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A note on the real inverse spectral problem for doubly stochastic matrices

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Abstract

The real (resp. symmetric) doubly stochastic inverse spectral problem is the problem of determining necessary and sufficient conditions for a real n-tuple $\lambda = (1, \lambda_2, ..., \lambda_n)$ to be the spectrum of an $n \times n$ (resp. symmetric) doubly stochastic matrix. If $\lambda_i \leq 0$ for all $i = 2, ..., n$ and the sum of all the entries in λ is nonnegative, then we refer to such λ as a normalized Suleimanova spectrum.

The purpose of this paper is to first fix an error in Theorem 9 of the recent paper [I. Adeli, M. Taheria, M.M Moghadama, A recursive method for constructing doubly stochastic matrices and inverse eigenvalue problem, Linear Algebra Appl. 537 (2018) 318-331], after giving a counterexample. Secondly, we give a negative answer to a question posed in [C.R. Johnson, P. Paparella, Perron spectratopes and the real nonnegative inverse eigenvalue problem, Linear Algebra Appl. 493 (2016) 281-300] concerning the realizability of normalized Suleimanova spectra for the case when n is odd. Some sufficient conditions for a positive answer to this question are given.

Keywords: Symmetric doubly stochastic matrices, Inverse eigenvalue problem, Normalized Suleimanova spectrum
2000 MSC: 15A12, 15A18, 15A51

1. Introduction

An $n \times n$ matrix A with real entries is said to be nonnegative if all of its entries are nonnegative. A nonnegative matrix A is said to be doubly stochastic if each of its row and column sum is equal to 1. Let I_n and J_n be the $n \times n$ identity matrix and the $n \times n$ matrix whose all entries are $\frac{1}{n}$ respectively. In addition, if $e_n = \frac{1}{\sqrt{n}}(1, ..., 1)^T$ then clearly an $n \times n$ nonnegative matrix A is doubly stochastic if and only if $Ae_n = e_n$ and $A^T e_n = e_n$ or equivalently $AJ_n = J_n A = J_n$.

The *real (resp. symmetric) doubly stochastic inverse eigenvalue problem* RDIEP (resp. SDIEP) is the problem of determining necessary and sufficient conditions for a real n-tuple $\sigma = \{1, \lambda_2, ..., \lambda_n\}$ to be the spectrum of an $n \times n$ (resp. symmetric) doubly stochastic matrix. If there exists a doubly stochastic matrix A whose spectrum is σ, then we shall say that σ is *realizable* and A *realizes* σ.

So far, the SDIEP has only been solved for the case $n = 3$ by Perfect and Mirsky [12] and remains open for the cases $n \geq 4$ (see [4, 6, 7, 8, 10, 12, 13] for a collection of most sufficient conditions for the SDIEP). Until now, there are two principal methods to solve the SDIEP. The first one relies on taking a diagonal matrix $\Lambda = \text{diag}(1, \lambda_2, ..., \lambda_n)$ and a nonsingular matrix P, and then exploring the conditions under which $A = P^{-1}\Lambda P$ is doubly stochastic. The second method falls into the category of constructing new doubly stochastic matrices from smaller size matrices with known spectra. Our paper here falls under the second category.

We call $\sigma = \{1, \lambda_2, ..., \lambda_n\} \subset \mathbb{R}$ a *normalized Suleimanova spectrum* if $1 + \lambda_2 + ... + \lambda_n \geq 0$ and $0 \geq \lambda_i \geq -1$ for all $i = 2, ..., n$. In [3], the following question was posed.

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Question 1. *If σ is a normalized Suleimanova spectrum, is σ realizable by a doubly stochastic matrix?*

The authors in [3] proved that the answer is yes for all Hadamard orders (i.e. $n = 2^k$) and in this case the realizing matrix is symmetric. We will denote by NS-SDIEP to be the problem which asks what sets of normalized Suleimanova spectrum occur as the spectrum of a symmetric doubly stochastic matrix.

This paper is organised as follows. In Section 2, we collect some preliminary results that will be used later. In section 3, we give a counterexample to Theorem 9 in [1] in the case where n is odd and we present an alternative statement though in this case it will not be an improvement of the results in [13] (as was claimed in [1]) but rather a new independent sufficient condition for the SDIEP. In Section 4, we give a negative answer to Question 1 in the case where n is odd and we present some sufficient conditions for the realizability of normalized Suleimanova spectrum.

2. Auxiliary materials

We present in this section auxiliary results concerning spectral properties of certain block doubly stochastic matrices. We start with the following theorem from [8, 9].

Theorem 1. *Let A be an $n \times n$ doubly stochastic matrix whose eigenvalues are given by $1, \lambda_2, ..., \lambda_n$ and let B be an $m \times m$ doubly stochastic matrix with eigenvalues $1, \mu_2, ..., \mu_m$. Then for any $\rho \geq 0$ and for any $r \geq 0$ such that ρ and r do not vanish simultaneously, the $(m+n) \times (m+n)$ matrix C defined by*

- *For $m \geq n$,*

$$C = \frac{1}{r + \frac{\rho m}{\sqrt{mn}}} \begin{pmatrix} rA & \rho e_n e_m^T \\ \rho e_m e_n^T & (r + \rho\frac{m-n}{\sqrt{mn}})B \end{pmatrix}$$

is doubly stochastic with eigenvalues

$$1, \frac{r\sqrt{mn} - \rho n}{r\sqrt{mn} + \rho m}, \frac{r}{r + \frac{\rho m}{\sqrt{mn}}}\lambda_2, ..., \frac{r}{r + \frac{\rho m}{\sqrt{mn}}}\lambda_n, \frac{r\sqrt{mn} + \rho(m-n)}{r\sqrt{mn} + \rho m}\mu_2, ..., \frac{r\sqrt{mn} + \rho(m-n)}{r\sqrt{mn} + \rho m}\mu_m.$$

- *For $n \geq m$,*

$$C = \frac{1}{r + \frac{\rho n}{\sqrt{mn}}} \begin{pmatrix} (r + \rho\frac{n-m}{\sqrt{mn}})A & \rho e_n e_m^T \\ \rho e_m e_n^T & rB \end{pmatrix}$$

is doubly stochastic with eigenvalues

$$1, \frac{r\sqrt{mn} - \rho m}{r\sqrt{mn} + \rho n}, \frac{r\sqrt{mn} + \rho(n-m)}{r\sqrt{mn} + \rho n}\lambda_2, ..., \frac{r\sqrt{mn} + \rho(n-m)}{r\sqrt{mn} + \rho n}\lambda_n, \frac{r}{r + \frac{\rho n}{\sqrt{mn}}}\mu_2, ..., \frac{r}{r + \frac{\rho n}{\sqrt{mn}}}\mu_m.$$

Remark 1. *It is worthy to mention here that the preceding theorem is a specialisation of a result in [9] that deals with k diagonal blocks instead of only two. Its proof relies on a result which is presented in Perfect [11] and is due to R. Rado. An alternative[1] proof can be achieved from Fiedler [2]. Indeed, the proof of Lemma 2.2 there, is essentially the same as the proof of Theorem 1. One simply replaces the symmetry assumption in Fiedler's paper by the property that the algebraic and geometric multiplicities of the eigenvalue 1 of any doubly stochastic matrix are the same.*

As a conclusion, we have the following two lemmas that are Theorem 1 and Theorem 2 in [1] and which constitute the basis for their results.

Lemma 1. *Let A be an $n \times n$ doubly stochastic matrix with eigenvalues $1, \lambda_2, ..., \lambda_n$. Then, for any $0 \leq r \leq 1$, there exists an $(n+1) \times (n+1)$ doubly stochastic matrix C with eigenvalues given by $1, 1 - \frac{n+1}{n}r, \frac{n-r}{n}\lambda_2, ..., \frac{n-r}{n}\lambda_n$.*

[1] Many thanks go to the reviewer for pointing out this.

Proof: It suffices to check that the matrix $C = \begin{pmatrix} 1-r & \frac{r}{\sqrt{n}}e_n^T \\ \frac{r}{\sqrt{n}}e_n & (\frac{n-r}{n})A \end{pmatrix}$ is doubly stochastic with eigenvalues $1, 1-\frac{n+1}{n}r, \frac{n-r}{n}\lambda_2, ..., \frac{n-r}{n}\lambda_n$. ■

Lemma 2. *Let A be an $n\times n$ doubly stochastic matrix with eigenvalues $1, \lambda_2, ..., \lambda_n$. In addition, let B be an $m\times m$ doubly stochastic matrix with eigenvalues $1, \mu_2, ..., \mu_m$. Then for any $0 \leq r \leq 1$ and $n \geq m$, there exists an $(n+m)\times(n+m)$ doubly stochastic matrix C with eigenvalues given by $1, \alpha+r-1, \alpha\lambda_2, ..., \alpha\lambda_n, r\mu_2, ..., r\mu_m$ where $\frac{1-\alpha}{m} = \frac{1-r}{n}$.*

Proof: It suffices to check that the matrix $C = \begin{pmatrix} \alpha A & \frac{(1-r)\sqrt{m}}{\sqrt{n}}e_n e_m^T \\ \frac{(1-r)\sqrt{m}}{\sqrt{n}}e_m e_n^T & rB \end{pmatrix}$ is doubly stochastic with eigenvalues $1, \alpha+r-1, \alpha\lambda_2, ..., \alpha\lambda_n, r\mu_2, ..., r\mu_m$. ■

For our purposes, the following result due to Soules [13], is also needed.

Theorem 2. *[13] If $1 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$ and*

$$\frac{1}{n} + \frac{n-m-1}{n(m+1)}\lambda_2 + \sum_{k=1}^{m} \frac{\lambda_{n-2k+2}}{(k+1)k} \geq 0, \tag{1}$$

holds with $n = 2m+2$ if n even and $n = 2m+1$ if n odd, then there exists an $n\times n$ symmetric doubly stochastic matrix D such that D has eigenvalues $1, \lambda_2, ..., \lambda_n$.

Remark 2. *The proof of the preceding theorem is done by constructing an $n\times n$ matrix whose off-diagonal entries are all nonnegative and its smallest diagonal entry is given by the left-hand side of (1). So if (1) is valid then we obtain the symmetric doubly stochastic matrix D mentioned in the preceding theorem and so its trace is nonnegative. Consequently, if $1 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$ and (1) is true, then $1+\lambda_2+...+\lambda_n \geq 0$.*

Finally, we have the following simple observation which is very useful for us and will be used later.

Lemma 3. *Let $n > 1$ be a positive integer. Then, the following statements hold.*

i) $\frac{1}{n} + \frac{1}{n(n-1)} + ... + \frac{1}{3\times 2} + \frac{1}{2\times 1} = 1$.

ii) Let p be any positive integer which is greater than 1. If $n > 2^p$, then

$$\frac{1}{n} + \frac{1}{n(n-1)} + ... + \frac{1}{2^p(2^p+1)} + \frac{2^p-1}{2^p} = 1.$$

iii) For any positive integer $m \geq 1$, it holds that $\sum_{k=1}^{m} \frac{1}{(k+1)k} = 1 - \frac{1}{m+1} = \frac{m}{m+1}$. Consequently, for any positive integer $m \geq 2$, we have $\sum_{k=2}^{m} \frac{1}{(k+1)k} = \frac{m}{m+1} - \frac{1}{2}$.

iv) For any positive even integer $m \geq 2$, it holds that $\sum_{k=\frac{m}{2}}^{m-1} \frac{1}{(k+1)k} = \frac{1}{m}$. Hence, for any positive even integer $m \geq 4$, we have $\sum_{k=\frac{m}{2}}^{m-2} \frac{1}{(k+1)k} = \frac{1}{m} - \frac{1}{m(m-1)}$.

Proof:

i) The left hand side in Part i) can be rewritten as

$$\frac{1}{n} + \sum_{k=1}^{n-1} \frac{1}{k(k+1)} = \frac{1}{n} + \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{n} + \left(\frac{1}{n-1} - \frac{1}{n}\right) + ... + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(1 - \frac{1}{2}\right) = 1.$$

ii) From Part i), we can write

$$\frac{1}{2^p(2^p-1)}+\frac{1}{(2^p-1)(2^p-2)}...+\frac{1}{3\times 2}+\frac{1}{2\times 1}=1-\frac{1}{2^p}=\frac{2^p-1}{2^p}.$$

It follows that

$$\frac{1}{n}+\frac{1}{n(n-1)}+...+\frac{1}{2^p(2^p+1)}+\frac{2^p-1}{2^p}=\frac{1}{n}+\frac{1}{n(n-1)}+...+\frac{1}{2^p(2^p+1)}+\frac{1}{2^p(2^p-1)}+...+\frac{1}{2\times 1}=1.$$

iii) It follows easily from the first part.

iv) For $m=2$, the assertion is true. Suppose $m>2$, then from Part i), we certainly know that $\sum\limits_{k=1}^{m-1}\frac{1}{(k+1)k}=1-\frac{1}{m}=\frac{m-1}{m}$. Therefore, $\sum\limits_{k=1}^{\frac{m}{2}-1}\frac{1}{(k+1)k}+\sum\limits_{k=\frac{m}{2}}^{m-1}\frac{1}{(k+1)k}=\frac{m-1}{m}$, and thus we obtain

$$\sum_{k=\frac{m}{2}}^{m-1}\frac{1}{(k+1)k}=\frac{m-1}{m}-\sum_{k=1}^{\frac{m}{2}-1}\frac{1}{(k+1)k}=\frac{m-1}{m}-\left(1-\frac{1}{\frac{m}{2}}\right)=\frac{1}{m}.$$

■

3. Main observations

In a recent paper [1], the authors used Lemma 1 and Lemma 2 to obtain a recursive method for constructing doubly stochastic matrices for the inverse eigenvalue problem by distinguishing between the cases n even and n odd. In addition, it is claimed that these new sufficient conditions improve Soules condition (1) in both cases.

Theorem 3. *[1] Let $n\geq 5$. If $1\geq\lambda_2\geq....\geq\lambda_n\geq -1$ and*

$$\frac{1}{n}+\frac{1}{n}\lambda_2+\frac{\frac{n}{2}-[\frac{n+2}{4}]}{(\frac{n}{2})\,[\frac{n+2}{4}]}\lambda_4+\sum_{k=1}^{[\frac{n+2}{4}]-1}\frac{\lambda_{n-4k+4}}{(k+1)k}\geq 0, \tag{2}$$

for n even ([.] stands for the integer part) and

$$\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{\frac{n+1}{2}-[\frac{n+3}{4}]}{(\frac{n+1}{2})\,[\frac{n+3}{4}]}\lambda_3+\sum_{k=1}^{[\frac{n+3}{4}]-1}\frac{\lambda_{n-4k+4}}{(k+1)k}\geq 0, \tag{3}$$

for n odd, holds. Then there exists an $n\times n$ symmetric doubly stochastic matrix C whose eigenvalues are $1,\lambda_2,\lambda_3,...,\lambda_n$.

