Survival criterion for a population subject to selection and mutations ; Application to temporally piecewise constant environments

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Abstract

We study a parabolic Lotka-Volterra type equation that describes the evolution of a population structured by a phenotypic trait, under the effects of mutations, and competition for resources modelled by a nonlocal feedback. The limit of small mutations is characterized by a Hamilton-Jacobi equation with constraint that describes the concentration of the population on some traits. This result was already established in [PB08, BMP09, LMP11] in a constant environment, when the asymptotic persistence of the population was ensured.

Here, we relax the assumptions on the growth rate and the initia data to extend the study to situations where the population goes extinct at the limit. For that purpose, we provide conditions on the initial data for the asymptotic fate of the population. Finally, we show how this study for a constant environment allows to consider temporally piecewise constant environments. This applies to several applications in biology such as the adaptation to a pharmacological treatment, and the interaction between two populations evolving on different ecological timescales.

Keywords:

1 Introduction

1.1 Model and motivations

In this paper, we study the asymptotic behaviour of solutions to Lotka-Volterra parabolic equations used to model the evolutionary dynamics of a population where individuals are characterized by a phenotypic trait $x \in \mathbb{R}^d$. The population density $(t,x) \mapsto n(t,x)$ satisfies the integro-differential problem

$$\begin{cases} \partial_t n(t,x) - \sigma \Delta n(t,x) = n(t,x) R(x, I(t)),\ x \in \mathbb{R}^d,\ t > 0, \\ I(t) = \int_{\mathbb{R}^d} \psi(x) n(t,x) \mathrm{d}x\,, \\ n(0,x) = n^0 \in L^1(\mathbb{R}^d),\ n^0 \geq 0\,. \end{cases} \tag{1}$$

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The population follows a selection-mutation dynamics, describing the interplay between ecology and evolution via the competition for resources. The term $R(x, I)$ models the growth rate of individuals with trait x depending on the nonlocal interaction term $I(t)$. This interaction term I represents the total consumption of a resource, with $\psi(x)$ being the trait-dependent consumption rate. The growth rate R is then naturally assumed decreasing in I. Mutations are described by the Laplace term and arise with rate σ. Such macroscopic selection-mutation models can in fact be obtained from stochastic individual-based population models in a large population limit (see [FM04, CFM08] and subsequent works).

We assume that mutations have a small effect, and we change the time scale to study the effect of mutations on the evolution of the population. More precisely, taking $\sigma = \varepsilon^2$ and making the change of variable $t \mapsto t/\varepsilon$, one obtains the rescaled problem

$$\begin{cases} \partial_t n_\varepsilon - \varepsilon \Delta n_\varepsilon = \frac{1}{\varepsilon} n_\varepsilon R(x, I_\varepsilon(t)),\ x \in \mathbb{R}^d,\ t > 0, \\ n_\varepsilon(t = 0) = n_\varepsilon^0 \in L^1(\mathbb{R}^d),\ n_\varepsilon^0 \geq 0\,, \end{cases} \tag{2}$$

$$I_\varepsilon(t) = \int_{\mathbb{R}^d} \psi(x) n_\varepsilon(t, x) \mathrm{d}x\,. \tag{3}$$

The study of the asymptotic solutions as $\varepsilon \to 0$ has been carried out using a Hamilton-Jacobi approach in several works, starting from [DJMP05], and followed by [PB08, BMP09, LMP11] for rigorous justifications. With this scaling, the selection is fast compared to the diversification of traits arising from mutations. As a consequence, we expect that as $\varepsilon \to 0$, $n_\varepsilon(t, \cdot)$ concentrates as a Dirac mass which evolves in time. A classical method to study such asymptotic solutions consists in making the Hopf-Cole transformation

$$n_\varepsilon(t, x) = e^{\frac{u_\varepsilon(t,x)}{\varepsilon}}\,.$$

The problem (2) then rewrites on u_ε as

$$\begin{cases} \partial_t u_\varepsilon - \varepsilon \Delta u_\varepsilon = |\nabla u_\varepsilon|^2 + R(x, I_\varepsilon(t)), \quad x \in \mathbb{R}^d,\ t > 0, \\ u_\varepsilon(t = 0) = \varepsilon \ln n_\varepsilon^0\,. \end{cases} \tag{4}$$

In [PB08, BMP09, LMP11], the authors establish the convergence, up to a subsequence, of $(u_\varepsilon)_\varepsilon$ towards a function u which is solution of a Hamilton-Jacobi equation in the viscosity sense [CILS92, Bar94], for different sorts of growth rates. In those earlier works, the assumptions ensured the persistence of the population at the limit: in [PB08, BMP09], the growth rate was bounded and everywhere positive for a non-zero small enough population size. In [LMP11], the growth rate was assumed concave, and the initial condition was taken such that the population was viable.

In this paper, we relax these assumptions to take into account more general growth functions and no strong constraint on the initial condition. As a consequence, the population at initial time may not be viable at initial time, so that the asymptotic extinction of the population may arise. This situation is of interest when considering brutal changes in the environment, since the population may become instantaneously not viable. We provide conditions on the initial condition for the asymptotic fate of the population in a constant environment. This extends the convergence result to weaker assumptions. Moreover, we can now obtain a rigorous justification of the limit $\varepsilon \to 0$ in the case where the environment is piecewise constant in time. This allows to extend the Hamilton-Jacobi framework to models involving sudden variations of the environment. For example, one can think of the introduction of a toxic substance in the nutritive medium of a population of microorganisms in vitro, or of a fungal disease treatment in plant ecology. Therefore, our model could help understanding how to deal with the effect of resistance

acquisition in order to control the population under study. Let us introduce the problem under study for a temporally piecewise constant environment.

For $\mathcal{E}$ a discrete space, consider $e : \mathbb{R}_+ \to \mathcal{E}$ a piecewise constant function describing the environment. It is equivalently defined by the increasing sequence $(T_i)_{i\in\mathbb{N}}$ in $\mathbb{R}_+$ and the sequence $(e_i)_{i\in\mathbb{N}}$ in $\mathcal{E}$, such that for all $t \in \mathbb{R}_+$, there exists $j \in \mathbb{N}$ such that $T_j \leq t < T_j + 1$ and $e(t) = e_j$. Now, while still assuming that mutations have a small effect by taking $\sigma = \varepsilon^2$ in (1), we also consider that the environment varies slowly compared to the birth and death events. As a consequence, the growth rate now writes $R(x, e(\varepsilon t), I(t))$. Therefore, using the same change of variables as before, $t \mapsto t/\varepsilon$, one obtains the rescaled problem in a temporally piecewise constant environment, that writes

$$\begin{cases} \partial_t n_\varepsilon - \varepsilon\Delta n_\varepsilon = \frac{1}{\varepsilon} n_\varepsilon R(x, e(t), I_\varepsilon(t)),\ x \in \mathbb{R}^d,\ t > 0, \\ I_\varepsilon(t) = \int_{\mathbb{R}^d} \psi(x) n_\varepsilon(t, x)\mathrm{d}x\,, \\ n_\varepsilon(t = 0) = n_\varepsilon^0 \in L^1(\mathbb{R}^d),\ n_\varepsilon^0 \geq 0\,, \end{cases} \tag{5}$$

and from the Hopf-Cole transformation $n_\varepsilon(t, x) = e^{\frac{u_\varepsilon(t,x)}{\varepsilon}}$, we can write the corresponding problem on u_ε:

$$\begin{cases} \partial_t u_\varepsilon - \varepsilon\Delta u_\varepsilon = |\nabla u_\varepsilon|^2 + R(x, e(t), I_\varepsilon(t)), \quad x \in \mathbb{R}^d,\ t > 0, \\ I_\varepsilon(t) = \int_{\mathbb{R}^d} \psi(x) n_\varepsilon(t, x)\mathrm{d}x\,, \\ u_\varepsilon(t = 0) = \varepsilon \ln n_\varepsilon^0\,. \end{cases} \tag{6}$$

In the past years, several articles have treated the evolutive dynamics of a population in time-varying environments. In [LCDH15], the authors consider a similar selection-mutation problem in dimension 1, in a time-fluctuating environment having a single maximum value, and enriched with a drift term in trait to take into account possible bias in phenotypic variations. The authors show that periodic solutions with Gaussian profiles exist, and study the effects of the mutation rate, the bias, the natural selection and the frequency of environmental oscillations on the phenotypical dynamics of the population. In [MPS15], the authors consider the same scaling as in (2), but assume that the environment oscillates in time with a rescaled period $1/\varepsilon$. They rigorously justify the Hamilton-Jacobi limit, where the limiting growth rate in the concave case derives from a homogeneization process. In the long-time asymptotics, the concentration trait maximizes this homogeneized growth rate. In [IM18], the authors first study the long time asymptotic of the selection-mutation problem in a time periodic environment, before studying the small mutations scaling. In this framework, the solutions converge towards a Dirac mass while the population size varies periodically in time.

Our work follows the approach of [MPS15], for a piecewise constant environment that varies at a time scale of order 1 in the rescaled problem. In [MPS15], the scaling describes the fact that the environment varies at the same scale as time. In [IM18], the limiting problem models the adaptation to the averaged environment over a period. In our case, the environmental variations are slower than the time acceleration, but fast enough with respect to mutations so that we can observe the adaptation to variations in the environment. This work therefore provides a new perspective in the study of evolutionary dynamics in time-fluctuating environments.

1.2 Assumptions

We first give the general set of assumptions that we consider. In these assumptions, the main novelty is that R is neither concave nor bounded (Assumption (H4)). We also consider a weaker regularity assumption on R (Assumption (H6)). Finally, we do not assume anything on the viability of the population at initial time.

1.2.1 General assumptions

Assumptions on ψ and R: there exist strictly positive constants ψ_m, ψ_M such that

$$0 < \psi_m < \psi < \psi_M < +\infty, \quad \psi \in W^{2,\infty}(\mathbb{R}^d). \tag{H1}$$

We also assume that there exists $I_M > 0$ such that

$$\max_{x\in\mathbb{R}^d} R(x, I_M) = 0\,, \tag{H2}$$

so that the population size can not grow too much. There also exist strictly positive constants K_i such that $\forall x \in \mathbb{R}^d$, $I \in \mathbb{R}_+$,

$$-K_1 \leq \frac{\partial R}{\partial I}(x, I) \leq -K_1^{-1} < 0, \tag{H3}$$

describing the negative effect of competition on reproduction. The three following assumptions are more general than the ones used in previous works. We have that

$$-K_2 - K_3|x|^2 \leq R(x, I) \leq K_0 \quad \forall 0 \leq I \leq 2I_M, \tag{H4}$$

$$\Delta(\psi R) \geq -K_3 \quad \forall x \in \mathbb{R}^d,\, I \in [0, 2I_M]\,, \tag{H5}$$

and furthermore,

$$\forall 0 \leq I \leq 2I_M,\, R(\cdot, I) \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^d)\,. \tag{H6}$$

Finally, we assume that the space of traits having a positive growth rate in the absence of competition is compact:

$$\{x \in \mathbb{R}^d,\, R(x, 0) \geq 0\} \text{ is compact in } \mathbb{R}^d\,. \tag{H7}$$

Assumptions on the initial condition: there exists I_m such that

$$0 < I_m \leq I_\varepsilon(0) \leq I_M, \tag{H8}$$

and there exist strictly positive constants $A_{1,2}$, $B_{1,2} > 0$, such that

$$n^0_\varepsilon(x) = e^{\frac{u^0_\varepsilon(x)}{\varepsilon}}, \text{ with } -A_1 - B_1|x|^2 \leq u^0_\varepsilon(x) \leq A_2 - B_2|x|\,. \tag{H9}$$

Note that the right-hand side is meant to control the initial data when $|x|$ is large.

1.2.2 Assumptions in a concave setting

We also give a convergence result in a concave setting, that provides enough regularity to fully characterize the dynamics. This framework relies on the uniform concavity of u^0_ε as well as on the concavity of R. In particular, it is possible under additional assumptions to derive the so-called *canonical equation* that describes the dynamics of the unique concentration point of the population over time.

Assumptions on ψ and R: we assume that ψ is smooth and strictly positive ; R is smooth

and satisfies (H2) and (H3). Assumption (H4) is refined to: there exist positive constants such that

$$-K_2 - K_3|x|^2 \leq R(x,I) \leq K_0 - K_1|x|^2 \text{ for } 0 \leq I \leq 2I_M\,. \tag{H10}$$

Moreover, we assume that

$$-2\underline{K}_2 \leq D^2R(x,I) \leq -2\overline{K}_2 < 0\,, \tag{H11}$$

and

$$\left|\frac{\partial^2 R}{\partial I \partial x_i}(x,I)\right| + \left|\frac{\partial^3 R}{\partial I \partial x_i \partial x_j}(x,I)\right| \leq K_3\,. \tag{H12}$$

Assumptions on the initial condition: u_ε^0 is strictly uniformly concave.

For some positive constants,

$$\begin{gathered} -\underline{L}_0 - \underline{L}_1|x|^2 \leq u_\varepsilon^0(x) \leq \overline{L}_0 + \overline{L}_1|x|^2\,, \\ -2\underline{L}_1 \leq D^2u_\varepsilon^0 \leq -2\overline{L}_1\,. \end{gathered} \tag{H13}$$

We also have that $(I_\varepsilon^0)_\varepsilon$ converges to I_0 with

$$0 < I_m \leq I_0 \leq I_M\,. \tag{H14}$$

Assumptions for the canonical equation: we also assume that

$$\| D^3R(\cdot,I) \|_{L^\infty(\mathbb{R}^d)} \leq K_4 \tag{H15}$$

and

$$\| D^3u_0(\cdot) \|_{L^\infty(\mathbb{R}^d)} \leq L_2\,. \tag{H16}$$

1.3 Main results and plan of the paper

We first prove the local uniform convergence (up to a subsequence) of $(u_\varepsilon)_\varepsilon$ towards a continuous function u, and the weak convergence in the sense of measures of $(n_\varepsilon)_\varepsilon$. Assuming convergence of $(I_\varepsilon)_\varepsilon$, we are also able to identify this limit as the viscosity solution of a Hamilton-Jacobi equation that may carry a constraint depending on the limiting function I. First, let

$$\Gamma_t := \{x \in \mathbb{R}^d,\, u(t,x) = 0\} \tag{7}$$

be the space of zeros of the limiting function u.

Theorem 1.1. *Under Assumptions* (H1)-(H9),

(i) after extraction of a subsequence, $(u_\varepsilon)_\varepsilon$ converges locally uniformly to a function $u \in \mathcal{C}((0,\infty)\times\mathbb{R}^d)$ with $u \leq 0$. Moreover, up to a subsequence, $(n_\varepsilon)_\varepsilon$ converges weakly in the sense of measures towards $n \in L^\infty(\mathbb{R}_+;\mathcal{M}^1(\mathbb{R}^d))$, with $\mathcal{M}^1(\mathbb{R}^d)$ the space of Radon measures in $\mathbb{R}^d$. Finally, a.e in t, Supp $n(t,\cdot) \subset \Gamma_t$.

(ii) Moreover, if $(u_\varepsilon^0)_\varepsilon$ is a sequence of locally uniformly lipschitz functions converging locally uniformly to u^0, then $u \in \mathcal{C}([0,T)\times\mathbb{R}^d)$ and $u(0,x) = u^0(x)$.

(iii) If $(I_\varepsilon)_\varepsilon$ converges to I on some time interval $[0,T)$ with $T \in (0,+\infty]$, then u is a viscosity solution of the following Hamilton-Jacobi equation on $[0,T)\times\mathbb{R}^d$:

$$\partial_t u = |\nabla u|^2 + R(x, I(t))\,. \tag{8}$$

While I is lower bounded by a strictly positive constant, for a.e $t \in [0,T)$, Equation (8) is complemented with the constraint

$$\max_{x\in\mathbb{R}^d} u(t,x) = 0\,, \tag{9}$$

and for any Lebesgue continuity point of I, we have that

$$\Gamma_t \subseteq \{x \in \mathbb{R}^d,\, R(x, I(t)) = 0\}\,.$$

Remark 1.2. *Assuming* (H6)*, if the lower bound on u_ε^0 in* (H9) *is relaxed to: $\exists x_0 \in \mathbb{R}^d$, L_0, $M_0 > 0$ such that*

$$\forall x \in B_{L_0}(x_0),\, u_\varepsilon^0(x) \geq -M_0\,, \tag{10}$$

then for all $L > 0$, for all $t_1 > 0$, there exist positive constants $D_{1,2}$ and $D_3(L)$ such that for $\varepsilon \leq 1$,

$$u_\varepsilon(t,x) \geq -\frac{D_1|x|^2 + D_2 + D_3(L)t}{t_1} \quad \text{for } (t,x) \in (t_1,+\infty)\times B_L(0)\,. \tag{11}$$

Theorem 1.1 shows that it is necessary to understand the asymptotic behaviour of $(I_\varepsilon)_\varepsilon$. Our main result is the following theorem, that uses the convergence of $(u_\varepsilon)_\varepsilon$ and the one of $(n_\varepsilon)_\varepsilon$, both obtained up to a subsequence, to provide conditions at initial time for the asymptotic extinction or persistence of the population.

Theorem 1.3. *Under Assumptions* (H1)-(H9)*,*

i) if

$$\mathit{Supp}\; n(0,\cdot) \cap \{x \in \mathbb{R}^d,\, R(x,0) > 0\} \neq \emptyset\,, \tag{12}$$

then for every $T > 0$, there exists $\underline{I}(T) > 0$ and $\varepsilon(T) > 0$ such that

$$I_\varepsilon(t) \geq \underline{I}(T), \qquad \forall t \in [0,T), \forall \varepsilon \leq \varepsilon(T)\,, \tag{13}$$

and $(I_\varepsilon)_\varepsilon$ converges up to a subsequence a.e in $t \in \mathbb{R}_+$ towards a BV function I.

ii) If

$$\exists C > 0 \text{ such that } \Gamma_0 \subseteq \{x \in \mathbb{R}^d,\, R(x,0) \leq -C\}\,, \tag{14}$$

then there exists a finite and positive constant $T_0 <$, such that $\lim_{\varepsilon\to 0} I_\varepsilon(t)|_{(0,T_0)} = 0$.

iii) If

$$\Gamma_0 \subseteq \{x \in \mathbb{R}^d,\, R(x,0) \leq 0\}\,, \tag{15}$$

then $\forall \delta > 0$, $\exists \varepsilon_\delta > 0$, $\forall \varepsilon < \varepsilon_\delta$, $\exists t_\varepsilon \in [0,T]$,such that

$$I_\varepsilon(t_\varepsilon) < \delta.$$

Note that these conditions are based on the viability of the traits initially present in the population, and not on the growth rate of the individuals at initial time, since we look at $R(\cdot,0)$ rather than at $R(\cdot, I_\varepsilon(0))$. Numerical simulations in section 5 will illustrates these situations.

The first assertion corresponds to the situation where at least a part of the initial population is viable for some strictly positive competition level. In this case we can ensure that the population size stays uniformly strictly positive, and the previously developped tools apply to prove the

convergence. We emphasize that the population size may initially decrease for ε small, until the competition weakens enough to be balanced by the demography (see Figure 1 (1b, 1c, 1d)).

The two other assertions concern the case of a population where all individuals have a negative growth rate for any strictly positive competition level. In this case, we show that at the limit, the population size reaches zero, either ponctually or during a time interval.

Remark 1.4. *Note that the condition i) for asymptotic persistence is stronger than the one for asymptotic extinction, since it involves Supp $n(0,\cdot)$ rather than Γ_0. Therefore, one situation is not described, namely when*

$$\textit{Supp } n(0,\cdot) \subseteq \{x \in \mathbb{R}^d,\, R(x,0) \leq 0\} \textit{ and } \Gamma_0 \cap \{x \in \mathbb{R}^d,\, R(x,0) > 0\} \neq \emptyset\,.$$

We illustrate this situation numerically in Section 5 (see Figure 3).

