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NON-REDUCED MODULI SPACES OF SHEAVES ON MULTIPLE CURVES

JEAN–MARC DRÉZET

Abstract. Some coherent sheaves on projective varieties have a non reduced versal deformation space. For example, this is the case for most unstable rank 2 vector bundles on $\mathbb{P}_2$ (cf. [18]). In particular, it may happen that some moduli spaces of stable sheaves are non reduced.

We consider the case of some sheaves on ribbons (double structures on smooth projective curves): the quasi locally free sheaves of rigid type. Let E be such a sheaf.

– Let $\mathcal{E}$ be a flat family of sheaves containing E. We find that it is a reduced deformation of E when some canonical family associated to $\mathcal{E}$ is also flat.

– We consider a deformation of the ribbon to reduced projective curves with two components, and find that E can be deformed in two distinct ways to sheaves on the reduced curves. In particular some components $\mathbf{M}$ of the moduli spaces of stable sheaves deform to two components of the moduli spaces of sheaves on the reduced curves, and $\mathbf{M}$ appears as the "limit" of varieties with two components, whence the non reduced structure of $\mathbf{M}$.

Mathematics Subject Classification : 14D20, 14B20

1. Introduction

Let Y be a projective variety over $\mathbb{C}$ and E a coherent sheaf on Y. Let $(V_E, v_0, \mathcal{E}_E, \alpha)$ be a semi-universal deformation of E ((V_E, v_0) is a germ of an analytic variety, $\mathcal{E}_E$ is a coherent sheaf on $Y \times V_E$ and $\alpha : \mathcal{E}_{E,s_0} \to E$ is an isomorphism, cf. [17]). It may happen that V_E is not reduced at s_0, for example in the case of unstable rank-2 vector bundles on $\mathbb{P}_2$ (cf. [18]). We will study here the case of some sheaves on multiple curves (in particular on *ribbons*, i.e. double structures on smooth projective curves), and try to see *why* V_E is not reduced, and similarly why some moduli spaces of stable sheaves on Y are not reduced.

If $\mathcal{F}$ is a flat family of sheaves on Y parametrised by the germ (T, t_0) of an analytic variety, and if $\mathcal{F}_{t_0} \simeq E$, there is a morphism $\phi : (T, t_0) \to (V_E, v_0)$ such that $(I_X \times \phi)^*(\mathcal{E}_E) \simeq \mathcal{F}$, with a uniquely defined tangent map

$$T_{t_0}T \longrightarrow T_{v_0}V_E = \mathrm{Ext}^1_{\mathcal{O}_X}(E, E) \ .$$

If T is reduced then ϕ can be factorized to $V_{E,red}$, the reduced germ associated to V_E: $\phi : T \xrightarrow{\phi'} V_{E,red} \hookrightarrow V_E$. It is then natural to ask:

(i) What is the tangent space $T_{v_0}V_{E,red} \subset \mathrm{Ext}^1_{\mathcal{O}_X}(E, E)$?
(ii) Under which conditions on $\mathcal{F}$, when T is non reduced, is ϕ a morphism to $V_{E,red}$?

and the more vague question

(iii) Why can V_E be non reduced ?

The problem can also be stated in terms of moduli spaces of stable sheaves: let $\mathcal{O}_Y(1)$ be an ample line bundle on Y and P_E the Hilbert polynomial of E. Let $\mathbf{M}$ be the moduli space of

sheaves on Y, stable with respect to $\mathcal{O}_X(1)$, and with Hilbert polynomial P_E. Suppose now that E is stable. We can then ask

(i') What is the tangent space $T_E\mathbf{M}_{red} \subset T_E\mathbf{M}$?
(ii') If $\mathcal{F}$ is a flat family of stable sheaves with Hilbert polynomial P_E, parametrised by a variety T, we get a morphism $\phi : T \to \mathbf{M}$. Under which conditions on $\mathcal{F}$, when T is non reduced, is ϕ a morphism to $\mathbf{M}_{red}$?
(iii') Why can $\mathbf{M}$ be non reduced ?

We suppose now that Y is a *primitive multiple curve* (i.e. a Cohen-Macaulay scheme whose associated reduced scheme is a projective smooth curve, and such that Y is locally embeddable in a smooth surface). In this case, there is a canonical filtration of E, and the first result of this paper (theorem 3.2.1) is that the associated reduced scheme $V_{E,red}$ of V_E corresponds to deformations of E such that this associated filtration deforms flatly together with a flat deformation of E. A similar result has been proved in [18] for unstable rank-2 vector bundles on $\mathbb{P}_2$.

Suppose now that Y is a *ribbon* (a primitive multiple curve of multiplicity 2). For suitable sheaves E, we have seen in [11] that the fact that V_E is not reduced can also be explained by considering deformations of Y to reduced curves with two irreducible components and the associated deformations of E: in fact E can be deformed in two distinct ways to sheaves on the deformations of Y. In other words the deformation space V_E can be seen as the "limit" of two sequences of deformation spaces of sheaves on the reduced curves with two irreducible components, whence the non reduced structure on V_E.

The second result of this paper (theorem 5.2.1) is a description of the part of the non reduced structure of V_E that comes from such a deformation of Y.

1.1. First result – Good families of sheaves on multiple curves

Let Y be a primitive multiple curve and $C = Y_{red}$ the associated smooth curve. Let $\mathcal{I}_C$ be the ideal sheaf of C in Y. The *multiplicity* of Y is the smallest integer n such that $\mathcal{I}_C^n = 0$. We have a filtration $C = C_1 \subset \cdots \subset C_n = Y$, where for $1 \leq i \leq n$, C_i is the subscheme corresponding to the ideal sheaf $\mathcal{I}_C^i$ (C_i is a primitive multiple curve of multiplicity i).

A coherent sheaf $\mathcal{E}$ on Y is called *quasi locally free* if there exist integers $m_1, \ldots, m_n$ such that $\mathcal{E}$ is locally isomorphic to $\bigoplus_{i=1}^{n} \mathcal{O}_{C_i} \otimes \mathbb{C}^{m_1}$. The sequence $(m_1, \ldots, m_n)$ is called the *type* of $\mathcal{E}$.

Let X be a connected algebraic variety, and $\mathbb{E}$ a coherent sheaf on $X \times Y$, flat on X, such that for every closed point $x \in X$, $\mathbb{E}_x$ is quasi locally free of type $(m_1, \ldots, m_n)$. For $0 \leq i \leq n$ let $\mathbb{E}_i = \mathcal{I}_C^i \mathbb{E}$ (so that $\mathbb{E}_i/\mathbb{E}_{i+1}$ is concentrated on $X \times C$). We say that $\mathbb{E}$ is a *good family* if for $0 \leq i < n$ the sheaf $\mathbb{E}_i/\mathbb{E}_{i+1}$ on $X \times C$ is flat on X (which is equivalent to say that it is locally free on C). Then we have the theorem 3.2.1:

Theorem: 1 – *The sheaf $\mathbb{E}$ is a good family if and only if it is locally isomorphic to $\bigoplus_{i=1}^{n} \mathcal{O}_{X\times C_i} \otimes \mathbb{C}^{m_i}$.*

2 – *If $\mathbb{E}$ is a good family on $X \times Y$, then for every $x \in X$ the image of the Kodaïra-Spencer morphism of $\mathbb{E}$*

$$\omega_x(\mathbb{E}) : TX_x \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_Y}(\mathbb{E}_x, \mathbb{E}_x)$$

is contained in $H^1(\mathcal{E}nd(\mathbb{E}_x))$.

For the sheaves on ribbons studied in 1.2, $H^1(\mathcal{E}nd(\mathbb{E}_x))$ is the tangent space of $V_{\mathbb{E}_x,red}$, and if $\mathbb{E}$ is a good family then the image of the canonical morphism to $V_{\mathbb{E}_x}$ associated to $\mathbb{E}$ at x is contained in $V_{\mathbb{E}_x,red}$.

1.2. Second result – The non-reduced structure of the deformation spaces of sheaves on a ribbon Y and its relations with the deformations of Y

Suppose that Y is a ribbon (a primitive multiple curve of multiplicity 2). Let $C = Y_{red}$. For vector bundles V_E is smooth (because in this case $\mathrm{Ext}^2_{\mathcal{O}_Y}(E,E) = \{0\}$). We will consider *quasi-locally free sheaves of rigid type* (defined and studied in [6]), i.e. coherent sheaves locally isomorphic to $r\mathcal{O}_Y \oplus \mathcal{O}_C$ for some integer r. Deformations of these sheaves are also quasi-locally free sheaves of rigid type, and $\deg(E_{|C})$ is invariant under deformation.

For these sheaves, for (i) we have from [6]: $T_{v_0}V_{E,red} = H^1(\mathcal{E}nd(E))$. Let L be the ideal sheaf of C in Y. It is a line bundle on C. We will prove in 3.1.3 that

$$T_{v_0}V_E/T_{v_0}V_{E,red} \;\simeq\; H^0(\mathcal{E}xt^1_{\mathcal{O}_Y}(E,E)) \;\simeq\; H^0(L^*)\ .$$

We suppose that $\deg(L) < 0$, and consider a *maximal reducible deformation* of Y, i.e. a flat morphism $\pi : \mathcal{C} \to S$, and $P \in S$, such that

- S is a smooth curve and $\mathcal{C}_P \simeq Y$.
- $\mathcal{C}$ is a reduced algebraic variety with two irreducible components $\mathcal{C}_1, \mathcal{C}_2$.
- For $i = 1, 2$, let $\pi_i : \mathcal{C}_i \to S$ be the restriction of π. Then $\pi_i^{-1}(P) = C$ and π_i is a flat family of smooth irreducible projective curves.
- For every $s \in S\backslash\{P\}$, the components $\mathcal{C}_{1,s}, \mathcal{C}_{2,s}$ of $\mathcal{C}_s$ meet transversally.

