# Semiconcavity results and sensitivity relations for the sub-Riemannian distance

Vincenzo Basco, Piermarco Cannarsa, Hélène Frankowska

# Semiconcavity Results and Sensitivity Relations for the Sub-Riemannian Distance

Vincenzo Basco[a,b], Piermarco Cannarsa[a], Hélène Frankowska[b]

[a] *Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata", Via della Ricerca Scientifica, 1 - 00133 Roma, Italy.*
[b] *IMJ-PRG, CNRS, Sorbonne Université, case 247, 4 place Jussieu, 75252 Paris, France.*

**Abstract**

Regularity properties are investigated for the value function of the Bolza optimal control problem with affine dynamic and end-point constraints. In the absence of singular geodesics, we prove the local semiconcavity of the sub-Riemannian distance from a compact set $\Gamma \subset \mathbb{R}^n$. Such a regularity result was obtained by the second author and L. Rifford in [Semiconcavity results for optimal control problems admitting no singular minimizing controls, *Annales de l'IHP Analyse non linéaire 25(4): 2008*] when $\Gamma$ is a singleton. Furthermore, we derive sensitivity relations for time optimal control problems with general target sets $\Gamma$, that is, without imposing any geometric assumptions on $\Gamma$.

## 1. Introduction

Regularity properties of the value function of optimal control problems with finite horizon, in the absence of state constraints, have been widely investigated. For the Mayer and Bolza problems it can be shown that the value function is continuous, Lipschitz continuous, or semiconcave in line with the problem data (see [3, 4, 5, 6, 7, 14]). Even for optimal exit time problems, regularity results are available under suitable controllability assumptions (see [10, 12, 13, 14]). More precisely, let $\Gamma$ be a

*Email addresses:* basco@mat.uniroma2.it (Vincenzo Basco), cannarsa@mat.uniroma2.it (Piermarco Cannarsa), helene.frankowska@imj-prg.fr (Hélène Frankowska)

This work has been partly supported by the INdAM National Group on Mathematical Analysis, Probability, and their Applications. The second author acknowledges support from the MIUR Excellence Department Project, awarded to the Department of Mathematics, University of Rome "Tor Vergata", CUP E83C18000100006. The third author was partially supported by the Air Force Office of Scientific Research under award number FA9550-18-1-0254 and also benefited from the support of the FJMH Program PGMO and from the support to this program from EDF-THALES-ORANGE-CRITEO under grant PGMO 2018-0047H.

compact subset of $\mathbb{R}^n$ and consider the following time minimization problem

$$\begin{cases} \text{minimize } \theta_\Gamma(x(\cdot), u(\cdot)) \\ \text{over all trajectory-control pairs } (x,u)(\cdot) \text{ satisfying} \\ x'(s) = f(x(s), u(s)) \quad \text{for a.e. } s \geqslant 0, \quad x(0) = x_0 \\ u \in L^2(\mathbb{R}^+; \mathbb{R}^m), \end{cases}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a given function, $x_0 \in \mathbb{R}^m$, and $\theta_\Gamma(x(\cdot), u(\cdot)) := \inf\{s \geqslant 0 \,|\, x_{x_0,u}(s) \in \Gamma\}$ is the so-called *transfer time* (to $\Gamma$) along the trajectory $x(\cdot)$ starting from $x_0$ and associated with the control $u(\cdot)$. For any $x_0 \in \mathbb{R}^n$ and any control $u(\cdot)$ we denote by $x_{x_0,u}(\cdot)$ the solution of the Cauchy problem $x'(s) = f(x(s), u(s))$ for a.e. $s \geqslant 0$, $x(0) = x_0$. By convention $\theta_\Gamma(x_{x_0,u}(\cdot), u(\cdot)) = +\infty$ if $x_{x_0,u}(s) \notin \Gamma$ for all $s \geqslant 0$. The set $\Gamma$ is called the *target set* and the value function $\tau_\Gamma(x_0) = \inf\{\theta_\Gamma(x_{x_0,u}(\cdot), u(\cdot)) \,|\, u \in L^2(\mathbb{R}^+; \mathbb{R}^m)\}$ is the *minimum time function*. It is well known (see [14, Chapter 8]) that $\tau_\Gamma$ is locally Lipschitz continuous on the set $\mathscr{A} = \{x_0 \in \mathbb{R}^n \,|\, \exists\, u \in L^2(\mathbb{R}^+; \mathbb{R}^m),\ \theta_\Gamma(x_{x_0,u}(\cdot), u(\cdot)) < \infty\}$ provided that $\Gamma$ satisfies Petrov's condition: there exists $r > 0$ such that for any $y \in \partial\Gamma$ and any proximal unit vector $\nu$ to $\Gamma$ at $y$ we can find $u \in \mathbb{R}^m$ satisfying $\langle f(y,u), \nu \rangle < -r$. In addition, if the target set fulfils the *uniform inner ball property*, then $\tau_\Gamma(\cdot)$ is locally semiconcave on $\mathscr{A}\backslash\Gamma$ and it is locally Lipschitz continuous on $\mathscr{A}\backslash\Gamma$ if and only if $\Gamma$ satisfies Petrov's condition.

