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PLANE BIPOLAR ORIENTATIONS AND QUADRANT WALKS

MIREILLE BOUSQUET-MÉLOU, ÉRIC FUSY, AND KILIAN RASCHEL

Abstract. Bipolar orientations of planar maps have recently attracted some interest in combinatorics, probability theory and theoretical physics. Plane bipolar orientations with n edges are known to be counted by the nth Baxter number $b(n)$, which can be defined by a linear recurrence relation with polynomial coefficients. Equivalently, the associated generating function $\sum_n b(n)t^n$ is *D-finite*. In this paper, we address a much refined enumeration problem, where we record for every r the number of faces of degree r. When these degrees are bounded, we show that the associated generating function is given as the constant term of a multivariate rational series, and thus is still D-finite. We also provide detailed asymptotic estimates for the corresponding numbers.

The methods used earlier to count all plane bipolar orientations, regardless of their face degrees, do not generalize easily to record face degrees. Instead, we start from a recent bijection, due to Kenyon *et al.*, that sends bipolar orientations onto certain lattice walks confined to the first quadrant. Due to this bijection, the study of bipolar orientations meets the study of walks confined to a cone, which has been extremely active in the past 15 years. Some of our proofs rely on recent developments in this field, while others are purely bijective. Our asymptotic results also involve probabilistic arguments.

À Christian Krattenthaler à l'occasion de son 60e *anniversaire,*
avec admiration, reconnaissance et amitié

1. Introduction

A *planar map* is a connected planar multigraph embedded in the plane, and taken up to orientation preserving homeomorphism (Figure 1). The enumeration of planar maps is a venerable topic in combinatorics, which was born in the early sixties with the pioneering work of William Tutte [82, 83]. It is also studied in theoretical physics, where planar maps are seen as a discrete model of *quantum gravity* [22, 8]. The enumeration of maps also has connections with factorizations of permutations, and hence representations of the symmetric group [50, 51]. Finally, 40 years after the first enumerative results of Tutte, planar maps crossed the border between combinatorics and probability theory, where they are now studied as random metric spaces [2, 24, 57, 63]. The limit behaviour of large random planar maps is now well understood, and gave birth to a variety of limiting objects, either continuous [26, 58, 59, 69], or discrete [2, 23, 27, 68].

The enumeration of maps equipped with some additional structure (*e.g.*, a spanning tree, a proper colouring, a self-avoiding-walk, a configuration of the Ising model ...) has

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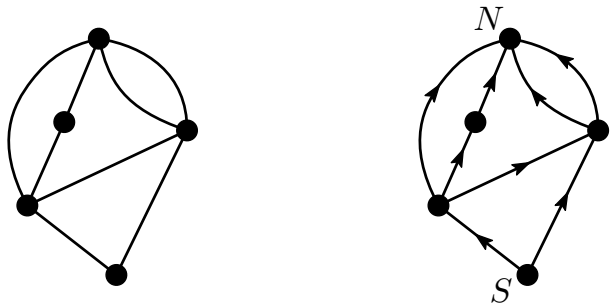


FIGURE 1. Left: a planar map. Right: the same map, equipped with a bipolar orientation with source S and sink N.

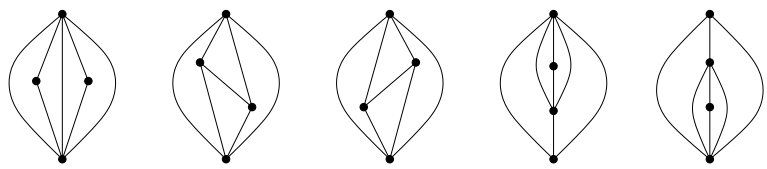
attracted the interest of both combinatorialists and theoretical physicists since the early days of this study [35, 52, 71, 85, 84]. This paper is devoted to the enumeration of planar maps equipped with a *bipolar orientation*: an acyclic orientation of its edges, having a unique source and a unique sink, both incident to the outer face (Figure 1).

The number of bipolar orientations of a given multigraph is an important invariant in graph theory [28, 73]. Given a multigraph G with a directed edge (S, N), the number of bipolar orientations of G with source S and sink N is (up to a sign) the derivative of the chromatic polynomial $\chi_G(\lambda)$, evaluated at $\lambda = 1$. It is also the coefficient of x^1y^0 in the Tutte polynomial $\mathrm{T}_G(x, y)$ [49, 56].

In fact, the first enumerative result on plane bipolar orientations, due to Tutte in 1973, was stated in terms of the derivative of the chromatic polynomial [85] (the interpretation in terms of orientations was only discovered 10 years later). One of Tutte's main results gives the number of bipolar orientations of triangulations of a digon having $k + 2$ vertices (equivalently, $2k$ inner faces, or $3k + 1$ edges), as

$$a(k) = \frac{2(3k)!}{k!\,(k+1)!\,(k+2)!} \sim \frac{\sqrt{3}}{\pi} 27^k k^{-4}. \tag{1}$$

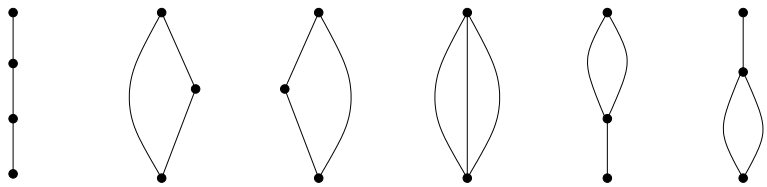
For instance, the 5 oriented triangulations explaining Tutte's result for $k = 2$ are the following ones, where all edges are implicitly oriented upwards.



A more recent result, due to (Rodney) Baxter [4], gives the number of bipolar orientations of general planar maps with n edges as

$$b(n) = \frac{2}{n(n+1)^2} \sum_{m=1}^{n} \binom{n+1}{m-1}\binom{n+1}{m}\binom{n+1}{m+1} \sim \frac{32}{\sqrt{3}\pi} 8^n n^{-4}, \tag{2}$$

where the asymptotic estimate can be obtained by standard techniques for the asymptotics of sums [72]. For instance, the 6 bipolar orientations obtained for $n = 3$ are the following ones, where all edges are implicitly oriented upwards.



Baxter stated his result in terms of the Tutte polynomial, and was apparently unaware of its interpretation in terms of bipolar orientations. Amusingly, he was also unaware of the fact that the above numbers $b(n)$ were known in the combinatorics literature as ... Baxter numbers (after another Baxter, *Glen* Baxter).

Tutte's and Baxter's proofs both rely on a recursive description of the chromatic (or Tutte) polynomial, which gives a functional equation defining the generating function of maps equipped with a bipolar orientation. Both solutions were based on a guess-and-check approach, but these equations can now be solved in a more systematic way [16, 15]. Moreover, several bijective proofs of Baxter's result have been found, by constructing bijections between plane bipolar orientations and various objects known to be counted by Baxter numbers, like Baxter permutations, pairs of twin trees, or configurations of three non-intersecting lattice paths [1, 9, 40, 44].

Tutte's and Baxter's results share some common features, for instance the exponent -4 occurring in the asymptotic estimate. Moreover, both sequences $a(k)$ and $b(n)$ are *polynomially recursive*, that is, they satisfy a linear recurrence relation with polynomial coefficients:

$$(k+1)(k+2)a(k) = 3(3k-1)(3k-2)a(k-1),$$
$$(n+2)(n+3)b(n) = (7n^2+7n-2)b(n-1) + 8(n-1)(n-2)b(n-2).$$

Equivalently, the associated generating functions, namely $A(t) = \sum_{k\geqslant 0} a(k)t^k$ and $B(t) = \sum_{n\geqslant 0} b(n)t^n$ are *D-finite*, meaning that they satisfy a linear differential equation with polynomial coefficients.

In this paper, we prove universality of these features (detailed statements will be given in Section 3): for any finite set $\Omega \subset \{2,3,\dots\}$ and for any integer $e \geqslant 2$, the generating function of plane bipolar orientations such that all inner faces have their degree in Ω and the outer face has degree e is a D-finite series, given as the constant term of an explicit multivariate rational function (for maps not carrying an orientation, the corresponding series are known to be systematically algebraic [5, 20]). For instance, if we consider bipolar orientations of quadrangulations of a digon (with the above notation, $\Omega = \{4\}$ and $e = 2$), having $k+2$ vertices (equivalently, k inner faces, or $2k+1$ edges), then the corresponding numbers $c(k)$ satisfy

$$(k+2)(k+1)^2c(k) = 4(2k-1)(k+1)(k-1)c(k-1) + 12(2k-1)(2k-3)(k-1)c(k-2),$$

their asymptotic behaviour is

$$c(k) \sim \frac{9}{4\sqrt{3}\pi} 12^k k^{-4}, \tag{3}$$

and their generating function $\sum_{k\geqslant 0} c(k)t^{2k}$ is the constant term (in x and z) of the rational series

$$\frac{(1-\bar{x}^2z^2-2\bar{x}z^3)(1+3\bar{x}^4-\bar{x}^2/t)}{1-t(x\bar{z}+\bar{x}^2+\bar{x}z+z^2)}, \tag{4}$$

expanded as a series in t whose coefficients are Laurent polynomials in x and z (where we used the short notations $\bar{x} := 1/x$ and $\bar{z} := 1/z$). The counterpart of the latter result for triangulations is

$$\sum_{k\geqslant 0} a(k)t^{3k} = [x^0z^0]\frac{(1-\bar{x}z^2)(1+2\bar{x}^3-\bar{x}^2/t)}{1-t(x\bar{z}+\bar{x}+z)}, \tag{5}$$

where the operator $[x^0z^0]$ extracts the constant term in x and z. Constant terms (or diagonals) of multivariate rational functions form an important subclass of D-finite series, for which specific methods have been developed, for instance to determine their asymptotic behaviour [65, 74], possibly automatically [64, 66], or the recurrence relations satisfied by their coefficients [12, 13, 55].

However, this paper is not only a paper on the enumerative properties of (decorated) maps. It is also a paper on the enumerative and probabilistic properties of lattice walks confined to a cone. The reason for that is that the first ingredient in our approach is a recent beautiful bijection by Kenyon, Miller, Sheffield and Wilson [53] (denoted KMSW), which encodes plane bipolar orientations by lattice walks confined to the first quadrant of the plane. Among all known bijections that transform bipolar orientations into different objects [1, 9, 40, 44, 53], the KMSW one seems to be the only one that naturally keeps track of the degree distribution of the faces. For instance, the above numbers $c(k)$ that count oriented quadrangulations also count quadrant walks starting and ending at the origin, and consisting of $2k$ steps taken in $\{(-2,0),(-1,1),(0,2),(1,-1)\}$ (Figure 2).

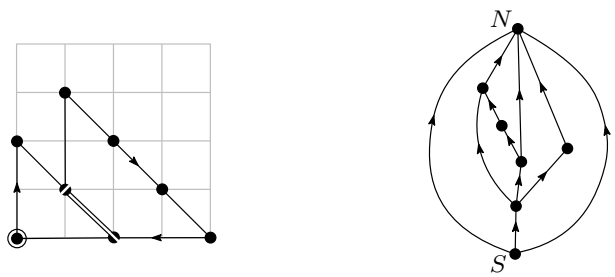


FIGURE 2. A walk in the quadrant and the corresponding bipolar orientation, through the Kenyon–Miller–Sheffield–Wilson (KMSW) bijection.

As it happens, the enumeration of lattice walks confined to a cone is at the moment a very active topic in enumerative combinatorics [7, 11, 14, 18, 33, 54, 76]. These efforts have in the past 15 years led to a very good understanding of quadrant walk enumeration — provided that all allowed steps are *small*, that is, belong to $\{-1,0,1\}^2$. As shown by the above example of quadrangulations, this is not the case for walks coming from bipolar orientations (unless all faces have degree 2 or 3). It is only very recently that an approach was designed for arbitrary steps, by the first author and two collaborators [10]. This approach will not work with *any* collection of steps, but it *does* work for the well structured step sets involved in the KMSW bijection, which we call *tandem steps*. In fact, the enumeration of bipolar orientations provides a beautiful application, with arbitrarily large steps, of the method of [10]. Hence this paper solves an enumerative problem on maps, *and* a quadrant walk problem. Moreover, in order to work out the asymptotic behaviour of the number of bipolar orientations, we go through a probabilistic study of the corresponding quadrant walks, for which we derive local limit theorems and harmonic functions.

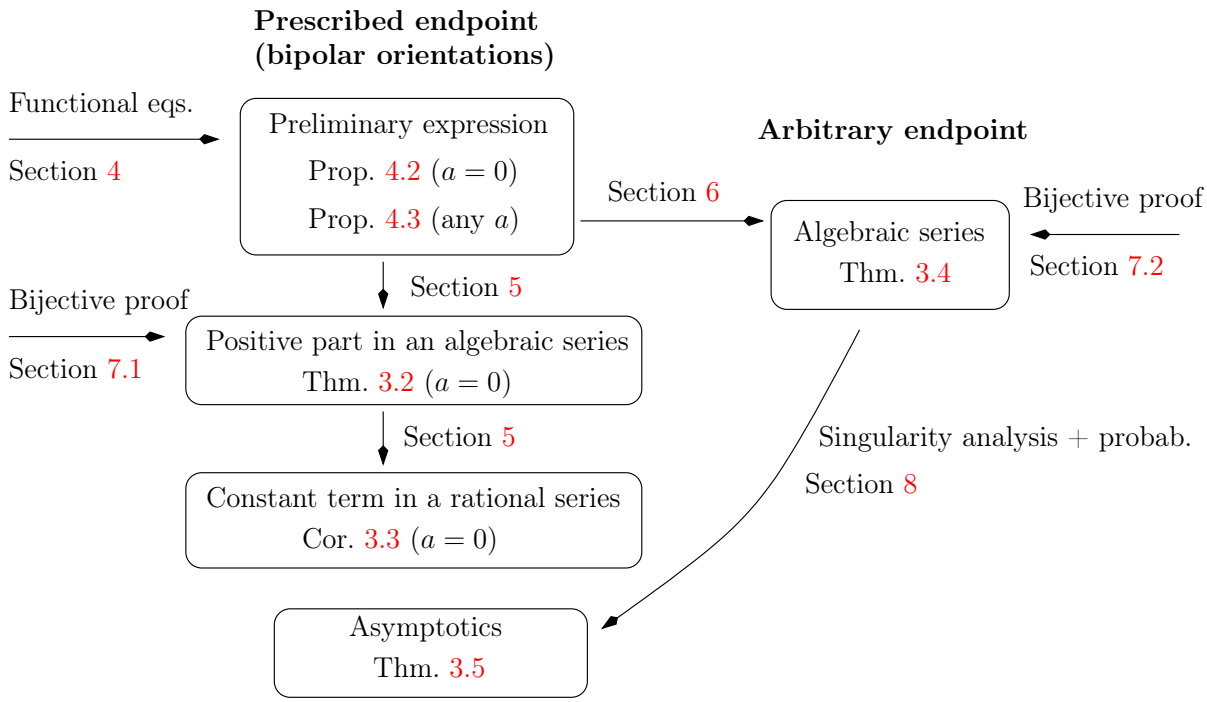


FIGURE 3. Enumeration of tandem walks in the quadrant starting at (a, b): our results and proofs.

Outline of the paper (see also Figure 3). Our main enumerative results (both exact, and asymptotic) are stated in Section 3, after a preliminary section where we describe the KMSW bijection and recall its main properties (Section 2). We prove our exact results in Sections 4 to 6, using the general approach of [10]. Section 7 is a bijective *intermezzo*, where we provide a combinatorial explanation of our results in terms of bipolar orientations, using the KMSW bijection. These combinatorial proofs are more elegant than the algebraic approach used in the earlier sections, but they are also completely *ad hoc*, while the approach of [10] is far more robust. In Section 8 we are back to quadrant walks, this time in a probabilistic setting. By combining our enumerative results and probabilistic tools inspired by a recent paper of Denisov and Wachtel [29], we obtain detailed global and local limit theorems for random walks (related to bipolar orientations) conditioned to stay in the first quadrant. We also determine explicitly the associated discrete harmonic function. This allows us to prove the asymptotic results stated in Section 3. We conclude in Section 9 with some complements — among others, a combinatorial proof of Baxter's result (2) based on the KMSW walks, and a discussion on random generation, which leads to the uniform random bipolar orientation of Figure 4. This figure suggests that drawing planar maps at random according to the number of their bipolar orientations creates a bias in favour of "fatter" maps. This has been recently confirmed by Ding and Gwynne [32, Fig. 2], who showed that the number of points in a ball of radius r of a large bipolar-oriented map grows like r^d, with $2.8 \leqslant d \leqslant 3.3$, instead of r^4 for uniform maps. One expects the corresponding diameter of maps of size n to scale like $n^{1/d}$.

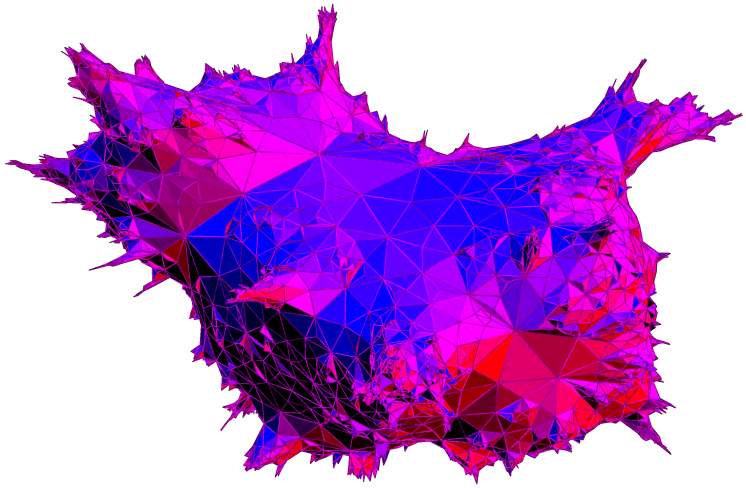


FIGURE 4. A random bipolar orientation of a triangulation with 5000 faces (© Jérémie Bettinelli). The orientations of the edges are not shown.

2. The Kenyon–Miller–Sheffield–Wilson bijection

In this section, we recall a few definitions on planar maps, and describe the KMSW bijection between bipolar orientations and certain lattice walks.

A *planar map* is a proper embedding of a connected multigraph in the plane, taken up to orientation-preserving homeomorphism. A (planar) map has naturally vertices and edges, but defines also *faces*, which are the connected components of the complement of the underlying multigraph. One of the faces, surrounding the map, is unbounded. We call it the *outer* face. The other faces are called *inner* faces. The *degree* of a vertex or face is the number of edges incident to it, counted with multiplicity. The degree of the outer face is the *outer degree*. A (plane) *bipolar orientation* is a planar map endowed with an acyclic orientation of its edges, having a unique source and a unique sink, both incident to the outer face. We denote them by S and N respectively, as illustrated in Figure 5(a). We will usually draw the source S at the bottom of the map, the sink N at the top, and orient all edges upwards.

It is known [28] that bipolar orientations are characterized by two local properties (see Figure 5(b)):

- the edges incident to a given vertex $v \notin \{S, N\}$ are partitioned into a non-empty sequence of consecutive outgoing edges and a non-empty sequence of consecutive ingoing edges (in cyclic order around v),
- in a dual way, the contour of each inner face f is partitioned into a non-empty sequence of consecutive edges oriented clockwise around f and a non-empty sequence of consecutive edges oriented counterclockwise; these are called the *left boundary* and *right boundary* of f, respectively. The edges of the outer face form two oriented paths going from S to N, called *left* and *right outer boundaries* following an obvious convention.

If an inner face f has $i+1$ clockwise edges and $j+1$ counterclockwise edges, then f is said to be *of type* (i, j).

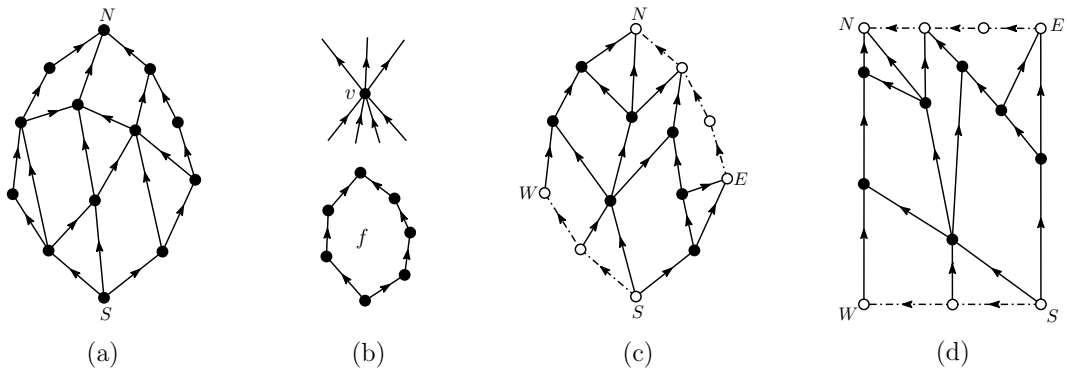


FIGURE 5. (a) A plane bipolar orientation. (b) Local properties of plane bipolar orientations. (c)–(d) A marked bipolar orientation, drawn in two different ways.

A *marked bipolar orientation* is a bipolar orientation where the right (respectively left) outer boundary carries a distinguished vertex $E \neq S$ (respectively $W \neq N$), such that:

- each vertex between E and N along the right outer boundary (N excluded) has outdegree 1 and the unique outgoing edge has an inner face on its left,
- similarly each vertex between W and S along the left outer boundary (S excluded) has indegree 1 and the unique ingoing edge has an inner face on its right.

See Figure 5(c) for an example. Note that a bipolar orientation can be identified with a marked bipolar orientation, upon declaring E to be N and W to be S. The *upper right boundary* (respectively *lower right boundary*) is the path from E to N (respectively from S to E) along the right outer boundary; and similarly the *lower left boundary* (respectively *upper left boundary*) is the path from S to W (respectively from W to N) along the left outer boundary. Note that the upper right and lower left boundaries do not share any vertex. A vertex or edge is called *plain* if it does not belong to the upper right nor to the lower left boundary. In our figures, plain vertices are shown in black and plain edges in solid lines. The non-plain vertices are white, and the non-plain edges are dashed.

At the end of 2015, Kenyon *et al.* [53] introduced a bijection between certain 2-dimensional walks and marked bipolar orientations. These walks, which we call *tandem walks*[1], are defined as sequences of steps of two types: south-east steps $(1, -1)$ (called SE steps for short) and steps of the form $(-i, j)$ with $i, j \geqslant 0$, which we call *face steps*; the *level* of such a step is the integer $r = i + j$. Note that a walk, being defined as a sequence of steps, has no prescribed starting point nor endpoint. In other words, it is defined up to translation. We often say that it is *non-embedded*. When we will embed walks in the plane, we will specify a starting point explicitly. Given a tandem walk w with successive steps $s_1, \ldots, s_n$ in $\mathbb{Z}^2$, the bijection builds a marked bipolar orientation as follows. We start with the marked bipolar orientation O_0 consisting of a single edge $e = \{S, N\}$ with $E = N$ and $W = S$. The marked vertex W will remain the same all along the construction, but the source S will move from vertex to vertex. Then, for k from 1 to n, we construct a marked bipolar orientation O_k from O_{k-1} and the kth step s_k. Two cases may occur:

[1]The name *tandem* originates from the case where the only steps are $(1, -1)$, $(-1, 0)$ and $(0, 1)$. Then it can be seen that such walks, starting at $(0, 0)$ and restricted to the quadrant, describe the evolution of two queues in series — or in *tandem*.

- If s_k is a SE step (Figure 6), we push E one step up; if $E \neq N$ in O_{k-1}, this means that one dashed edge of O_{k-1} becomes plain in O_k; otherwise we also push $N = E$ one step up, thereby creating a new (plain) edge that is both on the left and right outer boundary of the orientation. In this case, we still have $E = N$ in O_k.
- If s_k is a face step $(-i, j)$, we first glue a new inner face f of type (i, j) to the right outer boundary in such a way that the upper vertex of f is E and the lower vertex of f lies on the right outer boundary of O_k (Figure 7); more precisely, if $i+1$ does not exceed the length of the lower right boundary of O_{k-1}, then the $i+1$ left edges of f are identified with the top $i+1$ edges on this boundary; otherwise, the lowest left edges of f are dashed and become part of the lower left boundary in O_k, while the lower vertex of f becomes the source of O_k. We finally choose E to be the end of the first edge along the right boundary of f.

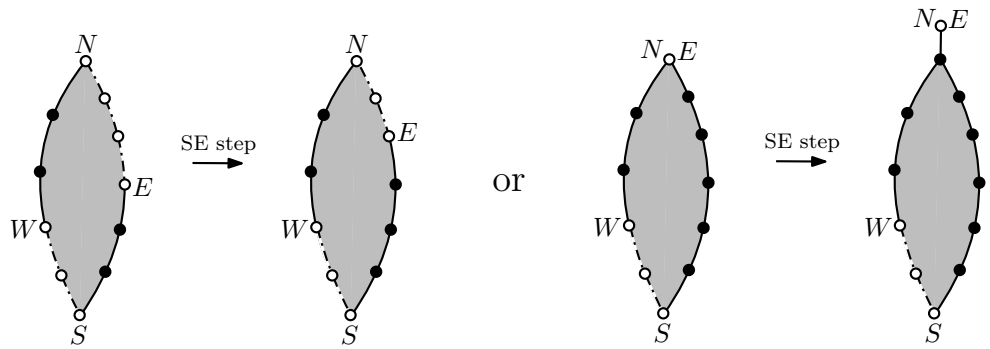


FIGURE 6. Updating the marked bipolar orientation when a SE step is read.

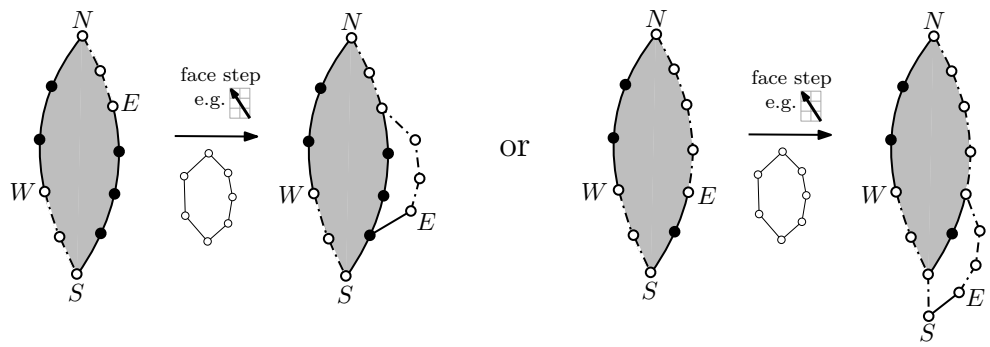


FIGURE 7. Updating the marked bipolar orientation when a face step $(-2, 3)$ is read.

We denote the marked bipolar orientation constructed from w by $\Phi(w) := O_n$. A complete example is detailed in Figure 8.