Finally, we give additional results in a concave setting that was first studied in [LMP11] in the case where the population persists at the limit. It was proved that if R is concave and $(u_\varepsilon^0)_\varepsilon$ is uniformly strictly concave, then u is strictly concave. This property provides enough regularity to better understand the dynamics at play. In particular, for $\overline{x}(t) := \operatorname{argmax}_{x\in\mathbb{R}^d} u(t,x)$ the most adapted trait in the population at time t, we have that Supp $n(t,\cdot) = \Gamma_t = \{\overline{x}(t)\}$ with $\overline{x}(0) = x_0$. The time evolution of $\overline{x}$ can be described by the so-called *canonical equation*. Moreover, in [MR16], the authors show uniqueness for u, allowing to have a strong convergence. We combine these results with Theorem 1.3 to extend the study to situations where the initial population is not necessarily viable, and provide an estimate of the maximal duration of extinction, after which the population grows again. Let us define the function $h : t \mapsto R(\overline{x}(t),0)$, and $\overline{T} := \sup\{t > 0,\, R(\overline{x}(t),0) < 0\}$ the first positive time at which $R(\overline{x}(t),0)$ is no longer negative.

Theorem 1.5. *Assume* (H2)-(H3) *together with* (H10)-(H14).

i) If $\Gamma_0 \subseteq \{R(\cdot,0) > 0\}$, then u is strictly concave and $(I_\varepsilon)_\varepsilon$ converges to $I > 0$ with

$$(u, I) \in L^\infty_{loc}(\mathbb{R}_+; W^{2,\infty}_{loc}(\mathbb{R}^d)) \cap W^{1,\infty}_{loc}(\mathbb{R}_+; L^\infty_{loc}(\mathbb{R}^d)) \times W^{1,\infty}(\mathbb{R}_+)$$

the classical solution of (8)-(9). *Moreover, weakly in the sense of measures, for $(t,x) \in [0,T]\times\mathbb{R}^d$,*

$$n_\varepsilon(t,x) \underset{\varepsilon\to 0}{\rightharpoonup} \overline{\rho}(t)\delta(x - \overline{x}(t))$$

with $\overline{\rho}(t) := \frac{I(t)}{\psi(\overline{x}(t))}$, where I is implicitly defined by

$$R(\overline{x}(t), I(t)) = 0\,.$$

Finally, assuming additionally (H15)-(H16), *we have that $\overline{x} \in W^{1,\infty}(\mathbb{R}_+;\mathbb{R}^d)$ and satisfies the canonical equation*

$$\dot{\overline{x}}(t) = (-D^2 u(t,\overline{x}(t)))^{-1} \cdot \nabla_x R(\overline{x}(t), I(t))\,, \quad \overline{x}(0) = x_0\,. \tag{16}$$

ii) If $\Gamma_0 \subseteq \{R(\cdot,0) < 0\}$, then on $(0,\overline{T})$, $(I_\varepsilon)_\varepsilon$ converges to 0 and u is solution of

$$\partial_t u(t,x) = |\nabla u(t,x)|^2 + R(x,0)\,. \tag{17}$$

Moreover, $R(\overline{x}(\overline{T}),0) = 0$, and assuming additionally (H15)-(H16), *we have that $\overline{x} \in W^{1,\infty}([0,\overline{T});\mathbb{R}^d)$ and satisfies the canonical equation*

$$\dot{\overline{x}}(t) = (-D^2 u(t,\overline{x}(t)))^{-1} \cdot \nabla_x R(\overline{x}(t), 0)\,, \quad \overline{x}(0) = x_0\,. \tag{18}$$

Finally, we have that if $\max_{x\in\mathbb{R}^d} R(x,0) < 0$, *then* $\overline{T} = +\infty$ *and* h *is non-decreasing on* $(0,+\infty)$. *If* $\max_{x\in\mathbb{R}^d} R(x,0) \geq 0$, *then* h *is increasing on* $(0,\overline{T})$ *and for some constant* $A_{1,2} > 0$ *related to the concavity assumptions on* R *and on the* u_ε^0, *one has the lower bound*

$$\overline{T} \geq A_1 \frac{-R(\overline{x}(0),0)}{|\nabla_x R(\overline{x}(0),0)|^2}\,,$$

complemented with the upper bound

$$\overline{T} \leq A_2 \frac{-R(\overline{x}(0),0)}{|\nabla_x R(\overline{x}(\overline{T}),0)|^2}\,,$$

when $\max_{x\in\mathbb{R}^d} R(x,0) > 0$.

Remark 1.6. *If* $\Gamma_0 \subseteq \{x \in \mathbb{R}^d,\, R(x,0) = 0\}$, *then the dynamics can not be identified: depending on the shape of* R, *both extinction or persistence can occur.*

Example 1.7. *Let us give an example in the concave case that leads to an explicit formulation for* u *and* $\overline{x}$ *in dimension 1. First, following [MR16], it is possible to derive approximations of* u_ε *and* n_ε. *More precisely, let us write* $u_\varepsilon = u + \varepsilon v + o(\varepsilon)$, *so that*

$$n_\varepsilon = \frac{1}{\sqrt{2\pi\varepsilon}} \exp\left(\frac{u(t,x)}{\varepsilon} + v(t,x) + o(1)\right).$$

Then, thanks to the concavity of u, *it is possible to derive moment estimates on the solution by using the Laplace method. We obtain*

$$\mu_\varepsilon(t) = \frac{1}{\rho_\varepsilon(t)} \int_{\mathbb{R}} x n_\varepsilon(t,x)dx = \overline{x}(t) - \frac{\varepsilon}{u''(\overline{x}(t))}(3u''' + v')(\overline{x}(t)) + o(\varepsilon)\,,$$
$$V_\varepsilon(t) = \frac{1}{\rho_\varepsilon(t)} \int_{\mathbb{R}} (x-\mu_\varepsilon(t))^2 n_\varepsilon(t,x)dx = \frac{\varepsilon}{-u''(\overline{x}(t))} + o(\varepsilon)\,.$$

Moreover, u *writes*

$$u(t,x) = -\frac{\alpha(t)}{2}(x - \overline{x}(t))^2\,, \tag{19}$$

for $\alpha(t) = -u''(\overline{x}(t))$. *Consider now a growth rate given by*

$$R(x,I) = r - g(x-x_m)^2 - \kappa I \tag{20}$$

for r, g, x_m, $\kappa > 0$, *and an initial condition given by* $n_\varepsilon^0(x) = \rho_0 \delta_{(x)}$. *Then, the canonical equation rewrites*

$$\dot{\overline{x}}(t) = \frac{-2g}{\alpha(t)}(x - \overline{x}(t))^2$$

Using that one can compute

$$V_\varepsilon(t) = \varepsilon\sqrt{g}^{-1} \tanh(2\sqrt{g}t)\,,$$

the solution writes

$$\overline{x}(t) = x_m\left(1 - \frac{1}{\cosh(2\sqrt{g}t)}\right). \tag{21}$$

We apply Theorem 1.3 and Theorem 1.5 to the asymptotic study of (6) for the evolution of a population in a temporally piecewise constant concave environment. More precisely, the following theorem focuses on the case where the growth rate in (5) is defined by

$$R(x,e(t),I_\varepsilon(t)) = R_1(x,I_\varepsilon(t))\mathbb{1}_{[0,T_1)} + R_2(x,I_\varepsilon(t))\mathbb{1}_{[T_1,T_2)}\,,$$

for some $T_{1,2} > 0$, and $R_{1,2}$ smooth and concave. In this situation, we can naturally define $u_\varepsilon(T_1, x)$ and $I_\varepsilon(T_1)$ by

$$u_\varepsilon(T_1, x) := \lim_{t \to T_1^-} u_\varepsilon(t, x)\,, \quad I_\varepsilon(T_1) := \lim_{t \to T_1^-} I_\varepsilon(t)\,.$$

The following result follows from Theorem 1.5 (and Remark 1.6) to deal with the discontinuity at time T_1 and determine the asymptotic fate of the population in the new environment.

Theorem 1.8. *Assume that $R_{1,2}$ satisfy* (H2)-(H3) *together with* (H10)-(H11)*, and that the initial condition verifies* (H13)-(H14)*. Then in $[0, T_2)$, $(u_\varepsilon)_\varepsilon$ converges towards a continuous function u, and $(n_\varepsilon)_\varepsilon$ converges weakly in the sense of measures towards n, such that*

i) on $[0, T_1)$, $(I_\varepsilon)_\varepsilon$ converges to I such that

$$(u, I) \in L^\infty_{loc}([0, T_1); W^{2,\infty}_{loc}(\mathbb{R}^d)) \cap W^{1,\infty}_{loc}([0, T_1); L^\infty_{loc}(\mathbb{R}^d)) \times W^{1,\infty}([0, T_1))$$

where u is strictly concave and is the solution of the constrained Hamilton-Jacobi equation (8) *associated with the growth rate R_1, I is defined implicitly by $R(\overline{x}(t), I(t)) = 0$, and one has $n(t, x) = \overline{\rho}(t)\delta_{(x-\overline{x}(t))}$ with $\overline{\rho}(t) = \frac{I(t)}{\psi(\overline{x}(t))}$. Moreover, assuming additionally that R_1 and $(u^0_\varepsilon)_\varepsilon$ satisfy Assumption* (H15)-(H16)*, we have that $\overline{x} \in W^{1,\infty}([0, T_1); \mathbb{R}^d)$ and satisfies the canonical equation* (16) *in $(0, T_1)$ starting from $\overline{x}(0)$.*

ii) If $R_2(\overline{x}(T_1), 0) \leq 0$, then there is asymptotic extinction of the population, either at a single time point or on a time interval.

iii) If $R_2(\overline{x}(T_1), 0) > 0$, then $(I_\varepsilon)_\varepsilon$ converges towards $I > 0$ on $[T_1, T_2)$ and u is strictly concave with

$$(u, I) \in L^\infty_{loc}([T_1, T_2); W^{2,\infty}_{loc}(\mathbb{R}^d)) \cap W^{1,\infty}_{loc}((T_1, T_2); L^\infty_{loc}(\mathbb{R}^d)) \times W^{1,\infty}((T_1, T_2))\,.$$

Moreover, if R_2 and the $u_\varepsilon(T_1, x)$ verifies (H15)-(H16)*, $\overline{x} \in W^{1,\infty}((T_1, T_2); \mathbb{R}^d)$ and satisfies the canonical equation* (16) *in (T_1, T_2) starting from $\overline{x}(T_1)$.*

We illustrate numerically this situation in Section 5. The plan of the paper is as follows. In Section 2, we prove Theorem 1.1 that shows the convergence up to a subsequence of $(u_\varepsilon)_\varepsilon$ and $(n_\varepsilon)_\varepsilon$. In Section 3, we prove Theorem 1.3, the main contribution of the paper, that give criteria for the asymptotic fate of the population. The proof of Theorem 1.5 is given in Section 4, and mainly focuses on the description of the situation of extinction. Finally, in Section 5, we perform numerical simulations of the model, for constant and piecewise constant environments.

2 Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1 on the convergence up to a subsequence of $(u_\varepsilon)_\varepsilon$ and $(n_\varepsilon)_\varepsilon$. In the following, we detail important bounds and regularity results that allow to pass to the limit and whose proofs have been adapted due to our weaker assumptions compared with previous works. The proofs of the convergence of $(u_\varepsilon)_\varepsilon$ and $(n_\varepsilon)_\varepsilon$, and of the identification of the limiting problem follow classical steps and are detailed in Appendix C.

2.1 Preliminary estimates

We first establish estimates on I_ε and ρ_ε.

Proposition 2.1. *Assume* (H1)-(H6)*, and that* $0 \leq I_\varepsilon(0) \leq I_M + C\varepsilon^2$*. Then,* $\exists \varepsilon_0 > 0$ *such that* $\forall \varepsilon < \varepsilon_0$*,* $\forall t \in \mathbb{R}_+$*,*

$$0 \leq I_\varepsilon(t) \leq 2I_M \,. \tag{22}$$

Moreover, the solution n_ε *is nonnegative for all time and there exists* $\rho_M > 0$ *such that* $\forall \varepsilon < \varepsilon_0$*,* $\forall t \in R_+$*,*

$$0 \leq \rho_\varepsilon(t) = \int_{\mathbb{R}^d} n_\varepsilon(t,x)\,dx \leq \rho_M \,. \tag{23}$$

The proof derives from [PB08] using weaker assumptions on R. The main difference is that we can not obtain a strictly positive lower bound anymore. We detail the proof in Appendix A.

2.2 Regularity of $(u_\varepsilon)_\varepsilon$

We prove now a regularity result for u_ε. For $T > 0$, let us define $D(T) = \sqrt{A_2 + (B_2^2 + K_0)T}$, and the additional sequence $(v_\varepsilon := \sqrt{2D(T)^2 - u_\varepsilon})_\varepsilon$.

Proposition 2.2. *Assume* (H1)-(H9)*. Then,*

i) *the sequences* $(u_\varepsilon)_\varepsilon$ *as well as* $(v_\varepsilon)_\varepsilon$ *are locally uniformly bounded: there exist positive constants* F_i*,* $i \in \{1, ..., 4\}$*, such that for* $(t,x) \in [0,T] \times \mathbb{R}^d$*,*

$$-F_1 T - F_2|x|^2 \leq u_\varepsilon(t,x) \leq F_3 T - F_4|x| \,. \tag{24}$$

ii) *for all* $t_0 > 0$*,* $(u_\varepsilon)_\varepsilon$ *is locally equicontinuous in time and* $(v_\varepsilon)_\varepsilon$ *is locally uniformly Lipschitz in* $[t_0, T] \times \mathbb{R}^d$*, with for some* $L > 1$ *and any* $(t,x) \in [t_0,T] \times B_L(0)$*,*

$$|\nabla v_\varepsilon| \leq C(T,L)\left(1 + \frac{1}{\sqrt{t_0}}\right) . \tag{25}$$

iii) *if* $(\nabla u_\varepsilon^0)_\varepsilon$ *is locally uniformly bounded, then* $(u_\varepsilon)_\varepsilon$ *is locally equicontinuous in time and* $(v_\varepsilon)_\varepsilon$ *is locally uniformly Lipschitz on* $[0,T] \times \mathbb{R}^d$*, for some* $L > 1$*,* $a \leq 1$ *and any* $(t,x) \in [0,T] \times B_L(0)$*,*

$$|\nabla v_\varepsilon|(t,x) \leq C(T,L)\left(1 + \frac{1}{\sqrt{t+a}}\right) .$$

Proof. The proof follows arguments used in [BMP09], see also [LMP11, IM18]. The novelty here lies in the weaker assumptions on R, that is neither bounded nor concave. As a consequence, the Lipschitz bounds are more difficult to obtain. In order to prove the above proposition, we combine techniques from both [BMP09] and [LMP11]. We detail here the proof of the local Lipschitz bounds, while the rest of the proof is postponed to Appendix B.

Regularizing effect in space. Neglecting the subscript ε here for the simplification of notations, denote $v = \sqrt{2D(T)^2 - u}$, that satisfies on $[0,T] \times \mathbb{R}^d$

$$\partial_t v - \varepsilon\Delta v - \left[\varepsilon\frac{1}{v} - 2v\right]|\nabla v|^2 = -\frac{R(x,I)}{2v} \,. \tag{26}$$

Write $p = \nabla v$. Differentiating (26) with respect to x, multiplying by $\frac{p}{|\nabla p|}$, we obtain that

$$\partial_t|p| - \varepsilon\Delta|p| - 2\left[\varepsilon\frac{1}{v} - 2v\right]p \cdot \nabla|p| + \left[\varepsilon\frac{1}{v^2} + 2\right]|p|^3 - \frac{1}{2v^2}R(x,I)|p| + \frac{1}{2v}\nabla_x R \cdot \frac{p}{|p|} \leq 0 \,. \tag{27}$$

Now, we use Assumption (H6) to obtain a local lower-bound on $\nabla_x R$: for $L > 0$, there exists $\overline{K}_L > 0$ such that for any $0 \leq I \leq 2I_M$, $\| \nabla_x R(\cdot, I) \|_{L^\infty(B_L(0))} \leq \overline{K}_L$, so that on $[0,T] \times B_L(0)$,

$$\frac{1}{2v} \nabla_x R \cdot \frac{p}{|p|} \geq -\frac{\overline{K}_L}{2v} > -\frac{\overline{K}_L}{2D(T)}.$$

Moreover, thanks to (H4) and to (24) yielding the bounds (59) on v_ε (see Appendix B), we have that

$$-\frac{1}{2v^2} R|p| > -\frac{K_0}{2v^2}|p| > -\frac{K_0}{2D(T)^2}|p|.$$

As a consequence, for $A_3 > 0$, $D_1(T) > 0$, using the bounds (59) on v_ε, and for $\theta(T,L)$ large enough, we have that $\forall (t,x) \in [0,T] \times B_L(0)$,

$$\partial_t |p| - \varepsilon \Delta |p| - [A_3|x| + D_1(T)] \, ||p| \cdot \nabla |p|| + 2(|p| - \theta(T,L))^3 \leq 0. \tag{28}$$

We are now going to find a strict supersolution for Equation (28) to obtain an upper-bound for $|v_\varepsilon|$ on $[0,T] \times B_L(0)$. For t_1, $A_4 > 0$, and $(t,x) \in (t_1, T] \times B_L(0)$, define

$$z(t,x) = \frac{1}{2\sqrt{t - t_1}} + \frac{A_4 L^2}{(L^2 - |x|^2)} + \theta(T,L).$$

We prove that for A_4 large enough, z is a strict supersolution of (28) in $(t_1, T] \times B_L(0)$. Indeed, we compute

$$\begin{aligned}
\partial_t z - \varepsilon \Delta z - [A_3|x| + D_1(T)] z \nabla z + 2(z - \theta(T,L))^3 &= -\frac{1}{4\sqrt{t-t_1}(t-t_1)} - \varepsilon \left(\frac{2A_4 L^2 d}{(L^2 - |x|^2)^2} + \frac{8A_4 L^2 |x|^2}{(L^2 - |x|^2)^3} \right) \\
&\quad - [A_3|x| + D_1(T)] \left(\frac{1}{2\sqrt{t-t_1}} + \frac{A_4 L^2}{L^2 - |x|^2} + \theta(T,L) \right) \frac{2A_4 L^2 x}{(L^2 - |x|^2)^2} + 2 \left(\frac{1}{2\sqrt{t-t_1}} + \frac{A_4 L^2}{L^2 - |x|^2} \right)^3 \\
&\geq -\varepsilon \left(\frac{2A_4 L^2 d}{(L^2 - |x|^2)^2} + \frac{8A_4 L^4}{(L^2 - |x|^2)^3} \right) - [A_3 L + D_1(T)] \left(\frac{1}{2\sqrt{t-t_1}} + \frac{A_4 L^2}{L^2 - |x|^2} + \theta(T,L) \right) \frac{2A_4 L^3}{(L^2 - |x|^2)^2} \\
&\quad + \frac{3}{\sqrt{t-t_1}} \frac{A_4^2 L^4}{(L^2 - |x|^2)^2} + 2 \frac{A_4^3 L^6}{(L^2 - |x|^2)^3},
\end{aligned}$$

using that $|x| \leq L$. It can be shown that for $L > 1$, $\varepsilon \leq 1$, $t_1 \leq 1$ and $A_4 = A_4(T)$ large enough, the right-hand side of the inequality is strictly positive, so that z is a strict supersolution of (28) in $(t_1, T] \times B_L(0)$ and for $\varepsilon \leq 1$. We next prove that

$$|p(t,x)| \leq z(t,x) \quad \text{in } (t_1, T] \times B_L(0)\,.$$

First, note that for $|x| \to L$ or $t \to t_1$, $z(t,x)$ goes to infinity, so that $|p|(t,x) - z(t,x)$ attains its maximum at an interior point of $(t_1, T] \times B_L(0)$. Define now $t_m \leq T$ such that

$$\max_{(t,x) \in (t_1, t_m] \times B_L(0)} \{ |p|(t,x) - z(t,x) \} = 0.$$

If such t_m does not exist, then the result is proved. Otherwise, take x_m such that $\forall (t,x) \in (0, t_m) \times B_L(0)$,

$$|p|(t,x) - z(t,x) \leq |p|(t_m, x_m) - z(t_m, x_m) = 0.$$

Then, we have at this point that

$$\begin{gathered}
\partial_t (|p|(t_m, x_m) - z(t_m, x_m)) \geq 0, \quad -\Delta (|p|(t_m, x_m) - z(t_m, x_m)) \geq 0, \\
|p|(t_m, x_m) \nabla |p|(t_m, x_m) = z(t_m, x_m) \nabla z(t_m, x_m).
\end{gathered}$$

As a consequence, since $|p|$ (resp. z) is a subsolution (resp. strict supersolution) of (28), we obtain from this that

$$2(|p|(t_m, x_m) - \theta(T, L))^3 - 2(z(t_m, x_m) - \theta(T, L))^3 < 0\,,$$

and we deduce that

$$|p|(t_m, x_m) < z(t_m, x_m)\,,$$

in contradiction with the definition of (t_m, x_m). Therefore, in $(t_1, T] \times B_L(0)$, for $L > 1$, we have that

$$|p|(t, x) \leq z(t, x) = \frac{1}{2\sqrt{t - t_1}} + \frac{A_4(T)L^2}{(L^2 - |x|^2)} + \theta(T, L).$$

Finally, we deduce that on $(2t_1, T] \times B_{\frac{L}{2}}(0)$, we have that

$$|p|(t, x) \leq C(T, L)\left(1 + \frac{1}{\sqrt{t_1}}\right),$$

leading to the result for $t_0 = 2t_1$.

