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Viability of an Open Set for Stochastic Control Systems*

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Abstract

The problem of compatibility of a stochastic control system and a set of constraints - the so called viability property - has been widely investigated during the last three decades. Given a stochastic control system, the question is to characterize sets A such that for any initial condition in A there exists a control ensuring that the associated stochastic process remains forever almost surely in A (this is called the viability property of A). When A is closed and the dynamics is continuous, the viability property has been characterized in the literature through several equivalent geometric conditions involving A, the drift and the diffusion of the control system. In this article we give a necessary and sufficient condition involving the boundary of an open set A ensuring the viability property of A, whenever A has a $C^{2,1}$ boundary and the dynamics are Lipschitz. If moreover a classical convexity condition on the con trol dynamics holds true, we show that the viability of an open set A is equivalent to the viability of its closure. This last result is rather surprising, because several very elementary examples in the deterministic framework show that, in general, there is no such equivalence for a general open set A. We will also discuss examples illustrating that the above equivalence is wrong when either the boundary of A does not have enough regularity, or the dynamics are not Lipschitz continuous.

Introduction

Let us consider a stochastic control system described by the following differential equation:

$$\left\{\begin{array}{l} dX(s) = b(X(s), u(s))ds + \sigma(X(s), u(s))dW(s),\ u(s) \in U,\ s \geq 0, \\ X(0) = x, \end{array}\right. \tag{1}$$

where U is a compact metric space, $b : \mathbb{R}^n \times U \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}$, and $W = (W(s),\ s \in [0, +\infty))$ is a d-dimensional standard $(\mathcal{F}_s)$-Brownian motion on some complete stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_s), P)$. We denote by $\mathcal{U} = \mathcal{U}(\Omega, (\mathcal{F}_s), P)$ the set of all U-valued $(\mathcal{F}_s)$-progressively measurable processes $u(\cdot)$ called admissible controls. Let $t \mapsto X_t^{x,u}$ stand for the solution of (1) corresponding to a control $u \in \mathcal{U}$ (whose existence and uniqueness is ensured by classical assumptions given later on).

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A set $K \subset \mathbb{R}^n$ is said to be viable for (1) if and only if, for all $x \in K$, there exists a control $u \in \mathcal{U}$ such that, P almost surely, $X_t^{x,u} \in K$ for all $t \geq 0$. This property has important applications in several fields (cf. [3, 4] for many examples).

When K is closed, the viability property of K for (1) can be characterized by conditions involving only b, σ and the geometry of K. It is worth pointing out that these conditions do not require any regularity property of the closed set K. In [23, 5, 6] such a chararacterization of viability is given in terms of the stochastic tangent cones to K, while in [11, 13] it is given through the distance function d_K to K (cf also [18] for smooth sets). Further, in [10] a characterization is obtained through the first and second derivatives of real valued C^2 functions having a local maximum in K. The viability property has been also studied for random sets [6], for time dependent sets [13], for infinite dimensional systems [14], as well as for backward stochastic differential equations [12] (see also [8, 24, 25]).

It is worth pointing out that the above viability property differs from the so-called invariance property: A set $K \subset \mathbb{R}^n$ is said to be invariant for (1) whenever for all $x \in K$ and for any control $u \in \mathcal{U}$, $X_t^{x,u} \in K$ for all $t \geq 0$, P almost surely. Of course, for stochastic uncontrolled differential equations enjoying the existence and uniqueness of solutions, the invariance and the viability properties do coincide. The viability property is weaker than invariance and also more difficult to investigate because the controller needs to construct for every initial point a control such that the associate trajectory is viable, while for the invariance, controls are given a priori and it remains only to check that the associated trajectories stay in K. The invariance property can be characterized as well by conditions involving only b, σ and the geometry of K (cf [1, 5, 7, 15, 17, 19, 28]).

When a set K is the closure of its interior $Int(K)$, it may happen that K is viable but the set $Int(K)$ is not necessarily viable as shown by the following one dimensional example

$$dX(x) = 2\sqrt{X(s)}dW(s), \ s \geq 0.$$

Here $d = n = 1$ and $K = [0, +\infty)$. In this case K is viable but $Int(K) = (0, +\infty)$ is not viable. This example will be discussed in detail in the last section.

In the present paper, we restrict our attention to the sets K with $C^{2,1}$ boundary and to stochastic control systems with Lipschitz dynamics. Our main goal is to give a characterization of the viability property of the interior $Int(K)$ using a geometric condition. We also show that it is equivalent to the so-called $\varepsilon-$viability introduced in [11]. The question if similar results can be obtained for the stochastic control systems with less regular sets K remains open.