Theorem 2.1 (KENYON ET AL. [53]). *The mapping Φ is a bijection between (non-embedded) tandem walks with n steps and marked bipolar orientations with $n+1$ plain edges. It transforms SE steps into plain vertices, and face steps of level r into inner faces of degree $r+2$.*

The boundary lengths of the orientation $\Phi(w)$ are also conveniently translated through this bijection. Let us denote by a (respectively $b+1$, $c+1$, d) the length of the lower

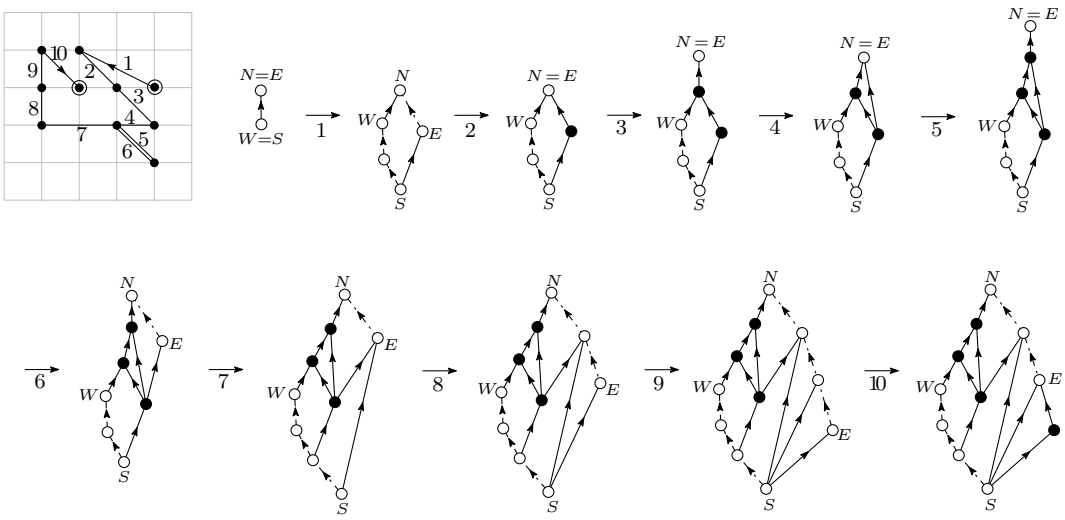


FIGURE 8. A tandem walk of length 10 and the associated marked bipolar orientation, which is constructed step by step.

left (respectively upper left, lower right, upper right) boundary of $\Phi(w)$ (see Figure 9, right). We call the 4-tuple $(a, b; c, d)$ the *signature* of the marked bipolar orientation. Let us embed the walk w in the plane so that it starts at some point $(x_{\text{start}}, y_{\text{start}})$. Let $x_{\min}$ and $y_{\min}$ be the minimal x- and y-coordinates along the walk, and let x_{end} and y_{end} be the x- and y-coordinates of the final point of w. Then one easily checks that

$$\begin{aligned} a &= x_{\text{start}} - x_{\min}, & b &= y_{\text{start}} - y_{\min}, \\ c &= x_{\text{end}} - x_{\min}, & d &= y_{\text{end}} - y_{\min}, \end{aligned} \tag{6}$$

as illustrated in Figure 9. Indeed these quantities are initially all equal to 0 when we start constructing $\Phi(w)$ (that is, for the initial orientation O_0 and the empty walk), and then the parameters in each pair (e.g., a and $x_{\text{start}} - x_{\min}$) change in the same way at each step of the construction (see Figures 6 and 7).

If we embed w in the plane so that $x_{\min} = y_{\min} = 0$, then it becomes a tandem walk in the quadrant $\{x \geqslant 0, y \geqslant 0\}$ starting at (a, b), ending at (c, d), constrained to visit at

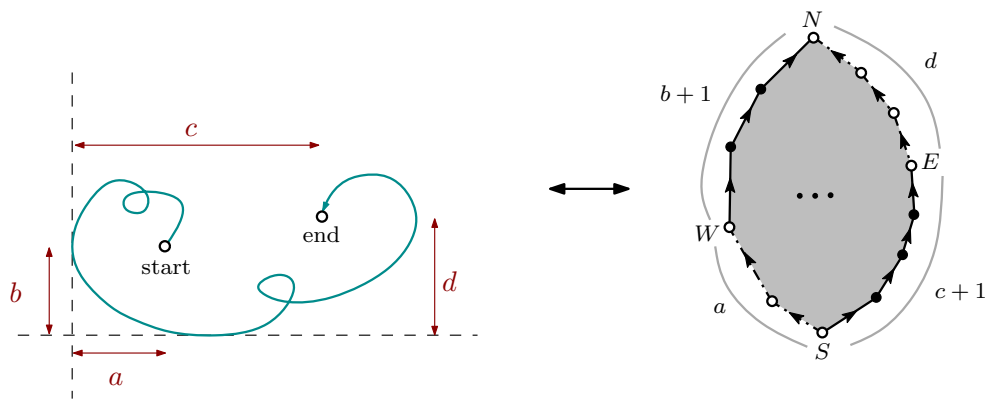


FIGURE 9. The correspondence between the coordinates of the endpoints and the signature in the KMSW bijection.

least once the x-axis and the y-axis. For unmarked bipolar orientations ($a = d = 0$), the constraint holds automatically, and the following corollary [53, Thm. 2.2] is obtained.

Corollary 2.2. *The mapping Φ specializes to a bijection between tandem walks of length n in the quadrant, starting at $(0, b)$ and ending at $(c, 0)$, and bipolar orientations with $n + 1$ edges, having $b + 1$ edges on the left outer boundary and $c + 1$ edges on the right outer boundary.*

Specializing further to excursions*, that is, walks starting and ending at $(0, 0)$, we obtain, upon erasing the two outer edges, a bijection between excursions of length n and bipolar orientations with $n - 1$ edges.*

We now define two involutions on marked bipolar orientations.

Definition 2.3. *Let O be a marked bipolar orientation of signature $(a, b; c, d)$. We define $\rho(O)$ as the marked bipolar orientation obtained by reversing all edge directions in O. This exchanges the roles of N and S on the one hand, of E and W on the other hand. The signature of $\rho(0)$ is $(d, c; b, a)$.*

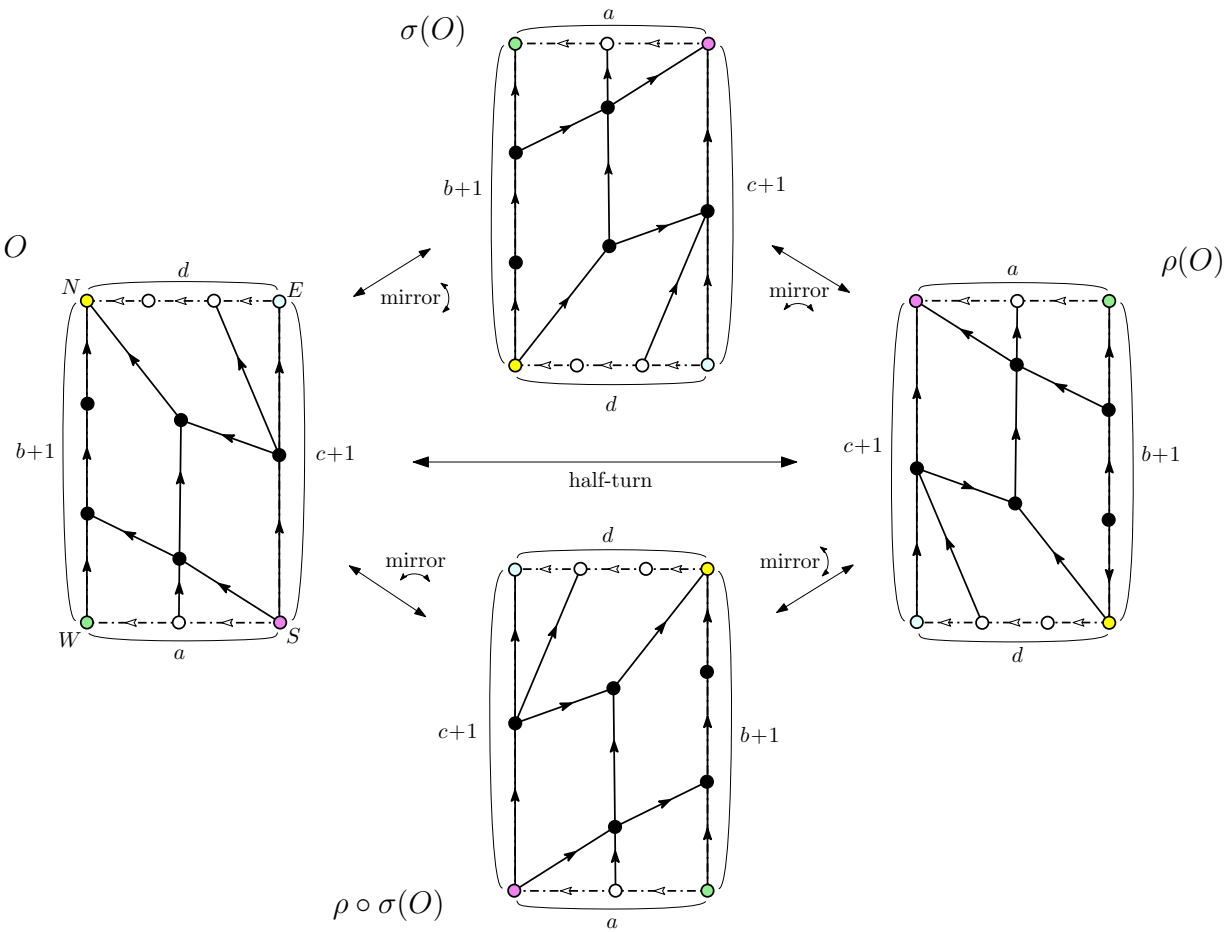


FIGURE 10. The orbit of a marked bipolar orientation under the action of the two involutions σ and ρ; the dashed edges are drawn as horizontal segments, which makes it easier to see the mirror-effect of σ and $\rho \circ \sigma$.

We define $\sigma(O)$ by first reflecting O in a mirror, then reversing the edge directions of plain edges only. The new points S', N', W' and E' in $\sigma(O)$ correspond to E, W, N and S, respectively. The signature of $\sigma(O)$ is $(d, b; c, a)$.

This description clearly shows that ρ and σ are involutions. Moreover, the marked bipolar orientations $(\rho\circ\sigma)(O)$ and $(\sigma\circ\rho)(O)$ are both obtained by reflecting O in a mirror and reversing the directions of all dashed edges, and thus they coincide. Hence ρ and σ generate a dihedral group of order 4. Their effect is perhaps better seen if we draw marked orientations with the rectangular convention adopted on the right of Figure 5: all plain edges go upward, while dashed edges go left. Then we can forget edge directions, ρ corresponds to a half-turn rotation, σ to a reflection in a horizontal mirror, and $\rho\circ\sigma$ to a reflection in a vertical mirror. This is illustrated in Figure 10.

It is easy to describe the involution on tandem walks induced by ρ. This description (which we will not exploit) is used in the proof of Theorem 2.2 in [53].

Proposition 2.4. *Let $w = s_1, \dots, s_n$ be a tandem walk, and $O = \Phi(w)$ the corresponding marked bipolar orientation. Let $\widetilde{s}_k$ be $(-j, i)$ if $s_k = (-i, j)$, for any $i, j \in \mathbb{Z}^2$, and define $\widetilde{w} = \widetilde{s}_n, \dots, \widetilde{s}_1$. Then $\Phi(\widetilde{w}) = \rho(O)$.*

It seems more difficult to directly describe the involution on tandem walks induced by σ. This involution will be used in Section 7.2 to prove bijectively some of our enumerative results.

3. Counting tandem walks in the quadrant

The KMSW bijection described in the previous section relates two topics that are actively studied at the moment in combinatorics and probability theory: planar maps, here equipped with a bipolar orientation, and walks confined to a cone, here the first quadrant. In this section, we state our main results on the enumeration of these objects.

The enumeration of walks confined to the quadrant is well understood when the walk consists of *small steps*, that is, when the steps are taken in $\{-1, 0, 1\}^2$. This is not the case here, unless we only consider orientations with inner faces of degree 2 and 3. Recently, the first author and two of her collaborators developed an approach to count quadrant walks with larger steps, generalizing in particular the definition of a certain group that plays a key role in the small step case [10]. This approach does not apply to all possible step sets; in particular, it requires that the group (or what has replaced it for large steps, namely a certain *orbit*) is finite. This is the case for tandem walks, and we will count them using the approach of [10].

Given two points (a, b) and (c, d) in the first quadrant, we denote by

$$Q_{c,d}^{a,b} \equiv Q_{c,d}^{a,b}(t, z_0, z_1, \dots)$$

the generating function of tandem walks going from (a, b) to (c, d) in the quadrant, where every edge is weighted by t, and every face step of level r by z_r (which we take as an indeterminate). For instance, the walk of Figure 8, once translated so that it becomes a quadrant walk visiting both coordinate axes, contributes $t^{10} z_1^3 z_2^2 z_3$ to the series $Q_{1,2}^{3,2}$.

Returning to bipolar orientations, it follows from Corollary 2.2 that $tQ_{c,0}^{0,b}$ counts bipolar orientations with left (respectively right) outer boundary of length $b+1$ (respectively $c+1$) with a weight t per edge, and z_r per inner face of degree $r+2$. Also, $\frac{1}{t}(Q_{0,0}^{0,0} - 1)$, specialized to $z_r = 1$ for all r, simply counts bipolar orientations by edges. As recalled

in the introduction, the number of bipolar orientations having n edges is the nth Baxter number $b(n)$, given by (2). We will recover this result using the bijection with tandem walks in Section 9.1.

If we want to count *marked* bipolar orientations of signature $(a, b; c, d)$, we must recall that they are in bijection with tandem walks in the quadrant, joining (a, b) to (c, d) and constrained to visit both coordinate axes. An inclusion-exclusion argument gives their generating function as

$$t\left(Q_{c,d}^{a,b} - Q_{c,d-1}^{a,b-1} - Q_{c-1,d}^{a-1,b} + Q_{c-1,d-1}^{a-1,b-1}\right).$$

Here, every *plain* edge is weighted by t, and every inner face of degree $r+2$ by z_r.

Keeping a and b fixed, we group all the $Q_{c,d}^{a,b}$ into a bigger generating function that counts quadrant tandem walks starting at (a, b):

$$Q^{(a,b)}(x, y) := \sum_{c,d \geqslant 0} Q_{c,d}^{a,b}\, x^c y^d.$$

By Corollary 2.2, we are especially interested in the series $Q^{(0,b)}(x, 0)$, since $txQ^{(0,b)}(x, 0)$ counts bipolar orientations with a left boundary of length $b+1$, by edges (t), face degrees (z_r for each inner face of degree $r+2$) and length of the lower right boundary (x).

3.1. Preliminaries

3.1.1. ***Walk generating functions.*** It may be a bit unusual to involve infinitely many variables in generating functions, as we do with the z_r's. Hence let us clarify in which ring these series live.

Many of the series that we consider count (sometimes implicitly) embedded tandem walks, not necessarily confined to the quadrant, and record with variables x and y the coordinates of their endpoint. Then a natural option is to work with formal power series in infinitely many variables $t, z_0, z_1, \ldots$ with coefficients in $\mathbb{Q}[x, 1/x, y, 1/y]$, the ring of Laurent polynomials in x and y. However, it will sometimes be convenient to handle a *finite* collection of steps, and moreover to assign real values to the z_r's. This is why we usually assume that $z_r = 0$ for $r > p$, for some arbitrary p, and take our series in the ring of formal power series in t with coefficients in $\mathbb{Q}[x, 1/x, y, 1/y, z_0, z_1, \ldots, z_p]$. We call this specialization the *p-specialization*, and the corresponding walks, *p-tandem walks*. Both points of view can be reconciled by letting $p \to \infty$. Indeed, if a walk starting at (a, b) and ending at (c, d) uses a face step of level r, then $(d-c)-(b-a) \geqslant r - 2(n-1)$ (look at the projection of the walk on a line of slope -1). That is, $r \leqslant (d-c)-(b-a)+2(n-1)$. Hence, when a, b, c, d, n are fixed, a walk of length n going from (a, b) to (c, d), cannot use arbitrarily large steps. This means that for p large enough, the coefficient of $t^n x^c y^d$ in any walk generating function (with a fixed starting point) is a polynomial in the z_r's which is independent of p.

3.1.2. ***Periodicities.*** Throughout the paper, we will meet periodicity conditions, describing which points can be reached from say, the origin, in a fixed number of steps. So let us clarify this right now. For a step set $\mathcal{S}$, we call a walk consisting of steps taken in $\mathcal{S}$ an *$\mathcal{S}$-walk*. The following terminology is borrowed from Spitzer [80, Chap. 1.5]. Take a finite step set $\mathcal{S} \subset \mathbb{Z}^2$, and denote the lattice spanned by $\mathcal{S}$ in $\mathbb{Z}^2$ by Λ. We say that $\mathcal{S}$ is *strongly aperiodic* if, for any $(i, j) \in \Lambda$, the lattice generated by $(i, j) + \mathcal{S}$ coincides with Λ. In this case, for $(i, j) \in \Lambda$, there exists $N_0 \in \mathbb{N}$ such that for all $n > N_0$, there exists an $\mathcal{S}$-walk of

length n going from $(0,0)$ to (i,j). We say that $\mathcal{S}$ has *period* 1. Otherwise, there exists an integer $p>1$ (the *period*), such that for all $(i,j)\in\Lambda$ there exists $r\in[\![0,p-1]\!]$ such that for n large enough, there exists an $\mathcal{S}$-walk of length n from $(0,0)$ to (i,j) if and only if n equals r modulo p.

Lemma 3.1. *Let D be a non-empty finite subset of $\mathbb{N}$, not reduced to $\{0\}$, and define*

$$\iota := \gcd(r+2, r\in D).$$

Let $\mathcal{S}_D$ be the set of steps

$$\mathcal{S}_D = \{(1,-1)\}\cup\bigcup_{r\in D}\{(-r,0),(-r+1,1),\dots,(0,r)\}.$$

Then the lattice Λ_D spanned by $\mathcal{S}_D$ is $\mathbb{Z}^2$ if ι is odd, and $\{(i,j): i+j \text{ even}\}$ otherwise.

If there exists an n-step walk from $(0,0)$ to (i,j) with steps in $\mathcal{S}_D$, then $i-j\equiv 2n \bmod \iota$. Conversely, if n satisfies this condition and is large enough, there exists an $\mathcal{S}_D$-walk from $(0,0)$ to (i,j). This means that the step set $\mathcal{S}_D$ has period ι if ι is odd, $\iota/2$ otherwise. In particular, $\mathcal{S}_D$ is strongly aperiodic if and only if $\iota\in\{1,2\}$.

Proof. If ι is odd, then there exists an odd r in D, say $r=2s+1$ with $s\geqslant 0$. Then $(-s,s+1)$ belongs to $\mathcal{S}_D$, and

$$(-s,s+1)+s(1,-1)=(0,1).$$

Hence Λ_D contains the vectors $(0,1)$ and $(1,-1)$ (which is always in $\mathcal{S}_D$), and thus coincides with $\mathbb{Z}^2$.

If ι is even, then every $r\in D$ is even, and for every step $(i,j)\in\mathcal{S}_D$ the difference $i-j$ is even: equal to 2 for a SE step, to $-r$ for a step $(-i,r-i)$. Hence the same holds necessarily for any point (i,j) of Λ_D, which is thus included in $\{(i,j): i+j \text{ even}\}$. Now take $r=2s\in D$ with $s\geqslant 1$. Then $(-s-1,s-1)\in\mathcal{S}_D$, and

$$(-s-1,s-1)+s(1,-1)=(-1,-1).$$

Hence Λ_D contains $(-1,-1)$ and $(1,-1)$, and thus all points (i,j) such that $i+j$ is even. We have thus proved the first statement of the lemma.

Now consider an $\mathcal{S}_D$-walk of length n going from $(0,0)$ to (i,j), and let (i_k,j_k) be the point reached after k steps. Then $(i_k-j_k)-(i_{k-1}-j_{k-1})$ equals 2 mod ι for every k. Hence after n steps, we find $i-j\equiv 2n \bmod \iota$.

Let us now prove the next result for $(i,j)=(0,0)$. The set G of lengths n such that there exists an n-step walk starting and ending at $(0,0)$ (we call such walks *excursions*) is an additive semi-group of $\mathbb{N}$. The structure of semi-groups of $\mathbb{N}$ is well understood: there exists an integer p (the period), such that $G\subset p\mathbb{N}$ and $mp\in G$ for all large enough m. Clearly $p=\gcd(G)$. By the previous result, all elements n of G satisfy $2n\equiv 0 \bmod \iota$, that is, $\iota\mid 2n$. Hence the period p is a multiple of ι if ι is odd, and of $\iota/2$ otherwise. Now, saying that for any large enough n such that $\iota\mid 2n$ there exists an n-step excursion, is equivalent to saying that p *equals* ι if ι is odd, and $\iota/2$ otherwise. So let us first prove that $p\mid\iota$. For each $r\in D$, there exists an excursion of length $r+2$ (consisting of the steps $(0,r)$ and $(-r,0)$ followed by r SE steps). Hence $D+2\subset G$, and thus $p:=\gcd(G)$ divides $\iota:=\gcd(D+2)$. This proves that $p=\iota$ if ι is odd, but if ι is even, we can still have $p=\iota$ or $p=\iota/2$. So assume that ι is even. Then each $r\in D$ is even, and there exists an excursion of length $1+r/2$ (consisting of the step $(-r/2,r/2)$ followed by $r/2$ SE steps).

Hence $1 + D/2 \subset G$, and thus $p = \gcd(G)$ divides $\iota/2 = \gcd(1 + D/2)$. This concludes the proof when $(i, j) = (0, 0)$.

Once the period p is determined, the extension to general points (i, j) is standard. See for instance the proof of [10, Prop. 9], and references therein. ∎

Remark. The period was already determined in the original paper [53, Thm. 2.6], where it is described as

$$\gcd\left(\{r + 1 : 2r \in D\} \cup \{2r + 3 : 2r + 1 \in D\}\right).$$

Both descriptions are of course equivalent. The reason why we prefer to introduce ι is that this is the quantity that naturally arises in asymptotic estimates (see for instance Corollary 3.6).

3.1.3. ***Some definitions and notation on formal power series.*** Let $\mathbb{A}$ be a commutative ring and let x be an indeterminate. We denote by $\mathbb{A}[x]$ (respectively $\mathbb{A}[[x]]$) the ring of polynomials (respectively formal power series) in x with coefficients in $\mathbb{A}$. If $\mathbb{A}$ is a field, then $\mathbb{A}(x)$ denotes the field of rational functions in x, and $\mathbb{A}((x))$ the set of Laurent series in x, that is, series of the form

$$\sum_{n \geqslant n_0} a_n x^n,$$

with $n_0 \in \mathbb{Z}$ and $a_n \in \mathbb{A}$. The coefficient of x^n in a series $F(x)$ is denoted by $[x^n]F(x)$.

This notation is generalized to polynomials, fractions and series in several indeterminates. For instance, the generating function of bipolar orientations, counted by edges (variable t) and faces (variable z) belongs to $\mathbb{Q}[z][[t]]$. For a multivariate series, say $F(x, y) \in \mathbb{Q}[[x, y]]$, the notation $[x^i]F(x, y)$ stands for the *series* $F_i(y)$ such that $F(x, y) = \sum_i F_i(y)x^i$. It should not be mixed up with the coefficient of $x^i y^0$ in $F(x, y)$, which we denote by $[x^i y^0]F(x, y)$. If $F(x, x_1, \dots, x_d)$ is a series in the x_i's whose coefficients are Laurent series in x, say

$$F(x, x_1, \dots, x_d) = \sum_{i_1, \dots, i_d} x_1^{i_1} \cdots x_d^{i_d} \sum_{n \geqslant n_0(i_1, \dots, i_d)} a(n, i_1, \dots, i_d)x^n,$$

then the *nonnegative part of F in x* is the following formal power series in $x, x_1, \dots, x_d$:

$$[x^{\geqslant}]F(x, x_1, \dots, x_d) = \sum_{i_1, \dots, i_d} x_1^{i_1} \cdots x_d^{i_d} \sum_{n \geqslant 0} a(n, i_1, \dots, i_d)x^n.$$

We denote the reciprocals of variables using bars: to be precise, $\bar{x} = 1/x$, so that $\mathbb{A}[x, \bar{x}]$ is the ring of Laurent polynomials in x with coefficients in $\mathbb{A}$.

If $\mathbb{A}$ is a field, a power series $F(x) \in \mathbb{A}[[x]]$ is *algebraic* (over $\mathbb{A}(x)$) if it satisfies a non-trivial polynomial equation $P(x, F(x)) = 0$ with coefficients in $\mathbb{A}$. It is *differentially finite* (or *D-finite*) if it satisfies a non-trivial linear differential equation with coefficients in $\mathbb{A}(x)$. For multivariate series, D-finiteness requires the existence of a differential equation *in each variable*. We refer to [60, 61] for general results on D-finite series.

For a series F in several variables, we denote the derivative of F with respect to the ith variable by F_i'.

In the next three subsections we state our main enumerative results, both exact and asymptotic.

3.2. Quadrant tandem walks with prescribed endpoints

We give here an explicit expression for the generating function $Q^{(0,b)}(x,y)$ that counts tandem walks starting at height b on the y-axis. We define the *step generating function* $S(x,y)$, which counts all tandem steps, as

$$S(x,y) := x\bar{y} + \sum_{r\geqslant 0} z_r \sum_{i=0}^{r} \bar{x}^{r-i} y^i, \tag{7}$$

and we let $K(x,y) := 1-tS(x,y)$. In the p-specialization, $S(x,y)$ is a (Laurent) polynomial.

We let $Y_1 \equiv Y_1(x)$ be the unique power series in t satisfying $K(x,Y_1)=0$, that is,

$$Y_1 = t\Big(x + Y_1 \sum_{r\geqslant 0} z_r \sum_{i=0}^{r} \bar{x}^{r-i} Y_1^i\Big). \tag{8}$$

This series has coefficients in $\mathbb{Q}[x,\bar{x},z_0,z_1,\ldots]$, and starts by

$$Y_1 = tx + t^2 x \sum_{r\geqslant 0} z_r \bar{x}^r + O(t^3).$$

In the p-specialization, this series is algebraic. We observe that $H(x) := \frac{Y_1(x)}{tx}$ is the generating function of tandem walks starting at the origin, ending on the x-axis and *staying in the upper half-plane* $\{y \geqslant 0\}$, where as usual t marks the length, x marks the final abscissa and z_r marks the number of face steps of level r. Indeed, upon considering the first step, say $(-r+i,i)$, of such a walk, and the first time it comes back to the x-axis, it is standard [62, Ch. 11] to establish

$$H = 1 + \sum_{r\geqslant 0} \sum_{i=0}^{r} (t z_r \bar{x}^{r-i})(tx)^i H^{i+1},$$

which is equivalent to (8) with $txH = Y_1$.

Theorem 3.2. *Let $Y_1 \equiv Y_1(x)$ and $K(x,y)$ be defined as above. The generating function $Q^{(0,b)}(x,y)$ can be expressed as the nonnegative part in x of an explicit series*[2]*:*

$$Q^{(0,b)}(x,y) = [x^{\geqslant}] \frac{-Y_1}{yK(x,y)} (Y_1^b + \cdots + \bar{x}^b)\Big(1 - \frac{1}{tx^2} + \sum_{r\geqslant 0} z_r (r+1)\bar{x}^{r+2}\Big), \tag{9}$$

where the argument of $[x^{\geqslant}]$ is expanded as an element of $\mathbb{Q}[x,\bar{x},z_0,z_1,\ldots]((t))[[y]]$. In particular, the generating function of bipolar orientations of left boundary length $b+1$ is $txQ^{(0,b)}(x,0)$, where

$$Q^{(0,b)}(x,0) = [x^{\geqslant}] \frac{Y_1}{tx} (Y_1^b + \cdots + \bar{x}^b)\Big(1 - \frac{1}{tx^2} + \sum_{r\geqslant 0} z_r (r+1)\bar{x}^{r+2}\Big). \tag{10}$$

In the p-specialization, these series are D-finite in all their variables.

