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► **To cite this version:**

Patrick Bernard, Stefan Suhr. Cauchy and uniform temporal functions of globally hyperbolic cone fields. 2019. hal-02129166v1

**HAL Id: hal-02129166**

**<https://hal.science/hal-02129166v1>**

Preprint submitted on 14 May 2019 (v1), last revised 25 Mar 2020 (v2)

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# Cauchy and uniform temporal functions of globally hyperbolic cone fields.

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**Abstract.** We study a class of time functions called uniform temporal functions in the general context of globally hyperbolic closed cone fields. We prove some existence results for uniform temporal functions, and prove the density of uniform temporal functions in Cauchy causal functions.

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**Résumé.** Nous étudions une classe de fonctions temps, appelées fonctions temporeles uniformes, adans le contexte général d'un champ de cone fermé globalement hyperbolique. Nous prouvons des résultats d'existence pour ces fonctions temporeles uniformes, et nous prouvons la densité des fonctions temporeles uniformes dans les fonctions causales de Cauchy.

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This research has been supported by the SFB/TRR 191 “Symplectic Structures in Geometry, Algebra and Dynamics”, funded by the Deutsche Forschungsgemeinschaft

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# 1 Introduction

Let us first recall some definitions from [4], see also [9]. We say that  $C \subset E$  is a convex cone in the vector space  $E$  if it is a non empty convex set  $C$  contained in an open half space which is positively homogeneous:  $tx \in C$  if  $t > 0$  and  $x \in C$  (this definition is slightly more restrictive than in [4], here we do not admit singular or degenerate points). A cone field is a subset  $\mathcal{C} \subset TM$  such that  $\mathcal{C}(x) := \mathcal{C} \cap T_x M$  is a convex cone for each  $x \in M$ . We say that  $\mathcal{C}$  is a closed cone field if it is a cone field such that  $\mathcal{C} \cup T^0 M$  is closed in  $TM$ , where  $T^0 M$  is the zero section of the tangent bundle. Our definitions imply that cone fields are disjoint from  $T^0 M$ . We say that  $\mathcal{E}$  is an open cone field if it is a cone field open in  $TM$ . The main example of closed cone fields we have in mind is the set of future pointing non spacelike vectors associated to an oriented Lorentzian metric. Our setting is more general, but our main results are new even in this important case. The main example of open cone field is the set of future pointing timelike vectors associated to an oriented Lorentzian metric.

For a closed cone field  $\mathcal{C}$ , the curve  $\gamma : I \rightarrow M$  is said to be  $\mathcal{C}$ -causal (or just causal) if it is locally Lipschitz and if the inclusion  $\dot{\gamma}(t) \in \mathcal{C}(\gamma(t)) \cup \{0\}$  holds for almost all  $t \in I$ . The strictly causal future  $\mathcal{J}_{\mathcal{C}}^{++}(x)$  of  $x$  is the set of points  $y \in M$  such that there exists a nonconstant causal curve  $\gamma : [0, T] \rightarrow M$  satisfying  $\gamma(0) = x$  and  $\gamma(T) = y$ . The causal future  $\mathcal{J}_{\mathcal{C}}^+(x)$  of  $x$  is the set  $\mathcal{J}_{\mathcal{C}}^+(x) = \mathcal{J}_{\mathcal{C}}^{++}(x) \cup \{x\}$ . The causal past  $\mathcal{J}_{\mathcal{C}}^-(x)$  is the set of points  $x' \in M$  such that  $x \in \mathcal{J}_{\mathcal{C}}^+(x')$ . More generally, for each subset  $A \subset M$ , we denote by  $\mathcal{J}_{\mathcal{C}}^{\pm}(A) := \cup_{x \in A} \mathcal{J}_{\mathcal{C}}^{\pm}(x)$  the causal future and past of  $A$ . We have the inclusion  $\mathcal{J}_{\mathcal{C}}^+(y) \subset \mathcal{J}_{\mathcal{C}}^+(x)$  if  $y \in \mathcal{J}_{\mathcal{C}}^+(x)$ .

We say that the closed cone field  $\mathcal{C}$  is acausal if  $x \notin \mathcal{J}^{++}(x)$  for each  $x \in M$ . We say that  $\mathcal{C}$  is globally hyperbolic if it is acausal and if moreover  $\mathcal{J}^+(K) \cap \mathcal{J}^-(K')$  is compact for each compact sets  $K, K' \subset M$ .

In the case where  $\mathcal{C}$  contains an open cone field (for example in the case of an oriented Lorentzian metric), the above definition is equivalent to the usual condition that  $\mathcal{C}$  is acausal and that the diamonds  $\mathcal{J}^+(x) \cap \mathcal{J}^-(x')$  are all compact, as was proved in [4].

A function  $\tau : M \rightarrow \mathbb{R}$  is said :

- causal if  $\tau \circ \gamma$  is non decreasing for each causal curve  $\gamma$ ;
- a time function if  $\tau \circ \gamma$  is increasing for each causal curve  $\gamma$ ;
- temporal if it is smooth and if  $d\tau_x \cdot v > 0$  for each  $(x, v) \in \mathcal{C}$ .

A smooth function is causal if and only  $d\tau_x \cdot v \geq 0$  for each  $(x, v) \in \mathcal{C}$ . A temporal function is a time function, and a time function is causal.

A causal curve  $\gamma : ]a, b[ \rightarrow M$  is said *inextensible* if neither  $\lim_{t \uparrow b} \gamma(t)$  nor  $\lim_{t \downarrow a} \gamma(t)$  exist. Each causal curve can be extended to an inextensible causal curve, see [4, Corollary 2.15].

The causal function  $\tau$  is said Cauchy if  $\tau \circ \gamma : ]a, b[ \rightarrow \mathbb{R}$  is onto for each inextensible causal curve  $\gamma : ]a, b[ \rightarrow M$ .