When the control system (1) reduces to an uncontrolled differential equation, in [16], some (weaker than in the present paper) regularity assumptions were imposed on the closed set K and it was shown that invariance of K is equivalent to the invariance of its interior. In the present paper we exhibit a deterministic example illustrating that without sufficient regularity of boundary, the equivalence between the viability of K and its interior may fail. This example, of course, does not satisfy assumptions of [16].

Even when K is sufficiently regular, an additional convexity assumption like in [22] has to be imposed on the controlled stochastic system for us to prove the equivalence between the invariance of K and of its interior. This convexity assumption is needed to show that the limit of stochastic processes solution to (1) is still a solution to (1). This convexity condition, used to obtain the existence of optimal controls, cf. [22], is very strong and is not satisfied even by diffusions depending on controls in a linear way. This is the reason why the $\varepsilon-$viability seems to be a more convenient notion for this type of investigations. Under the convexity assumption, it is known from [11] that viability and $\varepsilon-$viability coincide.

This paper can be considered as a first attempt to deal with viability of the interior of K for control systems in the stochastic framework (while the invariance of the interior of K for uncontrolled systems was studied in [16]). Once again the novelty of the results of the paper concerns the viability property, the invariance property is somehow simpler because it requires to study the existence of a constrained solution of the stochastic differential equation (1) *for any fixed control*, while the main difficulty of the viability property consists *in the construction of a control* such that the corresponding solution to (1) satisfies the constraint. Let us note that the viability property of the interior of a set has many important applications in optimal control of the deterministic systems under state constraints: regularity of the value function, uniqueness of solutions to Hamilton-Jacobi equations, maximum pri nciple and sensitivity relations were investigated

with its help, cf. [9, 21]. Having in mind similar applications, it is of a crucial importance to have sufficient conditions for the viability of the interior of K for the stochastic control systems as well.

Let us describe now how our paper is organized. After a section devoted to notations and preliminaries, we state and prove our main result in section 2. Section 3 is devoted to the discussion of several examples and counterexamples.

1 Preliminaries

We denote by B the closed unit ball of $\mathbb{R}^n$ and by $Int(B)$ its interior. As usual, $B(x,r) = x + rB$. The notation $|x|$ stands for the Euclidean norm of $x \in \mathbb{R}^n$. The set of continuous real valued function on $\mathbb{R}^n$ is denoted by $C(\mathbb{R}^n)$, while $C^1(\mathbb{R}^n)$ and $C^2(\mathbb{R}^n)$ are the sets of continuously differentiable and twice continously differentiable functions, respectively.

1.1 Assumptions on the dynamics

In what follows, we assume that the coefficients of the control system satisfy the following conditions:

(H1) b and σ are continuous in (x,v);

(H2) $|\sigma(x,v) - \sigma(x',v)| \leq C_0|x-x'|$ for all $x, x' \in \mathbb{R}^n$, for all $v \in U$;

(H3) $|b(x,v) - b(x',v)| \leq C_0|x-x'|$ for all $x, x' \in \mathbb{R}^n$, for all $v \in U$;

where $C_0 > 0$ is a given constant.

It is well known that under (H1-H3), for every initial condition x and every admissible control $u \in \mathcal{U}$ there exists a unique adapted continuous solution $t \mapsto X_t^{x,u}$ of (1) defined on $[0, +\infty)$.

Consider the second order Kolmogorov operator $\mathcal{L}$ associated with the stochastic control system (1), defined for all $\phi \in C^2(\mathbb{R}^n)$ by

$$\mathcal{L}_{x,v}\phi := \langle b(x,v), \nabla\phi(x)\rangle + \frac{1}{2}\mathrm{Tr}\left(D^2\phi(x)\sigma(x,v)\sigma^*(x,v)\right),\ (x,v) \in \mathbb{R}^n \times U.$$

1.2 Set of constraints

Let $K \subset \mathbb{R}^n$ be a nonempty closed set of state constraints. Its interior is denoted by $Int(K)$, and its boundary by ∂K. We introduce the distance function from K

$$d_K(x) := \inf_{y \in K} |x-y|,\ x \in \mathbb{R}^n,$$

$d_{\partial K}$ is the distance to the boundary of K, and the oriented distance function from ∂K is defined by

$$\delta_K(x) \quad := \quad \begin{cases} d_{\partial K}(x) & \text{if } x \in K, \\ -d_{\partial K}(x) & \text{else.} \end{cases}$$

We also define the set of projections of x onto ∂K by $\Pi_{\partial K}(x) := \{y \in \partial K,\ d_{\partial K}(x) = |y-x|\}$. For any $\varepsilon > 0$, $K_\varepsilon := K + \varepsilon Int(B)$ is the ε-neighborhood of K.

