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DIRICHLET THEOREM FOR JACOBI-DUNKL EXPANSIONS

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ABSTRACT. The purpose of this paper is to study the pointwise convergence of the Jacobi-Dunkl series. Indeed, we recall some properties of the Jacobi-Dunkl coefficients. Then, we establish a Dirichlet type theorem for expansions in term of Jacobi-Dunkl polynomials.

1. INTRODUCTION

The Fourier series is named in honour of Jean-Baptiste Joseph Fourier (1768-1830) who introduced it for the purpose of solving the heat equation in a metal plate, publishing his initial results in [6], and pursuing his study in [7]. The question whether the Fourier series of a periodic function converges to a given function is well researched and an extensive literature exists on this subject. We mention here for example [9, 5, 12]. Indeed, mathematicians studied pointwise, absolute, uniform, quadratic convergences... It is well known that many problems for partial differential equations are reduced to a power series expansion of the desired solution in terms of special functions or orthogonal polynomials. In particular, by using the properties of Jacobi polynomials ([15]), the Fourier-Jacobi series has been studied extensively by many authors and several results concerning the approximation of functions by partial sums of these series are proved (see e.g. [1, 11, 13, 14, 16, 17, 18]).

In this paper, we also discuss this subject. More precisely, we are interested in Jacobi-Dunkl expansions.

In [2], the author defined the Jacobi-Dunkl coefficients associated with Jacobi-Dunkl polynomials given by

$$\psi_n^{(\alpha,\beta)}(\theta) := \begin{cases} R_{|n|}^{(\alpha,\beta)}(\cos(2\theta)) + \frac{i\lambda_n^{(\alpha,\beta)}}{4(\alpha+1)} \sin(2\theta) R_{|n|-1}^{(\alpha+1,\beta+1)}(\cos(2\theta)) & \text{if } n \in \mathbb{Z} \setminus \{0\}, \\ 1 & \text{if } n = 0, \end{cases} \quad (1)$$

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where $R_m^{(\alpha,\beta)}(x)$, $m \in \mathbb{N}$, is the normalized Jacobi polynomial of degree m such that $R_m^{(\alpha,\beta)}(1) = 1$, and $\lambda_n^{(\alpha,\beta)}$ is given by

$$\lambda_n^{(\alpha,\beta)} := 2 \operatorname{sgn}(n) \sqrt{|n|(|n| + \rho)}, \quad n \in \mathbb{Z},$$

with

$$\alpha \geq \beta \geq -\frac{1}{2}; \quad \alpha \neq -\frac{1}{2}, \text{ and } \rho := \alpha + \beta + 1 > 0.$$

In the second section, we will give some preliminaries concerning these polynomials. Then, we will see more properties of the Jacobi-Dunkl coefficients in the third section. In section 4, we state a theorem about Jacobi-Dunkl convergence in quadratic mean. Finally, we will focus on pointwise convergence. We establish a Dirichlet type theorem which generalizes the classical one, see [10]. The proof is based on the asymptotic behaviour of Jacobi and Jacobi-Dunkl polynomials studied in [3] and [4].

2. PRELIMINARIES

In this section, we will recall some properties of Jacobi and Jacobi-Dunkl polynomials. We denote by

$$(a)_n := \begin{cases} a(a+1)\dots(a+n-1) & \text{if } n \in \mathbb{N} \setminus \{0\}, \\ 1 & \text{if } n = 0. \end{cases}$$

$(a)_n$ is called the Pochhammer symbol.

${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function, given by

$$\forall a, b \in \mathbb{C}, \forall c \in \mathbb{C} \setminus \mathbb{Z}_-, \forall z \in \mathbb{C}; |z| < 1, \quad {}_2F_1(a, b; c; z) := \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{n! (c)_n} z^n.$$

The Jacobi polynomials $\varphi_m^{(\alpha,\beta)}(\theta)$, $m \in \mathbb{N}$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, are defined by

$$\varphi_m^{(\alpha,\beta)}(\theta) := R_m^{(\alpha,\beta)}(\cos(2\theta)) = {}_2F_1(-m, m + \rho; \alpha + 1; (\sin \theta)^2).$$

The Jacobi operator $\Delta_{\alpha,\beta}$ defined on $C^2\left(\left]0, \frac{\pi}{2}\right[\right)$ is given by

$$\Delta_{\alpha,\beta} := \frac{d^2}{d\theta^2} + \frac{A'_{\alpha,\beta}}{A_{\alpha,\beta}} \frac{d}{d\theta},$$

where

$$A_{\alpha,\beta}(\theta) := \begin{cases} 2^{2\rho} (\sin |\theta|)^{2\alpha+1} (\cos \theta)^{2\beta+1} & \text{if } \theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\setminus \{0\}, \\ 0 & \text{if } \theta = 0. \end{cases}$$