Next we give a counterexample to the preceding theorem in the case when n is odd. Consider the following list of real numbers $\sigma=(1,1,1,-\frac{2}{3},-\frac{2}{3},-\frac{2}{3},-1)$. It is easy to see that σ satisfies (3). Hence, by Theorem 3, there exists a 7×7 symmetric doubly stochastic matrix A with spectrum σ. On the other hand, σ cannot be the spectrum of any 7×7 nonnegative matrix since otherwise by the Perron-Frobenius theorem (see [5]), there would exist a reducible nonnegative matrix with spectrum σ. However, there is no way of splitting σ into 3 subsets where each subset has a nonnegative sum. That leads to a contradiction.

More explicitly, the authors in [1] claimed (without justification) that when n is odd, and (3) is verified, then we have

$$\frac{1}{\frac{n+1}{2}}+\frac{\frac{n+1}{2}-[\frac{n+3}{4}]}{(\frac{n+1}{2})\,[\frac{n+3}{4}]}\frac{1}{\alpha}\lambda_3+\sum_{k=1}^{[\frac{n+3}{4}]-1}\frac{1}{\alpha}\frac{\lambda_{n-4k+4}}{(k+1)k}\geq 0, \tag{4}$$

and

$$\frac{1}{\frac{n-1}{2}}+\frac{\frac{n-1}{2}-\left[\frac{n+1}{4}\right]}{\left(\frac{n-1}{2}\right)\left[\frac{n+1}{4}\right]}\frac{1}{r}\lambda_4+\sum_{k=1}^{\left[\frac{n+1}{4}\right]-1}\frac{1}{r}\frac{\lambda_{n-4k+3}}{(k+1)k}\geq 0, \tag{5}$$

where $\alpha=\frac{(n-1)\lambda_2+(n+1)}{2n}$ and $r=\frac{(n+1)\lambda_2+(n-1)}{2n}$. In fact, if (3) is valid then it holds that

$$\begin{aligned}0&\leq \frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{\frac{n+1}{2}-\left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]}\lambda_3+\sum_{k=1}^{\left[\frac{n+3}{4}\right]-1}\frac{\lambda_{n-4k+4}}{(k+1)k}\\&=\frac{(n-1)\lambda_2+(n+1)}{2n}\left(\frac{1}{\frac{n+1}{2}}+\frac{\frac{n+1}{2}-\left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]}\frac{1}{\alpha}\lambda_3+\sum_{k=1}^{\left[\frac{n+3}{4}\right]-1}\frac{1}{\alpha}\frac{\lambda_{n-4k+4}}{(k+1)k}\right)\\&=\alpha\left(\frac{1}{\frac{n+1}{2}}+\frac{\frac{n+1}{2}-\left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]}\frac{1}{\alpha}\lambda_3+\sum_{k=1}^{\left[\frac{n+3}{4}\right]-1}\frac{1}{\alpha}\frac{\lambda_{n-4k+4}}{(k+1)k}\right).\end{aligned}$$

As α is positive, then this shows that if (3) is valid then inequality (4) is always true. However, inequality (5) may not necessarily be true when (3) is valid, as can be easily checked for the list $\sigma=(1,1,1,-\frac{2}{3},-\frac{2}{3},-\frac{2}{3},-1)$ that satisfies (3) and does not verify (5).

Next, we shall present an alternative sufficient condition for the odd case by using the same recursive method for constructing doubly stochastic matrices. However, this will not be an improvement of Soules condition, but rather an independent condition as we shall prove. Before doing so, we need the following auxiliary lemma.

Lemma 4. *Let $n\geq 5$ be odd and let $1\geq\lambda_2\geq....\geq\lambda_n\geq -1$. Then*

$$\begin{aligned}&\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{\frac{n+1}{2}-\left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]}\lambda_4+\sum_{k=1}^{\left[\frac{n+3}{4}\right]-1}\frac{\lambda_{n-4k+4}}{(k+1)k}\\&\qquad\leq\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{4}{(n-1)(n+1)}\lambda_2+\frac{\frac{n-1}{2}-\left[\frac{n+1}{4}\right]}{\left(\frac{n-1}{2}\right)\left[\frac{n+1}{4}\right]}\lambda_4+\sum_{k=1}^{\left[\frac{n+1}{4}\right]-1}\frac{\lambda_{n-4k+3}}{(k+1)k}.\end{aligned}$$

Proof: We distinguish between the cases $n=4m+1$ and $n=4m+3$ (since n here takes only odd values).

For $n = 4m + 1$, we have

$$\begin{aligned}
&\frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{m}{(2m+1)(m+1)}\lambda_4 + \sum_{k=1}^{m} \frac{\lambda_{n-4k+4}}{(k+1)k} \\
&\leq \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{m}{(2m+1)(m+1)}\lambda_4 + \sum_{k=1}^{m} \frac{\lambda_{n-4k+3}}{(k+1)k} \qquad \text{(since } \lambda_{n-4k+4} \leq \lambda_{n-4k+3}\text{)} \\
&= \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{m}{(2m+1)(m+1)}\lambda_4 + \frac{1}{m(m+1)}\lambda_4 + \sum_{k=1}^{m-1} \frac{\lambda_{n-4k+3}}{(k+1)k} \\
&= \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{m+1}{m(2m+1)}\lambda_4 + \sum_{k=1}^{m-1} \frac{\lambda_{n-4k+3}}{(k+1)k} \\
&= \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \left(\frac{1}{2m(2m+1)} + \frac{1}{2m}\right)\lambda_4 + \sum_{k=1}^{m-1} \frac{\lambda_{n-4k+3}}{(k+1)k} \\
&\leq \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{1}{2m(2m+1)}\lambda_2 + \frac{1}{2m}\lambda_4 + \sum_{k=1}^{m-1} \frac{\lambda_{n-4k+3}}{(k+1)k} \qquad \text{(since } \lambda_4 \leq \lambda_2\text{)} \\
&= \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{4}{(n-1)(n+1)}\lambda_2 + \frac{\frac{n-1}{2} - \left[\frac{n+1}{4}\right]}{\left(\frac{n-1}{2}\right)\left[\frac{n+1}{4}\right]}\lambda_4 + \sum_{k=1}^{\left[\frac{n+1}{4}\right]-1} \frac{\lambda_{n-4k+3}}{(k+1)k}.
\end{aligned}$$

For $n = 4m + 3$, we have

$$\begin{aligned}
&\frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{1}{2m+2}\lambda_4 + \sum_{k=1}^{m} \frac{\lambda_{n-4k+4}}{(k+1)k} \\
&= \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \left(\frac{1}{(2m+2)(2m+1)} + \frac{m}{(m+1)(2m+1)}\right)\lambda_4 + \sum_{k=1}^{m} \frac{\lambda_{n-4k+4}}{(k+1)k} \\
&\leq \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{1}{(2m+2)(2m+1)}\lambda_2 + \frac{m}{(m+1)(2m+1)}\lambda_4 + \sum_{k=1}^{m} \frac{\lambda_{n-4k+3}}{(k+1)k} \\
&= \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{4}{(n-1)(n+1)}\lambda_2 + \frac{\frac{n-1}{2} - \left[\frac{n+1}{4}\right]}{\left(\frac{n-1}{2}\right)\left[\frac{n+1}{4}\right]}\lambda_4 + \sum_{k=1}^{\left[\frac{n+1}{4}\right]-1} \frac{\lambda_{n-4k+3}}{(k+1)k}.
\end{aligned}$$

Thus the proof is complete.

■

Theorem 4. *Let $n \geq 5$ be odd and let $1 \geq \lambda_2 \geq .... \geq \lambda_n \geq -1$. If*

$$\frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]}\lambda_4 + \sum_{k=1}^{\left[\frac{n+3}{4}\right]-1} \frac{\lambda_{n-4k+4}}{(k+1)k} \geq 0, \tag{6}$$

then there exists an $n \times n$ symmetric doubly stochastic matrix C with eigenvalues $1, \lambda_2, \lambda_3, ..., \lambda_n$.

Proof: As n is odd, then by (6), we know that

$$
\begin{aligned}
0 &\leq \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]}\lambda_4 + \sum_{k=1}^{\left[\frac{n+3}{4}\right]-1} \frac{\lambda_{n-4k+4}}{(k+1)k} \\
&\leq \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]}\lambda_3 + \sum_{k=1}^{\left[\frac{n+3}{4}\right]-1} \frac{\lambda_{n-4k+4}}{(k+1)k} \qquad \text{(since } \lambda_4 \leq \lambda_3 \text{ and its coefficient is } > 0\text{)} \\
&= \left(\frac{(n-1)\lambda_2 + (n+1)}{2n}\right)\frac{1}{\frac{n+1}{2}} + \frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]}\lambda_3 + \sum_{k=1}^{\left[\frac{n+3}{4}\right]-1} \frac{\lambda_{n-4k+4}}{(k+1)k} \\
&= \alpha\left(\frac{1}{\frac{n+1}{2}} + \frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]}\frac{1}{\alpha}\lambda_3 + \sum_{k=1}^{\left[\frac{n+3}{4}\right]-1} \frac{1}{\alpha}\frac{\lambda_{n-4k+4}}{(k+1)k}\right),
\end{aligned}
$$

where $\alpha = \frac{(n-1)\lambda_2+(n+1)}{2n} > 0$. In order to apply Theorem 2, we first show by contradiction that $\frac{\lambda_n}{\alpha} \geq -1$ and $\frac{\lambda_3}{\alpha} \leq 1$ (since the λ_i are in the decreasing order). Suppose that $\frac{\lambda_n}{\alpha} < -1$, that is, $2n\lambda_n+(n-1)\lambda_2+(n+1) < 0$. It then follows that, $\frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{2}{n+1}\lambda_n < 0$, and hence (6) implies that

$$
\frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]}\lambda_4 + \sum_{k=1}^{\left[\frac{n+3}{4}\right]-1} \frac{\lambda_{n-4k+4}}{(k+1)k} \geq 0 > \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{2}{n+1}\lambda_n,
$$

that is,

$$
\frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]}\lambda_4 + \sum_{k=2}^{\left[\frac{n+3}{4}\right]-1} \frac{\lambda_{n-4k+4}}{(k+1)k} + \frac{1}{2}\lambda_n - \frac{2}{n+1}\lambda_n > 0.
$$

Since $1 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$, then we can write

$$
\begin{aligned}
0 &< \frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]}\lambda_2 + \sum_{k=2}^{\left[\frac{n+3}{4}\right]-1} \frac{\lambda_2}{(k+1)k} + \frac{1}{2}\lambda_n - \frac{2}{n+1}\lambda_n \\
&= \left(\frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]} + \sum_{k=2}^{\left[\frac{n+3}{4}\right]-1} \frac{1}{(k+1)k}\right)\lambda_2 + \left(\frac{1}{2} - \frac{2}{n+1}\right)\lambda_n \\
&= \left(\frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]} + \frac{\left[\frac{n+3}{4}\right]-1}{\left[\frac{n+3}{4}\right]} - \frac{1}{2}\right)\lambda_2 + \left(\frac{1}{2} - \frac{2}{n+1}\right)\lambda_n \quad \text{(using Lemma 3, Part iii) with } m = \left[\tfrac{n+3}{4}\right]-1\text{)} \\
&= \frac{n-3}{2(n+1)}(\lambda_2 + \lambda_n).
\end{aligned}
$$

So $\lambda_2+\lambda_n > 0$, and this implies that $0 > 2n\lambda_n+(n-1)\lambda_2+(n+1) = n(\lambda_2+\lambda_n)+n(1+\lambda_n)+1-\lambda_2 > 0$, and we get a contradiction. Therefore, $\frac{\lambda_n}{\alpha} \geq -1$. Suppose now that $\frac{\lambda_3}{\alpha} > 1$, that is, $2n\lambda_3-(n-1)\lambda_2-(n+1) > 0$. But this implies that $2n\lambda_3 - (n-1)\lambda_2 - (n+1) = (n-1)(\lambda_3 - \lambda_2) + (n+1)(\lambda_3 - 1) > 0$, and we also get a contradiction. Therefore, $\frac{\lambda_3}{\alpha} \leq 1$ and hence by Soules condition (1), there exists an $(\frac{n+1}{2}) \times (\frac{n+1}{2})$ symmetric doubly stochastic matrix $\widehat{A}$ with eigenvalues $1, \frac{1}{\alpha}\lambda_3, \frac{1}{\alpha}\lambda_5, ..., \frac{1}{\alpha}\lambda_n$. On the other hand, by (6) and

Lemma 4, we have

$$
\begin{aligned}
0 &\le \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]}\lambda_4 + \sum_{k=1}^{\left[\frac{n+3}{4}\right]-1} \frac{\lambda_{n-4k+4}}{(k+1)k} \\
&\le \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{4}{(n-1)(n+1)}\lambda_2 + \frac{\frac{n-1}{2} - \left[\frac{n+1}{4}\right]}{\left(\frac{n-1}{2}\right)\left[\frac{n+1}{4}\right]}\lambda_4 + \sum_{k=1}^{\left[\frac{n+1}{4}\right]-1} \frac{\lambda_{n-4k+3}}{(k+1)k} \\
&= \left(\frac{(n+1)\lambda_2 + (n-1)}{2n}\right)\frac{1}{\frac{n-1}{2}} + \frac{\frac{n-1}{2} - \left[\frac{n+1}{4}\right]}{\left(\frac{n-1}{2}\right)\left[\frac{n+1}{4}\right]}\lambda_4 + \sum_{k=1}^{\left[\frac{n+1}{4}\right]-1} \frac{\lambda_{n-4k+3}}{(k+1)k} \\
&= r\left(\frac{1}{\frac{n-1}{2}} + \frac{\frac{n-1}{2} - \left[\frac{n+1}{4}\right]}{\left(\frac{n-1}{2}\right)\left[\frac{n+1}{4}\right]}\frac{1}{r}\lambda_4 + \sum_{k=1}^{\left[\frac{n+1}{4}\right]-1} \frac{1}{r}\frac{\lambda_{n-4k+3}}{(k+1)k}\right),
\end{aligned}
$$

where $r = \frac{(n+1)\lambda_2 + (n-1)}{2n}$. Next, we prove that r is positive. Indeed, by (6), we know that

$$
\begin{aligned}
0 &\le \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]}\lambda_4 + \sum_{k=1}^{\left[\frac{n+3}{4}\right]-1} \frac{\lambda_{n-4k+4}}{(k+1)k} \\
&\le \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]}\lambda_2 + \sum_{k=1}^{\left[\frac{n+3}{4}\right]-1} \frac{\lambda_2}{(k+1)k} \quad \text{(since } \lambda_2 \ge \lambda_4 \ge ..., \text{ and their coefficients are } > 0) \\
&= \frac{1}{n} + \left(\frac{n-1}{n(n+1)} + \frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]} + \sum_{k=1}^{\left[\frac{n+3}{4}\right]-1} \frac{1}{(k+1)k}\right)\lambda_2 \\
&= \frac{1}{n} + \left(\frac{n-1}{n(n+1)} + \frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]} + 1 - \frac{1}{\left[\frac{n+3}{4}\right]}\right)\lambda_2 \\
&= \frac{1}{n} + \frac{n-1}{n}\lambda_2.
\end{aligned}
$$