We recall that $A \subset R^n$ is a closed domain of class $C^{2,1}$ if A it is a connected set such that for any $x \in \partial A$ there exists $r > 0$ and a $C^{2,1}$ function (i.e. a twice differentiable function with locally Lipschitz second order derivative) $\phi : B(x,r) \mapsto \mathbb{R}$ with nonvanishing gradient on $\partial A \cap B(x,r)$ such that

$$A \cap B(x,r) = \{y \in B(x,r),\ \phi(y) \leq 0\}.$$

Recall, cf [20], that A is a closed domain of class $C^{2,1}$ with compact boundary if and only if ∂A is compact and there exists $\bar{r} > 0$ such that the oriented distance function δ_A is of class $C^{2,1}$ on $\partial A + \bar{r}B$. Then for all $x \in (\partial A + \bar{r}B) \cap A$,

$$\left\{ \begin{array}{l} \text{there exists a unique } \bar{x} \in \partial A \text{ such that } \delta_A(x) = |x - \bar{x}| \\ \text{and } \nabla\delta_A(x) = \nabla\delta_A(\bar{x}) = -n_A(\bar{x}), \end{array} \right. \tag{2}$$

where $n_A(\bar{x})$ denotes the outward unit normal to A at $\bar{x}$.

We introduce the following smoothness assumption on K which will be needed to investigate the viability of open sets.

(H4) The set $K \subset \mathbb{R}^n$ is a compact domain of class $C^{2,1}$.

1.3 Viability and ε-viability of a closed set

We first recall some useful definition from [11].

Definition 1.1. *Let K be a nonempty subset of $\mathbb{R}^n$.*

The set K is viable for (1), if and only if for all $x \in K$, there is an admissible control $v(\cdot) \in \mathcal{U}$ such that the solution $X^{x,v(\cdot)}$ to (1) satisfies

$$P\big[X_s^{x,v(\cdot)} \in K,\ \forall s \in [0,\infty)\big] = 1.$$

The set K is ε-viable if and only if for some $\lambda > 0$ and for all $x \in K$ and for all $\varepsilon > 0$, there is an admissible control $v(\cdot) \in \mathcal{U}$ such that

$$E[\int_0^{+\infty} e^{-\lambda s} d_K^2(X_s^{x,v(\cdot)})ds] \leq \varepsilon..$$

Remark 1.2. *Observe that the ε-viability of K is equivalent to the fact that the following value function*

$$W(x) := \inf_{u \in \mathcal{U}} E[\int_0^{\infty} e^{-\lambda s} d_K^2(X_s^{x,u})ds],\ x \in \mathbb{R}^n. \tag{3}$$

vanishes on K.

We recall the following characterization of the viability property of a closed set.

Proposition 1.3. *(cf [10] Theorem A.1, and [11]) Let K be a nonempty closed subset of $\mathbb{R}^n$. Let us assume that the conditions* (H1)−(H3) *are satisfied by the stochastic control system* (1)*. Then the following statements are equivalent:*

(i) *The set K is ε-viable for (1).*

(ii) *For all $x \in \partial K$ and any $\mathcal{C}^2$ function $\varphi : \mathbb{R}^n \to \mathbb{R}$ with a local maximum on K at x, there exists $v \in U$ with*

$$\sigma(x,v)^* D\varphi(x) = 0, \text{ and } (\mathcal{L}_{x,v}\varphi) \leq 0.$$

Moreover if the following condition

(H5) *The set $\left\{\left(\frac{1}{2}\sigma\sigma^*(x,v), b(x,v)\right),\ v \in U\right\}$ is convex for all $x \in \mathbb{R}^n$,*

holds true, then i) and ii) are also equivalent to the viability of K.

2 Viability of the interior of K

We now state our main result:

Theorem 2.1. *Let $K \subset \mathbb{R}^n$ be a nonempty compact set. Assume* (H1)-(H4)*. Then the following statements are equivalent:*

(i) For all $x \in \partial K$ there exists some $v \in U$ such that

$$\sigma(x,v)^* D\big(\delta_K\big)(x) = 0, \text{ and } \mathcal{L}_{x,v}(\delta_K) \geq 0\,.$$

(ii) The set $Int(K)$ is viable for (1), i.e., for all $x \in Int(K)$, there is an admissible control $v(\cdot) \in \mathcal{U}$ such that the solution $X^{x,v(\cdot)}$ to (1) *satisfies, P-a.s.,*

$$X_s^{x,v(\cdot)} \in Int(K),\ \forall s \geq 0.$$

The above theorem is a direct consequence of the following proposition having an extra (but more heavy) equivalence relation.