For all $m \in \mathbb{N}$, $\varphi_m^{(\alpha,\beta)}$ is the unique even C^∞ -solution on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ of the differential equation

$$\begin{cases} \Delta_{\alpha,\beta} u & = -\lambda_m^2 u, \\ u(0) & = 1, \\ u'(0) & = 0. \end{cases}$$

The Jacobi-Dunkl operator $\Lambda_{\alpha,\beta}$ is defined by

$$\Lambda_{\alpha,\beta}f(\theta) := \frac{d}{d\theta}f(\theta) + \frac{A'_{\alpha,\beta}(\theta)}{A_{\alpha,\beta}(\theta)} \frac{f(\theta) - f(-\theta)}{2}, \quad f \in C^1 \left(\left] -\frac{\pi}{2}, \frac{\pi}{2} \right[\right),$$

with $\frac{A'_{\alpha,\beta}(\theta)}{A_{\alpha,\beta}(\theta)} = (2\alpha + 1) \cot \theta - (2\beta + 1) \tan \theta$, $\theta \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[\setminus \{0\}$.

According to [2], the differential-difference equation

$$\begin{cases} \Lambda_{\alpha,\beta}u(\theta) &= i\lambda_n^{(\alpha,\beta)}u(\theta); \quad n \in \mathbb{Z}, \\ u(0) &= 1, \end{cases}$$

admits a unique C^∞ -solution on $\left] -\frac{\pi}{2}, \frac{\pi}{2} \right[$ given by (1), which is related to the Jacobi polynomial and to its derivative by

$$\psi_n^{(\alpha,\beta)}(\theta) := \begin{cases} \varphi_{|n|}^{(\alpha,\beta)}(\theta) - \frac{i}{\lambda_n^{(\alpha,\beta)}} \frac{d}{d\theta} \varphi_{|n|}^{(\alpha,\beta)}(\theta) & \text{if } n \in \mathbb{Z} \setminus \{0\}, \\ 1 & \text{if } n = 0, \end{cases} \quad (2)$$

and satisfies

$$\forall n \in \mathbb{Z}, \forall \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right], \quad |\psi_n^{(\alpha,\beta)}(\theta)| \leq 1.$$

For all $n, p \in \mathbb{Z}$, we have the following orthogonality property

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi_n^{(\alpha,\beta)}(\theta) \overline{\psi_p^{(\alpha,\beta)}(\theta)} A_{\alpha,\beta}(\theta) d\theta = (h_n^{(\alpha,\beta)})^{-1} \delta_{n,p}, \quad (3)$$

where $h_n^{(\alpha,\beta)} = \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\psi_n^{(\alpha,\beta)}(\theta)|^2 A_{\alpha,\beta}(\theta) d\theta \right)^{-1}$: $h_0^{(\alpha,\beta)} = \frac{\Gamma(\rho + 1)}{2^{2\rho} \Gamma(\alpha + 1) \Gamma(\beta + 1)}$

and

$$\forall n \in \mathbb{Z} \setminus \{0\}, \quad h_n^{(\alpha,\beta)} = \frac{(2|n| + \rho) \Gamma(\alpha + |n| + 1) \Gamma(\rho + |n|)}{2^{2\rho+1} (\Gamma(\alpha + 1))^2 \Gamma(|n| + 1) \Gamma(\beta + |n| + 1)}.$$

Let $p \in [1, +\infty]$. We denote by

• $L_{\alpha,\beta}^p = L^p \left(\left[-\frac{\pi}{2}, \frac{\pi}{2} \right], A_{\alpha,\beta}(\theta) d\theta \right)$: the space of measurable functions f on $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$ such that

$$\begin{cases} \|f\|_{p,\alpha,\beta} = \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f(\theta)|^p A_{\alpha,\beta}(\theta) d\theta \right)^{\frac{1}{p}} < +\infty & \text{if } 1 \leq p < +\infty, \\ \|f\|_{\infty,\alpha,\beta} = \operatorname{ess\,sup}_{\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]} |f(\theta)| < +\infty & \text{if } p = +\infty. \end{cases}$$

• $\tilde{L}_{\alpha,\beta}^p = L^p \left(\left[0, \frac{\pi}{2} \right], A_{\alpha,\beta}(\theta) d\theta \right)$ the space of measurable functions g on $\left[0, \frac{\pi}{2} \right]$ such that

$$\begin{cases} \left(\int_0^{\frac{\pi}{2}} |g(\theta)|^p A_{\alpha,\beta}(\theta) d\theta \right)^{\frac{1}{p}} < +\infty & \text{if } 1 \leq p < +\infty, \\ \operatorname{ess\,sup}_{\theta \in \left[0, \frac{\pi}{2} \right]} |g(\theta)| < +\infty & \text{if } p = +\infty. \end{cases}$$