Recovering the local semiconcavity property for the minimum time function, associated with the above problem, when the target set does not satisfy the uniform inner ball property, becomes quite challenging. Indeed, let us suppose that $f(x,u) = \sum_{i=1}^m u_i f_i(x)$ and $u(\cdot)$ takes values in the $m$-dimensional closed unit ball, with $f_1, ..., f_m$ smooth ($C^\infty$ or $C^\omega$) vector fields on $\mathbb{R}^n$ and $1 \leqslant m \leqslant n$. Then Petrov's condition may be not satisfied and, if $\Gamma$ is a singleton, the uniform ball property fails. Nevertheless, the minimum time to reach a point is equal to the sub-Riemannian distance $d_{SR}$ from such a point associated with the distribution $\Delta = \text{span}\{f_1, ..., f_m\}$ on the manifold $M = \mathbb{R}^n$ (see [2, 20, 22]). Regularity properties of $d_{SR}$ were obtained for subanalytic structures (see [1, 25], and Section 4). In particular, if the Lie algebra generated by $\Delta$ is regular everywhere, i.e., it satisfies Hörmander's condition (see [19] and Section 4), then for any $x_0$ there exists a dense subset $S_{x_0}$ of $\mathbb{R}^n$ such that for all $y \in S_{x_0}$ the function $d_{SR}(x_0, \cdot)$ is Lipschitz continuous on a suitable open neighborhood of $y$ (see [24, Chapter 2]). One can show (see [11]), assuming furthermore that any geodesics associated with $\Delta$ connecting $x$ to $x_0 \neq x$ is not singular (see Section 2 for the definition), that the function $d_{SR}(x_0, \cdot)$ is locally semiconcave on $\mathbb{R}^n\backslash\{x_0\}$. So, under such assumptions, it follows that for any compact set $\Lambda \subset \mathbb{R}^n$ and any $y \in \mathbb{R}^n\backslash\Lambda$ the function $d_{SR}(y, \cdot)$ is $C(y)$-semiconcave on $\Lambda$. Such a property does not suffice to guarantee the local semiconcavity of $d_{SR}(\Gamma, \cdot) = \inf_{y\in\Gamma} d_{SR}(y, \cdot)$ on $\mathbb{R}^n\backslash\Gamma$, because the semiconcavity constant $C(y)$ might blow up with $y \in \Gamma$. Nevertheless, in this paper we analyze the local semiconcavity property of the function $\inf_{y\in\Gamma} d_{SR}(y, \cdot)$ obtaining uniform bounds on the semiconcavity constant $C(y)$ as $y$ lies in a compact set (see Section 4). More precisely, we will show that for any compact

set $\Gamma \subset \mathbb{R}^n \backslash \Lambda$ there exists a nonnegative constant $C = C(\Lambda, \Gamma)$ such that $d_{SR}(x_0, \cdot)$ is $C$-semiconcave on $\Lambda$ for every $x_0 \in \Gamma$.