Additional assumption on the initial condition. We consider now that $(\nabla u_\varepsilon^0)_\varepsilon$ is locally uniformly bounded, and follow the previous proof. For a, $A_5 > 0$, define

$$z(t, x) = \frac{1}{2\sqrt{t + a}} + \frac{A_5 L^2}{(L^2 - |x|^2)} + \theta(T, L).$$

We can prove similarly that for A_5 large enough, z is a strict supersolution of (28) in $[0, T] \times B_L(0)$. We prove now that in $[0, T] \times B_L(0)$,

$$|p(t, x)| \leq z(t, x).$$

First, note that the inequality is trivially satisfied for $|x| \to L$ since $z(t, x)$ then goes to infinity. For $t = 0$, we use the uniform boundedness of (∇u_ε^0): there exists $C > 0$ such that $\forall x \in \mathbb{R}^d$,

$$|\nabla v(0, x)| \leq C \leq \frac{1}{2\sqrt{a}} \leq z(0, x)$$

where the inequality in the middle holds for a small enough. Define $t_m \leq T$ such that

$$\max_{(t,x) \in [0, t_m] \times B_L(0)} \{|p|(t, x) - z(t, x)\} = 0.$$

If such t_m does not exist, then the result is proved. Otherwise, take x_m such that $\forall (t, x) \in (0, t_m) \times B_L(0)$,

$$|p|(t, x) - z(t, x) \leq |p|(t_m, x_m) - z(t_m, x_m) = 0.$$

Then, we have at this point that

$$\partial_t(|p|(t_m, x_m) - z(t_m, x_m)) \geq 0, \quad -\Delta(|p|(t_m, x_m) - z(t_m, x_m)) \geq 0,$$
$$|p|(t_m, x_m)\nabla|p|(t_m, x_m) = z(t_m, x_m)\nabla z(t_m, x_m).$$

As a consequence, since $|p|$ (resp. z) is a subsolution (resp. strict supersolution) of (28), we obtain that

$$2(|p|(t_m, x_m) - \theta(T, L))^3 - 2(z(t_m, x_m) - \theta(T, L))^3 < 0,$$

so that

$$|p|(t_m, x_m) < z(t_m, x_m),$$

in contradiction with the definition of (t_m, x_m). We deduce that in $[0,T] \times B_L(0)$, for $L > 1$, we have

$$|p|(t,x) \leq z(t,x) = \frac{1}{2\sqrt{t+a}} + \frac{A_5(T)L^2}{(L^2 - |x|^2)} + \theta(T,L).$$

Finally, we deduce that on $[0,T] \times B_{\frac{L}{2}}(0)$, we have that

$$|p|(t,x) \leq C(T,L)\left(1 + \frac{1}{\sqrt{t+a}}\right), \tag{29}$$

and that ends the proof. □

Proof of Remark 1.2 Take $L > 0$. From (H6), we have that there exists $K_L > 0$ such that

$$\partial n_\varepsilon(t,x) - \varepsilon \Delta n_\varepsilon(t,x) \geq -\frac{K_L}{\varepsilon} n_\varepsilon(t,x)\,.$$

Following [BMP09, AC17], we look for a supersolution of the form $g_\varepsilon(t) w_\varepsilon(t,x)$ with w_ε solution of the heat equation starting from $n_\varepsilon^0(x)$, and $g_\varepsilon(0) = 1$. A straightforward computation shows that it is sufficient to have $g_\varepsilon(t) = e^{-\frac{K_L t}{\varepsilon}}$. As a consequence, for $(t,x) \in (0,+\infty) \times B_L(0)$, we have that

$$\begin{aligned} n_\varepsilon(t,x) &\geq \left(\frac{\varepsilon}{4\pi t}\right)^{d/2} \int_{|y-x_0| \leq L_0} e^{-\frac{|x-y|^2}{4\varepsilon t}} e^{-\frac{M_0 + K_L t}{\varepsilon}} \mathrm{d}y\,, \\ &\geq \left(\frac{\varepsilon}{4\pi t}\right)^{d/2} |B_{L_0}(x_0)| e^{-\frac{2|x|^2 + 2(|x_0| + L_0)^2}{4\varepsilon t}} e^{-\frac{M_0 + K_L t}{\varepsilon}}\,, \end{aligned}$$

leading to

$$u_\varepsilon(t,x) \geq \varepsilon \log\left(\left(\frac{\varepsilon}{4\pi t}\right)^{d/2} |B_{L_0}(x_0)|\right) - \frac{2|x|^2 + 2(|x_0| + L_0)^2}{4\varepsilon t} - \frac{M_0 + K_L t}{\varepsilon}$$

and (11) follows for $\varepsilon \leq 1$ and any $t_1 > 0$.

3 Proof of Theorem 1.3

In this part, we prove Theorem 1.3. We start by treating the asymptotic persistent case *i*). Then, we deal with the asymptotic extinction of the population on a time interval *ii*), before giving the proof in the case of the asymptotic extinction at a time point *iii*).

3.1 Asymptotic persistence

We prove now the first statement of Theorem 1.3. Take $T < \infty$, and consider Assumptions (H1)-(H9), as well as (12). First, we show that in this case, there exists a strictly positive uniform lower bound for $(I_\varepsilon)_\varepsilon$. Then, we use this property to show that I_ε has bounded variations, which leads to the convergence up to a subsequence of $(I_\varepsilon)_\varepsilon$.

3.1.1 Strictly positive uniform lower bound on $(I_\varepsilon)_\varepsilon$

In this section we aim at proving (13). We start by proving an inequality related to $\frac{\mathrm{d}I_\varepsilon(t)}{\mathrm{d}t}$. Let us remark that

$$\frac{d}{dt}I_\varepsilon(t) = \varepsilon \int \Delta\phi(x) n_\varepsilon(t,x)dx + J_\varepsilon(t),$$

where

$$J_\varepsilon(t) = \frac{1}{\varepsilon}\int \psi(x) n_\varepsilon(t,x) R(x, I_\varepsilon(t))\mathrm{d}x\,. \tag{30}$$

We now consider the evolution of the negative part of J_ε.

Lemma 3.1. *Under assumptions* (H1)-(H9)*, there exist two positive constants G_1 and G_2 such that*

$$(J_\varepsilon(0))_- \leq \frac{G_1}{\varepsilon}\,, \tag{31}$$

and

$$\frac{d}{dt}(J_\varepsilon(t))_- \leq G_2 - \frac{K_1}{\varepsilon} I_\varepsilon(t)(J_\varepsilon(t))_-\,, \tag{32}$$

for every $t \in \mathbb{R}$ such that $(J_\varepsilon(t))_-$ is differentiable.

Proof. We start by deriving (31). From (H4),(H8) and (H9), we have that

$$R(x, I_\varepsilon(0)) \geq -K_2 - K_3|x|^2 \text{ and } n_\varepsilon(0,x) \leq e^{\frac{A_2 - B_2|x|^2}{\varepsilon}}.$$

We choose a positive constant M such that $A_2 - B_2M^2 < 0$, and then we obtain

$$\begin{aligned}\varepsilon J_\varepsilon(0) &= \int_{\{|x|<M\}} \psi(x) n_\varepsilon(x,0) R(x, I_\varepsilon(0)\mathrm{d}x + \int_{\{|x|\geq M\}} \psi(x) n_\varepsilon(x,0) R(x, I_\varepsilon(0)\mathrm{d}x\,,\\ &\geq -I_M(K_2 + K_3M^2) - \psi_M \int_{\{|x|\geq M\}} e^{\frac{A_2 - B_2|x|^2}{\varepsilon}}(K_2 + K_3|x|^2)\mathrm{d}x\,.\end{aligned}$$

The second term of the right-hand-side is of order $\varepsilon^{d/2}$ for M chosen above, and we obtain (31) for small ε. To obtain (32), we compute the time derivative of J_ε, to get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}J_\varepsilon(t) = \underbrace{\frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} \psi R^2 n_\varepsilon \mathrm{d}x}_{\geq 0} + \int_{\mathbb{R}^d} \psi R \Delta n_\varepsilon \mathrm{d}x + \left(\int_{\mathbb{R}^d} \Delta\psi n_\varepsilon \mathrm{d}x\right)\left(\int_{\mathbb{R}^d} \psi n_\varepsilon \frac{\partial R}{\partial I}\mathrm{d}x\right)\\ + \frac{J_\varepsilon(t)}{\varepsilon}\int_{\mathbb{R}^d} \psi n_\varepsilon \frac{\partial R}{\partial I}\mathrm{d}x.\end{aligned}$$

Now, $\int_{\mathbb{R}^d} \psi R \Delta n_\varepsilon \mathrm{d}x = \int_{\mathbb{R}^d} \Delta(\psi R) n_\varepsilon \mathrm{d}x \geq -K_3\rho_\varepsilon(t) \geq -K_3\rho_M$ using (H5) and Proposition (2.1). Moreover, from (H1) and (H3),

$$-K_1 I_\varepsilon(t) \leq \int_{\mathbb{R}^d} \psi n_\varepsilon \frac{\partial R}{\partial I}\mathrm{d}x \leq -K_1^{-1} I_\varepsilon(t) \quad \text{and} \quad \left|\int_{\mathbb{R}^d} \Delta\psi n_\varepsilon \mathrm{d}x\right| \leq \rho_\varepsilon(t) \parallel \Delta\psi \parallel_\infty,$$

so that Proposition (2.1) leads to

$$\left(\int_{\mathbb{R}^d} \Delta\psi n_\varepsilon \mathrm{d}x\right)\left(\int_{\mathbb{R}^d} \psi n_\varepsilon \frac{\partial R}{\partial I}\mathrm{d}x\right) \geq -K_1\rho_\varepsilon(t) I_\varepsilon(t) \parallel \Delta\psi \parallel_\infty \geq -2K_1 \parallel \Delta\psi \parallel_\infty I_M \rho_M.$$

Therefore, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}J_\varepsilon(t) \geq -G + \frac{J_\varepsilon(t)}{\varepsilon}\int_{\mathbb{R}^d} \psi n_\varepsilon \frac{\partial R}{\partial I}\mathrm{d}x \quad \text{for some } G \geq 0.$$

Finally, considering $(J_\varepsilon(t))_- = \max(0, -J_\varepsilon(t))$ the negative part of J_ε, and using (H3)-(H4) permit to conclude. □

Let us now define for $I \in (0, I_M)$,

$$\Omega_I := \{x \in \mathbb{R}^d,\, R(x, I) > 0\}\,, \tag{33}$$

and for $\varepsilon > 0$ and $t \in (0, T)$,

$$\widetilde{\Omega}_\varepsilon(t) := \{x \in \mathbb{R}^d,\, R(x, I_\varepsilon(t)) > 0\}\,.$$

We first prove the following Lemma.

Lemma 3.2. *Assume* (H1)-(H9), *and* (12). *Then, there exists* $\varepsilon_0 > 0$, $I_0 \in (0, I_M)$ *and* $I_* \in (0, I_0)$ *such that for all* $\varepsilon < \varepsilon_0$,

$$I_\varepsilon(0) \geq \int_{\Omega_{I_*}} \psi(x) n_\varepsilon(0, x)\, dx \geq I_0 > 0\,. \tag{34}$$

Proof. The left-hand-side inequality always holds true. We prove that the assertion is true for some $I_* \in (0, I_M)$. We proceed by contradiction. Assume that $\forall I \in (0, I_M)$, there exists $(\varepsilon_k)_k \to 0$ such that

$$\lim_{k\to+\infty} \int_{\Omega_I} \psi(x) n_{\varepsilon_k}(x, 0)\mathrm{d}x = 0\,. \tag{35}$$

Consider $x_0 \in \text{Supp } n(0, \cdot) \cap \{x,\, R(x, 0) > 0\}$ which is non-empty thanks to (12). Then, we deduce that $R(x_0, 0) > 0$, and from (H3) that there exists $I_* \in (0, I_M)$ such that $R(x_0, I_*) \geq \frac{R(x_0,0)}{2} > 0$. As a consequence,

$$\text{Supp } n(0, \cdot) \cap \Omega_{I_*} \neq \emptyset\,,$$

and therefore

$$\int_{\Omega_{I_*}} n(x, 0)dx > 0.$$

Since $\psi \geq \psi_m > 0$, and $(n_{\varepsilon_k})_{k\to+\infty}$ converges weakly in the sense of measures towards n, we obtain that for every ε small enough

$$\int_{\Omega_{I_*}} \psi(x) n_\varepsilon(x, 0)\mathrm{d}x \geq \frac{\psi_m}{2}\int_{\Omega_{I_*}} n(x, 0)\mathrm{d}x = I_0 > 0$$

which contradicts (35).
Finally, we know from Assumption (H3) that $\forall I_1 < I_2$, we have $\Omega_{I_2} \subset \Omega_{I_1}$, leading to

$$\int_{\Omega_{I_2}} \psi(x) n_\varepsilon(0, x)\mathrm{d}x < \int_{\Omega_{I_1}} \psi(x) n_\varepsilon(0, x)\mathrm{d}x.$$

As a consequence, taking a smaller I_* does not change the inequality, and we can assume $I_* < I_0$. □

We now introduce two ε-dependent times that also depend on a small and fixed parameter δ:

$$t_\varepsilon = \inf\{t > 0, I_\varepsilon(t) \leq I_* - \delta\}\,, \tag{36}$$

$$s_\varepsilon = \inf\{t > t_\varepsilon, I_\varepsilon(t) \leq I_* - 2\delta\}\,. \tag{37}$$

Lemma 3.2 yields that these times are positive.

The proof of (13) in Theorem 1.3i) is separated in different cases.

Case A: if $s_\varepsilon \geq T$ for ε small enough, then by definition, $\forall t \leq T$, $I_\varepsilon(t) \geq I_* - 2\delta$.

Case B1: if, up to a subsequence and for $1 - \alpha/2 < \beta < 1$ to be adjusted,

$$s_\varepsilon < T \text{ and } s_\varepsilon - t_\varepsilon \geq \varepsilon^\beta, \tag{38}$$

then for all $t \in (0, s_\varepsilon)$, $I_\varepsilon(t) > I_* - 2\delta$, and we deduce from Lemma 3.1 that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(J_\varepsilon(t)\right)_- \leq G_2 - \frac{K_1(I_* - 2\delta)}{\varepsilon}\left(J_\varepsilon(t)\right)_-.$$

Denoting $G = K_1(I_* - 2\delta)$, we obtain from the Gronwall inequality that

$$\left(J_\varepsilon(s_\varepsilon)\right)_- \leq \left(J_\varepsilon(0)\right)_- e^{-Gs_\varepsilon/\varepsilon} + \varepsilon\frac{G_2}{G} \leq H\varepsilon, \tag{39}$$

where the last inequality holds for some $H > 0$ and a small ε thanks to (31), and since $\beta < 1$. Now, let us show that I_ε is bounded by below on (s_ε, T). For $t \in [s_\varepsilon, T]$,

$$\begin{aligned} I_\varepsilon(t) &= I_\varepsilon(s_\varepsilon) + \varepsilon\int_{s_\varepsilon}^t\int \Delta\psi(x) n_\varepsilon(x,u)\mathrm{d}x\mathrm{d}u + \int_{s_\varepsilon}^t J_\varepsilon(u)\mathrm{d}u\,, \\ &\geq I_\varepsilon(s_\varepsilon) - I_M||\Delta\psi||_\infty\varepsilon T - \int_{s_\varepsilon}^t \left(J_\varepsilon(u)\right)_-\mathrm{d}u\,, \\ &\geq \frac{I_* - 2\delta}{2} - \int_{s_\varepsilon}^t \left(J_\varepsilon(u)\right)_-\mathrm{d}u, \end{aligned}$$

for ε small enough, using (H1). To conclude the proof it is sufficient to prove that

$$\int_{s_\varepsilon}^t \left(J_\varepsilon(u)\right)_-\mathrm{d}u < \frac{I_* - 2\delta}{4}.$$

We proceed by contradiction and assume that up to a subsequence, there exists $T_\varepsilon < T$ such that

$$\int_{s_\varepsilon}^{T_\varepsilon} \left(J_\varepsilon(u)\right)_-\mathrm{d}u = \frac{I_* - 2\delta}{4}.$$

Now using Lemma 3.1 again, $\forall t \in [s_\varepsilon, T_\varepsilon]$

$$\begin{aligned} \frac{d}{dt}\left(J_\varepsilon(t)\right)_- &\leq G_2 - \frac{K_1 I_\varepsilon(t)}{\varepsilon}\left(J_\varepsilon(t)\right)_-\,, \\ &\leq G_2 - \frac{K_1\left[\frac{I_*-2\delta}{2} - \int_{s_\varepsilon}^t \left(J_\varepsilon(u)\right)_- du\right]}{\varepsilon}\left(J_\varepsilon(t)\right)_-\,, \\ &\leq G_2 - \frac{K_1(I_* - 2\delta)}{4\varepsilon}\left(J_\varepsilon(t)\right)_-\,. \end{aligned}$$

As a consequence, $\forall s_\varepsilon \leq t \leq T_\varepsilon$,

$$\left(J_\varepsilon(t)\right)_- \leq \left[\left(J_\varepsilon(s_\varepsilon)\right)_- - \frac{4G_2\varepsilon}{K_1(I_* - 2\delta)}\right] e^{-\frac{K_1(I_*-2\delta)}{4\varepsilon}(t-s_\varepsilon)} + \frac{4G_2\varepsilon}{K_1(I_* - 2\delta)}.$$

Using (39), we deduce that there exists a positive constant G_3 such that $\forall t \in [s_\varepsilon, T_\varepsilon]$, $0 \leq \left(J_\varepsilon(t)\right)_- \leq G_3\varepsilon$. As a consequence, we have that

$$0 \leq \int_{s_\varepsilon}^{T_\varepsilon} \left(J_\varepsilon(s)\right)_-\mathrm{d}s = \frac{I_* - 2\delta}{4} \leq G_3\varepsilon T$$

which leads to a contradiction as $\varepsilon \to 0$. The result is then proved in this case.