Proposition 2.2. *Let $K \subset \mathbb{R}^n$ be a nonempty compact set. Assume* (H1)-(H4)*. Then the following statements are equivalent:*

(a) *The set K is ε-viable for (1).*

(b) *For all $x \in \partial K$ there exists some $v \in U$ such that*

$$\sigma(x,v)^* D\big(\delta_K\big)(x) = 0, \text{ and } \mathcal{L}_{x,v}(\delta_K) \geq 0\,.$$

(c) *The set $Int(K)$ is viable for (1).*

Proof : We proceed by proving (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

From our assumptions on the set K, one can find (cf [20]) $r_1 > 0$ small enough and a function $g : \mathbb{R}^n \mapsto \mathbb{R}$ of class $C^{2,1}$ with

$$\begin{cases} 0 \leq g(x) \leq 1 & \text{for} \quad x \in K \\ 0 < g(x) & \text{for} \quad x \in Int(K) \\ g(x) = \delta_K(x) & \text{for} \quad x \in K \cap (\partial K + r_1 B) \\ g(x) < 0 & \text{for} \quad x \notin K. \end{cases} \tag{4}$$

We first prove that (a) implies (b). Fix $x \in \partial K$. Then clearly the C^2 function $y \mapsto \varphi(y) := -g(y)$ has a local maximum on K at x. So from Proposition 1.3-(ii) we deduce the desired conclusion (b).

Let us prove now that (b) implies (c). Suppose that (b) holds true, and define the function

$$V(x) = -\ln g(x), \text{ for } x \in Int(K).$$

Clearly $0 < V(x) < +\infty$ for any $x \in Int(K)$.

Observe that for any $x \in Int(K)$ and $u \in U$

$$(\mathcal{L}_{x,u} V) = -\frac{1}{g(x)}\,(\mathcal{L}_{x,u} g) + \frac{1}{2}\frac{1}{g^2(x)}|\sigma^\star(x,u)\nabla g(x)|^2. \tag{5}$$

We claim that there exists a constant $L > 0$ such that for all $x \in Int(K)$ there is $u \in U$ satisfying

$$-\frac{1}{g(x)}\,(\mathcal{L}_{x,u} g) + \frac{1}{2}\frac{1}{g^2(x)}|\sigma^\star(x,u)\nabla g(x)|^2 \leq L. \tag{6}$$

Fix $\varepsilon_1 > 0$ small enough. The above relation is clear when $g(x) \geq \varepsilon_1$ because g does not vanish on $Int(K)$. So we need only to prove the claim when $x \in Int(K)$ and $g(x) < \varepsilon_1$.

Consider such a point $x \in Int(K)$ and let $\bar{x} \in \Pi_{\partial K}(x)$. By (b), there exists $u \in U$ with $\sigma(\bar{x}, u)^* \nabla g(\bar{x}) = 0$ and $(\mathcal{L}_{\bar{x},u} g) \geq 0$. Thus,

$$|\sigma^\star(x,u)\nabla g(x)| = |\sigma^\star(x,u)\nabla g(x) - \sigma^\star(\bar{x},u)\nabla g(\bar{x})| \leq C|x-\bar{x}| \quad = Cd_{\partial K}(x) = Cg(x),$$

where C is a Lipschitz constant of the map $y \mapsto \sigma^\star(y,u)\nabla g(y)$ on K. Hence

$$\frac{1}{2}\frac{1}{g(x)^2}|\sigma^\star(x,u)\nabla g(x)|^2 \leq C^2. \tag{7}$$

Moreover since the function $y \mapsto (\mathcal{L}_{y,u} g)$ is Lipschitz continuous on K with some constant $c > 0$, we have

$$-(\mathcal{L}_{x,u} g) \leq (\mathcal{L}_{\bar{x},u} g) - (\mathcal{L}_{x,u} g) \leq c|x-\bar{x}| = cd_{\partial K}(x) = cg(x).$$

Hence

$$-\frac{1}{g(x)}(\mathcal{L}_{x,u_i} g) \leq c \text{ for } x \in Int(K) \text{ such that } g(x) < \varepsilon_1. \tag{8}$$

Since c and C in the estimates (7) and (8) do not depend on $x \in Int(K)$ with $g(x) \leq \varepsilon_1$, we obtain our claim (6).