For every $s \in S\backslash\{P\}$, $\mathcal{C}_{1,s}$ and $\mathcal{C}_{2,s}$ meet in exactly $-\deg(L)$ points. It is proved in [10] that such a deformation exists if L can be written as $L = \mathcal{O}_C(-P_1 - \cdots - P_k)$, for distinct points $P_1, \ldots, P_k$ of C. In this case we can construct $\mathcal{C}$ such that if $\mathcal{Z}$ is the closure of the set of intersections points of the components $\mathcal{C}_{1,s}$, $\mathcal{C}_{2,s}$ of $\mathcal{C}_s$, $s \neq P$, we have $\mathcal{Z} \cap C = \{P_1, \ldots, P_k\}$.

We denote by $\Delta_{\mathcal{C}} \subset H^0(L^*)$ the 1-dimensional subspace generated by the isomorphism $L \simeq \mathcal{O}_C(-P_1 - \cdots - P_k)$. Note that this parameter $\Delta_{\mathcal{C}}$ of $\mathcal{C}$ lies in the vector space $T_{v_0}V_E/T_{v_0}V_{E,red}$ which measures the non-reduceness of V_E.

To relate the deformations of E and those of Y we use the *Kodaïra-Spencer elements* associated to pairs of sheaves on $\mathcal{C}$ (cf. 2.2). Let $\mathcal{E}_1$, $\mathcal{E}_2$ be coherent sheaves on $\mathcal{C}$, flat on S, and such that $\mathcal{E}_{1,P} = \mathcal{E}_{2,P} = E$. Then we define

$$\omega_{\mathcal{E}_1,\mathcal{E}_2} \;\in\; \mathrm{Ext}^1_{\mathcal{O}_Y}(E,E)$$

On the second neighbourhood Y_2 of Y in $\mathcal{C}$ we have exact sequences

$$0 \longrightarrow E \longrightarrow \mathcal{E}_{i|Y_2} \longrightarrow E \longrightarrow 0$$

for $i = 1, 2$, and associated elements $\sigma_i \in \mathrm{Ext}^1_{\mathcal{O}_{Y_2}}(E,E)$. The difference $\omega_{\mathcal{E}_1,\mathcal{E}_2} = \sigma_1 - \sigma_2$ lies in $\mathrm{Ext}^1_{\mathcal{O}_Y}(E,E)$.

Let $\mathcal{E}$ be a coherent sheaf on $\mathcal{C}$, flat on S and such that $\mathcal{E}_{|Y} = E$. Recall that E is locally isomorphic to $r\mathcal{O}_Y \oplus \mathcal{O}_C$ with $r > 0$. Then from [11] there are two possibilities: for every $s \in S\backslash\{P\}$ in a neighbourhood of P

1) $\mathcal{E}_{s|\mathcal{C}_{1,s}\backslash\mathcal{Z}}$ is locally free of rank r and $\mathcal{E}_{s|\mathcal{C}_{2,s}\backslash\mathcal{Z}}$ is locally free of rank $r+1$,
2) $\mathcal{E}_{s|\mathcal{C}_{1,s}\backslash\mathcal{Z}}$ is locally free of rank $r+1$ and $\mathcal{E}_{s|\mathcal{C}_{2,s}\backslash\mathcal{Z}}$ is locally free of rank r,

(that is: $\mathcal{E}$ is of rank r on one of the components of $\mathcal{C}$ and of rank $r+1$ on the other). So we see that E can be deformed in two distinct ways to sheaves on the reduced curves with two components.

We then consider two such coherent sheaves $\mathcal{E}^{[1]}$, $\mathcal{E}^{[2]}$, on $\mathcal{C}$, and we have two cases:

Case A

– $\mathcal{E}_s^{[1]}$ is of rank r on $\mathcal{C}_{2,s}\backslash\mathcal{Z}$ and of rank $r+1$ on $\mathcal{C}_{1,s}\backslash\mathcal{Z}$.
– $\mathcal{E}_s^{[2]}$ is of rank $r+1$ on $\mathcal{C}_{2,s}\backslash\mathcal{Z}$ and of rank r on $\mathcal{C}_{1,s}\backslash\mathcal{Z}$.

Case B

– $\mathcal{E}_s^{[1]}$ and $\mathcal{E}_s^{[2]}$ are of rank r on $\mathcal{C}_{2,s}\backslash\mathcal{Z}$ and of rank $r+1$ on $\mathcal{C}_{1,s}\backslash\mathcal{Z}$.

Let

$$\phi : \mathrm{Ext}^1_{\mathcal{O}_Y}(E,E) \longrightarrow H^0(\mathcal{E}xt^1_{\mathcal{O}_Y}(E,E)) \simeq H^0(L^*)$$

be the canonical morphism. Recall that its kernel $H^1(\mathcal{E}nd(E))$ corresponds to good deformations of E, or deformations parametrised by a reduced variety (cf. 1.1). Then we have (theorem 5.2.1)

Theorem : **1** – *In case A, $\phi(\omega_{\mathcal{E}^{[1]},\mathcal{E}^{[2]}})$ generates $\Delta_{\mathcal{C}}$.*

2 – *in case B, we have $\phi(\omega_{\mathcal{E}^{[1]},\mathcal{E}^{[2]}}) = 0$.*

This means that the "non-reduced part" of the deformation of E to sheaves on $\mathcal{C}$ in $T_{v_0}V_E/T_{v_0}V_{E,red}$, corresponds to some parameter of the deformation of Y to reduced curves with two components. If E is stable, we have two canonical morphisms $f_i : U \to \mathbf{M}$, $i = 1, 2$, corresponding to $\mathcal{E}^{[i]}$, where U is a neighbourhood of P in S and $\rho : \mathbf{M} \to S$ is the relative moduli space of stable sheaves on $\mathcal{C}$ containing E, and we have $f_i(P) = E$. The image of $Tf_{1,P} - Tf_{2,P}$ is contained in $\mathrm{Ext}^1_{\mathcal{O}_Y}(E,E)$ (the tangent space of $\rho^{-1}(P)$ at E). The preceding theorem implies that in case A, the image of $Tf_{1,P} - Tf_{2,P}$ is contained in $\phi^{-1}(\Delta_{\mathcal{C}})$, and in case B, its image is contained in $T_{\mathbf{M}_{red}}$.

Remarks: – It is proved in [11] that given a quasi locally free sheaf E on Y (i.e. a sheaf locally isomorphic to a direct sum $a\mathcal{O}_Y \oplus b\mathcal{O}_C$), there exists a smooth curve S', $P' \in S'$, a morphism $f : S' \to S$ such that $f(P') = P$, with non zero tangent map at P', and a coherent sheaf $\mathcal{E}'$ on $f^*\mathcal{C}$ flat on S' and such that $\mathcal{E}'_{P'} = E$, i.e. E can be deformed to sheaves on the reduced curves with two components.

– In [6] many examples of non-empty moduli spaces of stable sheaves containing quasi locally free sheaves of rigid type are given.

Notation: In this paper, an *algebraic variety* is a quasi-projective scheme over $\mathbb{C}$.

1.3. Outline of the paper

Section 2 contains definitions and properties of *primitive multiple curves* of any multiplicity, with some particular results for ribbons. This section contains also a description of the Kodaïra-Spencer elements that are used here.

Section 3 is devoted to the study of quasi locally free sheaves of rigid type on a primitive multiple curve, and to their deformations. In particular we give an answer to question (ii). Some results are valid in any multiplicity.

In section 4 we recall some definitions concerning *maximal reducible deformations* of ribbons, i.e. deformations to reduced curves with two components intersecting transversally. We recall also some results about deformations of quasi locally free sheaves of rigid type on ribbons to sheaves on the reduced curves with two components.

In section 5 we prove the main result of this paper, i.e. the theorem 5.2.1.

2. Preliminaries

2.1. Primitive multiple curves and quasi locally free sheaves

(cf. [1], [2], [4], [5], [6], [7], [8], [12]).

2.1.1. *Definitions* – Let C be a smooth connected projective curve. A *multiple curve with support* C is a Cohen-Macaulay scheme Y such that $Y_{red} = C$.

Let n be the smallest integer such that $Y = C^{(n-1)}$, $C^{(k-1)}$ being the k-th infinitesimal neighbourhood of C, i.e. $\mathcal{I}_{C^{(k-1)}} = \mathcal{I}_C^k$. We have a filtration $C = C_1 \subset C_2 \subset \cdots \subset C_n = Y$ where C_i is the biggest Cohen-Macaulay subscheme contained in $Y \cap C^{(i-1)}$. We call n the *multiplicity* of Y. We say that Y is *primitive* if, for every closed point x of C, there exists a smooth surface S, containing a neighbourhood of x in Y as a locally closed subvariety. In this case, $L = \mathcal{I}_C/\mathcal{I}_{C_2}$ is a line bundle on C and we have $\mathcal{I}_{C_j} = \mathcal{I}_C^j$, $\mathcal{I}_{C_j}/\mathcal{I}_{C_{j+1}} = L^j$ for $1 \leq j < n$. We call L the line bundle on C *associated* to Y. Let $P \in C$. Then there exist elements y, t of $m_{S,P}$ (the maximal ideal of $\mathcal{O}_{S,P}$) whose images in $m_{S,P}/m_{S,P}^2$ form a basis, and such that for $1 \leq i < n$ we have $\mathcal{I}_{C_i,P} = (y^i)$.

We will write $\mathcal{O}_n = \mathcal{O}_{C_n}$ and we will see $\mathcal{O}_i$ as a coherent sheaf on C_n with schematic support C_i if $1 \leq i < n$.

2.1.2. *The canonical filtrations* – Let $\mathcal{E}$ be a coherent sheaf on Y. The *first canonical filtration* of $\mathcal{E}$, $0 = \mathcal{E}_n \subset \mathcal{E}_{n-1} \subset \cdots \subset \mathcal{E}_0 = \mathcal{E}$, is defined as follows: for $0 \leq i \leq n$, we have $\mathcal{E}_i = \mathcal{I}_C^i \mathcal{E}$. For $0 \leq i < n$, the sheaf $G_i(\mathcal{E}) = \mathcal{E}_i/\mathcal{E}_{i+1}$ is concentrated on C. The same definition applies if $\mathcal{E}$ is a coherent sheaf on a non-empty open subset of Y. The pair

$$\sigma(\mathcal{E}) \;=\; \big((\mathrm{rk}(G_0(\mathcal{E})), \ldots, \mathrm{rk}(G_{n-1}(\mathcal{E}))\ ,\ (\deg(G_0(\mathcal{E})), \ldots, \deg(G_{n-1}(\mathcal{E}))\big)$$

is called the *complete type* of $\mathcal{E}$.