So obviously $\lambda_2 \neq -1$, and it follows that $(n+1)\lambda_2 + (n-1) = 1 + (n-1)\lambda_2 + (n-2) + 2\lambda_2 > 0$ for $n > 4$. This implies that $r > 0$. Also, as $(n+1)\lambda_2 + (n-1) \le n + 1 + n - 1 = 2n$, then $r \le 1$. Next, in order to again apply Theorem 2, we shall show that $\frac{\lambda_n}{r} \ge -1$ and $\frac{\lambda_3}{r} \le 1$ by contradiction. Suppose that $\frac{\lambda_n}{r} < -1$, that is, $2n\lambda_n + (n+1)\lambda_2 + (n-1) < 0$. It follows that, $\frac{1}{n} + \frac{n+1}{n(n-1)}\lambda_2 + \frac{2}{n-1}\lambda_n < 0$. From (6), we get

$$
\frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]}\lambda_4 + \sum_{k=1}^{\left[\frac{n+3}{4}\right]-1} \frac{\lambda_{n-4k+4}}{(k+1)k} \ge 0 > \frac{1}{n} + \frac{n+1}{n(n-1)}\lambda_2 + \frac{2}{n-1}\lambda_n,
$$

that is,

$$
\left(\frac{n-1}{n(n+1)} - \frac{n+1}{n(n-1)}\right)\lambda_2 + \frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]}\lambda_4 + \sum_{k=2}^{\left[\frac{n+3}{4}\right]-1} \frac{\lambda_{n-4k+4}}{(k+1)k} + \frac{1}{2}\lambda_n - \frac{2}{n-1}\lambda_n > 0.
$$

Since $1 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$, then we can write

$$
\begin{aligned}
0 &< \left(\frac{n-1}{n(n+1)} - \frac{n+1}{n(n-1)}\right)\lambda_2 + \frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]}\lambda_2 + \sum_{k=2}^{\left[\frac{n+3}{4}\right]-1} \frac{\lambda_2}{(k+1)k} + \frac{1}{2}\lambda_n - \frac{2}{n-1}\lambda_n \\
&= \left(\frac{n-1}{n(n+1)} - \frac{n+1}{n(n-1)} + \frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]} + \sum_{k=2}^{\left[\frac{n+3}{4}\right]-1} \frac{1}{(k+1)k}\right)\lambda_2 + \left(\frac{1}{2} - \frac{2}{n-1}\right)\lambda_n \\
&= \left(\frac{n-1}{n(n+1)} - \frac{n+1}{n(n-1)} + \frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]} + \frac{\left[\frac{n+3}{4}\right]-1}{\left[\frac{n+3}{4}\right]} - \frac{1}{2}\right)\lambda_2 + \left(\frac{1}{2} - \frac{2}{n-1}\right)\lambda_n \text{ (by Lemma 3, Part iii))} \\
&= \left(\frac{n-1}{n(n+1)} - \frac{n+1}{n(n-1)} + \frac{1}{\left[\frac{n+3}{4}\right]} - \frac{1}{\frac{n+1}{2}} + 1 - \frac{1}{\left[\frac{n+3}{4}\right]} - \frac{1}{2}\right)\lambda_2 + \left(\frac{1}{2} - \frac{2}{n-1}\right)\lambda_n \\
&= \frac{n-5}{2(n-1)}(\lambda_2 + \lambda_n).
\end{aligned}
$$

Hence $\lambda_2 + \lambda_n > 0$, and it then follows that $0 > 2n\lambda_n + (n+1)\lambda_2 + (n-1) = n(\lambda_2 + \lambda_n) + (n-1)(1+\lambda_n) + (\lambda_2 + \lambda_n) > 0$, and we have a contradiction. Suppose now that $\frac{\lambda_3}{r} > 1$. This implies that $2n\lambda_3 - (n+1)\lambda_2 - (n-1) = (n+1)(\lambda_3 - \lambda_2) + (n-1)(\lambda_3 - 1) > 0$, and we again get a contradiction. Therefore, by Soules condition (1), there exists an $(\frac{n-1}{2}) \times (\frac{n-1}{2})$ symmetric doubly stochastic matrix $\widehat{B}$ with eigenvalues $1, \frac{1}{r}\lambda_4, \frac{1}{r}\lambda_6, ..., \frac{1}{r}\lambda_{n-1}$.

Finally, applying Lemma 2 with $A = \widehat{A}$ and $B = \widehat{B}$, we obtain an $n \times n$ symmetric doubly stochastic matrix C whose eigenvalues are $1, \lambda_2, ..., \lambda_n$.

■

Notation 1. *We will say that New condition 1 holds if (2) is satisfied when n is even, while (6) is satisfied if n is odd.*

Observation 1. *For n even, Soules implies New condition 1. For n odd, Soules and New condition 1 are independent.*

Proof: Let n be even. Suppose now that σ satisfies Soules condition. Since

$$\lambda_{n-4i} \geq \lambda_{n-2i}, \qquad i = 1, 2, ..., \left[\tfrac{n+2}{4}\right] - 2$$

and

$$\lambda_4 \geq \lambda_i, \qquad i = 6, 8, ..., n - \left[\tfrac{n+2}{2}\right] + 2,$$

then σ obviously satisfies New condition 1. For the second part, it suffices to check that for $n = 7$, the list $(1, 0, 0, 0, -\frac{2}{7}, -\frac{2}{7}, -\frac{2}{7})$ satisfies New condition 1 and does not satisfy Soules condition, while the list $(1, 0, 0, -\frac{3}{14}, -\frac{3}{14}, -\frac{3}{14}, -\frac{3}{14})$ does the opposite i.e. it satisfies Soules condition and does not satisfy New condition 1. ■

Next, we present the following auxiliary lemma.

Lemma 5. *Let $n = 4m + 1 \geq 9$ where m is a positive integer and $1 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$. Then*

$$
\begin{aligned}
&\frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{2(n-1)}{(n+1)(n+3)}\lambda_4 + \frac{\frac{n+3}{4} - \left[\frac{n+7}{8}\right]}{\left(\frac{n+3}{4}\right)\left[\frac{n+7}{8}\right]}\lambda_8 + \sum_{k=1}^{\left[\frac{n+7}{8}\right]-1} \frac{\lambda_{n-8k+8}}{(k+1)k} \\
&\qquad \leq \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{4}{(n-1)(n+1)}\lambda_2 + \frac{2}{n-1}\lambda_4 + \frac{\frac{n-1}{4} - \left[\frac{n+3}{8}\right]}{\left(\frac{n-1}{4}\right)\left[\frac{n+3}{8}\right]}\lambda_8 + \sum_{k=1}^{\left[\frac{n+3}{8}\right]-1} \frac{\lambda_{n-8k+7}}{(k+1)k}.
\end{aligned}
$$

Proof: As $n = 4m + 1$, we have

$$\begin{aligned}
&\frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{2(n-1)}{(n+1)(n+3)}\lambda_4 + \frac{\frac{n+3}{4} - \left[\frac{n+7}{8}\right]}{\left(\frac{n+3}{4}\right)\left[\frac{n+7}{8}\right]}\lambda_8 + \sum_{k=1}^{\left[\frac{n+7}{8}\right]-1} \frac{\lambda_{n-8k+8}}{(k+1)k} \\
&= \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \left(\frac{1}{2m(2m+1)} + \frac{m-1}{2m(m+1)}\right)\lambda_4 + \frac{\frac{n+3}{4} - \left[\frac{n+7}{8}\right]}{\left(\frac{n+3}{4}\right)\left[\frac{n+7}{8}\right]}\lambda_8 + \sum_{k=1}^{\left[\frac{n+7}{8}\right]-1} \frac{\lambda_{n-8k+8}}{(k+1)k} \\
&\leq \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{1}{2m(2m+1)}\lambda_2 + \frac{m-1}{2m(m+1)}\lambda_4 + \frac{\frac{n+3}{4} - \left[\frac{n+7}{8}\right]}{\left(\frac{n+3}{4}\right)\left[\frac{n+7}{8}\right]}\lambda_8 + \sum_{k=1}^{\left[\frac{n+7}{8}\right]-1} \frac{\lambda_{n-8k+7}}{(k+1)k}.
\end{aligned}$$

Next, we distinguish between the cases m odd and m even.
Case 1: For $m = 2p + 1$, then $n = 8p + 5$ and hence

$$\left[\frac{n+7}{8}\right] = \left[\frac{n+3}{8}\right] = p + 1.$$

Therefore,

$$\begin{aligned}
&\frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{1}{2m(2m+1)}\lambda_2 + \frac{m-1}{2m(m+1)}\lambda_4 + \frac{\frac{n+3}{4} - \left[\frac{n+7}{8}\right]}{\left(\frac{n+3}{4}\right)\left[\frac{n+7}{8}\right]}\lambda_8 + \sum_{k=1}^{\left[\frac{n+7}{8}\right]-1} \frac{\lambda_{n-8k+7}}{(k+1)k} \\
&= \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{1}{2m(2m+1)}\lambda_2 + \frac{m-1}{2m(m+1)}\lambda_4 + \frac{1}{2p+2}\lambda_8 + \sum_{k=1}^{p} \frac{\lambda_{n-8k+7}}{(k+1)k} \\
&= \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{1}{2m(2m+1)}\lambda_2 + \frac{m-1}{2m(m+1)}\lambda_4 + \left(\frac{1}{(2p+1)(2p+2)} + \frac{p}{(2p+1)(p+1)}\right)\lambda_8 + \sum_{k=1}^{p} \frac{\lambda_{n-8k+7}}{(k+1)k} \\
&\leq \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{1}{2m(2m+1)}\lambda_2 + \frac{m-1}{2m(m+1)}\lambda_4 + \frac{1}{(2p+1)(2p+2)}\lambda_4 + \frac{p}{(2p+1)(p+1)}\lambda_8 + \sum_{k=1}^{p} \frac{\lambda_{n-8k+7}}{(k+1)k} \\
&= \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{1}{2m(2m+1)}\lambda_2 + \frac{1}{2m}\lambda_4 + \frac{p}{(2p+1)(p+1)}\lambda_8 + \sum_{k=1}^{p} \frac{\lambda_{n-8k+7}}{(k+1)k} \\
&= \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{4}{(n-1)(n+1)}\lambda_2 + \frac{2}{n-1}\lambda_4 + \frac{\frac{n-1}{4} - \left[\frac{n+3}{8}\right]}{\left(\frac{n-1}{4}\right)\left[\frac{n+3}{8}\right]}\lambda_8 + \sum_{k=1}^{\left[\frac{n+3}{8}\right]-1} \frac{\lambda_{n-8k+7}}{(k+1)k}.
\end{aligned}$$

Case 2: For $m = 2p$, then $n = 8p + 1$ and in this case, we have

$$\left[\frac{n+7}{8}\right] = \left[\frac{n+3}{8}\right] + 1 = p + 1.$$

Therefore,

$$\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{1}{2m(2m+1)}\lambda_2+\frac{m-1}{2m(m+1)}\lambda_4+\frac{\frac{n+3}{4}-\left[\frac{n+7}{8}\right]}{\left(\frac{n+3}{4}\right)\left[\frac{n+7}{8}\right]}\lambda_8+\sum_{k=1}^{\left[\frac{n+7}{8}\right]-1}\frac{\lambda_{n-8k+7}}{(k+1)k}$$

$$=\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{1}{2m(2m+1)}\lambda_2+\frac{m-1}{2m(m+1)}\lambda_4+\frac{p}{(2p+1)(p+1)}\lambda_8+\sum_{k=1}^{p}\frac{\lambda_{n-8k+7}}{(k+1)k}$$

$$=\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{1}{2m(2m+1)}\lambda_2+\frac{m-1}{2m(m+1)}\lambda_4+\frac{p}{(2p+1)(p+1)}\lambda_8+\frac{1}{p(p+1)}\lambda_8+\sum_{k=1}^{p-1}\frac{\lambda_{n-8k+7}}{(k+1)k}$$

$$=\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{1}{2m(2m+1)}\lambda_2+\frac{m-1}{2m(m+1)}\lambda_4+\frac{p+1}{p(2p+1)}\lambda_8+\sum_{k=1}^{p-1}\frac{\lambda_{n-8k+7}}{(k+1)k}$$

$$=\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{1}{2m(2m+1)}\lambda_2+\frac{m-1}{2m(m+1)}\lambda_4+\left(\frac{1}{2p(2p+1)}+\frac{1}{2p}\right)\lambda_8+\sum_{k=1}^{p-1}\frac{\lambda_{n-8k+7}}{(k+1)k}$$

$$\leq\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{1}{2m(2m+1)}\lambda_2+\frac{m-1}{2m(m+1)}\lambda_4+\frac{1}{2p(2p+1)}\lambda_4+\frac{1}{2p}\lambda_8+\sum_{k=1}^{p-1}\frac{\lambda_{n-8k+7}}{(k+1)k}$$

$$=\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{1}{2m(2m+1)}\lambda_2+\frac{1}{2m}\lambda_4+\frac{1}{2p}\lambda_8+\sum_{k=1}^{p-1}\frac{\lambda_{n-8k+7}}{(k+1)k}$$

$$=\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{4}{(n-1)(n+1)}\lambda_2+\frac{2}{n-1}\lambda_4+\frac{\frac{n-1}{4}-\left[\frac{n+3}{8}\right]}{\left(\frac{n-1}{4}\right)\left[\frac{n+3}{8}\right]}\lambda_8+\sum_{k=1}^{\left[\frac{n+3}{8}\right]-1}\frac{\lambda_{n-8k+7}}{(k+1)k}.$$

Thus the proof is complete. ■

By substituting in Lemma 2, the doubly stochastic matrices obtained when New condition 1 holds, we have the following theorem. It is worthy to point out here that its proof (and the proofs of many other theorems below) may appear to be somewhat tedious, however the technique used in each case of the proof is similar to that of Theorem 4 with only minor difference, namely in the expressions of α and r in each case (see below).

