingale.

*Remark 4.1.* It is well known that the  $\mathbb{F}$ -optional projection of  $X$  exists and is an  $\mathbb{F}$ -special semimartingale under the strong condition that  $\mathbb{E}X_t^* < \infty$  for all  $t > 0$ , where  $X_t^* = \sup_{s \leq t} |X_s|$  see, e.g., [HWY92, Theorem 8.6].

We have the following canonical decompositions of  $X$  and  ${}^{o,\mathbb{F}}X$ :

$$X_t = X_0 + X_t^{c,\mathbb{G}} + \int_0^t \int_E x(\mu^{\mathbb{G}}(ds, dx) - \nu^{\mathbb{G}}(ds, dx)) + B_t^{\mathbb{G}} = X_0 + M_t^{\mathbb{G}} + B_t^{\mathbb{G}}, \quad (4.1)$$

$${}^{o,\mathbb{F}}X_t = X_0 + ({}^{o,\mathbb{F}}X)_t^{c,\mathbb{F}} + \int_0^t \int_E x(\mu^{\mathbb{F}}(ds, dx) - \nu^{\mathbb{F}}(ds, dx)) + B_t^{\mathbb{F}} = X_0 + M_t^{\mathbb{F}} + B_t^{\mathbb{F}}. \quad (4.2)$$

The following theorem presents computation of the first two  $\mathbb{F}$ -characteristics of  ${}^{o,\mathbb{F}}X$ . The computation of the third  $\mathbb{F}$ -characteristic of  ${}^{o,\mathbb{F}}X$  will be discussed later in a special setting of Proposition 4.3, as well as in some of the examples in Section 5 on the case by case basis.

**Theorem 4.2.** *Let  $X$  be a special  $\mathbb{G}$ -semimartingale with  $\mathbb{G}$ -characteristic triple  $(B^{\mathbb{G}}, C^{\mathbb{G}}, \nu^{\mathbb{G}})$ . Assume that:*

1. *The optional projection of  $X$  on  $\mathbb{F}$  exists and has jumps bounded by 1.*
2. *A1, A2 and B1 to B3 are satisfied.*
3.  *$M^{\mathbb{G}}$  is a square integrable martingale.<sup>1</sup>*

*Then the first two  $\mathbb{F}$ -characteristics  ${}^{o,\mathbb{F}}X$  are*

$$B_t^{\mathbb{F}} = \int_0^t {}^{o,\mathbb{F}}(b^{\mathbb{G}}a^{\mathbb{G}})_s ds, \quad (4.3)$$

$$C_t^{\mathbb{F}} = \int_0^t h_s^2 d\langle Z^c \rangle_s, \quad (4.4)$$

where

$$h_t = \mathbb{E} \left( \frac{d\langle M^{\mathbb{G}}, Z \rangle_t^{\mathbb{G}}}{d\langle Z \rangle_t^{\mathbb{G}}} \middle| \mathcal{F}_t \right) \quad d[Z] \otimes d\mathbb{P} \text{ a.e.} \quad (4.5)$$

*Proof.* In view of Assumption 3 above, we know that the process  $M^{\mathbb{G}}$  admits a Kunita-Watanabe decomposition of the form

$$M_t^{\mathbb{G}} = M_0^{\mathbb{G}} + \int_0^t H_s dZ_s + M_t^{\perp}, \quad (4.6)$$

where  $M^{\perp}$  is a square integrable  $\mathbb{G}$ -martingale orthogonal to  $Z$  satisfying  $M_0^{\perp} = 0$ , and  $H$  is a  $\mathbb{G}$ -predictable process such that  $\int_0^t H_s dZ_s$  is a square integrable  $\mathbb{G}$ -martingale (see e.g. [Sch01]). In particular, since  $Z$  is assumed to be square integrable, we have

$$d\langle M^{\mathbb{G}}, Z \rangle_t^{\mathbb{G}} \ll d\langle Z \rangle_t^{\mathbb{G}},$$

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<sup>1</sup>Note that in Assumption A3 we postulated that  $M^{\mathbb{G}}$  is a martingale, but not necessarily a square integrable martingale.

and, in particular,

$$H_t = \frac{d\langle M^\mathbb{G}, Z \rangle_t^\mathbb{G}}{d\langle Z \rangle_t^\mathbb{G}}. \quad (4.7)$$

We will show that  $\mathbb{E}(M_t^\perp | \mathcal{F}_t) = 0$  for every  $t \geq 0$ . Fix  $t \geq 0$ , note that the equality  $\mathbb{E}(M_t^\perp | \mathcal{F}_t) = 0$  is equivalent to  $\mathbb{E}(M_t^\perp \eta) = 0$  for any  $\mathcal{F}_t$ -measurable bounded random variable  $\eta$ . Let  $\eta_s = \mathbb{E}(\eta | \mathcal{F}_s)$ , for any  $s \in \mathbb{R}_+$ .

The fact that  $M^\perp$  is orthogonal to the martingale  $(\eta_s, s \geq 0)$ , due to B1 and B2, implies that  $\mathbb{E}(M_t^\perp \eta_t) = M_0^\perp \mathbb{E}(\eta_0) = 0$ . Thus, since  $\eta_t = \eta$ , we obtain  $\mathbb{E}(M_t^\perp \eta) = 0$ .

By linearity of  $\mathbb{F}$ -optional projections, we may write

$${}^{\circ, \mathbb{F}}X_t = X_0 + {}^{\circ, \mathbb{F}}M_t^\mathbb{G} + {}^{\circ, \mathbb{F}}B_t^\mathbb{G} = X_0 + {}^{\circ, \mathbb{F}}M_t^\mathbb{G} + {}^{\circ, \mathbb{F}}B_t^\mathbb{G} - \int_0^t {}^{\circ, \mathbb{F}}(b^\mathbb{G} a^\mathbb{G})_u du + \int_0^t {}^{\circ, \mathbb{F}}(b^\mathbb{G} a^\mathbb{G})_u du. \quad (4.8)$$

The process  $M_t^B = {}^{\circ, \mathbb{F}}B_t^\mathbb{G} - \int_0^t {}^{\circ, \mathbb{F}}(b^\mathbb{G} a^\mathbb{G})_u du$  is an  $\mathbb{F}$ -martingale (see Lemma 3.3). Invoking assumption B2, which, in fact, postulates the immersion between  $\mathbb{F}$  and  $\mathbb{G}$ , and recalling Lemma 3.3 again we see that this process is null. Hence, by uniqueness of canonical decomposition of  ${}^{\circ, \mathbb{F}}X$ , we have

$$M_t^\mathbb{F} = {}^{\circ, \mathbb{F}}M_t^\mathbb{G}. \quad (4.9)$$

Thus,

$${}^{\circ, \mathbb{F}}X_t = X_0 + {}^{\circ, \mathbb{F}}M_t^\mathbb{G} + \int_0^t {}^{\circ, \mathbb{F}}(b^\mathbb{G} a^\mathbb{G})_s ds = X_0 + M_t^\mathbb{F} + \int_0^t {}^{\circ, \mathbb{F}}(b^\mathbb{G} a^\mathbb{G})_s ds.$$

The above and the assumption that  ${}^{\circ, \mathbb{F}}X$  has jumps bounded by 1 imply that the first characteristic of  ${}^{\circ, \mathbb{F}}X$ , that is  $B^\mathbb{F}$ , is given by (4.3).

Now, since  $M^\mathbb{G}$  is square integrable then, invoking the Jensen inequality, we conclude that  $M^\mathbb{F} = {}^{\circ, \mathbb{F}}M^\mathbb{G}$  is square integrable. Next, invoking the predictable representation property we see that there exists an  $\mathbb{F}$ -predictable process  $h$  such that  $\mathbb{E} \int_0^t h_s^2 d[Z]_s < \infty$  and

$$M_t^\mathbb{F} = \int_0^t h_s dZ_s. \quad (4.10)$$

The continuous martingale part of  ${}^{\circ, \mathbb{F}}X$  is thus given as  $\int_0^t h_s dZ_s^c$ , where  $Z^c$  is continuous part of  $\mathbb{F}$ -martingale  $Z$ , so that

$$C_t^\mathbb{F} = \int_0^t h_s^2 d\langle Z^c \rangle_s.$$

We will now compute  $h$ . Towards this end, we fix  $t \geq 0$  and we observe using (4.9) that for any bounded  $\mathcal{F}_t$ -measurable random variable  $\gamma$  we have

$$\mathbb{E}(\gamma M_t^\mathbb{G}) = \mathbb{E}(\gamma M_t^\mathbb{F}). \quad (4.11)$$