The Jacobi coefficients (see [8]) of a function $g \in \widetilde{L}_{\alpha,\beta}^1$ are defined by

$$\forall m \in \mathbb{N}, \quad \mathcal{F}_{\alpha,\beta}(g)(m) = \int_0^{\frac{\pi}{2}} g(\theta) \varphi_m^{(\alpha,\beta)}(\theta) A_{\alpha,\beta}(\theta) d\theta.$$

The Jacobi-Dunkl coefficients (see [2]) of a function $f \in L_{\alpha,\beta}^1$ are defined by

$$\forall n \in \mathbb{Z}, \quad \mathcal{F}f(n) := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta) \overline{\psi_n^{(\alpha,\beta)}(\theta)} A_{\alpha,\beta}(\theta) d\theta,$$

and satisfy

$$\forall n \in \mathbb{Z}, \quad |\mathcal{F}f(n)| \leq \|f\|_{1,\alpha,\beta}.$$

Now, we consider the analog of the Fourier series given by

$$\sum_{n=-\infty}^{+\infty} \mathcal{F}f(n) \psi_n^{(\alpha,\beta)}(\theta) h_n^{(\alpha,\beta)}, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

For $n \in \mathbb{N}$, we denote its partial sum by

$$S_n^f(\theta) := \sum_{k=-n}^n \mathcal{F}f(k) \psi_k^{(\alpha,\beta)}(\theta) h_k^{(\alpha,\beta)}, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

3. JACOBI-DUNKL COEFFICIENTS

Let $f \in L_{\alpha,\beta}^1$. We put for all $k \in \mathbb{N}$,

$$a_k(f) := \mathcal{F}f(k) + \mathcal{F}f(-k),$$

and

$$b_k(f) := \begin{cases} -\frac{i}{\lambda_k^{(\alpha,\beta)}} [\mathcal{F}f(k) - \mathcal{F}f(-k)] & \text{if } k \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{if } k = 0. \end{cases}$$

Hence, by (2) we can write $S_n^f(\theta)$, for $n \in \mathbb{N} \setminus \{0\}$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, as

$$S_n^f(\theta) = \frac{a_0(f)}{2} h_0^{(\alpha,\beta)} + \sum_{k=1}^n \left(a_k(f) \varphi_k^{(\alpha,\beta)}(\theta) + b_k(f) \frac{d}{d\theta} \varphi_k^{(\alpha,\beta)}(\theta) \right) h_k^{(\alpha,\beta)}.$$

Remark 3.1.

For all $k \in \mathbb{N}$, we have these relations:

$$(1) \quad \mathcal{F}f(k) = \frac{a_k(f) + i\lambda_k^{(\alpha,\beta)} b_k(f)}{2}.$$

$$(2) \quad \mathcal{F}f(-k) = \frac{a_k(f) - i\lambda_k^{(\alpha,\beta)} b_k(f)}{2}.$$

Proposition 3.2.

For all $k \in \mathbb{N}$, we have the following integral representations:

$$(1) \quad a_k(f) = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta) \varphi_k^{(\alpha,\beta)}(\theta) A_{\alpha,\beta}(\theta) d\theta.$$

$$(2) \quad b_k(f) = \frac{2}{\left(\lambda_k^{(\alpha,\beta)}\right)^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta) \frac{d}{d\theta} \varphi_k^{(\alpha,\beta)}(\theta) A_{\alpha,\beta}(\theta) d\theta, \quad k \neq 0.$$

Proof.

$$(1) \quad a_k(f) = \mathcal{F}f(k) + \mathcal{F}f(-k) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta) \left[\overline{\psi_k^{(\alpha,\beta)}(\theta)} + \overline{\psi_{-k}^{(\alpha,\beta)}(\theta)} \right] A_{\alpha,\beta}(\theta) d\theta.$$

Since we know that

$$\overline{\psi_k^{(\alpha,\beta)}(\theta)} + \overline{\psi_{-k}^{(\alpha,\beta)}(\theta)} = 2\Re\left(\psi_k^{(\alpha,\beta)}(\theta)\right) = 2\varphi_k^{(\alpha,\beta)}(\theta),$$

then, we obtain the result.

$$(2) \quad b_k(f) = \frac{i}{\lambda_k^{(\alpha,\beta)}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta) \left[\overline{\psi_{-k}^{(\alpha,\beta)}(\theta)} - \overline{\psi_k^{(\alpha,\beta)}(\theta)} \right] A_{\alpha,\beta}(\theta) d\theta, \quad k \neq 0.$$

As we have

$$\overline{\psi_{-k}^{(\alpha,\beta)}(\theta)} - \overline{\psi_k^{(\alpha,\beta)}(\theta)} = 2i\Im\left(\psi_k^{(\alpha,\beta)}(\theta)\right) = -\frac{2i}{\lambda_k^{(\alpha,\beta)}} \frac{d}{d\theta} \varphi_{|k|}^{(\alpha,\beta)}(\theta),$$

then, we get the equality. □

Remarks 3.3.