Theorem 5. *Let $n \geq 8$ and let $1 \geq \lambda_2 \geq .... \geq \lambda_n \geq -1$. If*

$$\frac{1}{n}+\frac{1}{n}\lambda_2+\frac{2}{n}\lambda_4+\frac{\frac{n}{4}-\left[\frac{n+4}{8}\right]}{\left(\frac{n}{4}\right)\left[\frac{n+4}{8}\right]}\lambda_8+\sum_{k=1}^{\left[\frac{n+4}{8}\right]-1}\frac{\lambda_{n-8k+8}}{(k+1)k}\geq 0, \tag{7}$$

for $n = 4m$,

$$\frac{1}{n}+\frac{1}{n}\lambda_2+\frac{2(n-2)}{n(n+2)}\lambda_4+\frac{\frac{n+2}{4}-\left[\frac{n+6}{8}\right]}{\left(\frac{n+2}{4}\right)\left[\frac{n+6}{8}\right]}\lambda_8+\sum_{k=1}^{\left[\frac{n+6}{8}\right]-1}\frac{\lambda_{n-8k+8}}{(k+1)k}\geq 0, \tag{8}$$

for $n = 4m + 2$,

$$\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{2}{n+1}\lambda_4+\frac{\frac{n+1}{4}-\left[\frac{n+5}{8}\right]}{\left(\frac{n+1}{4}\right)\left[\frac{n+5}{8}\right]}\lambda_8+\sum_{k=1}^{\left[\frac{n+5}{8}\right]-1}\frac{\lambda_{n-8k+8}}{(k+1)k}\geq 0, \tag{9}$$

for $n = 4m + 3$, and

$$\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{2(n-1)}{(n+1)(n+3)}\lambda_4+\frac{\frac{n+3}{4}-\left[\frac{n+7}{8}\right]}{\left(\frac{n+3}{4}\right)\left[\frac{n+7}{8}\right]}\lambda_8+\sum_{k=1}^{\left[\frac{n+7}{8}\right]-1}\frac{\lambda_{n-8k+8}}{(k+1)k}\geq 0, \tag{10}$$

for $n = 4m + 1$ hold, then there exists an $n \times n$ symmetric doubly stochastic matrix with eigenvalues $1, \lambda_2, ..., \lambda_n$.

Proof: The proof is similar to that of Theorem 4. First suppose that $n = 4m$. By (7), we have

$$\alpha \left(\frac{2}{n} + \frac{2}{n}\frac{1}{\alpha}\lambda_4 + \frac{\frac{n}{4} - [\frac{n+4}{8}]}{(\frac{n}{4})\,[\frac{n+4}{8}]}\frac{1}{\alpha}\lambda_8 + \sum_{k=1}^{[\frac{n+4}{8}]-1} \frac{1}{\alpha}\frac{\lambda_{n-8k+8}}{(k+1)k} \right) \geq 0,$$

where $\alpha = \frac{1+\lambda_2}{2} \geq 0$. We show by contradiction that $\alpha > 0$. Suppose that $\alpha = 0$. Then, $\lambda_2 = -1$, which implies that $\lambda_3 = ... = \lambda_n = -1$. It follows that

$$\frac{1}{n} + \frac{1}{n}\lambda_2 + \frac{2}{n}\lambda_4 + \frac{\frac{n}{4} - [\frac{n+4}{8}]}{(\frac{n}{4})\,[\frac{n+4}{8}]}\lambda_8 + \sum_{k=1}^{[\frac{n+4}{8}]-1} \frac{\lambda_{n-8k+8}}{(k+1)k} < 0,$$

and this contradicts (7). Equivalently, we have

$$\frac{1}{\frac{n}{2}} + \frac{1}{\frac{n}{2}}\frac{1}{\alpha}\lambda_4 + \frac{\frac{n}{4} - [\frac{n+4}{8}]}{(\frac{n}{4})\,[\frac{n+4}{8}]}\frac{1}{\alpha}\lambda_8 + \sum_{k=1}^{[\frac{n+4}{8}]-1} \frac{1}{\alpha}\frac{\lambda_{n-8k+8}}{(k+1)k} \geq 0,$$

and hence

$$\frac{1}{\frac{n}{2}} + \frac{1}{\frac{n}{2}}\frac{1}{\alpha}\lambda_3 + \frac{\frac{n}{4} - [\frac{n+4}{8}]}{(\frac{n}{4})\,[\frac{n+4}{8}]}\frac{1}{\alpha}\lambda_7 + \sum_{k=1}^{[\frac{n+4}{8}]-1} \frac{1}{\alpha}\frac{\lambda_{n-8k+7}}{(k+1)k} \geq 0.$$

We first show by contradiction that $\frac{\lambda_n}{\alpha} \geq -1$. The proof of this, is virtually the same as earlier with only minor difference. Indeed, suppose that $\frac{\lambda_n}{\alpha} < -1$, that is, $2\lambda_n + \lambda_2 + 1 < 0$. It follows that, $\frac{1}{n} + \frac{1}{n}\lambda_2 + \frac{2}{n}\lambda_n < 0$. From condition (7), we conclude that

$$\frac{1}{n} + \frac{1}{n}\lambda_2 + \frac{2}{n}\lambda_4 + \frac{\frac{n}{4} - [\frac{n+4}{8}]}{(\frac{n}{4})\,[\frac{n+4}{8}]}\lambda_8 + \sum_{k=1}^{[\frac{n+4}{8}]-1} \frac{\lambda_{n-8k+8}}{(k+1)k} \geq 0 > \frac{1}{n} + \frac{1}{n}\lambda_2 + \frac{2}{n}\lambda_n,$$

that is,

$$\frac{2}{n}\lambda_4 + \frac{\frac{n}{4} - [\frac{n+4}{8}]}{(\frac{n}{4})\,[\frac{n+4}{8}]}\lambda_8 + \sum_{k=2}^{[\frac{n+4}{8}]-1} \frac{\lambda_{n-8k+8}}{(k+1)k} + \frac{1}{2}\lambda_n - \frac{2}{n}\lambda_n > 0.$$

Since $1 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$, then we clearly have

$$\begin{aligned}
0 &< \left(\frac{2}{n} + \frac{\frac{n}{4} - [\frac{n+4}{8}]}{(\frac{n}{4})\,[\frac{n+4}{8}]} + \sum_{k=2}^{[\frac{n+4}{8}]-1} \frac{1}{(k+1)k} \right) \lambda_2 + \frac{1}{2}\lambda_n - \frac{2}{n}\lambda_n \\
&= \left(\frac{2}{n} + \frac{\frac{n}{4} - [\frac{n+4}{8}]}{(\frac{n}{4})\,[\frac{n+4}{8}]} + \frac{[\frac{n+4}{8}] - 1}{[\frac{n+4}{8}]} - \frac{1}{2} \right) \lambda_2 + \frac{1}{2}\lambda_n - \frac{2}{n}\lambda_n \quad \text{(using Lemma 3, Part iii) with } m = [\tfrac{n+4}{8}] - 1) \\
&= \frac{n-4}{2n}(\lambda_2 + \lambda_n).
\end{aligned}$$

Therefore $\lambda_2 + \lambda_n > 0$. It follows that $0 > 2\lambda_n + \lambda_2 + 1 = (\lambda_2 + \lambda_n) + (1 + \lambda_n) > 0$, and we have a contradiction. Next, we show that $\frac{\lambda_3}{\alpha} \leq 1$ by contradiction. If $\frac{\lambda_3}{\alpha} > 1$, then clearly $2\lambda_3 - \lambda_2 - 1 > 0$, and we get a contradiction. Thus, by New condition 1, there are $(\frac{n}{2}) \times (\frac{n}{2})$ symmetric doubly stochastic matrices $\widehat{A}$ and $\widehat{B}$ with eigenvalues $1, \frac{1}{\alpha}\lambda_4, \frac{1}{\alpha}\lambda_6, ..., \frac{1}{\alpha}\lambda_n$ and $1, \frac{1}{\alpha}\lambda_3, \frac{1}{\alpha}\lambda_5, ..., \frac{1}{\alpha}\lambda_{n-1}$ respectively. Now applying Lemma

2 with $A = \widehat{A}$, $B = \widehat{B}$ and $r = \alpha = \frac{1+\lambda_2}{2}$, we obtain an $n \times n$ symmetric doubly stochastic matrix C whose eigenvalues are $1, \lambda_2, ..., \lambda_n$.

For $n = 4m + 2$, using (8), we conclude that

$$\frac{1}{\frac{n}{2}} + \frac{\frac{n}{2}-1}{\frac{n}{2}(\frac{n}{2}+1)}\frac{1}{\alpha}\lambda_4 + \frac{\frac{n+2}{4} - [\frac{n+6}{8}]}{(\frac{n+2}{4})[\frac{n+6}{8}]}\frac{1}{\alpha}\lambda_8 + \sum_{k=1}^{[\frac{n+6}{8}]-1}\frac{1}{\alpha}\frac{\lambda_{n-8k+8}}{(k+1)k} \geq 0,$$

and

$$\frac{1}{\frac{n}{2}} + \frac{\frac{n}{2}-1}{\frac{n}{2}(\frac{n}{2}+1)}\frac{1}{\alpha}\lambda_3 + \frac{\frac{n+2}{4} - [\frac{n+6}{8}]}{(\frac{n+2}{4})[\frac{n+6}{8}]}\frac{1}{\alpha}\lambda_7 + \sum_{k=1}^{[\frac{n+6}{8}]-1}\frac{1}{\alpha}\frac{\lambda_{n-8k+7}}{(k+1)k} \geq 0,$$

where $\alpha = \frac{1+\lambda_2}{2} > 0$. We show that $\frac{\lambda_n}{\alpha} \geq -1$ by a similar argument as in the previous case. Suppose that $\frac{\lambda_n}{\alpha} < -1$, that is, $2\lambda_n + \lambda_2 + 1 < 0$. It follows that, $\frac{1}{n} + \frac{1}{n}\lambda_2 + \frac{2}{n}\lambda_n < 0$. Then by (8), it holds that

$$\frac{1}{n} + \frac{1}{n}\lambda_2 + \frac{2(n-2)}{n(n+2)}\lambda_4 + \frac{\frac{n+2}{4} - [\frac{n+6}{8}]}{(\frac{n+2}{4})[\frac{n+6}{8}]}\lambda_8 + \sum_{k=1}^{[\frac{n+6}{8}]-1}\frac{\lambda_{n-8k+8}}{(k+1)k} \geq 0 > \frac{1}{n} + \frac{1}{n}\lambda_2 + \frac{2}{n}\lambda_n,$$

that is,

$$\frac{2(n-2)}{n(n+2)}\lambda_4 + \frac{\frac{n+2}{4} - [\frac{n+6}{8}]}{(\frac{n+2}{4})[\frac{n+6}{8}]}\lambda_8 + \sum_{k=2}^{[\frac{n+6}{8}]-1}\frac{\lambda_{n-8k+8}}{(k+1)k} + \frac{1}{2}\lambda_n - \frac{2}{n}\lambda_n > 0.$$

Since $1 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$, we have

$$\begin{aligned}
0 &< \left(\frac{2(n-2)}{n(n+2)} + \frac{\frac{n+2}{4} - [\frac{n+6}{8}]}{(\frac{n+2}{4})[\frac{n+6}{8}]} + \sum_{k=2}^{[\frac{n+6}{8}]-1}\frac{1}{(k+1)k}\right)\lambda_2 + \frac{1}{2}\lambda_n - \frac{2}{n}\lambda_n \\
&= \left(\frac{2(n-2)}{n(n+2)} + \frac{\frac{n+2}{4} - [\frac{n+6}{8}]}{(\frac{n+2}{4})[\frac{n+6}{8}]} + \frac{[\frac{n+6}{8}]-1}{[\frac{n+6}{8}]} - \frac{1}{2}\right)\lambda_2 + \frac{1}{2}\lambda_n - \frac{2}{n}\lambda_n \quad \text{(using Lemma 3, Part iii) with } m = [\tfrac{n+6}{8}] - 1) \\
&= \frac{n-4}{2n}(\lambda_2 + \lambda_n).
\end{aligned}$$

Hence $\lambda_2 + \lambda_n > 0$, and it then follows that $0 > 2\lambda_n + \lambda_2 + 1 = (\lambda_2 + \lambda_n) + (1 + \lambda_n) > 0$, and we have a contradiction. Note that $\frac{\lambda_3}{\alpha} \leq 1$ by a similar argument as in the previous case. Thus, by New condition 1, there are $(\frac{n}{2}) \times (\frac{n}{2})$ symmetric doubly stochastic matrices $\widehat{A}$ and $\widehat{B}$ with eigenvalues $1, \frac{1}{\alpha}\lambda_4, \frac{1}{\alpha}\lambda_6, ..., \frac{1}{\alpha}\lambda_n$ and $1, \frac{1}{\alpha}\lambda_3, \frac{1}{\alpha}\lambda_5, ..., \frac{1}{\alpha}\lambda_{n-1}$ respectively. Now by applying Lemma 2 with $A = \widehat{A}$, $B = \widehat{B}$ and $r = \alpha = \frac{1+\lambda_2}{2}$, the proof of the second part can be achieved.

For the case $n = 4m + 3$, using (9) we can write

$$\begin{aligned}
0 &\leq \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{2}{n+1}\lambda_4 + \frac{\frac{n+1}{4} - [\frac{n+5}{8}]}{(\frac{n+1}{4})[\frac{n+5}{8}]}\lambda_8 + \sum_{k=1}^{[\frac{n+5}{8}]-1}\frac{\lambda_{n-8k+8}}{(k+1)k} \\
&\leq \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{2}{n+1}\lambda_3 + \frac{\frac{n+1}{4} - [\frac{n+5}{8}]}{(\frac{n+1}{4})[\frac{n+5}{8}]}\lambda_7 + \sum_{k=1}^{[\frac{n+5}{8}]-1}\frac{\lambda_{n-8k+8}}{(k+1)k} \\
&= \alpha\left(\frac{1}{\frac{n+1}{2}} + \frac{1}{\frac{n+1}{2}}\frac{1}{\alpha}\lambda_3 + \frac{\frac{n+1}{4} - [\frac{n+5}{8}]}{(\frac{n+1}{4})[\frac{n+5}{8}]}\frac{1}{\alpha}\lambda_7 + \sum_{k=1}^{[\frac{n+5}{8}]-1}\frac{1}{\alpha}\frac{\lambda_{n-8k+8}}{(k+1)k}\right),
\end{aligned}$$

where $\alpha = \frac{(n-1)\lambda_2+(n+1)}{2n} > 0$. As earlier, we shall show now that $\frac{\lambda_n}{\alpha} \geq -1$ by contradiction. Suppose that $\frac{\lambda_n}{\alpha} < -1$, that is, $2n\lambda_n + (n-1)\lambda_2 + (n+1) < 0$. It follows that $\frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{2}{n+1}\lambda_n < 0$. From (9), we conclude that

$$\frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{2}{n+1}\lambda_4 + \frac{\frac{n+1}{4} - [\frac{n+5}{8}]}{(\frac{n+1}{4})[\frac{n+5}{8}]}\lambda_8 + \sum_{k=1}^{[\frac{n+5}{8}]-1} \frac{\lambda_{n-8k+8}}{(k+1)k} \geq 0 > \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{2}{n+1}\lambda_n,$$

that is,

$$\frac{2}{n+1}\lambda_4 + \frac{\frac{n+1}{4} - [\frac{n+5}{8}]}{(\frac{n+1}{4})[\frac{n+5}{8}]}\lambda_8 + \sum_{k=2}^{[\frac{n+5}{8}]-1} \frac{\lambda_{n-8k+8}}{(k+1)k} + \frac{1}{2}\lambda_n - \frac{2}{n+1}\lambda_n > 0.$$