Case B2: if, up to a subsequence,

$$s_\varepsilon < T \text{ and } s_\varepsilon - t_\varepsilon < \varepsilon^\beta \,, \tag{40}$$

then we first prove that at some $\varepsilon-$dependent time, the resource consumption of individuals having a positive growth rate is uniformly bounded by below (see Lemma 3.3). Next, we prove that this is sufficient to conclude. Recalling the definition of Ω_I in (33), we introduce a family of test functions $\varphi_{\varepsilon,I}$ such that for a given $C' > 0$ and $0 < \alpha < 1$,

$$\begin{cases} \varphi_{\varepsilon,I}(x) = 1 & \text{for } x \in \Omega_{I+C'\varepsilon^{\alpha/2}} \\ \varphi_{\varepsilon,I}(x) = 0 & \text{for } x \in \Omega_I^c \\ \varphi_{\varepsilon,I}(x) \in (0,1) & \text{for } x \in \Omega_I \setminus \Omega_{I+C'\varepsilon^{\alpha/2}} \end{cases}$$

Moreover, we ask that $\varphi_{\varepsilon,I} \underset{\varepsilon\to 0}{\longrightarrow} \mathbb{1}_{\Omega_I}$ and that

$$||D^2\varphi_{\varepsilon,I}||_{L^\infty}(\mathbb{R}^d) \leq \frac{C}{\varepsilon^\alpha}, \tag{41}$$

for $C > 0$ and $0 < \alpha < 1$. Such a sequence of functions exists since using the assumptions on R, for any $x \in \Omega_{I+C\varepsilon^{\alpha/2}}$, then $d(x, \partial\Omega_I) \geq \tilde{C}\varepsilon^{\alpha/2}$ with $\tilde{C} > 0$.
Finally, we define

$$I_{\varepsilon,1}(t) = \int_{\mathbb{R}^d} \psi(x) n_\varepsilon(t,x) \varphi_{\varepsilon,I_\varepsilon(t)}(x) ds.$$

We start by proving the following Lemma.

Lemma 3.3. *Assume* (H1)-(H9), (12) *and* (40), *then there exists* $\varepsilon_1(\delta) > 0$ *and* $I_1 > 0$ *such that for a sequence* $(\tau_\varepsilon)_{\varepsilon<\varepsilon_1(\delta)}$ *with* $\tau_\varepsilon \in (0,T]$, *we have*

$$I_{\varepsilon,1}(\tau_\varepsilon) \geq I_1 \,,$$

and $I_\varepsilon \geq I_1$ *on* $[0, \tau_\varepsilon]$.

Proof. Let us introduce for $t \in (0,T)$,

$$I_{\varepsilon,2}(t) := \int_{\mathbb{R}^d} \psi(x) n_\varepsilon(t,x)(1 - \varphi_{\varepsilon,I_*-2\delta}) \mathrm{d}x \,,$$

an approximation of the consumption rate at time t of individuals that would have a negative growth rate for a competition level of $I_* - 2\delta$. We obtain the result depending on the situation at time t_ε.

a) First, assume that

$$I_{\varepsilon,2}(t_\varepsilon) < \frac{I_*}{8} \,.$$

In the following we prove that $I_{\varepsilon,2}(s_\varepsilon)$ is still small and hence $I_{\varepsilon,1}(s_\varepsilon) = I_\varepsilon(s_\varepsilon) - I_{\varepsilon,2}(s_\varepsilon)$ is bounded by below by a positive constant. To prove that, we derive an estimate on $I_{\varepsilon,2}$ for $t \in (t_\varepsilon, s_\varepsilon)$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} I_{\varepsilon,2}(t) &= \varepsilon \int_{\mathbb{R}^d} \Delta(\psi(1 - \varphi_{\varepsilon,I_*-2\delta})) n_\varepsilon(t,x) \mathrm{d}x + \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \psi n_\varepsilon (1 - \varphi_{\varepsilon,I_*-2\delta}) R(x, I_\varepsilon(t)) \mathrm{d}x \,, \\ &< C_1 \varepsilon^{1-\alpha} + \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \psi n_\varepsilon (1 - \varphi_{\varepsilon,I_*-2\delta}) R(x, I_\varepsilon(t)) \mathrm{d}x \,, \end{aligned}$$

where we used (H1), (22) and (41). Now, $1-\varphi_{\varepsilon,I_*-2\delta}$ has its support in $\Omega^c_{I_*-2\delta+C'\varepsilon^{\alpha/2}}$. Moreover, in $[t_\varepsilon, s_\varepsilon]$, we have that $I_\varepsilon(t) \geq I_* - 2\delta$, so that (H3) and (H4) yield

$$\begin{aligned} R(x, I_\varepsilon(t)) \leq R(x, I_* - 2\delta) &< R(x, I_* - 2\delta + C'\varepsilon^{\alpha/2}) + C'\varepsilon^{\alpha/2} K_1 \\ &< C' K_1 \varepsilon^{\alpha/2}, \quad \forall x \in \Omega^c_{I_*-2\delta+C'\varepsilon^{\alpha/2}}. \end{aligned}$$

As a consequence, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{\varepsilon,2}(t) < C_1 \varepsilon^{1-\alpha} + C_2 \varepsilon^{\alpha/2-1} I_{\varepsilon,2}(t)\,.$$

The Gronwall Lemma yields that

$$\begin{aligned} I_{\varepsilon,2}(s_\varepsilon) &\leq I_{\varepsilon,2}(t_\varepsilon) e^{C_2\varepsilon^{\alpha/2-1}(s_\varepsilon - t_\varepsilon)} + \frac{C_1\varepsilon^{1-\alpha}}{C_2\varepsilon^{\alpha/2-1}} \left(e^{C_2\varepsilon^{\alpha/2-1}(s_\varepsilon - t_\varepsilon)} - 1 \right), \\ &\leq \frac{I_*}{8} e^{C_2\varepsilon^{\beta+\alpha/2-1}} + \frac{C_1\varepsilon^{2-3\alpha/2}}{C_2} \left(e^{C_2\varepsilon^{\beta+\alpha/2-1}} - 1 \right), \end{aligned}$$

using the assumption on $I_{\varepsilon,2}(t_\varepsilon)$ and (40). It follows that since $\beta > 1 - \alpha/2$, then for ε small enough $I_{\varepsilon,2}(s_\varepsilon) \leq I_*/4$. Then,

$$\begin{aligned} I_{\varepsilon,1}(s_\varepsilon) &= I_\varepsilon(s_\varepsilon) - I_{\varepsilon,2}(s_\varepsilon)\,, \\ &\geq I_* - 2\delta - \frac{I_*}{4} > 0\,, \end{aligned}$$

for δ small enough, leading to the result for $\tau_\varepsilon = s_\varepsilon$.

b) In the other case, assume that

$$I_{\varepsilon,2}(t_\varepsilon) \geq \frac{I_*}{8}\,.$$

We first evaluate an approximation of $I_{\varepsilon,2}$ on $(0, t_\varepsilon)$, that enables us to deduce that t_ε is small. From this we will obtain that $I_{\varepsilon,1}(t_\varepsilon)$ is bounded by below by a positive constant. For $t \in (0, t_\varepsilon)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{\varepsilon,2}(t) = \varepsilon \int_{\mathbb{R}^d} \Delta(\psi(1 - \varphi_{\varepsilon,I_*-2\delta})) n_\varepsilon(t,x)\mathrm{d}x + \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \psi n_\varepsilon (1 - \varphi_{\varepsilon,I_*-2\delta}) R(x, I_\varepsilon(t))\mathrm{d}x\,.$$

Using (H1), (22) and (41), we have that the first term in the right-hand-side is smaller than $C_1\varepsilon^{1-\alpha}$ for some $C_1 > 0$. Moreover, note that $1 - \varphi_{\varepsilon,I_*-2\delta}$ has its support in $\Omega^c_{I_*-2\delta+C'\varepsilon^{\alpha/2}} \supset \Omega^c_{I_*-\delta}$ for ε small enough,. Therefore for $t \in (0, t_\varepsilon)$, $I_\varepsilon(t) > I_* - \delta > I_* - 2\delta + C'\varepsilon^{\alpha/2}$ for ε small enough. It follows that there exists $C_2 > 0$ such that $R(x, I_\varepsilon(t)) < -C_2$ on $\Omega^c_{I_*-2\delta+C'\varepsilon^{\alpha/2}}$. Note that the constant C_2 depends on δ but we forget this dependency for sake of simplicity. Using this information we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{\varepsilon,2}(t) < C_1\varepsilon^{1-\alpha} - \frac{C_2}{\varepsilon} I_{\varepsilon,2}(t)\,,$$

and the Gronwall Lemma combined with the estimate on $I_{\varepsilon,2}(t_\varepsilon)$ yields

$$\frac{I_*}{8} \leq I_{\varepsilon,2}(t_\varepsilon) \leq I_{\varepsilon,2}(0) e^{-\frac{C_2}{\varepsilon}t_\varepsilon} + \frac{C_1}{C_2}\varepsilon^{2-\alpha} \left(1 - e^{-C_2 t_\varepsilon/\varepsilon}\right) \leq I_M e^{-\frac{C_2}{\varepsilon}t_\varepsilon} + C_1'\varepsilon^{2-\alpha}\,,$$

where we have used (H8). It follows that necessarily, $t_\varepsilon \leq A\varepsilon$ for some $A > 0$.
Let us now deduce a lower bound for $I_{\varepsilon,1}(t_\varepsilon)$. We compute

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \psi(x) n_\varepsilon(t,x) \varphi_{\varepsilon,I_*-\delta}(x)\mathrm{d}x = \varepsilon \int_{\mathbb{R}^d} n_\varepsilon \Delta(\psi\varphi_{\varepsilon,I_*-\delta})\mathrm{d}x + \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \psi n_\varepsilon R \varphi_{\varepsilon,I_*-\delta}\mathrm{d}x\,.$$

Using (23) and (41), the first term on the right hand side is bounded by below by $-C_3\varepsilon^{1-\alpha}$ for some $C_3 > 0$. Moreover, φ_ε has its support in $\Omega_{I_*-\delta}$ that is included in a compact from (H7), so that $|R| < C_4$ for some $C_4 > 0$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \psi n_\varepsilon \varphi_{\varepsilon,I_*-\delta}\mathrm{d}x \geq -C_3\varepsilon^{1-\alpha} - \frac{C_4}{\varepsilon}\int_{\mathbb{R}^d} \psi n_\varepsilon \varphi_{\varepsilon,I_*-\delta}\mathrm{d}x\,,$$

and the Gronwall Lemma yields that

$$\int_{\mathbb{R}^d} \psi(x) n_\varepsilon(t_\varepsilon,x)\varphi_{\varepsilon,I_*-\delta}(x)\mathrm{d}x \geq \int_{\mathbb{R}^d} \psi(x) n_\varepsilon(0,x)\varphi_{\varepsilon,I_*-\delta}(x)\mathrm{d}x e^{-\frac{C_4}{\varepsilon}t_\varepsilon} - \frac{C_3}{C_4}\varepsilon^{2-\alpha}\left(1-e^{-\frac{C_4}{\varepsilon}t_\varepsilon}\right).$$

Now, we have that for ε small, $I_* - \delta + C'\varepsilon < I_*$ and using the definition of $\varphi_{\varepsilon,I_*-\delta}$ and Lemma 3.2:

$$\begin{aligned}\int_{\mathbb{R}^d} \psi(x) n_\varepsilon(0,x)\varphi_{\varepsilon,I_*-\delta}(x)\mathrm{d}x &\geq \int_{\Omega_{I_*-\delta+C'\varepsilon^{\alpha/2}}} \psi(x) n_\varepsilon(0,x)\mathrm{d}x\,,\\ &\geq \int_{\Omega_{I_*}} \psi(x) n_\varepsilon(0,x)\mathrm{d}x \geq I_0.\end{aligned}$$

Finally, from $t_\varepsilon \leq A\varepsilon$, we obtain that

$$\int_{\mathbb{R}^d} \psi(x) n_\varepsilon(t_\varepsilon,x)\varphi_{\varepsilon,I_*-\delta}(x)\mathrm{d}x \geq I_0 e^{-C_4 A} - \frac{C_3}{C_4}\varepsilon > 0$$

for ε small enough, leading to $I_{\varepsilon,1}(t_\varepsilon) > \underline{I} > 0$ for some $\underline{I}$, and the result is proved for $\tau_\varepsilon = t_\varepsilon$. □

We have derived a positive uniform lower bound for I_ε at some ε-dependent time interval. It remains to extend this result to obtain a uniform lower bound on the interval $[0,T]$. Write $E := \| \Delta\psi \|_\infty \rho_M$ and define

$$\nu_\varepsilon = \inf\{t \geq \tau_\varepsilon,\, J_\varepsilon(t) \geq -(E+1)\varepsilon\}\,.$$

Then, either $\nu_\varepsilon > T$ or $\nu_\varepsilon \leq T$, and we prove now the result in each situation.

i) If $\nu_\varepsilon > T$, then for all $t \in [\tau_\varepsilon, T]$, $J_\varepsilon(t) < -(E+1)\,\varepsilon$, so that

$$\frac{\mathrm{d}I_\varepsilon(t)}{\mathrm{d}t} = \varepsilon\int_{\mathbb{R}^d} \Delta\psi(x) n_\varepsilon(t,x)\mathrm{d}x + J_\varepsilon(t) < -\varepsilon < 0\,,$$

so that I_ε is strictly decreasing. From (H3), we deduce that for any $x \in \mathbb{R}^d$,

$$\frac{\partial}{\partial t}R(x,I_\varepsilon(t)) = \frac{\partial}{\partial I}R(x,I_\varepsilon(t)) \times \frac{\mathrm{d}I_\varepsilon(t)}{\mathrm{d}t} > K_1^{-1}\varepsilon > 0\,,$$

so that for all $t \in [\tau_\varepsilon, T]$,

$$\begin{aligned}R(x,I_\varepsilon(t)) &> R(x,I_\varepsilon(\tau_\varepsilon)) + (t-\tau_\varepsilon)K_1^{-1}\varepsilon\\ &\geq (t-\tau_\varepsilon)K_1^{-1}\varepsilon \quad \text{ on } \Omega_{I_\varepsilon(\tau_\varepsilon)}\,.\end{aligned}$$

Now, for $t \in [\tau_\varepsilon, T]$, let us introduce

$$I_{\varepsilon,3}(t) := \int_{\mathbb{R}^d} \psi n_\varepsilon(t,x)\varphi_{\varepsilon,I_\varepsilon(\tau_\varepsilon)}\mathrm{d}x\,.$$

In particular, note that $I_{\varepsilon,3}(\tau_\varepsilon) = I_{\varepsilon,1}(\tau_\varepsilon)$. We compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I_{\varepsilon,3}(t) &= \varepsilon\int_{\mathbb{R}^d} \Delta(\psi\varphi_{\varepsilon,I_\varepsilon(\tau_\varepsilon)}) n_\varepsilon(t,x)\mathrm{d}x + \frac{1}{\varepsilon}\int_{\mathbb{R}^d} \psi\varphi_{\varepsilon,I_\varepsilon(\tau_\varepsilon)} n_\varepsilon(t,x)R(x,I_\varepsilon(t))\mathrm{d}x\\ &> -C\varepsilon^{1-\alpha} + (t-\tau_\varepsilon)K_1^{-1}I_{\varepsilon,3}(t)\end{aligned}$$

for some $C > 0$, using (H1), (22) and (41). We deduce that for $t \in [\tau_\varepsilon, T]$,

$$\begin{aligned} I_{\varepsilon,3}(t) &\geq I_{\varepsilon,3}(\tau_\varepsilon) e^{\frac{K_1}{2}(t-\tau_\varepsilon)^2} - Ce^{\frac{K_1(t-\tau_\varepsilon)^2}{2}} \varepsilon^{1-\alpha} \int_{\tau_\varepsilon}^t e^{-\frac{K_1(s-\tau_\varepsilon)^2}{2}} \mathrm{d}s \,, \\ &\geq I_{\varepsilon,1}(\tau_\varepsilon) - CTe^{\frac{K_1 T^2}{2}} \varepsilon^{1-\alpha} \,, \\ &\geq I_1 - CTe^{K_1 T^2} \varepsilon^{1-\alpha} > 0 \end{aligned}$$

for $\varepsilon < \varepsilon_0(T)$ small enough. It follows that there exists $\underline{I}(T) > 0$ such that $\forall t \in (0,T)$, $I_\varepsilon(t) \geq I_{\varepsilon,3}(t) \geq \underline{I}(T)$.

ii) If $\nu_\varepsilon < T$, we can use the previous argument to show that

$$\exists \varepsilon_0(T) > 0,\ \underline{I}(T) > 0 \text{ such that } \forall \varepsilon < \varepsilon_0(T),\ \forall t \in [\tau_\varepsilon, \nu_\varepsilon),\ I_\varepsilon(t) \geq \underline{I}(T) \,.$$

Therefore, $I_\varepsilon(\nu_\varepsilon) \geq \underline{I}(T)$, and $J_\varepsilon(\nu_\varepsilon) \geq -(E+1)\varepsilon$. For $\varepsilon < \varepsilon_1(T)$ small enough and $t \in (\nu_\varepsilon, T)$, we have that

$$I_\varepsilon(t) = I_\varepsilon(\nu_\varepsilon) + \int_{\nu_\varepsilon}^t I'_\varepsilon(s)\mathrm{d}s = I_\varepsilon(\nu_\varepsilon) + \int_{\nu_\varepsilon}^t J_\varepsilon(s)\mathrm{d}s + \mathcal{O}(\varepsilon)(t-\nu_\varepsilon) \geq \frac{\underline{I}(T)}{2} - \int_{\nu_\varepsilon}^t (J_\varepsilon(s))_- \mathrm{d}s, \quad (42)$$

and we obtain from (32) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(J_\varepsilon(t))_- \leq G - \frac{K_1^{-1}}{\varepsilon} \left(\frac{\underline{I}(T)}{2} - \int_{\nu_\varepsilon}^t (J_\varepsilon(s))_- \mathrm{d}s \right) (J_\varepsilon(t))_- .$$

Now, we want to show that under some conditions, $\forall \varepsilon < \varepsilon_0(T)$, $\int_{\nu_\varepsilon}^T (J_\varepsilon(s))_- \mathrm{d}s \leq \frac{\underline{I}(T)}{4}$. Let us proceed by contradiction. Assume that this is not the case: there exists a sequence $(\varepsilon_k)_k$ in $(0, \varepsilon_1(T))$ with $\lim\limits_{k\to+\infty} \varepsilon_k = 0$, and $\forall k$, $\exists T'_{\varepsilon_k} < T$ such that $\int_{\nu_{\varepsilon_k}}^{T'_{\varepsilon_k}} (J_{\varepsilon_k}(s))_- \mathrm{d}s = \frac{\underline{I}(T)}{4}$. Then we have $\forall k \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(J_{\varepsilon_k}(t))_- \leq G - \frac{K_1^{-1}}{\varepsilon_k} \frac{\underline{I}(T)}{4} (J_{\varepsilon_k}(t))_- \quad \forall \nu_{\varepsilon_k} \leq t \leq T'_{\varepsilon_k}.$$

As a consequence, $\forall k \geq 0$, $\forall \nu_{\varepsilon_k} \leq t \leq T'_{\varepsilon_k}$,

$$(J_{\varepsilon_k}(t))_- \leq (J_{\varepsilon_k}(\nu_{\varepsilon_k}))_- e^{-\frac{\underline{I}(T)}{4K_1\varepsilon_k}(t-\nu_{\varepsilon_k})} + \frac{4GK_1\varepsilon_k}{\underline{I}(T)} \left(1 - e^{-\frac{\underline{I}(T)}{4K_1\varepsilon_k}(t-\nu_{\varepsilon_k})} \right),$$

or equivalently

$$(J_{\varepsilon_k}(t))_- \leq \left[(J_{\varepsilon_k}(\nu_{\varepsilon_k}))_- - \frac{4GK_1\varepsilon_k}{\underline{I}(T)} \right] e^{-\frac{\underline{I}(T)}{4K_1\varepsilon_k}(t-\nu_{\varepsilon_k})} + \frac{4GK_1\varepsilon_k}{\underline{I}(T)}.$$

Since $(J_{\varepsilon_k}(\nu_{\varepsilon_k}))_- \leq (E+1)\varepsilon_k$, we deduce that there exists a positive constant G_1 such that $\forall t \in [\nu_{\varepsilon_k}, T'_{\varepsilon_k}]$, $0 \leq (J_{\varepsilon_k}(t))_- \leq G_1\varepsilon_k$. As a consequence, we have that

$$0 \leq \int_{\nu_{\varepsilon_k}}^{T'_{\varepsilon_k}} (J_{\varepsilon_k}(s))_- \mathrm{d}s = \frac{\underline{I}(T)}{4} \leq G_1\varepsilon_k T \,,$$

which leads to a contradiction for k large enough. Therefore, for all $\varepsilon < \varepsilon_1(T)$, we have $\int_{\nu_\varepsilon}^T (J_\varepsilon(s))_- \mathrm{d}s \leq \frac{\underline{I}(T)}{4}$, and from the estimate (42), we have that $\forall \varepsilon < \varepsilon_1(T)$, for $t \in (\nu_\varepsilon, T)$,

$$I_\varepsilon(t) \geq \frac{\underline{I}(T)}{4} \,,$$

and the result is proved.