Let $k \in \mathbb{N}$.

(1) If the function f is even, then

$$b_k(f) = 0 \quad \text{and} \quad a_k(f) = 4 \int_0^{\frac{\pi}{2}} f(\theta) \varphi_k^{(\alpha,\beta)}(\theta) A_{\alpha,\beta}(\theta) d\theta.$$

(2) If the function f is odd, then

$$a_k(f) = 0 \quad \text{and} \quad b_k(f) = \frac{4}{\left(\lambda_k^{(\alpha,\beta)}\right)^2} \int_0^{\frac{\pi}{2}} f(\theta) \frac{d}{d\theta} \varphi_k^{(\alpha,\beta)}(\theta) A_{\alpha,\beta}(\theta) d\theta, \quad k \neq 0.$$

Proposition 3.4.

Let f be in $L_{\alpha,\beta}^1$, a real-valued function. For all $k \in \mathbb{N}$, we have these properties:

$$(1) \quad \mathcal{F}f(-k) = \overline{\mathcal{F}f(k)}.$$

$$(2) \quad a_k(f) = 2\Re(\mathcal{F}f(k)) \in \mathbb{R}.$$

$$(3) \quad b_k(f) = \frac{2}{\lambda_k^{(\alpha,\beta)}} \Im(\mathcal{F}f(k)) \in \mathbb{R}, \quad k \neq 0.$$

Proof.

$$(1) \quad \mathcal{F}f(-k) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta) \overline{\psi_{-k}^{(\alpha,\beta)}(\theta)} A_{\alpha,\beta}(\theta) d\theta = \overline{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta) \overline{\psi_k^{(\alpha,\beta)}(\theta)} A_{\alpha,\beta}(\theta) d\theta} = \overline{\mathcal{F}f(k)}.$$

$$(2) \ a_k(f) = \mathcal{F}f(k) + \mathcal{F}f(-k) = \mathcal{F}f(k) + \overline{\mathcal{F}f(k)} = 2\Re(\mathcal{F}f(k)).$$

(3) For all $k \in \mathbb{Z} \setminus \{0\}$, we have

$$b_k(f) = \frac{i}{\lambda_k^{(\alpha,\beta)}} [\mathcal{F}f(-k) - \mathcal{F}f(k)] = \frac{i}{\lambda_k^{(\alpha,\beta)}} [\overline{\mathcal{F}f(k)} - \mathcal{F}f(k)] = \frac{2}{\lambda_k^{(\alpha,\beta)}} \Im(\mathcal{F}f(k)).$$

□

In the following parts, we will study for a suitable given function f , the convergence of the series

$$\sum_{n=-\infty}^{+\infty} \mathcal{F}f(n) \psi_n^{(\alpha,\beta)}(\theta) h_n^{(\alpha,\beta)}.$$

4. CONVERGENCE IN QUADRATIC MEAN

Theorem 4.1. For all $f \in L_{\alpha,\beta}^2$, we have

$$\lim_{n \rightarrow +\infty} \|S_n^f - f\|_{2,\alpha,\beta} = 0.$$

Proof. Let $f \in L_{\alpha,\beta}^2$ and $n \in \mathbb{N}$.

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |S_n^f(\theta) - f(\theta)|^2 A_{\alpha,\beta}(\theta) d\theta &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (S_n^f(\theta) - f(\theta)) \overline{(S_n^f(\theta) - f(\theta))} A_{\alpha,\beta}(\theta) d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |S_n^f(\theta)|^2 A_{\alpha,\beta}(\theta) d\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} S_n^f(\theta) \overline{f(\theta)} A_{\alpha,\beta}(\theta) d\theta \\ &\quad - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta) \overline{S_n^f(\theta)} A_{\alpha,\beta}(\theta) d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f(\theta)|^2 A_{\alpha,\beta}(\theta) d\theta \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