By using integration by parts formula we may write the left-hand side of (4.11) as

$$\mathbb{E}(\gamma M_t^\mathbb{G}) = \mathbb{E}\left(\int_0^t \widehat{\gamma}_{s-} dM_s^\mathbb{G} + \int_0^t M_{s-}^\mathbb{G} k_s dZ_s + [\widehat{\gamma}, M^\mathbb{G}]_t\right), \quad (4.12)$$

where  $(\widehat{\gamma}_s)_{s \in [0, t]}$  is a bounded martingale defined (and represented) by

$$\widehat{\gamma}_s := \mathbb{E}(\gamma | \mathcal{F}_s) = \mathbb{E}(\gamma) + \int_0^s k_u dZ_u, \quad s \in [0, t]. \quad (4.13)$$

Now, let us note that from the above representation of  $\widehat{\gamma}$  as stochastic integral with respect to  $Z$  and from (4.6) we may write

$$[\widehat{\gamma}, M^\mathbb{G}]_t = \int_0^t k_s d[Z, M^\mathbb{G}]_s = \int_0^t k_s \left( H_s d[Z]_s + d[Z, M^\perp]_s \right).$$

Using this we obtain from (4.12)

$$\mathbb{E}(\gamma M_t^\mathbb{G}) = \mathbb{E} \left( \int_0^t \widehat{\gamma}_{s-} dM_s^\mathbb{G} + \int_0^t M_{s-}^\mathbb{G} k_s dZ_s + \int_0^t k_s H_s d[Z]_s + \int_0^t k_s d[Z, M^\perp]_s \right). \quad (4.14)$$

Now we prove that the stochastic integrals  $\int_0^\cdot \widehat{\gamma}_{s-} dM_s^\mathbb{G}$ ,  $\int_0^\cdot M_{s-}^\mathbb{G} k_s dZ_s$  and  $\int_0^\cdot k_s d[Z, M^\perp]_s$  in (4.14) are  $\mathbb{G}$ -martingales on  $[0, t]$ . The first stochastic integral, i.e.,  $\int_0^\cdot \widehat{\gamma}_{s-} dM_s^\mathbb{G}$ , is a  $\mathbb{G}$ -martingale since  $(\widehat{\gamma}_{s-})_{s \in [0, t]}$  is a bounded predictable process. Next, we prove that the local martingale

$$\int_0^u M_{s-}^\mathbb{G} k_s dZ_s, \quad u \in [0, t], \quad (4.15)$$

is a  $\mathbb{G}$ -martingale. Using [CE15, Lemma 16.2.5], [CE15, Theorem 16.2.6] and the Èmery inequality for BMO ([CE15, Theorem A.8.15.]) applied to local martingales (with  $p = 1$ ) we obtain

$$\mathbb{E}(M_-^\mathbb{G} \cdot (k \cdot Z))_t^* \leq C \mathbb{E}(M^\mathbb{G})_t^* \|(k \cdot Z)\|_{BMO}.$$

From the Doob maximal inequality, we obtain that  $\mathbb{E}(M^\mathbb{G})_t^* < \infty$ . Next, since  $\widehat{\gamma}$  is bounded, using [CE15, Remark A.8.3.] we see that  $\|(k \cdot Z)\|_{BMO} < \infty$ . Therefore the local martingale given by (4.15) is a martingale. Finally, we consider

$$\int_0^u k_s d[Z, M^\perp]_s, \quad u \in [0, t]. \quad (4.16)$$

Using Kunita-Watanabe's inequality and the Cauchy-Schwartz's inequality we obtain that

$$\mathbb{E}(k \cdot [Z, M^\perp])_t^* \leq \mathbb{E} \int_0^t |k_s| |d[Z, M^\perp]_s| \leq \left( \mathbb{E} \int_0^t k_s^2 d[Z]_s \right)^{1/2} (\mathbb{E}[M^\perp]_t)^{1/2} < +\infty.$$

So the process given by (4.16) is a martingale. Consequently, since the processes  $\int_0^\cdot \widehat{\gamma}_{s-} dM_s^\mathbb{G}$ ,  $\int_0^\cdot M_{s-}^\mathbb{G} k_s dZ_s$  and  $\int_0^\cdot k_s d[Z, M^\perp]_s$  in (4.14) are  $\mathbb{G}$ -martingales on  $[0, t]$ , the left hand side of (4.11) takes the form

$$\mathbb{E}(\gamma M_t^\mathbb{G}) = \mathbb{E} \left( \int_0^t k_s H_s d[Z]_s \right). \quad (4.17)$$

Now we deal with the right-hand side of (4.11). Invoking (4.9) and (4.13), and using integration by parts formula, we may write the right-hand side of (4.11) as

$$\mathbb{E}(\gamma M_t^\mathbb{F}) = \mathbb{E}(\gamma \circ^\mathbb{F} M_t^\mathbb{G}) = \mathbb{E} \left( \int_0^t \widehat{\gamma}_{s-} d^\mathbb{F} M_s^\mathbb{G} + \int_0^t \circ^\mathbb{F} M_{s-}^\mathbb{G} k_s dZ_s + [\widehat{\gamma}, \circ^\mathbb{F} M^\mathbb{G}]_t \right). \quad (4.18)$$

Next, let us note that from (4.9), (4.13) and (4.10) we get

$$[\hat{\gamma}, {}^o\mathbb{F}M^{\mathbb{G}}]_t = \int_0^t k_s d[Z, M^{\mathbb{F}}]_s = \int_0^t k_s h_s d[Z]_s.$$

Using this and (4.18) we obtain

$$\mathbb{E}(\gamma M_t^{\mathbb{F}}) = \mathbb{E}\left(\int_0^t \hat{\gamma}_{s-} dM_s^{\mathbb{F}} + \int_0^t M_{s-}^{\mathbb{F}} k_s dZ_s + \int_0^t k_s h_s d[Z]_s\right). \quad (4.19)$$

Applying reasoning analogous to the one that led to (4.17), and invoking (4.19) we conclude that

$$\mathbb{E}(\gamma M_t^{\mathbb{F}}) = \mathbb{E}\left(\int_0^t k_s h_s d[Z]_s\right). \quad (4.20)$$

Putting together (4.11), (4.17) and (4.20), we see that (4.11) is equivalent to

$$\mathbb{E}\left(\int_0^t H_s k_s d[Z]_s\right) = \mathbb{E}\left(\int_0^t h_s k_s d[Z]_s\right) \quad (4.21)$$

for any  $k$ , such that  $\int_0^t k_s dZ_s$  is bounded.

We will now show that (4.21) extends to any  $k$  bounded, a result that we will need in what follows. Towards this end let us take an arbitrary predictable and bounded  $k$  and define a square integrable random variable  $\psi$  by

$$\psi := \int_0^t k_s dZ_s.$$

The random variable  $\psi$  is a (point-wise) limit of the sequence  $\psi_n := \psi \wedge n$  of bounded random variables and hence  $\mathbb{E}\psi_n \rightarrow \mathbb{E}\psi = 0$ . Moreover, for each  $n$  we have the predictable representation  $\psi_n = \mathbb{E}(\psi_n) + \int_0^t k_s^n dZ_s$ , and thus

$$\begin{aligned} \mathbb{E}\left(\int_0^t (k_s^n - k_s)^2 d[Z]_s\right) &= \mathbb{E}\left(\int_0^t k_s^n dZ_s - \int_0^t k_s dZ_s\right)^2 = \mathbb{E}((\psi_n - \psi - \mathbb{E}(\psi_n))^2) \\ &\leq 2\mathbb{E}((\psi_n - \psi)^2) + 2(\mathbb{E}(\psi_n))^2 \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Using this and the Kunita-Watanabe inequality we obtain

$$\mathbb{E}\left(\int_0^t |H_s(k_s^n - k_s)| d[Z]_s\right) \leq \left(\mathbb{E}\left(\int_0^t |H_s|^2 d[Z]_s\right)\right)^{\frac{1}{2}} \left(\mathbb{E}\left(\int_0^t |k_s^n - k_s|^2 d[Z]_s\right)\right)^{\frac{1}{2}} \xrightarrow{n \rightarrow \infty} 0$$

and

$$\mathbb{E}\left(\int_0^t |h_s(k_s^n - k_s)| d[Z]_s\right) \leq \left(\mathbb{E}\left(\int_0^t |h_s|^2 d[Z]_s\right)\right)^{\frac{1}{2}} \left(\mathbb{E}\left(\int_0^t |k_s^n - k_s|^2 d[Z]_s\right)\right)^{\frac{1}{2}} \xrightarrow{n \rightarrow \infty} 0.$$

Using these two facts and (4.21) for  $k^n$ , we can pass to the limit in (4.21) and obtain that (4.21) holds for any bounded  $k$ . Thus, using [HWY92, Theorem 5.16], we have

$$\mathbb{E}\left(\int_0^t (\mathbb{E}(H_s|\mathcal{F}_s) - h_s) k_s d[Z]_s\right) = 0$$

for any bounded  $k$ . Hence taking

$$k_s = \mathbb{1}_{\{(\mathbb{E}(H_s|\mathcal{F}_s) - h_s) > 0\}} \quad \text{and} \quad k_s = \mathbb{1}_{\{(\mathbb{E}(H_s|\mathcal{F}_s) - h_s) < 0\}}$$

we obtain

$$h_s = \mathbb{E}(H_s|\mathcal{F}_s) \quad d[Z] \otimes d\mathbb{P} \text{ a.e. on } [0, t] \times \Omega. \quad (4.22)$$

This, together with formula (5.10) gives (4.5).  $\square$

Theorem 4.2 did not impose any special structural assumptions on the  $\mathbb{G}$ -special semimartingale  $X$ . Below, we shall provide a result, which deals with a special case of the problem studied in this section.