BV bound

We derive now a sub-Lipschitz bound as well as a BV bound on I_ε that allow to pass to the limit after extraction of a subsequence, ending the proof of the first point of Theorem 1.3.

Proposition 3.4. *With the assumptions* (H1)-(H9) *and assuming that* $\exists \underline{I}(T) > 0$, $\varepsilon_0(T) > 0$ *such that* $\forall \varepsilon < \varepsilon_0$, $\forall t \in [0,T]$, $I_\varepsilon(t) > \underline{I}(T)$, *we obtain the following locally uniform BV bound on* $[0,T]$. *For* $\varepsilon < \varepsilon_0 \leq 1$, *and* C_1, G *some positive constants, we have the sub-Lipschitz bound*

$$\frac{dI_\varepsilon}{dt}(t) \geq -\varepsilon \left[C_1 \rho_M + \frac{GK_1}{\underline{I}(T)} \right] - (J_\varepsilon(0))_- e^{-\frac{\underline{I}(T)}{K_1 \varepsilon} t}, \tag{43}$$

so that we obtain the BV bound

$$\int_0^T \left| \frac{dI_\varepsilon}{dt}(t) \right| dt \leq A + \frac{K_1}{\underline{I}(T)} \left(\int_{\mathbb{R}^d} \psi(x) n_\varepsilon(0,x) R(x, I_\varepsilon(0)) dx \right)_-, \tag{44}$$

with $A(T) = 2I_M + T\left(2C_1\rho_M + \frac{GK_1}{\underline{I}(T)}\right)$.
Consequently, after extraction of a subsequence, $(I_\varepsilon(t))_\varepsilon$ *converges a.e in* $\mathbb{R}_+$ *when* ε *goes to zero to a function* I *such that* $\forall T > 0$, *there exists* $\underline{I}(T) > 0$ *such that* $I(t) \geq \underline{I}$ *on* $[0,T]$.

Proof. We adapt the proof of [PB08]. We want to show that

$$\int_0^T \left| \frac{\mathrm{d}I_\varepsilon}{\mathrm{d}t} \right| (t) \mathrm{d}t < \overline{C}$$

for some positive constant $\overline{C}$. Writing

$$\left| \frac{\mathrm{d}I_\varepsilon}{\mathrm{d}t} \right| (t) = \left(\frac{\mathrm{d}I_\varepsilon}{\mathrm{d}t}(t) \right)_+ + \left(\frac{\mathrm{d}I_\varepsilon}{\mathrm{d}t}(t) \right)_- = \frac{\mathrm{d}I_\varepsilon}{\mathrm{d}t}(t) + 2\left(\frac{\mathrm{d}I_\varepsilon}{\mathrm{d}t}(t) \right)_-,$$

we obtain that

$$\int_0^T \left| \frac{\mathrm{d}I_\varepsilon}{\mathrm{d}t}(t) \right| \mathrm{d}t = I_\varepsilon(T) - I_\varepsilon(0) + 2\int_0^T \left(\frac{\mathrm{d}I_\varepsilon}{\mathrm{d}t}(t) \right)_- \mathrm{d}t\,.$$

Now, on $[0,T]$,

$$\frac{\mathrm{d}I_\varepsilon}{\mathrm{d}t}(t) = \varepsilon \int_{\mathbb{R}^d} \Delta\psi(x) n_\varepsilon(t,x) \mathrm{d}x + J_\varepsilon(t) \geq -C_1 \varepsilon \rho_\varepsilon(t) - (J_\varepsilon(t))_-\,, \tag{45}$$

so that

$$\left(\frac{\mathrm{d}I_\varepsilon}{\mathrm{d}t}(t) \right)_- \leq C_1 \varepsilon \rho_M + (J_\varepsilon(t))_-\,.$$

Therefore, we have that

$$\int_0^T \left| \frac{\mathrm{d}I_\varepsilon}{\mathrm{d}t}(t) \right| \mathrm{d}t \leq I_\varepsilon(T) - I_\varepsilon(0) + 2TC_1\varepsilon\rho_M + \int_0^T (J_\varepsilon(t))_- \mathrm{d}t\,,$$

and the uniform BV bound on I_ε relies on a uniform bound for $\int_0^T (J_\varepsilon(t))_- \mathrm{d}t$. We use Lemma 3.1 together with the lower bound on I_ε to get

$$\frac{\mathrm{d}}{\mathrm{d}t}(J_\varepsilon(t))_- \leq G - \frac{\underline{I}(T)}{\varepsilon K_1}(J_\varepsilon(t))_-\,,$$

leading to

$$(J_\varepsilon(t))_- \leq \varepsilon \frac{GK_1}{\underline{I}(T)} + e^{-\frac{\underline{I}(T)}{K_1 \varepsilon} t} (J_\varepsilon(0))_- \,.$$

This inequality combined with (45) give the sub-Lipschitz bound (43). Finally, we also obtain that

$$\begin{aligned}\int_0^T \left| \frac{\mathrm{d}I_\varepsilon}{\mathrm{d}t}(t) \right| \mathrm{d}t &\leq I_\varepsilon(T) - I_\varepsilon(0) + \varepsilon T \left(2C_1\rho_M + \frac{GK_1}{\underline{I}(T)} \right) \\ &\quad + \frac{K_1 \varepsilon}{\underline{I}(T)} \left(1 - e^{-\frac{\underline{I}(T)}{K_1 \varepsilon} T} \right) (J_\varepsilon(0))_- \,, \\ &\leq 2I_M + C\varepsilon^2 + \varepsilon T \left(2C_1\rho_M + \frac{GK_1}{\underline{I}(T)} \right) + \frac{K_1 \varepsilon}{\underline{I}(T)} \left(1 - e^{-\frac{\underline{I}(T)}{K_1 \varepsilon} T} \right) (J_\varepsilon(0))_- \,.\end{aligned}$$

Finally, the convergence in $\mathbb{R}_+$ of $(I_\varepsilon(t))_\varepsilon$ up to a subsequence follows, ending the proof of Proposition 3.4, and of the first assertion of Theorem 1.3. □

3.2 Asymptotic extinction on a time interval

We show now the second statement of Theorem 1.3. We recall the assumption (14) namely that $\exists C > 0$ such that

$$\Gamma_0 = \{x \in \mathbb{R}^d,\, u(0,x) = 0\} \subseteq \{x \in \mathbb{R}^d,\, R(x,0) \leq -C\} \,,$$

Let us define for $\delta > 0$,

$$\mathcal{O}_\delta := \{x \in \mathbb{R}^d,\, u(0,x) > -\delta\}$$

and recall that $\mathcal{O}_0 = \Gamma_0$ since $u \leq 0$ from Theorem 1.1 i). Using (24), $\mathcal{O}_\delta$ is bounded and from the local uniform continuity of u, $\mathcal{O}_\delta \subset A_{C/2} := \{x \in \mathbb{R}^d,\, R(x,0) \leq -C/2\}$ for δ small enough. Moreover, by the uniform continuity in time of u, there exists $T_0 > 0$ so that

$$\forall t \in [0, T_0),\, \{x \in \mathbb{R}^d,\, u(t,x) > -\frac{\delta}{2}\} \subseteq \mathcal{O}_\delta \,.$$

Now, since ϕ is positive and bounded from (H1), we write on $[0, T_0)$,

$$I_\varepsilon(t) \leq \psi_M \int_{\mathcal{O}_\delta} n_\varepsilon(t,x)\mathrm{d}x + \int_{(\mathcal{O}_\delta)^c} n_\varepsilon(t,x)\mathrm{d}x \,,$$

and we prove that each term separately goes to 0, starting with the second term.

From Proposition 2.2 (i), there exist positive constants F_i, $i \in \{1, ..., 4\}$, such that for $(t,x) \in [0,T] \times \mathbb{R}^d$, $\forall \varepsilon < \varepsilon_0$,

$$-F_1 T - F_2 |x|^2 \leq u_\varepsilon(t,x) \leq F_3 T - F_4 |x| \,.$$

Therefore there exist $r_0 > 0$ such that for all $t \in [0, T_0]$ and $|x| \geq r_0$, $u_\varepsilon(t,x) \leq -\frac{F_4}{2}|x|$. Now,

$$\begin{aligned}\int_{(\mathcal{O}_\delta)^c} n_\varepsilon(t,x)\mathrm{d}x &= \int_{(\mathcal{O}_\delta)^c \cap B(0,r_0)} n_\varepsilon(t,x)\mathrm{d}x + \int_{(\mathcal{O}_\delta)^c \cap B(0,r_0)^c} n_\varepsilon(t,x)\mathrm{d}x \\ &= \int_{(\mathcal{O}_\delta)^c \cap B(0,r_0)} n_\varepsilon(t,x)\mathrm{d}x + \int_{(\mathcal{O}_\delta)^c \cap B(0,r_0)^c} e^{\frac{-F_4 x}{2\varepsilon}} \mathrm{d}x\end{aligned} \tag{46}$$

Finally it remains to control the integral on $(\mathcal{O}_\delta)^c \cap B(0,r_0)$. Remark that on $(\mathcal{O}_\delta)^c$, $\forall t \ni [0,T_0)$, $u(t,\cdot) \leq -\frac{\delta}{2} < 0$, there exists $\varepsilon_0 > 0$ small enough so that $\forall \varepsilon < \varepsilon_0$, on $(\mathcal{O}_\delta)^c$, $u_\varepsilon(t,\cdot) \leq -\frac{\delta}{4}$. We then deduce that

$$0 \leq \int_{(\mathcal{O}_\delta)^c \cap B(0,r_0)} n_\varepsilon(t,x)\mathrm{d}x \leq \int_{(\mathcal{O}_\delta)^c \cap B(0,r_0)} e^{-\frac{\delta}{4\varepsilon}}\mathrm{d}x \leq |B(0,r_0)|e^{-\frac{\delta}{4\varepsilon}}\,.$$

Combining with (46), $\int_{(\mathcal{O}_\delta)^c} n_\varepsilon(t,x)\mathrm{d}x$ goes to 0 as $\varepsilon \to 0$ for every $t \in [0,T_0)$.
We now consider $\int_{\mathcal{O}_\delta} n_\varepsilon(t,x)\mathrm{d}x$ on $(0,T_0)$. Let $\varphi_\varepsilon \in \mathcal{C}^\infty_{c,+}(\mathcal{O}_\delta)$ be a test function such that $\varphi_\varepsilon \equiv 1$ in $\mathcal{O}_\delta$, and such that $\| D^2\varphi_\varepsilon \|_\infty < \frac{1}{\varepsilon}$. Then, for $t \in [0,T_0)$,

$$0 \leq \int_{\mathcal{O}_\delta} n_\varepsilon(t,x)\mathrm{d}x \leq \int_{\mathbb{R}^d} \varphi_\varepsilon(x) n_\varepsilon(t,x)\mathrm{d}x$$

Moreover,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \varphi_\varepsilon(x) n_\varepsilon(t,x)\mathrm{d}x &= \varepsilon \int_{\mathbb{R}^d} \Delta\varphi_\varepsilon n_\varepsilon(t,x)\mathrm{d}x + \frac{1}{\varepsilon}\int_{\mathbb{R}^d} \varphi_\varepsilon(x) n_\varepsilon(t,x) R(x, I_\varepsilon(t))\mathrm{d}x\,,\\
&< \varepsilon \,\|\, \Delta\varphi_\varepsilon \,\|_\infty\, \rho_M + \frac{1}{\varepsilon}\int_{\mathbb{R}^d} \varphi_\varepsilon(x) n_\varepsilon(t,x) R(x,0)\mathrm{d}x,\\
&< \rho_M - \frac{C}{\varepsilon}\int_{\mathbb{R}^d} \varphi_\varepsilon(x) n_\varepsilon(t,x)\mathrm{d}x\,,
\end{aligned}$$

using Assumption (H3) and that $\mathcal{O}_\delta \subset A_{C/2}$. It follows that on $(0,T_0)$, and for $\overline{C} > 0$ some constant whose value can change from line to line, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \varphi_\varepsilon(x) n_\varepsilon(t,x)\mathrm{d}x &< \int_{\mathbb{R}^d} \varphi_\varepsilon(x) n_\varepsilon(0,x)\mathrm{d}x e^{-\frac{C}{\varepsilon}t} + \frac{\rho_M}{C}\varepsilon\left(1 - e^{-\frac{C}{\varepsilon}t}\right)\,,\\
&< \overline{C}\left(e^{-\frac{C}{\varepsilon}t} + \varepsilon\right) \underset{\varepsilon\to 0}{\to} 0\,,
\end{aligned}$$

using Assumption (H8). This concludes the proof of Theorem 1.3 ii).

3.3 Asymptotic extinction at a time point

We show now the last assertion of Theorem 1.3, namely that if

$$\Gamma_0 \subseteq \{x \in \mathbb{R}^d,\, R(x,0) \leq 0\}\,,$$

then, for any $T < +\infty$,

$$\forall \lambda > 0,\, \exists \varepsilon_\lambda > 0,\, \forall \varepsilon < \varepsilon_\lambda,\, \exists t_\varepsilon \in [0,T],\, I_\varepsilon(t_\varepsilon) < \lambda\,. \tag{47}$$

Recall that the space of possible concentration points of the population at the limit writes $\Gamma_t := \{x \in \mathbb{R}^d,\, u(t,x) = 0\}$. We prove the result with a contradiction argument. If (47) is not true, then $\exists T,\, \lambda > 0,\, \exists (\varepsilon_k)_k \to 0$ such that $\forall t \in [0,T],\, I_{\varepsilon_k}(t) \geq \lambda > 0$. Then, following Proposition 3.4, $(I_{\varepsilon_k})_k$ converges on $(0,T)$ towards a function $I : t \mapsto I(t) \geq \lambda$, and by Theorem 1.1, u is then a viscosity solution of the following constrained Hamilton-Jacobi problem:

$$\begin{cases}
\partial_t u = |\nabla u|^2 + R(x, I(t))\,,\\
\max\limits_x u(t,x) = 0\,,\\
\text{Supp } n(t,\cdot) \subset \Gamma_t \subset \{R(\cdot, I(t)) = 0\} \text{ for all Lebesgue point of } I\,.
\end{cases}$$

Now, by assumption,

$$\forall x \in \Gamma_0,\ R(x,\lambda) < R(x,0) + \sup \partial_I R(x,I)\lambda \leq -K_1^{-1}\lambda < 0\,.$$

By continuity of u, there exists $\delta > 0$ small enough so that

$$\{u(0,\cdot) > -2\delta\} \subset \left\{R(\cdot,\lambda) < -\frac{\lambda}{2K_1}\right\}\,,$$

and there exists $t_1 > 0$ small enough so that

$$\forall t \in [0,t_1),\ \{u(t,\cdot) > -\delta\} \subset \{u(0,\cdot) > -2\delta\}\,.$$

Therefore, for $t \in (0,t_1)$ and $x \in \Gamma_t$, we have that

$$0 = R(x,I(t)) \leq R(x,\lambda) < -\frac{\lambda}{2K_1} < 0\,,$$

leading to a contradiction.

4 The concave case

In this part, we give the proofs of Theorem 1.5 as well as Theorem 1.8 that deal with a constant or piecewise constant concave environments. The main part of Theorem 1.5 is obtained from the combination of the results of [LMP11] and [MR16] with Theorem 1.3 to ensure the asymptotic fate of the population from the initial condition. We prove here the last part of point *ii)* regarding the maximal time of extinction, the canonical equation being obtained similarly as before. Then, we prove Theorem 1.8 that treats the case of a temporally piecewise constant concave environment, and that mainly relies on Theorem 1.5.

4.1 Proof of Theorem 1.5 - ii)

Let us first recall that we denote $\overline{x}(t) = \text{argmax}_{x\in\mathbb{R}^d} u(t,x)$ the unique maximum point of u at time t. Let us define

$$\overline{T} := \sup\{t > 0,\ R(\overline{x}(t),0) < 0\}\,.$$

By continuity of R and u, and since u is strictly concave, we deduce that $R(\overline{x}(\overline{T}),0) = 0$. We first show the following Lemma.

Lemma 4.1. *Under Assumptions* (H2)-(H3) *together with* (H10)-(H14)*, and if* $\Gamma_0 \subseteq \{R(\cdot,0) < 0\}$*, we have that*

$$\forall t \in (0,\overline{T}),\ \lim_{\varepsilon\to 0} I_\varepsilon(t) = 0\,.$$

Proof. Take any $t_0 \in (0,\overline{T})$. Then, $\exists C_0 > 0$ such that $R(\overline{x}(t_0),0) \leq -2C_0$. By the local uniform continuity of u, there exists $\delta > 0$ small enough such that

$$\mathcal{O}_\delta := \{u(t_0,\cdot) \geq u(t_0,\overline{x}(t_0)) - 2\delta\} \subseteq \{R(\cdot,0) \leq -C_0\} =: A_{C_0}\,.$$

Moreover, by the uniform continuity in time of u, there exists $h > 0$ small enough so that $\forall t \in [t_0,t_0+h)$, $\{u(t,\cdot) \geq u(t_0,\overline{x}(t_0)) - \delta\} \subseteq \mathcal{O}_\delta$ and $\max_{x\in\mathbb{R}^d} u(t,x) \geq u(t_0,\overline{x}(t_0)) - \frac{\delta}{2}$, so that $\overline{x}(t) \in \{u(t,\cdot) \geq u(t_0,\overline{x}(t_0)) - \delta\} \neq \emptyset$. Now, we write

$$I_\varepsilon(t) = \int_{\mathcal{O}_{\delta/2}} \psi(x) n_\varepsilon(t,x)\mathrm{d}x + \int_{\mathcal{O}_{\delta/2}^c} \psi(x) n_\varepsilon(t,x)\mathrm{d}x\,,$$

and we prove that each term in the right-hand side goes to 0 as $\varepsilon \to 0$, following the proof of Theorem 1.3-ii). First, on $\mathcal{O}^c_{\delta/2}$, we have that $u(t,x) < u(t_0, \overline{x}(t_0)) - \delta/2 < 0$ since $u \leq 0$, so that for ε small enough, $u_\varepsilon(t,x) \leq \frac{u(t_0,\overline{x}(t_0))-\delta/2}{2}$ on $\mathcal{O}^c_{\delta/2}$. Therefore, we can follow the proof of Lemma **??** and use Assumption H1 to deduce that on $[t_0, t_0+h)$,

$$\int_{\mathcal{O}^c_{\delta/2}} \psi(x) n_\varepsilon(t,x) \mathrm{d}x \underset{\varepsilon\to 0}{\to} 0\,.$$

We estimate now $\int_{\mathcal{O}_{\delta/2}} \psi(x) n_\varepsilon(t,x)\mathrm{d}x$. For that purpose, let $\varphi_\varepsilon \in \mathcal{C}^\infty_{c,+}(\mathcal{O}_\delta)$ be a test function such that $\varphi_\varepsilon \equiv 1$ in $\mathcal{O}_{\delta/2}$ and $0 < \varphi_\varepsilon(x)$ in $\mathcal{O}_\delta \setminus \mathcal{O}_{\delta/2}$, with $\| D^2\varphi_\varepsilon \|_\infty < \frac{1}{\varepsilon}$. Then, for $t \in (t_0, t_0+h)$, we can compute that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \varphi_\varepsilon(x) n_\varepsilon(t,x)\mathrm{d}x < \rho_M - \frac{C_0}{\varepsilon}\int_{\mathbb{R}^d} \varphi_\varepsilon(x) n_\varepsilon(t,x)\mathrm{d}x\,,$$

using Assumption (H3) and the fact that $\mathcal{O}_\delta \subset A_{C_0}$. It follows that on (t_0, t_0+h), and for $\overline{C} > 0$ some constant whose value can change from line to line, we have that