We have by the orthogonality property (3),

$$\begin{aligned} I_1 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\sum_{k=-n}^n \mathcal{F}f(k) \psi_k^{(\alpha,\beta)}(\theta) h_k^{(\alpha,\beta)}(\theta) \right) \overline{\left(\sum_{p=-n}^n \mathcal{F}f(p) \psi_p^{(\alpha,\beta)}(\theta) h_p^{(\alpha,\beta)}(\theta) \right)} A_{\alpha,\beta}(\theta) d\theta \\ &= \sum_{k=-n}^n \sum_{p=-n}^n \left(\mathcal{F}f(k) \overline{\mathcal{F}f(p)} h_p^{(\alpha,\beta)} h_k^{(\alpha,\beta)} \right) \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi_k^{(\alpha,\beta)}(\theta) \overline{\psi_p^{(\alpha,\beta)}(\theta)} A_{\alpha,\beta}(\theta) d\theta \right) \\ &= \sum_{k=-n}^n \sum_{p=-n}^n \mathcal{F}f(k) \overline{\mathcal{F}f(p)} h_p^{(\alpha,\beta)} h_k^{(\alpha,\beta)} \left(h_k^{(\alpha,\beta)} \right)^{-1} \delta_{k,p} \\ &= \sum_{k=-n}^n \mathcal{F}f(k) \overline{\mathcal{F}f(k)} h_k^{(\alpha,\beta)} \\ &= \sum_{k=-n}^n |\mathcal{F}f(k)|^2 h_k^{(\alpha,\beta)}. \end{aligned}$$

Furthermore

$$\begin{aligned}
I_2 &= - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\sum_{k=-n}^n \mathcal{F}f(k) \psi_k^{(\alpha,\beta)}(\theta) h_k^{(\alpha,\beta)} \right) \overline{f(\theta) A_{\alpha,\beta}(\theta)} d\theta \\
&= - \sum_{k=-n}^n \mathcal{F}f(k) h_k^{(\alpha,\beta)} \overline{\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta) \overline{\psi_k^{(\alpha,\beta)}(\theta) A_{\alpha,\beta}(\theta)} d\theta \right)} \\
&= - \sum_{k=-n}^n \mathcal{F}f(k) \overline{\mathcal{F}f(k)} h_k^{(\alpha,\beta)} \\
&= - \sum_{k=-n}^n |\mathcal{F}f(k)|^2 h_k^{(\alpha,\beta)} \\
&= -I_1.
\end{aligned}$$

We also have

$$I_3 = \overline{I_2} = I_2 = -I_1.$$

Then

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |S_n^f(\theta) - f(\theta)|^2 A_{\alpha,\beta}(\theta) d\theta = \|f\|_{2,\alpha,\beta}^2 - \sum_{k=-n}^n |\mathcal{F}f(k)|^2 h_k^{(\alpha,\beta)}.$$

By the Plancherel formula [2, Theorem 3.4], we obtain

$$\lim_{n \rightarrow +\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |S_n^f(\theta) - f(\theta)|^2 A_{\alpha,\beta}(\theta) d\theta = 0.$$

□

5. DIRICHLET TYPE CONVERGENCE

Notation 5.1.

For all $n \in \mathbb{N}$, $\theta, \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We denote by

$$D_n^{(\alpha,\beta)}(\theta, \phi) := \sum_{k=-n}^n \psi_k^{(\alpha,\beta)}(\theta) \overline{\psi_k^{(\alpha,\beta)}(\phi)} h_k^{(\alpha,\beta)}.$$

$D_n^{(\alpha,\beta)}(\theta, \phi)$ is the analog of the Dirichlet kernel associated with the Fourier series.

Proposition 5.2.

Let $f \in L_{\alpha,\beta}^1$, $n \in \mathbb{N}$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We have

$$S_n^f(\theta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\phi) D_n^{(\alpha,\beta)}(\theta, \phi) A_{\alpha,\beta}(\phi) d\phi.$$

Proof.

$$\begin{aligned}
S_n^f(\theta) &= \sum_{k=-n}^n \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\phi) \overline{\psi_k^{(\alpha,\beta)}(\phi)} A_{\alpha,\beta}(\phi) d\phi \right) \psi_k^{(\alpha,\beta)}(\theta) h_k^{(\alpha,\beta)} \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\phi) \left(\sum_{k=-n}^n \overline{\psi_k^{(\alpha,\beta)}(\phi)} \psi_k^{(\alpha,\beta)}(\theta) h_k^{(\alpha,\beta)} \right) A_{\alpha,\beta}(\phi) d\phi \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\phi) D_n^{(\alpha,\beta)}(\theta, \phi) A_{\alpha,\beta}(\phi) d\phi.
\end{aligned}$$

□

Proposition 5.3.