**Proposition 4.3.** *Assume that conditions B1-B3 hold. Additionally, assume that the process  $Z$  satisfies*

$$d[Z]_t = dY_t + \kappa_t dt, \quad t \geq 0, \quad (4.23)$$

where  $\kappa$  is  $\mathbb{F}$ -predictable and  $\kappa \neq 0$  outside an evanescent set. Let  $X$  be a square integrable  $\mathbb{G}$ -martingale with Galtchouk-Kunita-Watanabe decomposition of the form

$$X_t = \int_0^t H_u dZ_u + O_t, \quad t \geq 0, \quad (4.24)$$

where  $O$  is a  $\mathbb{G}$ -martingale orthogonal to  $Z$  and  $H$  is integrable with respect to  $Z$  and such that its predictable projection  ${}^{p,\mathbb{F}}H$  exists. Then

$$d^{o,\mathbb{F}}X_t = h_t dZ_t, \quad t \geq 0, \quad (4.25)$$

with  $h = {}^{p,\mathbb{F}}H$  outside an evanescent set, so that  $h_t(\omega) = \mathbb{E}(H_t|\mathcal{F}_{t-})(\omega)$  for  $(\omega, t)$  outside an evanescent set. In particular the  $\mathbb{F}$ -characteristics of  ${}^{o,\mathbb{F}}X$  are

$$B^{\mathbb{F}} = 0, \quad (4.26)$$

$$C^{\mathbb{F}} = \int_0^\cdot h_s^2 d\langle Z^c \rangle_s, \quad (4.27)$$

and

$$\nu^{\mathbb{F}}(A, dt) = \int_{\mathbb{R}} \mathbb{1}_{A \setminus \{0\}}(h_t x) \nu^{Z, \mathbb{F}}(dx, dt), \quad A \in \mathcal{B}(\mathbb{R}), \quad (4.28)$$

where  $\nu^{Z, \mathbb{F}}$  is the  $\mathbb{F}$ -compensator of the jump measure of  $Z$ .

*Proof.* The canonical decomposition of the  $\mathbb{F}$ -semimartingale  ${}^{o,\mathbb{F}}X$  is (cf. (4.2))

$${}^{o,\mathbb{F}}X_t = X_0 + ({}^{o,\mathbb{F}}X)_t^{c, \mathbb{F}} + \int_0^t \int_E x(\mu^{\mathbb{F}}(ds, dx) - \nu^{\mathbb{F}}(ds, dx)) + B_t^{\mathbb{F}} = X_0 + M_t^{\mathbb{F}} + B_t^{\mathbb{F}}.$$

Now, in view of (4.3) we see that  $B^{\mathbb{F}}$  is a null process. Consequently,  ${}^{o,\mathbb{F}}X$  is a square integrable  $\mathbb{F}$ -martingale. Thus, invoking assumption B1 we conclude that

$$d^{o,\mathbb{F}}X_t = h_t dZ_t, \quad t \geq 0,$$



for an  $\mathbb{F}$ -predictable process  $h$ . The process  $h$  is such that for any bounded  $\mathbb{F}$ -predictable process  $\phi$  and any  $t \geq 0$ , one has

$$\mathbb{E}\left(\int_0^t h_s dZ_s \int_0^t \phi_s dZ_s\right) = \mathbb{E}\left(\left(\int_0^t H_s dZ_s + O_t\right) \int_0^t \phi_s dZ_s\right),$$

or, equivalently,

$$\mathbb{E}\left[\int_0^\cdot h_s dZ_s, \int_0^\cdot \phi_s dZ_s\right]_t = \mathbb{E}\left[\int_0^\cdot H_s dZ_s + M_\cdot, \int_0^\cdot \phi_s dZ_s\right]_t.$$

Hence, since in view of B2 process  $Z$  is both an  $\mathbb{F}$ -semimartingale and a  $\mathbb{G}$ -semimartingale, by [HWY92, Theorem 9.15 part 3)] and the fact that  $M$  is orthogonal to  $Z$  we obtain that, for any bounded  $\mathbb{F}$ -predictable process  $\phi$  and any  $t \geq 0$ ,

$$\mathbb{E}\left(\int_0^t h_s \phi_s d[Z]_s\right) = \mathbb{E}\left(\int_0^t H_s \phi_s d[Z]_s\right),$$

which, by using (4.23), may be written as

$$\mathbb{E}\left(\int_0^t h_s \phi_s (dY_s + \kappa_s ds)\right) = \mathbb{E}\left(\int_0^t H_s \phi_s (dY_s + \kappa_s ds)\right).$$

This and the fact that  $Y$  is both an  $\mathbb{F}$  and a  $\mathbb{G}$ -martingale imply, with help of a localizing sequence of stopping times if needed, that

$$\mathbb{E}\left[\int_0^t h_s \phi_s \kappa_s ds\right] = \mathbb{E}\left[\int_0^t H_s \phi_s \kappa_s ds\right].$$

Theorem 5.16. part 2) in [HWY92] applied for  $S = 0$  and  $T = t$ , combined with [HWY92, Theorem 5.5] yield

$$\mathbb{E}\left[\int_0^t H_s \phi_s \kappa_s ds\right] = \mathbb{E}\left[\int_0^t {}^{p,\mathbb{F}}(H_s \phi_s \kappa_s) ds\right] = \mathbb{E}\left[\int_0^t \phi_s \kappa_s {}^{p,\mathbb{F}}H_s ds\right] = \mathbb{E}\left[\int_0^t \phi_s \kappa_s h_s ds\right].$$

Since  $\phi$  is arbitrary we obtain from the above that, outside an evanescent set,

$$\kappa_t h_t = \kappa_t {}^{p,\mathbb{F}}H_t$$

and thus, by the fact that  $\kappa$  does not vanish, we have  $h = {}^{p,\mathbb{F}}H$  outside an evanescent set. The above proves (4.25). Formulae (4.26) and (4.27) follow immediately from (4.25).

To finish the proof we need to justify (4.28). This formula is a consequence of the fact that  $\Delta^{\circ,\mathbb{F}}X_t = h_t \Delta Z_t$ , which entails that the jump measure of  ${}^{\circ,\mathbb{F}}X$  is the image of the jump measure of  $Z$  under the mapping  $(t, x) \rightarrow (t, x h_t \mathbb{1}_{\{h_t \neq 0\}})$ , and thus the  $\mathbb{F}$ -compensator of  ${}^{\circ,\mathbb{F}}X$  is the image of the  $\mathbb{F}$ -compensator of  $Z$ .

The proof is complete.  $\square$

## 5 Examples

Examples 5.1–5.4 below illustrate the results in the case when  $X$  is  $\mathbb{F}$ -adapted. In what follows the natural filtration of a process  $Y$  is denoted by  $\mathbb{F}^Y$ .

**Example 5.1.** Consider two one-point càdlàg processes  $Y^1$  and  $Y^2$  on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and let  $Y = (Y^1, Y^2)$ . That is,  $Y^i$  ( $i = 1, 2$ ) starts from 0 at time  $t = 0$  and jumps to 1 at some random time. Thus,  $Y$  can be identified with a pair of positive random variables  $T_1$  and  $T_2$  given by  $T_i := \inf\{t > 0 : Y_t^i = 1\}$ ,  $i = 1, 2$ . In other words,  $Y_t^i = \mathbb{1}_{\{T_i \leq t\}}$ ,  $i = 1, 2$ . We assume that, under  $\mathbb{P}$ , the probability distribution of  $(T_1, T_2)$  admits a density function  $f(u, v)$  which is continuous in both variables.

Now, let  $X = Y^1$ ,  $\mathbb{F} = \mathbb{F}^X$  and  $\mathbb{G} = \mathbb{F}^Y$ . Clearly,  $X$  is a special  $\mathbb{G}$ -semimartingale and a special  $\mathbb{F}$ -semimartingale on  $(\Omega, \mathcal{F}, \mathbb{P})$ .