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi_\varepsilon(x) n_\varepsilon(t,x)\mathrm{d}x &< \int_{\mathbb{R}^d} \varphi_\varepsilon(x) n_\varepsilon(0,x)\mathrm{d}x e^{-\frac{C_0}{\varepsilon}(t-t_0)} + \frac{\rho_M}{C_0}\varepsilon\left(1 - e^{-\frac{C_0}{\varepsilon}(t-t_0)}\right)\,,\\ &< \overline{C}\left(e^{-\frac{C_0}{\varepsilon}(t-t_0)} + \varepsilon\right) \underset{\varepsilon\to 0}{\to} 0\,,\end{aligned}$$

using Assumption (H8). Finally, $\forall t \in (t_0, t_0+h)$, using (H1) and the definition of φ_ε, we obtain that

$$\int_{\mathcal{O}_{\delta/2}} \psi(x) n_\varepsilon(t,x)\mathrm{d}x \leq \int_{\mathbb{R}^d} \psi(x)\varphi_\varepsilon(x) n_\varepsilon(t,x)\mathrm{d}x \underset{\varepsilon\to 0}{\to} 0\,,$$

which ends the proof. □

It follows that u is solution of Equation (17) on $(0, \overline{T})$, and that under the additional Assumptions (H15)-(H16), $\overline{x}$ satisfies the canonical Equation (18). Next, we study h defined by $h(t) = R(\overline{x}(t), 0)$. We compute

$$h'(t) = \dot{\overline{x}}(t)\cdot \nabla_x R(\overline{x}(t),0) = (-D^2 u(t,\overline{x}(t)))^{-1}\cdot |\nabla_x R(\overline{x}(t),0)|^2\,,$$

so that h is increasing while $|\nabla_x R(\overline{x}(t),0)| \neq 0$, and non-decreasing in the general case. As a consequence, if $\max_{x\in\mathbb{R}^d} R(x,0) < 0$, then $\overline{T} = +\infty$ by definition and h is non-decreasing on $(0,\overline{T})$. If $\max_{x\in\mathbb{R}^d} R(x,0) \geq 0$, then h is increasing on $(0,\overline{T})$. In this case, we provide some estimates for $\overline{T}$. From [MR16] (Theorem 1.1), and using the estimates (H11) and (H13), we have that on $[0,\overline{T}]\times\mathbb{R}^d$,

$$0 < \min(2\overline{L}_1, \sqrt{\overline{K}_2}) \leq -D^2 u(t,x) \leq \max(2\underline{L}_1, \sqrt{\underline{K}_2})\,.$$

Moreover, by the concavity assumption of R, we have that for $t \in (0,\overline{T})$,

$$|\nabla_x R(\overline{x}(\overline{T}),0)|^2 \leq |\nabla_x R(\overline{x}(t),0)|^2 \leq |\nabla_x R(\overline{x}(0),0)|^2\,.$$

It leads to

$$\frac{1}{\max(2\underline{L}_1, \sqrt{\underline{K}_2})}|\nabla_x R(\overline{x}(\overline{T}),0)|^2 \leq h'(t) \leq \frac{1}{\min(2\overline{L}_1, \sqrt{\overline{K}_2})}|\nabla_x R(\overline{x}(0),0)|^2\,,$$

the lower bound being equal to 0 when $\max_{x\in\mathbb{R}^d} R(x,0) = 0$. The result then follows from this estimate combined with the equality

$$\int_0^{\overline{T}} h'(t)\mathrm{d}t = -h(0)\,.$$

4.2 Proof of Theorem 1.8

In this part, we prove Theorem 1.8 that describes the dynamics of the population when a switch in the environment occurs. First, the convergence of $(u_\varepsilon)_\varepsilon$ and the one of $(n_\varepsilon)_\varepsilon$ in $[0, T_1)$ follows from Theorem 1.5, that also yields point (i) and that the $u_\varepsilon(T_1, x)$ are uniformly strictly concave and verify Assumption (H13). By definition of $u_\varepsilon(T_1, x)$ and $I_\varepsilon(T_1)$ and by continuity of u and I, we have that $I(T_1) = \lim_{t \to T_1^-} I(t)$ satisfies Assumption (H14), and that $u(T_1, x) = \lim_{t \to T_1} u(t, x)$ is well-defined and strictly concave. Therefore, Γ_{T_1} and $\overline{x}(T_1)$ are also well-defined, and the convergence of $(u_\varepsilon)_\varepsilon$ and of $(n_\varepsilon)_\varepsilon$ in $[T_1, T_2)$ follows from Theorem 1.5. Then, point (ii) follows from Theorem 1.3, and (iii) follows from Theorem 1.3 combined with Theorem 1.5.

5 Numerics and application to switching environments

In this part, we perform some numerical simulations of (2)-(3) and (5) to illustrate the selection-mutation dynamics in temporally constant and piecewise constant environments. For that purpose, we use a finite difference scheme with an implicit time discretization scheme, at the exception of the nonlinear term $I_\varepsilon(t)$ that is treated explicitly. The corresponding scheme can be found in Appendix E.

5.1 Constant environment

We begin with numerical simulations of the problem (2)-(3) that illustrate Theorem 1.3. We consider the growth rate given by

$$R(x, I_\varepsilon) = a(x) - I_\varepsilon \,,$$

with a to define.

Asymptotic persistence and extinction on a time interval

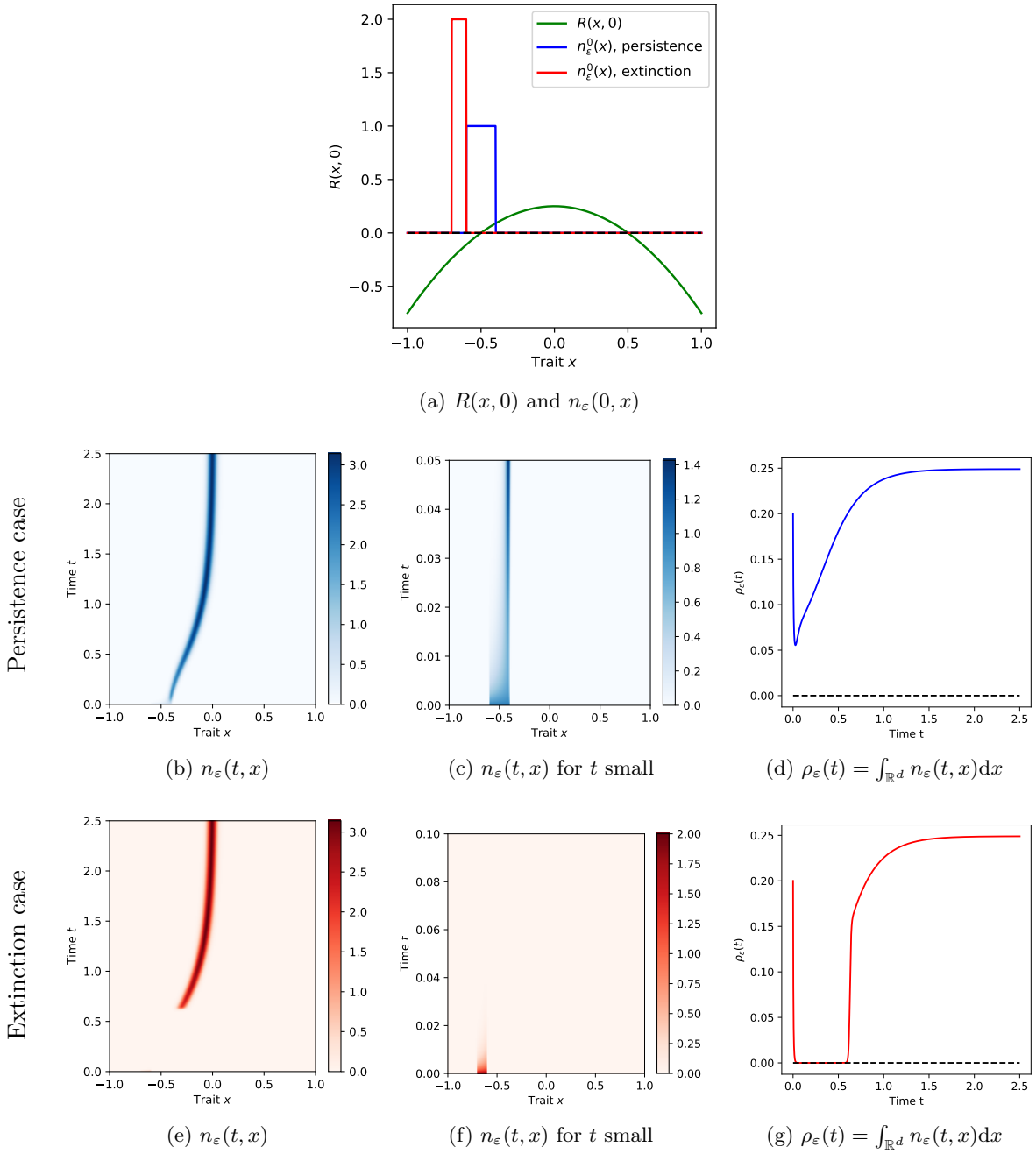
We choose a as a quadratic function with

$$a(x) = r - g x^2 \,, \tag{48}$$

for $r = 0.25$ and $g = 1$, which is strictly positive in $]-0.5, 0.5[$. We choose two expressions for the initial condition, in order to illustrate the two first cases of Theorem 1.3. They are given by

$$n_\varepsilon^0(x) = \frac{I_\varepsilon(0)}{b-a} \mathbb{1}_{[a,b]} \,, \tag{49}$$

where $I_\varepsilon(0) = 0.2$, and $(a, b) = (-0.6, -0.4)$ in the case of asymptotic persistence, while $(a, b) = (-0.7, -0.6)$ in the case of asymptotic extinction (see Figure 1a). In the asymptotic persistence situation, a part of the initial population is composed of individuals having a positive growth rate in the absence of competition, where the population survives for all times. One can see that n_ε evolves towards the best trait $x = 0$ (see Figure 1b). For small times, the mass drops (see Figure 1d) as a result of the extinction of the part of the population that is not viable, but it does not reach zero as the population seems to be sustained thanks to larger trait values (see Figure 1c). In the asymptotic extinction case, the initial population size vanishes near $t = 0$ (see Figures 1e and 1f). After some time, the population grows again at some trait values having a nonnegative growth rate in the absence of competition (see Figure 1g). This phenomenon is surprising, and shows a limitation of the Hamilton-Jacobi approach: from the modelling viewpoint, the model then ceases to be valid.



(a) $R(x,0)$ and $n_\varepsilon(0,x)$

(b) $n_\varepsilon(t,x)$ (c) $n_\varepsilon(t,x)$ for t small (d) $\rho_\varepsilon(t) = \int_{\mathbb{R}^d} n_\varepsilon(t,x)\mathrm{d}x$

(e) $n_\varepsilon(t,x)$ (f) $n_\varepsilon(t,x)$ for t small (g) $\rho_\varepsilon(t) = \int_{\mathbb{R}^d} n_\varepsilon(t,x)\mathrm{d}x$

Figure 1 – Numerical simulations of (2)-(3) for $R(x,0)$ given by (48) and initial conditions given by (49). Second line (blue): situation where Supp $n(0,\cdot) \cap \{x \in \mathbb{R}^d,\, R(x,0) > 0\} \neq \emptyset$. The population density evolves towards better trait values, while the population mass stays stricly positively bounded by below. Third line (red): situation where $\Gamma_0 \subseteq \{x \in \mathbb{R}^d,\, R(x,0) \leq 0\}$ for some $C > 0$. The population mass almost immediately drops to very small values, before growing again when the population density concentrates around trait values that have a positive growth rate. Parameters: $r = 0.25$, $g = 1$, $I_\varepsilon(0) = 0.2$; $\mathrm{d}x = \varepsilon = 10^{-3}$; $\mathrm{d}t = 10^{-4}$.

Asymptotic extinction: critical case

Figure 2 illustrates the third situation described in Theorem 1.3, occurring when $\Gamma_0 \subseteq \{R(\cdot,0) = 0\}$. For that purpose, we consider

$$R(x,0) = a(x) = -x^2(x-0.75)(x-2)\,. \tag{50}$$

Moreover, we consider two initial conditions given by

$$n_\varepsilon^{0,1}(x) = \frac{I_\varepsilon(0)}{\sqrt{2\pi\varepsilon}} e^{-\frac{x^2}{2\varepsilon}}\,, \qquad n_\varepsilon^{0,2}(x) = \frac{I_\varepsilon(0)}{\sqrt{2\pi\varepsilon}} e^{-\frac{(x-0.75)^2}{2\varepsilon}}\,, \tag{51}$$

with $I_\varepsilon(0) = 0.2$. Figure 2a shows the corresponding growth function as well as the two initial conditions that we consider. One can see in Figure 2 that the solutions behave differently. The solution issued from $n_\varepsilon^{0,1}$ (Figures 2b-2c-2d) keeps a density that is close to zero during a whole time interval, before increasing again very fast from a population density concentrated at trait values having a positive growth rate. Figures 2e-2f-2g illustrate the situation when the initial population is concentrated near values that have a positive growth rate in the absence of competition. In that case, the drop in population density is not sufficient to attain zero, since $\varepsilon \neq 0$, so that mutations allow reaching better traits fast enough to rescue the population. This illustrates the critical situation where asymptotic extinction occurs ponctually in time.

Case not treated by Theorem 1.3

Figure 3 illustrates the situation described in Remark 1.4 where Supp $n(0,\cdot) \subseteq \{R(\cdot,0) \leq 0\}$ and $\Gamma_0 \cap \{R(\cdot,0) > 0\} \neq \emptyset$. For that purpose, we consider again

$$a(x) = r - gx^2\,, \tag{52}$$

for $r = 0.25$, $g = 1$, and

$$n_\varepsilon^0(x) = \frac{0.2}{\sqrt{2\pi\varepsilon}} \exp\left(-\frac{(x+0.75)^2}{2\varepsilon}\right) + \frac{\varepsilon}{\sqrt{2\pi\varepsilon}} \exp\left(-\frac{x^2}{2\varepsilon}\right)\,, \tag{53}$$

so that $\Gamma_0 = \{-0.75, 0\}$ and Supp $n(0,\cdot) = \{-0.75\}$, with $R(0,0) > 0$ and $R(-0.75,0) \leq 0$. Therefore, Γ_0 meets a viable trait ($x = 0$), but the population mass at this point vanishes as ε goes to zero (see Figure 3a). In this case, the individuals holding traits with negative growth rates die quickly which leads to a rapid drop in population size. Afterwards, the population stabilizes around the best trait. However, this simulation suggests that the asymptotic extinction of the population indeed occurs at a time point near $t = 0$.

5.2 Piecewise constant environment

In this section, we illustrate different phenomena arising in a temporally piecewise constant environment. More precisely, in the case where the environment switches between two states, we consider the problem (5) for e a periodic function of time with period T such that for $t \in [0,T]$, $e(t) = \mathbb{1}_{[0,T/2)}(t) + 2\mathbb{1}_{[T/2,T)}(t)$. Then, for $(t,x) \in [0,T]\times\mathbb{R}^d$, we define the corresponding growth rate $R(x,e(t),I_\varepsilon(t))$ by $R_{e(t)}(x,I_\varepsilon(t))$. We investigate numerically situations where the period of fluctuations has an effect on the dynamics, wether it acts on the fate of the population, the mean fitness, or on the dynamics of the optimal trait.

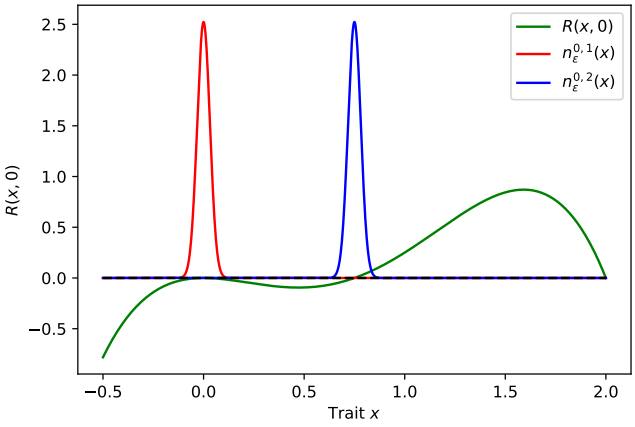
Effect of the period of fluctuations on the fate of the population

We consider here a periodic switch between two concave growth rates given by

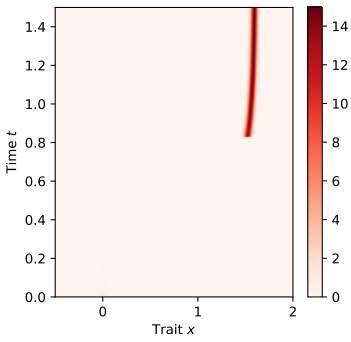
$$R_1(x,I_\varepsilon(t)) = r - g(x+\theta)^2 - I_\varepsilon(t) \quad \text{and} \quad R_2(x,I_\varepsilon(t)) \;\; = r - g(x-\theta)^2 - I_\varepsilon(t)\,, \tag{54}$$

with $g\theta^2 < r < 4g\theta^2$ to ensure that there are traits viable in both environments. Here, we take $\theta = 0.5$, $g = 1$ and $r = 0.5$. The initial condition is given by

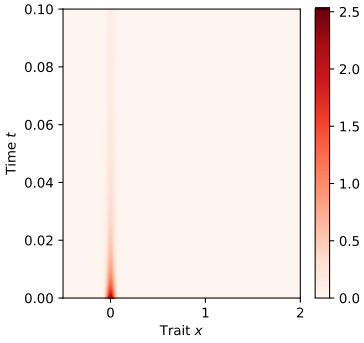
$$n_\varepsilon^0(x) = \rho_0 \frac{g^{\frac{1}{4}}}{\sqrt{2\pi\varepsilon}} e^{-\frac{\sqrt{g}(X-a)^2}{2\varepsilon}}\,, \tag{55}$$



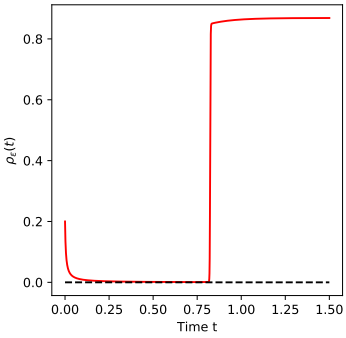
(a) $R(x,0)$ and $n_\varepsilon^0(x)$



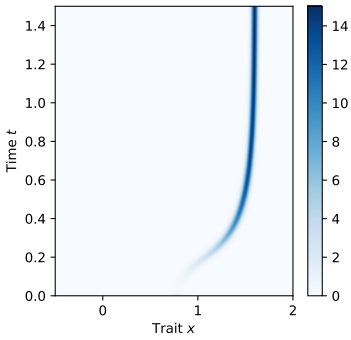
(b) $n_\varepsilon(t,x)$



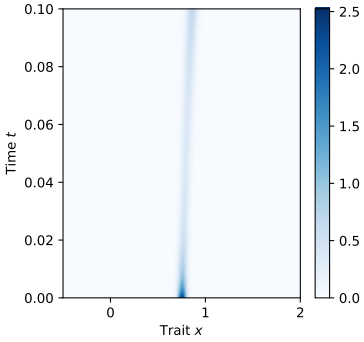
(c) $n_\varepsilon(t,x)$ for t small



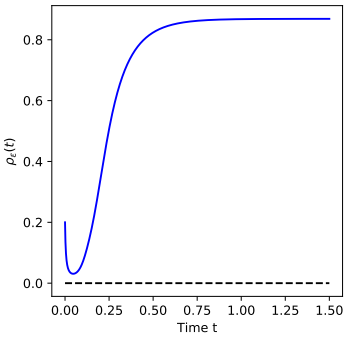
(d) $\rho_\varepsilon(t) = \int_{\mathbb{R}^d} n_\varepsilon(t,x)\mathrm{d}x$