In order to obtain the semiconcavity results we assume that there are no singular geodesics and we study the dependence of the semiconcavity constant with respect to the initial point, showing that it is bounded from the above when $x_0$ lies in a compact set. As it was the case in [11], a key point of the reasoning is to show the local invertibility of the end-point map $(x_0, u) \mapsto x_{x_0,u}(T)$, where $T > 0$, and to prove the $C^{1,1}$ regularity of its inverse function (Proposition 4.5). Then, we use a compactness result ensuring that all optimal controls are uniformly Lipschitz continuous and uniformly bounded. The final step consists in combining the local semiconcavity property of the cost functional with the $C^{1,1}$ regularity of the inverse of end-point map.

After establishing semiconcavity, we address sensitivity relations and transversality conditions for the minimum time function associated with an affine control system as above. Such relations are given in the form of the following inclusions

$$\begin{cases} -p(t) \in \partial^P \tau_\Gamma(x_{x_0,\bar{u}}(t)) & \forall\, t \in [0, \tau_\Gamma(x_0)) \\ p(\tau_\Gamma(x_0)) \in \operatorname{Lim\,sup}_{t \to \tau_\Gamma(x_0)-} N^P_{\mathbb{R}^n \backslash \Gamma_t}(x_{x_0,\bar{u}}(t)), \end{cases}$$

where $x_0 \in \mathbb{R}^n \backslash \Gamma$, $\bar{u}(\cdot)$ is an optimal control for $\tau_\Gamma$ at $x_0$, $\Gamma_t = \{y \in \mathbb{R}^n \,|\, \tau_\Gamma(y) \leqslant t\}$, and $p(\cdot)$ solves the adjoint equation $-p'(t) = \mathrm{d}_x f(x_{x_0,\bar{u}}(t), \bar{u}(t))^* p(t)$ for a.e. $t \in [0, \tau_\Gamma(x_0)]$. Sensitivity relations for the minimum time function to reach a set with the *inner ball property* were already investigated (see [4, 9, 18]). We recover, for time optimal control problems, sensitivity relations for the co-state in terms of proximal normal cones (see [26, 9] and Section 2). This is done under the assumption that there are no singular geodesics associated with $\Delta$ and the target set is merely compact. The analysis, that applies to any compact target, is based on the dynamic programming principle and further properties of viscosity solutions of the eikonal equation $|F(x)^* \nabla \tau_\Gamma(x)| - 1 = 0$ for $x \in \Gamma^c$, where $F(x)$ is the matrix which has $f_1(x), ..., f_m(x)$ as column vectors.

The outline of the paper is as follows. Section 2 recalls some basic notations and results from nonsmooth analysis and control theory. In Section 3, we state our main results. We give their proof in Section 4. Finally, in Section 5, we derive sensitivity relations for the minimum time function.

## 2. Preliminaries

Let $(X, |\cdot|_X)$ be a normed space. We denote by $B_X(z, r)$ the open ball centered at $z$ with radius $r > 0$ in $X$ (we write $B_r(z)$ in place of $B_{\mathbb{R}^n}(z, r)$ when no confusion arises) and we set $S^1 = \partial B_X(0, 1)$. For a subset $C \subset X$ we write $\operatorname{int} C$, $\overline{C}$, and $C^c$ for the *interior*, the *closure*, and the *complement* of $C$, respectively. We denote by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the Euclidean norm and the scalar product in $\mathbb{R}^n$, respectively. Let $A \subset X$ be a nonempty subset. The distance from $x$ to $A$ is defined by $d(x, A) = \inf \{|x - y|_X \,|\, y \in A\}$. A function $\varphi : A \subset X \to \mathbb{R}$ is said to be $C$-*semiconcave* (with

linear modulus) on $A$, with $C \geqslant 0$, if it is continuous and

$$\varphi(x+h)+\varphi(x-h)-2\varphi(x) \leqslant C\,|h|^2 \quad \forall\, x,h \in X,\ [x-h,x+h] \subset A.$$