Let $n \in \mathbb{N}$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D_n^{(\alpha,\beta)}(\theta, \phi) A_{\alpha,\beta}(\phi) d\phi = 1.$$

Proof.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D_n^{(\alpha,\beta)}(\theta, \phi) A_{\alpha,\beta}(\phi) d\phi = \sum_{k=-n}^n \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overline{\psi_k^{(\alpha,\beta)}(\phi)} A_{\alpha,\beta}(\phi) d\phi \right) \psi_k^{(\alpha,\beta)}(\theta) h_k^{(\alpha,\beta)}.$$

As we know, by the orthogonality property (3), that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overline{\psi_k^{(\alpha,\beta)}(\phi)} A_{\alpha,\beta}(\phi) d\phi = \left(h_0^{(\alpha,\beta)}\right)^{-1} \delta_{0,k},$$

then, we get

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D_n^{(\alpha,\beta)}(\theta, \phi) A_{\alpha,\beta}(\phi) d\phi = \psi_0^{(\alpha,\beta)}(\theta) = 1.$$

□

Proposition 5.4.

(1) $\forall n \in \mathbb{N}, \forall \theta, \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]; \theta \neq \pm\phi$, we have

$$\begin{aligned}
D_n^{(\alpha,\beta)}(\theta, \phi) &= \frac{\Gamma(\alpha + n + 2)\Gamma(\rho + n + 1)}{2^{2\rho-1}(\Gamma(\alpha + 1))^2(2n + \rho + 1)n!\Gamma(\beta + n + 1)} \times \frac{1}{\cos(2\theta) - \cos(2\phi)} \\
&\times \left[\varphi_{n+1}^{(\alpha,\beta)}(\theta)\varphi_n^{(\alpha,\beta)}(\phi) - \varphi_n^{(\alpha,\beta)}(\theta)\varphi_{n+1}^{(\alpha,\beta)}(\phi) + \frac{\lambda_n^{(\alpha,\beta)}\lambda_{n+1}^{(\alpha,\beta)}}{4(n+1)(n+\rho)} \right. \\
&\times \left. \left(\Im\psi_{n+1}^{(\alpha,\beta)}(\theta)\Im\psi_n^{(\alpha,\beta)}(\phi) - \Im\psi_n^{(\alpha,\beta)}(\theta)\Im\psi_{n+1}^{(\alpha,\beta)}(\phi) \right) \right],
\end{aligned}$$

$$\text{with } \Im\psi_n^{(\alpha,\beta)}(\theta) = \frac{\psi_n(\theta) - \psi_n(-\theta)}{2i}.$$

(2) $\forall n \in \mathbb{N}, \forall \theta, \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have

- (a) $D_n^{(\alpha,\beta)}(\theta, \phi) \in \mathbb{R}$.
- (b) $D_n^{(\alpha,\beta)}(\theta, \theta) > 0$.

$$(c) D_n^{(\alpha, \beta)}(\phi, \theta) = D_n^{(\alpha, \beta)}(\theta, \phi).$$

Proof.

- (1) The case $n = 0$ is obvious, and we have the result in [4, theorem 3.1], for $n \in \mathbb{N} \setminus \{0\}$.
 (2) (a) We deduce the result from (1), for $\theta \neq \pm\phi$. We also have

$$D_n^{(\alpha, \beta)}(\theta, \theta) = \sum_{k=-n}^n \left| \psi_k^{(\alpha, \beta)}(\theta) \right|^2 h_k^{(\alpha, \beta)} \in \mathbb{R},$$

and

$$\begin{aligned} D_n^{(\alpha, \beta)}(\theta, -\theta) &= \sum_{k=-n}^n \left(\psi_k^{(\alpha, \beta)}(\theta) \right)^2 h_k^{(\alpha, \beta)} \\ &= h_0^{(\alpha, \beta)} + \sum_{k=1}^n \left(\left(\psi_k^{(\alpha, \beta)}(\theta) \right)^2 + \left(\overline{\psi_k^{(\alpha, \beta)}(\theta)} \right)^2 \right) h_k^{(\alpha, \beta)} \\ &= h_0^{(\alpha, \beta)} + 2\Re \left(\sum_{k=1}^n \left(\psi_k^{(\alpha, \beta)}(\theta) \right)^2 \right) \in \mathbb{R}. \end{aligned}$$

$$(b) D_n^{(\alpha, \beta)}(\theta, \theta) = h_0^{(\alpha, \beta)} + \sum_{k=-n, k \neq 0}^n \left| \psi_k^{(\alpha, \beta)}(\theta) \right|^2 h_k^{(\alpha, \beta)} > 0.$$

$$(c) D_n^{(\alpha, \beta)}(\phi, \theta) = \overline{D_n^{(\alpha, \beta)}(\theta, \phi)} = D_n^{(\alpha, \beta)}(\theta, \phi).$$

□

Theorem 5.5.

Let f be a piecewise continuous function on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$ such that

- i) $f(-\theta) = f(\theta)$,
- ii) f is differentiable on θ and $-\theta$.