The  $\mathbb{G}$ -characteristics of  $X$  are  $(B^\mathbb{G}, 0, \nu^\mathbb{G})$ , where

$$B_t^\mathbb{G} = \int_0^t \kappa_s ds, \quad \nu^\mathbb{G}(ds, dx) = \delta_1(dx) \kappa_s ds,$$

$\delta_1$  is the Dirac measure at 1, and  $\kappa$  is given by (this result follows, for example, by application of [LB95, Theorem 4.1.11])

$$\kappa_s = \frac{\int_s^\infty f(s, v) dv}{\int_s^\infty \int_s^\infty f(u, v) du dv} \mathbb{1}_{\{s \leq T_1 \wedge T_2\}} + \frac{f(s, T_2)}{\int_s^\infty f(u, T_2) du} \mathbb{1}_{\{T_2 < s \leq T_1\}}, \quad s \geq 0.$$

Thus, according to Theorem 3.5, the  $\mathbb{F}$ -characteristics of  $X$  are  $(B^\mathbb{F}, 0, \nu^\mathbb{F})$ , where

$$B_t^\mathbb{F} = \int_0^t {}^{o, \mathbb{F}}(\kappa)_s ds, \quad \nu^\mathbb{F} = (\nu^\mathbb{G})^{p, \mathbb{F}}.$$

Now, we will provide explicit formulae for  $B^\mathbb{F}$  and  $\nu^\mathbb{F}$ ; for the latter, we only need to compute  $\nu^\mathbb{F}(dt, \{1\})$ . It can be easily shown that these computations boil down to computing the  $\mathbb{F}$ -optional projection of the process  $\kappa$ . Indeed, for an arbitrary  $\mathbb{F}$ -predictable, bounded function  $W$  on  $\Omega \times \mathbb{R}$  we have

$$\begin{aligned} \mathbb{E}\left(\int_{\mathbb{R}_+ \times \mathbb{R}} W(s, x) \nu^\mathbb{G}(ds, dx)\right) &= \mathbb{E}\left(\int_{\mathbb{R}_+} W(s, 1) \kappa_s ds\right) = \mathbb{E}\left(\int_{\mathbb{R}_+} {}^{p, \mathbb{F}}(W(\cdot, 1) \kappa)_s ds\right) \\ &= \mathbb{E}\left(\int_{\mathbb{R}_+} {}^{p, \mathbb{F}}(\kappa)_s W(s, 1) ds\right) = \mathbb{E}\left(\int_{\mathbb{R}_+ \times \mathbb{R}} {}^{p, \mathbb{F}}(\kappa)_s W(s, x) \delta_1(dx) ds\right), \end{aligned}$$

where  ${}^{p, \mathbb{F}}(\kappa)$  denotes the  $\mathbb{F}$ -predictable projection of  $\kappa$ . Next, we note that the measure  $\rho$  defined as

$$\rho(dt, dx) := {}^{p, \mathbb{F}}(\kappa)_t \delta_1(dx) dt$$

is  $\mathbb{F}$ -predictable, and thus, due to uniqueness of the dual predictable projections, we have  $\rho = (\nu^\mathbb{G})^{p, \mathbb{F}}$ , and so  $\nu^\mathbb{F} = \delta_1(dx) {}^{p, \mathbb{F}}(\kappa)$ . Finally, we note that, in view of the continuity assumptions on  $f$  and that fact that  $\kappa$  admits two jumps only, we have

$$\mathbb{E}\left(\int_{\mathbb{R}_+ \times \mathbb{R}} {}^{p, \mathbb{F}}(\kappa)_s W(s, x) \delta_1(dx) ds\right) = \mathbb{E}\left(\int_{\mathbb{R}_+ \times \mathbb{R}} {}^{o, \mathbb{F}}(\kappa)_s W(s, x) \delta_1(dx) ds\right),$$

where  ${}^{\circ, \mathbb{F}}(\kappa)$  denotes the  $\mathbb{F}$ -optional projection of  $\kappa$ . Using the key lemma (see e.g. [AJ17, Lemma 2.9]) we obtain

$$\begin{aligned} {}^{\circ, \mathbb{F}}(\kappa)_s &= \mathbb{E} \left( \frac{\int_s^\infty f(s, v) dv}{\int_s^\infty \int_s^\infty f(u, v) du dv} \mathbb{1}_{\{s \leq T_1 \wedge T_2\}} + \frac{f(s, T_2)}{\int_s^\infty f(u, T_2) du} \mathbb{1}_{\{T_2 < s \leq T_1\}} \middle| \mathcal{F}_s \right) \\ &= \frac{\int_0^\infty f(s, v) dv}{\int_s^\infty \int_0^\infty f(u, v) du dv} \mathbb{1}_{\{T_1 > s\}}. \end{aligned}$$

Consequently,

$$B_t^{\mathbb{F}} = \int_0^t \frac{\int_0^\infty f(s, v) dv}{\int_s^\infty \int_0^\infty f(u, v) du dv} \mathbb{1}_{\{T_1 > s\}} ds$$

and  $\nu^{\mathbb{F}}((0, t], \{1\})$  is given as

$$\nu^{\mathbb{F}}((0, t], \{1\}) = \int_0^t \frac{\int_0^\infty f(s, v) dv}{\int_s^\infty \int_0^\infty f(u, v) du dv} \mathbb{1}_{\{T_1 > s\}} ds.$$

We note that the last result agrees with the classical computation of intensity of  $T_1$  in its own filtration, which is given as  $\lambda_s^1 = \frac{f^1(s)}{1 - F^1(s)}$  with  $F^1(s) = \mathbb{P}(T_1 \leq s)$  and  $f^1(s) = \frac{\partial F^1(s)}{\partial s}$ .  $\square$

**Example 5.2.** Let  $X$  be a real-valued process on  $(\Omega, \mathcal{F}, \mathbb{P})$  satisfying

$$dX_t = m_t dt + \sum_{j=1}^2 \sigma_t^j dW_t^j + dM_t, \quad t \geq 0,$$

where  $W^j$ s are independent standard Brownian motions (SBMs), and  $M_t = \int_0^t \int_{\mathbb{R}} x(\mu(ds, dx) - \nu(ds, dx))$  is a pure jump martingale, with absolutely continuous compensating part, say  $\nu(dx, dt) = \eta(t, dx)dt$ . We assume that  $M$  is independent of  $W^j$ s. The coefficients  $m$  and  $\sigma^j > 0$ ,  $j = 1, 2$  are adapted to  $\mathbb{G} := \mathbb{F}^{W^1, W^2, M}$  and bounded.

Let  $\mathbb{F} = \mathbb{F}^X$ . Since  $M$  and  $\sigma^1 \cdot W^1 + \sigma^2 \cdot W^2$  are true  $\mathbb{G}$ -martingales, then  $X$  is a special semimartingale in  $\mathbb{G}$  and thus in  $\mathbb{F}$ .

The  $\mathbb{G}$ -characteristics of  $X$  are

$$B_t^{\mathbb{G}} = \int_0^t m_s ds, \quad C_t^{\mathbb{G}} = \int_0^t ((\sigma_s^1)^2 + (\sigma_s^2)^2) ds, \quad \nu^{\mathbb{G}}(dx, dt) = \eta(t, dx)dt.$$

Now, in view of Theorem 3.5, we conclude that the  $\mathbb{F}$ -characteristics of  $X$  are

$$B_t^{\mathbb{F}} = \int_0^t {}^{\circ, \mathbb{F}}(m)_s ds, \quad C_t^{\mathbb{F}} = \int_0^t ((\sigma_s^1)^2 + (\sigma_s^2)^2) ds, \quad \nu^{\mathbb{F}}(dx, dt) = (\eta(t, dx)dt)^{p, \mathbb{F}}. \quad \square$$

**Example 5.3.** In this example we consider time homogeneous Poisson process with values in  $\mathbb{R}^2$ . There is a one-to-one correspondence between any time homogeneous Poisson process with values in  $\mathbb{R}^2$ , say  $N = (N^1, N^2)$ , and a homogeneous Poisson measure, say  $\mu$ , on  $E := \{0, 1\}^2 \setminus \{(0, 0)\}$ .<sup>2</sup> See for instance discussion in [BJVV08].

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<sup>2</sup>We refer to [JS03] for the definition of the Poisson measure.

Let  $\mathbb{G} = \mathbb{F}^N$ , and let  $\nu$  denote the  $\mathbb{G}$ -dual predictable projection of  $\mu$ . The measure  $\nu$  is a measure on a finite set, so it is uniquely determined by its values on the atoms in  $E$ . Therefore the Poisson process  $N = (N^1, N^2)$  is uniquely determined by

$$\nu(dt, \{1, 0\}) = \lambda_{10}dt, \quad \nu(dt, \{0, 1\}) = \lambda_{01}dt, \quad \nu(dt, \{1, 1\}) = \lambda_{11}dt \quad (5.1)$$

for some positive constants  $\lambda_{10}$ ,  $\lambda_{01}$  and  $\lambda_{11}$ . Clearly, the Poisson process  $N = (N^1, N^2)$  is a  $\mathbb{G}$ -special semimartingale, and the  $\mathbb{G}$ -characteristic triple of  $N$  is  $(B, 0, \nu)$ , where

$$B_t = \begin{bmatrix} (\lambda_{10} + \lambda_{00})t \\ (\lambda_{01} + \lambda_{00})t \end{bmatrix}.$$

Let  $X = N^1$ . Then,  $X$  is a  $\mathbb{G}$ -special semimartingale, and the  $\mathbb{G}$ -characteristic triple of  $X$  is  $(B^{\mathbb{G}}, 0, \nu^{\mathbb{G}})$ , where

$$\nu^{\mathbb{G}}(dt, \{1\}) = \nu(dt, \{(1, 0)\}) + \nu(dt, \{(1, 1)\}) = \lambda_{10}dt + \lambda_{11}dt, \quad \nu^{\mathbb{G}}(dt, \{0\}) = 0,$$

and  $B_t^{\mathbb{G}} = (\lambda_{10} + \lambda_{00})t$ .