We will provide two different proofs of (10) (which easily implies (9) as explained in Remark 2 below): first in Sections 4 and 5 using the method developed in [10] for quadrant walks with large steps, and then in Section 7.1 using the KMSW bijection and local operations on marked bipolar orientations. This second approach explains combinatorially

[2]Throughout the paper, we use the notation $u^a + \cdots + v^a$ for $\sum_{k=0}^{a} u^k v^{a-k}$.

why the enumeration of tandem walks in the quadrant is related to the enumeration of tandem walks in the upper half-plane, that is, to the series $Y_1(x)$.

Remarks. **1.** We will give another D-finite expression for $Q^{(0,b)}(x,y)$, and more generally of $Q^{(a,b)}(x,y)$, in Section 4.6 (Propositions 4.2 and 4.3), again as the positive part of an algebraic generating function. In this alternative expression, the expansion has to be done (more classically) in t first.

2. Since $yK(x,y)$ is a formal power series in y, with constant term $(-tx)$, the expression (10) is clearly the special case $y=0$ of (9). Conversely, a simple argument involving factorizations of walks allows us to derive (9) from (10). Indeed, upon expanding the right-hand side of (9) in y, what we want to prove is that, for all $d \geqslant 0$,

$$Q_d^b(x) := [y^d]Q^{(0,b)}(x,y) = [x^{\geqslant}]\frac{Y_1}{tx}(Y_1^b + \cdots + \bar{x}^b)\Big(1 - \frac{1}{tx^2} + \sum_{r\geqslant 0} z_r(r+1)\bar{x}^{r+2}\Big)P_d, \quad (11)$$

where

$$P_d = [y^d]\frac{-tx}{yK(x,y)} = [y^d]\frac{1}{1 - y(\bar{x}/t - \sum_{r\geqslant 0} z_r \sum_{i=0}^r \bar{x}^{r-i+1}y^i)}.$$

Note that P_d is a polynomial in $1/t$, $\bar{x}$, and the z_r's, which can alternatively be described by the recurrence relation

$$P_0 = 1, \qquad P_{d+1} = \frac{\bar{x}}{t}P_d - \sum_{r\geqslant 0} z_r \sum_{i=0}^r \bar{x}^{r-i+1}P_{d-i} \quad \text{for all } d \geqslant 0, \quad (12)$$

where $P_d = 0$ for $d < 0$. We now prove (11) by induction on $d \geqslant 0$. The case $d = 0$ is precisely (10). Assume that (11) holds for $Q_0^b, \dots, Q_d^b$, and let us prove it for Q_{d+1}^b. A last step decomposition of quadrant tandem walks ending at height d gives

$$Q_d^b = \mathbb{1}_{d=b} + txQ_{d+1}^b + t[x^{\geqslant}]\sum_{r\geqslant 0} z_r \sum_{i=0}^r \bar{x}^{r-i}Q_{d-i}^b,$$

with $Q_d^b = 0$ for $d < 0$. Extracting Q_{d+1}^b, and observing that $[x^{\geqslant}](\bar{x}[x^{\geqslant}]G(x)) = [x^{\geqslant}](\bar{x}G(x))$, we see that

$$Q_{d+1}^b = [x^{\geqslant}]\Big(\frac{\bar{x}}{t}Q_d^b - \sum_{r\geqslant 0} z_r \sum_{i=0}^r \bar{x}^{r-i+1}Q_{d-i}^b\Big).$$

We now use the induction hypothesis (11) to replace $Q_0^b, \dots, Q_d^b$ by their respective expressions in terms of $P_0, \dots, P_d$. Observe that for $e \geqslant 0$ we have $[x^{\geqslant}](\bar{x}^e[x^{\geqslant}]G(x)) = [x^{\geqslant}](\bar{x}^e G(x))$. Finally we use the recurrence relation (12) to conclude that (11) holds for Q_{d+1}^b.

It is well known that algebraic series, in particular Y_1 and its powers, can be expressed as constant terms of rational functions. Hence we can also express $Q^{(0,b)}(x,y)$ in terms of a rational function, this time in three variables x, y, z. The following result is proved in Section 5.

Corollary 3.3. *As above, let $S(x,y)$ be defined by (7), and let $K(x,y) = 1 - tS(x,y)$. The series $Q^{(0,b)}(x,y)$ can alternatively be expressed as*

$$Q^{(0,b)}(x,y) = [x^{\geqslant}][z^0]\frac{tz^2}{y}\frac{S_2'(x,z)}{K(x,y)K(x,z)}(z^b + \cdots + \bar{x}^b)\Big(1 - \frac{\bar{x}^2}{t} + \sum_{r\geqslant 0} z_r(r+1)\bar{x}^{r+2}\Big),$$

where the argument of $[x^{\geqslant}][z^0]$ *is expanded as a series in* $\mathbb{Q}[x,\bar{x},z,\bar{z},z_0,z_1,\ldots]((t))[[y]]$.

In particular,

$$Q^{(0,b)}(x,0) = -[x^{\geqslant}][z^0]\frac{z^2}{x}\frac{S_2'(x,z)}{K(x,z)}(z^b+\cdots+\bar{x}^b)\Big(1-\frac{\bar{x}^2}{t}+\sum_{r\geqslant 0} z_r(r+1)\bar{x}^{r+2}\Big). \tag{13}$$

This result, specialized to p-tandem walks, yields an expression for $[x^c]Q^{(0,b)}(x,0)$ as the constant term in x and z of a rational expression of t, x, z and the z_r's. From expressions of this form, recent algorithms based on "creative telescoping" can efficiently construct polynomial recurrences satisfied by the coefficients [12, 13, 55]. For instance, let us specialize (13) to $x=0$, $b=0$, $z_p=1$ and $z_r=0$ if $r\neq p$. We obtain

$$Q^{(0,0)}(0,0) = -[x^0][z^0]\frac{z^2}{x}\frac{S_2'(x,z)}{K(x,z)}\Big(1-\frac{\bar{x}^2}{t}+(p+1)\bar{x}^{p+2}\Big). \tag{14}$$

By Corollary 2.2, the series $tQ^{(0,0)}(0,0)$ counts (by edges) bipolar orientations of outer degree 2 with all inner faces of degree $p+2$. By Lemma 3.1, such orientations have $n+1$ edges, where $(p+2)$ divides $2n$. By counting adjacencies between edges and faces, it is easy to see that these orientations have $\frac{2n}{p+2}$ inner faces. If p is odd, this number is necessarily even. Retaining only non-zero coefficients in $Q^{(0,0)}(0,0)$, we write

$$Q^{(0,0)}(0,0) = \sum_{k\geqslant 0} a(k)t^{ck(p+2)/2},$$

where $c=2$ if p is odd, and $c=1$ otherwise. In this way, $a(k)$ counts orientations with ck inner faces. In particular, when $p=3$ and $p=4$, we recover from (14) the expressions (5) and (4) given in the introduction. One can also derive the following recurrence relations, which were computed for us by Pierre Lairez (in all cases, $a(0)=1$), from the above expression.

- For $p=1$ (triangulations):

$$(k+3)(k+2)a(k+1) = 3(3k+2)(3k+1)a(k).$$

  This gives the number of bipolar triangulations with outer degree 2 and $2k$ inner faces (equivalently, $k+2$ vertices) as

$$a(k) = \frac{2(3k)!}{k!\,(k+1)!\,(k+2)!},$$

  which is Tutte's result (1).
- For $p=2$ (quadrangulations), $a(k)$ gives the number of bipolar orientations of a quadrangulated digon with k inner faces (denoted $c(k)$ in the introduction), and

$$(k+4)(k+3)^2a(k+2) = 4(2k+3)(k+3)(k+1)a(k+1)+12(2k+3)(2k+1)(k+1)a(k),$$

  as announced in the introduction. Using Petkovšek's algorithm [75], we proved that this recurrence relation has no hypergeometric solution. Still, one can derive from (14), specialized to $p=2$, an expression for $a(k)$ as a single sum involving multinomial coefficients.

- For $p = 3$ (pentagulations), $a(k)$ gives the number of bipolar orientations of a pentagulated digon with $2k$ inner faces, and

$$\begin{aligned}27(3k+8)(3k+4)(5k+3)(3k+5)^2(3k+7)^2(k+2)^2a(k+2) &= \\ 60(5k+7)(3k+5)(5k+9)(5k+6)(3k+4)(8+5k)(145k^3+532k^2+626k+233)a(k+1) \\ - 800(5k+6)(5k+1)(5k+7)(5k+2)(5k+3)(5k+9)(5k+4)(8+5k)^2a(k).\end{aligned}$$

  Again, there is no hypergeometric solution.

Starting from (13), similar constructions can be performed for a prescribed starting point $(0, b)$ and a prescribed endpoint $(c, 0)$, in order to count bipolar orientations of signature $(0, b+1; c+1, 0)$.

3.3. Quadrant tandem walks ending anywhere

We now consider the specialization $Q^{(a,b)}(1,1)$, which counts tandem walks in the quadrant starting at (a, b), and records the length (variable t), the number of face steps of each level r (variable z_r), but not the coordinates of the endpoint. For this problem, we can either consider $Q^{(a,b)}(1,1)$ as a series in infinitely many variables $t, z_0, z_1, \ldots$, or apply the p-specialization and count p-tandem walks only.

Let W be the unique formal power series in t satisfying

$$W = t\left(1 + \sum_{r\geqslant 0} z_r \left(W + \cdots + W^{r+1}\right)\right). \tag{15}$$

Note that $W = Y_1(1)$, where $Y_1 \equiv Y_1(x)$ is given by (8). In the p-specialization, this series is algebraic.

Theorem 3.4. *For $a, b \geqslant 0$, the generating function of quadrant tandem walks starting at (a, b) and ending anywhere in the quadrant is*

$$Q^{(a,b)}(1,1) = \frac{W}{t} \cdot \sum_{i=0}^{a} A_i \cdot \sum_{j=0}^{b} W^j,$$

where A_i is a series in W and the z_r's:

$$\begin{aligned} A_i &= [u^i]\,\frac{1}{W}\,\frac{\bar{u}-1}{S(\bar{u},W)-S(1,W)} \\ &= [u^i]\,\frac{1}{1-uW\sum_{i,k\geqslant 0} u^i W^k \sum_{r>i+k} z_r}, \end{aligned} \tag{16}$$

with $S(x, y)$ given by (7). In particular $Q^{(0,0)}(1,1) = W/t$.

In the p-specialization, each A_i, and thus the whole series $tQ^{(a,b)}(1,1)$, becomes a polynomial in W and $z_0, \ldots, z_p$.

We will provide a first proof in Section 6 using functional equations and algebraic manipulations. A bijective proof will then be given in Section 7.2. It involves the KMSW bijection and the involution σ, both described in Section 2.

Remarks. **1.** In our combinatorial proof, the term $\frac{W}{t}A_iW^j$ will be interpreted as the generating function of tandem walks that start at $(0, j)$, remain in the upper half-plane $\{y \geqslant 0\}$, touch the x-axis at least once and end on the line $\{y = i\}$ (see Lemma 7.3). In particular, for $a = b = 0$, this proof gives a length preserving bijection between tandem

walks in the quadrant that start at the origin, and tandem walks in the upper half-plane that start at the origin and end on the x-axis. Moreover, this bijection preserves the number of SE steps.

In the case where $z_p = 1$ and $z_r = 0$ for $r \neq p$, three such bijections already appear in the literature. The first two are only valid for $p = 1$: one is due to Gouyou-Beauchamps [48], and uses a simple correspondence between 1-tandem walks and standard Young tableaux with at most 3 rows, and then the Robinson–Schensted correspondence; the second, more recent one is due to Eu [38] (generalized in [39] to Young tableaux with at most k rows). The third bijection, due to Chyzak and Yeats [25], is very recent and holds for any p. It relies on certain automata rules to build (step by step) a half-plane walk ending on the x-axis from a quarter plane excursion. These three constructions do not seem to be equivalent to the correspondence presented in Section 7.2.

2. Let us define *double-tandem walks* as walks with steps N, W, SE, E, S, NW: these are the three steps involved in 1-tandem walks, and their reverses. With these steps too, it is known that walks in the quadrant that start at the origin are equinumerous with walks (of the same length) in the upper half-plane that start at the origin and end on the x-axis [18, Prop. 10]; see [70] for an intriguing refinement involving walks confined to a triangle. A bijection between these two families of walks was recently given by Yeats [87], and then reformulated using automata in [25]. We do not know of any bijection for these walks that would generalize the KMSW map, but we conjecture that there exists an involution on double-tandem walks having the same properties as the involution σ of Section 2 (once defined on 1-tandem walks). See the remark at the end of Section 7.2 for details.

3.4. Asymptotic number of quadrant walks with prescribed endpoints

We now fix $p \geqslant 1$, and focus on the asymptotic enumeration of p-tandem walks with prescribed endpoints confined to the quadrant. To be precise, we aim at finding an asymptotic estimate of the coefficients $[t^n]Q^{a,b}_{c,d}(t, z_0, \dots, z_p)$ as $n \to \infty$, for any prescribed a, b, c, d and nonnegative weights $z_0, \dots, z_p$ with $z_p > 0$. As it turns out, a detailed estimate can be derived by combining recent asymptotic results by Denisov and Wachtel [29] (or rather, a variant of these results that applies to our *periodic* walks) and the algebraic expression for $Q^{(a,b)}(1, 1)$ given in Theorem 3.4.

Let

$$D := \{r \in [\![0, p]\!],\ z_r > 0\}, \qquad \text{and} \qquad \iota := \gcd(r + 2, r \in D). \tag{17}$$

It follows from Lemma 3.1 that there can only exist a walk of length n from (a, b) to (c, d) if $c - d \equiv a - b + 2n \bmod \iota$ (and n is large enough). Our main asymptotic result is the following.

Theorem 3.5. *Fix $p \geqslant 1$. Let a, b, c, d be nonnegative integers and let $z_0, \dots, z_p$ be nonnegative weights with $z_p > 0$. Let ι be defined by (17). Then, as $n \to \infty$ conditioned on $c - d \equiv a - b + 2n \bmod \iota$, we have*

$$[t^n]Q^{a,b}_{c,d} \sim \kappa\, \gamma^n n^{-4},$$

where the growth rate γ is explicit and depends only on the weights z_r, while the multiplicative constant κ, also explicit, depends on these weights and on a, b, c, d as well.

The explicit values of κ and γ are given in Section 8.2, together with the proof of the theorem. When specialized to bipolar orientations ($a = d = 0$), this theorem will give (once

the constants are explicit) the following detailed asymptotic estimate for the number of orientations with prescribed face degrees and boundary lengths.

Corollary 3.6 (Bipolar orientations with prescribed face degrees). *Let $\Omega \subset \{2,3,4,\dots\}$ be a finite set such that $\max(\Omega) \geqslant 3$, and let ι be the gcd of all elements in Ω. Let α be the unique positive solution of the equation*

$$1 = \sum_{s\in\Omega} \binom{s-1}{2} \alpha^{-s},$$

and let

$$\gamma = \sum_{s\in\Omega} \binom{s}{2} \alpha^{-s+2}.$$

Then, for $2n \equiv b+c \bmod \iota$, the number $B_n^{(\Omega)}(b,c)$ of bipolar orientations with $n+1$ edges, left boundary length $b+1$, right boundary length $c+1$, and all inner face degrees in Ω, satisfies

$$B_n^{(\Omega)}(b,c) \sim \kappa \gamma^n n^{-4} \qquad \text{as } n \to \infty,$$

where the constant κ is

$$\kappa := \frac{\iota\gamma^2}{4\sqrt{3}\pi\alpha^4\sigma^4}(b+1)(b+2)(c+1)(c+2)\alpha^{-b-c},$$

with $\sigma^2 = \frac{\alpha^2}{\gamma}\sum_{s\in\Omega}\binom{s}{3}\alpha^{-s}$.

(The above quantity σ, which will be seen as a variance in Section 8, should not be mixed up with the transformation σ of Section 2. We hope that this will not cause any inconvenience.) The proof is given in Section 8.2. Specializing further to the case of bipolar d-angulations ($\Omega = \{d\}$, with $d \geqslant 3$), we have

$$\iota = d, \qquad \alpha = \binom{d-1}{2}^{1/d}, \qquad \gamma = \frac{d}{d-2}\binom{d-1}{2}^{2/d},$$

so that $\sigma^2 = (d-2)/3$. Hence the number of bipolar orientations having $n+1$ edges (for $2n-b-c$ divisible by d), left (respectively right) boundary length $b+1$ (respectively $c+1$), satisfies

$$B_n^{(d)}(b,c) \sim \frac{9(b+1)(b+2)(c+1)(c+2)}{4\sqrt{3}\pi d}\left(\frac{d}{d-2}\right)^{n+4}\binom{d-1}{2}^f n^{-4},$$

where $f = (2n-b-c)/d$ is the number of inner faces. When $b = c = 0$ and $d = 3$ or $d = 4$, this estimate is in agreement with (1) and (3).

4. A functional equation approach

Let $p \geqslant 1$. In this section, we apply the general approach to quadrant walk enumeration described in [10] to p-tandem walks confined to the quadrant. This approach consists of four steps, detailed here in Sections 4.2 to 4.5. Let us recall that it is not systematic, and does not work for all sets of steps. Moreover, the fact that our step set depends on p adds another difficulty. The first step of the approach (write a functional equation) is however simple and systematic. The second (compute the so-called *orbit* of the problem) is also systematic as long as the orbit is finite, and can even be performed automatically for a fixed (small) value of p [10, Sec. 3.2]. The last two steps (construction of a *section-free* equation and extraction of the generating function) definitely require more inventiveness.

At the end, the approach yields an expression for $Q^{(a,b)}(x,y)$ as the nonnegative part (in x and y) of an algebraic series. We state it in the first subsection below. This expression is *not* the one of Theorem 3.2, which will be derived later in Section 5.

4.1. First expression for $Q^{(a,b)}(x,y)$

The algebraic ingredient in this expression is a new series x_1, involving the indeterminates x, y and z_r, and is defined as follows.

Lemma 4.1. *Recall the definition (7) of $S(x,y)$. The equation $S(x,y) = S(X,y)$, when solved for X, admits $p+1$ roots $x_0 = x, x_1, \dots, x_p$, which can be taken as Laurent series in $\bar{y} := 1/y$ with coefficients in $\mathbb{C}[z_1, \dots, z_p, 1/z_p, x, \bar{x}]$. Exactly one of these roots, say x_1, contains some positive powers of y in its series expansion. It has coefficients in $\mathbb{Q}[z_1, \dots, z_p, x, \bar{x}]$ and reads $x_1 = z_p \bar{x} y^p (1 + O(\bar{y}))$. The other roots are formal power series in $\bar{y}$ with no constant term.*

Examples. For $p = 1$ we have

$$S(x,y) = x\bar{y} + z_0 + z_1(\bar{x} + y),$$

and the equation $S(X,y) = S(x,y)$ has two solutions, $x_0 = x$ and $x_1 = z_1 \bar{x} y$.

For $p = 2$ we have

$$S(x,y) = x\bar{y} + z_0 + z_1(\bar{x} + y) + z_2(\bar{x}^2 + \bar{x}y + y^2),$$

and the equation $S(X,y) = S(x,y)$ has three solutions. One of them is $x_0 = x$ and the other two satisfy a quadratic equation:

$$x^2 X^2 - yX(xz_1 + z_2(1 + xy)) - xyz_2 = 0.$$

Hence

$$x_{1,2} = \frac{xz_1 + z_2(1+xy) \pm \sqrt{(xz_1 + z_2(1+xy))^2 + 4x^3 \bar{y} z_2}}{2x^2 \bar{y}}.$$

(We take x_1 to correspond to the $+$ sign.) We expand both solutions as Laurent series in $\bar{y}$ (not y!), and find:

$$\begin{aligned} x_1 &= z_2 \bar{x} y^2 + \bar{x}(z_1 + \bar{x} z_2) y + \bar{y} - (z_1/z_2 + \bar{x})\bar{y}^2 + O(\bar{y}^3), \\ x_2 &= \qquad\qquad\qquad\qquad\qquad\qquad - \bar{y} + (z_1/z_2 + \bar{x})\bar{y}^2 + O(\bar{y}^3). \end{aligned}$$

□

We prove Lemma 4.1 in Section 4.3. It implies that $y_1 := \bar{x}_1 = 1/x_1$ is a power series in $\bar{y}$ whose coefficients lie in $\mathbb{Q}[z_1, \dots, z_p, 1/z_p, x, \bar{x}]$ (this comes from the monomial form of the first coefficient of x_1). We can now give our first expression for $Q^{(a,b)}(x,y)$, which we first state in the case $a = 0$.

Proposition 4.2. *Fix $p \geqslant 1$, and let $y_1 = 1/x_1$, where x_1 is defined in Lemma 4.1. The generating function of p-tandem walks confined to the first quadrant and starting at $(0,b)$ is the nonnegative part (in x and y) of an algebraic function:*

$$Q^{(0,b)}(x,y) = [x^{\geqslant} y^{\geqslant}] \frac{(1 - \bar{x}\bar{y})\, S_1'(x,y)}{1 - tS(x,y)} \sum_{k=0}^{b} \left(y^{k+1} - y_1^{k+1}\right) \bar{x}^{b-k},$$

where the argument of $[x^{\geqslant} y^{\geqslant}]$ is expanded as a series of $\mathbb{Q}[x, \bar{x}, z_0, \dots, z_p, 1/z_p]((\bar{y}))[[t]]$.

In particular, $Q^{(0,b)}(x,y)$ is D-finite.

The general case is more involved.

Proposition 4.3. *Fix $p \geqslant 1$, and let $y_1 = 1/x_1$, where x_1 is defined in Lemma 4.1. The generating function of p-tandem walks confined to the first quadrant and starting at (a,b) is the nonnegative part (in x and y) of an algebraic function:*

$$Q^{(a,b)}(x,y) = [x^{\geqslant} y^{\geqslant}] \frac{(1-\bar{x}\bar{y})S_1'(x,y)}{1-tS(x,y)} \big(\eta_b \zeta_a - \eta_{b-1}\zeta_{a-1}\big), \tag{18}$$

where $\eta_b := \sum_{k=0}^{b} (y^{k+1} - y_1^{k+1})\bar{x}^{b-k}$ and ζ_a is the Laurent polynomial in x and y (with coefficients that are polynomial in $z_1, \dots, z_p$) defined by

$$\begin{aligned}\sum_{a \geqslant 0} \zeta_a u^a &= \frac{1}{uy(1-u\bar{y})(S(\bar{u},y) - S(x,y))} \\ &= \frac{1}{(1-ux)(1-u\bar{y})(1-u\bar{x}y\sum_{i+j+k<p} z_{i+j+k+1} u^i \bar{x}^j y^k)}.\end{aligned}$$

The argument of $[x^{\geqslant} y^{\geqslant}]$ in (18) is meant as a series of $\mathbb{Q}[x, \bar{x}, z_0, \dots, z_p, 1/z_p]((\bar{y}))[[t]]$.

Propositions 4.2 and 4.3 will be proved in Sections 4.5 and 4.6, respectively.

4.2. A functional equation

The starting point of our approach is a functional equation that characterizes the series $Q(x,y) := Q^{(a,b)}(x,y)$, and simply relies on a step-by-step construction of quadrant tandem walks. It reads

$$Q(x,y) = x^a y^b + tS(x,y)Q(x,y) - tx\bar{y}Q(x,0) - t\sum_{r=1}^{p} z_r \sum_{i=1}^{r} \bar{x}^i y^{r-i}\big(Q_0(y) + \cdots + x^{i-1}Q_{i-1}(y)\big),$$

where $S(x,y)$ is the step polynomial given by (7), and $Q_i(y)$ counts quadrant tandem walks starting at (a,b) and ending at abscissa i. We call $Q(x,0)$ and the series $Q_i(y)$ *sections* of $Q(x,y)$. Equivalently,

$$K(x,y)Q(x,y) = x^a y^b - tx\bar{y}Q(x,0) - \sum_{j=1}^{p} \bar{x}^j G_j(y), \tag{19}$$

where $K(x,y) = 1 - tS(x,y)$ and

$$G_j(y) = t\sum_{r=j}^{p} z_r \big(Q_0(y)y^{r-j} + Q_1(y)y^{r-j-1} + \cdots + Q_{r-j}(y)y^0\big).$$

4.3. The orbit of p-tandem walks

The aim of this subsection is to prove the following result.

Proposition 4.4. *Let $x_0, \dots, x_p$ be defined as the roots of $S(X,y) = S(x,y)$, as in Lemma 4.1. Let us write $x_{p+1} = \bar{y}$. For $0 \leqslant i \leqslant p+1$, denote moreover $y_i = \bar{x}_i := 1/x_i$, so that in particular, $y_{p+1} = y$. Then, for $0 \leqslant i, j \leqslant p+1$ and $i \neq j$, we have*

$$S(x_i, y_j) = S(x,y).$$

In the terminology of [10], the pairs (x_i, y_j) with $i \neq j$ form the *orbit* of (x,y) for the step set $\mathcal{S}_p := \{(1,-1)\} \cup \{(-i,j) : i,j \geqslant 0, i+j \leqslant p\}$.

Our first task is to prove Lemma 4.1.

Proof of Lemma 4.1. Recall the expression (7) of the step polynomial $S(x,y)$. We refer the reader to [81, Ch. 6] for generalities on algebraic series. Clearly one of the roots of $S(X,y)=S(x,y)$ (solved for X) is $x_0=x$. The others satisfy

$$0=\frac{S(x,y)-S(X,y)}{x-X}=\bar{y}-\bar{x}\bar{X}\sum_{\substack{i,j,k\geqslant 0\\ i+j+k<p}} z_{i+j+k+1}\bar{x}^i\bar{X}^j y^k. \tag{20}$$

This expression is a polynomial in $\bar{X}$, and is thus well suited to determine the roots x_i such that $\bar{x}_i=1/x_i$ is a formal power series in $\bar{y}$ (or in a positive power of $\bar{y}$). The other roots $\bar{x}_i$ will involve positive powers of y, hence their reciprocals will be formal power series in (a positive power of) $\bar{y}$. More precisely, upon multiplying the above identity by $x\bar{y}^{p-1}$ and expanding in powers of $\bar{y}$, we have

$$0=x\bar{y}^p-z_p\bar{X}-\sum_{\ell=1}^{p-1}\bar{y}^\ell\sum_{i+j\leqslant\ell} z_{i+j+p-\ell}\,\bar{x}^i\bar{X}^{j+1}. \tag{21}$$

The coefficient of $\bar{y}^0$ is $-z_p\bar{X}$. It has degree 1 in $\bar{X}$, hence there is a unique $\bar{x}_i$, say $\bar{x}_1$, that only involves nonnegative powers of $\bar{y}$. This series can be computed iteratively from the above equation. It is a power series in $\bar{y}$, starts by

$$\bar{x}_1=\frac{x\bar{y}^p}{z_p}+O(\bar{y}^{p+1}),$$

and its coefficients belong to $\mathbb{Q}[z_1,\dots,z_p,1/z_p,x,\bar{x}]$. This proves the claimed properties of x_1.

To understand the nature of the other roots $x_2,\dots,x_p$, we now write $X=\bar{y}U$, and multiply (20) by $\bar{y}^pU^p$. Then

$$0=\bar{y}^{p+1}U^p-\bar{x}\sum_{i+j+k<p} z_{i+j+k+1}\bar{x}^iU^{p-1-j}\bar{y}^{p-1-j-k}. \tag{22}$$

The coefficient of $\bar{y}^0$ is

$$-\bar{x}z_p\sum_{k=0}^{p-1}U^k.$$

It has degree $p-1$ in U, hence (22) admits $p-1$ solutions $u_2,\dots,u_p$ that expand in nonnegative powers of $\bar{y}$ only. Their constant terms are the pth roots of unity distinct from 1. All of them are power series in $\bar{y}$, and their expansions can be computed recursively using (22). Their coefficients lie in $\mathbb{C}[z_1,\dots,z_p,1/z_p,x,\bar{x}]$ (in fact, in the lemma, we could replace $\mathbb{C}$ by the extension of $\mathbb{Q}$ generated by pth roots of unity). ∎

We then need the following symmetry properties of $S(x,y)$.