Since $1 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$, we have

$$0 < \left(\frac{2}{n+1} + \frac{\frac{n+1}{4} - [\frac{n+5}{8}]}{(\frac{n+1}{4})[\frac{n+5}{8}]} + \sum_{k=2}^{[\frac{n+5}{8}]-1} \frac{1}{(k+1)k} \right) \lambda_2 + \frac{1}{2}\lambda_n - \frac{2}{n+1}\lambda_n$$

$$= \left(\frac{2}{n+1} + \frac{\frac{n+1}{4} - [\frac{n+5}{8}]}{(\frac{n+1}{4})[\frac{n+5}{8}]} + \frac{[\frac{n+5}{8}] - 1}{[\frac{n+5}{8}]} - \frac{1}{2} \right) \lambda_2 + \frac{1}{2}\lambda_n - \frac{2}{n+1}\lambda_n \quad \text{(using Lemma 3, Part iii) with } m = [\tfrac{n+5}{8}] - 1)$$

$$= \frac{n-3}{2(n+1)}(\lambda_2 + \lambda_n).$$

Hence $\lambda_2 + \lambda_n > 0$, and it then follows that $0 > 2n\lambda_n + (n-1)\lambda_2 + (n+1) = n(\lambda_2+\lambda_n) + n(1+\lambda_n) + 1 - \lambda_2 > 0$, and again we get a contradiction. In addition, $\frac{\lambda_3}{\alpha} \leq 1$ by a similar argument as in the proof of Theorem 4. Therefore by New condition 1, there exists an $(\frac{n+1}{2}) \times (\frac{n+1}{2})$ symmetric doubly stochastic matrix $\widehat{A}$ with eigenvalues $1, \frac{1}{\alpha}\lambda_3, \frac{1}{\alpha}\lambda_5, ..., \frac{1}{\alpha}\lambda_n$. Now from (9), we can write

$$0 \leq \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{2}{n+1}\lambda_4 + \frac{\frac{n+1}{4} - [\frac{n+5}{8}]}{(\frac{n+1}{4})[\frac{n+5}{8}]}\lambda_8 + \sum_{k=1}^{[\frac{n+5}{8}]-1} \frac{\lambda_{n-8k+8}}{(k+1)k}$$

$$= \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \left(\frac{1}{(\frac{n+1}{2})(\frac{n-1}{2})} + \frac{\frac{n-3}{2}}{(\frac{n+1}{2})(\frac{n-1}{2})} \right) \lambda_4 + \frac{\frac{n+1}{4} - [\frac{n+5}{8}]}{(\frac{n+1}{4})[\frac{n+5}{8}]}\lambda_8 + \sum_{k=1}^{[\frac{n+5}{8}]-1} \frac{\lambda_{n-8k+8}}{(k+1)k}$$

$$\leq \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{1}{(\frac{n+1}{2})(\frac{n-1}{2})}\lambda_2 + \frac{\frac{n-3}{2}}{(\frac{n+1}{2})(\frac{n-1}{2})}\lambda_4 + \frac{\frac{n+1}{4} - [\frac{n+5}{8}]}{(\frac{n+1}{4})[\frac{n+5}{8}]}\lambda_8 + \sum_{k=1}^{[\frac{n+5}{8}]-1} \frac{\lambda_{n-8k+7}}{(k+1)k}$$

$$= r \left(\frac{1}{\frac{n-1}{2}} + \frac{\frac{n-3}{2}}{(\frac{n+1}{2})(\frac{n-1}{2})}\frac{1}{r}\lambda_4 + \frac{\frac{n+1}{4} - [\frac{n+5}{8}]}{(\frac{n+1}{4})[\frac{n+5}{8}]}\frac{1}{r}\lambda_8 + \sum_{k=1}^{[\frac{n+5}{8}]-1} \frac{1}{r}\frac{\lambda_{n-8k+7}}{(k+1)k} \right),$$

where $r = \frac{(n+1)\lambda_2+(n-1)}{2n}$. A virtually identical proof to that used in Theorem 4 for the same claim, shows that $0 < r \leq 1$. Next, in order to apply Lemma 2, we shall show as earlier that $\frac{\lambda_n}{r} \geq -1$ by contradiction. Suppose that $\frac{\lambda_n}{r} < -1$, that is, $2n\lambda_n + (n+1)\lambda_2 + (n-1) < 0$. It follows that $\frac{1}{n} + \frac{n+1}{n(n-1)}\lambda_2 + \frac{2}{n-1}\lambda_n < 0$, and hence using (9), we obtain

$$\frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{2}{n+1}\lambda_4 + \frac{\frac{n+1}{4} - [\frac{n+5}{8}]}{(\frac{n+1}{4})[\frac{n+5}{8}]}\lambda_8 + \sum_{k=1}^{[\frac{n+5}{8}]-1} \frac{\lambda_{n-8k+8}}{(k+1)k} \geq 0 > \frac{1}{n} + \frac{n+1}{n(n-1)}\lambda_2 + \frac{2}{n-1}\lambda_n,$$

that is,

$$\left(\frac{n-1}{n(n+1)}-\frac{n+1}{n(n-1)}\right)\lambda_2+\frac{2}{n+1}\lambda_4+\frac{\frac{n+1}{4}-\left[\frac{n+5}{8}\right]}{\left(\frac{n+1}{4}\right)\left[\frac{n+5}{8}\right]}\lambda_8+\sum_{k=2}^{\left[\frac{n+5}{8}\right]-1}\frac{\lambda_{n-8k+8}}{(k+1)k}+\frac{1}{2}\lambda_n-\frac{2}{n-1}\lambda_n>0.$$

Since $1\geq\lambda_2\geq ...\geq\lambda_n\geq -1$, then we have

$$\begin{aligned}0&<\left(\frac{n-1}{n(n+1)}-\frac{n+1}{n(n-1)}+\frac{2}{n+1}+\frac{\frac{n+1}{4}-\left[\frac{n+5}{8}\right]}{\left(\frac{n+1}{4}\right)\left[\frac{n+5}{8}\right]}+\sum_{k=2}^{\left[\frac{n+5}{8}\right]-1}\frac{1}{(k+1)k}\right)\lambda_2+\frac{1}{2}\lambda_n-\frac{2}{n-1}\lambda_n\\&=\left(\frac{n-1}{n(n+1)}-\frac{n+1}{n(n-1)}+\frac{2}{n+1}+\frac{\frac{n+1}{4}-\left[\frac{n+5}{8}\right]}{\left(\frac{n+1}{4}\right)\left[\frac{n+5}{8}\right]}+\frac{\left[\frac{n+5}{8}\right]-1}{\left[\frac{n+5}{8}\right]}-\frac{1}{2}\right)\lambda_2+\left(\frac{1}{2}-\frac{2}{n-1}\right)\lambda_n\\&=\frac{n-5}{2(n-1)}(\lambda_2+\lambda_n).\end{aligned}$$

It then follows that $0>2n\lambda_n+(n+1)\lambda_2+(n-1)=n(\lambda_2+\lambda_n)+(n-1)(1+\lambda_n)+(\lambda_2+\lambda_n)>0$, and we have a contradiction. In addition, the proof that $\frac{\lambda_3}{r}\leq 1$ can be easily done as that of Theorem 4. Therefore, by New condition 1, there exists an $(\frac{n-1}{2})\times(\frac{n-1}{2})$ symmetric doubly stochastic matrix $\widehat{B}$ with eigenvalues $1,\frac{1}{r}\lambda_4,\frac{1}{r}\lambda_6,...,\frac{1}{r}\lambda_{n-1}$. Now applying Lemma 2 with $A=\widehat{A}$ and $B=\widehat{B}$, we obtain an $n\times n$ symmetric doubly stochastic matrix C with eigenvalues $1,\lambda_2,...,\lambda_n$.

Finally, let $n=4m+1$. From (10), we conclude that

$$\begin{aligned}0&\leq\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{2(n-1)}{(n+1)(n+3)}\lambda_4+\frac{\frac{n+3}{4}-\left[\frac{n+7}{8}\right]}{\left(\frac{n+3}{4}\right)\left[\frac{n+7}{8}\right]}\lambda_8+\sum_{k=1}^{\left[\frac{n+7}{8}\right]-1}\frac{\lambda_{n-8k+8}}{(k+1)k}\\&\leq\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{2(n-1)}{(n+1)(n+3)}\lambda_3+\frac{\frac{n+3}{4}-\left[\frac{n+7}{8}\right]}{\left(\frac{n+3}{4}\right)\left[\frac{n+7}{8}\right]}\lambda_7+\sum_{k=1}^{\left[\frac{n+7}{8}\right]-1}\frac{\lambda_{n-8k+8}}{(k+1)k}\\&=\alpha\left(\frac{1}{\frac{n+1}{2}}+\frac{\frac{n+1}{2}-1}{\frac{n+1}{2}(\frac{n+1}{2}+1)}\frac{1}{\alpha}\lambda_3+\frac{\frac{n+3}{4}-\left[\frac{n+7}{8}\right]}{\left(\frac{n+3}{4}\right)\left[\frac{n+7}{8}\right]}\frac{1}{\alpha}\lambda_7+\sum_{k=1}^{\left[\frac{n+7}{8}\right]-1}\frac{1}{\alpha}\frac{\lambda_{n-8k+8}}{(k+1)k}\right),\end{aligned}$$

where $\alpha=\frac{(n-1)\lambda_2+(n+1)}{2n}>0$. By a similar argument as the one used in the previous case, we can show that $\frac{\lambda_n}{\alpha}\geq -1$ and $\frac{\lambda_3}{\alpha}\leq 1$. On the other hand, by (10) and Lemma 5 we also have

$$\begin{aligned}0&\leq\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{2(n-1)}{(n+1)(n+3)}\lambda_4+\frac{\frac{n+3}{4}-\left[\frac{n+7}{8}\right]}{\left(\frac{n+3}{4}\right)\left[\frac{n+7}{8}\right]}\lambda_8+\sum_{k=1}^{\left[\frac{n+7}{8}\right]-1}\frac{\lambda_{n-8k+8}}{(k+1)k}\\&\leq\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{4}{(n-1)(n+1)}\lambda_2+\frac{2}{n-1}\lambda_4+\frac{\frac{n-1}{4}-\left[\frac{n+3}{8}\right]}{\left(\frac{n-1}{4}\right)\left[\frac{n+3}{8}\right]}\lambda_8+\sum_{k=1}^{\left[\frac{n+3}{8}\right]-1}\frac{\lambda_{n-8k+7}}{(k+1)k}\\&=r\left(\frac{1}{\frac{n-1}{2}}+\frac{1}{\frac{n-1}{2}}\frac{1}{r}\lambda_4+\frac{\frac{n-1}{4}-\left[\frac{n+3}{8}\right]}{\left(\frac{n-1}{4}\right)\left[\frac{n+3}{8}\right]}\frac{1}{r}\lambda_8+\sum_{k=1}^{\left[\frac{n+3}{8}\right]-1}\frac{1}{r}\frac{\lambda_{n-8k+7}}{(k+1)k}\right),\end{aligned}$$

where $r=\frac{(n+1)\lambda_2+(n-1)}{2n}$ with $\frac{\lambda_n}{r}\geq -1$ and $\frac{\lambda_3}{r}\leq 1$ (again for this, a similar proof as above can be employed). Thus by New condition 1, there are $(\frac{n+1}{2})\times(\frac{n+1}{2})$ and $(\frac{n-1}{2})\times(\frac{n-1}{2})$ symmetric doubly stochastic matrices $\widehat{A}$

and $\widehat{B}$ with eigenvalues $1, \frac{1}{\alpha}\lambda_3, \frac{1}{\alpha}\lambda_5, ..., \frac{1}{\alpha}\lambda_n$ and $1, \frac{1}{r}\lambda_4, \frac{1}{r}\lambda_6, ..., \frac{1}{r}\lambda_{n-1}$ respectively and such that $0 < r \leq 1$ as earlier. Finally, applying Lemma 2 with $A = \widehat{A}$ and $B = \widehat{B}$, we obtain an $n \times n$ symmetric doubly stochastic matrix C whose eigenvalues are $1, \lambda_2, ..., \lambda_n$. ■

Notation 2. *The conditions of Theorem 5 are referred to as New condition 2.*

Observation 2. *New condition 2 and New condition 1 are independent. For $n = 14$, the list given by*

$$(1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, -\frac{2}{3})$$

satisfies New condition 1 and does not satisfy neither New condition 2 nor Soules condition (1). However the list

$$(1, 0, 0, 0, 0, -\frac{3}{50}, -\frac{3}{50}, -\frac{3}{35}, -\frac{3}{35}, -\frac{1}{10}, -\frac{1}{10}, -\frac{1}{10}, -\frac{1}{10}, -\frac{1}{10})$$

satisfies New condition 2 and does not satisfy neither New condition 1 nor Soules condition (1).

We can recursively apply the same process to obtain more general sufficient conditions for the SDIEP. Although, it seems that there is no systematic way of doing this for general n, however we put forward the following conjecture.

Conjecture 1. *Let k and n be any positive integers such that $n \geq 2^k$, and let $1 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$. For each $i = 1, ..., k$, denote by α_{i-1} to be the remainder of the Euclidean division of n by 2^{i-1} and define*

$$r_i^{(n)} = \begin{cases} 2^{i-1} - \alpha_{i-1} & \text{if } \alpha_{i-1} \neq 0 \\ 0 & \text{if } \alpha_{i-1} = 0. \end{cases}$$

If

$$\frac{1}{n} + \frac{n - \left[\frac{n+1}{2}\right]}{n\left[\frac{n+1}{2}\right]}\lambda_2 + \frac{\frac{n+r_2^{(n)}}{2} - \left[\frac{n+2+r_2^{(n)}}{4}\right]}{\left(\frac{n+r_2^{(n)}}{2}\right)\left[\frac{n+2+r_2^{(n)}}{4}\right]}\lambda_4 + \frac{\frac{n+r_3^{(n)}}{4} - \left[\frac{n+4+r_3^{(n)}}{8}\right]}{\left(\frac{n+r_3^{(n)}}{4}\right)\left[\frac{n+4+r_3^{(n)}}{8}\right]}\lambda_8 + ...$$

$$... + \frac{\frac{n+r_k^{(n)}}{2^{k-1}} - \left[\frac{n+2^{k-1}+r_k^{(n)}}{2^k}\right]}{\left(\frac{n+r_k^{(n)}}{2^{k-1}}\right)\left[\frac{n+2^{k-1}+r_k^{(n)}}{2^k}\right]}\lambda_{2^k} + \sum_{i=1}^{\left[\frac{n+2^{k-1}+r_k^{(n)}}{2^k}\right]-1} \frac{\lambda_{n-2^k i+2^k}}{(i+1)i} \geq 0,$$

then there exists a $n \times n$ symmetric doubly stochastic matrix with eigenvalues $1, \lambda_2, ..., \lambda_n$.