It is known that a spacetime is globally hyperbolic if and only if it admits a Cauchy temporal function, see [7, 8, 10]. This was extended in [4] to the present setting, see also [9].

Actually we introduced in [4] the more precise notion of uniform temporal functions (there called steep temporal functions):

**Definition 1.1.** *Given a metric  $g$ , the temporal function  $\tau$  is said  $g$ -uniform (or uniform with respect to  $g$ ) if  $d\tau_x \cdot v \geq |v|_x^g$  for each  $(x, v) \in \mathcal{C}$ .*

*The function  $\tau$  is said completely uniform if it is uniform with respect to some complete metric.*

It is easy to check that a completely uniform temporal function is Cauchy.

We proved in [4] that a globally hyperbolic cone fields admits uniform time functions with respect to any metric, hence completely uniform temporal functions. Conversely, the existence of a completely uniform time function implies global hyperbolicity. The existence of a completely uniform temporal function immediately implies that the globally hyperbolic cone field  $\mathcal{C}$  admits a globally hyperbolic enlargement (a globally hyperbolic closed cone field  $\mathcal{C}'$  which contains  $\mathcal{C}$  in its interior). Indeed, if  $g$  is a complete metric and  $\tau$  is  $g$ -uniform, then  $\tau$  is  $g/4$ -uniform for the enlargement  $\mathcal{C}' := \{(x, v) : d\tau_x \cdot v \geq |v|_x^g/2, v \neq 0\}$ . This extends [2] and [7, Theorem 1.2].

Not all Cauchy temporal functions are  $g$ -uniform. Moreover, as will be exemplified in Section 3, not all Cauchy temporal functions are completely uniform.

Our goal in the present paper is to clarify the differences between the notions of completely uniform temporal functions and of Cauchy temporal functions. In order to do so, we will prove some existence results of completely uniform temporal functions which mimic corresponding results for Cauchy temporal functions. We will deduce that each Cauchy temporal function can be uniformly approximated by completely uniform temporal functions.

The subset  $H \subset M$  is said to be :

- acausal if each causal curve intersects  $H$  at most once;
- Cauchy if each inextendible causal curve intersects  $H$  exactly once;
- a spacelike hypersurface if it is a hypersurface and if  $T_x H$  is disjoint from  $\mathcal{C}(x)$  for each  $x \in M$ ;
- a smooth Cauchy hypersurface if it is smooth, Cauchy, and spacelike.

Since each causal curve can be extended to an inextendible causal curve, a Cauchy hypersurface is acausal.

We state our main results in the convenient setting of closed cone fields, but they are also new in the more classical setting of genuine Lorentz metrics.

Each level set of a Cauchy temporal function is a smooth Cauchy hypersurface. Our first results is a converse which extends [3, Theorem 1.2] to the more general case of closed cone fields and to uniform temporal functions (instead of Cauchy temporal functions) :

**Theorem 1.** *Let  $H$  be a smooth spacelike Cauchy hypersurface, and  $g$  a Riemannian metric. Then there exists a  $g$ -uniform temporal function which is null on  $H$ .*

A proof is given in Section 2, it is different from the one in [3]. It is not much harder to find a Cauchy temporal function with two given level sets, but the following statement is more delicate:

**Theorem 2.** *Let  $H_0, H_1$  be disjoint smooth spacelike Cauchy hypersurfaces with  $H_1 \subset \mathcal{J}^+(H_0)$ . Then there exists a completely uniform temporal function  $\tau : M \rightarrow \mathbb{R}$  with  $\{\tau = i\} = H_i$  for  $i = 0, 1$ .*

The proof will be given in section 4. We deduce in Section 5:

**Theorem 3.** *Let  $\tau$  be a Cauchy causal function, and let  $\epsilon > 0$  be given. Then there exists a completely uniform temporal function  $v$  such that  $\sup |v - \tau| \leq \epsilon$ .*

We will however see in Section 3 an example of a Cauchy temporal function which is not completely uniform. These results for example imply the following version of [1, Theorem 4.2]:

**Corollary 1.2.** *If  $\tau$  is a Cauchy temporal function for  $\mathcal{C}$ , then there exists an enlargement  $\mathcal{C}'$  such that  $\tau$  is a Cauchy temporal function for  $\mathcal{C}'$ .*

PROOF. We consider a completely uniform temporal function  $v$  such that  $\tau - v$  is bounded. As explained above, there exists an enlargement  $\mathcal{C}_1$  of  $\mathcal{C}$  such that  $v$  is a uniform temporal function for  $\mathcal{C}_1$ . There exists also an enlargement  $\mathcal{C}_2$  such that  $\tau$  is temporal for  $\mathcal{C}_2$ . The intersection  $\mathcal{C}' := \mathcal{C}_1 \cap \mathcal{C}_2$  is an enlargement of  $\mathcal{C}$ . For  $\mathcal{C}'$ , the function  $\tau$  is temporal and the function  $v$  is completely uniform, hence Cauchy. Since  $v - \tau$  is bounded, the function  $\tau$  is also Cauchy.  $\square$

## 2 One Cauchy hypersurface

We prove Theorem 1 and then give a proposition on the extension of acausal compact sets to smooth Cauchy hypersurfaces.

PROOF OF THEOREM 1. We first prove the existence of a smooth causal function  $u$  which is null on  $H$  and  $g$ -uniform in a neighborhood  $U$  of  $H$ , see Proposition 2.1 below.

Then, we prove the existence of a smooth causal function  $v$  which is null on  $H$  and uniform outside of  $U$ , see Proposition 2.3 below. We deduce that  $\tau := u + v$  is a uniform temporal function null on  $H$ .  $\square$

The first step is a variant of [4, Proposition 5.1].