Let X be an algebraic variety and $\mathbb{E}$ a coherent sheaf on $X \times Y$, flat on X. We can also define the first canonical filtration of $\mathbb{E}$, $0 = \mathbb{E}_n \subset \mathbb{E}_{n-1} \subset \cdots \subset \mathbb{E}_0 = \mathbb{E}$, by $\mathbb{E}_i = p_Y^*(\mathcal{I}_C^i)\mathbb{E}$.

The *second canonical filtration* of $\mathcal{E}$, $0 = \mathcal{E}^{(0)} \subset \mathcal{E}^{(1)} \subset \cdots \subset \mathcal{E}^{(n)} = \mathcal{E}$, is defined as follows: for $0 \leq i \leq n$ and $P \in C$, $\mathcal{E}_P^{(i)}$ is the set of $u \in \mathcal{E}_P$ such that $\mathcal{I}_{C,P}^i u = 0$. For $1 \leq i \leq n$, the sheaf $G^{(i)}(\mathcal{E}) = \mathcal{E}^{(i)}/\mathcal{E}^{(i-1)}$ is concentrated on C.

2.1.3. *The case of double curves* – If $n = 2$, let $\mathcal{E}$ be a coherent sheaf on $Y = C_2$. Then we have canonical exact sequences

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_{|C} \longrightarrow 0 \ , \qquad 0 \longrightarrow \mathcal{E}^{(1)} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}^{(2)} \simeq \mathcal{E}_1 \otimes L^* \longrightarrow 0 \ .$$

2.1.4. *Quasi locally free sheaves* – Let $P \in C$ and M a $\mathcal{O}_{Y,P}$-module of finite type. We say that M is *quasi free* if there exist integers $m_i \geq 0$, $1 \leq i \leq n$, such that $M \simeq \oplus_{i=1}^n m_i \mathcal{O}_{i,P}$. These integers are uniquely determined. In this case we say that M is *of type* $(m_1, \ldots, m_n)$.

Let $\mathcal{E}$ be a coherent sheaf on a non-empty open subset $V \subset Y$. We say that $\mathcal{E}$ is *quasi locally free* at a point P of V if there exists a neighbourhood $U \subset V$ of P and integers $m_i \geq 0$, $1 \leq i \leq n$, such that for every $Q \in U$, $\mathcal{E}_Q$ is quasi free of type $m_1, \ldots, m_n$. The integers $m_1, \ldots, m_n$ are uniquely determined and depend only of $\mathcal{E}$, and $(m_1, \ldots, m_n)$ is called the *type of* $\mathcal{E}$.

We say that $\mathcal{E}$ is *quasi locally free* if it is quasi locally free at every point of V. The module $\mathcal{E}_P$ is quasi locally free at P if and only if the $\mathcal{O}_{C,P}$-modules $G_i(\mathcal{E}_P)$ are free, and $\mathcal{E}$ is quasi locally free if and only if the sheaves $G_i(\mathcal{E})$ are locally free on C.

2.2. Generalisation of the Kodaïra-Spencer morphism

Let S be a smooth curve and $s_0 \in S$ a closed point. Let $\rho : \mathcal{X} \to S$ be a flat projective morphism of algebraic varieties. Let $Y = \rho^{-1}(s_0)$. It is a projective variety. Let $\mathcal{E}$, $\mathcal{E}'$ be coherent sheaves on $\mathcal{X}$, flat on S, such that there exists an isomorphism $\mathcal{E}_{|Y} \simeq \mathcal{E}'_{|Y}$. Let $E = \mathcal{E}_{|Y}$. Let Y_2 be the second infinitesimal neighbourhood of Y in $\mathcal{X}$. If $s_0^{(2)}$ is the second infinitesimal neighbourhood of s_0 in S, we have $Y_2 = \rho^{-1}(s_0^{(2)})$. The ideal sheaf $\mathcal{I}_Y$ of Y in Y_2 is isomorphic to $\mathcal{O}_Y$. We have canonical exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E = E \otimes \mathcal{I}_Y & \xrightarrow{i} & \mathcal{E}_{|Y_2} & \longrightarrow & E & \longrightarrow & 0 \\ & & \| & & & & & & \\ 0 & \longrightarrow & E = E \otimes \mathcal{I}_Y & \xrightarrow{i'} & \mathcal{E}'_{|Y_2} & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

(the injectivity of i and i' follows easily from the flatness of $\mathcal{E}$, $\mathcal{E}'$ over S). Let $\sigma, \sigma' \in \mathrm{Ext}^1_{\mathcal{O}_{Y_2}}(E, E)$ correspond to these extensions.

Let $0 \to E \to \mathcal{F} \to E \to 0$ be an exact sequence of coherent sheaves on Y_2. The canonical morphism $\mathcal{F} \otimes \mathcal{I}_Y \to \mathcal{F}$ induces a morphism $E \otimes \mathcal{I}_Y \to E$, which vanishes if and only if $\mathcal{F}$ is concentrated on Y. In this way we get an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_Y}(E, E) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{Y_2}}(E, E) \xrightarrow{p} \mathrm{End}(E) \ .$$

We have $p(\sigma) = p(\sigma') = I_E$. So we have $\omega_{\mathcal{E},\mathcal{E}'} = \sigma - \sigma' \in \mathrm{Ext}^1_{\mathcal{O}_Y}(E, E)$. We call $\omega_{\mathcal{E},\mathcal{E}'}$ the *Kodaïra-Spencer element associated to* $\mathcal{E}$, $\mathcal{E}'$.

Connections with the Kodaïra-Spencer morphism – Suppose the $\mathcal{X}$ is the trivial family: $\mathcal{X} = Y \times S$. Let

$$\omega_{s_0} : T_{s_0}S \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_Y}(E, E)$$

be the Kodaïra-Spencer morphism of $\mathcal{E}$. Suppose that $\mathcal{E}'$ is the trivial family: $\mathcal{E}' = p_Y^*(E)$ (where p_Y is the projection $Y \times S \to S$). The isomorphism $\mathcal{I}_Y \simeq \mathcal{O}_Y$ is defined by the choice of a generator t of the maximal ideal of s_0 in S. Let u be the associated element of $T_{s_0}S$. Then we have $\omega_{\mathcal{E},\mathcal{E}'} = \omega_{s_0}(u)$.

3. Quasi locally free sheaves of rigid type

We keep the notations of 2.1.

3.1. Definitions and basic properties

A quasi locally free sheaf $\mathcal{E}$ on Y is called *of rigid type* if it is locally free or locally isomorphic to $a\mathcal{O}_n \oplus \mathcal{O}_k$ for some integers $a \geq 0$, $1 \leq k < n$. *The set of isomorphism classes of quasi locally free sheaves of rigid type of fixed complete type (cf. 2.1.2) is an open family* (cf. [6], 6-): let X be an algebraic variety, $\mathbb{E}$ a coherent sheaf on $X \times Y$, flat on X, and $x \in X$ a closed point. Suppose that $\mathbb{E}_x$ is quasi locally free of rigid type. Then there exists an open subset U of X containing x such that, for every $x' \in U$, $\mathbb{E}_{x'}$ is quasi locally free and $\sigma(\mathbb{E}_{x'}) = \sigma(\mathbb{E}_x)$.

More generally, let $\mathcal{E}$ be a quasi locally free sheaf on Y, locally isomorphic to $a\mathcal{O}_n \oplus b\mathcal{O}_k$, with $a, b > 0$, $1 \leq k < n$. By [6], proposition 5.1, there exists a vector bundle $\mathbb{E}$ on Y and a surjective morphism

$$\phi : \mathbb{E} \longrightarrow \mathcal{E}$$

inducing an isomorphism $\mathbb{E}_{|C} \simeq \mathcal{E}_{|C}$. Let $\mathcal{L} = \ker(\phi)$. By [6], lemme 5.2, $\mathcal{L}$ is a vector bundle of rank b on C_{n-k}. Let $P \in C$, and $z \in \mathcal{O}_{Y,P}$ an equation of C. At P the exact sequence $0 \to \mathcal{L} \to \mathbb{E} \to \mathcal{E} \to 0$ is isomorphic to the trivial one

$$0 \longrightarrow (z^k) \otimes \mathbb{C}^b \simeq \mathcal{O}_{n-k,P} \otimes \mathbb{C}^b \longrightarrow \mathcal{O}_{n,P} \otimes (\mathbb{C}^a \oplus \mathbb{C}^b) \longrightarrow (\mathcal{O}_{n,P} \otimes \mathbb{C}^a) \oplus (\mathcal{O}_{k,P} \otimes \mathbb{C}^b) \longrightarrow 0.$$

3.1.1. Lemma: *There is a canonical isomorphism*

$$\mathcal{L}_{|C_k} \simeq (\mathcal{E}^{(k)}/\mathcal{E}_{n-k}) \otimes \mathcal{I}_C^k .$$

Proof. Let $P \in C$, $z \in \mathcal{O}_{Y,P}$ an equation of C and $u \in \mathcal{E}_P^{(k)}$. Let $v \in \mathbb{E}_P$ be such that $\phi_P(v) = u$. Then we have $\phi_P(z^k v) = z^k u = 0$, hence $z^k v \in \mathcal{L}_P$. If $v' \in \mathbb{E}_P$ is such that $\phi_P(v') = u$, we have $w = v' - v \in \mathcal{L}_P$, hence the image of $z^k v' = z^k v + z^k w$ in $\mathcal{L}_P / z^k \mathcal{L}_P$ is the same as that of $z^k v$. By associating $z^k v$ to $u \otimes z^k$ we define a morphism $\overline{\theta} : \mathcal{E}^{(k)} \otimes \mathcal{I}_C^k \to \mathcal{L}_{|C_k}$. If $u \in \mathcal{E}_{n-k,P}$, let $u' \in \mathcal{E}_P$ be such that $u = z^{n-k}u'$. Let $v' \in \mathbb{E}_P$ be such that $\phi_P(v') = u'$. Then we can take $v = z^{n-k}v'$, and then $z^k v = 0$, hence $\overline{\theta}_P(u \otimes z^k) = 0$. It follows that $\overline{\phi}$ induces a morphism

$$\theta : (\mathcal{E}^{(k)}/\mathcal{E}_{n-k}) \otimes \mathcal{I}_C^k \longrightarrow \mathcal{L}_{|C_k} .$$

In the above description of the exact sequence $0 \to \mathcal{L} \to \mathbb{E} \to \mathcal{E} \to 0$,
we have $\mathcal{E}_P^{(k)}/\mathcal{E}_{n-k,P} = \mathcal{O}_{k,P} \otimes \mathbb{C}^b$, and $\theta_P = (\mathcal{O}_{k,P} \otimes \mathbb{C}^b) \otimes (z^k) \to (z^k) \otimes \mathbb{C}^b$ is the identity morphism. Hence ϕ is an isomorphism. □

The sheaf $\mathcal{L}_{|C_k}$ is a vector bundle of rank b, on C_k if $2k \leq n$, and on C_{n-k} if $2k > n$.