Then we have

$$\lim_{n \rightarrow +\infty} S_n^f(\theta) = f(\theta).$$

Proof.

Let $n \in \mathbb{N}$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$. By Proposition 5.3, we can write

$$f(\theta) - S_n^f(\theta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [f(\theta) - f(\phi)] D_n^{(\alpha, \beta)}(\theta, \phi) A_{\alpha, \beta}(\phi) d\phi.$$

From [4, Theorem 3.1], we have for all $\theta \neq \pm\phi$

$$\begin{aligned} f(\theta) - S_n^f(\theta) &= l_n^{(\alpha,\beta)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{f(\theta) - f(\phi)}{\cos(2\theta) - \cos(2\phi)} \left[\varphi_{n+1}^{(\alpha,\beta)}(\theta) \varphi_n^{(\alpha,\beta)}(\phi) - \varphi_n^{(\alpha,\beta)}(\theta) \varphi_{n+1}^{(\alpha,\beta)}(\phi) \right. \\ &\quad + \frac{\lambda_n^{(\alpha,\beta)} \lambda_{n+1}^{(\alpha,\beta)}}{4(n+1)(n+\rho)} \\ &\quad \left. \times (\mathfrak{S}\psi_{n+1}^{(\alpha,\beta)}(\theta) \mathfrak{S}\psi_n^{(\alpha,\beta)}(\phi) - \mathfrak{S}\psi_n^{(\alpha,\beta)}(\theta) \mathfrak{S}\psi_{n+1}^{(\alpha,\beta)}(\phi)) \right] A_{\alpha,\beta}(\phi) d\phi, \end{aligned}$$

where

$$l_n^{(\alpha,\beta)} := \frac{\Gamma(\alpha + n + 2) \Gamma(\rho + n + 1)}{2^{2\rho-1} (\Gamma(\alpha + 1))^2 (2n + \rho + 1) n! \Gamma(\beta + n + 1)}.$$

For all $\phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{\pm\theta\}$, we put

$$g_\theta(\phi) := \frac{f(\theta) - f(\phi)}{\cos(2\theta) - \cos(2\phi)}.$$

Since we have supposed that f is a piecewise continuous function on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then g_θ is also piecewise continuous on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{\pm\theta\}$.

Furthermore, we have

$$\lim_{\phi \rightarrow \theta} g_\theta(\phi) = -\frac{1}{2} \frac{1}{\sin(2\theta)} f'(\theta).$$

And from hypothese i) of our theorem, we deduce that

$$\lim_{\phi \rightarrow -\theta} g_\theta(\phi) = \frac{1}{2} \frac{1}{\sin(2\theta)} f'(-\theta).$$

Under the assumption ii) of the theorem, these limits exist and are finite.

We still call g_θ the extension of g_θ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus, $g_\theta \in L_{\alpha,\beta}^2$.

In the following, we denote by

$$\begin{aligned} \check{g}_\theta(\phi) &:= g_\theta(-\phi), \quad \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \\ g_\theta^1 &:= (g_\theta)|_{[0, \frac{\pi}{2}]}, \\ g_\theta^2 &:= (g_\theta)|_{[-\frac{\pi}{2}, 0]}, \\ \widehat{g}_\theta^2(\phi) &:= g_\theta^2(-\phi), \quad \phi \in \left[0, \frac{\pi}{2}\right]. \end{aligned}$$

Now, we write

$$f(\theta) - S_n^f(\theta) = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}
I_1 &:= l_n^{(\alpha,\beta)} \varphi_{n+1}^{(\alpha,\beta)}(\theta) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_\theta(\phi) \varphi_n^{(\alpha,\beta)}(\phi) A_{\alpha,\beta}(\phi) d\phi, \\
I_2 &:= -l_n^{(\alpha,\beta)} \varphi_n^{(\alpha,\beta)}(\theta) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_\theta(\phi) \varphi_{n+1}^{(\alpha,\beta)}(\phi) A_{\alpha,\beta}(\phi) d\phi, \\
I_3 &:= l_n^{(\alpha,\beta)} \frac{\lambda_n^{(\alpha,\beta)} \lambda_{n+1}^{(\alpha,\beta)}}{4(n+1)(n+\rho)} \mathfrak{S}\psi_{n+1}^{(\alpha,\beta)}(\theta) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_\theta(\phi) \mathfrak{S}\psi_n^{(\alpha,\beta)}(\phi) A_{\alpha,\beta}(\phi) d\phi, \\
I_4 &:= -l_n^{(\alpha,\beta)} \frac{\lambda_n^{(\alpha,\beta)} \lambda_{n+1}^{(\alpha,\beta)}}{4(n+1)(n+\rho)} \mathfrak{S}\psi_n^{(\alpha,\beta)}(\theta) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_\theta(\phi) \mathfrak{S}\psi_{n+1}^{(\alpha,\beta)}(\phi) A_{\alpha,\beta}(\phi) d\phi.
\end{aligned}$$