Lemma 4.5. *The step polynomial $S(x,y)$, defined by (7), satisfies $S(x,y)=S(\bar{y},\bar{x})$ and*

$$x\,\frac{S(x,y)-S(X,y)}{x-X}=-\bar{X}\,\frac{S(x,y)-S(x,\bar{X})}{y-\bar{X}}.$$

Proof. The first point is easy, using

$$S(x,y)=x\bar{y}+\sum_r z_r\frac{\bar{x}^{r+1}-y^{r+1}}{\bar{x}-y}.$$

For the second, we recall from (20) that

$$\frac{S(x,y)-S(X,y)}{x-X} = \bar{y} - \bar{x}\bar{X} \sum_{\substack{i,j,k\geqslant 0\\ i+j+k<p}} z_{i+j+k+1}\bar{x}^i\bar{X}^j y^k, \tag{23}$$

and we compute from (7) that

$$\frac{S(x,y)-S(x,\bar{X})}{y-\bar{X}} = -xX\bar{y} + \sum_{\substack{i,j,k\geqslant 0\\ i+j+k<p}} z_{i+j+k+1}\bar{x}^i\bar{X}^j y^k. \tag{24}$$

The result follows by comparing these two expressions. ∎

We can now prove Proposition 4.4.

Proof of Proposition 4.4. By definition of the series x_i, for $0 \leqslant i \leqslant p$, we have $S(x,y) = S(x_i,y)$. So the claimed identity holds for $j = p+1$. The first identity in Lemma 4.5 then gives

$$S(x_i,y) = S(\bar{y},\bar{x}_i),$$

hence the claimed identity holds as well for $i = p+1$.

Now we specialize the second identity in Lemma 4.5 to $x = x_i$, $X = x_j$, with $0 \leqslant i \neq j \leqslant p$. This reads

$$x_i\frac{S(x_i,y)-S(x_j,y)}{x_i-x_j} = -y_j\frac{S(x_i,y)-S(x_i,y_j)}{y-y_j}.$$

Since the left-hand side is zero, we conclude that

$$S(x_i,y_j) = S(x_i,y) = S(x,y)$$

for $0 \leqslant i \neq j \leqslant p$, which concludes the proof of Proposition 4.4. ∎

4.4. A section-free functional equation

In the functional equation (19), we can replace the pair (x,y) by any element (x_i,y_j) of the orbit. The series that occur in the resulting equation are series in t with algebraic coefficients in x and y (and the z_r's). By Proposition 4.4, the kernel $K(x,y) = 1-tS(x,y)$ takes the same value at all points of the orbit. We thus obtain $(p+1)(p+2)$ equations. Our aim is to form a linear combination of these equations in which the right-hand side does not contain any section $Q(x_i,0)$ nor $G_k(y_j)$. As soon as $p > 1$, the vector space of such linear combinations has dimension larger than 1. We choose here a section-free combination that only involves the pairs (x_i,y_j) for $j \in \{0,1,p+1\}$. That is, y_j will be either y, or $\bar{x}$, or $y_1 := 1/x_1$. We focus on the case $a = 0$ until Section 4.6.

Lemma 4.6. *Let $x_0, \dots, x_p$ be defined by Lemma 4.1, and take $x_{p+1} = \bar{y}$ as before. For $Q(x,y) \equiv Q^{(0,b)}(x,y)$, the following identity holds:*

$$Q(x,y) - \bar{x} \sum_{i=1}^{p} \left(x_i^p \, Q(x_i, y) \prod_{j \neq 0,i,p+1} \frac{1 - \bar{x} x_j}{x_i - x_j} \right)$$
$$- y_1 \bar{y} \prod_{i=2}^{p+1} (1 - \bar{x} x_i) \sum_{i \neq 1} \frac{x_i^p \, Q(x_i, y_1)}{\prod_{j \neq i,1} (x_i - x_j)}$$
$$+ \bar{x}^2 \bar{y} (x_1 - \bar{y}) \prod_{i=2}^{p} (1 - \bar{x} x_i) \sum_{i \neq 0} \frac{x_i^p \, Q(x_i, \bar{x})}{\prod_{j \neq i,0} (x_i - x_j)} = \frac{(1 - \bar{x}\bar{y}) \, S_1'(x,y)}{1 - tS(x,y)} \sum_{k=0}^{b} (y^{k+1} - y_1^{k+1}) \bar{x}^{b-k}.$$

In order to prove this lemma, we need two other lemmas that involve classical symmetric functions. We recall the definition of complete and elementary homogeneous symmetric functions of degree k in m variables $u_1, \dots, u_m$:

$$h_k(u_1, \dots, u_m) = \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant m} u_{i_1} \cdots u_{i_k}, \qquad e_k(u_1, \dots, u_m) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant m} u_{i_1} \cdots u_{i_k}.$$

In this subsection, we only apply the following lemma to polynomials $P(u, v_1, \dots, v_m) \in \mathbb{Q}[u]$, but we use it in full generality in the next subsection. This lemma extends Lemma 13 in [17].

Lemma 4.7. *Let $P(u, v_1, \dots, v_m) \in \mathbb{Q}[u, v_1, \dots, v_m]$ be a polynomial, symmetric in the v_i's. Take $m+1$ variables $u_0, u_1, \dots, u_m$, and define*

$$E(u_0, \dots, u_m) := \sum_{i=0}^{m} \frac{P(u_i, u_0, \dots, u_{i-1}, u_{i+1}, \dots, u_m)}{\prod_{j \neq i} (u_i - u_j)}. \tag{25}$$

Then $E(u_0, \dots, u_m)$ is a symmetric polynomial in $u_0, \dots, u_m$, of degree at most $\deg(P) - m$. In particular, $E(u_0, \dots, u_m) = 0$ if P has degree less than m.

If $P(u, v_1, \dots, v_m) = u^{m+a}$ with $a \geqslant 0$, then

$$E(u_0, \dots, u_m) = \sum_{i=0}^{m} \frac{{u_i}^{m+a}}{\prod_{j \neq i} (u_i - u_j)} = h_a(u_0, \dots, u_m), \tag{26}$$

with h_a the complete homogeneous symmetric function.

Finally, for $a \geqslant 0$ we have

$$\sum_{i=0}^{m} \frac{{u_i}^{-a-1}}{\prod_{j \neq i} (u_i - u_j)} = \frac{(-1)^m}{\prod_{i=0}^{m} u_i} h_a(1/u_0, \dots, 1/u_m).$$

Proof. Let $E(u)$ denote the expression (25), where we use the shorthand notation u for the $(m+1)$-tuple $(u_0, \dots, u_m)$. Multiplication of $E(u)$ by the Vandermonde determinant

$$\Delta(u) := \prod_{0 \leqslant i < j \leqslant m} (u_i - u_j)$$

gives a polynomial in the u_i's which is antisymmetric in the u_i's (that is, swapping u_i and u_j changes the sign of the expression): this comes from the fact that E is symmetric, while Δ is antisymmetric. Hence $E(u)\Delta(u)$, as a polynomial, must be divisible by the Vandermonde determinant, and $E(u)$ itself is a polynomial. Its degree is obviously $\deg(P) - m$ (at most).

Next, in order to prove (26), note that h_a is the Schur function of the Ferrers diagram consisting of a single row of length a, and therefore, by definition of Schur functions [79, Ch. 4], we have

$$h_a(u_0,\dots,u_m) = \frac{\det\left(u_i^{a\cdot\delta_{j,0}+m-j}\right)_{0\leqslant i,j\leqslant m}}{\Delta(u_0,\dots,u_m)}.$$

Upon expanding the determinant according to the first column ($j=0$), we find

$$h_a(u)\Delta(u) = \det\left(u_i^{a\cdot\delta_{j,0}+m-j}\right)_{0\leqslant i,j\leqslant m} = \sum_{i=0}^m (-1)^i u_i^{a+m}\Delta(u_0,\dots,u_{i-1},u_{i+1},\dots,u_m).$$

But for any $0\leqslant i\leqslant m$ we have

$$\Delta(u_0,\dots,u_{i-1},u_{i+1},\dots,u_m) = \frac{\Delta(u_0,\dots,u_m)}{(-1)^i\prod_{j\neq i}(u_i-u_j)},$$

which yields (26).

To prove the last statement, we let $v_i = 1/u_i$ and note that $\frac{1}{u_i-u_j} = -\frac{v_iv_j}{v_i-v_j}$, hence

$$\begin{aligned}\sum_{i=0}^m u_i^{-a-1}\prod_{j\neq i}\frac{1}{u_i-u_j} &= (-1)^m(v_0\cdots v_m)\sum_{i=0}^m v_i^{m+a}\prod_{j\neq i}\frac{1}{v_i-v_j}\\ &= (-1)^m(v_0\cdots v_m)h_a(v_0,\dots,v_m) \qquad \text{by (26).}\end{aligned}$$

■

Lemma 4.8. *Let $x_1,\dots,x_p$ be the series defined in Lemma 4.1. Their elementary symmetric functions are:*

$$e_\ell(x_1,\dots,x_p) = \begin{cases} 1, & \text{if } \ell=0,\\ (-1)^{\ell-1}\bar{x}y\displaystyle\sum_{\substack{i,k\geqslant 0\\ i+k\leqslant p-\ell}} z_{i+k+\ell}\,\bar{x}^iy^k, & \text{for } 1\leqslant\ell\leqslant p.\end{cases}$$

In particular, they are x-nonpositive and y-nonnegative (meaning that, in every monomial that they contain, x has a nonpositive exponent and y a nonnegative one). Moreover, every monomial $\bar{x}^iy^j$ occurring in them satisfies $i\geqslant j/p$.

Finally,

$$\prod_{i=1}^p(1-\bar{x}x_i) = yS_1'(x,y). \tag{27}$$

Proof. It follows from (23) that $(S(x,y)-S(X,y))/(x-X)$ is a polynomial in $\bar{X}$ with constant term $\bar{y}$. Hence

$$\frac{S(x,y)-S(X,y)}{x-X} = \bar{y}\prod_{i=1}^p(1-x_i\bar{X}) = \bar{y}\sum_{\ell=0}^p(-1)^\ell e_\ell(x_1,\dots,x_p)\bar{X}^\ell.$$

Comparison with (23) gives the expression for $e_\ell(x_1,\dots,x_p)$. The next statement is then obvious. Letting X tend to x in the above identity finally gives (27). ■

Proof of Lemma 4.6. As already noted, in the basic functional equation (19) we can replace the pair (x,y) by any element (x_i,y_j) of its orbit. By Proposition 4.4, this does not change the value of $K(x,y)=1-tS(x,y)$. Recall that for the moment, we take $a=0$.

Using the elements (x_i, y) of the orbit, with $0 \leqslant i \leqslant p$, we first construct a linear combination avoiding all series $G_j(y)$:

$$K(x,y)\sum_{i=0}^{p}\frac{x_i^p\,Q(x_i,y)}{\prod_{j\neq i,p+1}(x_i-x_j)} = y^b - t\bar{y}\sum_{i=0}^{p}\frac{x_i^{p+1}\,Q(x_i,0)}{\prod_{j\neq i,p+1}(x_i-x_j)}. \tag{28}$$

Lemma 4.7 explains the simplicity of the right-hand side, namely the fact that all sections $G_j(y)$ disappear, and that the constant term is just y^b. In the former case the lemma is applied with $P(u) = u^{p-j}$ and $m = p$, in the latter case with $P(u) = u^p$ and $m = p$ again.

More generally, for $0 \leqslant k \leqslant p+1$, we can similarly eliminate all series $G_j(y_k)$ in the equations obtained from the elements (x_i, y_k) of the orbit, for $i \neq k$. After multiplying by y_k, this gives

$$y_kK(x,y)\sum_{i\neq k}\frac{x_i^p\,Q(x_i,y_k)}{\prod_{j\neq i,k}(x_i-x_j)} = y_k^{b+1} - t\sum_{i\neq k}\frac{x_i^{p+1}\,Q(x_i,0)}{\prod_{j\neq i,k}(x_i-x_j)},$$

where the natural range of the indices i and j is $0,\dots,p+1$. Note that the equation can be rewritten as

$$y_kK(x,y)\sum_{i\neq k}\frac{x_i^p\,Q(x_i,y_k)}{\prod_{j\neq i,k}(x_i-x_j)} = y_k^{b+1} - t\sum_{i=0}^{p+1}\frac{x_i^{p+1}\,Q(x_i,0)}{\prod_{j\neq i}(x_i-x_j)}(x_i-x_k). \tag{29}$$

We now take an appropriate linear combination of three of these $p+2$ equations, namely those obtained for $k = p+1$, $k = 1$ and $k = 0$, with respective weights

$$x_0 - x_1, \qquad x_{p+1} - x_0, \qquad x_1 - x_{p+1}. \tag{30}$$

(Of course these weights can be written in a simpler way as $x - x_1, \bar{y} - x$ and $x_1 - \bar{y}$, but the above notation makes the symmetry clearer.) Then, writing the three equations as in (29), it is easy to show that all terms involving $Q(x_i, 0)$ vanish from the right-hand side. Hence the only remaining term on the right-hand side is

$$\begin{aligned}(x_0-x_1)y_{p+1}^{b+1} + (x_{p+1}-x_0)y_1^{b+1} + (x_1-x_{p+1})y_0^{b+1}\\ = x_0x_1x_{p+1}\left((y_1-y_0)y_{p+1}^{b+2} + (y_0-y_{p+1})y_1^{b+2} + (y_{p+1}-y_1)y_0^{b+2}\right).\end{aligned}$$

With the notation (25) and $P(u, v_1, v_2) = u^{b+2}$, this can be rewritten as

$$\begin{aligned}&-x_0x_1x_{p+1}\Delta(y_0,y_1,y_{p+1})E(y_0,y_1,y_{p+1})\\ &\qquad = -x_0x_1x_{p+1}\Delta(y_0,y_1,y_{p+1})h_b(y_0,y_1,y_{p+1}) \qquad \text{by (26)}\\ &\qquad = x_0x_1x_{p+1}(y_0-y_1)(y_0-y_{p+1})\sum_{k=0}^{b}(y_{p+1}^{k+1}-y_1^{k+1})y_0^{b-k}\\ &\qquad = (1-\bar{x}\bar{y})(x-x_1)\sum_{k=0}^{b}(y^{k+1}-y_1^{k+1})\bar{x}^{b-k}. \qquad (31)\end{aligned}$$

The left-hand side in our linear combination is

$$K(x,y)\left((x-x_1)y\sum_{i\neq p+1}\frac{x_i^p\,Q(x_i,y)}{\prod_{j\neq i,p+1}(x_i-x_j)}+(\bar{y}-x)y_1\sum_{i\neq 1}\frac{x_i^p\,Q(x_i,y_1)}{\prod_{j\neq i,1}(x_i-x_j)}\right.$$
$$\left.+(x_1-\bar{y})\bar{x}\sum_{i\neq 0}\frac{x_i^p\,Q(x_i,\bar{x})}{\prod_{j\neq i,0}(x_i-x_j)}\right).$$

The term $Q(x,y)$ only occurs in the first sum, with coefficient

$$x^p y\frac{K(x,y)(x-x_1)}{\prod_{j=1}^p(x-x_j)}=\frac{K(x,y)(x-x_1)}{S_1'(x,y)}, \tag{32}$$

by (27). Division of our linear combination by this expression gives Lemma 4.6. ∎

4.5. Extracting $Q^{(0,b)}(x,y)$: proof of Proposition 4.2

We now derive the expression for $Q^{(0,b)}(x,y)$ given in Proposition 4.2 from Lemma 4.6. In the identity of the lemma, both sides are power series in t whose coefficients are algebraic functions of x and y (and the z_r's). More precisely, these coefficients are written as *polynomials* in x, $\bar{x}$, y, $\bar{y}$, y_1 and in $x_1,\dots,x_p$ (due to Lemma 4.7), symmetric in $x_2,\dots,x_p$ (but not x_1). We think of them as Laurent series in $\mathbb{C}[z_0,\dots,z_p,1/z_p,x,\bar{x}]((\bar{y}))$ (Lemma 4.1). We extract from each coefficient the monomials that are nonnegative in x and y, and show that $Q(x,y)$ is the only contribution on the left-hand side — this is exactly what Proposition 4.2 says. We proceed line by line in the identity of Lemma 4.6.

In the first line, the term $Q(x,y)$ is clearly nonnegative in x and y. Then the coefficient of t^n in the sum over i is a polynomial in y, $\bar{x}$ and $x_1,\dots,x_p$, symmetric in the x_i's. Since the symmetric functions of the x_i's are x-nonpositive (Lemma 4.8), the second term of the first line only involves negative powers of x (because of the factor $\bar{x}$ before the sum), and thus the contribution of the first line reduces to $Q(x,y)$.

Let us show that the second line only involves negative powers of y. The coefficient of t^n in it is, up to a factor $\bar{y}$, a polynomial in y_1, $\bar{x}$, and $x_0=x,x_2,\dots,x_p,x_{p+1}=\bar{y}$, symmetric in the latter p variables $x_2,\dots,x_p,x_{p+1}$. By Proposition 4.4, the series $x_0=x,x_2,\dots,x_p,x_{p+1}=\bar{y}$ are the solutions of the equation $S(X,y_1)=S(x,y_1)$ (solved for X), $x_0=x$ being the trivial solution. By Lemma 4.8, applied with y replaced by y_1, the symmetric functions of $x_2,\dots,x_{p+1}$ are polynomials in $\bar{x}$ and y_1. In particular, they are y-nonpositive as y_1 itself, and so is the whole second line. It is even y-negative due to the factor $\bar{y}$. Hence the second line does not contribute in the extraction.

Let us finally consider the last term on the left-hand side. Things are a bit more delicate here: we are going to prove that every y-nonnegative monomial that occurs there is x-negative. We need the following lemma.

Lemma 4.9. *Let $\mathbb{A}=\mathbb{Q}(z_0,\dots,z_p)$. For a series $G(x,\bar{y})\in\mathbb{A}[x,\bar{x}]((\bar{y}))$, we say that G satisfies property $\mathcal{P}$ if all monomials $x^k\bar{y}^\ell$ (with $k,\ell\in\mathbb{Z}$) that occur in $G(x,\bar{y})$ satisfy $k\leqslant\ell/p$. Equivalently, $G(\bar{x}^p,x\bar{y})\in\mathbb{A}[x]((\bar{y}))$.*

Then the series x_1 defined in Lemma 4.1 satisfies $\mathcal{P}$, as well as all its (positive or negative) powers.

Note that Lemma 4.8 implies that the symmetric functions of $x_1,\dots,x_p$ satisfy $\mathcal{P}$. Also, any sum or product of series satisfying $\mathcal{P}$ still satisfies $\mathcal{P}$, and the "series" $\bar{x}$ and $\bar{y}$ satisfy $\mathcal{P}$.

We delay the proof of Lemma 4.9 to complete the proof of Proposition 4.2. We get back to the identity of Lemma 4.6. The coefficient of t^n in the sum

$$\sum_{i\neq 0} \frac{x_i^p\, Q(x_i,\bar{x})}{\prod_{j\neq i,0}(x_i-x_j)}$$

is a polynomial in $\bar{x}$ and $x_1, x_2, \ldots, x_{p+1} = \bar{y}$, symmetric in the latter $p+1$ variables. By the above observations, it satisfies $\mathcal{P}$. Now consider the product $\prod_{i=2}^{p}(1-\bar{x}x_i)$: it is a polynomial in $\bar{x}$ and $x_2, \ldots, x_p$, symmetric in the latter $p-1$ variables. If p_k is the kth power sum, we have of course

$$p_k(x_2,\ldots,x_p) = p_k(x_1,\ldots,x_p) - x_1^k.$$

We recall that power sums generate (as an algebra) all symmetric polynomials. Hence the above product is a polynomial in $\bar{x}, x_1$, and the elementary functions of $x_1,\ldots,x_p$. By the above observations and Lemma 4.9, it satisfies $\mathcal{P}$. So does the factor $(x_1-\bar{y})$. Hence every monomial occurring in the last part of the left-hand side in Lemma 4.6 is of the form $\bar{x}^2\bar{y}x^k\bar{y}^\ell$ with $k \leqslant \ell/p$. If it is nonnegative in y, it is x-negative, and thus cannot contribute in the extraction. ∎

Proof of Lemma 4.9. Recall that $X = x_1 = z_p\bar{x}y^p(1+O(\bar{y}))$ satisfies $S(x,y) = S(X,y)$, which we write as (21). Then we see that $G(x,\bar{y}) := \bar{X} = 1/x_1$ satisfies $\mathcal{P}$. Indeed, denoting $\tilde{G} = G(\bar{x}^p, x\bar{y})$, we have

$$z_p\tilde{G} = \bar{y}^p + \sum_{\ell=1}^{p}\sum_{\substack{i,j\geqslant 0\\ i+j\leqslant \ell}} z_{i+j+p-\ell}\, x^{ip+\ell}\bar{y}^\ell\tilde{G}^{j+1},$$

from which, by an inductive argument, it is clear that $\tilde{G} = \bar{y}^p/z_p(1+O(\bar{y}))$ is a series in $\bar{y}$ with polynomial coefficients in x. Moreover, since the first coefficient of $\tilde{G}$, being $1/z_p$, does not depend on x, property $\mathcal{P}$ holds as well for the reciprocal of G, which is $X = x_1$. ∎

4.6. Quadrant tandem walks starting at (a,b): the series $Q^{(a,b)}(x,y)$

We finally generalize the expression for $Q^{(0,b)}(x,y)$ given in Proposition 4.2 to quadrant tandem walks starting at an arbitrary position (a,b).

Proof of Proposition 4.3. We start from the functional equation (19), and adapt the solution presented earlier in this section to the case where a is not necessarily zero.

First, we generalize the section-free equation of Lemma 4.6. We follow step by step the proof of this lemma, given in Section 4.4. By Lemma 4.7, the linear combination (28) becomes

$$K(x,y)\sum_{i=0}^{p}\frac{x_i^p\, Q(x_i,y)}{\prod_{j\neq i,p+1}(x_i-x_j)} = h_a(x_0,\ldots,x_p)y^b - t\bar{y}\sum_{i=0}^{p}\frac{x_i^{p+1}\, Q(x_i,0)}{\prod_{j\neq i,p+1}(x_i-x_j)}. \tag{33}$$

Hence, denoting $\hat{\mathbf{x}}_k = (x_0,\ldots,x_{k-1},x_{k+1},\ldots,x_{p+1})$ for $0\leqslant k\leqslant p+1$, the counterpart of (29) is obtained by replacing y_k^{b+1} by $h_a(\hat{\mathbf{x}}_k)y_k^{b+1}$. We now rewrite the homogeneous symmetric functions $h_a(\hat{\mathbf{x}}_k)$ in a simpler fashion. Recall that x_{p+1} is defined to be $\bar{y}$, while $x_0, x_1, \ldots, x_p$ are the roots of $S(X,y)-S(x,y)$ (Lemma 4.1). In particular,

$$S(\bar{u},y) - S(x,y) = \bar{u}\bar{y}\prod_{i=0}^{p}(1-ux_i),$$

so that, for $k = 0, \dots, p+1$, we have

$$h_a(\hat{\mathbf{x}}_k) = [u^a] \frac{1 - ux_k}{\prod_{0 \leqslant i \leqslant p+1}(1 - ux_i)} = [u^a](1 - ux_k)C(u), \tag{34}$$

where we have defined

$$\begin{aligned} C(u) := \prod_{i=0}^{p+1} \frac{1}{1 - ux_i} &= \frac{1}{uy(1 - u\bar{y})(S(\bar{u}, y) - S(x, y))} \\ &= \frac{1}{(1 - ux)(1 - u\bar{y})(1 - u\bar{x}y \sum_{i+j+k<p} z_{i+j+k+1} u^i \bar{x}^j y^k)}, \end{aligned} \tag{35}$$

by (20).

Then we take the same linear combination of three equations as in the case $a = 0$, with weights given by (30). The left-hand side keeps the same form, while the right-hand side reads

$$\begin{aligned} &(x_0 - x_1)h_a(\hat{\mathbf{x}}_{p+1})y_{p+1}^{b+1} + (x_{p+1} - x_0)h_a(\hat{\mathbf{x}}_1)y_1^{b+1} + (x_1 - x_{p+1})h_a(\hat{\mathbf{x}}_0)y_0^{b+1} \\ &\quad = [u^a] \Big[\big((x_0 - x_1)(1 - ux_{p+1})y_{p+1}^{b+1} + (x_{p+1} - x_0)(1 - ux_1)y_1^{b+1} \\ &\qquad\qquad + (x_1 - x_{p+1})(1 - ux_0)y_0^{b+1} \big) C(u) \Big] \\ &\quad = \big((x_0 - x_1)y_{p+1}^{b+1} + (x_{p+1} - x_0)y_1^{b+1} + (x_1 - x_{p+1})y_0^{b+1} \big)[u^a]C(u) \\ &\qquad\qquad - \big((x_0 - x_1)y_{p+1}^{b} + (x_{p+1} - x_0)y_1^{b} + (x_1 - x_{p+1})y_0^{b} \big)[u^{a-1}]C(u). \end{aligned}$$

Let us write $\zeta_a := [u^a]C(u)$. We now return to the derivation (31) and conclude that the above expression is

$$(1 - \bar{x}\bar{y})(x - x_1)\big(\eta_b \zeta_a - \eta_{b-1}\zeta_{a-1}\big),$$

where $\eta_b := \sum_{k=0}^{b} (y^{k+1} - y_1^{k+1})\bar{x}^{b-k}$ as defined in Proposition 4.3 (note that $\eta_{-1} = 0$).

We then isolate $Q(x, y)$ by dividing the whole equation by (32), and thus obtain the counterpart of Lemma 4.6: the left-hand side is unchanged, while the right-hand side is

$$\frac{(1 - \bar{x}\bar{y})S_1'(x, y)}{K(x, y)} \big(\eta_b \zeta_a - \eta_{b-1}\zeta_{a-1}\big).$$

It remains to apply the operator $[x^{\geqslant} y^{\geqslant}]$. Again, the only term that survives on the left-hand side is $Q(x, y)$, and this concludes the proof. ∎

5. Final expressions for $Q^{(0,b)}(x, y)$

We still fix $p \geqslant 1$. Our aim now is to derive the expressions for $Q^{(0,b)}(x, y)$ given in Theorem 3.2 and Corollary 3.3 from Proposition 4.2. We begin with Theorem 3.2. As explained in the second remark following this proposition, it suffices to prove the case $y = 0$. By linearity (and Proposition 4.2), it is enough to prove the following lemma.

Lemma 5.1. *For $k \geqslant 0$, we have*

$$[y^0] \frac{(1 - \bar{x}\bar{y})S_1'(x, y)}{K(x, y)} (y^{k+1} - y_1^{k+1}) = \frac{Y_1^{k+1}}{tx} \left(1 - \frac{1}{tx^2} + \sum_{r=0}^{p} z_r (r+1) \bar{x}^{r+2} \right) + \frac{1}{tx^2} \mathbb{1}_{k=0}.$$

The proof that we will give is closely related to the proof of the equivalence of Propositions 18 and 19 in [10].

Recall that $Y_1 = xt + O(t^2)$ is the unique power series in t that cancels $K(x, Y)$. We will also need to handle the other roots of $K(x, Y) = 1 - tS(x, Y)$.