We say that $\varphi$ is *locally semiconcave on* $A$ if for any compact subset $K \subset A$ there exists $C_K \geqslant 0$ such that $\varphi$ is $C_K$-semiconcave on $K$. If $A$ is open, we say that $\varphi \in C^{1,1}$ or $\varphi \in C^{1,1}_{\rm loc}$ if $\varphi$ is continuously differentiable with Lipschitz continuous or locally Lipschitz continuous differential on $A$, respectively. It is well known that any $\varphi \in C^{1,1}_{\rm loc}$ is locally semiconcave. We say that $\phi : X \to X$ has a *sub-linear growth* if there exists $M \geqslant 0$ such that $|\phi(x)|_X \leqslant M(1+|x|_X)$ for all $x \in X$.

For $p \in \mathbb{N}^+$ we denote by $L^p(0,T;\mathbb{R}^n)$ the set of all Lebesgue measurable functions $g : [0,T] \to \mathbb{R}^n$ such that $\|g\|^p_{L^p} := \int_0^T |g(s)|^p\, ds < \infty$, by $C(0,T;\mathbb{R}^n)$ the space of all $\mathbb{R}^n$-valued continuous functions on $[0,T]$, and by $C^p(0,T;\mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued functions on $[0,T]$, $p$-times continuously differentiable.

Let $D \subset \mathbb{R}^n$ be nonempty and $\{A_h\}_{h\in D}$ be a family of nonempty subsets of $\mathbb{R}^n$. The *upper limit* (in the Kuratowski-Painlevé sense) of $A_h$ at $h_0 \in D$, written $\operatorname{Lim\,sup}_{h\to_D h_0} A_h$, is the set of all vectors $v \in \mathbb{R}^n$ such that $\liminf_{h\to_D h_0} d_{A_h}(v) = 0$. If $D = \mathbb{N}^+$, then $\operatorname{Lim\,sup}_{i\to\infty} S(i) := \operatorname{Lim\,sup}_{\substack{y\to 0\\ A}} G(y)$ where $A = \{1/i\}_{i\in\mathbb{N}^+}$ and $G(1/i) := S(i)$.

Let $E$ be a closed subset of $\mathbb{R}^n$ and $x \in E$. We denote by $E^-$ the *negative polar* of the set $E$, i.e. the set $\{y \in \mathbb{R}^n \,|\, \langle\, y, x\,\rangle \leqslant 0 \quad \forall x \in E\}$. The *proximal normal cone* to $E$ at $x$ is the set defined by

$$N^P_E(x) = \left\{p \in \mathbb{R}^n \,|\, \exists \sigma = \sigma(x,p) \geqslant 0 : \langle\, p, y-x\,\rangle \leqslant \sigma\,|y-x|^2 \ \forall y \in E\right\}.$$

Furthermore, $p \in N^P_E(x)$ if and only if there exists $\lambda > 0$ such that $B_{r|p|}(x+rp) \subset E^c$ for all $0 \leqslant r \leqslant \lambda$ (see [26]).

The *contingent cone* to $E$ at $x$ is the set defined by

$$T_E(x) = \{v \in \mathbb{R}^n \,|\, \exists\, t_i \to 0+,\ \exists\, v_i \to v,\ x + t_i v_i \in E\ \forall i\}.$$

It is known that $N^P_E(\xi) \subset T_E(\xi)^-$ for all $\xi \in \partial E$.