(e) $n_\varepsilon(t,x)$



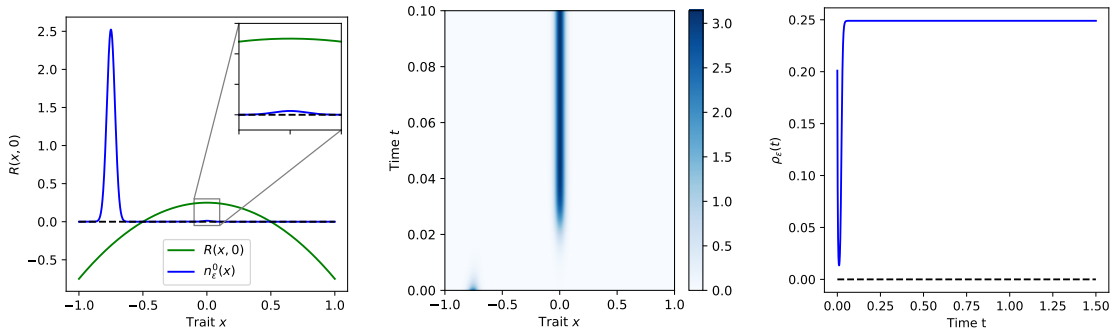
(f) $n_\varepsilon(t,x)$ for t small



(g) $\rho_\varepsilon(t) = \int_{\mathbb{R}^d} n_\varepsilon(t,x)\mathrm{d}x$

Figure 2 – Numerical simulations of (2)-(3) in the case where $\Gamma_0 \subseteq \{R(\cdot,0) = 0\}$, for $R(x,0)$ given by (50) and initial conditions given by (51). The population mass drops to very small values on some time interval when the initial population density concentrates far from viable traits (Figure 2d), whereas it increases away from zero faster when the population is concentrated near viable traits (Figure 2g). Parameters: $I_\varepsilon(0) = 0.2$; $\mathrm{d}x = \varepsilon = 10^{-3}$; $\mathrm{d}t = 10^{-4}$.

with $\rho_0 = 0.25$ and $a = 0$ (see Figure 4a), and allows initial persistence in both environments. Figures 4b and 4c show the evolution of n_ε and ρ_ε over the first environmental switch when the period of fluctuation is large ($T = 1$). On $[0, \frac{T}{2})$, the population concentrates on better traits relatively to the first environment. However, these traits have negative growth rates in



(a) $n_\varepsilon^0(x)$ and $R(x,0)$ (b) $n_\varepsilon(t,x)$ (c) $\rho_\varepsilon(t) = \int_{\mathbb{R}^d} n_\varepsilon(t,x)\mathrm{d}x$

Figure 3 – Numerical simulations of (2)-(3) in the case of Remark 1.4, where Supp $n(0,\cdot) \subseteq \{R(\cdot,0) \leq 0\}$ and $\Gamma_0 \cap \{R(\cdot,0) > 0\} \neq \emptyset$, with $a(x)$ given by (52) for $r = 0.25$ and $g = 1$, and an initial condition given by (53). The population density is split between a subpopulation concentrated at non viable trait values, and a subpopulation concentrated at viable trait values, but having a mass going to zero as $\varepsilon \to 0$. The mass drops immediately as the population carrying traits with negative growth rates disappears, but it quickly increases again towards a positive stationary value, as the population concentrates around the best possible trait. Parameters: $\mathrm{d}x = \varepsilon = 10^{-3}$; $\mathrm{d}t = 10^{-4}$.

the second environment. As a result, at switching time, the population is in a situation of asymptotic extinction.

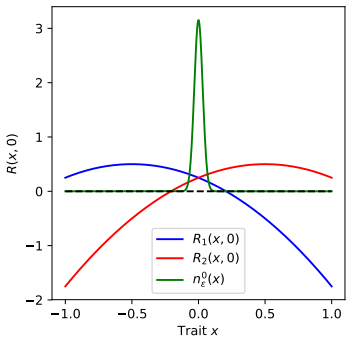
Then, we consider the case where the period of fluctuations is smaller ($T = 0.2$), for the same initial condition. In this situation, the period is small enough so that the population remains concentrated in traits having positive growth rates in the absence of competition in both environments. The population is therefore persistent, and n_ε is periodic in time (see Figures 4d and 4e). The fate of a population may be drastically affected by the timing of environmental fluctuations.

Effect of the period of fluctuations on the mean fitness

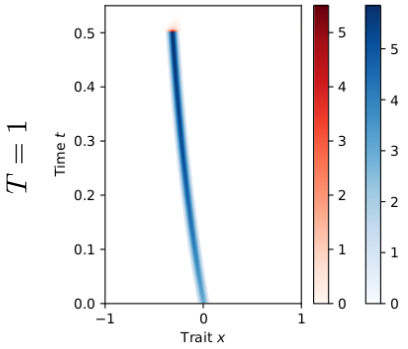
We are now interested in highlighting how the period of fluctuations between two environments can have an effect on the mean fitness of the population. To do so, we consider the growth rates given by (54) with $\theta = 1$, $r = 1$ and $g = 0.2$. This choice aims at considering environments where each respective optimal trait is also viable in the other environment and the asymptotic extinction of the population can not occur. The initial condition is given by (55) with $a = 0$ and $\rho_0 = 0.2$ (see Figure 5a). A natural indicator for the mean fitness in the population during a period is given by

$$\overline{\rho_\varepsilon}(T) := \int_0^T \rho_\varepsilon(t)\mathrm{d}t\,.$$

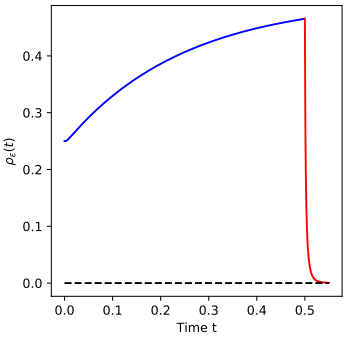
We perfom numerical simulations of the solution of (5) for T ranging in $[0.1, 5]$. Figure 5b shows several simulations of the time evolution of the mass of the population for increasing values of the period T, when a stationary regime is attained. It is observed that the mass drop gets larger and larger for an increasing time spent in each environment. However, Figure 5c shows that the stationary mean population mass over a period of fluctuations increases with the lenth of the period. This can be interpreted as follows. For a small period of fluctuations, the population never gets fully adapted to an environment, but does not suffer much either from



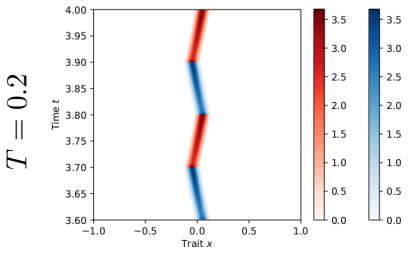
(a) $R_{1,2}(x,0)$ and $n_\varepsilon^0(x)$ from (55)



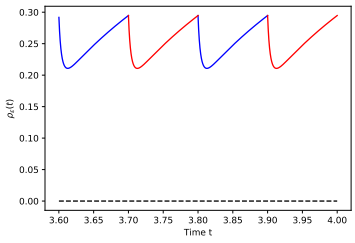
(b) $n_\varepsilon(t,x)$



(c) $\rho_\varepsilon(t) = \int_{\mathbb{R}^d} n_\varepsilon(t,x)\mathrm{d}x$



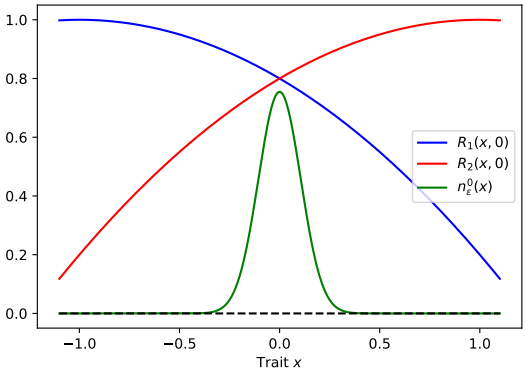
(d) $n_\varepsilon(t,x)$



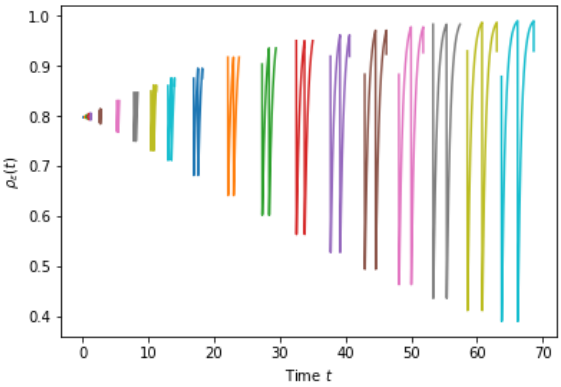
(e) $\rho_\varepsilon(t) = \int_{\mathbb{R}^d} n_\varepsilon(t,x)\mathrm{d}x$

Figure 4 – Numerical simulations of (5) for a growth rate given by (54) and an initial condition given by (55) with $\rho_0 = 0.25$ and $a = 0$. Up: situation when $T = 1$. There is asymptotic extinction at the first switching time, since the population is concentrated at trait values corresponding to positive growth rates in the first environment and to negative growth rates in the second one. Down: periodic solution when $T = 0.2$. The solution remains concentrated at trait values that correspond to positive growth rates in both environments, while the mass reaches a periodic time evolution. Parameters: $\mathrm{d}x = \varepsilon = 10^{-3}$; $\mathrm{d}t = 10^{-4}$.

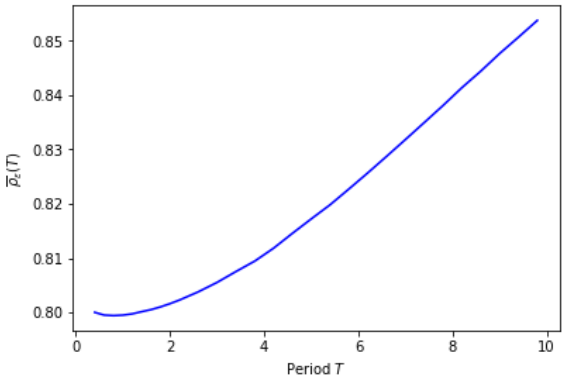
the fluctuations. If the period of fluctuations is larger, this allows the population to concentrate on the optimal trait in each environment. As a consequence, the population mass can drop significatively at switching time, which seems more costly. Understanding which situation is better from an evolutive point of vue is not intuitive. Our simulations show an example where larger periods of fluctuations are better from an evolutive point of vue, even if the population is less stable in the ecosystem.



(a) $n_\varepsilon^0(x)$, $R_1(x,0)$ and $R_2(x,0)$



(b) $\rho_\varepsilon(t)$ for increasing T from left to right



(c) $\overline{\rho_\varepsilon}(T)$

Figure 5 – Numerical simulations of (5) for a growth rate given by (54) with $\theta = 1$, $r = 1$ and $g = 0.2$ and an initial condition given by (55) with $\rho_0 = 0.2$ and $a = 0$. (a) growth rates for each environment and initial condition. (b) evolution of $\rho_\varepsilon(t)$ over a period for different period values T. Drops in population mass become larger when the period of fluctuations increases. (c) stationary mean population mass over a period of environmental fluctuations as a function of the duration T of the period. The mean fitness of the population increases as the period of fluctuations gets larger. Parameters: $\mathrm{d}x = \varepsilon = 10^{-3}$; $\mathrm{d}t = 10^{-4}$.

Effect of the period of fluctuations on the concentration trait

Finally, we illustrate in Figure 6 the situation where the period of fluctuations affects the trait value at which the population concentrates. For that purpose, we consider an environmental switch between a concave and a symmetric bimodal shape. More precisely, consider

$$R_1(x, I_\varepsilon(t)) = 0.7 - \frac{1}{5}x^2 - I_\varepsilon(t) \quad \text{and} \quad R_2(x, I_\varepsilon(t)) = 0.2 - \frac{2}{3}x^4 + \frac{4}{5}x^2 - I_\varepsilon(t)\,, \tag{56}$$

with an initial condition given by

$$n_\varepsilon^0(x) = \rho_0 \frac{g^{\frac{1}{4}}}{\sqrt{2\pi\varepsilon}} e^{-\sqrt{g}\frac{(x-1)^2}{2\varepsilon}}\,. \tag{57}$$

Both growth rates and the initial condition are shown in Figure 6a. Figures 6b-6c (resp. 6e-6f) show the evolution of the solution n_ε and its mass ρ_ε during two periods, when a stationary

periodic solution is attained, for a short period (resp. large period). One can see that when the period of fluctuations is small, the population remains monomorphic and mostly adapted to the bimodal environment. When the period of fluctuations gets larger, the population has enough time to adapt to the unimodal environment. In this situation, the population becomes dimorphic in the bimodal environment. This may be an effect of the fact that $\varepsilon \neq 0$, so that very small mutations can have an effect of the population dynamics. Overall, these simulations show that when the environment switches between very different phenotypic landscapes, complex phenomena may appear and the trait at which the population concentrates may by hard to predict.

A Proof of Proposition 2.1

Proof. We adapt the proof of [PB08].

1. **Trivial lower bounds on I_ε and ρ_ε.** It is clear from Equation (2) that for a nonnegative initial condition, the solution n_ε stays nonnegative on $\mathbb{R}_+$. A direct consequence is that $\forall t \geq 0$, $\rho_\varepsilon(t) \geq 0$, and $I_\varepsilon(t) \geq \psi_m \rho_\varepsilon(t) \geq 0$ using (H1).
2. **Upper bound on $\mathbf{I_\varepsilon(t)}$.** First, we use Equation (2) to compute

$$\frac{\mathrm{d}}{\mathrm{d}t} I_\varepsilon(t) = \varepsilon \int_{\mathbb{R}^d} \psi(x) \Delta n_\varepsilon(t,x) \mathrm{d}x + \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \psi(x) n_\varepsilon(t,x) R(x, I_\varepsilon(t)) \mathrm{d}x.$$

   Now, for $L > 0$, define χ_L a smooth function with compact support such that $\chi_L \mid_{B(0,L)} \equiv 1$ and $\chi_L \mid_{\mathbb{R}^d \setminus B(0,2L)} \equiv 0$ and define $\psi_L = \chi_L \cdot \psi \in W^{2,\infty}_c(\mathbb{R}^d)$. Using an integration by parts on $\int_{\mathbb{R}^d} \psi_L(x) \Delta n_\varepsilon(t,x) \mathrm{d}x$, and since ψ_L converges to ψ in $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$ as $L \to +\infty$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} I_\varepsilon(t) = \varepsilon \int_{\mathbb{R}^d} \Delta\psi(x) n_\varepsilon(t,x) \mathrm{d}x + \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \psi(x) n_\varepsilon(t,x) R(x, I_\varepsilon(t)) \mathrm{d}x. \tag{58}$$

   Now, writing $C = \frac{\|\Delta\psi\|_\infty K_1}{\psi_m}$, we have for all $x \in \mathbb{R}^d$,

$$R(x, I_M + C\varepsilon^2) \leq \underbrace{R(x, I_M)}_{=0 \text{ by } H2} + C\varepsilon^2 \sup_{I \in \mathbb{R}} \frac{\partial}{\partial I} R(x, I) < -\frac{C\varepsilon^2}{K_1} = -\varepsilon^2 \frac{\| \Delta\psi \|_\infty}{\psi_m},$$

   from (H3). As a consequence, if at some t_0, $I_\varepsilon(t_0) = I_M + C\varepsilon^2$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t} I_\varepsilon(t)|_{t_0} \leq \varepsilon \frac{\| \Delta\psi \|_\infty}{\psi_m} I_\varepsilon(t_0) - \varepsilon^2 \frac{\| \Delta\psi \|_\infty}{\psi_m} \frac{1}{\varepsilon} I_\varepsilon(t_0) = 0,$$

   and the result is proved for ε small enough so that $I_M + C\varepsilon^2 \leq 2 I_M$.
3. **Upper bound on $\rho_\varepsilon(\mathbf{t})$.** It follows directly from the last estimate that $0 \leq \rho_\varepsilon(t) \leq \frac{1}{\psi_m}(I_M + C\varepsilon^2) =: \rho_M$.

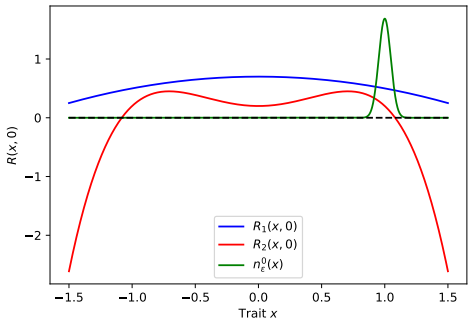
□

B Proof of Proposition 2.2

Proof. **(i) Upper bound on $(\mathbf{u}_\varepsilon)_\varepsilon$.**

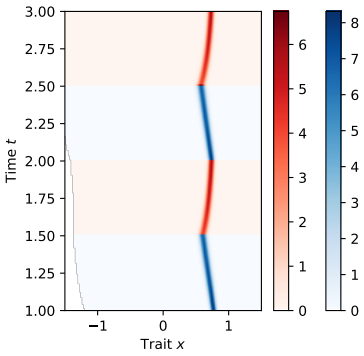
For $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, let us define $\overline{u}(t,x) = -B_2|x| + A_2 + (B_2^2 + K_0)t$. Then by (H9), we have that $\forall \varepsilon > 0$, $\forall x \in \mathbb{R}^d$, $u^0_\varepsilon(x) \leq \overline{u}(0,x)$, and for I_ε defined by (3) and a.e $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$\partial_t \overline{u} - \varepsilon \Delta \overline{u} - |\nabla \overline{u}|^2 - R(x, I_\varepsilon(t)) \geq B_2^2 + K_0 + \varepsilon B_2 \frac{d-1}{|x|} - B_2^2 - K_0 \geq 0,$$

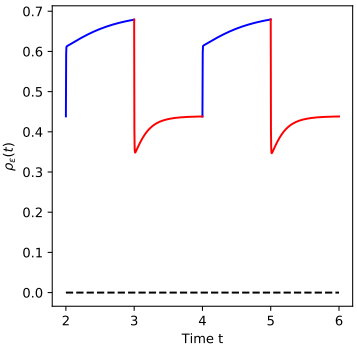


(a) $n_\varepsilon^0(x)$ and $R_1(x,0)$, $R_2(x,0)$ from (56)

$T = 1$

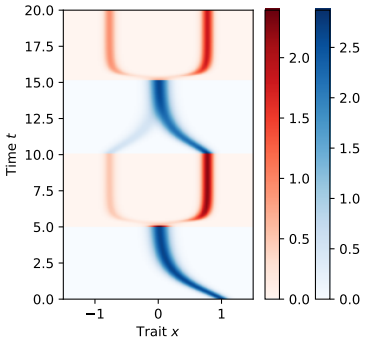


(b) $n_\varepsilon(t,x)$ at stationary state

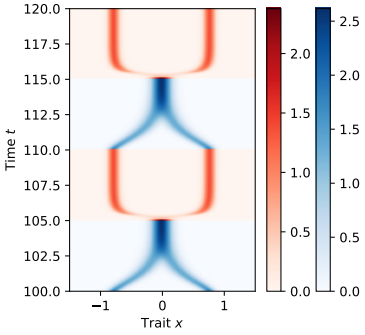


(c) $\rho_\varepsilon(t) = \int_{\mathbb{R}^d} n_\varepsilon(t,x)\mathrm{d}x$ at stationary state

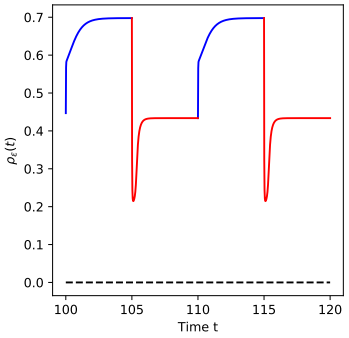
$T = 10$



(d) $n_\varepsilon(t,x)$ for t small



(e) $n_\varepsilon(t,x)$ at stationary state



(f) $\rho_\varepsilon(t) = \int_{\mathbb{R}^d} n_\varepsilon(t,x)\mathrm{d}x$ at stationary state

Figure 6 – Numerical simulations of (5) for a growth rate given by (56) and the initial condition given by (57). Up ((b) & (c)): stationary periodic solution and mass evolution when $T = 1$: the switching frequency is high enough to maintain the trait in the same subpart of the phenotypic landscape. Down: stationary periodic solution when $T = 10$. The switching frequency is low enough to allow for an exploration of the bimodal environment. The apparition of a dimorphic population may occur only for $\varepsilon \neq 0$. Parameters: $\mathrm{d}x = \varepsilon = 10^{-3}$; $\mathrm{d}t = 10^{-4}$.

using (H4). As a consequence, $\overline{u}$ is a supersolution to (4). Using a comparison principle in the class of L^2 functions, we obtain that for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$u_\varepsilon(t,x) \leq -B_2|x| + A_2 + (B_2^2 + K_0)t\,.$$

Lower bound on $(\mathbf{u}_\varepsilon)_\varepsilon$.
Denote $M_1 = \max\left(\frac{\sqrt{K_3}}{2}, B_1\right)$ and define for $(t,x) \in [0,T]\times\mathbb{R}^d$, $\underline{u_\varepsilon}(t,x) = -A_1 - M_1|x|^2 - (2d\varepsilon M_1 + K_2)t$. From (H9), $\underline{u_\varepsilon}(0,x) \le u_\varepsilon^0(x)$ on $\mathbb{R}^d$, and (H4) yields that

$$\partial_t \underline{u_\varepsilon} - |\nabla \underline{u_\varepsilon}|^2 - R(x, I_\varepsilon) - \varepsilon\Delta\underline{u_\varepsilon} \le -K_2 - 4M_1^2|x|^2 + K_2 + K_3|x|^2 \le 0,$$

and the lower bound on u_ε follows. This leads to

$$D(T) \le v_\varepsilon(t,x) \le C|x| + B(T)\,, \quad \text{for } (t,x) \in [0,T]\times\mathbb{R}^d\,, \tag{59}$$

with B, C positive constants.