Consequently, there exists some constant $L > 0$ such that

$$\inf_{u \in U}(\mathcal{L}_{x,u} V) \leq L, \text{ for all } x \in Int(K). \tag{9}$$

We claim that there exists a Borel measurable function $\bar{u} : Int(K) \mapsto U$ such that

$$\inf_{u \in U}(\mathcal{L}_{x,u} V) = \left(\mathcal{L}_{x,\bar{u}(x)} V\right), \forall\, x \in Int(K).$$

Les us prove this claim. Denote by $L(x) := \inf_{u \in U}(\mathcal{L}_{x,u} V)$ (which is continuous on $Int(K)$) and define

$$F(x) := \{u \in U,\ (\mathcal{L}_{x,u} V) = L(x)\}$$

which is a closed nonempty set. We claim that F is weakly Borel measurable in the sense that, for any open set $O \subset U$, the set

$$R := \{x \in Int(K),\ F(x) \cap O \neq \emptyset\}$$

is a Borel set. Indeed, fix an open subset $O \subset U$, and consider an increasing sequence of closed subsubsets $O_n \subset O,\ n \geq 1$, such that $O = \cup_{n \geq 1} \uparrow O_n$. Then the complement of R in $Int(K)$ is

$$\begin{aligned} R^c &= \{x \in Int(K),\ F(x) \cap O = \emptyset\} \\ &= \{x \in Int(K),\ (\mathcal{L}_{x,u} V) > L(x),\ \forall u \in O\} = \cap_{n \geq 1} \downarrow R'_n, \end{aligned}$$

with

$$R'_n := \{x \in Int(K),\ \inf_{u \in O_n}(\mathcal{L}_{x,u} V) > L(x)\},\ n \geq 1.$$

However, as the functions $x \mapsto \inf_{u \in O_n}(\mathcal{L}_{x,u} V)$ and $x \mapsto L(x)$ are both continuous in $Int(K)$, R'_n is a Borel set, for all $n \geq 1$, and consequently, so are R^c and R.

Therefore, denoting by $\mathcal{B}$ the set of all Borel subsets of $Int(K)$, the Kuratowski and Ryll-Nardzeski measurable selection theorem [29] applied to the measurable space $(Int(K), \mathcal{B})$, yields that there exists a Borel measurable selection $\bar{u} : Int(K) \mapsto U$ such that

$$\bar{u}(x) \in F(x),\ \forall x \in Int(K).$$

This proves our claim.

Now we consider an increasing sequence of compact subsets $Q_i \subset Int(K)$ with $\bigcup Q_i = Int(K)$ and $dist(Q_i, \partial K) \geq \rho_i > 0$. Since the function $(x,u) \mapsto (\mathcal{L}_{x,u}V)$ is uniformly continuous on the compact set $Q_i \times U$, there exist $\varepsilon_1 \geq \varepsilon_2 \ldots \geq \varepsilon_i \ldots > 0$ satisfying the following relation:

$$\forall u \in U, \forall x, y \in Q_i, \text{ with } |x-y| < \varepsilon_i, \ |(\mathcal{L}_{x,u}V) - (\mathcal{L}_{y,u}V)| \leq 1. \tag{10}$$

Let $x \in Int(K)$ and $i > 0$ be such that $x \in Int(Q_i)$. We set $\xi_{i,0} = \bar{u}(x)$ and $u^{i,0} = \xi_{i,0} I_{[0,\infty)} \in \mathcal{U}$.

Step I: construction of a suitable control u^i for Q_i. Let us define the following stopping times

$$\tau_{i,0} = 0,$$
$$\tau_{i,1} = \inf\{s \geq \tau_{i,0},\ X_s^{x,u^{i,0}} \notin Q_i \text{ or } |X_s^{x,u^{i,0}} - X_{\tau_{i,0}}^{x,u^{i,0}}| \geq \varepsilon_i\}.$$

We set $\xi_{i,1} = \bar{u}(X_{\tau_{i,1}}^{x,u^{i,0}}) I_{\{\tau_{i,1} < \infty\}} + \xi_{i,0} I_{\{\tau_{i,1} = \infty\}}$ and $u^{i,1} = \xi_{i,0} I_{[0,\tau_{i,1})} + \xi_{i,1} I_{[\tau_{i,1},+\infty)} \in \mathcal{U}$, and we define the new stopping time

$$\tau_{i,2} = \inf\{s \geq \tau_{i,1},\ X_s^{x,u^{i,1}} \notin Q_i \text{ or } |X_s^{x,u^{i,1}} - X_{\tau_{i,1}}^{x,u^{i,1}}| \geq \varepsilon_i\}.$$