Let $\varphi$ be a real valued function on $E$. The *superdifferential* $D^+\varphi(x)$ of $\varphi$ at $x \in E$ is defined as the set of all $p \in \mathbb{R}^n$ satisfying $\operatorname{lim\,sup}_{\substack{y\to x\\ E}} \dfrac{\varphi(y)-\varphi(x)-\langle\, p, y-x\,\rangle}{|y-x|} \leqslant 0$. Moreover, if $\varphi$ is locally semiconcave, then for all $x \in \operatorname{int} E$ holds the following property (see [14, Theorem 3.3.6])

$$\operatorname{co}\, D^*\varphi(x) = D^+\varphi(x), \tag{1}$$

where $D^*\varphi(x) := \{\xi \in \mathbb{R}^n \,|\, \exists x_i \to x,\ \nabla\varphi(x_i) \to \xi\}$ and "co" stands for the convex hull. The *proximal* and *horizontal proximal supergradient* of $\varphi$ at $x$ are the sets defined, respectively, by

$$\partial^P\varphi(x) = \left\{\xi \in \mathbb{R}^n \,|\, (-\xi,1) \in N^P_{\operatorname{hypo}\varphi}(x,\varphi(x))\right\},$$
$$\partial^{\infty,P}\varphi(x) = \left\{\xi \in \mathbb{R}^n \,|\, (-\xi,0) \in N^P_{\operatorname{hypo}\varphi}(x,\varphi(x))\right\},$$

where $\operatorname{hypo}\varphi$ denotes the *hypograph* of the function $\varphi$. For further properties of superdifferentials and proximal cones we refer to [16, 26].

## 3. Main Result

Let $1 \leqslant m \leqslant n$ be two natural numbers. Consider the optimal control problem

$$\text{minimize } \int_0^t L(x_{x_0,u}(s), u(s))\, ds \tag{2}$$

over all controls $u \in L^2(0,t;\mathbb{R}^m)$ such that the solution $x_{x_0,u}(\cdot)$ of the affine control system

$$\begin{cases} x'(s) = \sum_{i=1}^m u_i(s)\, f_i(x(s)) \quad \text{for a.e. } s \in [0,t] \\ x(0) = x_0 \end{cases} \tag{3}$$

satisfies the end-point constraint

$$x_{x_0,u}(t) = y, \tag{4}$$

where $(t,y) \in [0,\infty) \times \mathbb{R}^n$ and $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are given functions. We say that a control $v \in L^2(0,t;\mathbb{R}^m)$ *steers* $x_0$ *to* $y$ *in time* $t$ if $x_{x_0,v}(t) = y$. The infimum of the cost functional in (2) over all controls steering $x_0$ to $y$ in time $t$ is denoted by $V_{x_0}(t,y)$ (if there are no controls steering $x_0$ to $y$ in time $t$, we set $V_{x_0}(t,y) = +\infty$). The function $V_{x_0} : [0,\infty) \times \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is called the *value function* of the problem (2)-(4) with starting point $x_0$. A control $v \in L^2(0,t;\mathbb{R}^m)$ is said to be an *optimal control* or a *minimizer* (for the problem (2)-(4)) at $(x_0,t,y)$ if $x_{x_0,v}(t) = y$ and $V_{x_0}(t,y) = \int_0^t L(x_{x_0,v}(s), v(s))\, ds$. We denote by $U_{x_0}(s,y)$ the (possibly empty) set of all optimal controls steering $x_0$ to $y$ in time $s$.