Lemma 5.2. *The equation $tS(x, Y) = 1$, when solved for Y, admits $p+1$ roots $Y_1, Y_2, \dots, Y_{p+1}$, taken as Puiseux series in t. Only Y_1 is a power series in t. The other roots are Laurent series in $t^{1/p}$ that contain some negative powers in t. They have coefficients in $\mathbb{C}[z_0, \dots, z_{p-1}, z_p^{1/p}, 1/z_p, x, \bar{x}]$.*

Proof. The equation $tS(x, Y) = 1$ reads

$$Y = t\left(x + \sum_{i+j\leqslant p} z_{i+j}\bar{x}^i Y^{j+1}\right).$$

If $t = 0$ this reduces to $Y = 0$, hence Y_1 is the unique power series solution. Its expansion in t can be computed iteratively from the equation, and its coefficients lie in $\mathbb{Q}[z_0, \dots, z_p, x, \bar{x}]$. The other roots $Y_2, \dots, Y_{p+1}$ thus involve negative powers of t. Writing $V = 1/Y$, the equation $tS(x, Y) = 1$, once multiplied by V^{p+1}, reads

$$V^p = txV^{p+1} + tz_p + t\sum_{i+j\leqslant p, j<p} z_{i+j}\bar{x}^i V^{p-j}. \tag{36}$$

The Newton polygon method [42, pp. 498–500] allows us to conclude that the p solutions $V_2, \dots, V_{p+1}$ read

$$V_j = \xi^j z_p^{1/p} t^{1/p}\left(1 + o(1)\right),$$

where ξ is a primitive pth root of unity, and have coefficients in $\mathbb{C}[z_0, \dots, z_{p-1}, z_p^{1/p}, 1/z_p, x, \bar{x}]$. The claimed properties of $Y_j = 1/V_j$ follow. ∎

Going back to Lemma 5.1, we need to extract the constant term in y from a series of the form $N(x, y)/K(x, y)$, where $N(x, y)$ is a Laurent series in $\bar{y}$. In our case

$$N(x, y) = (1 - \bar{x}\bar{y})S_1'(x, y)(y^{k+1} - y_1^{k+1}), \tag{37}$$

but, in the following lemma, we first focus on the case where N is a monomial in y.

Lemma 5.3. *Upon expanding $1/K(x, y)$ as a power series in t with coefficients in the ring $\mathbb{Q}[z_0, \dots, z_p, x, \bar{x}, y, \bar{y}]$, for $k \geqslant 0$ we have*

$$[y^0]\frac{y^k}{K(x, y)} = -\frac{1}{tz_p}Y_1^k \prod_{j\neq 1}\frac{1}{Y_1 - Y_j},$$

while for $k < 0$ we have

$$[y^0]\frac{y^k}{K(x, y)} = \frac{1}{tz_p}\sum_{i=2}^{p+1} Y_i^k \prod_{j\neq i}\frac{1}{Y_i - Y_j},$$

where the natural range of j is $[\![1, p+1]\!]$.

Proof. The partial fraction decomposition of $1/K(x,y)$ reads

$$\frac{1}{K(x,y)} = -\frac{y}{tz_p}\prod_j \frac{1}{y-Y_j} = -\frac{1}{tz_p}\sum_i \frac{Y_i}{y-Y_i}\prod_{j\neq i}\frac{1}{Y_i-Y_j}$$
$$= -\frac{1}{tz_p}\frac{\bar{y}Y_1}{1-\bar{y}Y_1}\prod_{j\neq 1}\frac{1}{Y_1-Y_j} + \frac{1}{tz_p}\sum_{i=2}^{p+1}\frac{1}{1-yY_i^{-1}}\prod_{j\neq i}\frac{1}{Y_i-Y_j}.$$

Recall that $Y_1 = O(t)$ while, for $i \geqslant 2$, $1/Y_i = O(t^{1/p})$. Hence, in the ring of series in $t^{1/p}$,

$$\frac{1}{K(x,y)} = -\frac{1}{tz_p}\sum_{k\geqslant 1} Y_1^k \bar{y}^k \prod_{j\neq 1}\frac{1}{Y_1-Y_j} + \frac{1}{tz_p}\sum_{k\geqslant 0}\sum_{i=2}^{p+1} Y_i^{-k} y^k \prod_{j\neq i}\frac{1}{Y_i-Y_j}.$$

For $k \neq 0$ this gives the claimed expression for $[y^0](y^k/K) = [\bar{y}^k](1/K)$. For $k=0$ it gives

$$[y^0]\frac{1}{K} = \frac{1}{tz_p}\sum_{i=2}^{p+1}\prod_{j\neq i}\frac{1}{Y_i-Y_j}.$$

However, by Lemma 4.7, we have $\sum_{i=1}^{p+1}\prod_{j\neq i}\frac{1}{Y_i-Y_j} = 0$, hence the claimed expression also holds for $k=0$. ∎

Proof of Lemma 5.1. For $N(x,y)$ a Laurent series in $\bar{y}$, we denote by $N_<(x,y) := [y^<]N(x,y)$ the negative part of N in y, and by $N_\geqslant(x,y) := [y^\geqslant]N(x,y)$ the nonnegative part. Then Lemma 5.3 gives

$$[y^0]\frac{N(x,y)}{K(x,y)} = -\frac{1}{tz_p}N_\geqslant(x,Y_1)\prod_{j\neq 1}\frac{1}{Y_1-Y_j} + \frac{1}{tz_p}\sum_{i=2}^{p+1}N_<(x,Y_i)\prod_{j\neq i}\frac{1}{Y_i-Y_j}. \tag{38}$$

For $N(x,y)$ given by (37), it is easy to express $N_<$ and $N_\geqslant$. Recall that $S_1'(x,y)$ has valuation -1 and degree $p-1$ in y, while $y_1 = x\bar{y}^p/z_p(1+O(\bar{y}))$ by Lemma 4.1. This gives

$$N_\geqslant(x,y) = y^{k+1}(1-\bar{x}\bar{y})S_1'(x,y) + \mathbb{1}_{k=0}\,\bar{x}\bar{y},$$
$$N_<(x,y) = -y_1^{k+1}(1-\bar{x}\bar{y})S_1'(x,y) - \mathbb{1}_{k=0}\,\bar{x}\bar{y}.$$

According to (38), we have to evaluate $N_<(x,y)$ at $y=Y_i$, for $i\geqslant 2$, and hence to evaluate the series $y_1 = 1/x_1$ at $y=Y_i$. In the following lemma, we emphasize the fact that y_1 depends on y (it is a power series in $\bar{y}$) with the notation $y_1(y)$.

Lemma 5.4. *Fix $i \in [\![2, p+1]\!]$. The series $1/Y_i$ is a power series in $t^{1/p}$ with no constant term. Hence $y_1(Y_i)$ is a formal power series in $t^{1/p}$, which in fact equals Y_1.*

Proof. The first statement follows from Lemma 5.2, so it remains to identify $y_1(Y_i)$. Recall that $S(x,y_1(y)) = S(x,y)$. Replacing y by Y_i gives $S(x,y_1(Y_i)) = S(x,Y_i)$. Since Y_i is a root (in y) of $K(x,Y) = 1-tS(x,Y)$, it follows that $K(x,y_1(Y_i)) = 1-tS(x,y_1(Y_i)) = 0$ as well. Hence $y_1(Y_i)$ is one of the Y_j's. But Y_1 is the only Y_j that does not contain negative powers of t (Lemma 5.2), and we conclude that $y_1(Y_i) = Y_1$. ∎

We can now apply (38). This gives

$$[y^0]\frac{N(x,y)}{K(x,y)} = -\frac{1}{tz_p}\sum_{i=1}^{p+1}\left(Y_1^{k+1}(1-\bar{x}/Y_i)S_1'(x,Y_i) + \mathbb{1}_{k=0}\,\bar{x}/Y_i\right)\prod_{j\neq i}\frac{1}{Y_i-Y_j}.$$

We will evaluate this sum using Lemma 4.7. The Laurent polynomial $P(y) := (1 - \bar{x}/y)S_1'(x,y)$ has degree $p-1$ in y, and valuation -2. Moreover,

$$P_{-2} := [\bar{y}^2]P(y) = -\bar{x}, \qquad \text{and} \qquad P_{-1} := [\bar{y}]P(y) = 1 + \sum_{r=0}^{p} z_r r \bar{x}^{r+2}.$$

Hence, by Lemma 4.7,

$$[y^0]\frac{N(x,y)}{K(x,y)} = -\frac{1}{tz_p}\frac{(-1)^p}{\prod_i Y_i}\left(Y_1^{k+1}P_{-1} + \bar{x}\mathbb{1}_{k=0} + Y_1^{k+1}P_{-2}h_1(1/Y_1,\dots,1/Y_{p+1})\right).$$

The elementary symmetric functions of $1/Y_1,\dots,1/Y_{p+1}$ are easily computed using the fact that each of them is a root V of (36). One finds

$$e_{p+1}(1/Y_1,\dots,1/Y_{p+1}) = \frac{1}{\prod_i Y_i} = (-1)^{p-1}\bar{x}z_p$$

and

$$e_1(1/Y_1,\dots,1/Y_{p+1}) = \frac{1}{tx}\left(1 - t\sum_{r=0}^{p} z_r\bar{x}^r\right).$$

Since $e_1 = h_1$, this gives the expression of Lemma 5.1. ∎

Proof of Corollary 3.3. We start from the expression for $Q^{(0,b)}(x,y)$ given in Theorem 3.2. By linearity, it suffices to prove that, for $0 \leqslant k \leqslant b$,

$$-Y_1^{k+1} = [z^0]\frac{tz^{k+2}S_2'(x,z)}{K(x,z)}. \tag{39}$$

The numerator occurring on the right-hand side is a polynomial in z, because $S_2'(x,z)$ has valuation -2 in z. Hence the first part of Lemma 5.3 tells us that

$$[z^0]\frac{tz^{k+2}S_2'(x,z)}{K(x,z)} = -\frac{1}{z_p}Y_1^{k+2}S_2'(x,Y_1)\prod_{j\neq 1}\frac{1}{Y_1 - Y_j}. \tag{40}$$

Upon writing

$$S(x,z) - 1/t = S(x,z) - S(x,Y_1) = -\frac{1}{t}K(x,z) = z_p\bar{z}\prod_{i=1}^{p+1}(z - Y_i),$$

we can compute

$$S_2'(x,Y_1) = \frac{z_p}{Y_1}\prod_{i=2}^{p+1}(Y_1 - Y_i),$$

which, combined with (40), gives (39). An alternative, purely combinatorial proof of (39) in terms of one-dimensional lattice walks is given in Section 9.4.2. ∎

6. Quadrant walks with arbitrary endpoint: algebraic solution

We are now going to prove Theorem 3.4, which gives an explicit algebraic expression for the series $Q^{(a,b)}(1,1)$. We still work in the p-specialization, for $p \geqslant 1$. We write $Q(x,y) \equiv Q^{(a,b)}(x,y)$.

The proof that we give generalizes the proof given for $p = 1$ in [18, Sec. 5.2]. In our basic functional equation (19), let us replace the pair (x, y) by $(x_i, \bar{x})$, with $1 \leqslant i \leqslant p$. By Proposition 4.4, $K(x_i, \bar{x}) = K(x, y)$. We thus obtain

$$K(x,y)Q(x_i,\bar{x}) = x_i^a \bar{x}^b - t x_i x Q(x_i, 0) - \sum_{k=1}^{p} \bar{x}_i^k G_k(\bar{x}). \tag{41}$$

We now consider the linear combination (33). It involves, on the right-hand side, the series $Q(x, 0)$, and then $Q(x_i, 0)$ for $1 \leqslant i \leqslant p$. By taking a linear combination with (41), for $1 \leqslant i \leqslant p$, we can eliminate the latter p series — upon introducing the series $G_k(\bar{x})$. More precisely,

$$\begin{aligned} K(x,y)\left(y \sum_{i=0}^{p} \frac{x_i^p\, Q(x_i,y)}{\prod_{j\neq i,p+1}(x_i - x_j)} - \bar{x} \sum_{i=1}^{p} \frac{x_i^p\, Q(x_i,\bar{x})}{\prod_{j\neq i,p+1}(x_i - x_j)} \right) \\ = h_a(x_0,\dots,x_p) y^{b+1} - t\, \frac{x^{p+1} Q(x,0)}{\prod_{j\neq 0,p+1}(x - x_j)} \\ - \bar{x} \sum_{i=1}^{p} \frac{x_i^p}{\prod_{j\neq i,p+1}(x_i - x_j)} \left(x_i^a \bar{x}^b - \sum_{k=1}^{p} \bar{x}_i^k G_k(\bar{x}) \right). \end{aligned} \tag{42}$$

We now simplify the right-hand side. The coefficient of $Q(x, 0)$ can be rewritten in terms of $S_1'(x, y)$ due to (27). The sums over i can be evaluated in closed form using (27) again, and Lemma 4.7. First, since $a \geqslant 0$,

$$\begin{aligned} \sum_{i=1}^{p} \frac{x_i^{p+a}}{\prod_{j\neq i,p+1}(x_i - x_j)} &= \sum_{i=0}^{p} \frac{x_i^{p+a}}{\prod_{j\neq i,p+1}(x_i - x_j)} - \frac{x^{p+a}}{\prod_{j=1}^{p}(x - x_j)} \\ &= h_a(x_0, x_1, \dots, x_p) - \frac{x^{p+a}}{\prod_{j=1}^{p}(x - x_j)} && \text{by Lemma 4.7} \\ &= h_a(x_0, x_1, \dots, x_p) - \frac{x^a \bar{y}}{S_1'(x,y)} && \text{by (27).} \end{aligned}$$

Similarly, for $1 \leqslant k \leqslant p$,

$$\begin{aligned} \sum_{i=1}^{p} \frac{x_i^{p-k}}{\prod_{j\neq i,p+1}(x_i - x_j)} &= \sum_{i=0}^{p} \frac{x_i^{p-k}}{\prod_{j\neq i,p+1}(x_i - x_j)} - \frac{x^{p-k}}{\prod_{j=1}^{p}(x - x_j)} \\ &= -\frac{x^{p-k}}{\prod_{j=1}^{p}(x - x_j)} && \text{by Lemma 4.7} \\ &= -\frac{\bar{x}^k \bar{y}}{S_1'(x,y)} && \text{by (27).} \end{aligned}$$

Hence the right-hand side of (42) simplifies to

$$h_a(x_0,\dots,x_p)\left(y^{b+1} - \bar{x}^{b+1}\right) + \frac{\bar{x}\bar{y}}{S_1'(x,y)} \left(x^{a-b} - t x^2 Q(x,0) - \sum_{k=1}^{p} \bar{x}^k G_k(\bar{x}) \right).$$

The rightmost term can be expressed in terms of $Q(x,\bar{x})$. Indeed, specializing the main functional equation (19) to the case $y=\bar{x}$, we obtain

$$K(x,\bar{x})Q(x,\bar{x}) = x^{a-b} - tx^2Q(x,0) - \sum_{k=1}^{p} \bar{x}^k G_k(\bar{x}).$$

We can thus rewrite the right-hand side of (42) as

$$h_a(x_0,\dots,x_p)\left(y^{b+1}-\bar{x}^{b+1}\right) + \frac{\bar{x}\bar{y}}{S_1'(x,y)}K(x,\bar{x})Q(x,\bar{x}). \tag{43}$$

Recall from (34) and (35) that

$$\begin{aligned} h_a(x_0,\dots,x_p) = h_a(\hat{\mathbf{x}}_{p+1}) &= [u^a]\frac{1}{uy(S(\bar{u},y)-S(x,y))} \\ &= [u^a]\frac{1}{(1-ux)(1-u\bar{x}y\sum_{i+j+k<p} z_{i+j+k+1}u^i\bar{x}^jy^k)} \\ &:= D_a(x,y). \end{aligned} \tag{44}$$

Equation (42), with its right-hand side written as (43), holds for indeterminates x and y, where $x_1,\dots,x_p$ are the roots of $S(X,y)=S(x,y)$ distinct from x. By Lemma 4.4, the x_i's can also be described as the roots X of $S(x,1/X)=S(x,y)$ distinct from $\bar{y}$. Observe that we have not used the fact that we usually take them as Puiseux series in $\bar{y}$. We can choose them in any algebraic closure of $\mathbb{Q}(z_1,\dots,z_p,x,y)$, and (42) still holds. We now specialize y to Y_1, which is the unique power series in t satisfying $K(x,y)=1-tS(x,y)=0$. Then the corresponding values $x_1,\dots,x_p$ are the roots X of $S(x,1/X)=S(x,Y_1)$ distinct from $1/Y_1$, or equivalently the roots X of $K(x,1/X)=0$ distinct from $1/Y_1$. We choose to take them as Puiseux series in t, hence they are in fact the series $1/Y_2,\dots,1/Y_{p+1}$, with the Y_i's defined in Lemma 5.2. Note that each $1/Y_i$, for $i\geqslant 2$, is a formal power series in $t^{1/p}$ with no constant term.

With these values of $x_1,\dots,x_p$, the series $Q(x_i,Y_1)$ and $Q(x_i,\bar{x})$ occurring in (42) (specialized to $y=Y_1$) are well defined power series in $t^{1/p}$. But since $K(x,Y_1)=0$, the left-hand side of (42) vanishes, and so does its right-hand side, which we have simplified to (43). We thus obtain

$$K(x,\bar{x})Q(x,\bar{x}) = D_a(x,Y_1)\left(\bar{x}^{b+1}-Y_1^{b+1}\right)xY_1S_1'(x,Y_1). \tag{45}$$

We now want to express $S_1'(x,Y_1)$. Since the series $x_1,\dots,x_p$ are $1/Y_2,\dots,1/Y_{p+1}$ when $y=Y_1$, the specialization of (27) at $y=Y_1$ reads

$$Y_1S_1'(x,Y_1) = \prod_{i=2}^{p+1}(1-\bar{x}/Y_i).$$

On the other hand, $K(x,y)$ factors as

$$K(x,y) = \frac{tx}{Y_1}(1-\bar{y}Y_1)\prod_{i=2}^{p+1}(1-y/Y_i),$$

so that

$$K(x,\bar{x}) = \frac{tx}{Y_1}(1-xY_1)\prod_{i=2}^{p+1}(1-\bar{x}/Y_i).$$

With these two identities, Equation (45) gives

$$Q(x,\bar{x}) = \frac{Y_1}{tx} D_a(x,Y_1) \frac{\bar{x}^{b+1} - Y_1^{b+1}}{\bar{x} - Y_1}.$$

If $x = 1$, the series Y_1 specializes to W, and we obtain

$$Q(1,1) = \frac{W}{t} D_a(1,W) \frac{1 - W^{b+1}}{1 - W},$$

where $D_a(x,y)$ is defined by (44). This gives the expression of Theorem 3.4. ∎

Remark. We have first obtained an expression for the series $Q(x,\bar{x})$, which counts tandem walks starting at (a,b) with a weight x^{i-j} for walks ending at (i,j). Then we have specialized this expression to $x = 1$. There is no loss of information in this specialization. Indeed, since every SE step lets $i-j$ increase by 2 and each face step of level r lets $i-j$ decrease by r, the series $x^{b-a}Q(x,\bar{x})$ is equal to $Q(1,1)$ where t is replaced by tx^2 and z_r by $z_r x^{-r-2}$.

7. Bijective proofs

In this section we give combinatorial proofs of the expression for $Q^{(0,b)}(x,0)$ given in Theorem 3.2 and of the expression for $Q^{(a,b)}(1,1)$ given in Theorem 3.4. In both cases we use the KMSW bijection to interpret our series as generating functions of marked bipolar orientations, and perform simple transformations on these orientations to establish the identities.

7.1. The expression for $Q^{(0,b)}(x,0)$

For $i,b \geqslant 0$ let us denote by $Q_i^{(b)} := [x^i]Q^{(0,b)}(x,0)$ the generating function of tandem walks starting at $(0,b)$, staying in the quadrant, and ending at $(i,0)$. The variable t records the number of steps, x the final x-coordinate, and z_r the number of face steps of level r. Similarly, let $H_i^{(b)}$ be the series that counts tandem walks starting at $(0,b)$, staying in the half-plane $\{y \geqslant 0\}$, and ending at $(i,0)$. As explained after the definition (8) of Y_1, we have

$$Y_1 = tx \sum_{i\in\mathbb{Z}} x^i H_i^{(0)},$$

so that Y_1 counts tandem walks starting at the origin, ending on the line $\{y = -1\}$, but staying on or above the x-axis until this last step. By concatenating $b+1$ such walks, and translating the resulting walk b steps up, we see that Y_1^{b+1} counts tandem walks starting at $(0,b)$, ending on the line $\{y = -1\}$ and staying in $\{y \geqslant 0\}$ up to the last step. In other words,

$$Y_1^{b+1} = tx \sum_{i\in\mathbb{Z}} x^i H_i^{(b)} =: txH^{(b)}(x). \tag{46}$$

Through the KMSW bijection described in Section 2 (see in particular Figure 9), the series $H_i^{(b)}$ counts marked bipolar orientations of signature $(a,b;a+i,0)$, for some $a \geqslant 0$, where t records the number of plain edges minus 1 — called the *size* of the orientation — and z_r the number of inner faces of degree $r+2$. These orientations have no dashed edge on their right boundary. The series $Q_i^{(b)}$ counts those that, in addition, have no dashed edge on the left boundary. That is, those for which $a = 0$.

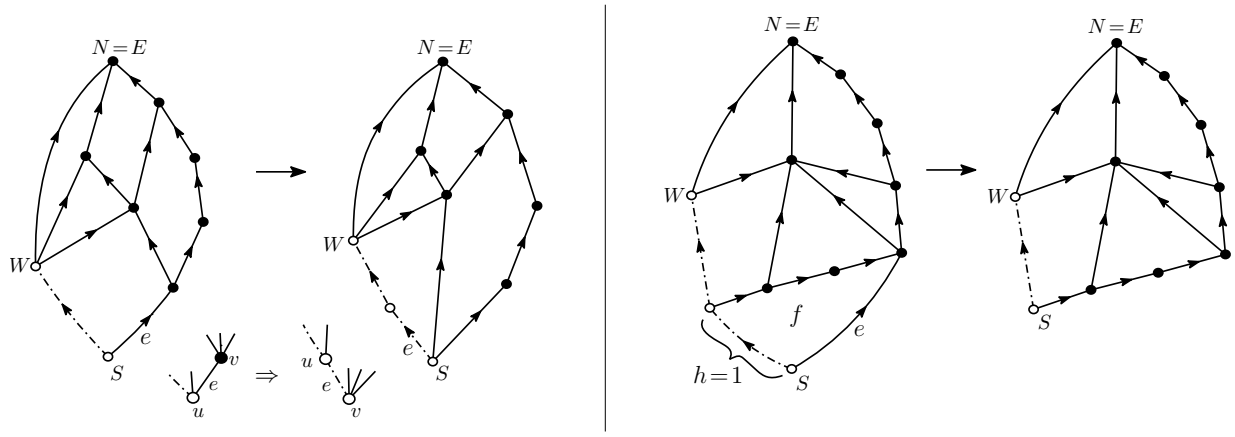


FIGURE 11. Left: sliding the bottom-right boundary edge from right to left. Right: erasing the inner face incident to the bottom-right edge.

We now consider an orientation O counted by $H_i^{(b)}$. Let e be the bottom edge of the right outer boundary of O, directed from S to a vertex v. Then e is necessarily plain. Let f be the face on its left. Three cases occur:

(0) f is the outer face,
(1) f is an inner face and e is the unique ingoing edge at v,
(2) f is an inner face and there are several ingoing edges at v.

Accordingly, O will be said to be of type 0, 1 or 2. The associated generating functions are denoted $H_{0,i}^{(b)}$, $H_{1,i}^{(b)}$ and $H_{2,i}^{(b)}$. Clearly,

$$H_i^{(b)} = H_{0,i}^{(b)} + H_{1,i}^{(b)} + H_{2,i}^{(b)}. \tag{47}$$

Lemma 7.1. *For $i \geqslant 0$, the above defined series satisfy:*

$$H_{0,i}^{(b)} = \mathbb{1}_{i=b=0} + tQ_{i-1}^{(b-1)}, \tag{48}$$

$$H_{1,i+2}^{(b)} = t\big(H_i^{(b)} - Q_i^{(b)}\big), \tag{49}$$

$$H_{2,i}^{(b)} = t\sum_{r\geqslant 0}(r+1)z_r H_{i+r}^{(b)}. \tag{50}$$

Proof. We adopt the notation $(e,\, f)$ used above in the definition of the types.

Let O be an orientation of type 0, counted by $H_{0,i}^{(b)}$. By definition e is plain, so that $a = 0$. Erasing e leaves a bipolar orientation of signature $(0, b-1; i-1, 0)$, or just a single point if O is reduced to the edge e (in which case $b = i = 0$). This gives the first identity.

Now let O be a marked bipolar orientation of type 1 counted by $H_{1,i+2}^{(b)}$. Its signature is of the form $(a, b; a+i+2, 0)$ for some $a \geqslant 0$. Let O' be obtained by reversing the orientation of e, and transforming it into a dashed edge (Figure 11, left). Graphically, we "slide" the bottom edge e from the right to the left boundary. It is easy to see that we thus obtain a marked bipolar orientation with source v, of signature $(a+1, b; a+1+i, 0)$, containing at least one dashed edge (on the left boundary). Conversely, let us start from a marked bipolar orientation O' of signature $(\alpha, b; \alpha+i, 0)$ for some $\alpha \geqslant 1$. Such orientations are counted by $H_i^{(b)} - Q_i^{(b)}$. Since the vertex lying just above the source on the left boundary has indegree 1, we can reverse the direction of the bottom dashed edge, and make it plain:

this gives a marked bipolar orientation O of type 1, of signature $(a, b; a+i+2, 0)$ with $a = \alpha - 1$, and thus counted by $H^{(b)}_{1,i+2}$. This correspondence yields the second identity of the lemma.

Finally let O be a marked bipolar orientation of type 2 counted by $H^{(b)}_{2,i}$, of signature $(a, b; i+a, 0)$ for some $a \geqslant 0$. Let $r+2$ be the degree of the face f and let $h \geqslant 0$ be the number of dashed edges on the left boundary of f. Since v has indegree at least 2, the top edge of the left boundary of f cannot be dashed, by definition of marked bipolar orientations. Hence $h \leqslant r$. If $h > 0$, then the h dashed edges of f form a path P of length h on the left boundary of f, starting from S; we call P the *bottom-left path* of f.

Let O' be obtained by erasing e and P, thereby choosing the top vertex of P as the new source (Figure 11, right). The size has decreased by one, and the signature of O' is $(a-h, b; a-h+i+r, 0)$. Thus O' is a marked bipolar orientation counted by $H^{(b)}_{i+r}$.