Combining the fact that

$$l_n^{(\alpha,\beta)} \underset{+\infty}{\sim} \frac{1}{2^{2\rho} (\Gamma(\alpha+1))^2} n^{2\alpha+1},$$

and the result (35) of [3], we get

$$l_n^{(\alpha,\beta)} \varphi_{n+1}^{(\alpha,\beta)}(\theta) \underset{+\infty}{\sim} n^{\alpha+\frac{1}{2}} \frac{\cos[(2n+2+\rho)|\theta| - (2\alpha+1)\frac{\pi}{4}]}{\sqrt{\pi}\Gamma(\alpha+1)A_{\frac{2\alpha-1}{4}, \frac{2\beta-1}{4}}(\theta)}.$$

Moreover, we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_\theta(\phi) \varphi_n^{(\alpha,\beta)}(\phi) A_{\alpha,\beta}(\phi) d\phi = \mathcal{F}_{\alpha,\beta} \left(g_\theta^1 + \overset{\vee}{g_\theta^2} \right) (n).$$

From the Parseval formula for the Jacobi coefficients (see [2]), we obtain

$$\mathcal{F}_{\alpha,\beta} \left(g_\theta^1 + \overset{\vee}{g_\theta^2} \right) (n) = o\left(n^{-(\alpha+\frac{1}{2})}\right).$$

Thus, $\lim_{n \rightarrow +\infty} I_1 = 0$.

We use the same proof as for I_1 to show that

$$\begin{aligned}
\lim_{n \rightarrow +\infty} I_2 &= \lim_{n \rightarrow +\infty} - \left[n^{\alpha+\frac{1}{2}} \frac{\cos[(2n+\rho)|\theta| - (2\alpha+1)\frac{\pi}{4}]}{\sqrt{\pi}\Gamma(\alpha+1)A_{\frac{2\alpha-1}{4}, \frac{2\beta-1}{4}}(\theta)} \mathcal{F}_{\alpha,\beta} \left(g_\theta^1 + \overset{\vee}{g_\theta^2} \right) (n+1) \right] \\
&= 0.
\end{aligned}$$

Otherwise, we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_\theta(\phi) \mathfrak{S}\psi_n^{(\alpha,\beta)}(\phi) A_{\alpha,\beta}(\phi) d\phi = \frac{1}{2i} \mathcal{F} \left(\overset{\vee}{g_\theta} - g_\theta \right) (n).$$

By [2, corollary 3.5], we have

$$\mathcal{F} \left(\overset{\vee}{g_\theta} - g_\theta \right) (n) = o\left(n^{-(\alpha+\frac{1}{2})}\right).$$

Furthermore, we get, from [3, Theorem 4.7], that

$$\mathfrak{S}\psi_{n+1}^{(\alpha,\beta)}(|\theta|) \underset{+\infty}{\sim} \frac{2^{2\rho}\Gamma(\alpha+1)}{\sqrt{\pi}} \frac{n^{-(\alpha+\frac{1}{2})}}{A_{\frac{2\alpha-1}{4}, \frac{2\beta-1}{4}}(\theta)} \sin \left[(2n+2+\rho)|\theta| - (2\alpha+1)\frac{\pi}{4} \right].$$

Since we have

$$\lim_{n \rightarrow +\infty} \frac{\lambda_n^{(\alpha,\beta)} \lambda_{n+1}^{(\alpha,\beta)}}{4(n+1)(n+\rho)} = 1,$$

then, we get

$$\lim_{n \rightarrow +\infty} I_3 = 0.$$

We use the same reasons as for I_3 to show that

$$\begin{aligned} \lim_{n \rightarrow +\infty} I_4 &= \lim_{n \rightarrow +\infty} -l_n^{(\alpha,\beta)} \frac{\lambda_n^{(\alpha,\beta)} \lambda_{n+1}^{(\alpha,\beta)}}{4(n+1)(n+\rho)} \mathfrak{S}\psi_n^{(\alpha,\beta)}(\theta) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_\theta(\phi) \mathfrak{S}\psi_{n+1}^{(\alpha,\beta)}(\phi) A_{\alpha,\beta}(\phi) d\phi \\ &= 0. \end{aligned}$$

Hence, we obtain

$$\lim_{n \rightarrow +\infty} [f(\theta) - S_n^f(\theta)] = \lim_{n \rightarrow +\infty} I_1 + I_2 + I_3 + I_4 = 0.$$

Which achieves the proof. \square

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