**Proposition 2.1.** *Let  $H \subset M$  be a smooth spacelike Cauchy hypersurface,  $g$  be a Riemannian metric, and  $V$  be a neighborhood of  $H$ . Given  $\alpha \in ]0, 1[$ , there exists a causal function  $u : M \rightarrow [-1, 1]$  with the following properties:*

- $H = u^{-1}(0)$ .
- The function  $u$  is  $g$ -uniform on  $u^{-1}(] - \alpha, \alpha[)$  and temporal on  $u^{-1}(] - 1, 1[)$
- The function  $u$  is equal to  $+1$  on  $\mathcal{J}^+(H) - V$  and to  $-1$  on  $\mathcal{J}^-(H) - V$ .

PROOF. We can work in a tubular neighborhood of  $H$  and therefore assume that  $M = H \times \mathbb{R}$  (the function built locally can be extended globally by the constant values  $\pm 1$ ). There exists a smooth positive function  $y \mapsto \delta(y)$  on  $H$  such that

$$\mathcal{C}(y, 0) \subset \{(v_y, v_z) : v_z \geq 3\delta(y)\|v_y\|, v_z \geq 3\delta(y)\|(v_y, v_z)\|\}$$

for each  $y \in H$ . Here we have used the notation  $\|v_y\| := \|(v_y, 0)\|_{(y,z)}$ , where the norms are measured with respect to the metric  $g$  on  $M$ . Then, there exists a smooth positive function  $\epsilon : H \rightarrow ]0, 1[$  such that

$$\mathcal{C}(y, z) \subset \{(v_y, v_z) : v_z \geq 2\delta(y)\|v_y\|, v_z \geq 2\delta(y)\|(v_y, v_z)\|\}$$

provided  $|z| \leq \epsilon(y)$ . We also assume that  $\epsilon \leq \delta$ .

Let  $f : H \rightarrow \mathbb{R}$  be a smooth positive function such that  $\|df_y\| + f(y) \leq \epsilon(y)$  for all  $y \in H$ , (see for example [4, Lemma 5.1] for the existence of such a function). We set

$$u(y, z) = \phi(z/f(y)),$$

where  $\phi : \mathbb{R} \rightarrow [-1, 1]$  is a smooth nondecreasing function which has positive derivative on  $] - 1, 1[$  and is equal to  $1$  on  $[1, \infty)$  and to  $-1$  on  $(-\infty, -1]$ . We moreover assume that  $\phi(t) = t$  on  $[-\alpha, \alpha]$ .

On the open set  $\{(y, z) : |z| < f(y)\} = u^{-1}(] - 1, 1[)$ , we have

$$du_{(y,z)} \cdot (v_y, v_z) = \frac{\phi'(z/f(y))}{f(y)} (v_z - \frac{z}{f(y)} df_y \cdot v_y) \geq \frac{\phi'(z/f(y))}{2f(y)} v_z$$

for  $(v_y, v_z) \in \mathcal{C}(y, z)$  since  $|(z/f(y))df_y \cdot v_y| \leq \delta(y)\|v_y\| \leq v_z/2$ . This implies that  $u$  is temporal on this open set. Outside of this open set,  $u$  is locally constant, hence it is a causal function.

Finally, on  $u^{-1}(] - \alpha, \alpha[)$ , we have  $\phi'(z/f(y)) = 1$  hence

$$du_{(y,z)} \cdot (v_y, v_z) \geq (1/2f(y))v_z \geq (\delta(y)/f(y))\|(v_y, v_z)\|_{(y,z)} \geq \|(v_y, v_z)\|_{(y,z)}$$

for each  $(v_y, v_z) \in \mathcal{C}(y, z)$ . In other words, the causal function  $f$  is uniform on  $u^{-1}(] - \alpha, \alpha[)$ .  $\square$

We need a Lemma before the second step.

**Lemma 2.2.** *Let  $\mathcal{C}$  be a globally hyperbolic cone field, and  $F \subset M$  be a closed past subset, i.e.  $F = \mathcal{J}^-(F)$ . The complement  $U$  of  $F$  is then globally hyperbolic. If  $\tau : U \rightarrow \mathbb{R}$  is a Cauchy temporal function, then there exists a smooth causal function  $v : M \rightarrow [0, \infty)$  which is null on  $F$ , temporal on  $U$ , and equal to  $\tau$  on  $\{\tau \geq 1\}$ . The function  $v$  is of the form  $\eta \circ \tau$  on  $U$ .*

PROOF. The global hyperbolicity of  $(U, \mathcal{C}|_U)$  follows easily from the definition. Let  $\eta_k : \mathbb{R} \rightarrow [0, \infty)$  be a smooth function such that  $\eta_k(t) = 0$  for  $t \leq -k$ ,  $\eta'_k(t) > 0$  for  $t > -k$ , and  $\eta_k(t) = t$  for  $t \geq 1$ . Each of the functions  $\eta_k \circ \tau$  extends by 0 to a smooth function on  $M$ . If  $\{a_k\}_{k \in \mathbb{N}}$  is an appropriate sequence of positive numbers (see [6, Lemma 3.2] for the existence of such a sequence) then  $v := \sum a_k \eta_k \circ \tau$  is a smooth function on  $M$ , which is null on  $F$ , and which is positive and temporal on its complement.  $\square$

**Proposition 2.3.** *Let  $H \subset M$  be a Cauchy hypersurface,  $U$  an open neighborhood of  $H$ , and  $g$  a Riemannian metric. Then there exists a smooth function  $v : M \rightarrow \mathbb{R}$  satisfying  $\{v = 0\} = H$ , which is temporal outside of  $H$  and  $g$ -uniform outside of  $U$ .*

PROOF. For each compact set  $K \subset \mathcal{J}^+(H)$ , the set  $\mathcal{J}^-(K) \cap \mathcal{J}^+(H)$  is compact. We can therefore build a sequence of compact sets  $K_n \subset \mathcal{J}^+(H)$  such that  $\mathcal{J}^+(H) = \cup_n K_n$ ,  $\mathcal{J}^-(K_n) \cap \mathcal{J}^+(H) = K_n$ , and  $K_{n-1}$  is contained in the interior  $\overset{\circ}{K}_n$  of  $K_n$  in  $\mathcal{J}^+(H)$  for each  $n \geq 1$ . We take  $K_0 = \emptyset$ .