3.1.2. Corollary: *There is a canonical isomorphism*

$$\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{E},\mathcal{E}) \ \simeq \ \mathcal{H}om((\mathcal{E}^{(k)}/\mathcal{E}_{n-k}) \otimes \mathcal{I}^k_C, \mathcal{E}^{(k)}/\mathcal{E}_{n-k}) \ .$$

Proof. From the exact sequence $0 \to \mathcal{L} \to \mathbb{E} \to \mathcal{E} \to 0$, we deduce the exact sequence

$$\mathcal{H}om(\mathbb{E},\mathcal{E}) \longrightarrow \mathcal{H}om(\mathcal{L},\mathcal{E}) \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{E},\mathcal{E}) \longrightarrow 0 \ ,$$

and the result follows easily using local isomorphisms of $\mathcal{E}$ with $a\mathcal{O}_n \oplus b\mathcal{O}_k$ and lemma 3.1.1. □

If $\mathcal{E}$ is of rigid type (i.e. if $b = 1$), then $\mathcal{E}^{(k)}/\mathcal{E}_{n-k}$ is a line bundle on $C_{\inf(k,n-k)}$, and it follows that

$$\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{E},\mathcal{E}) \ \simeq \ \mathcal{H}om(\mathcal{I}^k_{C|C_{\inf(k,n-k)}}, \mathcal{O}_{\inf(k,n-k)}) \ .$$

It follows that we have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(\mathcal{E}nd(\mathcal{E})) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{E},\mathcal{E}) & \longrightarrow & \mathrm{Hom}(\mathcal{I}^k_{C|C_{\inf(k,n-k)}}, \mathcal{O}_{\inf(k,n-k)}) & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & H^0(\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{E},\mathcal{E})) & & \end{array}$$

3.1.3. *The case of double curves* – If $n = 2$ we have $k = 1$ and from corollary 3.1.2

$$\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{E},\mathcal{E}) \ \simeq \ L^* \ .$$

3.1.4. Remark: Let Let X be an algebraic variety, $\mathbb{E}$ a coherent sheaf on $X \times Y$, flat on X, such that for every closed point $x \in X$, $\mathbb{E}_x$ is quasi locally free of rigid type. Let $x \in X$, and

$$\omega_x(\mathbb{E}) : TX_x \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_Y}(\mathbb{E}_x, \mathbb{E}_x)$$

the Kodaïra-Spencer morphism of $\mathbb{E}$. Let X_{red} be the reduced subscheme associated to X. Then the image of Kodaïra-Spencer morphism of $\mathbb{E}_{X_{\mathrm{red}}\times Y}$

$$\omega_x(\mathbb{E}_{X_{\mathrm{red}}\times Y}) : TX_{\mathrm{red},x} \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_Y}(\mathbb{E}_x, \mathbb{E}_x)$$

is contained in $H^1(\mathcal{E}nd(\mathbb{E}_x))$. Suppose that $\mathbb{E}$ is a complete deformation of $\mathbb{E}_x$ (i.e. $\omega_x(\mathbb{E})$ is surjective), and that $\mathbb{E}_x$ is simple. Then $\mathrm{im}(\omega_x(\mathbb{E}_{|X_{\mathrm{red}}\times Y})) = H^1(\mathcal{E}nd(\mathbb{E}_x))$ (cf. [6], théorème 6.10, corollaire 6.11).

3.2. Families of quasi locally free sheaves of rigid type

Let $m_1, \ldots, m_n$ be non negative integers. Let X be a connected algebraic variety, $U \subset X \times Y$ an open subset such that $p_X(U) = X$ (where p_X is the projection $X \times Y \to X$) and $\mathbb{E}$ a coherent sheaf on U, flat on X, such that for every closed point $x \in X$, $\mathbb{E}_x$ is quasi locally free of type $(m_1, \ldots, m_n)$. We say that $\mathbb{E}$ is a *good family* if for $0 \leq i < n$ the sheaf $\mathbb{E}_i/\mathbb{E}_{i+1}$ on $(X \times C) \cap U$ is flat on X (where $0 = \mathbb{E}_n \subset \mathbb{E}_{n-1} \subset \cdots \subset \mathbb{E}_0 = \mathbb{E}$ is the first canonical filtration of $\mathbb{E}$). If $\mathbb{E}$ is a good family then by [13], exposé IV, proposition 1.1, for $0 \leq i < n$, $\mathbb{E}_i$ is a flat family of sheaves on C_{n-i}, and by [16], lemma 1.27, $\mathbb{E}_i/\mathbb{E}_{i+1}$ is a vector bundle on $X \times C$.

3.2.1. Theorem: **1 –** *The sheaf $\mathbb{E}$ is a good family if and only if it is locally isomorphic to $\bigoplus_{i=1}^n \mathcal{O}_{X\times C_i}\otimes\mathbb{C}^{m_i}$.*

2 – *If $\mathbb{E}$ is a good family on $X\times Y$, then for every $x\in X$ the image of the Kodaïra-Spencer morphism of $\mathbb{E}$*

$$\omega_x(\mathbb{E}) : TX_x \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_Y}(\mathbb{E}_x,\mathbb{E}_x)$$

is contained in $H^1(\mathcal{E}nd(\mathbb{E}_x))$.

Proof. Suppose that $\mathbb{E}$ is locally isomorphic to $\bigoplus_{i=1}^n \mathcal{O}_{X\times C_i}\otimes\mathbb{C}^{m_i}$. Then it is obvious that the sheaves $\mathbb{E}_i/\mathbb{E}_{i+1}$ are vector bundles on $(X\times C)\cap U$, hence they are flat on X and $\mathbb{E}$ is a good family. On the other hand, the local structure of $\mathbb{E}$ does not vary when x varies, hence for every $P\in C$, the image of $\mathrm{im}(\omega_x(\mathbb{E}))$ in $\mathrm{Ext}^1_{\mathcal{O}_{Y,P}}(\mathbb{E}_{x,P},\mathbb{E}_{x,P})$ must be 0, so $\mathrm{im}(\omega_x(\mathbb{E}))\subset H^1(\mathcal{E}nd(\mathbb{E}_x))$.

Conversely, suppose that $\mathbb{E}$ is a good family. The proof that $\mathbb{E}$ is locally isomorphic to $\bigoplus_{i=1}^n \mathcal{O}_{X\times C_i}\otimes\mathbb{C}^{m_i}$ is similar to that of théorème 6.5 of [6]. We make an induction on n. The result for $n=1$ follows from [16], lemma 1.27. Suppose that it is true for $n-1\geq 1$. We make an induction on m_n.

Suppose that $m_n=0$. Let k be the smallest integer such that $m_q=0$ for $k+1\leq q\leq n$. Then we have $k<n$, and for every $x\in X$, $\mathbb{E}_x$ is concentrated on C_k. Then $\mathbb{E}$ is concentrated on $(X\times C_k)\cap U$: this follows easily by induction on k from the exact sequence $0\to\mathbb{E}_1\to\mathbb{E}\to\mathbb{E}_{(X\times C)\cap U}\to 0$, using the fact that $\mathbb{E}_1$ is flat on X. By the induction hypothesis (on $(X\times C_k)\cap U$), $\mathbb{E}$ is locally isomorphic to $\bigoplus_{i=1}^n \mathcal{O}_{X\times C_i}\otimes\mathbb{C}^{m_i}$.