Similarly, by iteration, we define a nondecreasing sequence of stopping times $(\tau_{i,k})_{k\geq 0}$ and a sequence of random variables $(\xi_{i,k})_{k\geq 0}$. Now for an arbitrary $u_0 \in U$ define

$$\Theta_i := \lim_{k \uparrow +\infty} \uparrow \tau_{i,k},\ u^i := \sum_{k\geq 0} \xi_{i,k} I_{[\tau_{i,k}, \tau_{i,k+1})} + u_0 I_{[\Theta_i, +\infty)} \in \mathcal{U}.$$

We claim that

$$\Theta_i = \inf\{s \geq 0,\ X_s^{x,u^i} \notin Q_i\}. \tag{11}$$

From the very definition of Θ_i and of the sequence $(\tau_{i,k})_{k\geq 0}$, to prove our claim it is enough to show that the sequence $(\tau_l)_{l\geq 0}$ defined below satisfies $\lim_l \tau_l = +\infty$. To prove (11), fix $u \in \mathcal{U}$ and an arbitrary $\varepsilon > 0$. Let us introduce the (auxiliary) stopping times

$$\tau_0 = 0$$
$$\tau_{l+1} = \inf\{s \geq \tau_l,\ |X_s^{x,u} - X_{\tau_l}^{x,u}| \geq \varepsilon\}, \text{ for } l \geq 0\ .$$

Observe that for the particular choice $u = u^i$ we have $\tau_{i,k} = \tau_k \wedge \Theta_i$. Thus to prove (11), it suffices to show that $\tau_k \uparrow +\infty$ as $k \to \infty$.

Let $T > 0$ be arbitrarily chosen. Then for all $t \in [0, T \wedge 1]$, we deduce from the (conditional) Chebychev inequality and from standard estimates on the trajectory $t \mapsto X_t^{x,u}$ that

$$P[\tau_{l+1} - \tau_l \leq t | \mathcal{F}_{\tau_l}] = P[\sup_{s\in[0,t]} |X_{\tau_l+s}^{x,u} - X_{\tau_l}^{x,u}| \geq \varepsilon | \mathcal{F}_{\tau_l}]$$
$$\leq \frac{1}{\varepsilon^3} E[\sup_{s\in[0,t]} |X_{\tau_l+s}^{x,u} - X_{\tau_l}^{x,u}|^3 | \mathcal{F}_{\tau_l}] \leq C\frac{t^{3/2}}{\varepsilon^3},\ P\text{-a.s. on } \{\tau_l < +\infty\}.$$

Consequently, since b and σ are bounded on K, for all $N \geq 1$,

$$P[\tau_N \leq T] = P[\sum_{l=0}^{N-1} \tau_{l+1} - \tau_l \leq T,\ \tau_j \leq T, \forall j = 0, 1 \ldots N-1]$$
$$\leq \sum_{l=0}^{N-1} P[\tau_{l+1} - \tau_l \leq \frac{T}{N}, \tau_l \leq T] = \sum_{l=0}^{N-1} E[I_{\{\tau_{l+1}-\tau_l \leq \frac{T}{n}\}}\ \ I_{\{\tau_l \leq T\}}]$$
$$= \sum_{l=0}^{N-1} E\{E[I_{\{\tau_{l+1}-\tau_l \leq \frac{T}{N}\}} I_{\{\tau_l \leq T\}} | \mathcal{F}_{\tau_l}]\} = \sum_{l=0}^{N-1} E[P\big(\tau_{l+1} - \tau_l \leq \frac{T}{N} | \mathcal{F}_{\tau_l}\big) I_{\tau_l \leq T}]$$
$$\leq CN\frac{(T/N)^{3/2}}{\varepsilon^3} \to 0, \text{ as } N \to \infty.$$

Thus, $\tau_N \uparrow +\infty$, P-a.s., as $N \to \infty$. Our claim (11) is proved. Moreover, observe that we have obtained

$$P[\bigcup_{k\geq 0} \uparrow \{\tau_{i,k} = \Theta_i,\ \Theta_i < +\infty\}] = P[\Theta_i < +\infty]. \tag{12}$$