Note that for $k = 1$, $k = 2$ and $k = 3$, we obtain respectively Soules condition, New condition 1 and New condition 2. In order to support this conjecture, we shall verify it for the case $n = 26$ and we refer to the new condition in this case as *New condition 3*.

Example 1. *Let $\sigma = \{1, \lambda_2, ..., \lambda_{26}\}$ be a list of 26 real numbers such that $1 \geq \lambda_2 \geq ... \geq \lambda_{26} \geq -1$. In this case, New condition 3 states*

$$\begin{aligned} 0 &\leq \frac{1}{26} + \frac{1}{26}\lambda_2 + \frac{6}{(13)(7)}\lambda_4 + \frac{3}{28}\lambda_8 + \frac{1}{4}\lambda_{16} + \frac{1}{2}\lambda_{26} \\ &= \alpha\left(\frac{1}{13} + \frac{6}{(13)(7)}\frac{1}{\alpha}\lambda_4 + \frac{3}{28}\frac{1}{\alpha}\lambda_8 + \frac{1}{4}\frac{1}{\alpha}\lambda_{16} + \frac{1}{2}\frac{1}{\alpha}\lambda_{26}\right) \end{aligned}$$

with $\alpha = \frac{1+\lambda_2}{2}$. *By (10) of Theorem 5, there exists a* 13×13 *symmetric doubly stochastic matrix* $\widehat{A}$ *with eigenvalues* $1, \frac{\lambda_4}{\alpha}, \frac{\lambda_6}{\alpha}, ..., \frac{\lambda_{26}}{\alpha}$. *On the other hand, since* $1 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$, *we conclude that*

$$\frac{1}{13} + \frac{6}{(13)(7)}\frac{1}{\alpha}\lambda_3 + \frac{3}{28}\frac{1}{\alpha}\lambda_7 + \frac{1}{4}\frac{1}{\alpha}\lambda_{15} + \frac{1}{2}\frac{1}{\alpha}\lambda_{25} \geq 0.$$

Hence, there exists a 13×13 *symmetric doubly stochastic matrix* $\widehat{B}$*with eigenvalues* $1, \frac{\lambda_3}{\alpha}, \frac{\lambda_5}{\alpha}, ..., \frac{\lambda_{25}}{\alpha}$. *Again applying Lemma 2 with* $A = \widehat{A}$ *and* $B = \widehat{B}$ *and* $r = \alpha = \frac{1+\lambda_2}{2}$, *we obtain a* 26×26 *symmetric doubly stochastic matrix* C *with eigenvalues* $1, \lambda_2, ..., \lambda_{26}$. *Note that the list given by*

$$(1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-\frac{1}{13},-\frac{1}{13},-\frac{1}{13},-\frac{1}{13},-\frac{1}{13},-\frac{1}{13},-\frac{1}{13},-\frac{1}{13},-\frac{1}{13})$$

satisfies New condition 3 and does not satisfy New condition 1 nor New condition 2.

Notation 3. *For convenience, we shall denote by* S, N *the sets of all* n*-tuples* σ *satisfying Soules condition, and New condition 1, respectively.*

4. On the realizability of normalized Suleimanova spectra

4.1. A negative answer to Question 1

We begin with a lemma whose proof can be found in [5].

Lemma 6. *Let* A *be an* $n \times n$ *indecomposable doubly stochastic matrix. If* A *has exactly* r *roots of unit modulus, then these are the* r*-roots of unity. If* $r > 1$, *then* r *is a divisor of* n. *Moreover,* A *is cogredient to a matrix of the form*

$$\begin{pmatrix} 0 & A_1 & 0 & \dots & 0 \\ 0 & 0 & A_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & A_{r-1} \\ A_r & 0 & 0 & \dots & 0 \end{pmatrix}, \tag{11}$$

where the A_i *are doubly stochastic of order* $\frac{n}{r} \times \frac{n}{r}$, $i = 1, \dots, r$.

Making use of the above lemma we have the following:

Proposition 1. *If* n *is odd and* $\lambda = (1, \lambda_2, ..., \lambda_{n-1}, -1)$ *with* $|\lambda_i| < 1$ *for all* $i = 2, ..., n-1$, *then* λ *cannot be the spectrum of any* $n \times n$ *doubly stochastic matrix.*

Proof: Suppose that $\lambda = (1, \lambda_2, \dots, \lambda_{n-1}, -1)$ is the spectrum of an $n \times n$ doubly stochastic matrix A. Now as A has 2 eigenvalues of unit modulus which are 1 and -1, then by virtue of the preceding lemma, 2 must be a divisor of n which is a contradiction as n is odd.■

A direct conclusion is the following corollary that gives a negative answer for Question 1 in the case when n is odd.

Corollary 1. *If* n *is odd, then* $\lambda = (1, 0, ..., 0, -1)$ *cannot be the spectrum of any* $n \times n$ *doubly stochastic matrix.*

It should be noted here that for n odd, $\lambda = (1, 0, ..., 0, -1)$ is not the only normalized Suleimanova spectrum that is not realizable by an $n \times n$ doubly stochastic matrix. Indeed, for $n = 3$ there exists an infinite number of such points. To see this, we first recall the following result from Perfect and Mirsky[12].

Theorem 6. *[12] Let $1 \geq \lambda_2 \geq \lambda_3 \geq -1$, and $1+\lambda_2+\lambda_3 \geq 0$. There exists a 3×3 doubly stochastic matrix with spectrum $(1,\lambda_2,\lambda_3)$ if and only if $2+\lambda_2+3\lambda_3 \geq 0$.*

As a conclusion, we have the following.

Corollary 2. *Let $1 \geq \lambda_2 \geq \lambda_3 \geq -1$,and $1+\lambda_2+\lambda_3 \geq 0$. There exists a 3×3 doubly stochastic matrix with spectrum $(1,\lambda_2,\lambda_3)$ if and only if $(1,\lambda_2,\lambda_3)$ is in $Convhull[(1,1,1),(1,1,-1),(1,-1/2,-1/2)]$ where Convhull stands for convex hull.*

Thus we have the following conclusion.

Corollary 3. *The region of $\mathbb{R}^3$ that contains all decreasingly ordered normalized Suleimanova spectra that are realizable by doubly stochastic matrices, is*

$$Convhull[(1,-1/2,-1/2),(1,0,0),(1,0,-2/3)],$$

and the region of all decreasingly ordered normalized Suleimanova spectra that are not realizable by doubly stochastic matrices, is

$$Convhull[(1,-1/2,-1/2),(1,0,-1),(1,0,-2/3)] \setminus [(1,-1/2,-1/2),(1,0,-2/3)],$$

where $[(1,-1/2,-1/2),(1,0,-2/3)]$ is the line-segment joining $(1,-1/2,-1/2)$ to $(1,0,-2/3)$.

4.2. Some improvement

Here we will derive new sufficient conditions for a normalized Suleimanova spectrum to be the spectrum of a symmetric doubly stochastic matrix. We shall start with the following definition.

Definition 1. *An $n\times n$ matrix H is a Hadamard matrix if $h_{ij} \in \{\pm 1\}$ and $HH^T = nI_n$. If n is a positive integer such that an $n\times n$ Hadamard matrix exists, then n is said to be a Hadamard order.*

Let $H_0 = (1)$, $H_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and for $n \in \mathbb{N}\backslash\{0,1\}$ (where $\mathbb{N}$ is the set of natural numbers), define

$$H_n := H_1 \otimes H_{n-1} = \begin{pmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{pmatrix} \in M_{2^n}(\mathbb{R}).$$

It is well-known that H_n is a Hadamard matrix for every $n \in \mathbb{N}$, and the matrix H_n obtained from the previous construction is known as the Walsh matrix of order 2^n. Note that Walsh matrices satisfy the following additional well-known properties:

- $H_n^T = H_n$;
- $H_n^{-1} = \frac{1}{2^n} H_n$.

Next, we recall the following result which is due to Johnson and Paparella [3].

Theorem 7. *[3] For any positive integer k and for any $n = 2^k$, let $\lambda = (1,\lambda_2,\lambda_3,...,\lambda_n)$ be a list of n real numbers with*

$$1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant ... \geqslant \lambda_n \geqslant -1,$$
$$\lambda_i \leq 0 \quad for \quad i = 2,...,n,$$

and

$$1 + \sum_{i=2}^{n} \lambda_i \geq 0.$$

Then there exists an $n\times n$ symmetric doubly stochastic matrix with spectrum λ.

It is worth mentioning here that the proof of the preceding theorem is constructive. Indeed, with the same notation as above, the authors prove that $H_k \text{diag}(1, \lambda_2, \lambda_3, ..., \lambda_n) H_k^{-1}$ is actually an $n \times n$ symmetric doubly stochastic matrix with spectrum $(1, \lambda_2, \lambda_3, ..., \lambda_n)$.

Our next objective is to exploit the recursive method described earlier in order to find new families of sufficient conditions for NS-SDIEP that improve Soules condition, and New condition 1 for the case of normalized Suleimanova spectra.

Theorem 8. *Let p and n be two positive integers such that $p \geq 2$ and $n \geq 2^p$, and let $0 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$. If*

$$1 + \lambda_2 + ... + \lambda_{n-1} + \lambda_n \geq 0, \tag{12}$$

for $n = 2^p$, and

$$\frac{1}{n} + \frac{1}{n(n-1)}\lambda_2 + ... + \frac{1}{2^p(2^p+1)}\lambda_{n-2^p+1} + \frac{1}{2^p}(\lambda_{n-2^p+2} + ... + \lambda_{n-1} + \lambda_n) \geq 0, \tag{13}$$

for $n > 2^p$, then there is an $n \times n$ symmetric doubly stochastic matrix with normalized Suleimanova spectrum $1, \lambda_2, ..., \lambda_n$.

Proof: We will proceed by induction. For $n = 2^p$, this is true by Theorem 7. Let $n > 2^p$ and suppose that the assertion is true for $n - 1$. Since

$$\begin{aligned}
0 &\leq \frac{1}{n} + \frac{1}{n(n-1)}\lambda_2 + ... + \frac{1}{2^p(2^p+1)}\lambda_{n-2^p+1} + \frac{1}{2^p}\left(\lambda_{n-2^p+2} + ... + \lambda_{n-1} + \lambda_n\right) \\
&= \frac{n-1+\lambda_2}{n(n-1)} + \frac{1}{(n-1)(n-2)}\lambda_3 + ... + \frac{1}{2^p(2^p+1)}\lambda_{n-2^p+1} + \frac{1}{2^p}\left(\lambda_{n-2^p+2} + ... + \lambda_{n-1} + \lambda_n\right) \\
&= \theta\left(\frac{1}{n-1} + \frac{1}{(n-1)(n-2)}\frac{\lambda_3}{\theta} + ... + \frac{1}{2^p(2^p+1)}\frac{\lambda_{n-2^p+1}}{\theta} + \frac{1}{2^p}\left(\frac{\lambda_{n-2^p+2}}{\theta} + ... + \frac{\lambda_{n-1}}{\theta} + \frac{\lambda_n}{\theta}\right)\right),
\end{aligned}$$

with $\theta = \frac{n-1+\lambda_2}{n} > 0$, and consequently $\frac{\lambda_i}{\theta} \leq 0$ for $i = 3, ..., n$. Next, we show that $\frac{\lambda_i}{\theta} \geq -1$ by contradiction. Suppose that $\frac{\lambda_n}{\theta} < -1$, then $\frac{1}{n} + \frac{1}{n(n-1)}\lambda_2 + \frac{1}{n-1}\lambda_n < 0$. In view of (13), we conclude that

$$\frac{1}{n} + \frac{1}{n(n-1)}\lambda_2 + ... + \frac{1}{2^p(2^p+1)}\lambda_{n-2^p+1} + \frac{1}{2^p}(\lambda_{n-2^p+2} + ... + \lambda_{n-1} + \lambda_n) \geq 0 > \frac{1}{n} + \frac{1}{n(n-1)}\lambda_2 + \frac{1}{n-1}\lambda_n,$$

that is,

$$\frac{1}{(n-1)(n-2)}\lambda_3 + ... + \frac{1}{2^p(2^p+1)}\lambda_{n-2^p+1} + \frac{1}{2^p}(\lambda_{n-2^p+2} + ... + \lambda_{n-1}) + \left(\frac{1}{2^p} - \frac{1}{n-1}\right)\lambda_n > 0.$$

As $n > 2^p$ and all the $\lambda_i \leq 0$, we get a contradiction. Thus, we arrive at the following inequality:

$$\frac{1}{n-1} + \frac{1}{(n-1)(n-2)}\frac{\lambda_3}{\theta} + ... + \frac{1}{2^p(2^p+1)}\frac{\lambda_{n-2^p+1}}{\theta} + \frac{1}{2^p}\left(\frac{\lambda_{n-2^p+2}}{\theta} + ... + \frac{\lambda_{n-1}}{\theta} + \frac{\lambda_n}{\theta}\right) \geq 0.$$

Therefore, by the induction hypothesis, there exists an $(n-1) \times (n-1)$ symmetric doubly stochastic matrix $\widehat{A}$ with eigenvalues $(1, \frac{\lambda_3}{\theta}, ..., \frac{\lambda_n}{\theta})$. Next, let $r = \frac{n-1}{n}(1 - \lambda_2) = 1 - \frac{1+(n-1)\lambda_2}{n}$. In order to apply Lemma 1, we shall prove that $0 \leq r \leq 1$. First observe that $n \geq 1 + (n-1)\lambda_2$. Moreover, using (13), we can write

$$\begin{aligned}
0 &\leq \frac{1}{n} + \frac{1}{n(n-1)}\lambda_2 + ... + \frac{1}{2^p(2^p+1)}\lambda_{n-2^p+1} + \frac{1}{2^p}(\lambda_{n-2^p+2} + ... + \lambda_{n-1} + \lambda_n) \\
&\leq \frac{1}{n} + \left(\frac{1}{n(n-1)} + ... + \frac{1}{2^p(2^p+1)} + \frac{2^p-1}{2^p}\right)\lambda_2 \\
&= \frac{1}{n} + \frac{n-1}{n}\lambda_2.
\end{aligned}$$

Hence $1+(n-1)\lambda_2 \geq 0$ and therefore $0 \leq r \leq 1$. Now applying Lemma 1 with this r and with $A = \widehat{A}$, we obtain an $n \times n$ symmetric doubly stochastic matrix C with eigenvalues $1, \lambda_2, ..., \lambda_n$. ■

Next, we present the following remark.

Remark 3. *In the sequel, we shall make the convention that a summation over the empty index set is defined to be zero.*

Now substituting in Lemma 2, the doubly stochastic matrices obtained from Theorem 8, we arrive at the following theorem.