Conversely, consider a marked bipolar orientation O' counted by $H^{(b)}_{i+r}$, and let $(\alpha, b; \alpha + i + r, 0)$ be its signature. For $0 \leqslant h \leqslant r$ consider the operation of attaching a path P of h dashed edges below the source S of O', choosing the bottom vertex of P as the new source, and then adding a new edge e so that it becomes the bottom-edge of the right outer boundary and encloses an inner face of degree $r+2$. We obtain a marked bipolar orientation O counted by $H^{(b)}_{2,i}$, of signature $(a, b; a+i, 0)$, where $a = \alpha + h$. The resulting correspondence thus gives

$$H^{(b)}_{2,i} = t \sum_{r \geqslant 0} \sum_{h=0}^{r} z_r H^{(b)}_{i+r} = t \sum_{r \geqslant 0} (r+1) z_r H^{(b)}_{i+r},$$

which concludes the proof of the lemma. ∎

Proof of Theorem 3.2. We now establish the expression (10) of $Q^{(0,b)}(x, 0)$. As explained below Theorem 3.2, this suffices to prove the entire proposition. For $i, b \geqslant 0$,

$$\begin{aligned} Q^{(b)}_i = H^{(b)}_i - (H^{(b)}_i - Q^{(b)}_i) &= H^{(b)}_i - \frac{1}{t} H^{(b)}_{1,i+2} && \text{by (49)} \\ &= H^{(b)}_i - \frac{1}{t}\Big(H^{(b)}_{i+2} - tQ^{(b-1)}_{i+1} - H^{(b)}_{2,i+2}\Big) && \text{by (47) and (48)} \\ &= Q^{(b-1)}_{i+1} + H^{(b)}_i - \frac{1}{t} H^{(b)}_{i+2} + \sum_{r \geqslant 0} z_r (r+1) H^{(b)}_{i+r+2} && \text{by (50).} \end{aligned}$$

By convention, $Q^{(-1)}_{i+1} = 0$. Recall that $Q^{(0,b)}(x, 0) = \sum_{i \geqslant 0} x^i Q^{(b)}_i$. Recall the definition of $H^{(b)}(x)$ in (46), and let $A(x) := 1 - \frac{1}{tx^2} + \sum_{r \geqslant 0} z_r (r+1) \bar{x}^{r+2}$. Multiplying the above identity by x^i, and summing over $i \geqslant 0$, we get

$$\begin{aligned} Q^{(0,b)}(x, 0) &= [x^{\geqslant}] \left(\bar{x} Q^{(0,b-1)}(x, 0) + A(x) H^{(b)}(x)\right) \\ &= [x^{\geqslant}] \left(\bar{x} Q^{(0,b-1)}(x, 0) + A(x) \frac{Y_1^{b+1}}{tx}\right). \end{aligned}$$

The case $y = 0$ of Theorem 3.2 then follows by induction on $b \geqslant 0$, due to the property, already used in Section 3, that $[x^{\geqslant}](\bar{x}[x^{\geqslant}]G(x)) = [x^{\geqslant}](\bar{x}G(x))$ for any series G. ∎

Remark. There is an analogy between the combinatorial proof given above and an argument used by Bouttier and Guitter in [21, Sec. 3.3]. Their aim is to determine the generating function M_i of rooted planar maps with outer degree i, with t recording the number of

edges and z_r the number of inner faces of degree r (the notation is ours). They consider the generating function N_i of a larger class: rooted planar maps of outer degree i with an additional marked vertex v (with the condition that the root-vertex minimizes the distance to v among all outer vertices). The generating function N_i is easy to compute using a bijection with certain labelled trees (called *mobiles*). Then they express M_i as $N_i - (N_i - M_i)$, and determine $N_i - M_i$ using a local operation consisting in "opening" the first edge on the leftmost geodesic path from the root-vertex to the marked vertex (Figure 6 in [21]).

Similarly, to determine the generating function $Q_i^{(b)}$ of bipolar orientations, we consider the larger class of *marked* bipolar orientations counted by $H_i^{(b)}$. We express $Q_i^{(b)}$ as $H_i^{(b)} - (H_i^{(b)} - Q_i^{(b)})$. Then $H_i^{(b)}$ is computable due to the bijection with tandem walks in the upper half-plane, while $H_i^{(b)} - Q_i^{(b)}$ is determined via the local operation consisting in "sliding" the lower left outer edge to the right boundary (Figure 11, left).

7.2. The expression for $Q^{(a,b)}(1,1)$

We will now give a combinatorial proof of Theorem 3.4 using the involution σ on marked bipolar orientations defined in Definition 2.3, and illustrated in Figure 10. We recall the expression of Theorem 3.4 here, but for convenience we exchange the roles of the indices (i,j) and (a,b): for $i,j \geqslant 0$,

$$Q^{(i,j)}(1,1) = \frac{W}{t} \cdot \sum_{a=0}^{i} A_a \cdot \sum_{b=0}^{j} W^b, \tag{51}$$

where $W = Y_1(1)$ satisfies (15) and A_a is the series in W and the z_r's defined by (16).

Let $\mathcal{Q}^{(i,j)}$ be the family of tandem walks starting at (i,j) and staying in the quadrant. For $a,b \geqslant 0$, let $\mathcal{H}^{b\to a}$ be the family of tandem walks starting at $(0,b)$, staying in the upper half-plane $\{y \geqslant 0\}$, reaching the x-axis at least once, and ending on the line $\{y = a\}$.

Recall from Section 2 that for a walk w, the signature of the marked bipolar orientation $\Phi(w)$, where Φ is the KMSW bijection, is given by (6). Recall also the definition of the involution σ on marked bipolar orientations (Definition 2.3).

Proposition 7.2. *The mapping $\Phi^{-1} \circ \sigma \circ \Phi$ is an involution on non-embedded tandem walks (seen as sequences of steps), which exchanges $a = x_{\text{start}} - x_{\min}$ and $d = y_{\text{end}} - y_{\min}$, while preserving $b = y_{\text{start}} - y_{\min}$ and $c = x_{\text{end}} - x_{\min}$. It also preserves the length, the number of SE steps, and the number of face steps of each level r.*

Upon embedding walks appropriately, it induces a bijection between $\mathcal{Q}^{(i,j)}$ and $\bigcup_{0\leqslant a\leqslant i}\bigcup_{0\leqslant b\leqslant j}\mathcal{H}^{b\to a}$, preserving the same statistics.

Proof. The first part directly follows from the properties of Φ and σ (see Theorem 2.1, Eq. (6) and Definition 2.3).

For the second part, we simply fix i and j and restrict $\Phi^{-1} \circ \sigma \circ \Phi$ to paths w such that $a = x_{\text{start}} - x_{\min} \leqslant i$ and $b = y_{\text{start}} - y_{\min} \leqslant j$. We embed them so that they start at (i,j): then they are exactly the walks of $\mathcal{Q}^{(i,j)}$. Then we embed the walks obtained by applying $\Phi^{-1} \circ \sigma \circ \Phi$ so that they start at $(0,b)$. This gives the announced result, illustrated by Figure 12. ∎

Due to the above proposition, in order to prove (51) it now suffices to prove a half-plane result dealing with $\mathcal{H}^{b\to a}$. But half-plane problems are in essence problems of walks on a half-line, and hence much simpler and perfectly understood.

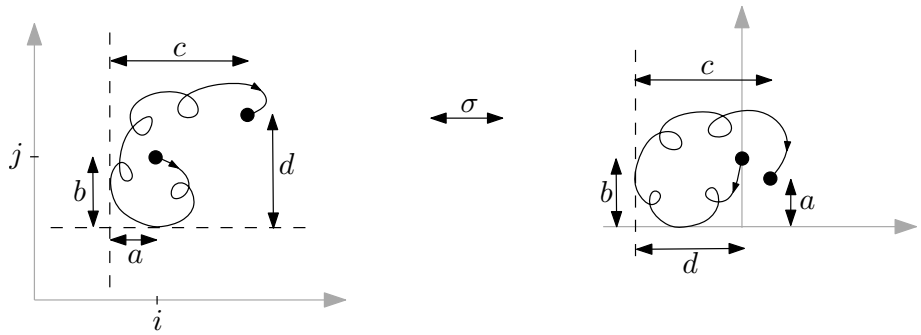


FIGURE 12. The bijection of Proposition 7.2 (derived from σ) between $\mathcal{Q}^{(i,j)}$ and $\bigcup_{0\leqslant a\leqslant i}\bigcup_{0\leqslant b\leqslant j}\mathcal{H}^{b\to a}$.

Lemma 7.3. *For $a, b \geqslant 0$ the generating function of $\mathcal{H}^{b\to a}$ (with t recording the length and z_r the number of face steps of level r) is $\frac{W}{t}A_aW^b$, with A_a defined by (16).*

Proof. Each walk of $\mathcal{H}^{b\to a}$ can be uniquely factored into a walk of $\mathcal{H}^{b\to 0}$ hitting the x-axis *only once*, followed by a walk of $\mathcal{H}^{0\to a}$. With the notation used at the beginning of Section 7.1, walks of $\mathcal{H}^{b\to 0}$ hitting the x-axis only once are counted by $txH^{(b-1)}(x) = Y_1(x)^b$ if we keep track of the abscissa of the endpoint (with the variable x), and thus by $W^b = Y_1(1)^b$ if we do not. It thus suffices to prove that half-plane walks going from the origin to ordinate a are counted by $\frac{W}{t}A_a$. However, it is a classical one-dimensional result [3, 19, 45], obtained in one line using the so-called *kernel method*, that their generating function is

$$\sum_{a\geqslant 0} H^{0\to a}u^a = \frac{1-\bar{u}W}{K(1,u)}. \tag{52}$$

However, we have

$$\begin{aligned} K(1,u) &= K(1,u) - K(1,W) \\ &= t(S(1,W) - S(1,u)) \\ &= t(u-W)\left(\frac{\bar{u}}{W} - \sum_{i,j,k\geqslant 0} z_{i+j+k+1}W^j u^k\right) \qquad \text{by (24).} \end{aligned}$$

Combined with (52), this gives

$$\sum_{a\geqslant 0} H^{0\to a}u^a = \frac{W}{t}\cdot\frac{1}{1-uW\sum_{j,k\geqslant 0}W^j u^k \sum_{r>j+k} z_r}, \tag{53}$$

which concludes the proof of the lemma and of Theorem 3.4. We have relied on a classical one-dimensional result to obtain (53), but we also give a purely combinatorial proof of the latter identity in Section 9.4.1. ∎

Remark. Let us return to the *double-tandem walks* defined just before Section 3.4. We believe that the first statement of Proposition 7.2 also holds for these walks, or more precisely, that there exists a length-preserving involution on (non-embedded) double-tandem walks that exchanges $a = x_{\text{start}} - x_{\min}$ and $d = y_{\text{end}} - y_{\min}$, while preserving $b = y_{\text{start}} - y_{\min}$ and $c = x_{\text{end}} - x_{\min}$. Moreover, this involution would preserve the total number of steps in $\{N, W, SE\}$.

We have tested this conjecture numerically as follows. First, we embed walks canonically so as to have $x_{\min} = y_{\min} = 0$. That is, walks are now confined to the quadrant and touch both boundaries. We let $\widetilde{D}[a,b,c,d;\ell,m]$ be the number of such walks, with double-tandem steps, that go from (a,b) to (c,d), have ℓ steps in $\{N,W,SE\}$ and m steps in $\{S,E,NW\}$. Our conjecture translates into

$$\widetilde{D}[a,b,c,d;\ell,m] = \widetilde{D}[d,b,c,a;\ell,m]. \tag{54}$$

In order to compute these numbers, we let $D[a,b,c,d;\ell,m]$ be the corresponding numbers for quadrant walks that do not necessarily touch both boundaries. Then

$$\begin{aligned}\widetilde{D}[a,b,c,d;\ell,m] = D[a,b,c,d;\ell,m] - D[a-1,b,c-1,d;\ell,m]\\ - D[a,b-1,c,d-1;\ell,m] + D[a-1,b-1,c-1,d-1;\ell,m].\end{aligned}$$

The reflection principle [47], or equivalently the approach of [18] for quadrant walks with small steps (see in particular Prop. 10), gives

$$D[a,b,c,d;\ell,m] = \\ [x^c y^d s^\ell t^m]\frac{x^{a+1}y^{b+1} - \bar{x}^{a+1}y^{a+b+2} + \bar{x}^{a+b+2}y^{a+1} - \bar{x}^{b+1}\bar{y}^{a+1} + x^{a+1}\bar{y}^{a+b+2} - x^{a+b+2}\bar{y}^{b+1}}{1 - s(\bar{x}+y+x\bar{y}) - t(x+\bar{y}+\bar{x}y)},$$

which yields an efficient computation method to test (54). It also yields a formula for $\widetilde{D}[a,b,c,d;\ell,m]$, as a sum involving multinomial coefficients. We have not tried to derive (54) from it, but this may be possible.

For $a = b = 0$, this involution would give a new bijective proof for the equi-enumeration of double-tandem walks in the quadrant ending anywhere and double-tandem walks in the upper half-plane ending on the x-axis (see the final remark in Section 3.3). It would also extend the involution of Proposition 7.2 for $p = 1$ (by taking all steps in $\{N,W,SE\}$).

We have not found any counterpart of this conjecture for $p \geqslant 2$.

8. Asymptotic enumeration

This section is devoted to the proof of Theorem 3.5 on the asymptotic enumeration of p-tandem walks with prescribed starting point (a,b) and endpoint (c,d), with a weight z_r for face steps of level r (for $0 \leqslant r \leqslant p$). We proceed by a reduction to a random walk model with zero drift. We first compute (due to the expression for $Q^{(a,b)}(1,1)$ given in Theorem 3.4) an asymptotic estimate of the probability that a random walk starting from (a,b) stays in the quadrant at least up to time n. Then we adapt recent results of Denisov and Wachtel [29] to our setting to derive an asymptotic estimate of the probability that a random walk starting at (a,b) stays in the quadrant at least up to time n *and* ends at a prescribed point (c,d). The reason why an adaptation is required is that the results of Denisov and Wachtel require an aperiodicity condition which does not hold for p-tandem walks.

An alternative approach would be to apply analytic combinatorics in several variables (ACSV) [74] to the explicit expression for $Q^{(a,b)}(x,y)$ given in Proposition 4.3 as the non-negative part of an algebraic series; this would yield, in theory, full asymptotic expansions, but these techniques are highly involved, especially with the complicated algebraic expression that we have. If $a = 0$, a more convenient starting point would be Corollary 3.3, where $Q^{(0,b)}(x,y)$ is expressed in terms of a *rational* function. For instance, starting from the expression (14) for $Q^{(0,0)}(0,0)$, Marni Mishna was able to work out the asymptotic number

of excursions when all face steps have level p, for $p = 3$ and $p = 4$, and it is possible that this could be extended to arbitrary p (personal communication). We refer to [10, 65, 67] for recent applications of ACSV to the enumeration of walks confined to cones.

In Section 8.1 we state our probabilistic results. We then derive from them asymptotic estimates for the weighted number of tandem walks in Section 8.2. We finally prove the results of Section 8.1 in Sections 8.3 and 8.4.

8.1. Random tandem walks and discrete harmonic functions

In this section, we fix a $(p+2)$-tuple $(z, z_0, \dots, z_p)$ of nonnegative reals (with $z_p > 0$) such that

$$z + \sum_{r=0}^{p} (r+1) z_r = 1. \tag{55}$$

We define a step distribution in $\mathbb{Z}^2$ by

$$\mathbb{P}((X,Y) = (i,j)) = \begin{cases} z, & \text{if } (i,j) = (1,-1), \\ z_r, & \text{if } j = i + r \text{ and } 0 \leqslant j \leqslant r, \\ 0, & \text{otherwise.} \end{cases} \tag{56}$$

To avoid trivialities, we assume throughout the section that $p \geqslant 1$. We then consider the *random tandem walk* that starts at (a,b) and takes each step independently under the above distribution. The point reached by the walk after n steps is denoted by $S^{(a,b)}(n)$, and we let $\tau^{(a,b)} \in \mathbb{N} \cup \{\infty\}$ denote the first time that the random walk exits the quadrant. The *drift* $(\mathbb{E}(X), \mathbb{E}(Y))$ of this walk is given by

$$\mathbb{E}(X) = -\mathbb{E}(Y) = z - \sum_{r=0}^{p} z_r \sum_{i=0}^{r} i = z - \sum_{r=0}^{p} z_r \binom{r+1}{2}.$$

Hence the drift vanishes if and only if

$$z = \sum_{r=1}^{p} z_r \binom{r+1}{2}. \tag{57}$$

Lemma 8.1. *Under the zero-drift assumption (57), the covariance matrix of the step distribution is*

$$M = \begin{pmatrix} \mathbb{E}(X^2) & \mathbb{E}(XY) \\ \mathbb{E}(XY) & \mathbb{E}(Y^2) \end{pmatrix} = \sigma^2 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{58}$$

where

$$\sigma^2 = \sum_{r=1}^{p} z_r \binom{r+2}{3}. \tag{59}$$

Proof. Using (57), we compute

$$\mathbb{E}(X^2) = \mathbb{E}(Y^2) = z + \sum_{r=0}^{p} z_r \sum_{i=0}^{r} i^2 = \sum_{r=0}^{p} z_r \sum_{i=0}^{r} (i + i^2) = 2 \sum_{r=0}^{p} z_r \binom{r+2}{3} = 2\sigma^2,$$

and

$$-\mathbb{E}(XY) = z + \sum_{r=0}^{p} z_r \sum_{i=0}^{r} i(r-i) = \sum_{r=0}^{p} z_r \sum_{i=0}^{r} i(r+1-i) = \sum_{r=0}^{p} z_r \binom{r+2}{3} = \sigma^2.$$

■

Our first result is an estimate for the probability that the walk remains in the quadrant until time n at least.

Proposition 8.2. *Let $a, b \geqslant 0$. Under the zero-drift assumption (57),*

$$\mathbb{P}\left(\tau^{(a,b)} > n\right) \sim \frac{1}{4\sqrt{\pi}} V(a,b)\, n^{-3/2} \qquad \text{as } n \to \infty,$$

where the constants $V(a,b)$ have generating function

$$\mathcal{V}(u,v) := \sum_{a,b\geqslant 0} V(a,b) u^a v^b = \frac{2}{\sigma} \cdot \frac{1-uv}{(1-u)^3(1-v)^3\Lambda(u)}, \tag{60}$$

with

$$\Lambda(u) = \sum_{k=0}^{p-1} u^k \sum_{r=k+1}^{p} z_r \binom{r-k+1}{2}$$

and $\sigma > 0$ defined by (59).

We will prove Proposition 8.2 in Section 8.3. The proof consists in applying Flajolet and Odlyzko's *singularity analysis* [41, 42] to the expression for $Q^{(a,b)}(1,1)$ given in Theorem 3.4.

Remarks. **1.** We admit that the factor 2 in the expression for $V(a,b)$ looks strange in sight of the denominator 4 occurring in the estimate of $\mathbb{P}\left(\tau^{(a,b)} > n\right)$. However, this is the right convention in terms of the limit behaviour of $V(a,b)$. See the discussion in Section 9.3.2.

2. By considering the first step of a tandem walk starting at (a,b), we see that the function $V(a,b)$ of Proposition 8.2 has to satisfy

$$V(a,b) = z\, V(a+1,b-1) + \sum_{r=0}^{p} z_r \sum_{i=0}^{r} V(a-i, b+r-i), \tag{61}$$

with the convention that $V(a,b) = 0$ if $a < 0$ or $b < 0$. In other words, $V(a,b)$ is equal to the expected value of $V(\cdot,\cdot)$ over the neighbours of (a,b) after one random step, i.e.,

$$V(a,b) = \mathbb{E}\left(V((a,b)+(X,Y))\mathbb{1}_{\tau^{(a,b)}>1}\right).$$

Such a function is called a *discrete harmonic function* for the walk model (here random p-tandem walks in the quadrant). We discuss more aspects of the above property in Section 9.3.1.

3. Except in a few particular cases, it is very rare to obtain an explicit expression for a discrete harmonic function (or for its generating function). The most remarkable features of the above result are the following:

- it deals with a random walk with large steps, thus going beyond the results of [77] which only apply to walks with steps in $\{-1,0,1\}^2$,
- the generating function (60) is rational, and moreover its denominator factors as a product of two univariate polynomials: this implies that $V(a,b)$ admits a polynomial-exponential expression.

Let us give two examples, with $z_r = 0$ unless $r = p$, for some fixed p. Then the conditions (55) and (57) force

$$z_p = \frac{2}{(p+1)(p+2)} \qquad \text{and} \qquad z = \frac{p}{p+2},$$

so that $\sigma^2 = p/3$. If $p = 1$, then $z = z_1 = 1/3$, and

$$\mathcal{V}(u,v) = \frac{6\sqrt{3}\,(1-uv)}{(1-u)^3(1-v)^3},$$

which gives

$$V(a,b) = 3\sqrt{3}\,(a+1)(b+1)(a+b+2). \tag{62}$$

If $p = 2$, then $z_2 = 1/6$, $z = 1/2$ and

$$\mathcal{V}(u,v) = \frac{2\sqrt{6}\,(1-uv)}{(1-u)^3(1-v)^3(1+u/3)},$$

which gives

$$V(a,b) = \frac{3\sqrt{6}}{4}(b+1)\left((a+1)(a+b+2) + \frac{a}{2} + \frac{b}{4} + \frac{5}{8} - \frac{2b+1}{8}\left(-\frac{1}{3}\right)^{a+1}\right). \tag{63}$$

4. The form of the estimate in Proposition 8.2 follows from a general result of Denisov and Wachtel [29, Thm. 1], which implies that for a zero-drift quadrant walk, $\mathbb{P}(\tau^{(a,b)} > n) \sim \widetilde{V}(a,b)n^{-q/2}$ for some discrete harmonic function $\widetilde{V}(a,b)$. The exponent q is $\pi/\arccos(-\rho)$, where ρ is the *correlation factor*

$$\rho := \frac{\mathbb{E}(XY)}{\sqrt{\mathbb{E}(X^2)\mathbb{E}(Y^2)}}.$$

From Lemma 8.1 we see that $\rho = -1/2$ and $q = 3$ for any tandem step distribution. However, the results of [29] do not seem to yield any explicit expression for $\widetilde{V}(a,b)$ (nor of its generating function).

5. Note that $\Lambda(u)$ has nonnegative coefficients, and $\Lambda(1) = \sigma^2$. Hence, as $a, b \to \infty$, $V(a,b)$ admits the asymptotic estimate

$$V(a,b) \sim \frac{2}{\sigma}\cdot\frac{1}{\Lambda(1)}[u^a][v^b]\frac{1-uv}{(1-u)^3(1-v)^3} = \frac{1}{\sigma^3}(a+1)(b+1)(a+b+2) \sim \frac{ab(a+b)}{\sigma^3}. \tag{64}$$

The asymptotic behaviour of $V(a,b)$ can be interpreted as a scaling limit of the discrete harmonic function; the limit function $V_\infty(a,b) := ab(a+b)$ is continuous harmonic in the classical sense:

$$\mathbb{E}(X^2)\frac{\partial^2 V_\infty}{\partial a^2} + 2\mathbb{E}(XY)\frac{\partial^2 V_\infty}{\partial a \partial b} + \mathbb{E}(Y^2)\frac{\partial^2 V_\infty}{\partial b^2} = 2\sigma^2\left(\frac{\partial^2 V_\infty}{\partial a^2} - \frac{\partial^2 V_\infty}{\partial a \partial b} + \frac{\partial^2 V_\infty}{\partial b^2}\right) = 0,$$

where we have used (58). In Section 9.3.2, we discuss the connections between V and V_∞ further.

We now turn to the probability that $S^{(a,b)}(n)$ reaches the point (c,d) at time n, without having ever left the quadrant. Recall that this is only possible if $2n \equiv (c-d)-(a-b) \bmod \iota$, where the *periodicity index* ι is defined by (17).

Proposition 8.3. *Let $a, b, c, d \in \mathbb{N}$. Under the zero-drift assumption (57), for $2n \equiv (c-d)-(a-b) \bmod \iota$ we have*

$$\mathbb{P}\left(S^{(a,b)}(n) = (c,d),\ \tau^{(a,b)} > n\right) \sim \frac{\iota}{4\sqrt{3}\pi\sigma^2}\frac{V(a,b)V(d,c)}{n^4} \qquad \text{as } n \to \infty,$$

where σ^2 is given by (59), ι is the periodicity index, and $V(a,b)$ is the discrete harmonic function of Proposition 8.2.

Again, the form of the above estimate resembles a general formula of Denisov and Wachtel [29, Thm. 6]. However, our result is more precise because all constants are explicit, and moreover Theorem 6 in [29] requires a *strong aperiodicity* assumption, which does not hold in general for tandem walks (Lemma 3.1). We will explain how to adapt the main arguments of [29] to our periodic walk in Section 8.3. This is also briefly discussed in [36, p. 3].

8.2. Asymptotic enumeration of weighted tandem walks

Our aim here is to derive the asymptotic result of Theorem 3.5 from the above probabilistic results, and to make all constants in this proposition explicit. For convenience, we change the notation, and replace the weights $z_0, \dots, z_p$ of Theorem 3.5 by $w_0, \dots, w_p$ (SE steps have weight 1). The *weight* of a p-tandem walk having n_r face steps of level r for $0 \leqslant r \leqslant p$ is defined as $\prod_{r=0}^{p} w_r^{n_r}$; furthermore we denote the weighted "number" of walks of length n staying in the quadrant, starting at (a,b) and ending at (c,d), by $q_n(a,b;c,d)$. With the notation introduced at the beginning of Section 3,

$$q_n(a,b;c,d) = [t^n x^c y^d] Q^{(a,b)}(x,y) = [t^n] Q^{(a,b)}_{c,d}.$$

By a suitable normalization, we will now relate $q_n(a,b;c,d)$ to a probability of the form $P(S_n^{(a,b)} = (c,d),\ \tau^{(a,b)} > n)$, as considered in Proposition 8.3. For two positive parameters α, γ (to be fixed later), we let

$$z = \alpha^2/\gamma, \quad z_r = w_r \alpha^{-r}/\gamma \quad \text{for } 0 \leqslant r \leqslant p. \tag{65}$$

As in Section 8.1, we want these values to describe the step distribution of a random walk with zero drift. Then the probability of a step (i,j) will be its weight, multiplied by α^{i-j}/γ. The zero-drift condition (57) is satisfied if and only if

$$\alpha^2 = \sum_{r=1}^{p} \binom{r+1}{2} w_r \alpha^{-r}. \tag{66}$$

This equation in α has a unique solution; indeed, as α increases from 0 to $+\infty$, the left-hand side increases from 0 to $+\infty$, while the right-hand side decreases from $+\infty$ to 0. Once α has been adjusted, the normalization condition (55) forces

$$\gamma = \alpha^2 + \sum_{r=0}^{p} (r+1) w_r \alpha^{-r} = \sum_{r=0}^{p} \binom{r+2}{2} w_r \alpha^{-r}. \tag{67}$$

With this choice of α and γ, the values $z, z_0, \dots, z_p$ define indeed a probability distribution on tandem steps, having zero drift. Now the probability of a quadrant walk of length n going from (a,b) to (c,d) is equal to its weight, multiplied by $\alpha^{(c-a)-(d-b)}/\gamma^n$. Hence,

$$\mathbb{P}\big(S^{(a,b)}(n) = (c,d),\ \tau^{(a,b)} > n\big) = \frac{\alpha^{(c-a)-(d-b)}}{\gamma^n}\, q_n(a,b;c,d).$$

Theorem 3.5 now follows from Proposition 8.3.

Theorem 3.5 (with explicit constants). *Let a,b,c,d be nonnegative integers and let $w_0, \dots, w_p$ be nonnegative weights with $w_p > 0$. Define $\alpha > 0$ and γ by (66) and (67). Let*

$$D = \{r \in [\![0,p]\!],\ w_r > 0\} \qquad \text{and} \qquad \iota = \gcd(r+2,\ r \in D\}.$$

Then, as $n \to \infty$ conditioned on $c - d \equiv a - b + 2n \bmod \iota$, we have

$$q_n(a, b; c, d) \sim \kappa\, \gamma^n n^{-4}, \tag{68}$$

where

$$\kappa := \frac{\iota}{4\sqrt{3}\pi\sigma^2} V(a, b) V(d, c) \alpha^{(d-b)-(c-a)},$$

with $V(\cdot, \cdot)$ the harmonic function of Proposition 8.2 and σ^2 given by (59), both taken with $z_r = w_r \alpha^{-r}/\gamma$.

We can now go back to the number of bipolar orientations with prescribed face degrees.