Now, let us set  $\mathbb{F} = \mathbb{F}^X$ . To find the  $\mathbb{F}$ -characteristics of  $X$  we use Theorem 3.5. Since the  $\mathbb{G}$ -characteristics of  $X$  are deterministic we have

$${}^{o, \mathbb{F}}(b_u^{\mathbb{G}} a_u^{\mathbb{G}}) = {}^{o, \mathbb{F}}(\lambda_{10} + \lambda_{00}) = \lambda_{10} + \lambda_{00}.$$

Analogously

$$(K_t^{\mathbb{G}}(dx) a_t^{\mathbb{G}} dt)^{p, \mathbb{F}} = ((\lambda_{10} + \lambda_{11})\delta_1(dx)dt)^{p, \mathbb{F}} = (\lambda_{10} + \lambda_{11})\delta_1(dx)dt.$$

Thus the  $\mathbb{F}$ -characteristics of  $X$  are the same as its  $\mathbb{G}$ -characteristics, that is

$$(B^{\mathbb{F}}, 0, \nu^{\mathbb{F}}) = (B^{\mathbb{G}}, 0, \nu^{\mathbb{G}}).$$

□

**Example 5.4.** Let  $Y = (Y^1, Y^2)^{\top}$  be given as the strong solution of the SDE

$$dY(t) = m(Y(t))dt + \Sigma(Y(t))dW(t), \quad Y(0) = (1, 1)^{\top}, \quad (5.2)$$

where  $W = (W_1, W_2)^{\top}$  is a two dimensional SBM process on  $(\Omega, \mathcal{F}, \mathbb{P})$ , and where

$$m(y^1, y^2) = (m_1(y^1, y^2), m_2(y^1, y^2))^{\top}, \quad \Sigma(y^1, y^2) = \begin{pmatrix} \sigma_{11}(y^1, y^2) & \sigma_{12}(y^1, y^2) \\ \sigma_{21}(y^1, y^2) & \sigma_{22}(y^1, y^2) \end{pmatrix}$$

are bounded. Next, let us set  $\mathbb{G} = \mathbb{F}^Y$ ,  $X = Y^1$  and  $\mathbb{F} = \mathbb{F}^X$ . Hence

$$dX(t) = m_1(X(t), Y^2(t))dt + \sigma_{1,1}(X(t), Y^2(t))dW^1(t) + \sigma_{1,2}(X(t), Y^2(t))dW^2(t) \quad (5.3)$$

Suppose that function  $\Sigma$  satisfies the following condition

$$\sigma_{11}^2(y^1, y^2) + \sigma_{12}^2(y^1, y^2) = \sigma_1^2(y^1), \quad (y^1, y^2) \in \mathbb{R}^2,$$

for some function  $\sigma_1 > 0$ , and suppose that function  $m_1$  satisfies

$$m_1(y^1, y^2) = \mu_1(y^1) \quad (y^1, y^2) \in \mathbb{R}^2.$$

Then (5.3) takes form

$$dX(t) = \mu_1(X(t))dt + \sigma_1(X(t))dZ(t), \quad X(0) = 1,$$

where

$$Z(t) = \int_0^t \frac{\sigma_{1,1}(X(t), Y^2(t))}{\sigma_1(X(t))} dW_1(t) + \int_0^t \frac{\sigma_{1,2}(X(t), Y^2(t))}{\sigma_1(X(t))} dW_2(t)$$

is a  $\mathbb{G}$ -adapted process, which is a continuous  $\mathbb{G}$ -local martingale. Since  $(Z^2(t) - t)_{t \geq 0}$  is a local martingale we obtain by Lévy's characterization theorem that  $Z$  is a standard Brownian motion in the filtration  $\mathbb{G}$ . Thus using continuity of paths of  $X$  we conclude that  $X$  has the  $\mathbb{G}$ -characteristic triple given as  $(B^\mathbb{G}, C^\mathbb{G}, 0)$ , where

$$B_t^\mathbb{G} = \int_0^t \mu_1(X_u) du, \quad C_t^\mathbb{G} = \int_0^t \sigma_1^2(X_u) du, \quad t \geq 0.$$

We will now apply Theorem 3.5 so to compute the  $\mathbb{F}$ -characteristics of  $X$ . Since  $X$  is  $\mathbb{F}$ -adapted the  $\mathbb{F}$ -characteristics of  $X$  are

$$B_t^\mathbb{F} = \int_0^t \circ_{\mathbb{F}}(\mu_1(X_u)) du = \int_0^t \mu_1(X_u) du, \quad C_t^\mathbb{F} = C_t^\mathbb{G} = \int_0^t \sigma_1^2(X_u) du$$

Finally by continuity of paths

$$\nu^\mathbb{F}(dt, dx) = (\nu^\mathbb{G}(dt, dx))^{p, \mathbb{F}} = (0)^{p, \mathbb{F}} = 0.$$

So we conclude that  $(B^\mathbb{G}, C^\mathbb{G}, 0) = (B^\mathbb{F}, C^\mathbb{F}, 0)$ . □

The remaining examples refer to the case when  $X$  is not  $\mathbb{F}$ -adapted. As discussed in Section 4, the Theorem 4.2 addresses computation of the first two  $\mathbb{F}$ -characteristics of the optional projection of  $X$  on the filtration  $\mathbb{F}$ . As of now, we do not have a generic formula that would allow for the computation of the third characteristic, that is the computation of  $\nu^\mathbb{F}$ . This, in general, needs to be done on the case by case basis. Nevertheless, in the special set-up that fits Proposition 4.3, the complete story about the  $\mathbb{F}$ -characteristics of the optional projection of  $X$  on the filtration  $\mathbb{F}$  can be told.

**Example 5.5.** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be the underlying probability space and let  $\mathbb{G}$  be the filtration generated by a Brownian motion  $B$  and a time inhomogeneous Poisson process  $N$  with deterministic compensator  $\nu(t) = \int_0^t \lambda(s) ds$ ,  $t \in \mathbb{R}_+$ , so that  $(\nu(t))_{t \in \mathbb{R}_+}$  is the unique continuous deterministic function such that  $(M_t)_{t \in \mathbb{R}_+} := (N_t - \nu(t))_{t \in \mathbb{R}_+}$  is an  $(\mathbb{F}^N, \mathbb{P})$ -martingale. We assume that  $B$  and  $N$  are independent under  $\mathbb{P}$ , hence  $M$  is also a  $(\mathbb{G}, \mathbb{P})$ -martingale. Let  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$  and  $\alpha : \mathbb{R}_+ \rightarrow ]0, \infty[$  be two deterministic functions, with  $\int_0^t \alpha^2(s) ds < \infty$ ,  $t \geq 0$ . Let  $i(t) = \mathbb{1}_{\{\phi(t)=0\}}$ , and set

$$\lambda(t) = \begin{cases} \alpha^2(t)/\phi^2(t) & \text{if } \phi(t) \neq 0, \\ 0 & \text{if } \phi(t) = 0, \end{cases} \quad t \in \mathbb{R}_+.$$

We assume that  $\lambda$  is such that  $\lim_{t \rightarrow \infty} \nu(t) = \infty$  and  $\nu(t) < \infty$ ,  $t \geq 0$ .

By Proposition 4 in [É89] the process  $V$  given by

$$dV_t = i(t)dB_t + \frac{\phi(t)}{\alpha(t)}(dN_t - \lambda(t)dt), \quad t \geq 0, \quad V_0 = 0, \quad (5.4)$$

is the unique strong solution of the following structure equation

$$d[V]_t = dt + \frac{\phi(t)}{\alpha(t)}dV_t, \quad t \geq 0, \quad V_0 = 0. \quad (5.5)$$

By Proposition 3 ii) in [É89] the process  $V$  has the predictable representation property in  $\mathbb{F}$ . Taking  $Z_t = \int_0^t \alpha(s)dV_s$ ,  $t \geq 0$ , as in [JP02] we see that  $Z$  satisfies

$$dZ_t = i(t)\alpha(t)dB_t + \phi(t)(dN_t - \lambda(t)dt), \quad t \geq 0, \quad Z_0 = 0. \quad (5.6)$$

and

$$d[Z]_t = \alpha^2(t)dt + \phi(t)dZ(t), \quad t \geq 0, \quad Z_0 = 0, \quad (5.7)$$

The process  $Z$  is obviously a square integrable  $(\mathbb{G}, \mathbb{P})$ -martingale. We shall denote by  $\mathbb{F}$  the natural filtration of  $Z$ , which is the same as the natural filtration of  $V$  and satisfies  $\mathbb{F} \subset \mathbb{G}$  so that  $Z$  is a square integrable  $(\mathbb{F}, \mathbb{P})$ -martingale. Clearly,  $Z$  has predictable representation property in  $\mathbb{F}$  since  $\alpha > 0$  and  $V$  has predictable representation property in  $\mathbb{F}$ .