For each  $n \geq 0$ , we set  $F_n = K_n \cup \mathcal{J}^-(H)$ . By Lemma 2.2, there exists a smooth causal function  $\tau_n$  which is null on  $F_n$  and which is positive and temporal outside of this set.

Since the set  $K_2 - U$  is compact and disjoint from  $\mathcal{J}^-(H)$ , there exists a positive constant  $a_0$  such that  $a_0\tau_0$  is uniform on  $K_2 - U$ . Then for each  $n \geq 1$ , since the set  $K_{n+2} - (U \cup \overset{\circ}{K}_{n+1})$  is compact and disjoint from  $\mathcal{J}^-(H) \cup K_n$ , there exists a positive number  $a_n$  such that  $a_n\tau_n$  is uniform on  $K_{n+2} - (\overset{\circ}{K}_{n+1} \cup U)$ . The locally finite sum  $\tau^+ := \sum_{n \geq 0} a_n\tau_n$  is a smooth function which is null on  $\mathcal{J}^-(H)$ , temporal outside of this set, and uniform on  $\mathcal{J}^+(H) - U$ .

We prove similarly the existence of a time function  $\tau^-$  which is null on  $\mathcal{J}^+(H)$ , negative and temporal outside of this set, and uniform on  $\mathcal{J}^-(H) - U$ . The function  $v := \tau^+ + \tau^-$  is as claimed.  $\square$

Theorem 1 is proved. We immediately deduce:

**Corollary 2.4.** *If  $H$  is a smooth Cauchy hypersurface for the cone field  $\mathcal{C}$ , then there exists a globally hyperbolic enlargement  $\mathcal{C}'$  of  $\mathcal{C}$  such that  $H$  is a smooth Cauchy hypersurface for  $\mathcal{C}'$ .*

Let us also mention, for later use:

**Lemma 2.5.** *Let  $\tau$  be a Cauchy temporal function, and let  $a \in \mathbb{R}$ . Let  $g$  be a Riemannian metric. Then there exists a Cauchy temporal function  $\tau_1$  such that  $\tau_1 = \tau$  on  $\{\tau \geq a+1/3\} \cup \{\tau = a\} \cup \{\tau \leq a-1/3\}$ , and which is  $g$ -uniform on  $\tau_1^{-1}(]a-1/10, a+1/10[)$ .*

PROOF. We first apply Lemma 2.1 in the hyperbolic strip  $\tau^{-1}(]a - 1/3, a + 1/3[)$  to the hypersurface  $\{\tau = a\}$  and the metric  $16g$ . We get causal function  $u : M \rightarrow [-1, 1]$  which is  $16g$ -uniform on  $u^{-1}(]a - 3/4, a + 3/4[)$ , which is equal to 0 on  $\{\tau = a\}$ , and which is equal to 1 on  $\{\tau \geq a + 1/3\}$  and to  $-1$  on  $\{\tau \leq a - 1/3\}$ .

Second, we consider a smooth increasing diffeomorphism  $\eta : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\eta(a) = a$ ,  $\eta(t) = t - 1/4$  on  $[a + 1/3, \infty)$  and  $\eta(t) = t + 1/4$  on  $(-\infty, a - 1/3]$ .

Finally we set  $\tau_1 = u/4 + \eta \circ \tau$ . This function is equal to  $a$  on  $\{\tau = a\}$ , it is  $g$ -uniform on  $\tau_1^{-1}(]a - 1/10, a + 1/10[) \subset u^{-1}(]a - 3/4, a + 3/4[)$  because  $u/4$  is.  $\square$

It is not much more difficult to find a Cauchy temporal function with prescribed values on two smooth Cauchy hypersurfaces than on one Cauchy hypersurface. More precisely:

**Proposition 2.6.** *Let  $H_0, H_1$  be disjoint smooth spacelike Cauchy hypersurfaces with  $H_1 \subset \mathcal{J}^+(H_0)$ , and let  $g$  be a complete Riemannian metric on  $M$ . Then there exists a Cauchy temporal function  $\tau$  such that  $\{\tau = i\} = H_i$  for  $i = 0, 1$ , and which is  $g$ -uniform in  $\{\tau \geq 9/10\}$  and in  $\{\tau \leq 1/10\}$ .*

In view of this result, the remaining difficulty in the proof of Theorem 2 is to interpolate between  $H_0$  and  $H_1$  in a uniform way. This will be done in Section 4.

PROOF. The open future  $\mathcal{J}^{++}(H_0)$  is globally hyperbolic and contains  $H_1$ . As a consequence, there exists a  $4g$ -uniform temporal function  $\tilde{\tau}_1 : \mathcal{J}^{++}(H_0) \rightarrow \mathbb{R}$  which is equal to 1 on  $H_1$ . In view of Lemma 2.2, there exists a smooth function  $\eta : \mathbb{R} \rightarrow ]0, \infty)$  with positive derivative such that  $\eta(t) = t$  for  $t \geq 1/2$  and such that  $\eta \circ \tilde{\tau}_1$  extends (with the value 0) to a smooth function on  $M$ , that will be denoted by  $\tau_1$ .

By a similar method in the globally hyperbolic manifold  $\{\tau_1 < 3/4\}$ , there exists a smooth function  $\tau_2 : M \rightarrow (-\infty, 1]$  which is equal to 1 on  $\{\tau_1 \geq 3/4\}$ , temporal in  $\{\tau_1 < 3/4\}$ , null on  $H_0$ , and  $4g$ -uniform in  $\{\tau_2 \leq 1/2\}$ .