Theorem 9. *Let p and n be two positive integers such that $p \geq 2$ and $n \geq 2^{p+1}$, and let $0 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$. If*

$$\frac{1}{n} + \frac{n-k-1}{n(k+1)}\lambda_2 + \sum_{i=2^p}^{k} \frac{\lambda_{n-2i+2}}{(i+1)i} + \frac{1}{2^p}\sum_{i=1}^{2^p-1} \lambda_{n-2i+2} \geq 0, \tag{14}$$

holds with $n = 2k+2$ for n even and $n = 2k+1$ for n odd, then there exists an $n \times n$ symmetric doubly stochastic matrix with eigenvalues $1, \lambda_2, ..., \lambda_n$.

Proof: Case 1: For $n = 2k+2$, Inequality (14) can be rewritten as

$$\begin{aligned} 0 &\leq \frac{1}{n} + \frac{1}{n}\lambda_2 + \sum_{i=2^p}^{k} \frac{\lambda_{n-2i+2}}{(i+1)i} + \frac{1}{2^p}\sum_{i=1}^{2^p-1} \lambda_{n-2i+2} \\ &= \alpha\left(\frac{1}{k+1} + \sum_{i=2^p}^{k} \frac{1}{\alpha}\frac{\lambda_{n-2i+2}}{(i+1)i} + \frac{1}{2^p}\frac{1}{\alpha}\sum_{i=1}^{2^p-1} \lambda_{n-2i+2}\right), \end{aligned}$$

with $\alpha = \frac{1+\lambda_2}{2} \geq 0$. We show by contradiction that $\alpha > 0$. Suppose that $\alpha = 0$. Then, $\lambda_2 = -1$, which implies that $\lambda_3 = ... = \lambda_n = -1$. It follows that

$$\frac{1}{n} + \frac{1}{n}\lambda_2 + \sum_{i=2^p}^{k} \frac{\lambda_{n-2i+2}}{(i+1)i} + \frac{1}{2^p}\sum_{i=1}^{2^p-1} \lambda_{n-2i+2} < 0,$$

and this contradicts (14). As before, we show next that $\frac{\lambda_n}{\alpha} \geq -1$ by contradiction. Suppose that $\frac{\lambda_n}{\alpha} < -1$, that is, $2\lambda_n + \lambda_2 + 1 < 0$ so that $\frac{1}{n} + \frac{1}{n}\lambda_2 + \frac{2}{n}\lambda_n < 0$. From condition (14), we conclude that

$$\frac{1}{n} + \frac{1}{n}\lambda_2 + \sum_{i=2^p}^{k} \frac{\lambda_{n-2i+2}}{(i+1)i} + \frac{1}{2^p}\sum_{i=1}^{2^p-1} \lambda_{n-2i+2} \geq 0 > \frac{1}{n} + \frac{1}{n}\lambda_2 + \frac{2}{n}\lambda_n,$$

that is,

$$\sum_{i=2^p}^{k} \frac{\lambda_{n-2i+2}}{(i+1)i} + \frac{1}{2^p}\sum_{i=2}^{2^p-1} \lambda_{n-2i+2} + \left(\frac{1}{2^p} - \frac{2}{n}\right)\lambda_n > 0.$$

Again as $n \geq 2^{p+1}$ and all $\lambda_i \leq 0$, we have a contradiction. Hence by Theorem 8, there exists a $(k+1)\times(k+1)$ symmetric doubly stochastic matrix $\widehat{A}$ with eigenvalues $1, \frac{1}{\alpha}\lambda_4, ..., \frac{1}{\alpha}\lambda_{2k+2}$. Moreover, since $1 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$, then

$$\frac{1}{k+1} + \sum_{i=2^p}^{k} \frac{1}{\alpha}\frac{\lambda_{n-2i+1}}{(i+1)i} + \frac{1}{2^p}\frac{1}{\alpha}\sum_{i=1}^{2^p-1} \lambda_{n-2i+1} \geq 0,$$

and therefore, there exists a $(k+1) \times (k+1)$ symmetric doubly stochastic matrix $\widehat{B}$ with eigenvalues $1, \frac{1}{\alpha}\lambda_3, ..., \frac{1}{\alpha}\lambda_{2k+1}$. Applying now Lemma 2, with $A = \widehat{A}$, $B = \widehat{B}$ and $\alpha = r$, we obtain an $n \times n$ symmetric

doubly stochastic matrix C with eigenvalues $1, \lambda_2, ..., \lambda_n$.
Case 2: For $n = 2k+1$, we have

$$0 \leq \frac{1}{2k+1} + \frac{k}{(2k+1)(k+1)}\lambda_2 + \sum_{i=2^p}^{k} \frac{\lambda_{n-2i+2}}{(i+1)i} + \frac{1}{2^p}\sum_{i=1}^{2^p-1} \lambda_{n-2i+2}$$
$$= \alpha\left(\frac{1}{k+1} + \sum_{i=2^p}^{k} \frac{1}{\alpha}\frac{\lambda_{n-2i+2}}{(i+1)i} + \frac{1}{2^p}\frac{1}{\alpha}\sum_{i=1}^{2^p-1} \lambda_{n-2i+2}\right),$$

with $\alpha = \frac{(k+1)+k\lambda_2}{2k+1} = \frac{(n+1)+(n-1)\lambda_2}{2n} = \frac{2+(n-1)(1+\lambda_2)}{2n} > 0$. Again, at this point we shall show by contradiction that $\frac{\lambda_n}{\alpha} \geq -1$. Suppose that $\frac{\lambda_n}{\alpha} < -1$, that is, $2n\lambda_n + (n-1)\lambda_2 + (n+1) < 0$. It follows that $\frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{2}{n+1}\lambda_n < 0$, and hence by (14), we conclude that

$$\frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \sum_{i=2^p}^{k} \frac{\lambda_{n-2i+2}}{(i+1)i} + \frac{1}{2^p}\sum_{i=1}^{2^p-1} \lambda_{n-2i+2} \geq 0 > \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{2}{n+1}\lambda_n,$$

that is,

$$\sum_{i=2^p}^{k} \frac{\lambda_{n-2i+2}}{(i+1)i} + \frac{1}{2^p}\sum_{i=2}^{2^p-1} \lambda_{n-2i+2} + \left(\frac{1}{2^p} - \frac{2}{n+1}\right)\lambda_n > 0.$$

As $n \geq 2^{p+1}$ and all the $\lambda_i \leq 0$, this leads to a contradiction. Hence, by Theorem 8, there exists a $(k+1)\times(k+1)$ symmetric doubly stochastic matrix with eigenvalues $1, \frac{1}{\alpha}\lambda_3, ..., \frac{1}{\alpha}\lambda_{2k+1}$. On the other hand, we know that

$$0 \leq \frac{1}{2k+1} + \frac{k}{(2k+1)(k+1)}\lambda_2 + \sum_{i=2^p}^{k} \frac{\lambda_{n-2i+2}}{(i+1)i} + \frac{1}{2^p}\sum_{i=1}^{2^p-1} \lambda_{n-2i+2}$$
$$\leq \frac{1}{2k+1} + \frac{k}{(2k+1)(k+1)}\lambda_2 + \frac{1}{k(k+1)}\lambda_2 + \sum_{i=2^p}^{k-1} \frac{\lambda_{n-2i+1}}{(i+1)i} + \frac{1}{2^p}\sum_{i=1}^{2^p-1} \lambda_{n-2i+1}$$
$$= r\left(\frac{1}{k} + \sum_{i=2^p}^{k-1} \frac{1}{r}\frac{\lambda_{n-2i+1}}{(i+1)i} + \frac{1}{2^p}\frac{1}{r}\sum_{i=1}^{2^p-1} \lambda_{n-2i+1}\right),$$

where $r = \frac{(k+1)\lambda_2+k}{2k+1}$. In order to apply Lemma 2, we need to show that $0 < r \leq 1$. Now since r can be rewritten as $r = \frac{(k+1)\lambda_2+k}{2k+1} = \frac{(n+1)\lambda_2+(n-1)}{2n} = \frac{(n-1)\lambda_2+1+2\lambda_2+n-2}{2n}$, then it is enough to show that $1 + (n-1)\lambda_2 \geq 0$ (as $n > 4$). From (14), we conclude that

$$0 \leq \frac{1}{2k+1} + \frac{k}{(2k+1)(k+1)}\lambda_2 + \sum_{i=2^p}^{k} \frac{\lambda_{n-2i+2}}{(i+1)i} + \frac{1}{2^p}\sum_{i=1}^{2^p-1} \lambda_{n-2i+2}$$
$$\leq \frac{1}{2k+1} + \left(\frac{k}{(2k+1)(k+1)} + \sum_{i=2^p}^{k} \frac{1}{(i+1)i} + \frac{1}{2^p}\sum_{i=1}^{2^p-1} 1\right)\lambda_2$$
$$= \frac{1}{2k+1} + \left(\frac{k}{(2k+1)(k+1)} + \frac{k}{k+1} - \frac{2^p-1}{2^p} + \frac{2^p-1}{2^p}\right)\lambda_2$$
$$= \frac{1}{2k+1} + \frac{2k}{2k+1}\lambda_2$$
$$= \frac{1}{n} + \frac{n-1}{n}\lambda_2.$$

Hence $1+(n-1)\lambda_2 \geq 0$ and so $1+(n-1)\lambda_2+2\lambda_2+n-2 \geq 1+(n-1)\lambda_2-2+n-2>0$. Hence, $r>0$. Also, as $(n+1)\lambda_2+(n-1) \leq n+1+n-1=2n$, then $r \leq 1$.

As earlier, we next show by contradiction that $\frac{\lambda_n}{r} \geq -1$. Suppose that $\frac{\lambda_n}{r}<-1$, that is, $2n\lambda_n+(n+1)\lambda_2+(n-1)<0$. It follows that $\frac{1}{n}+\frac{n+1}{n(n-1)}\lambda_2+\frac{2}{n-1}\lambda_n<0$, and therefore by (14), we conclude that

$$\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\sum_{i=2^p}^{k}\frac{\lambda_{n-2i+2}}{(i+1)i}+\frac{1}{2^p}\sum_{i=1}^{2^p-1}\lambda_{n-2i+2} \geq 0 > \frac{1}{n}+\frac{n+1}{n(n-1)}\lambda_2+\frac{2}{n-1}\lambda_n,$$

that is,

$$\left(\frac{n-1}{n(n+1)}-\frac{n+1}{n(n-1)}\right)\lambda_2+\sum_{i=2^p}^{k}\frac{\lambda_{n-2i+2}}{(i+1)i}+\frac{1}{2^p}\sum_{i=2}^{2^p-1}\lambda_{n-2i+2}+\left(\frac{1}{2^p}-\frac{2}{n-1}\right)\lambda_n>0.$$

Since $1 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$, then we can write

$$\begin{aligned}
0 &< \left(\frac{n-1}{n(n+1)}-\frac{n+1}{n(n-1)}+\sum_{i=2^p}^{k}\frac{1}{(i+1)i}+\frac{1}{2^p}\sum_{i=2}^{2^p-1}1\right)\lambda_2+\left(\frac{1}{2^p}-\frac{2}{n-1}\right)\lambda_n \\
&= \left(\frac{n-1}{n(n+1)}-\frac{n+1}{n(n-1)}+\frac{k}{k+1}-\frac{2^p-1}{2^p}+\frac{2^p-2}{2^p}\right)\lambda_2+\left(\frac{1}{2^p}-\frac{2}{n-1}\right)\lambda_n \\
&= \left(\frac{n-1}{n(n+1)}-\frac{n+1}{n(n-1)}+\frac{n-1}{n+1}-1+\frac{1}{2^p}+1-\frac{2}{2^p}\right)\lambda_2+\left(\frac{1}{2^p}-\frac{2}{n-1}\right)\lambda_n \qquad \left(\text{since } k=\frac{n-1}{2}\right) \\
&= \left(\frac{n-5}{n-1}+\frac{2}{n-1}-\frac{1}{2^p}\right)\lambda_2+\left(\frac{1}{2^p}-\frac{2}{n-1}\right)\lambda_n \\
&= \frac{n-5}{n-1}\lambda_2+\left(\frac{1}{2^p}-\frac{2}{n-1}\right)(\lambda_n-\lambda_2).
\end{aligned}$$

Since $n=2k+1$ and $n \geq 2^{p+1}$, then obviously $n \geq 2^{p+1}+1$. However, this leads to a contradiction as $\lambda_2 \leq 0$ and $(\lambda_n-\lambda_2) \leq 0$. Therefore, by Theorem 8, there exists a $k \times k$ symmetric doubly stochastic matrix with eigenvalues $1, \frac{1}{r}\lambda_4, ..., \frac{1}{r}\lambda_{2k}$. Now the proof can be easily completed by applying Lemma 2, with $A=\widehat{A}$ and $B=\widehat{B}$. ■

Again, substituting in Lemma 2 the doubly stochastic matrices obtained from Theorem 9, we obtain the following theorem for which its proof is virtually identical to that of Theorem 4. So only, a sketch of its proof is given here.

Theorem 10. *Let p and n be two positive integers such that $p \geq 2$ and $n \geq 2^{p+2}$, and let $0 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$. Suppose that*

$$\frac{1}{n}+\frac{1}{n}\lambda_2+\frac{\frac{n}{2}-\left[\frac{n+2}{4}\right]}{\left(\frac{n}{2}\right)\left[\frac{n+2}{4}\right]}\lambda_4+\sum_{i=2^p}^{\left[\frac{n+2}{4}\right]-1}\frac{\lambda_{n-4i+4}}{(i+1)i}+\frac{1}{2^p}\sum_{i=1}^{2^p-1}\lambda_{n-4i+4} \geq 0, \tag{15}$$

for n even and

$$\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{\frac{n+1}{2}-\left[\frac{n+3}{4}\right]}{\left(\frac{n+1}{2}\right)\left[\frac{n+3}{4}\right]}\lambda_4+\sum_{i=2^p}^{\left[\frac{n+3}{4}\right]-1}\frac{\lambda_{n-4i+4}}{(i+1)i}+\frac{1}{2^p}\sum_{i=1}^{2^p-1}\lambda_{n-4i+4} \geq 0, \tag{16}$$

for n odd, hold. Then there exists an $n \times n$ symmetric doubly stochastic matrix D whose eigenvalues are $1, \lambda_2, ..., \lambda_n$.