Clearly, conditions B1-B3 are satisfied here. Moreover, the process  $Z$  satisfies condition (4.23) with  $dY_t = \phi(t)dZ_t$  and  $\kappa(t) = \alpha^2(t)$ .

Let now  $X$  be a square integrable  $\mathbb{G}$ -martingale. Since  $\mathbb{G}$  is the filtration generated by  $B$  and  $N$ , the martingale  $X$  can be represented in the form

$$X_t = X_0 + \int_0^t \gamma_s dB_s + \int_0^t \Delta_s dM_s$$

for some  $\mathbb{G}$ -predictable processes  $\gamma$  and  $\Delta$ . Thus as a  $\mathbb{G}$  semimartingale,  $X$  has characteristics  $(0, C^{\mathbb{G}}, \nu^{\mathbb{G}})$  where

$$C^{\mathbb{G}} = \int_0^\cdot \gamma_t^2 dt,$$

and

$$\nu^{\mathbb{G}}(A, dt) = \left( \int_{\mathbb{R}} \mathbb{1}_{A \setminus \{0\}}(x) \delta_{\Delta_t}(dx) \right) \lambda(t) dt, \quad A \in \mathcal{B}(\mathbb{R}).$$

On the other hand  $X$  can be written in the form of the Galtchouk-Kunita-Watanabe decomposition with respect to  $Z$  as

$$X_t = X_0 + \int_0^t H_u dZ_u + O_t, \quad t \geq 0, \quad (5.8)$$

where  $O$  is a  $\mathbb{G}$ -martingale orthogonal to  $Z$  and  $H$  is a  $\mathbb{G}$ -predictable process.

Thus, for  $t \geq 0$  we have

$$O_t = \int_0^t (\gamma_s - H_s i(s) \alpha(s)) dB_s + \int_0^t (\Delta_s - H_s \phi(s)) dM_s. \quad (5.9)$$

The orthogonality of  $O$  and  $Z$  implies that  $\langle Z, O \rangle_t = 0$  for any  $t \geq 0$ . This together with (5.9) imply that for any  $t \geq s \geq 0$ , the process  $H$  satisfies

$$\int_s^t \left( (\gamma_t - H_t i(t) \alpha(t)) i(t) \alpha(t) + (\Delta_t - H_t \phi(t)) \phi(t) \lambda(t) \right) dt = 0.$$

Thus, for any  $t \geq 0$  we have

$$H_t = \frac{\gamma_t i(t) \alpha(t) + \Delta_t \phi(t) \lambda(t)}{i^2(t) \alpha^2(t) + \phi^2(t) \lambda(t)} = \frac{\gamma_t}{\alpha(t)} i(t) + (1 - i(t)) \frac{\Delta_t}{\phi(t)} \quad dt \otimes d\mathbb{P} \text{ a.e.} \quad (5.10)$$

Now, according to Proposition 4.3, we have

$$d^{o, \mathbb{F}} X_t = h_t dZ_t, \quad t \geq 0, \quad (5.11)$$

with  $h_t = {}^{p, \mathbb{F}} H_t$  for  $t \geq 0$  outside of an evanescent set, so that  $h_t = \mathbb{E}(H_t | \mathcal{F}_{t-})$  for  $t > 0$  outside of an evanescent set. In particular, in view of Proposition 4.3 again, this implies that the  $\mathbb{F}$ -characteristics of  ${}^{o, \mathbb{F}} X$  are

$$B^{\mathbb{F}} = 0, \quad (5.12)$$

$$C^{\mathbb{F}} = \int_0^\cdot h_s^2 i^2(s) \alpha^2(s) ds = \int_0^\cdot (\mathbb{E}(\gamma_s | \mathcal{F}_{s-}))^2 i(s) ds, \quad (5.13)$$

and, for any  $A \in \mathcal{B}(\mathbb{R})$ ,

$$\begin{aligned} \nu^{\mathbb{F}}(A, dt) &= \left( \int_{\mathbb{R}} \mathbb{1}_{A \setminus \{0\}}(h_t x) \delta_{\phi(t)}(dx) \right) \lambda(t) dt \\ &= \left( \int_{\mathbb{R}} \mathbb{1}_{A \setminus \{0\}}(\mathbb{E}(H_t | \mathcal{F}_{t-}) x) \delta_{\phi(t)}(dx) \right) (1 - i(t)) \lambda(t) dt \\ &= \left( \int_{\mathbb{R}} \mathbb{1}_{A \setminus \{0\}} \left( (1 - i(t)) \frac{\mathbb{E}(\Delta_t | \mathcal{F}_{t-})}{\phi(t)} x \right) \delta_{\phi(t)}(dx) \right) (1 - i(t)) \lambda(t) dt \\ &= \left( \int_{\mathbb{R}} \mathbb{1}_{A \setminus \{0\}}(x) \delta_{\mathbb{E}(\Delta_t | \mathcal{F}_{t-})}(dx) \right) \lambda(t) dt. \end{aligned}$$

**Example 5.6.** Consider a Poisson process  $N$  with intensity  $\lambda$  and an independent standard Brownian motion  $W$ . Take  $X = N$ ,  $\mathbb{G} = \mathbb{F}^X \vee \mathbb{F}^W$ . Clearly,  $X$  is a special  $\mathbb{G}$ -semimartingale, and we have

$$X_t = M_t^{\mathbb{G}} + B_t^{\mathbb{G}}, \quad t \geq 0,$$

where

$$M_t^{\mathbb{G}} = X_t - \lambda t, \quad B_t^{\mathbb{G}} = \lambda t, \quad t \geq 0.$$

The  $\mathbb{G}$ -characteristic triple of  $X$  is  $(B^{\mathbb{G}}, 0, \nu^{\mathbb{G}})$ , where

$$B_t^{\mathbb{G}} = \lambda t, \quad \nu^{\mathbb{G}}(dt, dx) = \lambda \delta_1(dx) dt, \quad t \geq 0.$$

In particular,  $b_t^{\mathbb{G}} = \lambda$  and  $a_t^{\mathbb{G}} = 1$  for  $t \geq 0$ .

Take  $\mathbb{F} = \mathbb{F}^W$ . We will use Theorem 4.2 taking  $Z = W$  to compute the first two  $\mathbb{F}$ -characteristics of  ${}^{o, \mathbb{F}} X$ . Then, we see that  $M^\perp = M^{\mathbb{G}}$  (cf. (4.6)), and thus, in view of (4.5), we conclude that  $h = 0$ . Consequently, using Theorem 4.2, we obtain

$$B_t^{\mathbb{F}} = \lambda t, \quad C_t^{\mathbb{F}} = 0, \quad t \geq 0.$$

In order to compute the third  $\mathbb{F}$ -characteristic of  ${}^{o,\mathbb{F}}X$  we note that

$${}^{o,\mathbb{F}}X_t = \mathbb{E}(X_t) = \lambda t, \quad t \geq 0, \quad (5.14)$$

and so,  $\nu^{\mathbb{F}} = 0$ . Observe that formula (5.14) also renders the first two  $\mathbb{F}$ -characteristics of  ${}^{o,\mathbb{F}}X$ :  $B_t^{\mathbb{F}} = \lambda t$ ,  $C_t^{\mathbb{F}} = 0$ ,  $t \geq 0$ , which (of course) agrees with the formulae we derived using Theorem 4.2.  $\square$

We will now present an example where  $X$  is continuous special  $\mathbb{G}$ -semimartingale, and  ${}^{o,\mathbb{F}}X$  is a purely discontinuous special  $\mathbb{F}$ -semimartingale.