The function  $\tau := (\tau_1 + \tau_2)/2$  is temporal, it is  $g$ -uniform in  $\{\tau \geq 9/10\} \subset \{\tau_1 \geq 4/5\}$  because  $\tau_1/2$  is  $g$ -uniform on this set, and it is  $g$ -uniform on  $\{\tau \leq 1/10\} \subset \{\tau_2 \leq 1/5\}$  because  $\tau_2/2$  is  $g$ -uniform in this set.  $\square$

We finish this section with an extension result which is a generalization of [3, Theorem 1.1].

**Proposition 2.7.** *Let  $\Sigma$  be a smooth spacelike hypersurface for the globally hyperbolic cone field  $\mathcal{C}$  and  $K$  be a compact and acausal subset of  $\Sigma$ . Then the set  $K$  is contained in a smooth Cauchy hypersurface  $G$ .*

*Moreover, if  $N^-, N^+$  are two smooth Cauchy hypersurfaces such that  $\Sigma \subset \mathcal{J}^+(N^-) \cap \mathcal{J}^-(N^+)$ , then the smooth Cauchy hypersurface  $H$  containing  $K$  can be chosen such that  $H \subset \mathcal{J}^+(N^-) \cap \mathcal{J}^-(N^+)$  and such that  $H$  is equal to  $N^-$  at infinity.*

PROOF. We prove the second part of the Proposition, the first part easily follows from it.

By Corollary 2.4, there exist globally hyperbolic enlargements  $\mathcal{C}^\pm$  of  $\mathcal{C}$  such that  $N^\pm$  is a Cauchy hypersurface for  $\mathcal{C}^\pm$ .

We associate to  $K$  the transverse open cone field  $\mathcal{E}_K$ , which is defined by:

$$\mathcal{E}_K(x) = T_x M \text{ if } x \notin K \text{ and}$$

$\mathcal{E}_K(x)$  is the component of the set of vectors transverse to  $T_x \Sigma$  which contains  $\mathcal{C}(x)$  if  $x \in K$ .

By definition, this open cone field contains  $\mathcal{C}$ , hence there exists an enlargement  $\mathcal{C}'$  of  $\mathcal{C}$  which is contained in  $\mathcal{E}_K$  (by [4, Lemma 2.8]). The intersection  $\mathcal{C}_1 := \mathcal{C}^+ \cap \mathcal{C}^- \cap \mathcal{C}'$  is a globally hyperbolic closed cone field,  $N^\pm$  are smooth Cauchy hypersurfaces of  $\mathcal{C}_1$ , and  $\Sigma$  is spacelike for  $\mathcal{C}_1$  at each point of  $K$ .



Note that the set of points of  $\Sigma$  at which  $\Sigma$  is spacelike is relatively open in  $\Sigma$  (hence a hypersurface of  $M$ ) and contains  $K$ . We can thus assume (by possibly reducing it), that  $\Sigma$  is spacelike for  $\mathcal{C}_1$ .

We can finally assume, by possibly reducing  $\mathcal{C}_1$ , that  $K$  is acausal for  $\mathcal{C}_1$ , and even that there exists a compact subset  $Z \subset \Sigma$ , which contains  $K$  in its relative interior, and which is acausal for  $\mathcal{C}_1$ .

To prove this last claim, we consider a decreasing sequence  $Z_n$  of compact subsets of  $\Sigma$  each of which is contained in the relative interior of the previous one, and such that  $K = \bigcap Z_n$ . We also consider a decreasing sequence  $\mathcal{C}_n$  of enlargements of  $\mathcal{C}$ , each of which is an enlargement of the next one, and such that  $\mathcal{C} = \bigcap \mathcal{C}_n$  (the existence of such a sequence of enlargements is proved in [4, Lemma 2.9]). We assume by contradiction that for each  $n$  there exists a  $\mathcal{C}_n$ -causal curve  $\gamma_n : I_n \rightarrow M$ ,  $I_n \subset [0, 1]$  starting and ending in  $Z_n$  and parametrized by  $\tau$ , *i.e.* such that  $\tau \circ \gamma_n(t) = t$ . Then at the limit (along a subsequence), we obtain a  $\mathcal{C}$ -causal curve  $\gamma : I \rightarrow M$  starting and ending in  $K$ . The acausality of  $K$  implies that this curve is reduced to a point  $\kappa \in K$ . A local study at  $\kappa$  leads to a contradiction.

We consider a Cauchy temporal function  $\tau$  for  $\mathcal{C}_1$  such that  $N^- = \tau^{-1}(0)$  and  $N^+ = \tau^{-1}(1)$ , according to Proposition 2.6. The cone field  $\mathcal{C}_1$  contains an open cone, hence a smooth vector field  $X$ . Each inextensible orbit of  $X$  intersects  $N^-$  exactly once. This defines a smooth map  $\pi : M \rightarrow N^-$  which, to each point  $x$ , associates the intersection of its orbit with  $N^-$ . The map

$$M \ni x \mapsto (\pi(x), \tau(x)) \in N^- \times \mathbb{R}$$

is a smooth diffeomorphism. In other words, we can assume that  $M = N^- \times \mathbb{R}$ ,  $N^- = N^- \times \{0\}$ ,  $N^+ = N^- \times \{1\}$ , that  $\mathcal{C}_1(x)$  contains the vector  $(0, 1)$  at each point, that the vertical coordinate  $\tau$  is a Cauchy temporal function for  $\mathcal{C}_1$ , and that  $\tau(\Sigma) \in [0, 1]$ .