Suppose that the result is true for $m_n-1\geq 0$. Let $P\in C$, $x\in X$ such that $Q=(x,P)\in U$. For every open subset $V\subset X\times Y$ and $x'\in X$, let $V_{x'}=V\cap(\{x\}\times Y)$. Let $Z\subset U$ be an open affine subset containing Q such that there is an isomorphism

$$\mathbb{E}_{x|Z_x} \;\simeq\; \bigoplus_{i=1}^n \mathcal{O}_{i|Z_x}\otimes\mathbb{C}^{m_i}\ ,$$

and that $\Delta=p_Y^*(\mathcal{I}_C)$ (which is a line bundle on $(X\times C_{n-1})\cap U$) is trivial on $(X\times C_{n-1})\cap Z$. Let $\zeta\in H^0(\Delta)$ be a section inducing an isomorphism $\Delta\simeq\mathcal{O}_{(X\times C_{n-1})\cap Z}$. Let $\sigma\in H^0(\mathbb{E}_{x|Z_x})$ be defined by some non zero element of $\mathbb{C}^{m_n}$, and $\overline{\sigma}\in H^0(\mathbb{E}_{|Z})$ extending σ. Then $\zeta^{n-1}\overline{\sigma}\in H^0(Z,\mathbb{E}_{n-1})$, $\mathbb{E}_{n-1}$ is a vector bundle on $(X\times C)\cap Z$, and $s=\zeta^{n-1}\overline{\sigma}_{|Z_x}$ does not vanish on Z_x. Let $T\subset Z$ be the open subset where s does not vanish. Let $x'\in X$ be such that $T_{x'}\neq\emptyset$, and $W\subset T_{x'}$ an open subset such that $\mathbb{E}_{|W}\simeq\bigoplus_{i=1}^n\mathcal{O}_{i|W}\times\mathbb{C}^{m_i}$. Then $\overline{\sigma}_{|W}:\mathcal{O}_{n|W}\to\mathcal{O}_{n|W}\to\mathbb{C}^{m_n}$ does not vanish at any point. It follows that

$$\mathrm{coker}(\overline{\sigma}_{|W}) \;\simeq\; \left(\bigoplus_{i=1}^{n-1}\mathcal{O}_{i|W}\otimes\mathbb{C}^{m_i}\right)\oplus\left(\mathcal{O}_{n|W}\otimes\mathbb{C}^{m_n-1}\right)\ .$$

From [13], exposé IV, corollaire 5.7, $\mathcal{F}=\mathrm{coker}(\overline{\sigma}_{|T})$ is flat on X. It is a family of quasi locally free sheaves of type $(m_1,\ldots,m_{n-1},m_n-1)$, and it is easy to verify that it is a good family. From the induction hypothesis we can assume, by replacing T with a smaller affine neighbourhood of Q, that

$$\mathcal{F} \;\simeq\; \left(\bigoplus_{i=1}^{n-1}\mathcal{O}_{(X\times C_i)\cap T}\otimes\mathbb{C}^{m_i}\right)\oplus\left(\mathcal{O}_T\otimes\mathbb{C}^{m_n-1}\right)\ .$$

Hence we have an exact sequence

$$0 \longrightarrow \mathcal{O}_T \longrightarrow \mathbb{E}_{|T} \longrightarrow \left(\bigoplus_{i=1}^{n-1} \mathcal{O}_{(X\times C_i)\cap T} \otimes \mathbb{C}^{m_i} \right) \oplus \left(\mathcal{O}_T \otimes \mathbb{C}^{m_n-1} \right) \longrightarrow 0 \ .$$

Now we have $\mathrm{Ext}^1_{\mathcal{O}_T}(\mathcal{O}_{(X\times C_i)\cap T}, \mathcal{O}_T) = \{0\}$ for $1 \leq i \leq n$: it suffices to prove that $\mathcal{E}xt^1_{\mathcal{O}_T}(\mathcal{O}_{(X\times C_i)\cap T}, \mathcal{O}_T) = 0$. This follows easily from the resolution

$$\cdots \longrightarrow \mathcal{O}_T \xrightarrow{\times\zeta^i} \mathcal{O}_T \xrightarrow{\times\zeta^{n-i}} \mathcal{O}_T \xrightarrow{\times\zeta^i} \mathcal{O}_T \longrightarrow \mathcal{O}_{(X\times C_i)\cap T} \ .$$

Hence $\mathbb{E}_{|T} \simeq \bigoplus_{i=1}^{n} \mathcal{O}_{(X\times C_i)\cap T} \otimes \mathbb{C}^{m_i}$, and the result is proved for m_n. □

Let $\mathbb{E}$ be a good family of quasi locally free sheaves of rigid type parametrised by X, and $x \in X$ a closed point. Suppose that $\mathbb{E}_x$ is simple. Let $(S, s_0, \mathcal{E}, \alpha)$ be a semi-universal deformation of $\mathbb{E}_x$. Let $f : S(x) \to S$ be the morphism induced by $\mathbb{E}$ (where $S(x)$ is the germ defined by $\mathbb{E}$ around x). Then TS_{s_0} is canonically isomorphic to $\mathrm{Ext}^1_{\mathcal{O}_Y}(\mathbb{E}_x, \mathbb{E}_x)$, and by [6], théorème 6.10 and corollaire 6.11, we have

$$TS_{\mathrm{red},s_0} \ = \ H^1(\mathcal{E}nd(\mathbb{E}_x)) \ .$$

It follows easily from theorem 3.2.1 that the image of $Tf_x : TX_x \to TS_{s_0}$ is contained in TS_{red,s_0} and that the image of f is contained in S_{red}. Hence if $\boldsymbol{M}$ is the moduli space of stable sheaves corresponding to $\mathbb{E}_x$ and X is connected, the image of the canonical morphism $f_{\mathbb{E}} : X \to \boldsymbol{M}$ associated to $\mathbb{E}$ is contained in $\boldsymbol{M}_{\mathrm{red}}$.

4. Coherent sheaves on reducible deformations of primitive double curves

4.1. Maximal reducible deformations

(cf. [9], [10], [11])

Let C be a projective irreducible smooth curve and $Y = C_2$ a primitive double curve, with underlying smooth curve C, and associated line bundle L on C. Let S be a smooth curve, $P \in S$ and $\pi : \mathcal{C} \to S$ a *maximal reducible deformation of* Y (cf. [9]), i.e. a deformation $\mathcal{C}$ of Y as in 1.2.

For every $z \in S\backslash\{P\}$, $\mathcal{C}_{1,z}$ and $\mathcal{C}_{2,z}$ meet in exactly $-\deg(L)$ points.

Let $\mathcal{Z} \subset \mathcal{C}$ be the closure in $\mathcal{C}$ of the locus of the intersection points of the components of $\pi^{-1}(z)$, $z \neq P$. Since S is a curve, $\mathcal{Z}$ is a curve of $\mathcal{C}_1$ and $\mathcal{C}_2$. It intersects C in a finite number of points. If $x \in C$, let r_x be the number of branches of $\mathcal{Z}$ at x and s_x the sum of the multiplicities of the intersections of these branches with C. If $x \in \mathcal{Z}$, then the branches of $\mathcal{Z}$ at x intersect transversally with C, and we have $r_x = s_x$. We have

$$L \ \simeq \ \mathcal{O}_C(-\sum_{x\in\mathcal{Z}\cap C} r_x x) \ \simeq \ \mathcal{I}_{\mathcal{Z}\cap C, C} \ .$$

For every $\in C$, there exists an unique integer $p > 0$ such that $\mathcal{I}_{C,x}/\langle(\pi_1, \pi_2)\rangle$ is generated by the image of $(\pi_1^p\lambda_1, 0)$, for some $\lambda_1 \in \mathcal{O}_{\mathcal{C}_1,x}$ not divisible by π_1. Moreover $(\pi_1^p\lambda_1, 0)$ is a generator of the ideal $\mathcal{I}_{\mathcal{C}_1,\mathcal{C},x}$ of $\mathcal{C}_2$ in $\mathcal{C}$, and λ_1 is a generator of the ideal of $\mathcal{Z}$ in $\mathcal{C}_1$ at x. The integer p

does not depend on x. Of course we have a symmetric result: $\mathcal{I}_{C,x}/\langle(\pi_1,\pi_2)\rangle$ is generated by the image of $(0,\pi_2^p\lambda_2)$, for some $\lambda_2 \in \mathcal{O}_{\mathcal{C}_2,x}$ not divisible by π_2. Moreover $(0,\pi_2^p\lambda_2)$ is a generator of the ideal $\mathcal{I}_{\mathcal{C}_2,\mathcal{C},x}$ of $\mathcal{C}_1$ in $\mathcal{C}$, and λ_2 is a generator of the ideal of $\mathcal{Z}$ in $\mathcal{C}_2$ at x. We can even assume that $(\lambda_1,\lambda_2) \in \mathcal{O}_{\mathcal{C},x}$.

In this paper we will always assume that $p = 1$.

Let $\mathcal{Z}_0 = \mathcal{C}_1 \cap \mathcal{C}_2 \subset \mathcal{C}$. We have then $\mathcal{Z}_0 = \mathcal{Z} \cup C$. The ideal sheaf $\mathbb{L}_1 = \mathcal{I}_{Z_0,\mathcal{C}_1}$ (resp. $\mathbb{L}_2 = \mathcal{I}_{Z_0,\mathcal{C}_2}$) of $\mathcal{Z}_0$ in $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) at x is generated by $\lambda_1\pi_1$ (resp. $\lambda_2\pi_2$). Hence $\mathbb{L}_1$ (resp. $\mathbb{L}_2$) is a line bundle on $\mathcal{C}_1$ (resp. $\mathcal{C}_2$). The ideal sheaf $\mathcal{I}_{Z,\mathcal{C}_1}$ (resp. $\mathcal{I}_{Z,\mathcal{C}_2}$) of $\mathcal{Z}$ in $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) is canonically isomorphic to $\mathbb{L}_1$ (resp. $\mathbb{L}_2$). We have also canonical isomorphisms $\mathbb{L}_{1|C} \simeq \mathbb{L}_{2|C} \simeq L$. It is also possible, by replacing S with a smaller neighbourhood of P, to assume that $\mathbb{L}_{1|\mathcal{Z}_0} \simeq \mathbb{L}_{2|\mathcal{Z}_0}$. Let $\mathbb{L} = \mathbb{L}_{1|\mathcal{Z}_0} = \mathbb{L}_{2|\mathcal{Z}_0}$. We have $\mathcal{I}_{\mathcal{C}_1,\mathcal{C}} = \mathbb{L}_2$ and $\mathcal{I}_{\mathcal{C}_2,\mathcal{C}} = \mathbb{L}_1$.

There exists a maximal reducible deformation of Y either if $\deg(L) = 0$, or if $\deg(L) < 0$ and there exists $-\deg(L)$ distinct points $P_1,\dots,P_d$ of C (with $d = -\deg(L)$) such that $L = \mathcal{O}_C(-P_1 - \cdots - P_d)$. And in the second case we can even assume that $\mathcal{Z} \cap C = \{P_1,\dots,P_d\}$. For such a deformation $\mathcal{C}$ we denote by $\Delta_{\mathcal{C}}$ the line in $H^0(L^*)$ generated by the canonical section of $L^* = \mathcal{O}_C(P_1 + \cdots + P_d)$.