Proof: If n is even, then by (15) and Theorem 9, there are $(\frac{n}{2}) \times (\frac{n}{2})$ symmetric doubly stochastic matrices $\widehat{A}$ and $\widehat{B}$ with eigenvalues $1, \frac{1}{\alpha}\lambda_4, \frac{1}{\alpha}\lambda_6, ..., \frac{1}{\alpha}\lambda_n$ and $1, \frac{1}{\alpha}\lambda_3, \frac{1}{\alpha}\lambda_5, ..., \frac{1}{\alpha}\lambda_{n-1}$ respectively, where $\alpha = \frac{1+\lambda_2}{2}$. Apply now Lemma 2, with $A = \widehat{A}$ and $B = \widehat{B}$, then we obtain an $n \times n$ symmetric doubly stochastic C with eigenvalues $1, \lambda_2, ..., \lambda_n$.
If n is odd, then by (16) and Theorem 9, there are $(\frac{n+1}{2}) \times (\frac{n+1}{2})$ and $(\frac{n-1}{2}) \times (\frac{n-1}{2})$ symmetric doubly stochastic matrices $\widehat{A}$ and $\widehat{B}$ with eigenvalues $1, \frac{1}{\alpha}\lambda_3, \frac{1}{\alpha}\lambda_5, ..., \frac{1}{\alpha}\lambda_n$ and $1, \frac{1}{r}\lambda_4, \frac{1}{r}\lambda_6, ..., \frac{1}{r}\lambda_{n-1}$ respectively with $\alpha = \frac{(n-1)\lambda_2+(n+1)}{2n}$ and $r = \frac{(n+1)\lambda_2+(n-1)}{2n}$. Applying Lemma 2 (the reason we can apply it, can be justified as earlier) with $A = \widehat{A}$ and $B = \widehat{B}$, we obtain an $n \times n$ symmetric doubly stochastic matrix C with eigenvalues $1, \lambda_2, .., \lambda_n$. ■

Notation 4. *For convenience, we denote by M_p the condition of Theorem 8, S_p the condition of Theorem 9 and N_p the condition of Theorem 10.*

With these new notations, we have the following observations.

Observation 3. *Let $\sigma = \{1, \lambda_2, ..., \lambda_n\}$ be a normalized Suleimanova spectrum with $n \geq 8$. If σ satisfies S then σ satisfies S_2. Consequently, with a slight abuse of notation, we conclude that $S \subset S_2$.*

Proof: Suppose that σ satisfies S. If $n = 2k+1$ for n odd and $n = 2k+2$ for n even, then clearly we have

$$\begin{aligned}
0 &\leq \frac{1}{n} + \frac{n-k-1}{n(k+1)}\lambda_2 + \sum_{i=4}^{k} \frac{\lambda_{n-2i+2}}{(i+1)i} + \frac{1}{12}\lambda_{n-4} + \frac{1}{6}\lambda_{n-2} + \frac{1}{2}\lambda_n \\
&\leq \frac{1}{n} + \frac{n-k-1}{n(k+1)}\lambda_2 + \sum_{i=4}^{k} \frac{\lambda_{n-2i+2}}{(i+1)i} + \frac{1}{12}\lambda_{n-4} + \frac{1}{6}\lambda_{n-4} + \frac{1}{4}\lambda_{n-2} + \frac{1}{4}\lambda_n \\
&= \frac{1}{n} + \frac{n-k-1}{n(k+1)}\lambda_2 + \sum_{i=4}^{k} \frac{\lambda_{n-2i+2}}{(i+1)i} + \frac{1}{4}\lambda_{n-4} + \frac{1}{4}\lambda_{n-2} + \frac{1}{4}\lambda_n.
\end{aligned}$$

Therefore, σ satisfies S_2. ■

Observation 4. *Let $\sigma = \{1, \lambda_2, ..., \lambda_n\}$ be a normalized Suleimanova spectrum with $n \geq 16$. If σ satisfies N then σ satisfies N_2. Thus, with a slight abuse of notation, we have $N \subset N_2$.*

Proof: If n is even and σ satisfies N, then

$$\begin{aligned}
0 &\leq \frac{1}{n} + \frac{1}{n}\lambda_2 + \frac{\frac{n}{2} - [\frac{n+2}{4}]}{(\frac{n}{2})[\frac{n+2}{4}]}\lambda_4 + \sum_{i=4}^{[\frac{n+2}{4}]-1} \frac{\lambda_{n-4i+4}}{(i+1)i} + \frac{1}{12}\lambda_{n-8} + \frac{1}{6}\lambda_{n-4} + \frac{1}{2}\lambda_n \\
&\leq \frac{1}{n} + \frac{1}{n}\lambda_2 + \frac{\frac{n}{2} - [\frac{n+2}{4}]}{(\frac{n}{2})[\frac{n+2}{4}]}\lambda_4 + \sum_{i=4}^{[\frac{n+2}{4}]-1} \frac{\lambda_{n-4i+4}}{(i+1)i} + \frac{1}{12}\lambda_{n-8} + \frac{1}{6}\lambda_{n-8} + \frac{1}{4}\lambda_{n-4} + \frac{1}{4}\lambda_n \\
&= \frac{1}{n} + \frac{1}{n}\lambda_2 + \frac{\frac{n}{2} - [\frac{n+2}{4}]}{(\frac{n}{2})[\frac{n+2}{4}]}\lambda_4 + \sum_{i=4}^{[\frac{n+2}{4}]-1} \frac{\lambda_{n-4i+4}}{(i+1)i} + \frac{1}{4}\lambda_{n-8} + \frac{1}{4}\lambda_{n-4} + \frac{1}{4}\lambda_n.
\end{aligned}$$

Therefore, σ satisfies N_2. A similar argument can also be employed, for the case when n is odd. ■

Theorem 11. *Let $n \geq 4$ be an integer, and let $0 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$. Define w_p, v_p, z_p and y_p as follows.*

- *For any positive integer $p \geq 2$ with $2^p \leq n$, let*

$$w_p := \frac{1}{n} + \frac{1}{n(n-1)}\lambda_2 + ... + \frac{1}{2^p(2^p+1)}\lambda_{n-2^p+1} + \frac{1}{2^p}(\lambda_{n-2^p+2} + ... + \lambda_{n-1} + \lambda_n).$$

- *For $n = \begin{cases} 2k+1 & \text{if } n \text{ is odd} \\ 2k+2 & \text{if } n \text{ is even} \end{cases}$, and for all positive integers $p \geq 2$ with $2^{p+1} \leq n$, let*

$$v_p := \frac{1}{n} + \frac{n-k-1}{n(k+1)}\lambda_2 + \sum_{i=2^p}^{k} \frac{\lambda_{n-2i+2}}{(i+1)i} + \frac{1}{2^p}\sum_{i=1}^{2^p-1} \lambda_{n-2i+2}.$$

- *For n even, and for all positive integers $p \geq 2$ with $2^{p+2} \leq n$, let*

$$z_p := \frac{1}{n} + \frac{1}{n}\lambda_2 + \frac{\frac{n}{2} - [\frac{n+2}{4}]}{(\frac{n}{2})\,[\frac{n+2}{4}]}\lambda_4 + \sum_{i=2^p}^{[\frac{n+2}{4}]-1} \frac{\lambda_{n-4i+4}}{(i+1)i} + \frac{1}{2^p}\sum_{i=1}^{2^p-1} \lambda_{n-4i+4}.$$

- *For n odd, and for any positive integer $p \geq 2$ with $2^{p+2} \leq n$, define*

$$y_p := \frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{\frac{n+1}{2} - [\frac{n+3}{4}]}{(\frac{n+1}{2})\,[\frac{n+3}{4}]}\lambda_4 + \sum_{i=2^p}^{[\frac{n+3}{4}]-1} \frac{\lambda_{n-4i+4}}{(i+1)i} + \frac{1}{2^p}\sum_{i=1}^{2^p-1} \lambda_{n-4i+4}.$$

Then, for $p > 2$,

1) $w_p - w_{p-1} \geq 0$,
2) $v_p - v_{p-1} \geq 0$,
3) $z_p - z_{p-1} \geq 0$,
4) $y_p - y_{p-1} \geq 0$,

Proof: We shall only give the proof of Part 1), as the rest can be proved similarly. For simplicity, let us denote $m = 2^p$. Then,

$$w_p = \frac{1}{n} + \sum_{i=m}^{n-1} \frac{\lambda_{n-i+1}}{i(i+1)} + \frac{1}{m}\sum_{i=1}^{m-1} \lambda_{n-i+1},$$

and

$$w_{p-1} = \frac{1}{n} + \sum_{i=\frac{m}{2}}^{n-1} \frac{\lambda_{n-i+1}}{i(i+1)} + \frac{1}{\frac{m}{2}}\sum_{i=1}^{\frac{m}{2}-1} \lambda_{n-i+1}.$$

It follows that for $p > 2$,

$$\begin{aligned}
w_p - w_{p-1} &= -\sum_{i=\frac{m}{2}}^{m-1} \frac{\lambda_{n-i+1}}{i(i+1)} + \frac{1}{m}\sum_{i=1}^{m-1}\lambda_{n-i+1} - \frac{1}{\frac{m}{2}}\sum_{i=1}^{\frac{m}{2}-1}\lambda_{n-i+1} \\
&= \sum_{i=\frac{m}{2}}^{m-1}\left(\frac{1}{m} - \frac{1}{i(i+1)}\right)\lambda_{n-i+1} + \left(\frac{1}{m} - \frac{1}{\frac{m}{2}}\right)\sum_{i=1}^{\frac{m}{2}-1}\lambda_{n-i+1} \\
&= \sum_{i=\frac{m}{2}}^{m-2}\left(\frac{1}{m} - \frac{1}{i(i+1)}\right)\lambda_{n-i+1} + \left(\frac{1}{m} - \frac{1}{m(m-1)}\right)\lambda_{n-m+2} - \frac{1}{m}\sum_{i=1}^{\frac{m}{2}-1}\lambda_{n-i+1} \\
&= \frac{1}{m}\left(\sum_{i=\frac{m}{2}}^{m-2}\lambda_{n-i+1} - \sum_{i=1}^{\frac{m}{2}-1}\lambda_{n-i+1}\right) + \left(\frac{1}{m} - \frac{1}{m(m-1)}\right)\lambda_{n-m+2} - \sum_{i=\frac{m}{2}}^{m-2}\frac{1}{i(i+1)}\lambda_{n-i+1} \\
&= \frac{1}{m}\left(\sum_{i=\frac{m}{2}}^{m-2}\lambda_{n-i+1} - \sum_{i=1}^{\frac{m}{2}-1}\lambda_{n-i+1}\right) + \sum_{i=\frac{m}{2}}^{m-2}\frac{1}{i(i+1)}\lambda_{n-m+2} - \sum_{i=\frac{m}{2}}^{m-2}\frac{1}{i(i+1)}\lambda_{n-i+1} \text{ (using Lemma 3, Part iv)} \\
&= \frac{1}{m}\left(\sum_{i=\frac{m}{2}}^{m-2}\lambda_{n-i+1} - \sum_{i=1}^{\frac{m}{2}-1}\lambda_{n-i+1}\right) + \sum_{i=\frac{m}{2}}^{m-2}\frac{1}{i(i+1)}\left(\lambda_{n-m+2} - \lambda_{n-i+1}\right) \geq 0,
\end{aligned}$$

since $0 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$. Thus the proof is complete.

■

As a result, we have the following observation (keeping in mind a slight abuse of notation).

Observation 5. *For a normalized Suleimanova spectrum, we have*

- $M_2 \subset M_3 \subset ... \subset M_p \subset M_{p+1} \subset ....$
- $S_2 \subset S_3 \subset ... \subset S_p \subset S_{p+1} \subset ....$
- $N_2 \subset N_3 \subset ... \subset N_p \subset N_{p+1} \subset ....$

Example 2. *Consider the list $\sigma = (1, 0, 0, 0, 0, -\frac{2}{3})$. It is easy to see that σ does not verify Soules condition (1) but it verifies condition M_2. Therefore, it is the spectrum of a 6×6 symmetric doubly stochastic matrix.*

Example 3. *For $n = 9$, consider the list $\sigma = (1, 0, 0, 0, 0, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5})$. A simple check shows that σ does not verify Soules condition (1). However, σ verifies conditions S_2 and M_3, and therefore it is the spectrum of a 9×9 symmetric doubly stochastic matrix.*

Finally, by making use of Theorem 1, we end our discussion with the following sufficient condition concerning realizable Suleimanova spectra of even order.

Theorem 12. *For any two nonnegative powers of 2, n and m with $m \geq n$, let $\sigma = \{1, \lambda_2, ..., \lambda_{n+m}\} \subset \mathbb{R}$, such that $0 \geq \lambda_2 \geq ... \geq \lambda_{n+m} \geq -1$. If there exists a permutation π on $\{\lambda_2, ..., \lambda_{n+m}\}$ such that $\pi(\lambda_2, ..., \lambda_{n+m}) = (\mu_2, ..., \mu_{n+m})$ and*

$$\mu_2 > -\frac{n}{m},$$

$$\mu_3 + ... + \mu_{n+1} \geq -\frac{m\mu_2 + n}{m+n},$$

and

$$\mu_{n+2} + ... + \mu_{n+m} \geq -\frac{n\mu_2 + m}{m+n},$$

then σ is realizable by an $(n+m) \times (n+m)$ symmetric doubly stochastic matrix.

Proof: Since $0 \geq \frac{m+n}{m\mu_2+n}\mu_3 + ... + \frac{m+n}{m\mu_2+n}\mu_{n+1} \geq -1$, and n is a nonnegative power of 2, then in view of Theorem 7, there exists an $n \times n$ symmetric doubly stochastic matrix $\widehat{A}$ with eigenvalues $(1, \frac{m+n}{m\mu_2+n}\mu_3, ..., \frac{m+n}{m\mu_2+n}\mu_{n+1})$. Similarly, since $0 \geq \frac{m+n}{n\mu_2+m}\lambda_{n+2} + ... + \frac{m+n}{n\mu_2+m}\mu_{n+m} \geq -1$ and m is a nonnegative power of 2, then by Theorem 7, there exists an $m \times m$ symmetric doubly stochastic matrix $\widehat{B}$ with spectrum $(1, \frac{m+n}{n\mu_2+m}\mu_{n+2}, ..., \frac{m+n}{n\mu_2+m}\mu_{n+m})$. In Theorem 1, taking $r = 1$, $\rho = \frac{\sqrt{mn}(1-\mu_2)}{m\mu_2+n}$ (as $\mu_2 > -\frac{n}{m}$, then $\rho > 0$), $A = \widehat{A}$ and $B = \widehat{B}$, then we get an $(n+m) \times (n+m)$ symmetric doubly stochastic matrix D with eigenvalues $1, \lambda_2, \lambda_3, ..., \lambda_{n+m}$. ■

Example 4. *Consider the list of 6 real numbers $\sigma = \{1, 0, 0, -\frac{1}{6}, -\frac{1}{6}, -\frac{11}{18}\}$. It is easy to check that σ does not verify S nor N. Moreover, σ does not verify M_2 (note that S_2 and N_2 cannot be applied). However, taking $\mu_2 = -\frac{1}{6}$, $\mu_3 = -\frac{1}{6}$, $\mu_4 = 0$, $\mu_5 = 0$, and $\mu_6 = -\frac{11}{18}$, then σ verifies Theorem 12 for the case of order $6 = m + n$ with $m = 4$ and $n = 2$. Therefore, it is realizable by a 6×6 symmetric doubly stochastic matrix.*

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