The set  $F := \mathcal{J}_{\mathcal{C}_1}^-(Z) \cup \{\tau \leq 0\}$  is a past set for  $\mathcal{C}_1$ , *i.e.*  $F = \mathcal{J}_{\mathcal{C}_1}^-(F)$ , and it is equal to  $\{\tau \leq 0\}$  at infinity (meaning, outside of a compact set of  $M$ ). Its complement is therefore a trapping domain for  $\mathcal{C}$  in the terminology of [4]. Since  $Z$  is acausal, it is contained in the boundary of  $F$ . In the identification  $M = N^- \times \mathbb{R}$ , the boundary of  $F$  is the graph of a compactly supported Lipschitz function  $\tilde{f} : N^- \rightarrow [0, 1]$ , which is smooth in a neighborhood of  $\pi(K)$ . We can then regularize  $\tilde{f}$  to a compactly supported smooth function  $f : N^- \rightarrow [0, 1]$  which is equal to  $\tilde{f}$  near  $\pi(K)$ , and the graph  $H$  of which is spacelike for  $\mathcal{C}_1$ . We can use for example [4, Proposition 4.5] or a more standard smoothing procedure.

The graph  $H$  is the desired smooth Cauchy hypersurface. □

### 3 A Cauchy temporal function which is not completely uniform

Let us consider the plane  $\mathbb{R}^2$  endowed with the constant cone field

$$\mathcal{C}(x) = \{(v_x, v_y) : v_y \geq |v_x|\},$$

*i.e.* the Minkowsky plane. In this plane, we consider the strip  $U = \mathbb{R} \times ]-1, 1[$ . The restriction of  $\mathcal{C}$  to  $U$  is globally hyperbolic.

let  $\varphi : ]-1, 1[ \rightarrow \mathbb{R}$  be a smooth increasing function onto  $\mathbb{R}$ . Then the function  $\varphi \circ y$  is a Cauchy time function on  $U$ .

An easy computation shows that the function  $\tau(x, y) := y/(1+x^2)$  is a temporal function on  $U$ . Indeed

$$|\partial_x \tau| < \frac{2|x|}{(1+x^2)^2} \leq \frac{1}{1+x^2} = \partial_y \tau.$$

The function  $h(x, y) = \tau(x, y) + \varphi(y)$  is thus a Cauchy temporal function.



**Proposition 3.1.** *If  $\varphi'(0) = 0$ , then  $h$  is not completely uniform.*

PROOF. We prove that, if  $g$  is a metric such that  $h$  is  $g$ -uniform, then  $g$  is not complete.

We consider the constant vector-fields  $V^+ = (1, 1)$  and  $V^- = (-1, 1)$ , they belong to the cone field. For the metric  $g$ , we have

$$|V^+|_{(x,0)} + |V^-|_{(x,0)} \leq dh_{(x,0)} \cdot (V^+ + V^-) = 2\partial_y h(x, 0) = \frac{2}{1+x^2}.$$

The  $g$ -length  $\ell$  of the axis  $y = 0$  thus satisfies

$$\ell = \int_{\mathbb{R}} \frac{1}{2} |V^+ - V^-|_{(x,0)} dx \leq \int_{\mathbb{R}} \frac{|V^+|_{(x,0)} + |V^-|_{(x,0)}}{2} dx \leq \int \frac{1}{1+x^2} dx < \infty.$$

This implies that the metric  $g$  on  $U$  is not complete.  $\square$

## 4 Two Cauchy Hypersurfaces

We prove Theorem 2. It is convenient in this section to fix once and for all a complete Riemannian metric  $g$  on  $M$  (but we shall be led to consider also other metrics). In view of Proposition 2.6, Theorem 2 follows from:

**Proposition 4.1.** *Let  $\tau$  be a smooth Cauchy temporal function, and  $g$  be a complete Riemannian metric on  $M$ . We assume that there exists  $a < b$  in  $]0, 1[$  such that  $\tau$  is  $g$ -uniform on  $\tau^{-1}(]0, a[ \cap ]b, 1[)$ . Then there exists a smooth Cauchy temporal function  $\theta$  and a complete Riemannian metric  $h$  on  $M$  such that:*

*The function  $\theta$  is  $h$ -uniform on  $\tau^{-1}(]0, 1[)$  and equal to  $\tau$  outside this strip.*

*The metric  $h$  is equal to  $g$  outside of the strip  $\tau^{-1}(]0, 1[)$ , and it satisfies*

$$d_h(\tau^{-1}(0), \tau^{-1}(1)) \geq 1/3.$$

PROOF. We introduce intermediate values  $a_1 \in ]0, a[$  and  $b_1 \in ]b, 1[$ , and we assume without loss of generality that  $a < 1/10$ ,  $b > 9/10$ .

We consider a smooth Cauchy hypersurface  $G$  which oscillates between the hypersurfaces  $\{\tau = a_1\}$  and  $\{\tau = b_1\}$ :

**Lemma 4.2.** *There exists a smooth Cauchy hypersurface  $G \subset \tau^{-1}([a_1, b_1])$  such that :*

*The intersection  $\mathcal{J}^{++}(G) \cap \{\tau < b_1\}$  is the disjoint union of a countable family of bounded sets  $F_n, n \geq 0$  such that  $d_g(F_n, F_m) \geq 1$  if  $n \neq m$ .*

*The intersection  $\mathcal{J}^{--}(G) \cap \{\tau > a_1\}$  is the disjoint union of a countable family of bounded sets  $P_n, n \geq 0$  such that  $d_g(P_n, P_m) \geq 1$  if  $n \neq m$ .*

Assuming for the moment this Lemma, we finish the proof of Proposition 4.1.

We consider a smooth modification  $\eta \circ \tau$  of  $\tau$ , where  $\eta : \mathbb{R} \rightarrow \mathbb{R}$  is an increasing diffeomorphism satisfying  $\eta(t) = t - 1/3$  for  $t > 9/10$  and  $\eta(t) = t + 1/3$  for  $t < 1/10$ .