**Example 5.7.** Consider a standard Brownian motion  $W$ . Let  $X = W$  and take  $\mathbb{G} = \mathbb{F}^X$ . The  $\mathbb{G}$ -characteristics triple of  $W$  is  $(0, C_t^{\mathbb{G}}, 0)$ , where  $C_t^{\mathbb{G}} = t$ . In particular, we have  $b^{\mathbb{G}} = 0$  and  $a^{\mathbb{G}} = 1$ . Next, define the filtration  $\mathbb{F}$  as

$$\mathcal{F}_t = \mathcal{F}_n^X, \quad t \in [n, n+1), \quad n = 0, 1, 2, \dots$$

The optional projection of  $X$  on  $\mathbb{F}$  exists and is given as

$${}^{o,\mathbb{F}}X_t = X_n, \quad t \in [n, n+1), \quad n = 0, 1, 2, \dots$$

In order to compute the  $\mathbb{F}$ -characteristics of  ${}^{o,\mathbb{F}}X$  we first observe that the canonical semimartingale representation of  ${}^{o,\mathbb{F}}X$ , with respect to the standard truncation function, is given as

$${}^{o,\mathbb{F}}X = x * \mu = (x \mathbb{1}_{|x| \leq 1}) * \nu + (x \mathbb{1}_{|x| \leq 1}) * (\mu - \nu) + (x \mathbb{1}_{|x| > 1}) * \mu, \quad (5.15)$$

where

$$\mu(dt, dx) = \sum_{n \geq 1} \delta_{(n, X_n - X_{n-1})}(dt, dx),$$

and

$$\nu(\omega, dt, dx) = \sum_{n \geq 1} \delta_n(dt) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

From (5.15) we obtain

$$B_t^{\mathbb{F}} = \int_0^t \int_{|x| \leq 1} x \nu(dt, dx) = 0.$$

Thus, the  $\mathbb{F}$ -characteristics triple of  ${}^{o,\mathbb{F}}X$  is  $(B^{\mathbb{F}}, 0, \nu^{\mathbb{F}})$ , where

$$\nu^{\mathbb{F}} = \nu.$$

$\square$

**Example 5.8.** Let us consider the case where  $\mathbb{F}$  is a Brownian filtration,  $\mathbb{G}$  its progressive enlargement with a strictly positive random time  $\tau$ . Taking  $X_t = \mathbb{1}_{\{\tau \leq t\}}$ ,  $t \geq 0$  we have (cf. [AJ17]),

$$\mathbb{G} = \mathbb{F} \nabla \mathbb{F}^X,$$

where  $\mathbb{F} \nabla \mathbb{F}^X$  is the smallest right-continuous filtration which contains  $\mathbb{F}$  and  $\mathbb{F}^X$ . Now, we define the Azéma supermartingale  $A$  by

$$A_t = \mathbb{P}(\tau > t | \mathcal{F}_t), \quad t \geq 0,$$



and we write its Doob-Meyer decomposition as

$$A_t = m_t - b_t, \quad t \geq 0,$$

where  $m$  is an  $\mathbb{F}$ -martingale, and  $b$  is an  $\mathbb{F}$ -predictable, increasing process which is the  $\mathbb{F}$ -dual predictable projection of  $X$ . We assume that  $\tau$  satisfies the following Jacod's absolute continuity assumption

$$\mathbb{P}(\tau > s | \mathcal{F}_t) = \int_s^\infty \alpha_t(u) du, \quad s, t \geq 0, \quad (5.16)$$

where, for any  $u \geq 0$ , the process  $\alpha_\cdot(u)$  is a positive continuous  $\mathbb{F}$ -martingale and the map  $(\omega, t, u) \rightarrow \alpha_t(\omega; u)$  is  $\tilde{\mathcal{P}}_\mathbb{F}$ -measurable. Using the fact that  $\int_0^\infty \alpha_t(u) du = \mathbb{P}(\tau > 0) = 1$  and  $\alpha_\cdot(u)$  is a martingale, it is shown in Proposition 4.1 in [EKJJ10] that

$$db_t = \alpha_t(t) dt$$

and

$$m_t = \mathbb{E}\left(\int_0^\infty \alpha_u(u) du | \mathcal{F}_t\right) = 1 + \int_0^t \alpha_u(u) du - \int_0^t \alpha_t(u) du. \quad (5.17)$$

Note that in the above set-up, the process  $A$  is continuous.

The process  $X$  is a special  $\mathbb{G}$ -semimartingale and we know (cf. Corollary 5.27 in [AJ17]) that its canonical decomposition is given as

$$X = M^\mathbb{G} + B^\mathbb{G},$$

and its  $\mathbb{G}$ -characteristics are  $(B^\mathbb{G}, 0, \nu^\mathbb{G})$ , where

$$B_t^\mathbb{G} = \int_0^t (1 - X_s) \frac{db_s}{A_s} = \int_0^t \frac{(1 - X_s)\alpha_s(s)}{A_s} ds, \quad t \geq 0$$

and

$$\nu^\mathbb{G}(dt, dx) = \delta_1(dx) \frac{(1 - X_{t-})\alpha_t(t)}{A_t} dt.$$

In particular, note that here we have  $b_t^\mathbb{G} = \frac{(1 - X_t)\alpha_t(t)}{A_t}$  and  $a_t^\mathbb{G} = 1$ .

Now, using Lemma 3.3 and observing that  ${}^{o,\mathbb{F}}X = 1 - A$  we can easily compute the first  $\mathbb{F}$ -characteristic of  ${}^{o,\mathbb{F}}X$ ,

$$B_t^\mathbb{F} = \int_0^t {}^{o,\mathbb{F}} \left( \frac{(1 - X_s)\alpha_s(s)}{A_s} \right) ds = \int_0^t \alpha_s(s) ds.$$

Next, recalling that  $A$  is a continuous process we conclude that  $\nu^\mathbb{F} = 0$ . Moreover, we see that  $C^\mathbb{F} = \langle m \rangle$ . This completes the computation of the  $\mathbb{F}$ -characteristics of  ${}^{o,\mathbb{F}}X$  which are  $(B^\mathbb{F}, \langle m \rangle, 0)$ .  $\square$

The next example is, in a sense, opposite to Example 5.7: here,  $X$  is a purely discontinuous special  $\mathbb{G}$ -semimartingale, and  ${}^{o,\mathbb{F}}X$  is a continuous special  $\mathbb{F}$ -semimartingale.

**Example 5.9.** Let  $\mathbb{F}$  be a Brownian filtration and  $\mathbb{G}$  its progressive enlargement with a strictly positive random time  $\tau \in \mathcal{F}_\infty$  satisfying Jacod's absolute continuity assumption (5.16) with some density  $\alpha_t(u)$ ,  $t, u \geq 0$ . Such a random time can be defined as  $\tau := \psi(\int_0^\infty f(t)dB_t)$ , where  $\psi$  is a differentiable, positive and strictly increasing function, and  $B$  is a real valued standard  $\mathbb{F}$ -Brownian motion (see [EKJJZ14]). Let  $\hat{X}$  be the compensated martingale

$$\hat{X}_t = \mathbb{1}_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \frac{\alpha_s(s)}{A_s} ds, \quad t \geq 0.$$

We see that its  $\mathbb{G}$ -characteristic triple is  $(0, 0, \nu^\mathbb{G})$  where, as in the previous example,

$$\nu^\mathbb{G}(dt, dx) = \delta_1(dx) \frac{\mathbb{1}_{\{t < \tau\}} \alpha_t(t)}{A_t} dt.$$

The  $\mathbb{F}$ -optional projection of  $\hat{X}$ , say  $v$ , is a continuous martingale, which is not constant. Indeed, note that if  $v$  were constant then  $v_\infty = v_0 = 0$ . Given that, one has  $\hat{X}_\infty = 1 - \int_0^\tau \frac{\alpha_s(s)}{A_s} ds \in \mathcal{F}_\infty$  and  $v_\infty = \hat{X}_\infty$ . But since  $v_\infty = 0$ , then  $\hat{X}_\infty = 0$ , and  $\hat{X}$  being a martingale would be null, which it is not. This is a contradiction, showing that  $v$  is not constant. Consequently, its  $\mathbb{F}$  characteristic triple is  $(0, C^\mathbb{F}, 0)$ , with  $C^\mathbb{F} \neq 0$ .

## 6 Conclusion and open problems for future research

As stated in the Introduction this paper is meant to initiate a systematic study of the change of properties of semimartingales under shrinkage of filtrations and, when appropriate, under respective projections. The paper does not aim at a complete and comprehensive study of the topic. Rather, we analyze in some special settings a selection of relevant research problems. Our study contributes, we believe, to understanding and solution of these problems.

Given its pioneering nature the study originated here leads to numerous open problems and calls for extensions in numerous directions. Below, we indicate some such open problems and suggestions for continuation of the research presented in this paper.

The results presented in this paper use several non-trivial assumptions. A natural direction for continuation of the present work will be to try to eliminate some of these assumptions.