4.2. Coherent sheaves on reduced reducible curves

(cf. [11], 4-)

Let D be a projective curve with two components D_1, D_2 intersecting transversally, and $Z = D_1 \cap D_2$. Let $\mathcal{E}$ be a coherent sheaf on D. Then the following conditions are equivalent:

(i) $\mathcal{E}$ is pure of dimension 1.
(ii) $\mathcal{E}$ is of depth 1.
(iii) $\mathcal{E}$ is locally free at every point of X belonging to only one component, and if $x \in Z$, then there exist integers a, a_1, $a_2 \geq 0$ and an isomorphism
$$\mathcal{E}_x \simeq a\mathcal{O}_{X,x} \oplus a_1\mathcal{O}_{D_1,x} \oplus a_2\mathcal{O}_{D_2,x} .$$
(iv) $\mathcal{E}$ is torsion free, i.e. for every $x \in X$, every element of $\mathcal{O}_{X,x}$ which is not a zero divisor in $\mathcal{O}_{X,x}$ is not a zero divisor in $\mathcal{E}_x$.
(v) $\mathcal{E}$ is reflexive.

Let $E_i = \mathcal{E}_{|D_i}/T_i$, where T_i is the torsion subsheaf. It is a vector bundle on D_i. Let $x \in Z$. Then there exists a finite dimensional vector space W, surjective maps $f_i : E_{i,x} \to W$, such that the $\mathcal{O}_{D,x}$-module $\mathcal{E}_x$ is isomorphic to $\{(\phi_1,\phi_2) \in E_1(x) \times E_2(x); f_1(\phi_1(x)) = f_2(\phi_2(x))\}$ (where $E_i(x)$ is the fibre at x of the sheaf E_i, and $E_{i,x}$ the fibre of the corresponding vector bundle). We have then
$$\mathcal{E}_x \simeq (W \times \mathcal{O}_{D,x}) \oplus (\ker(f_1) \otimes \mathcal{O}_{D_1,x}) \oplus (\ker(f_2) \otimes \mathcal{O}_{D_2,x}) .$$
We say that the sheaf is *linked* at x if W has the maximal possible dimension, i.e. $\dim(W) = \inf(rk(E_1), \mathrm{rk}(E_2))$ (i.e. if in (iii) $a_1 = 0$ or $a_2 = 0$). We say that $\mathcal{E}$ is *linked* if it is linked at every point of Z.

4.3. Regular sheaves

A coherent sheaf $\mathcal{E}$ on $\mathcal{C}$ is called *regular* if it is locally free on $\mathcal{C}\backslash\mathcal{Z}_0$, and if for every $x \in \mathcal{Z}_0$ there exists a neighbourhood of x in $\mathcal{C}$, a vector bundle $\mathbb{E}$ on U, $i \in \{1,2\}$, and a vector bundle F on $U \cap \mathcal{C}_i$, such that $\mathcal{E}_{|U} \simeq \mathbb{E} \oplus F$.

Let $\mathcal{E}$ be a coherent sheaf on $\mathcal{C}$. Then by [11], proposition 6.4.3, the following assertions are equivalent:

(i) $\mathcal{E}$ is regular (with $i = 1$).
(ii) There exists an exact sequence $0 \to E_2 \to \mathcal{E} \to E_1 \to 0$, where for $j = 1, 2$, E_j is a vector bundle on $\mathcal{C}_j$, such that the associated morphism $E_{1|\mathcal{Z}_0} \to \mathbb{L}^* \otimes E_{2|\mathcal{Z}_0}$ is surjective on a neighbourhood of C.
(iii) There exists an exact sequence $0 \to E_1 \to \mathcal{E} \to E_2 \to 0$, where for $j = 1, 2$, E_j is a vector bundle on $\mathcal{C}_j$, such that the associated morphism $E_{2|\mathcal{Z}_0} \to \mathbb{L}^* \otimes E_{1|\mathcal{Z}_0}$ is injective (as a morphism of vector bundles) on a neighbourhood of C.

If we restrict the exact sequence of (ii) to Y we get the canonical one

$$0 \longrightarrow (\mathcal{E}_{|Y})_1 \longrightarrow \mathcal{E}_{|Y} \longrightarrow (\mathcal{E}_{|Y})_{|C} \longrightarrow 0 \ ,$$

(cf. 2.1) and if we restrict the exact sequence of (iii) to Y we get

$$0 \longrightarrow (\mathcal{E}_{|Y})^{(1)} \longrightarrow \mathcal{E}_{|Y} \longrightarrow (\mathcal{E}_{|Y})^{(2)} = (\mathcal{E}_{|Y})_1 \otimes L^* \longrightarrow 0 \ .$$

In particular $\mathcal{E}_{|Y}$ is quasi locally free, and for $s \in \backslash\{P\}$ in a neighbourhood of P, $\mathcal{E}_s$ is a linked torsion free sheaf.

We have a similar result by taking $i = 2$.

For example, let $\mathcal{E}$ be a coherent sheaf on $\mathcal{C}$, flat on S. Suppose that for every $s \in S$, $\mathcal{E}_s$ is torsion free, and that $\mathcal{E}_{|Y}$ is quasi locally free of rigid type (cf. 3). Then $\mathcal{E}$ is regular ([11], proposition 6.4.5).

5. Kodaïra-Spencer elements

We keep the notations of 4.1.

5.1. Self-extensions of $\mathcal{O}_{C,x}$ on Y

We will need in 5.2 a description of the extensions

$$0 \longrightarrow \mathcal{O}_{C,x} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{X,x} \longrightarrow 0 \tag{1}$$

on Y.

Let $x \in C$. Let $z \in \mathcal{O}_{Y,x}$ be an equation of C and $t \in \mathcal{O}_{Y,x}$ over a generator of the maximal ideal of $\mathcal{O}_{C,x}$. The extensions (1) are parametrised by $\mathrm{Ext}^1_{\mathcal{O}_{Y,x}}(\mathcal{O}_{C,x}, \mathcal{O}_{C,x})$, which is isomorphic to $\mathcal{O}_{C,x}$. This can be seen easily by using the free resolution of $\mathcal{O}_{C,x}$ on Y:

$$\cdots \longrightarrow \mathcal{O}_{Y,x} \xrightarrow{\times z} \mathcal{O}_{Y,x} \xrightarrow{\times z} \mathcal{O}_{C,x} \longrightarrow 0$$

For every positive integer n, let

$$\mathcal{I}_{Y,n} \;=\; (z, t^n) \;\subset\; \mathcal{O}_{Y,x}\ , \qquad \mathcal{I}_{C,n} \;=\; (t^n) \;\subset\; \mathcal{O}_{C,x}$$

(the ideals of nx). Then we have an obvious extension

$$0 \longrightarrow (z) \simeq \mathcal{O}_{C,x} \longrightarrow \mathcal{I}_{Y,n} \longrightarrow \mathcal{I}_{C,n} \simeq \mathcal{O}_{C,x} \longrightarrow 0$$

and it is easy to see that it is associated to $t^n \in \mathrm{Ext}^1_{\mathcal{O}_{Y,x}}(\mathcal{O}_{C,x}, \mathcal{O}_{C,x})$.

5.2. Proof of the main result

Let $\mathcal{E}^{[1]}$, $\mathcal{E}^{[2]}$ be coherent sheaves on $\mathcal{C}$, flat on S. Suppose that $\mathcal{E}^{[1]}_{|Y}$, $\mathcal{E}^{[2]}_{|Y}$ are isomorphic. Let

$$E \;=\; \mathcal{E}^{[1]}_{|Y} \;=\; \mathcal{E}^{[2]}_{|Y}\ .$$

Suppose that E is quasi locally free of rigid type, and that for every $s \in S$, $\mathcal{E}^{[1]}_s$ and $\mathcal{E}^{[2]}_s$ are torsion free. Then $\mathcal{E}^{[1]}$ and $\mathcal{E}^{[2]}$ are regular (cf. 4.3). Suppose that for every $s \in S\backslash\{P\}$, $\mathcal{E}^{[1]}_s$ and $\mathcal{E}^{[2]}_s$ are linked (this is always true on a neighbourhood of P). It follows that there exists an integer r such that for $i = 1, 2$, for every $s \in S\backslash\{P\}$, $\mathcal{E}^{[i]}_s$ is of rank r on $\mathcal{C}_{1,s}\backslash\mathcal{Z}$ and $r+1$ on $\mathcal{C}_{2,s}\backslash\mathcal{Z}$, or of rank r on $\mathcal{C}_{2,s}\backslash\mathcal{Z}$ and $r+1$ on $\mathcal{C}_{1,s}\backslash\mathcal{Z}$. We suppose that $r > 0$. We will consider two cases:

Case A

- $\mathcal{E}^{[1]}_s$ is of rank r on $\mathcal{C}_{2,s}\backslash\mathcal{Z}$ and of rank $r+1$ on $\mathcal{C}_{1,s}\backslash\mathcal{Z}$.
- $\mathcal{E}^{[2]}_s$ is of rank $r+1$ on $\mathcal{C}_{2,s}\backslash\mathcal{Z}$ and of rank r on $\mathcal{C}_{1,s}\backslash\mathcal{Z}$.

Case B

- $\mathcal{E}^{[1]}_s$ and $\mathcal{E}^{[2]}_s$ are of rank r on $\mathcal{C}_{2,s}\backslash\mathcal{Z}$ and of rank $r+1$ on $\mathcal{C}_{1,s}\backslash\mathcal{Z}$.

We want to study $\omega_{\mathcal{E}^{[1]},\mathcal{E}^{[2]}} \in \mathrm{Ext}^1_{\mathcal{O}_Y}(E, E)$ (cf. 2.2).

Recall that $\mathcal{E}xt^1_{\mathcal{O}_Y}(E, E) \simeq H^0(L^*)$ (cf. 3.1.2, 3.1.3). From 4.1, $\mathcal{C}$ induces a one dimensional subspace $\Delta_{\mathcal{C}} \subset H^0(L^*)$.