Proof of Corollary 3.6. It follows from the KMSW bijection that $B_n^{(\Omega)}(b, c) = q_n(0, b; c, 0)$, taken for $w_r = 1$ if $r + 2 \in \Omega$, and $w_r = 0$ otherwise. With the notation of the above proposition, this gives $D = \Omega - 2$,

$$1 = \sum_{s \in \Omega} \binom{s-1}{2} \alpha^{-s} \qquad \text{and} \qquad \gamma = \sum_{s \in \Omega} \binom{s}{2} \alpha^{-s+2},$$

which fits with the values of α and γ given in Corollary 3.6. One easily checks that the value (59) of σ^2 is also in agreement with the corollary. Then we only need to determine the values $V(0, b)$ and $V(0, c)$. By Proposition 8.2,

$$\mathcal{V}(0, v) = \sum_{i \geqslant 0} V(0, i) v^i = \frac{2}{\sigma} \frac{1}{(1 - v)^3 \Lambda(0)},$$

and, by (57) and then (65),

$$\Lambda(0) = \sum_{r=1}^{p} z_r \binom{r+1}{2} = z = \frac{\alpha^2}{\gamma}.$$

Hence

$$V(0, i) = \frac{1}{\sigma \Lambda(0)} (i + 1)(i + 2) = \frac{\gamma}{\sigma \alpha^2} (i + 1)(i + 2).$$

The proof is then completed by putting all pieces together. ∎

8.3. The probability to stay in the quadrant till time n: proof

We return to the probabilistic setting of Proposition 8.2, where we consider a zero-drift random tandem walk with step distribution given by (56). The probability that the walk $S^{(a,b)}$ remains in the quadrant till time n is closely related to the coefficient of t^n in the series $Q^{(a,b)}(1, 1)$ given in Theorem 3.4. However, we need to incorporate a positive weight z for each SE step: this amounts to replacing t by zt and z_r by z_r/z for $0 \leqslant r \leqslant p$. Let us emphasize the dependence of $Q^{(a,b)}$ in the variables t and z_r by writing

$$Q^{(a,b)}(x, y) \equiv Q^{(a,b)}(t, \boldsymbol{z}; x, y),$$

with $\boldsymbol{z} = (z_0, \ldots, z_p)$. Then

$$\mathbb{P}\left(\tau^{(a,b)} > n\right) = [t^n] Q^{(a,b)}(tz, \boldsymbol{z}/z; 1, 1).$$

It now follows from Theorem 3.4 that

$$\sum_{n \geqslant 0} \mathbb{P}\left(\tau^{(a,b)} > n\right) t^n = \frac{\mathsf{W}}{t} \sum_{i=0}^{a} \mathsf{A}_i \sum_{j=0}^{b} \mathsf{W}^j, \tag{69}$$

where $\mathsf{W} = W(tz, \boldsymbol{z}/z)$ (see (15)) is the unique series in t satisfying $\mathsf{W} = t\phi(\mathsf{W})$, with

$$\phi(w) = z + \sum_{r=0}^{p} z_r(w + \cdots + w^{r+1}), \tag{70}$$

and $\mathsf{A}_i = \frac{1}{z}A_i(tz, \boldsymbol{z}/z)$ is a polynomial in W:

$$\mathsf{A}_i = [u^i]\frac{1}{z - u\mathsf{W}\sum_{i+k<r\leqslant p} u^i\mathsf{W}^k z_r}.$$

Starting from (69), we observe that Proposition 8.2 follows by linearity if we can prove the following lemma.

Lemma 8.4. *For $i, j \geqslant 0$, we have, as n tends to infinity,*

$$[t^n]\frac{\mathsf{W}}{t}\mathsf{A}_i\mathsf{W}^j \sim \frac{1}{4\sqrt{\pi}}\, U(i,j)n^{-3/2},$$

where

$$\sum_{i,j\geqslant 0} U(i,j)u^iv^j = \frac{2}{\sigma}\cdot\frac{1-uv}{(1-u)^2(1-v)^2\Lambda(u)},$$

with σ and $\Lambda(u)$ as in Proposition 8.2.

Proof. We will prove this lemma using Flajolet and Odlyzko's *singularity analysis* [41, 42].

Our first task is to determine the dominant singularities, and the singular behaviour of $\mathsf{W} \equiv \mathsf{W}(t)$. The equation $\mathsf{W} = t\phi(\mathsf{W})$, with ϕ defined by (70), fits in the *smooth aperiodic inverse function schema* of [42, Thm. VII.2, p. 453] (see also Thm. IV.6 on p. 404 in the same reference; aperiodicity comes for instance from the term $z_p w$ in the transformation ϕ). Consequently, W has a unique singularity $t_c > 0$ on its circle of convergence, and t_c is the unique positive solution of $\phi(t_c) = t_c\phi'(t_c)$. Recall that z and the z_i's satisfy the normalization condition (55), and that we assume that the zero-drift condition (57) holds. This means that $\phi(1) = \phi'(1) = 1$, so that the radius of W is $t_c = 1$. Moreover, $\mathsf{W}(t_c) = t_c = 1$ as well. Still using the above cited results of [42], we conclude that W admits a square-root singular expansion around $t_c = 1$, namely

$$\mathsf{W} = 1 - d\sqrt{1-t} + O(1-t),$$

where $d = \sqrt{2/\phi''(1)} = 1/\sigma$, with σ^2 given by (59). Consequently, for any $k \geqslant 1$, the series W^k also has a unique singularity on its circle of convergence, and

$$\mathsf{W}^k = 1 - \frac{k}{\sigma}\sqrt{1-t} + O(1-t),$$

and more generally, for any polynomial $P(w)$ having nonnegative coefficients,

$$P(\mathsf{W}) = 1 - \frac{1}{\sigma}P'(1)\sqrt{1-t} + O(1-t). \tag{71}$$

Recall that the series $\mathsf{A}_i\mathsf{W}^{j+1}$ can be expressed as a polynomial in W. More precisely, $\mathsf{A}_i\mathsf{W}^{j+1} = P_{i,j}(\mathsf{W})$, where

$$P_{i,j}(w) := [u^iv^j]\left(A(u,w)B(v,w)\right),$$

with

$$A(u,w) := \frac{1}{z - uw\sum_{i+k<r\leqslant p} u^iw^kz_r} \qquad \text{and} \qquad B(v,w) := \frac{w}{1-vw}.$$

It then follows from (71) that

$$\mathsf{A}_i\mathsf{W}^{j+1} = 1 - \frac{1}{\sigma}\, P'_{i,j}(1)\sqrt{1-t} + O(1-t),$$

with

$$P'_{i,j}(1) = [u^i v^j]\, \frac{\partial\left(A(u,w)B(v,w)\right)}{\partial w}\bigg|_{w=1}. \tag{72}$$

Since $t_c = 1$, we have the same singular expansion for $\mathsf{A}_i\mathsf{W}^{j+1}/t$, and thus, using the transfer theorem of [42, Cor. VI.1, p. 392], we find

$$[t^n]\frac{1}{t}\mathsf{A}_i\mathsf{W}^{j+1} \sim \frac{1}{2\sigma\sqrt{\pi}}\, P'_{i,j}(1)n^{-3/2} \sim \frac{1}{4\sqrt{\pi}}\, U(i,j)n^{-3/2}, \tag{73}$$

where we define

$$U(i,j) := 2\, P'_{i,j}(1)/\sigma. \tag{74}$$

It remains to express $P'_{i,j}(1)$, that is, the derivative in (72). Clearly, we have

$$B(v,1) = \frac{1}{1-v} \qquad \text{and} \qquad B'_2(v,1) = \frac{1}{(1-v)^2}.$$

Moreover,

$$A(u,1) = \frac{1}{z - u\sum_{0\leqslant i<r\leqslant p} u^i z_r(r-i)}.$$

Upon rewriting z as in (57), this gives

$$A(u,1) = \frac{1}{\sum_{0\leqslant i<r\leqslant p}(1-u^{i+1})z_r(r-i)} = \frac{1}{(1-u)\Lambda(u)},$$

where $\Lambda(u)$ is defined as in Proposition 8.2. Finally

$$A'_2(u,1) = A(u,1)^2\, u\left(\sum_{i+k<r\leqslant p} u^i(k+1)z_r\right) = A(u,1)^2 u\Lambda(u) = \frac{u}{(1-u)^2\Lambda(u)}.$$

Hence, getting back to (72) and (74), we have

$$\begin{aligned} P'_{i,j}(1) = \frac{\sigma}{2}\cdot U(i,j) &= [u^i v^j]\left(A(u,1)B'_2(v,1) + A'_2(u,1)B(v,1)\right) \\ &= [u^i v^j]\frac{1-uv}{(1-u)^2(1-v)^2\Lambda(u)}, \end{aligned}$$

as claimed in the lemma. ∎

8.4. The local limit theorem: proof

We now explain how to go from Proposition 8.2 to Proposition 8.3. We follow the steps in [29], with which some familiarity is assumed, and explain how to deal with periodicity issues. We refer to Spitzer's book [80] for several classical results on random walks.

Let $S(n)$ denote the point reached after n steps of the random tandem walk (starting at the origin). Recall that $S^{(a,b)}(n) := (a,b) + S(n)$ denotes the point attained after n steps, when starting from (a,b).

8.4.1. ***Local limit theorem for unconstrained walks.*** Lemma 8.1 gives the covariance matrix of the step distribution. By the central limit theorem (in its vectorial formulation), the random variable $\frac{1}{\sigma\sqrt{n}}S(n)$ converges in law to the random variable on $\mathbb{R}^2$ of density

$$f(x,y) = \frac{1}{2\pi\sqrt{3}} \exp\big(-\tfrac{1}{3}(x^2+y^2+xy)\big).$$

Obviously, the same limit holds for the random variable $\frac{1}{\sigma\sqrt{n}}S^{(a,b)}(n)$, for any fixed starting point (a,b). Let us now state the corresponding Gnedenko local limit theorem. Let $R^{(a,b)}(n)$ be the sublattice

$$R^{(a,b)}(n) = \{(i,j) \in \mathbb{Z}^2 : i - j \equiv 2n + (a-b) \mod \iota\}. \tag{75}$$

Note that it only depends on n through $n \bmod \iota$. According to Lemma 3.1, any point reachable from (a,b) in n steps lies in $R^{(a,b)}(n)$, and any point in $R^{(a,b)}(n)$ can be reached in n steps starting from (a,b), for n large enough. Then, with (a,b) fixed and n tending to infinity,

$$\sup_{(i,j)\in R^{(a,b)}(n)} \left| n \cdot \mathbb{P}\big(S^{(a,b)}(n) = (i,j)\big) - \frac{\iota}{\sigma^2} f\left(\frac{i}{\sigma\sqrt{n}}, \frac{j}{\sigma\sqrt{n}}\right)\right| \to 0. \tag{76}$$

The proof is classically done by a saddle-point argument, after diagonalization of the covariance matrix (we refer to Proposition P9 in [80, pp. 75–77], and the remark that follows it); moreover, as discussed in Example 2 in [80, pp. 78–79], a periodicity $\iota \geqslant 2$ of the step set results in ι saddle-points, each giving the same asymptotic contribution, with the effect that the asymptotic constant is multiplied by ι for points of the reachable sublattice.

8.4.2. ***Local limit theorem for walks confined in the quadrant.*** We begin with the following central limit theorem under the quadrant constraint.

Proposition 8.5. *The random variable $\frac{1}{\sigma\sqrt{n}}S^{(a,b)}(n)$ conditioned on $\{\tau^{(a,b)} > n\}$ converges in law to the random variable on $\mathbb{R}_+^2$ of density*

$$g(x,y) = \frac{1}{\sqrt{3\pi}} xy(x+y) \exp\big(-\tfrac{1}{3}(x^2+y^2+xy)\big). \tag{77}$$

There is a natural link between the function $xy(x+y)$ occurring in the density above, and the limit V_∞ of the discrete harmonic function (see (64)), which we discuss further in Section 9.3.2.

Proof. We use a normalization that transforms the random walk $S^{(a,b)}(n)$ into a walk with uncorrelated x- and y-projections. This also transforms the quarter plane into a different cone, to which we then apply [29, Thm. 3]. This classical argument has been used recently in a similar context in [29, Sec. 1.5] and [14, Thm. 4].

Let us now give details. Let $\mathcal{L}$ be the linear mapping of matrix

$$\frac{\sqrt{2}}{\sigma\sqrt{3}} \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

Then it is easy to check that if (X,Y) is the step distribution of our random tandem walk, given by (56), then the covariance matrix of $\mathcal{L}(X,Y)$ is the identity. Moreover, $\mathcal{L}$ maps the quadrant $\{re^{i\theta} : r \geqslant 0 \text{ and } \theta \in [0,\pi/2]\}$ to the cone $K_{\pi/3} = \{re^{i\theta} : r \geqslant$

0 and $\theta \in [0, \pi/3]\}$. Let $\tilde{S}^{(a,b)} := \mathcal{L}(S^{(a,b)})$ denote the transformed walk. By [29, Thm. 3], $\tilde{S}^{(a,b)}(n)/\sqrt{n}$, conditioned on $\{\tau^{(a,b)} > n\}$, converges in law to the random variable $\widetilde{S}$ having the following density (expressed in polar coordinates):

$$\rho(r,\theta) = H_0 \cdot \mathbb{1}_{\theta\in[0,\pi/3]} \cdot u(r,\theta) \cdot r \exp(-r^2/2),$$

where H_0 is the normalizing constant and the function $u(r,\theta)$ is given by [29, Eq. (3)]. This function may be interpreted as the unique harmonic function that is nonnegative in the cone $K_{\pi/3}$ and vanishes on the boundary.

Let us now show that $u(r,\theta) = r^3 \sin(3\theta)$. By Eq. (3) in [29], we need to solve the eigenvalue problem stated in Eq. (2) of that same paper. In dimension $d = 2$, the Laplace–Beltrami operator $L_{\mathbb{S}^{d-1}}$ involved in this problem is simply $\frac{\partial^2}{\partial\theta^2}$, and the eigenvalue problem becomes $\frac{\partial^2 m_j}{\partial\theta^2} = -\lambda_j m_j$. This is easily solved in $m_j(\theta) = a\cos(\sqrt{\lambda_j}\theta) + b\sin(\sqrt{\lambda_j}\theta)$ for arbitrary constants a and b. The boundary conditions $m_j(0) = m_j(\pi/3) = 0$ yield $a = 0$ and force $\lambda_j = (3j)^2$ (as in [29], we take $0 < \lambda_1 < \lambda_2 \leqslant \cdots$). In particular, $\lambda_1 = 9$, the function m_1 reads $m_1(\theta) = b\sin(3\theta)$, which together with [28, Eq. (3)] proves that (up to a multiplicative constant) $u(r,\theta) = r^3\sin(3\theta)$. The value of the normalizing constant H_0 in the above expression for $\rho(r,\theta)$ is then found to be $1/\sqrt{2\pi}$, since ρ must be a density.

In (x,y)-coordinates, we have

$$u(r,\theta) = r^3\sin(3\theta) = y(3x^2 - y^2),$$

hence the density of $\widetilde{S}$, expressed in Cartesian coordinates, is

$$\widetilde{g}(x,y) = \frac{1}{\sqrt{2\pi}} \cdot \mathbb{1}_{(x,y)\in K_{\pi/3}} \cdot y(3x^2 - y^2) \cdot \exp\bigl(-\tfrac{1}{2}(x^2+y^2)\bigr).$$

Getting back to $S^{(a,b)}(n)$ and the quadrant, we conclude that $\frac{1}{\sigma\sqrt{n}} S^{(a,b)}(n)$ conditioned on $\{\tau^{a,b} > n\}$ converges to the random variable $\frac{1}{\sigma}\mathcal{L}^{-1}(\widetilde{S})$, and by a routine change of variable we find that it has density $g(x,y)$. ∎

Let us now state the corresponding Gnedenko local limit theorem, that is, the counterpart of (76). In the aperiodic case this would be Theorem 5 in [29].

Proposition 8.6. *Let $R^{(a,b)}_{\geqslant}(n) := R^{(a,b)}(n) \cap \mathbb{N}^2$. For a, b fixed and n tending to infinity, we have*

$$\sup_{(i,j)\in R^{(a,b)}_{\geqslant}(n)} \left| n^{5/2} \cdot \mathbb{P}\bigl(S^{(a,b)}(n) = (i,j),\ \tau^{(a,b)} > n\bigr) - \frac{\iota V(a,b)}{4\sqrt{\pi}\sigma^2}\, g\left(\frac{i}{\sigma\sqrt{n}}, \frac{j}{\sigma\sqrt{n}}\right) \right| \to 0.$$

Proof (sketch). Given the estimate of $\mathbb{P}(\tau^{(a,b)} > n)$ given in Proposition 8.2, the above proposition is equivalent to

$$\sup_{(i,j)\in R^{(a,b)}_{\geqslant}(n)} \left| n \cdot \mathbb{P}\bigl(S^{(a,b)}(n) = (i,j) \mid \tau^{(a,b)} > n\bigr) - \frac{\iota}{\sigma^2}\, g\left(\frac{i}{\sigma\sqrt{n}}, \frac{j}{\sigma\sqrt{n}}\right) \right| \to 0.$$

The proof mimics the proof of [29, Thm. 5]. In the typical case where i and j are of the order of $\sqrt{n}$, it proceeds by a well chosen splitting of quadrant walks from (a,b) to (i,j) into two parts; to the first part, we apply Proposition 8.5 (convergence in law of the constrained walk to the density (77)), and to the second part, we apply the domain-unconstrained local limit theorem (76). The factor ι propagates from (76) to the final result. ∎

8.4.3. ***Conclusion.*** We now derive Proposition 8.3 from Proposition 8.6, as Theorem 6 is derived from Theorem 5 in [29]. For $n \geqslant 0$ let $n_1 := \lfloor n/2 \rfloor$ and $n_2 := \lceil n/2 \rceil$, so that $n = n_1 + n_2$. For two points (a,b) and (i,j) in the quadrant, let $\mathcal{E}_{i,j}^{(a,b)}(n)$ denote the event $\{S^{(a,b)}(n) = (i,j),\ \tau^{(a,b)} > n\}$. Observe that the step distribution (56) is such that the laws of $(-X,-Y)$ and (Y,X) coincide. In other words, time-reversal has the same effect as exchanging the roles of x and y. Hence, for any quadrant points (a,b), (c,d) and (i,j), we have

$$\begin{aligned}\mathbb{P}\big(\mathcal{E}_{c,d}^{(a,b)}(n),\ S^{(a,b)}(n_1) = (i,j)\big) &= \mathbb{P}\big(\mathcal{E}_{i,j}^{(a,b)}(n_1)\big) \cdot \mathbb{P}\big(\mathcal{E}_{c,d}^{(i,j)}(n_2)\big) \\ &= \mathbb{P}\big(\mathcal{E}_{i,j}^{(a,b)}(n_1)\big) \cdot \mathbb{P}\big(\mathcal{E}_{j,i}^{(d,c)}(n_2)\big).\end{aligned}$$

We now assume that $(c,d) \in R_{\geqslant}^{(a,b)}(n)$, and sum the above identity over all (i,j) in $R_{\geqslant}^{(a,b)}(n_1)$. In the first line below, we use Proposition 8.6 to estimate the probability of the two $\mathcal{E}$-events, and in the second line, we use the symmetry of $g(x,y)$ in x and y. This yields

$$\begin{aligned}\mathbb{P}\left(\mathcal{E}_{c,d}^{(a,b)}(n)\right) &\sim \sum_{(i,j)\in R_{\geqslant}^{(a,b)}(n_1)} \frac{\iota V(a,b)}{4\sqrt{\pi} n_1^{5/2}\sigma^2} g\left(\frac{i}{\sigma\sqrt{n_1}}, \frac{j}{\sigma\sqrt{n_1}}\right) \cdot \frac{\iota V(d,c)}{4\sqrt{\pi} n_2^{5/2}\sigma^2} g\left(\frac{j}{\sigma\sqrt{n_2}}, \frac{i}{\sigma\sqrt{n_2}}\right) \\ &\sim V(a,b)V(d,c)\frac{2\iota^2}{\pi n^5\sigma^4} \sum_{(i,j)\in R_{\geqslant}^{(a,b)}(n_1)} g\left(\frac{i}{\sigma\sqrt{n/2}}, \frac{j}{\sigma\sqrt{n/2}}\right)^2, \qquad (78)\end{aligned}$$

since $n_1 \sim n_2 \sim n/2$. Then we classically approximate the sum by an integral, using the description (75) of $R^{(a,b)}(n)$,

$$\frac{\iota}{(\sigma\sqrt{n/2})^2} \sum_{(i,j)\in R_{\geqslant}^{(a,b)}(n_1)} g\left(\frac{i}{\sigma\sqrt{n/2}}, \frac{j}{\sigma\sqrt{n/2}}\right)^2 \sim \iint_{(x,y)\in\mathbb{R}_+^2} g(x,y)^2 \mathrm{d}x\,\mathrm{d}y = \frac{1}{4\sqrt{3}}.$$

Returning to (78), this gives

$$\mathbb{P}\left(\mathcal{E}_{c,d}^{(a,b)}(n)\right) = \mathbb{P}\left(S^{(a,b)}(n) = (c,d),\ \tau^{(a,b)} > n\right) \sim \frac{\iota}{4\sqrt{3}\pi\sigma^2}\frac{V(a,b)V(d,c)}{n^4},$$

as stated in Proposition 8.3. ∎

9. Complements and final comments

9.1. Link with non-intersecting triples of directed walks

It is known that plane bipolar orientations can be encoded by certain non-intersecting triples of *directed* walks, that is, walks on the square lattice consisting of north and east steps (Figure 13, right) [1, 9, 44, 40]. We explain here how such a bijection can be obtained from the KMSW bijection with tandem walks. In fact, our construction is more general, as it extends to *marked* bipolar orientations. In particular, it gives a closed form expression for the number of marked bipolar orientations with prescribed signature, number of (plain) edges, and number of inner faces (see Proposition 9.1 and the paragraph just following it). However, this bijection does not allow us to record face degrees.

Consider a tandem walk w of length n going from (a,b) to (c,d), staying in the quadrant, with successive steps $s_1, \dots, s_n$. We associate with w a triple of directed walks D_1, D_2, D_3 as

follows. We initialize D_1, D_2, D_3 to be the empty walks starting (and ending) at $(0, -a-1)$, $(0,0)$, and $(0, b+1)$, respectively. Then we let D_1, D_2, D_3 grow by reading the successive steps of w. To be precise, for m from 1 to n:

- If s_m is a SE step, we add a north step to D_2, and leave D_1 and D_3 unchanged.
- If s_m is a face step $(-i, j)$ we append an east step to D_2, and we append the walk EN^i to D_1 and append the walk N^jE to D_3.

See Figure 13 for an example. One can see that at stage m the walks D_1, D_2, D_3 have the same number of east steps (which is the number of face steps among $s_1, \dots, s_m$), and D_2 has exactly m steps. In addition, if we denote by (i_m, j_m) the point of w reached after m steps (that is, $(i_m, j_m) = s_1 + \cdots + s_m$), and by $y_1^{(m)}, y_2^{(m)}, y_3^{(m)}$ the ordinates of the endpoints of D_1, D_2, D_3 at stage m, then $y_2^{(m)} - y_1^{(m)} = i_m + 1$ and $y_3^{(m)} - y_2^{(m)} = j_m + 1$. In particular, saying that w stays in the quadrant is equivalent to saying that the three walks D_1, D_2, D_3 do not intersect. By construction, the first step of D_1 (respectively the last step of D_3) is necessarily an east step, hence can be deleted without loss of information. We denote by $(\hat{D}_1, D_2, \hat{D}_3)$ the resulting triple of walks. This gives the following proposition.

Proposition 9.1. *The above mapping is a bijection Ψ between tandem walks of length n with k face steps, staying in the quadrant, starting at (a,b) and ending at (c,d), and non-intersecting triples of directed walks from $A_1 = (1, -a-1)$, $A_2 = (0,0)$, $A_3 = (0, b+1)$ to $B_1 = (k, n-k-c-1)$, $B_2 = (k, n-k)$, $B_3 = (k-1, n-k+d+1)$.*

Therefore, by the Lindström–Gessel–Viennot lemma [46], the number of such walks is given by

$$q_{n,k}(a,b,c,d) = \begin{vmatrix} \binom{n+a-c-1}{k-1} & \binom{n+a}{k-1} & \binom{n+a+d}{k-2} \\ \binom{n-c-1}{k} & \binom{n}{k} & \binom{n+d}{k-1} \\ \binom{n-b-c-2}{k} & \binom{n-b-1}{k} & \binom{n-b+d-1}{k-1} \end{vmatrix}.$$

As discussed at the beginning of Section 3, the number

$$\begin{aligned}\widetilde{q}_{n,k}(a,b,c,d) := q_{n,k}(a,b,c,d) &- q_{n,k}(a-1,b,c-1,d) \\ &- q_{n,k}(a,b-1,c,d-1) + q_{n,k}(a-1,b-1,c-1,d-1)\end{aligned}$$

counts marked bipolar orientations with $n+1$ plain edges, k inner faces, and signature $(a,b;c,d)$.

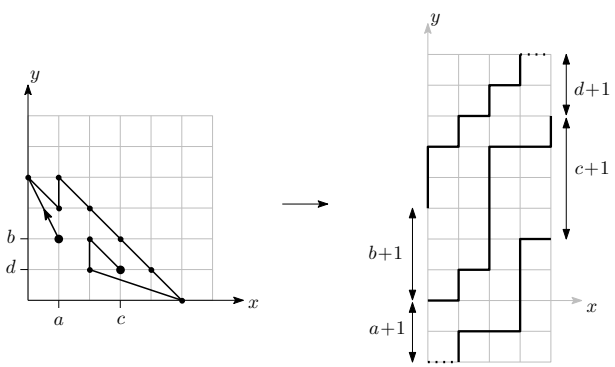


FIGURE 13. A tandem walk in the quadrant, and the corresponding non-intersecting triple of directed lattice walks.

Coming back to the involutions ρ and σ of Definition 2.3, we see that Ψ behaves simply with respect to ρ (it amounts to rotate $\Psi(w)$ by a half-turn), but that the transformation induced by σ is not simple. Accordingly, the underlying symmetry $\widetilde{q}_{n,k}(a,b,c,d) = \widetilde{q}_{n,k}(d,b,c,a)$ is not clear from the above determinant.

When $a = d = 0$, we have $q_{n,k}(0,b,c,0) = \widetilde{q}_{n,k}(0,b,c,0)$, and the map Ψ, composed with (the reverse of) the KMSW bijection Φ, sends bipolar orientations with $n+1$ edges, k inner faces, left outer boundary of length $b+1$, right outer boundary of length $c+1$, bijectively onto non-intersecting triples of directed walks joining $A_1 = (1,-1)$, $A_2 = (0,0)$, $A_3 = (0,b+1)$ to $B_1 = (k, n-k-c-1)$, $B_2 = (k, n-k)$, $B_3 = (k-1, n-k+1)$. (Another bijection between these two families can also be deduced from the correspondence between plane bipolar orientations and twin pairs of binary trees given in [40], and from the encoding of twin binary trees given in [34], upon taking the mirror encoding for the second binary tree.)