We apply Proposition 2.1 to the hypersurface  $G$  in the globally hyperbolic open set  $\tau^{-1}(]0, 1[)$  and get a smooth causal function  $u$  on  $M$  which is equal to  $1/6$  in  $\{\tau \geq 1\}$ , to  $-1/6$  in  $\{\tau \leq 0\}$ , and which is  $g$ -uniform on  $u^{-1}(]-1/7, 1/7[)$ . We set

$$V := \tau^{-1}((-\infty, a_1]) \cup u^{-1}(]-1/10, 1/10[) \cup \tau^{-1}(]b_1, \infty)),$$

the temporal function  $\eta \circ \tau + u$  is  $g$ -uniform in a neighborhood of  $\bar{V}$ , hence there exists a smooth function  $\lambda : M \rightarrow ]0, 1]$ , which is equal to one on  $V$ , and such that  $\eta \circ \tau + u$  is  $\lambda g$ -uniform on  $V$ . The problem is that  $\lambda g$  is not necessarily complete. To solve it, we define

$$\theta := \eta \circ \tau + 2u \quad , \quad h := \lambda g + du \otimes du.$$

The function  $\theta$  is the sum of the function  $\eta \circ \tau + u$ , which is  $\lambda g$ -uniform, and  $u$ , which is  $du \otimes du$ -uniform. It is therefore  $h$ -uniform. It is moreover equal to  $\tau$  outside of the strip  $\tau^{-1}([0, 1])$ .

We now prove that the metric  $h$  is complete, this implies that  $\theta$  is completely uniform. We denote by  $A$  the complement of  $V$ , and set  $A^+ := A \cap \mathcal{J}^+(G)$  and  $A^- := A \cap \mathcal{J}^-(G)$ . The closed set  $A^+$  is the disjoint union of the compact sets  $A_n^+ := A^+ \cap F_n$ , and the closed set  $A^-$  is the disjoint union of the compact sets  $A_n^- := A^- \cap P_n$ . We have

$$d_h(A^+, A^-) \geq 1/5.$$

Indeed, the  $h$ -length  $L$  of a curve  $\gamma$  satisfies  $L \geq \int_0^1 |du_{\gamma(t)} \cdot \dot{\gamma}(t)| dt \geq |u(\gamma(1)) - u(\gamma(0))|$  and  $u|_{A^+} \geq 1/10$ ,  $u|_{A^-} \leq -1/10$ . This remark also proves that  $d_h(\{\tau = 0\}, \{\tau = 1\}) \geq 1/3$ .

Since  $\lambda = 1$  on  $V$ , Lemma 4.2 implies, for each  $n$ , that

$$d_h(A_n^+, \cup_{m \neq n} A_m^+) \geq d_g(A_n^+, \cup_{m \neq n} A_m^+) \geq 1 \quad , \quad d_h(A_n^-, \cup_{m \neq n} A_m^-) \geq d_g(A_n^-, \cup_{m \neq n} A_m^-) \geq 1.$$

In view of these inequalities, a curve  $\gamma$  of finite  $h$ -length  $L$  visits only finitely many of the sets  $A_i^\pm$ . If  $K$  is the union of these finitely many visited sets, then  $K$  is compact, and  $\gamma$  is contained in the compact set  $\{x : d_g(K, x) \leq L\}$  (note that  $g$  is a complete metric).  $\square$

PROOF OF LEMMA 4.2: The proof consists of repeated applications of Proposition 2.7. We denote by  $N^+$  the hypersurface  $\{\tau = b_1\}$ , and by  $N^-$  the hypersurface  $\{\tau = a_1\}$ . We denote by  $S$  the open strip  $\tau^{-1}([a_1, b_1])$  and by  $\bar{S}$  its closure  $\tau^{-1}([a_1, b_1])$ . We denote by  $B(F, r)$  the set  $\{y : d(y, F) \leq r\}$ .

By Proposition 2.7, applied to  $-\mathcal{C}$ , there exists a smooth Cauchy hypersurface  $G_0^+ \subset \bar{S}$  which contains a point  $x_0 \in N^-$  and is equal to  $N^+$  at infinity. We set  $F_0 := \mathcal{J}^{++}(G_0^+) \cap S$ .

Then, by Proposition 2.7, there exists a smooth Cauchy hypersurface  $G_0^- \subset \bar{S}$  which contains  $G_0^+ \cap \mathcal{J}^+(B(F_0, 1))$  and is equal to  $N^-$  at infinity. Note then that

$$\mathcal{J}^{++}(G_0^-) \cap S \cap B(F_0, 1) = F_0.$$

Indeed, we have  $\mathcal{J}^-(G_0^+) \cap B(F_0, 1) \subset \mathcal{J}^-(G_0^-)$  hence  $\mathcal{J}^-(G_0^+) \subset \mathcal{J}^-(G_0^-) \cup B^c(F_0, 1)$ . Taking the complements in  $S$ , we get the inclusion  $\mathcal{J}^{++}(G_0^-) \cap S \cap B(F_0, 1) \subset F_0$ . The opposite inclusion is obvious. We set  $P_0 := \mathcal{J}^{--}(G_0^-) \cap S$ .

We construct inductively sequences  $G_n^\pm$  of smooth Cauchy hypersurfaces, and increasing sequences  $\mathcal{F}_n, \mathcal{P}_n$  of bounded subsets of  $S$  such that :

1.  $G_n^\pm$  is equal to  $N^\pm$  at infinity for each  $n$ ;
2.  $\mathcal{F}_n = S \cap \mathcal{J}^{++}(G_n^+)$ ,  $\mathcal{P}_n = S \cap \mathcal{J}^{--}(G_n^-)$ ;
3.  $G_{n+1}^+$  contains  $G_n^- \cap \mathcal{J}^-(B(\mathcal{F}_n \cup \mathcal{P}_n, 1) \cup B(x_0, 2n))$ ;
4.  $G_{n+1}^-$  contains  $G_{n+1}^+ \cap \mathcal{J}^+(B(\mathcal{F}_{n+1} \cup \mathcal{P}_n, 1) \cup B(x_0, 2n+2))$ .