(b) & (c) Regularizing effect in time Finally, we show the local uniform continuity in time of $(u_\varepsilon)_\varepsilon$ on either $[t_0,T]\times B_{L/2}(0)$ or $[0,T]\times B_{L/2}(0)$, depending on the hypothesis on the initial condition. In the following, we work with the general notation $[t_i,T]\times B_L(0)$.
Let us show that $\forall\eta>0$, $\exists\theta>0$ such that $\forall(t,s,x)\in[t_i,T]^2\times B_{L/2}(0)$ with $0\le t-s\le\theta$, for all $\varepsilon<\varepsilon_0$, we have that

$$|u_\varepsilon(t,x) - u_\varepsilon(s,x)| \le 2\eta.$$

We follow the proofs from [BBL02, Lemma 9.1] and [BMP09, sec. 3.4]. It consists in using the local uniform L^∞ bounds on $(u_\varepsilon)_\varepsilon$ and the uniform continuity in space to obtain the uniform local time continuity. Take $(s,x)\in[t_i,T[\times B_{L/2}(0)$, and define for $(t,y)\in[s,T[\times B_L(0)$ and any $\eta>0$,

$$\zeta(t,y) = u_\varepsilon(s,x) + \eta + A|x-y|^2 + B(t-s),$$

with A and B constants to be defined. We show that for A and B large enough, ζ is a strict supersolution to (4) on $[s,T]\times B_L(0)$, and $\zeta(t,y) > u_\varepsilon(t,y)$ on $\{s\}\times B(0,L)\cup[s,T]\times\partial B_L(0)$. First, using point *(a)*, $(u_\varepsilon)_\varepsilon$ is locally uniformly bounded, so that we can take A such that $\forall\varepsilon<\varepsilon_0$,

$$\frac{8\parallel u_\varepsilon\parallel_{L^\infty([t_i,T]\times B_L(0))}}{L^2} \le A\,.$$

With this choice, for $(t,y)\in[s,T]\times\partial B_L(0)$ and any $\eta>0$, $B>0$, $\zeta(t,y) > u_\varepsilon(t,y)$. Now, on $\{s\}\times B(0,L)$, we need to show that for A large enough, $\zeta(s,y) > u_\varepsilon(s,y)$. Let us proceed by contradiction. If there exists $\eta>0$, $\forall A>0$, $\exists y_{A,\varepsilon}\in B(0,L)$ such that $\zeta(s,y_{A,\varepsilon}) \le u_\varepsilon(s,y_{A,\varepsilon})$, or equivalently

$$u_\varepsilon(s,y_{A,\varepsilon}) - u_\varepsilon(s,x) \ge \eta + A|x-y_{A,\varepsilon}|^2, \tag{60}$$

then we obtain that

$$|x-y_{A,\varepsilon}| \le \sqrt{\frac{u_\varepsilon(s,y_{A,\varepsilon}) - u_\varepsilon(s,x) - \eta}{A}} \le \sqrt{\frac{2M}{A}}$$

with M a uniform upper bound on $\parallel u_\varepsilon\parallel_{L^\infty([t_i,T]\times B(0,L))}$. As a consequence, for all $\varepsilon>0$, $\lim_{A\to\infty}|x-y_{A,\varepsilon}| = 0$. Since $(u_\varepsilon)_\varepsilon$ is uniformly continuous in space on $B(0,L)$, there exists $h>0$ such that for all $\varepsilon>0$, if $|x-y_{A,\varepsilon}|\le h$, then $|u_\varepsilon(s,x) - u_\varepsilon(s,y_{A,\varepsilon})| < \frac{\eta}{2}$. This contradicts (60), and we deduce that $\zeta(s,y) > u_\varepsilon(s,y)$ on $B(0,L)$. Finally, we have that in $[s,T]\times B(0,L)$, for B large enough and $C \ge \sup_{I_m\le I_\varepsilon\le 2I_M}\parallel R(y,I_\varepsilon(t))\parallel_{L^\infty([s,T]\times B(0,L))}$,

$$\partial_t\zeta(t,y) - \varepsilon\Delta\zeta(t,y) - |\nabla\zeta(t,y)|^2 - R(y,I_\varepsilon(t)) \ge B - 2Ad\varepsilon - 9A^2L^2 - C \ge 0$$

and ζ is a supersolution of (4). Now since u_ε is a solution of (4), we deduce that for all $(t,y) \in [s,T] \times B(0,L)$,

$$u_\varepsilon(t,y) \leq \zeta(t,y) = u_\varepsilon(s,x) + \eta + A[x-y]^2 + B(t-s).$$

We can prove similarly that, up to changing A and B, $u_\varepsilon(t,y) - u_\varepsilon(s,x) \geq -\eta - A|x-y|^2 - B(t-s)$. We conclude by taking $x=y$ and $\theta < \frac{\eta}{B}$ in both inequalities.

□

C Proof of Theorem 1.1 (i)-(ii)

We use now the regularity properties obtained in Proposition 2.2 to prove the convergence of $(u_\varepsilon)_\varepsilon$ and of $(n_\varepsilon)_\varepsilon$.

Convergence of $(u_\varepsilon)_\varepsilon$

From Proposition 2.2, we know that $(u_\varepsilon)_\varepsilon$ is locally uniformly bounded and continuous. We use the Arzelà-Ascoli theorem to deduce that up to a subsequence, $(u_\varepsilon)_\varepsilon$ converges locally uniformly to a continuous function u in $(0,T) \times \mathbb{R}^d$. If moreover $(\nabla u_\varepsilon^0)_\varepsilon$ is locally uniformly bounded, then $(u_\varepsilon)_\varepsilon$ is locally uniformly bounded and continuous on $[0,T] \times \mathbb{R}^d$, and the Arzelà-Ascoli theorem applied near $t=0$ shows that $u \in \mathcal{C}([0,\infty) \times \mathbb{R}^d)$. In particular, $u(0,x) = \lim_{\varepsilon \to 0} u_\varepsilon(0,x) = u^0(x)$.

Proof of $u \leq 0$

Assume that for some (t,x), there exists b such that $0 < b \leq u(t,x)$. Then, by continuity, there exists $r > 0$ such that $\forall (t,y) \in B(x,r)$, $u(t,y) \geq \frac{b}{2}$. As a consequence, on $B(x,r)$, $n_\varepsilon(t,y) \to +\infty$ when $\varepsilon \to 0$, which contradicts the upper bound (22) on I_ε.

Proof of the convergence of $(n_\varepsilon)_\varepsilon$

We know from Proposition 2.1 that for a.e t, $\rho_\varepsilon(t)$ is bounded, so that up to a subsequence, $(n_\varepsilon(t))_\varepsilon$ converges weakly in the space of Radon measures in $\mathbb{R}^d$ towards a measure $n(t,\cdot)$. Moreover, since $\| n_\varepsilon(\cdot,x) \|_{L^1_x(\mathbb{R}^d)}$ is bounded in $L^\infty_t(\mathbb{R}_+)$, we deduce the L^∞-weak-$*$ convergence in $\mathbb{R}_+$ towards $n \in L^\infty(\mathbb{R}_+; \mathcal{M}^1(\mathbb{R}^d))$.

Proof of supp $(n(t,\cdot)) \subseteq \{u(t,\cdot) = 0\}$

Assume that there exists $x^* \in \text{supp } n(t^*,\cdot)$ such that $u(t^*,x^*) < 0$. Then, since $(u_\varepsilon)_\varepsilon$ is uniformly continuous on a neighborhood of (t^*,x^*), we obtain that for ε small enough, there exists a, $\delta > 0$ such that on $V_\delta := (t^*-\delta, t^*+\delta) \times B(x^*,\delta)$, we have that $u_\varepsilon(t,x) \leq -\frac{a}{2} < 0$. We deduce that

$$\int_{V_\delta} n \mathrm{d}t\mathrm{d}x = \int_{V_\delta} \lim_{\varepsilon \to 0} e^{\frac{u_\varepsilon(t,x)}{\varepsilon}} \mathrm{d}t\mathrm{d}x = 0$$

leading to the result for almost every t, $supp(n(t,\cdot)) \subset \{u(t,\cdot) = 0\}$.

D Proof of Theorem 1.1 (iii)

In this section, we identify $u = \lim_{\varepsilon \to 0} u_\varepsilon$, assuming that $(I_\varepsilon)_\varepsilon$ converges to a function I.

Identification of the Hamilton-Jacobi equation (8)

We define $\Phi_\varepsilon(t,x) := u_\varepsilon(t,x) - \int_0^t \nabla R(x, I_\varepsilon(s))\mathrm{d}s$. From (4), we deduce that

$$\partial_t\Phi_\varepsilon(t,x) - \varepsilon\Delta\Phi_\varepsilon(t,x) - |\nabla\Phi_\varepsilon(t,x)|^2 - 2\nabla\Phi_\varepsilon(t,x)\cdot\int_0^t \nabla R(x,I_\varepsilon(s))\mathrm{d}s$$
$$= \varepsilon\int_0^t \Delta R(x,I_\varepsilon(s))\mathrm{d}s + \left|\int_0^t \nabla R(x,I_\varepsilon(s))\mathrm{d}s\right|^2 .$$

Our goal is to pass to the limit $\varepsilon \to 0$. Since $I \mapsto R(x,I)$ is smooth, we obtain the locally uniform limits on $[0,T]$:

$$\lim_{\varepsilon\to 0}\int_0^t R(x,I_\varepsilon(s))\mathrm{d}s = \int_0^t R(x,I(s))\mathrm{d}s\,,$$
$$\lim_{\varepsilon\to 0}\int_0^t \nabla R(x,I_\varepsilon(s))\mathrm{d}s = \int_0^t \nabla R(x,I(s))\mathrm{d}s\,,$$
$$\lim_{\varepsilon\to 0}\int_0^t \Delta R(x,I_\varepsilon(s))\mathrm{d}s = \int_0^t \Delta R(x,I(s))\mathrm{d}s\,.$$

These limiting functions are continuous. Moreover, since, $(u_\varepsilon)_\varepsilon$ converges locally uniformly to the continuous function u as ε goes to zero, it follows that $(\Phi_\varepsilon)_\varepsilon$ converges locally uniformly to Φ with $\Phi(t,x) = u(t,x) - \int_0^t R(x,I(s))\mathrm{d}s$ as ε goes to 0, and this function is continuous. Next, let us show that Φ is a viscosity solution of

$$\partial_t\Phi(t,x) - |\nabla\Phi(t,x)|^2 - 2\nabla\Phi(t,x)\cdot\int_0^t \nabla R(x,I(s))\mathrm{d}s = \left|\int_0^t \nabla R(x,I(s))\mathrm{d}s\right|^2 . \tag{61}$$

Then, it will be straightforward that u is a viscosity solution of

$$\partial_t u(t,x) = |\nabla u(t,x)|^2 + R(x,I(t))\,, \tag{62}$$

and the proof will be complete. To show this result, take $\psi \in \mathcal{C}^\infty((0,T)\times\mathbb{R}^d)$, and suppose that $\Phi - \psi$ has a strict local maximum at a point $(t_1,x_1) \in (0,T)\times\mathbb{R}^d$. Now, since $\nabla\Phi_\varepsilon(t_{\varepsilon_j}x_{\varepsilon_j}) = \nabla\psi(t_{\varepsilon_j}x_{\varepsilon_j})$, $\partial_t\Phi_\varepsilon(t_{\varepsilon_j}x_{\varepsilon_j}) = \partial_t\psi(t_{\varepsilon_j}x_{\varepsilon_j})$, and $-\Delta\Phi_\varepsilon(t_{\varepsilon_j}x_{\varepsilon_j}) \geq -\Delta\psi(t_{\varepsilon_j}x_{\varepsilon_j})$, we deduce that

$$\partial_t\psi(t_{\varepsilon_j}x_{\varepsilon_j}) - |\nabla\psi(t_{\varepsilon_j}x_{\varepsilon_j})|^2 - 2\nabla\psi(t_{\varepsilon_j}x_{\varepsilon_j})\cdot\int_0^{t_{\varepsilon_j}} \nabla R(x,I(s))\mathrm{d}s - \left|\int_0^{t_{\varepsilon_j}} \nabla R(x,I(s))\mathrm{d}s\right|^2$$
$$= \partial_t\Phi_\varepsilon(t_{\varepsilon_j}x_{\varepsilon_j}) - |\nabla\Phi_\varepsilon(t_{\varepsilon_j}x_{\varepsilon_j})|^2 - 2\nabla\Phi_\varepsilon(t_{\varepsilon_j}x_{\varepsilon_j})\cdot\int_0^{t_{\varepsilon_j}} \nabla R(x,I(s))\mathrm{d}s - \left|\int_0^{t_{\varepsilon_j}} \nabla R(x,I(s))\mathrm{d}s\right|^2 ,$$
$$= \varepsilon_j\Delta\Phi_\varepsilon + \varepsilon_j\int_0^t \Delta R(x,I(s))\mathrm{d}s\,,$$
$$\leq \varepsilon_j\Delta\psi + \varepsilon_j\int_0^t \Delta R(x,I(s))\mathrm{d}s$$

For $\varepsilon_j \to 0$, and since ψ it smooth, it leads to

$$\partial_t\psi(t_1,x_1) - |\nabla\psi(t_1,x_1)|^2 - 2\nabla\psi(t_1,x_1)\cdot\int_0^{t_1} \nabla R(x,I(s))\mathrm{d}s - \left|\int_0^{t_1} \nabla R(x,I(s))\mathrm{d}s\right|^2 \leq 0\,.$$

The case of a local minimum can be treated similarly. Finally, we have proved that Φ is a viscosity solution of (61). It follows that u is a viscosity solution of (62).

Constraint when I is strictly positively lower bounded on $(0,T)$

We assume that there exists $\underline{I} > 0$ such that for $t \in (0,T)$, $I(t) \geq \underline{I}$. In that case, we show that

$$\max_{x\in\mathbb{R}^d} u(t,x) = 0 \, \forall t \in (0,T)\,.$$

Using the upper bound in (24), there exist positive constants such that

$$u_\varepsilon(t,x) \leq -A|x| + B + Ct\,.$$

Therefore, for M large enough, we obtain that

$$\lim_{\varepsilon\to 0} \int_{|x|>M} n_\varepsilon(t,x)\mathrm{d}x \leq \lim_{\varepsilon\to 0} \int_{|x|>M} e^{\frac{-A|x|+B+Ct}{\varepsilon}}\mathrm{d}x = 0\,,$$

so that we can write

$$\lim_{\varepsilon\to 0} \int_{|x|\leq M} n_\varepsilon(t,x)\mathrm{d}x \geq \frac{\underline{I}}{\psi_M}\,. \tag{63}$$

Now, if for all $|x| < M$, we have that $u(t,x) < 0$, then $\lim_{\varepsilon\to 0} e^{\frac{u_\varepsilon(t,x)}{\varepsilon}} = 0$, so that $\lim_{\varepsilon\to 0}\int_{|x|\leq M} n_\varepsilon(t,x)\mathrm{d}x = 0$, which contradicts (63), and the result follows.

Proof of $\Gamma_t \subseteq \{x \in \mathbb{R}^d,\, R(x,I(t)) = 0\}$

Take t any continuity point of I, and $\overline{x} \in \Gamma_t$. Using the definition of a viscosity solution at this point with the null function as a test function, we obtain that

$$R(\overline{x}, I(t)) \geq 0\,.$$

Next for the other inequality, integrate Equation (8) in time on $(t, t+s)$ with $s > 0$ small enough, at the fixed point $\overline{x}$, and divide by s. We obtain that

$$0 \geq \frac{u(t+s,\overline{x}) - u(t,\overline{x})}{s} \geq \frac{1}{s}\int_0^s R(\overline{x}, I(t+u))\mathrm{d}u\,,$$

and for $s \to 0^+$, since I is continuous in t, we have that

$$0 \geq \liminf_{s\to 0^+} \frac{u(t+s,\overline{x}) - u(t,\overline{x})}{s} \geq R(\overline{x}, I(t))\,,$$

and that concludes the proof of Theorem 1.1.

E Numerical scheme

Let us denote $\Omega_0 = (-X_{\max}, X_{\max})$ the trait space domain, discretized in M modes, so that $\Delta x = \frac{2X_{\max}}{M-1}$. We write $\{x_i\}_{1\leq i\leq M}$ for the nodes coordinates. For Δt the time step and N_t the number of iterations, we denote $\{t_k\}_{1\leq k\leq N_t}$. The discretized solution writes now $\{(n_\varepsilon^{k,i}, \rho_\varepsilon^k)\}_{1\leq k\leq N_t, 1\leq i\leq M}$, with $n_\varepsilon^{k,i} = n_\varepsilon(t_k, x_i)$ and $\rho_\varepsilon^k = \rho_\varepsilon(t_k)$. The discretization (far from the boundary) writes

$$\varepsilon\frac{n_\varepsilon^{k+1,i} - n_\varepsilon^{k,i}}{\Delta t} - \varepsilon^2\frac{n_\varepsilon^{k+1,i+1} - 2n_\varepsilon^{k+1,i} + n_\varepsilon^{k+1,i-1}}{\Delta x^2} - n_\varepsilon^{k+1,i}[a(x_i) - I_\varepsilon^k] = 0\,,$$

leading to

$$\begin{aligned}
\left(1+\varepsilon\frac{\Delta t}{\Delta x^2}-\frac{\Delta t}{\varepsilon}[a(x_1)-I_\varepsilon^k]\right)n_\varepsilon^{k+1,1}-\varepsilon\frac{\Delta t}{\Delta x^2}n_\varepsilon^{k+1,2}&=0\,,\\
-\varepsilon\frac{\Delta t}{\Delta x^2}n_\varepsilon^{k+1,i-1}+\left(1+2\varepsilon\frac{\Delta t}{\Delta x^2}-\frac{\Delta t}{\varepsilon}[a(x_i)-I_\varepsilon^k]\right)n_\varepsilon^{k+1,i}-\varepsilon\frac{\Delta t}{\Delta x^2}n_\varepsilon^{k+1,i+1}&=0\,,\quad 2\le i<M\,,\\
-\varepsilon\frac{\Delta t}{\Delta x^2}n_\varepsilon^{k+1,M-1}+\left(1+\varepsilon\frac{\Delta t}{\Delta x^2}-\frac{\Delta t}{\varepsilon}[a(x_M)-I_\varepsilon^k]\right)n_\varepsilon^{k+1,M}&=0\,.
\end{aligned}\tag{64}$$

The linear problem to solve writes $A_k n_\varepsilon^{k+1}=n_\varepsilon^k$ with A_k the matrix associated to (64).

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