The further specialization $a = b = c = d = 0$ gives a bijection for bipolar orientations of a digon having $n+1$ edges and k inner faces. Upon deleting the two outer edges (for $n \geqslant 3$), we obtain a bipolar orientation having $n-1$ edges and $k-2$ inner faces. In the corresponding tandem walk, the first step is necessarily of the form $(0,j)$, and the last one of the form $(-i,0)$. Hence, in the associated non-intersecting triple $\hat{D}_1, D_2, \hat{D}_3$, the first steps of $\hat{D}_1$ and D_2, and the last steps of D_2 and $\hat{D}_3$, are always east, hence these four steps can be deleted. We thus recover the fact that plane bipolar orientations with $n-1$ edges and $k-2$ inner faces are in bijection with non-intersecting triples of lattice walks from $(2,-1), (1,0), (0,1)$ to $(k, n-k-1), (k-1, n-k), (k-2, n-k+1)$, which are counted by the Baxter summand (see (2)):

$$q_{n,k}(0,0,0,0) = \frac{2}{n^2(n-1)} \binom{n}{k-2} \binom{n}{k-1} \binom{n}{k}.$$

9.2. Random generation of tandem walks in the quadrant

For $p \geqslant 1$, let z and $z_0, \dots, z_p$ be step probabilities satisfying (55) and (57). We let $\mathcal{Q}_n$ denote the family of p-tandem walks of length n in the quadrant starting at the origin, and $\mathcal{E}_n$ the subfamily of those that end at the origin (*excursions*). We consider the problem of generating a random walk in $\mathcal{Q}_n$ (or $\mathcal{E}_n$) such that each walk w occurs with probability proportional to $z^k \prod_r z_r^{n_r}$, with k the number of SE steps in w and n_r the number of face steps of level r. With $\boldsymbol{z} = (z, z_0, \dots, z_p)$, we refer to such random walks as *$\boldsymbol{z}$-distributed*.

Regarding $\mathcal{Q}_n$, the bijection of Proposition 7.2 reduces this problem to the random generation of $\boldsymbol{z}$-distributed tandem walks in the upper half-plane, starting at the origin and ending on the x-axis. Projected to the y-axis, these walks correspond to a model of critical Galton–Watson trees, and can thus be randomly generated in linear time [31].

We thus focus on $\mathcal{E}_n$. We begin with the special case $p = 1$ and $z_0 = 0$. Then $z = z_1 = 1/3$, and the $\boldsymbol{z}$-distribution is simply uniform. This case is particularly simple due to the existence of closed form expressions. Indeed, the number of walks of length $n = 3m+2i+j$ starting at the origin and ending at (i,j) is (see [18, Prop. 9])

$$q_n(0,0;i,j) = \frac{(i+1)(j+1)(i+j+2)(3m+2i+j)!}{m!\,(m+i+1)!\,(m+i+j+2)!}.$$

This makes it easy to draw a uniform excursion in $\mathcal{E}_n$ step by step in time $O(n)$ (in reverse order, that is, from the last one to the first one). This is how we generated the tandem walk encoding the random bipolar orientation with triangular faces of Figure 4.

For the general case $p \geqslant 1$, we propose an almost linear algorithm under two relaxations:

(i) the length is not exactly prescribed, but lies in a linear-size window $[\![2n, 3n]\!]$,
(ii) the distribution conditioned to a given size $m \in [\![2n, 3n]\!]$ coincides with the desired distribution only asymptotically (meaning that the total variation distance between the actual distribution and the $\boldsymbol{z}$-distribution is $o(1)$).

To obtain a random excursion w we generate two random walks w_1, w_2 in the quadrant, of respective lengths n and $2n$, using the above random generators for $\mathcal{Q}_n$ and $\mathcal{Q}_{2n}$. Let (a, b) be the ending point of w_1. If w_2 does not visit (b, a) between times n and $2n$, then we declare a failure situation and restart generating w_2 until we obtain a walk visiting (b, a) between times n and $2n$. Then let $n' \in [\![n, 2n]\!]$ be the index of the last visit of w_2 to (b, a). Let w_3 be the prefix of length n' of w_2, and let $\overleftarrow{w}_3$ be obtained by reversing time in w_3, and applying an x/y-symmetry. That is, if the rth step of w_3 is (i, j) then the $(n'-r+1)$th step of $\overleftarrow{w}_3$ is $(-j, -i)$. As already used in Section 8.4, $\overleftarrow{w}_3$ is also a tandem walk. Then the concatenation w of w_1 and $\overleftarrow{w}_3$ is a random excursion in $\mathcal{E}_m$, where $m = n + n' \in [\![2n, 3n]\!]$.

A random excursion in $\mathcal{E}_m$ is called *n-twisted* if it is $\boldsymbol{z}$-distributed *once conditioned on the point (X_n, Y_n) visited after n steps*. Clearly the random excursion w constructed by the above procedure is n-twisted. Moreover, by Proposition 8.5, as $n \to \infty$ the rescaled random point $\frac{1}{\sigma\sqrt{n}}(X_n, Y_n)$ converges to the law of density

$$g(x, y) = \frac{1}{\sqrt{3\pi}}\, xy(x + y) \exp\big(- \tfrac{1}{3}(x^2 + y^2 + xy)\big),$$

and a local limit statement also holds, following from Proposition 8.6. Let us compare this behaviour to the limit density of $\frac{1}{\sigma\sqrt{n}}(X_n, Y_n)$ in a $\boldsymbol{z}$-distributed excursion of $\mathcal{E}_m$. Let $q_n(i, j) := \mathbb{P}(S^{(0,0)}(n) = (i, j), \tau^{(0,0)} > n)$ denote the $\boldsymbol{z}$-weighted number of quadrant walks of length n starting at the origin and ending at (i, j). Then the probability that a $\boldsymbol{z}$-distributed excursion of $\mathcal{E}_m$ is at (i, j) after n steps is proportional to $q_n(i, j)q_{m-n}(j, i)$. This, combined with Proposition 8.6, implies that the rescaled random point $\frac{1}{\sigma\sqrt{n}}(X_n, Y_n)$ in a $\boldsymbol{z}$-distributed excursion of $\mathcal{E}_m$ asymptotically follows the law of density

$$c_\alpha\, g(x, y) g(x/\sqrt{\alpha}, y/\sqrt{\alpha}),$$

where $\alpha := (m - n)/n \in [1, 2]$, for some constant c_α. Moreover a local limit statement also holds.

We would now like to twist the way we produce the excursion w so that the distribution of its nth point approaches the above distribution. This is classically done by adding a rejection step (see, e.g., [30, II.3]). To be precise, let $g_0 \approx 0.267$ be the maximal value of $g(x, y)$ (attained at $(x, y) = (\sqrt{6}/2, \sqrt{6}/2)$ and at two other points). Then we repeat calling the random sampler above, at each attempt producing an n-twisted random excursion of length $m \in [\![2n, 3n]\!]$ and then flipping a coin with success probability $\frac{1}{g_0} g(\frac{X_n}{\sigma\sqrt{\alpha n}}, \frac{Y_n}{\sigma\sqrt{\alpha n}})$, where $\alpha = (m - n)/n$, and (X_n, Y_n) is the point reached after n steps. We return the excursion for the first successful attempt. With this additional rejection step we obtain a random sampler that is asymptotically $\boldsymbol{z}$-distributed.

Let us discuss (heuristically) the time complexity of the sampler. The coin-flipping probability of success is $\Theta(1)$ (uniformly over $\alpha \in [1, 2]$), hence the time complexity is of the same order as the one of the $\boldsymbol{z}$-twisted random excursion sampler. The generation of w_1 takes time $O(n)$ as already mentioned. Moreover it is well known [37] that, in the simple random 2-dimensional walk of length n, the number of distinct points visited by the walk is of the order of $n/\log(n)$. Hence we can expect that the number of attempts

needed to generate w_2 should be of the order of $\log(n)$. We can thus expect the overall time complexity of the sampler to be of the order of $n\log(n)$.

9.3. Some remarks on the discrete harmonic function $V(a,b)$

The harmonic function $V(a,b)$ of p-tandem walks in the quadrant, given by (60), is ubiquitous throughout Section 8, as it expresses the dependence of various asymptotic behaviours in terms of the starting point (a,b) of the random walk; see Propositions 8.2 and 8.3. The notion of discrete harmonic function is intrinsically interesting, as it is related to many probabilistic problems (Doob's h-transform and (non-)uniqueness problem, to quote a few of them). In this section we present some key features of $V(a,b)$.

9.3.1. ***The discrete harmonic function and Tutte's invariants.*** Recall that the values $V(a,b)$ must be related by the identity (61), which follows from a first step decomposition of random quadrant walks. This translates into a functional equation for the associated generating function $\mathcal{V}(u,v)$ defined by (60),

$$\left(\overline{S}(u,v)-1\right)\mathcal{V}(u,v) = z\bar{u}v\mathcal{V}(0,v) + \sum_{0<j\leqslant r\leqslant p} z_r u^{r-j}\bar{v}^j \sum_{k=0}^{j-1} v^k \mathcal{V}_k(u), \tag{79}$$

where

$$\overline{S}(u,v) = z\bar{u}v + \sum_{0\leqslant j\leqslant r\leqslant p} z_r u^{r-j}\bar{v}^j,$$

and $\mathcal{V}_k(u)$ is the coefficient of v^k in $\mathcal{V}(u,v)$. Observe that $\overline{S}(u,v)$ is closely related to the step polynomial $S(x,y)$ defined by (7). In fact, $\overline{S}(u,v)$ is $S(\bar{u},\bar{v})$, taken at $t=1$ and with a weight z on the first step. This comes from the fact that the recurrence relation (61) satisfied by the harmonic function is, in some sense, dual to the recurrence corresponding to the enumeration of walks.

Let $U_0, U_1, \dots, U_p$ be the $p+1$ solutions to $\overline{S}(u,v)=1$ (when solved for u). They are algebraic functions of v, z and the z_r's. Upon writing

$$\overline{S}(u,v) = z\bar{u}v + \sum_{r=0}^{p} z_r \frac{u^{r+1}-\bar{v}^{r+1}}{u-\bar{v}}, \tag{80}$$

we see that the equation $\overline{S}(u,v)=1$ implies

$$I_0(u) = I_0(\bar{v}), \qquad \text{with} \qquad I_0(u) = u + z\bar{u} - \sum_{r=0}^{p} z_r u^{r+1}. \tag{81}$$

In particular, all the functions $I_0(U_i)$ are equal: using the terminology of Tutte [86], we say that the rational function $I_0(u)$ is an *invariant*. We refer to [6, 43] for recent applications of invariants to quadrant problems.

It is shown in [77] how, in the case of small steps, the notion of invariants can be used to determine discrete harmonic functions in the quadrant. Let us illustrate this for the case $p=1$ of tandem walks (we take moreover $z_0=0$, so that $z=z_1=1/3$ as discussed in the examples following Proposition 8.2). In this case, the functional equation (79) reads

$$3uv\left(\overline{S}(u,v)-1\right)\mathcal{V}(u,v) = v^2\mathcal{V}(0,v) + u\mathcal{V}(u,0).$$

In [77], it is explained how one can give an analytic meaning to $\mathcal{V}(u,v)$ (even without knowing that it is a simple rational function), such that u can be specialized to U_0 and $U_1 = U_p$ in the above equation. Then the left hand-side vanishes, and it follows that

$$0 = v^2\mathcal{V}(0,v) + U_0\mathcal{V}(U_0,0) = v^2\mathcal{V}(0,v) + U_1\mathcal{V}(U_1,0),$$

so that the function $I(u) := u\mathcal{V}(u,0)$ is another invariant (if $p = 1$). Remarkably, this property, together with some properties peculiar to harmonic functions, eventually characterizes $I(u)$, and gives an expression for it in terms of the rational invariant $I_0(u)$ defined by (81):

$$I(u) = u\mathcal{V}(u,0) = \frac{C}{I_0(u) - I_0(1)},$$

for some normalizing constant C that can be determined (and is $2\sqrt{3}$ here).

We now return to general values of p. No counterpart of the analytic framework of [77] has been developed for large steps, and the role of invariants remains to be worked out in this setting. However, it follows from the exact value of $\mathcal{V}(u,v)$, given in Proposition 8.2, that the above identity still holds.

Proposition 9.2. *For p-tandem walks in the quadrant, the generating function $\mathcal{V}(u,v)$ of Proposition 8.2 satisfies*

$$u\mathcal{V}(u,0) = \frac{2}{\sigma} \cdot \frac{1}{I_0(u) - I_0(1)},$$

where $I_0(u)$ is the rational invariant defined by (81) and σ is given by (59). In particular, $u\mathcal{V}(u,0)$ is also an invariant.

The proof is a simple calculation, using the normalizing identity (55) and the zero-drift identity (57). We leave it to the reader. ∎

9.3.2. ***Probabilistic features.*** The aim of this section is twofold: we first comment on the links between the discrete harmonic function $V(a,b)$ and its continuous counterpart $V_\infty(a,b)$ (see (64)); then we give an estimation of $S^{(a,b)}(\tau^{(a,b)})$, the location of the random walk at its first exit (Proposition 9.3).

The asymptotic result (64) as both a and b tend to infinity,

$$V(a,b) \sim \frac{V_\infty(a,b)}{\sigma^3},$$

can be derived from purely probabilistic arguments: it is proved in [29, Lem. 13] that, when the covariance matrix is the identity, the discrete harmonic function $V(a,b)$ is asymptotically equivalent to the continuous harmonic function $V_\infty(a,b) = ab(a+b)$. The above factor $\frac{1}{\sigma^3}$ is thus due to a different normalization. It is remarkable that this factor captures all the dependence in the model in the parameters z and z_r. The function $V_\infty(a,b) = ab(a+b)$ can thus be viewed as a universal harmonic function for our class of models. This universality is also visible in the expression for the covariance matrix in Lemma 8.1.

In fact we can go further and state an exact formula relating V and V_∞.

Proposition 9.3. *Let us define the following shifted version of V_∞:*

$$V_\infty^{\mathrm{s}}(a,b) := V_\infty(a+1,b+1) = (a+1)(b+1)(a+b+2).$$

Then

$$V_\infty^{\mathrm{s}}(a,b) - \sigma^3 V(a,b) = \mathbb{E}\left(V_\infty^{\mathrm{s}}(S^{(a,b)}(\tau^{(a,b)}))\right), \tag{82}$$

where $S^{(a,b)}(\tau^{(a,b)})$ is the position where the random tandem walk started at (a,b) leaves the quadrant for the first time.

Remarks. **1.** If $p=1$ then $S^{(a,b)}(\tau^{(a,b)}))$ is necessarily of the form $(-1,j)$ or $(i,-1)$, for some integers $i,j \geqslant 0$. Since V^{s}_{∞} vanishes at these points, the expectation in (82) is zero, which is consistent with (62).

2. If $p \geqslant 2$, there are again two possibilities. The random walk may exit through the horizontal boundary, in which case $S^{(a,b)}(\tau^{(a,b)})$ is still of the form $(i,-1)$, where V^{s}_{∞} vanishes. It may also exit through the vertical boundary, and then $S^{(a,b)}(\tau^{(a,b)}) = (-i,j)$, for $1 \leqslant i \leqslant p$ and $j \geqslant 0$ and $V^{\mathrm{s}}_{\infty}(-i,j) = (1-i)(1+j)(j-i+2)$ does not necessarily vanish. Hence we can interpret (82) as an estimation of the square of the vertical coordinate, when an exit occurs through the vertical boundary.

Proof of Proposition 9.3. Let us define the function f by

$$f(a,b) = \mathbb{E}\left(V^{\mathrm{s}}_{\infty}((a,b)+(X,Y))\right) - V^{\mathrm{s}}_{\infty}(a,b). \tag{83}$$

Then it follows from [29, Eq. (5)] that

$$\sigma^3 V(a,b) = V^{\mathrm{s}}_{\infty}(a,b) - \mathbb{E}\left(V^{\mathrm{s}}_{\infty}(S^{(a,b)}(\tau^{(a,b)}))\right) + \mathbb{E}\left(\sum_{m=0}^{\tau^{(a,b)}-1} f(S^{(a,b)}(m))\right).$$

There are two slight differences between this formula and Eq. (5) in [29]: first, for the same reasons as above, we need the factor σ^3; second, the right-hand side of our formula involves V^{s}_{∞} rather than V_{∞} because the boundary axes of [29] are shifted by $(1,1)$ (more precisely, the walk is killed on the coordinate axes).

The function f defined by (83) can be computed explicitly, and is found to be 0. This completes the proof of the proposition. We compute f below in an independent lemma, which shows again the rich structure of tandem walks. ∎

Lemma 9.4. *The functions V_{∞} (and V^{s}_{∞}) are discrete harmonic functions for the random walk in $\mathbb{Z}^2$ with increments (X,Y) satisfying (56) (with no killing), in the sense that for all $(a,b) \in \mathbb{Z}^2$,*

$$\begin{aligned} V_{\infty}(a,b) &= \mathbb{E}\left(V_{\infty}((a,b)+(X,Y))\right) \\ &= zV_{\infty}(a+1,b-1) + \sum_{r=0}^{p} z_r \sum_{i+j=r} V_{\infty}(a-i,b+j), \end{aligned}$$

and similarly for V^{s}_{∞}.

Proof. Recall that $V_{\infty}(a,b) = ab(a+b)$. Hence

$$\mathbb{E}\left(V_{\infty}((a,b)+(X,Y))\right) = z(a+1)(b-1)(a+b) + \sum_{r=0}^{p} z_r \sum_{i+j=r} (a-i)(b+j)(a-i+b+j).$$

We write $j = r-i$, and sum over $i = 0,\dots,r$ for r fixed. Upon putting apart the cubic terms, this transforms the above expression into

$$zab(a+b) + \sum_{r=0}^{p} z_r (r+1)ab(a+b) + z(b-a-1)(a+b) + \sum_{r=0}^{p} z_r \frac{r(r+1)}{2}(a+b)(a-b+1).$$

Because of the drift condition (57), the sum of the third and fourth term vanishes. Moreover, the sum of the first two terms equals $ab(a+b) = V_\infty(a,b)$, due to the normalization condition (55). This completes the proof for V_∞. Finally, since $V_\infty^{\mathrm{s}}(a,b) = V_\infty(a+1,b+1)$, this function is also harmonic. ∎

9.4. More combinatorics for 1D walks

In this section, by a *1D walk* we mean any walk with steps in $\{-1\}\cup\mathbb{N} = \{-1,0,1,\dots\}$. Such walks are classically represented as directed 2D walks, upon drawing each step i as $(1,i)$. For $\mathcal{F}$ a family of 1D walks, we denote by F the associated generating function in $\mathbb{Q}[w_{-1},w_0,w_1,\dots][[t]]$, where the variable t records the length, and w_i the number of steps i, for $i \geqslant -1$. For such walks, we give here a combinatorial proof of two enumerative results that we have used in the paper. The first one deals with nonnegative walks going from height 0 to height a, and this result has been used in Section 7.2. The only published proofs that we know for it are algebraic [3, 19, 45]. The second one deals with nonnegative walks going from height k to height 0, and has been established algebraically when proving Corollary 3.3. (In practise we translate these walks so that they start at 0, end at $-k$, and never visit any vertex of height less than $-k$.)

We begin with preliminary arguments that have already been used in Section 7. For $a \geqslant 0$ we let $\mathcal{H}_a$ be the family of 1D walks going from height 0 to height a with nonnegative height all along. Furthermore, for $k \geqslant 0$ we let $\mathcal{D}_k$ be the family of 1D walks going from height 0 to height $-k$ and with height at least $-k$ all along (note that $\mathcal{D}_0 = \mathcal{H}_0$), and we let $\mathcal{L}_k$ be the subfamily of those that reach height $-k$ only at their ending point. By considering the first step i of a walk, we see that $Y := L_1$ is the unique power series in t satisfying

$$Y = t\sum_{i\geqslant -1} w_i Y^{i+1}.$$

Moreover, for $k \geqslant 1$ a decomposition of walks of $\mathcal{L}_k$ at their first visits to heights $-1,\dots,-k+1$ shows that $L_k = Y^k$. Since, clearly, $L_k = tw_{-1}D_{k-1}$, this gives, for $k \geqslant 0$,

$$D_k = \frac{1}{tw_{-1}} Y^{k+1}.$$

This is the analogue of (46).

9.4.1. *Nonnegative walks from 0 to a.*

The classical *kernel method* [3, 19] shows that

$$\sum_{a\geqslant 0} H_a u^a = \frac{1-\bar{u}Y}{1-tS(u)},$$

where $\bar{u} = 1/u$ and $S(u) = \sum_{i\geqslant -1} w_i u^i$ is the generating function of the steps. As argued in the proof of Lemma 7.3, this can be rewritten as

$$H_a = \frac{Y}{tw_{-1}}[u^a]\frac{1}{1-\frac{uY}{w_{-1}}\sum_{j,k\geqslant 0} w_{j+k+1}u^k Y^j}. \tag{84}$$

We now prove this identity combinatorially via standard path decompositions. As we will show, it implies the expression (53) for the series $H^{0\to a}$ counting tandem walks in the upper half-plane ending at ordinate a.

In a 1D walk a *record* is a point whose height is strictly smaller than the height of all subsequent points (by convention the endpoint is always considered as a record). For

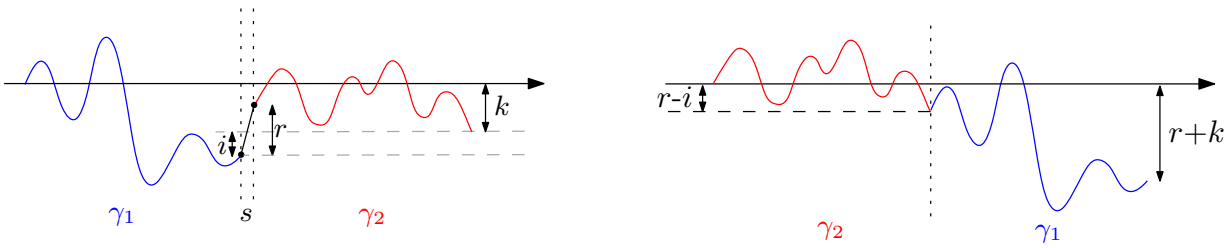


FIGURE 14. The bijection between $\mathcal{N}_{k,r,i}$ (for any fixed $i \leqslant r$) and $\mathcal{W}_{k+r}$.

a walk in $\mathcal{H}_a$ the heights of the successive records form an increasing sequence $0, i_1, i_1 + i_2, \ldots, i_1 + \cdots + i_e$, where $i_1, \ldots, i_e$ are positive and $i_1 + \cdots + i_e = a$; the sequence $(i_1, \ldots, i_e)$ is called the *record-sequence* of the walk (note that the record-sequence is empty if and only if $a = 0$). For $k \geqslant 1$ we let U_k be the generating function of those walks in $\mathcal{H}_k$ whose only records are the endpoints. This means that all points, except the starting point, have height at least k. Clearly, the generating function of walks in $\mathcal{H}_a$ that have record-sequence $(i_1, \ldots, i_e)$ is then equal to $H_0 \prod_{s=1}^{e} U_{i_s}$. Hence, by summing over all possible record-sequences, and using $H_0 = \frac{1}{tw_{-1}} Y$, we find:

$$H_a = \frac{Y}{tw_{-1}} [u^a] \frac{1}{1 - \sum_{k \geqslant 1} U_k u^k}.$$

We finally express the series U_k for $k \geqslant 1$. A walk counted by U_k consists of a first step $h > k$, followed by a (translated) walk of $\mathcal{D}_{h-k}$. The associated generating function is thus $tw_h D_{h-k} = \frac{w_h}{w_{-1}} Y^{h-k+1}$. Hence we have

$$U_k = \frac{Y}{w_{-1}} \sum_{h \geqslant k \geqslant 1} w_h Y^{h-k},$$

so that

$$\sum_{k \geqslant 1} U_k u^k = \frac{uY}{w_{-1}} \sum_{j,k \geqslant 0} w_{j+k+1} u^k Y^j,$$

which gives (84).

This result can now be applied to tandem walks. With the notation of Section 7.2, a tandem walk in $\mathcal{H}^{0 \to a}$, once projected on the vertical axis, becomes a 1D walk in $\mathcal{H}_a$: every SE step projects onto a -1 step, and for $r \geqslant 0$, a step r in the 1D walk can arise from any step $(-i, r)$ with $i \geqslant 0$. Hence, under the specialization $\{w_{-1} = 1, w_s = \sum_{r \geqslant s} z_r \text{ for } s \geqslant 0\}$, we have $Y = W$ and $H_a = H^{0 \to a}$. We thus recover (53).

9.4.2. ***Nonnegative walks from k to 0.*** Let $k \geqslant 1$. In this section we give a combinatorial proof of the following constant term expression for L_k:

$$L_k = Y^k = -t[y^0] y^{1+k} \frac{S'(y)}{K(y)}, \tag{85}$$

where $S(y) = \sum_{i \geqslant -1} w_i y^i$ and $K(y) := 1 - tS(y)$. Note that it directly yields the identity (39) used to prove Corollary 3.3, by taking the specialization for tandem walks: $\{w_{-1} = x, w_s = \sum_{r \geqslant s} \bar{x}^{r-s} z_r \text{ for } s \geqslant 0\}$.

Our combinatorial argument is borrowed from [78] (see also the interesting discussion therein regarding the fact that, for walks with steps ± 1, the argument is originally due to Désiré André, to whom the reflection principle was wrongly attributed).

Let $\mathcal{W}_k$ be the family of 1D walks ending at height $-k$, and let $\mathcal{N}_k = \mathcal{W}_k \backslash \mathcal{L}_k$ be the subset of those walks that visit height $-k$ before their final point. For $\gamma \in \mathcal{N}_k$, we define the *marked step* of γ as the last step that starts at height $\leqslant -k$. For $0 \leqslant i \leqslant r$, let $\mathcal{N}_{k,r,i}$ be the subfamily of $\mathcal{N}_k$ where the marked step is a step $+r$ that starts at height $-k-i$ (if $i = r$ then the marked step is always the last step).We refer to Figure 14 for an illustration. We claim that $\mathcal{N}_{k,r,i}$ and $\mathcal{W}_{k+r}$ are in bijection. To $\gamma \in \mathcal{N}_{k,r,i}$, written as $\gamma_1 s \gamma_2$ with s the marked step, we associate $\gamma_2\gamma_1 \in \mathcal{W}_{k+r}$. Conversely, for $\gamma \in \mathcal{W}_{k+r}$, we let γ_2 be the prefix of γ ending at the first visit to height $i-r$ (if $i = r$ then γ_2 is empty), and we take γ_1 as the corresponding suffix of γ; we associate to γ the walk $\gamma_1 s \gamma_2$ where s is a $+r$ step. In terms of generating functions this gives $N_{k,r,i} = t w_r W_{k+r}$. Hence

$$N_k = \sum_{r \geqslant 0} \sum_{i=0}^{r} N_{k,r,i} = \sum_{r \geqslant 0} \sum_{i=0}^{r} t w_r W_{k+r} = t \sum_{r \geqslant 0} (r+1) w_r W_{k+r}.$$

Moreover, for all $j \geqslant 1$, we clearly have

$$W_j(t) = [y^{-j}] \frac{1}{K(y)} = [y^0] \frac{y^j}{K(y)}.$$

Since $L_k = W_k - N_k$, for $k \geqslant 1$ we obtain

$$\begin{aligned} L_k &= [y^0] \frac{y^k}{K(y)} \cdot \Big(1 - t \sum_{r \geqslant 0} (r+1) w_r y^r \Big) \\ &= [y^0] \frac{y^k}{K(y)} \cdot \Big(K(t,y) + t w_{-1} y^{-1} - t \sum_{r \geqslant 0} r w_r y^r \Big) \\ &= t [y^0] \frac{y^k}{K(y)} \cdot \Big(w_{-1} y^{-1} - \sum_{r \geqslant 0} r w_r y^r \Big). \end{aligned}$$

Since $yS'(y) = -y^{-1} w_{-1} + \sum_{r \geqslant 0} r w_r y^r$, this gives the expression (85) for L_k.

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CNRS, Laboratoire Bordelais de Recherche en Informatique, Université de Bordeaux, France
Email address: mireille.bousquet-melou@u-bordeaux.fr

CNRS, Laboratoire d'Informatique de l'École Polytechnique, France
Email address: fusy@lix.polytechnique.fr

CNRS, Institut Denis Poisson, Université de Tours et Université d'Orléans, France
Email address: raschel@math.cnrs.fr