Let Y_2 be the second infinitesimal neighbourhood of Y in $\mathcal{C}$. Let $t \in \mathcal{O}_{S,P}$ be a generator of the maximal ideal. We will also denote by π (resp. π_i, $i = 1, 2$) the regular function $t \circ \pi$ (resp. $t \circ \pi_i$) defined on a neighbourhood of C. Then Y_2 is defined in a neighbourhood of Y by the equation $\pi^2 = 0$. Let $\mathcal{I}$ be the ideal sheaf of Y in Y_2. We have $\mathcal{I} \simeq \mathcal{O}_Y$. For $i = 1, 2$ we have a canonical exact sequence

$$0 \longrightarrow E \otimes \mathcal{I} \simeq E \longrightarrow \mathcal{E}^{[i]}_{|Y_2} \longrightarrow E \longrightarrow 0\ , \tag{2}$$

associated to $\sigma_i \in \mathrm{Ext}^1_{\mathcal{O}_{Y_2}}(E \otimes \mathcal{I}, E)$.

Given an extension $0 \to E \otimes \mathcal{I} \to \mathcal{F} \to E \to 0$ on Y_2, the canonical morphism $\mathcal{F} \otimes \mathcal{I} \to \mathcal{F}$ induces an endomorphism of E. In this way we get a canonical morphism

$\mathrm{Ext}^1_{\mathcal{O}_{Y_2}}(E, E\otimes\mathcal{I}) \to \mathrm{End}(E)$, whose kernel corresponds to extensions such that $\mathcal{F}$ is concentrated on Y. Hence we have an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_Y}(E\otimes\mathcal{I}, E) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{Y_2}}(E\otimes\mathcal{I}, E) \xrightarrow{\theta} \mathrm{End}(E)\ .$$

The image of σ_i, $i=1,2$, is I_E. Hence, by using the action of $\mathrm{Aut}(E)$, we see that θ is surjective, and that we have an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_Y}(E\otimes\mathcal{I}, E) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{Y_2}}(E\otimes\mathcal{I}, E) \xrightarrow{\theta} \mathrm{End}(E) \longrightarrow 0\ .$$

Recall that $\omega_{\mathcal{E}^{[1]},\mathcal{E}^{[2]}} = \sigma_1 - \sigma_2$. Let

$$\phi : \mathrm{Ext}^1_{\mathcal{O}_Y}(E,E) \longrightarrow H^0(\mathcal{E}xt^1_{\mathcal{O}_Y}(E,E))$$

be the canonical morphism.

5.2.1. Theorem : **1** – *In case A, $\phi(\omega_{\mathcal{E}^{[1]},\mathcal{E}^{[2]}})$ generates $\Delta_{\mathcal{C}}$.*

2 – *in case B, we have $\phi(\omega_{\mathcal{E}^{[1]},\mathcal{E}^{[2]}}) = 0$.*

Proof. We will only prove **1**. The proof of **2** follows easily.

Let $x\in C$. Since $E_x \simeq r\mathcal{O}_{Y,x}\oplus\mathcal{O}_{C,x}$, we have $\mathrm{Ext}^1_{\mathcal{O}_{Y,x}}(E_x,E_x)\simeq \mathrm{Ext}^1_{\mathcal{O}_{Y,x}}(\mathcal{O}_{C,x},\mathcal{O}_{C,x})$. We will give an explicit description of the extension

$$0\longrightarrow \mathcal{O}_{C,x}\longrightarrow \mathcal{V}\longrightarrow \mathcal{O}_{C,x}\longrightarrow 0$$

corresponding to $\phi(\omega_{\mathcal{E}^{[1]},\mathcal{E}^{[2]}})(x)\in \mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{O}_{C,x},\mathcal{O}_{C,x})$, and from 5.1, **1** will follow from the fact that $\mathcal{V}\simeq\mathcal{O}_{Y,x}$ if $x\notin\mathcal{Z}\cap C$, and $\mathcal{V}\simeq\mathcal{I}_{Y,1}$ if $x\in\mathcal{Z}\cap C$.

Let $\tau_1,\tau_2\in\mathrm{Ext}^1_{\mathcal{O}_{Y_2,x}}(E_x,E_x)$ be the images of σ_1,σ_2 respectively. We have also an exact sequence

$$0\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{Y,x}}(E_x,E_x)\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{Y_2,x}}(E_x,E_x)\longrightarrow \mathrm{End}(E_x)\ ,$$

and $\tau_1-\tau_2\in\mathrm{Ext}^1_{\mathcal{O}_{Y,x}}(E_x,E_x)$. In a neighbourhood of x in $\mathcal{C}$, $\mathcal{E}^{[1]}$ is isomorphic to $r\mathcal{O}_{\mathcal{C}}\oplus\mathcal{O}_{\mathcal{C}_1}$, and $\mathcal{E}^{[2]}$ is isomorphic to $r\mathcal{O}_{\mathcal{C}}\oplus\mathcal{O}_{\mathcal{C}_2}$. We can suppose that these isomorphisms are the same on Y. The exact sequence (2) is the canonical exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & r\mathcal{O}_{Y,x}\oplus\mathcal{O}_{C,x} & \longrightarrow & r\mathcal{O}_{Y_2,x}\oplus\mathcal{O}_{\mathcal{C}_i\cap Y_2,x} & \longrightarrow & r\mathcal{O}_{Y,x}\oplus\mathcal{O}_{C,x} & \longrightarrow & 0\\
 & & \| & & \| & & \| & & \\
 & & E_x & & \mathcal{E}^{[i]}_x & & E_x & &
\end{array}$$

Note that $\mathcal{O}_{\mathcal{C}_i\cap Y_2,x}=\mathcal{O}_{\mathcal{C}_i,x}/(\pi_1^2)$. We have $\tau_1-\tau_2\in\mathrm{Ext}^1_{\mathcal{O}_{Y_2,x}}(\mathcal{O}_{C,x},\mathcal{O}_{C,x})\subset\mathrm{Ext}^1_{\mathcal{O}_{Y_2,x}}(E_x,E_x)$. We have $\tau_1-\tau_2=\eta_1-\eta_2$, where $\eta_i\in\mathrm{Ext}^1_{\mathcal{O}_{Y_2,x}}(\mathcal{O}_{C,x},\mathcal{O}_{C,x})$ is associated to the canonical exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{C,x} & \longrightarrow & \mathcal{O}_{\mathcal{C}_i,x}/(\pi_i^2) & \longrightarrow & \mathcal{O}_{C,x} & \longrightarrow & 0\\
 & & \| & & & & \| & & \\
 & & \pi_i.\mathcal{O}_{\mathcal{C}_i,x}/(\pi_i^2) & & & & \mathcal{O}_{\mathcal{C}_i,x}/(\pi_i) & &
\end{array}$$

If $\beta \in \mathcal{O}_{\mathcal{C}_i,x}$, the image of $\pi_i\beta$ in $\mathcal{O}_{\mathcal{C}_i,x}/(\pi_i^2)$ depends only on $\beta_{|C}$. So for every $\alpha \in \mathcal{O}_{C,x}$ we can define $\pi_i\alpha \in \mathcal{O}_{\mathcal{C}_i,x}/(\pi_i^2)$. Let

$$\mathcal{N} \;=\; \big\{(\pi_1\alpha, -\pi_2\alpha) \in \mathcal{O}_{\mathcal{C}_i,x}/(\pi_1^2) \times \mathcal{O}_{\mathcal{C}_2,x}/(\pi_2^2)\ ;\ \alpha \in \mathcal{O}_{C,x}\big\}\ ,$$

which is a sub-$\mathcal{O}_{\mathcal{C},x}$-module of $\mathcal{O}_{\mathcal{C}_i,x}/(\pi_1^2) \times \mathcal{O}_{\mathcal{C}_2,x}/(\pi_2^2)$. Let

$$\mathcal{U} \;=\; \big[\mathcal{O}_{\mathcal{C}_i,x}/(\pi_1^2) \times \mathcal{O}_{\mathcal{C}_2,x}/(\pi_2^2)\big]/\mathcal{N}\ .$$

The morphism

$$\begin{aligned} \Phi : \mathcal{U} &\longrightarrow \mathcal{O}_{C,x} \times \mathcal{O}_{C,x} \\ (\alpha_1, \alpha_2) &\longmapsto (\alpha_{1|C}, \alpha_{2|C}) \end{aligned}$$

is surjective. We have $\ker(\Phi) = \big\{(\pi_1\lambda_1, \pi_2\lambda_2); \lambda_1, \lambda_2 \in \mathcal{O}_{C,x}\big\}/\mathcal{N}$. We have $\ker(\Phi) \simeq \mathcal{O}_{C,x}$, the isomorphism being defined by

$$\begin{aligned} \nu : \mathcal{O}_{C,x} &\longrightarrow \ker(\Phi) \\ \alpha &\longmapsto (\pi_1\alpha, 0) = (0, \pi_2\alpha). \end{aligned}$$

Hence we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{C,x} \longrightarrow \mathcal{U} \longrightarrow \mathcal{O}_{C,x} \oplus \mathcal{O}_{C,x} \longrightarrow 0\ . \tag{3}$$

We have an inclusion

$$\begin{aligned} \mu_1 : \mathcal{O}_{\mathcal{C}_1,x}/(\pi_1^2) &\longrightarrow \mathcal{U} \\ \alpha_1 &\longmapsto (\alpha_1, 0), \end{aligned}$$

and similarly $\mu_2 : \mathcal{O}_{\mathcal{C}_2,x}/(\pi_2^2) \hookrightarrow \mathcal{U}$. We have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{C,x} \arrow[r] \arrow[d, equal] & \mathcal{O}_{\mathcal{C}_1,x}/(\pi_1^2) \arrow[r] \arrow[d, hook, "\mu_1"] & \mathcal{O}_{C,x} \arrow[r] \arrow[d, hook, "\mu"] & 0 \\ 0 \arrow[r] & \mathcal{O}_{C,x} \arrow[r, "\nu"] & \mathcal{U} \arrow[r] & \mathcal{O}_{C,x} \oplus \mathcal{O}_{C,x} \arrow[r] & 0\ , \end{tikzcd}$$

where μ is the inclusion in the first factor.