Step II: Construction of a suitable control u for $Int(K)$. We iterate the process of Step I as follows for the set Q_{i+1}. We put

$$\begin{aligned}
&\tau_{i+1,0} := \Theta_i,\ \xi_{i+1,0} := \bar{u}(X^{x,u^i}_{\tau_{i+1,0}})I_{\{\tau_{i+1,0}<\infty\}} + \xi_{i,0}I_{\{\tau_{i+1,0}=\infty\}},\\
&u^{i+1,0} := u^i I_{[0,\Theta_i)} + \xi_{i+1,0}I_{[\Theta_i,+\infty)} \in \mathcal{U},\\
&\tau_{i+1,1} := \inf\{s \geq \tau_{i+1,0},\ X^{x,u^{1+i,0}}_s \notin Q_{i+1} \text{ or } |X^{x,u^{i+1,0}}_s - X^{x,u^{i+1,0}}_{\tau_{i+1,0}}| \geq \varepsilon_{i+1}\}\\
&\xi_{i+1,1} := \bar{u}(X^{x,u^{i+1,0}}_{\tau_{i+1,1}})I_{\{\tau_{i+1,1}<\infty\}} + \xi_{i+1,0}I_{\{\tau_{i+1,1}=\infty\}},\\
&u^{i+1,1} := u^i I_{[0,\Theta_i)} + \xi_{i+1,0}I_{[\tau_{i+1,0},\tau_{i+1,1})} + \xi_{i+1,1}I_{[\tau_{i+1,1},+\infty)} \in \mathcal{U},
\end{aligned}$$

etc.

By analogy with step I, we define a control

$$u^{i+1} := u^i I_{[0,\Theta_i)} + \sum_{k\geq 0} \xi_{i+1,k} I_{[\tau_{i+1,k},\tau_{i+1,k+1})} + u_0 I_{[\Theta_{i+1},+\infty)} \in \mathcal{U},$$

and the stopping time

$$\Theta_{i+1} = \lim_{k\uparrow+\infty} \uparrow \tau_{i+1,k} = \inf\{s \geq 0,\ X^{0,u^{i+1}}_s \notin Q_{i+1}\}.$$

Notice that

$$P[\bigcup_{k\geq 0} \uparrow \{\tau_{i+1,k} = \Theta_{i+1},\ \Theta_{i+1} < \infty\}] = P[\Theta_{i+1} < +\infty].$$

So we are now able to define the control u by setting

$$u := \sum_{m\geq i} u^m I_{[\Theta_{m-1},\Theta_m)} + u_0 I_{[\Theta,+\infty)}, \tag{13}$$

where

$$\Theta_{i-1} := 0 \text{ and } \Theta := \lim_m \uparrow \Theta_m.$$

Observe that due to the above construction,

$$\begin{aligned}
\Theta_m &= \inf\{s \geq 0,\ X^{x,u}_s \notin Q_m\},\ \text{for } m \geq i,\\
\Theta &= \inf\{s \geq 0,\ X^{x,u}_s \in \partial K\}.
\end{aligned}$$

Moreover, for $s \in [\tau_{m,k}, \tau_{m,k+1})$ ($m \geq i$, $k \geq 0$) we have $|X^{x,u}_s - X^{x,u}_{\tau_{m,k}}| \leq \varepsilon_m$. Hence,

$$\left(\mathcal{L}_{X^{x,u}_s,u(s)}V\right) = \left(\mathcal{L}_{X^{x,u}_s,\xi_{m,k}}V\right) \leq \left(\mathcal{L}_{X^{x,u}_{\tau_{m,k}},\xi_{m,k}}V\right) + 1 \leq L + 2,$$

in view of (9) and (10). From Itô's formula we obtain, for all $m \geq i$, $t \geq 0$,

$$E[V(X^{x,u}_{t\wedge\Theta_m})] = V(x) + E[\int_0^{t\wedge\Theta_m} \left(\mathcal{L}_{X^{x,u}_s,u(s)}V\right) ds] \leq V(x) + (L+2)t.$$

Consequently, letting $m \to \infty$, Fatou's lemma yields

$$E[\liminf_{m\to\infty} V(X^{x,u}_{t\wedge\Theta_m})] \leq V(x) + (L+2)t,\ t \in [0,+\infty). \tag{14}$$

Recalling that $V(x) = -\ln g(x)$ for $x \in Int(K)$, we observe that $\liminf_{m\to\infty} V(X^{x,u}_{t\wedge\Theta_m}) = +\infty$ on the set $\{\Theta(= \lim_m \uparrow \Theta_m) \leq t\}$. Thus (14) implies that $P(\Theta \leq t) = 0$, i.e., P-a.s. we have $X^{x,u}_s \in Int(K)$ for all $s \in [0,t]$, or in other words

$$P[X^{x,i}_s \in Int(K),\, s \in [0,t]\,] = 1,\, t \geq 0.$$

Replacing t by an increasing sequence $t_n \uparrow +\infty$, from the monotone continuity property of the probability measure P we deduce the viability property of $Int(K)$ on $[0,+\infty)$, our desired result (c).