Recall the decomposition (4.8)

$${}^{o,\mathbb{F}}X_t = X_0 + {}^{o,\mathbb{F}}M_t^\mathbb{G} + {}^{o,\mathbb{F}}B_t^\mathbb{G} = X_0 + {}^{o,\mathbb{F}}M_t^\mathbb{G} + {}^{o,\mathbb{F}}B_t^\mathbb{G} - \int_0^t {}^{o,\mathbb{F}}(b^\mathbb{G}a^\mathbb{G})_u du + \int_0^t {}^{o,\mathbb{F}}(b^\mathbb{G}a^\mathbb{G})_u du.$$

As it was shown in the proof of Theorem 4.2, if the immersion hypothesis B2 is postulated, then the martingale  $M_t^B = {}^{o,\mathbb{F}}B_t^\mathbb{G} - \int_0^t {}^{o,\mathbb{F}}(b^\mathbb{G}a^\mathbb{G})_u du$  is null. Therefore it does not intervene in the representation of the  $\mathbb{F}$ -characteristics of  ${}^{o,\mathbb{F}}X$ . If however the martingale  $M^B$  is not null, then the computation of the  $\mathbb{F}$ -characteristics of  ${}^{o,\mathbb{F}}X$  in terms of the  $\mathbb{G}$ -characteristics of  $X$  is much more challenging, and perhaps may not be doable.

The immersion hypothesis B2 postulated Theorem 4.2 is also heavily exploited in computation of the second  $\mathbb{F}$ -characteristic of  $X$ , that is in computation of  $C^\mathbb{F}$ . In fact, computation

of  $C^\mathbb{F}$  in terms of  $\mathbb{G}$ -canonical decomposition appears to be much more difficult, or even impossible, without the hypothesis B2, as the following reasoning shows: Assume that  $\mathbb{F}$  is a Brownian filtration generated by  $W$ , so that  $W$  enjoys the predictable representation property in  $\mathbb{F}$ . Also, take  $\mathbb{G}$  to be the progressive enlargement of  $\mathbb{F}$  by a random time  $\tau$ .<sup>3</sup> Assume that there exists  $\mu$ , a  $\mathbb{G}$ -predictable integrable process such that  $W^\mathbb{G}$  defined for any  $t \in \mathbb{R}_+$  as

$$W_t^\mathbb{G} := W_t + \int_0^t \mu_s ds$$

is a  $\mathbb{G}$ -martingale (hence, a  $\mathbb{G}$ -Brownian motion). Then, any  $\mathbb{G}$ -martingale  $X$  can be written as

$$X_t = X_0 + \int_0^t \psi_s dW_s^\mathbb{G} + M_t^\perp, \quad t \geq 0,$$

where  $\psi$  is a  $\mathbb{G}$ -predictable process and  $M^\perp$  a  $\mathbb{G}$ -martingale orthogonal to  $W^\mathbb{G}$  (in fact, it is a purely discontinuous martingale). Moreover, one can show (using the same methodology as in [GJW19]) that  ${}^{o,\mathbb{F}}X$ , which is an  $\mathbb{F}$ -martingale, has the form

$${}^{o,\mathbb{F}}X_t = {}^{o,\mathbb{F}}X_0 + \int_0^t \gamma_s dW_s, \quad t \geq 0,$$

where  $\gamma$  satisfies  $\gamma_t = \mathbb{E}(\psi_t + \mu_t X_t | \mathcal{F}_t)$ . So here we have that

$$C_t^\mathbb{F} = \int_0^t (\mathbb{E}(\psi_s + \mu_s X_s | \mathcal{F}_s))^2 ds,$$

$$C_t^\mathbb{G} = \int_0^t \psi_s^2 ds.$$

Clearly,  $C^\mathbb{G}$  alone does not suffice to compute  $C^\mathbb{F}$ , unless  $\mu \equiv 0$  – i.e.,  $\mathbb{F}$  is immersed in  $\mathbb{G}$ . In fact, it is not clear at all, how to compute the  $C^\mathbb{F}$  characteristic of  $X$  in terms of the canonical decomposition and  $\mathbb{G}$  characteristics of  $X$ .

The discussion above points to an important open problem: extend, if possible, the results of Theorem 4.2 to the case when the immersion hypothesis B2 is abandoned, and extend the result of [GJW19] to the case of general continuous semi-martingales.

Another challenging problem for future research is weakening of the predictable representation property condition B1, and replacing it with the postulate of the weak predictable representation property condition for  $Z$ , that is with the postulate that every local  $\mathbb{F}$ -martingale  $Y$  admits the representation

$$Y_t = Y_0 + \psi \cdot Z_t^c + \xi * \tilde{\mu}_t^Z, \quad t \geq 0,$$

where  $\psi$  is an  $\mathbb{F}$ -predictable process,  $\xi$  is a  $\tilde{\mathcal{P}}_\mathbb{F}$ -measurable function,  $Z^c$  is the continuous martingale part of  $Z$ , and  $\tilde{\mu}^Z$  is the  $\mathbb{F}$ -compensated measure of jumps of  $Z$ .

Finally, one would like to explicitly compute  $h$  showing in (4.4) in terms of  $c^\mathbb{G}$  and  $b^\mathbb{G}$ . A possible starting point for this may be the following “master equation”

$$C_t^\mathbb{G} = \int_0^t H_s^2 d\langle Z^c \rangle_s + \langle M^{\perp,c} \rangle_t = \int_0^t c_s^\mathbb{G} a_s^\mathbb{G} ds$$

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<sup>3</sup>See e.g. Chapter 5 in [AJ17] for the concept of the progressive enlargement of filtrations.

From this equation one would try to compute  $H$  in terms of  $c^{\mathbb{G}}$  and  $a^{\mathbb{G}}$  and then use (4.5) to compute  $h$ . In special cases, such as the one presented in Example 5.5, the quasi-explicit computation of  $h$  can be done. But, in general, this remains a challenging open problem.

## References

- [AJ17] Anna Aksamit and Monique Jeanblanc. Enlargements of Filtrations with Finance in view. Springer, 2017.
- [BJNng] Tomasz R. Bielecki, Jacek Jakubowski, and Mariusz Niewegłowski. Fundamentals of Theory of Structured Dependence between Stochastic Processes. Encyclopedia of Mathematics and its Applications. Cambridge Univ. Press, 2019 (forthcoming).
- [BJVV08] Tomasz R. Bielecki, Jacek Jakubowski, Andrea Vidozzi, and Luca Vidozzi. Study of dependence for some stochastic processes. Stoch. Anal. Appl., 26(4):903–924, 2008.
- [BY78] Pierre Brémaud and Marc Yor. Changes of Filtrations and of Probability Measures. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 45:269–295, 1978.
- [CE15] Samuel N. Cohen and Robert J. Elliott. Stochastic calculus and applications. Probability and its Applications. Springer, Cham, second edition, 2015.
- [É89] Michel Émery. On the azéma martingales. Séminaire de probabilités de Strasbourg, 23:66–87, 1989.
- [EKJJ10] Nicole El Karoui, Monique Jeanblanc, and Ying Jiao. What happens after a default: the conditional density approach. Stochastic Process. Appl., 120(7):1011–1032, 2010.
- [EKJJZ14] Nicole El Karoui, Monique Jeanblanc, Ying Jiao, and Behnaz Zargari. Conditional default probability and density. In Inspired by finance, pages 201–219. Springer, Cham, 2014.
- [GJW19] Pavel V Gapeev, Monique Jeanblanc, and Dongli Wu. Projections in enlargements of filtrations under Jacod’s hypothesis and examples. preprint, <https://hal.archives-ouvertes.fr/hal-0212034>, May 2019.
- [HWY92] Sheng Wu He, Jia Gang Wang, and Jia An Yan. Semimartingale Theory and Stochastic Calculus. Kexue Chubanshe (Science Press), Beijing, 1992.
- [Jac79] Jean Jacod. Calcul Stochastique et Problèmes de Martingales. Springer, 1979.
- [JP02] M. Jeanblanc and N. Privault. A complete market model with Poisson and Brownian components. In R.C. Dalang, M. Dozzi, and F. Russo, editors, Proceedings of the Ascona 99 Seminar on Stochastic Analysis, Random Fields and Applications, volume 52 of Progress in Probability, pages 189–204. Birkhäuser Verlag, 2002.

- [JS03] Jean Jacod and Albert N. Shiryaev. Limit theorems for stochastic processes, volume 288 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, second edition, 2003.
- [LB95] Günter Last and Andreas Brandt. Marked point processes on the real line. Probability and its Applications (New York). Springer-Verlag, New York, 1995. The dynamic approach.
- [LS89] Robert Liptser and Albert N. Shiryaev. Theory of Martingales, volume 288 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Kluwer Academic Publishers, Berlin, second edition, 1989.
- [LS01] R. Liptser and A. N. Shiryaev. Statistics of Random Processes., volume I: General Theory. Springer, second edition, 2001.
- [RW00] L. C. G. Rogers and David Williams. Diffusions, Markov processes, and martingales. Vol. 2. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. Itô calculus, Reprint of the second (1994) edition.
- [Sch01] Martin Schweizer. A guided tour through quadratic hedging approaches. In Option pricing, interest rates and risk management, Handb. Math. Finance, pages 538–574. Cambridge Univ. Press, Cambridge, 2001.
- [Str77] Christophe Stricker. Quasimartingales, martingales locales, semimartingales et filtration naturelle. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 39(1):55–63, 1977.