We denote  $P_{n+1} := \mathcal{P}_{n+1} - \mathcal{P}_n$  and  $F_{n+1} := \mathcal{F}_{n+1} - \mathcal{F}_n$ .

Note at each step that  $G_{n+1}^+ \subset \mathcal{J}^+(G_n^-) \cap \mathcal{J}^-(G_n^+)$  and  $G_{n+1}^- \subset \mathcal{J}^-(G_{n+1}^+) \cap \mathcal{J}^+(G_n^-)$ .

Denoting  $B_n := B(\mathcal{F}_n \cup \mathcal{P}_n, 1) \cup B(x_0, 2n)$  and  $B_n^c$  its complement, point 3 implies that  $\mathcal{J}^+(G_n^-) \cap B_n \subset \mathcal{J}^+(G_{n+1}^+)$ . This implies that  $\mathcal{J}^+(G_n^-) \subset \mathcal{J}^+(G_{n+1}^+) \cup B_n^c$ . Taking the complements and intersecting with  $S$  yields  $\mathcal{J}^{--}(G_{n+1}^+) \cap B_n \cap S \subset \mathcal{J}^{--}(G_n^-) \cap S = \mathcal{P}_n$  hence

$$\mathcal{P}_{n+1} \cap B_n = \mathcal{P}_n,$$

and in particular  $d(\mathcal{P}_{n+1}, \mathcal{P}_n) \geq 1$ , hence  $d(\mathcal{P}_n, \mathcal{P}_m) \geq 1$  for  $n \neq m$ . Similarly,

$$\mathcal{F}_{n+1} \cap (B(\mathcal{F}_n \cup \mathcal{P}_{n-1}, 1) \cup B(x_0, 2n+2)) = \mathcal{F}_n.$$

Finally, we observe that  $G_{n+1}^+ = G_n^- = G_n^+$  on  $B(x_0, 2n)$ . As a consequence, the sequence  $G_n^+$  is stabilizing on any bounded set, and has a limit  $G$ , which is a smooth Cauchy hypersurface, such that  $\mathcal{J}^{++}(G) \cap S = \cup F_n$  and  $\mathcal{J}^{--}(G) \cap S = \cup P_n$ . To prove that  $G$  is Cauchy, observe that the intersection  $\gamma \cap S$  of an inextensible causal curve  $\gamma$  with the strip  $S$  is bounded, hence the sequence  $G_n^+ \cap \gamma$  stabilizes and is equal to a single point.  $\square$

## 5 Density of completely uniform temporal functions in Cauchy temporal functions

We prove Theorem 3. We fix a complete metric  $g$ .

Since Cauchy temporal functions are dense in Cauchy causal functions, see [5, Corollary 9], we can assume without loss of generality that  $\tau$  is a Cauchy temporal function.

Notice that  $au$  is a uniform temporal function if  $u$  is and if  $a > 0$ . In view of this remark, it is enough to prove the result with  $\epsilon = 1$ , which will simplify notations.

We will show the existence of a completely uniform function  $v$  such that  $v^{-1}([k, k+1]) = \tau^{-1}([k, k+1])$  for each  $k \in \mathbb{Z}$ , this implies that  $\sup |v - \tau| \leq 1$ .

By repeated applications of Lemma 2.5, we can replace the Cauchy temporal function  $\tau$  by a Cauchy temporal function  $\tilde{\tau}$  such that  $\tilde{\tau}^{-1}([k, k+1]) = \tau^{-1}([k, k+1])$  for each  $k \in \mathbb{Z}$  and which moreover is  $g$ -uniform on  $\tilde{\tau}^{-1}(B)$ , with

$$B := \{t \in \mathbb{R} : d(t, \mathbb{Z}) < 1/10\}.$$

By repeated use of Proposition 4.1, we inductively construct a sequence  $v_n$ ,  $n \geq 0$ , of smooth temporal functions and a sequence  $h_n$ ,  $n \geq 0$  of complete metrics with the following properties:

- $h_0 = g$ ,  $v_0 = \tilde{\tau}$
- For each  $n \geq 1$ ,  $h_n = h_{n-1}$  and  $u_n = u_{n-1}$  outside of  $\tau^{-1}([-n, 1-n] \cup [n-1, n])$ .
- $u_n$  is  $h_n$ -uniform on  $u_n^{-1}([-n, n] \cup B)$ .
- $u_n^{-1}([k, k+1]) = \tau^{-1}([k, k+1])$  for each  $n$  and  $k$ .
- $d_{h_n}(\tau^{-1}(k), \tau^{-1}(k+1)) \geq 1/3$  for each  $k$  in  $\{-n, 1-n, \dots, n-2, n-1\}$ .

Since the sequences  $u_n$  and  $h_n$  stabilize on any compact set they converge respectively to a smooth temporal function  $u$  and a Riemannian metric  $h$  such that  $u$  is  $h$ -uniform. We just have to verify that  $h$  is a complete metric. Let  $\gamma$  be a curve of finite  $h$ -length. Since  $d_h(\tau^{-1}(k), \tau^{-1}(k+1)) \geq 1/3$  for each  $k$ , the curve  $\gamma$  crosses only a finite number of integral level sets of  $\tau$ . As a consequence, it is contained in  $\tau^{-1}([-N, N])$  for some large  $N$ . Then, the  $h_N$ -length of  $\gamma$  is equal its  $h$ -length, hence it is finite. Since  $h_N$  is complete, we deduce that  $\gamma$  is relatively compact in  $M$ .  $\square$

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