Let $\gamma \in \mathrm{Ext}^1_{Y_2,x}(\mathcal{O}_{C,x} \oplus \mathcal{O}_{C,x}, \mathcal{O}_{C,x}) = \mathrm{Ext}^1_{Y_2,x}(\mathcal{O}_{C,x}, \mathcal{O}_{C,x}) \oplus \mathrm{Ext}^1_{Y_2,x}(\mathcal{O}_{C,x}, \mathcal{O}_{C,x})$ associated to (3). From the preceding diagram and proposition 4.3.1 of [3], the first component of γ is η_1. Similarly the second component of γ is η_2. So we have $\gamma = (\eta_1, \eta_2)$. It follows that $\eta_1 - \eta_2$ corresponds to the top exact sequence in the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{C,x} \arrow[r] \arrow[d, equal] & \mathcal{V} \arrow[r] \arrow[d, hook] & \mathcal{O}_{C,x} \arrow[r] \arrow[d, "\psi"] & 0 \\ 0 \arrow[r] & \mathcal{O}_{C,x} \arrow[r] & \mathcal{U} \arrow[r] & \mathcal{O}_{C,x} \oplus \mathcal{O}_{C,x} \arrow[r] & 0 \end{tikzcd}$$

where $\psi : \alpha \mapsto (\alpha, -\alpha)$, and $\mathcal{V} = \big\{(\alpha, \beta) \in \mathcal{U}; \alpha_{|C} + \beta_{|C} = 0\big\}$.

If $(u_1, u_2) \in \mathcal{O}_{\mathcal{C}_i,x}/(\pi_1^2) \times \mathcal{O}_{\mathcal{C}_2,x}/(\pi_2^2)$ is such that $u_{1|C} + u_{2|C} = 0$, we will denote by $[u_1, u_2]$ the corresponding element of $\mathcal{V}$.

We have $\pi\mathcal{V} = \{0\}$, i.e. the top exact sequence is a sequence of $\mathcal{O}_{Y,x}$-modules.

The case $x \notin \mathcal{Z} \cap C$ – We have then $\mathcal{O}_{\mathcal{E},x} = \big\{(\alpha_1,\alpha_2) \in \mathcal{O}_{\mathcal{E}_1,x} \times \mathcal{O}_{\mathcal{E}_2,x}; \alpha_{1|C} = \alpha_{2|C}\big\}$. Let

$$\begin{aligned} \mathbf{f} : \mathcal{O}_{\mathcal{E},x} &\longrightarrow \mathcal{V} \\ 1 &\longmapsto [1,-1] \ . \end{aligned}$$

We now prove that $\mathbf{f}$ induces an isomorphism $\mathcal{O}_{Y,x} \simeq \mathcal{V}$. It is obvious that $\mathbf{f}$ is surjective and that $(\pi) \subset \ker(\mathbf{f})$. Suppose that $(\alpha_1,\alpha_2) \in \mathcal{O}_{\mathcal{E},x}$ is such that $\mathbf{f}(\alpha_1,\alpha_2) = 0$. We can then write

$$(\alpha_1,-\alpha_2) \ = \ (\pi_1\beta_1,-\pi_2\beta_2) + (\pi_1^2\epsilon_1,-\pi_2\epsilon_2^2) \ ,$$

with $\beta_{1|C} = \beta_{2|C}$. Hence $(\beta_1,\beta_2) \in \mathcal{O}_{\mathcal{E},x}$, and

$$(\alpha_1,\alpha_2) \ = \ \pi.\big[(\beta_1,\beta_2) + (\pi_1\epsilon_1,\pi_2\epsilon_2)\big] \ .$$

We have $(\beta_1,\beta_2) + (\pi_1\epsilon_1,\pi_2\epsilon_2) \in \mathcal{O}_{\mathcal{E}_x}$, hence $(\alpha_1,\alpha_2) \in (\pi)$.

The case $x \in \mathcal{Z} \cap C$ – We have then an isomorphism

$$\theta : \mathcal{O}_{\mathcal{E}_1,x}/(\pi_1\lambda_1) \longrightarrow \mathcal{O}_{\mathcal{E}_2,x}/(\pi_2\lambda_2)$$

such that $\theta(\pi_1) = \pi_2$, $\theta(\alpha)_{|C} = \alpha_{|C}$ for every $\alpha \in \mathcal{O}_{\mathcal{E}_1,x}/(\pi_1\lambda_1)$ (cf. 4.1). The restrictions $\lambda_{i|C}$, $i = 1,2$ are generators of the maximal ideal of $\mathcal{O}_{C,x}$. We can also assume that $\theta(\lambda_1) = \lambda_2$. We have then

$$\mathcal{O}_{\mathcal{E},x} \ = \ \big\{(\alpha_1,\alpha_2) \in \mathcal{O}_{\mathcal{E}_1,x} \times \mathcal{O}_{\mathcal{E}_2,x} \ ; \ \theta(\alpha_1) = \alpha_2\big\} \ . \tag{4}$$

We now prove that $\mathcal{V} \simeq \mathcal{I}_x$ (the ideal sheaf of $\{x\}$ in Y). Let $z = (\pi_1\lambda_1, 0)$, $t = (\lambda_1,\lambda_2)$ in $\mathcal{O}_{\mathcal{E},x}$ (cf. (4)). We have $\mathcal{I}_x = (t,z)$, z is an equation of C (in Y) and $t_{|C}$ is a generator of the maximal ideal of $\mathcal{O}_{C,x}$. Then there exists a unique morphism $\rho : \mathcal{I}_x \to \mathcal{V}$ such that

$$\rho(t) \ = \ [1,-1] \ , \qquad \rho(z) \ = \ [\pi_1,0] \ .$$

To prove this we have only to show that if $\alpha,\beta \in \mathcal{O}_{Y,x}$ are such that $\alpha z + \beta t = 0$, then we have $\beta[1,-1] + \alpha[\pi_1,0] = 0$ in $\mathcal{V}$. We have $\alpha z + \beta t = 0$ if and only if we can write $\alpha = \epsilon t + \gamma z$, $\beta = -\epsilon z$, with $\epsilon,\gamma \in \mathcal{O}_{Y,x}$. We have then

$$\begin{aligned} \beta[1,-1] + \alpha[\pi_1,0] \ &= \ -\epsilon z[1,-1] + (\epsilon t + \gamma z)[\pi_1,0] \\ &= \ \epsilon(\lambda_1,\lambda_2)[\pi_1,0] - \epsilon(\pi_1\lambda_1,0)[1,-1] + \gamma[\pi_1^2\lambda_1,0] \\ &= \ 0 \ . \end{aligned}$$

Now we show that ρ is injective. Suppose that $(\alpha_1,\alpha_2),(\beta_1,\beta_2) \in \mathcal{O}_{\mathcal{E},x}$ are such that

$$\rho((\alpha_1,\alpha_2)z + (\beta_1,\beta_2)t) \ = \ 0 \ .$$

Then we have

$$(\alpha_1,\alpha_2)[\pi_1,0] + (\beta_1,\beta_2)[1,-1] \ = \ [\alpha_1\pi_1 + \beta_1, -\beta_2] \ = \ 0 \ .$$

Hence we can write

$$(\alpha_1\pi_1 + \beta_1, -\beta_2) \ = \ (\pi_1\tau_1, -\pi_2\tau_2) + (\pi_1^2\theta_1, \pi_2^2\theta_2) \ ,$$

for some $\tau_i,\theta_i \in \mathcal{O}_{\mathcal{E}_i,x}$ such that $\tau_{1|C} = \tau_{2|C}$, i.e.

$$\alpha_1\pi_1 + \beta_1 \ = \ \pi_1\tau_1 + \pi_1^2\theta_1 \ , \qquad \beta_2 \ = \pi_2\tau_2 - \pi_2^2\theta_2 \ . \tag{5}$$

Let

$$u \ = \ (\alpha_1,\alpha_2)z + (\beta_1,\beta_2)t \ = \ \big(\lambda_1(\alpha_1\pi_1 + \beta_1), \lambda_2\beta_2\big) \ .$$

From (5) we see that β_1 is a multiple of π_1, and β_2 a multiple of π_2: $\beta_1 = \pi_1\beta_1'$, $\beta_2 = \pi_2\beta_2'$. We have then $u = \big(\pi_1\lambda_1(\alpha_1 + \beta_1'), \pi_2\lambda_2\beta_2'\big)$. We have

$$\alpha_1 + \beta_1' \ = \ \tau_1 + \pi_1\theta_1 \ , \ \beta_2' \ = \ \tau_2 - \pi_2\theta_2 \ .$$

Hence $(\alpha_1 + \beta_1')_{|C} = \beta_{2|C}'$ and $\big(\lambda_1(\alpha_1 + \beta_1'), \lambda_2\beta_2'\big) \in \mathcal{O}_{\mathcal{C},x}$. It follows that $u = 0$ in $\mathcal{O}_{Y,x}$.

Now we show that ρ is surjective. Let $[\alpha, \beta] \in \mathcal{V}$. Then $\alpha_{|C} = -\beta_{|C}$. Let $\mu \in \mathcal{O}_{\mathcal{C}_1,x}$ be such that $(\mu, -\beta) \in \mathcal{O}_{\mathcal{C},x}$. We have

$$[\alpha, \beta] - (\mu, -\beta)[1, -1] \ = \ [\alpha - \mu, 0] \ .$$

We can write $\alpha - \mu = \pi_1\zeta$, $\zeta \in \mathcal{O}_{\mathcal{C}_1,x}$. Let $\delta \in \mathcal{O}_{\mathcal{C}_2,x}$ be such that $(\zeta, \delta) \in \mathcal{O}_{\mathcal{C},x}$. We have then

$$[\alpha, \beta] \ = \ (\mu, -\beta)[1, -1] + (\zeta, \delta)[\pi_1, 0] \ = \ \rho((\mu, -\beta)t + (\zeta, \delta)z) \ .$$

□

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