Let us now prove that (c) implies (a). As in [11], we introduce the value function W defined by (3) for $\lambda > 0$ large enough but fixed. With the assumptions (H1)$-$(H3) one can easily show that W is continuous with polynomial growth (cf [11] and references therein).

Fix $x \in K$, and consider a sequence $(x_k)_{k\geq 0}$ such that

$$x_k \in Int(K),\ \text{for all } k \geq 0, \text{ and } \lim_k x_k = x\ .$$

From (c) we know that for all $k \geq 0$ there exists $u_k \in \mathcal{U}$ with $X^{x_k,u_k}_s \in Int(K)$ for all $s \geq 0$, P-a.s. This yields $W(x_k) = 0$. From the continuity of W we deduce that $W(x) = \lim_k W(x_k) = 0$. Hence in view of Remark 1.2, our desired result (a) is obtained. □

Now we state our second main result under the additional assumption (H5).

Proposition 2.3. *Let $K \subset \mathbb{R}^n$ be a nonempty closed set. Assume* (H1)-(H5)*. Then the set K is viable for (1) if and only if its interior $Int(K)$ is viable for (1).*

Proof : This result can be deduced from Proposition 2.2 by noticing that under $(H5)$ the ε- viability of K is equivalent[1] to the viability of K.

Indeed a viable set K is obviously ε-viable. Therefore $Int(K)$ is viable by Proposition 2.2.

Conversely suppose that $Int(K)$ is viable, thus K is ε-viable. Fix $x \in K$. From (H5), we know by [22] that the infimum in (3) is attained and so there is an admissible control $v(\cdot) \in \mathcal{U}$ on a possibly enlarged probability space such that

$$W(x) = E[\int_0^\infty e^{-\lambda s} d^2_K(X^{x,v}_s)ds].$$

So Remark 1.2 implies that $W(x) = 0$. This yields that $X^{x,v}_s \in K$ for all $s \geq 0$, P-a.e. The proof is complete. □

3 Examples

In this section we discuss two examples where the equivalence between the viability of K and $Int(K)$ fails.

For our result we need in a crucial way the smoothness of the constraint set K. This fact is illustrated in the following

Example 3.1. *Without sufficient regularity of K, it is possible to have the viability property for K while its interior $Int(K)$ is not viable. Indeed, consider the two-dimensional deterministic control problem with constraint set*

$$K := \{(x,y) \in \mathbb{R}^2,\, 0 \leq x \leq \frac{1}{2},\quad 0 \leq y \leq x^2\ \} \cup \big([-1,0] \times \mathbb{R}\big) \subset \mathbb{R}^2,$$

and

$$\begin{cases} dX(t) = -(1 + X(t))dt \\ dY(t) = -\frac{3}{4}u(t)dt, \quad u(t) \in [0,1]\,. \end{cases}$$

[1] This equivalence between ε- viability and viability is valid under (H5) for an arbitrary closed set K, even in the absence of the regularity assumption (H4).

By using Proposition 1.3, one can easily check that K is viable for the above control system.

Moreover, since $X(t) = e^{-t}(1 + X(0)) - 1$, any viable trajectory starting from some $(x_0, y_0) \in K$ with $x_0 > 0$ must reach $(0, 0)$ which does not belong to $Int(K)$. Consequently, $Int(K)$ is not viable.

Note that the lack of regularity of K occurs only at the point $(0, 0)$.

The Example below shows that without Lipschitz continuity of the dynamics Proposition 2.3 fails.

Example 3.2. *Consider the following dynamics on $\mathbb{R}$ (here $n = d = 1$)*

$$dX(s) = 2\sqrt{X(s)}dW(s),\ X(0) = x > 0. \tag{15}$$

with the constraint set $K := [0, \infty)$.

Observe that X is the squared Bessel process. We know that it touches 0 in finite time and 0 is absorbing (cf [27] chap. XI). So K is viable but $Int(K)$ is not viable.

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