Let us denote by (H) the following assumptions:

(H) (i) $f_1, ..., f_m$ are $C^2$ vector fields on $\mathbb{R}^n$ with sub-linear growth and Lipschitz continuous differential;

(ii) $L \in C^2$ and $\nabla_u^2 L(x,u) > 0$ for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$;

(iii) $G$ is a given nonempty compact subset of $\mathbb{R}^n$ and the following set is nonemepty

$$\mathscr{D}_G := \{(t,x) \in [0,\infty) \times \mathbb{R}^n \,|\, V_{x_0}(t,x) < +\infty \ \forall x_0 \in G\}\,;$$

(iv) there exists a nonempty open subset $\Omega_G \subset [0,\infty) \times \mathbb{R}^n$ such that $\Omega_G \subset \mathscr{D}_G$;

(v) there exist $c \geqslant 0$ and a function $\phi : [0,\infty) \to \mathbb{R}^+$ such that

$$\liminf_{r\to\infty} \phi(r)/r^2 > 0 \quad \& \quad L(x,u) \geqslant \phi(|u|) - c \quad \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^m,$$

and for any $r > 0$

$$\sup \left\{ \frac{|\nabla_x L(x,u)|}{\phi(|u|)} \,|\, x \in B_r(0),\ u \in \mathbb{R}^m \right\} < \infty.$$

**Remark 3.1.** Assume (H) and that $|L(x,u)| \leqslant \varphi(x)(1+|u|^2)$, where $\varphi(\cdot)$ is a locally bounded function on $\mathbb{R}^n$. Then $(t,x) \in \mathscr{D}_G$ if for any $x_0 \in G$ there exists a square integrable control $u : [0,t] \to \mathbb{R}^m$ steering $x_0$ to $x$ in time $t$.

Let $T > 0$ and $\mathscr{W} \subset L^2(0,T;\mathbb{R}^m)$ be such that all solutions of (3), with $u \in \mathscr{W}$, are well defined on $[0,T]$. The *end-point map* associated to the system (3) at time $T$, written $E_T$, is the function given by

$$E_T(x_0,u) = x_{x_0,u}(T) \qquad \forall\, (x_0,u) \in \mathbb{R}^n \times \mathscr{W}.$$

It can be proved that, if the vector fields $f_1, ..., f_m$ are smooth, then $\mathscr{W}$ can be chosen to be open (see [2, 22]).

**Definition 3.2.** A control $u \in L^2(0,t;\mathbb{R}^m)$ is said to be *singular* at $x_0$ if $\mathrm{d}E_t(x_0,u)(0,\cdot)$ is not a surjective map on $L^2(0,t;\mathbb{R}^m)$.

**Definition 3.3.** If (H)(iii)-(iv) hold true, we say that the problem (2)-(4) does *not admit singular minimizers* (on $G$) if any $u \in U_{x_0}(t,y)$ is not singular whenever $(t,y) \in \Omega_G$, $y \notin G$, and $x_0 \in G$.

We state next the main result.

**Theorem 3.4.** *Assume* (H) *and suppose that the problem* (2)-(4) *does not admit singular minimizers.*

*Then, for any compact subset $\Gamma \subset \Omega_G$, there exists a constant $C = C(G,\Gamma) \geqslant 0$ such that the value function $V_{x_0}(\cdot,\cdot)$ is $C$-semiconcave on $\Gamma$ for all $x_0 \in G$.*

Now, let us denote by (H)′ the following assumptions on $f_1, ..., f_m$:

(H)′ (i) $f_1, ..., f_m$ are smooth vector fields ($C^\infty$ or $C^\omega$) satisfying Hörmander's condition, i.e.,

$$\operatorname{span}\left\{X^i(x)\right\}_{i\geqslant 1} = \mathbb{R}^n \quad \forall\, x \in \mathbb{R}^n,$$

where $X^1(x) = \{f_1(x), ..., f_m(x)\}$, $X^{i+1}(x) = X^i(x) \cup \{[f,g](x) \,|\, f \in X^1(x), g \in X^i(x)\}$ for all $i \in \mathbb{N}^+$ ($[\cdot,\cdot]$ denotes the Lie bracket);

(ii) $f_1, ..., f_m$ have sub-linear growth, Lipschitz continuous differential, and $f_1(x), ..., f_m(x)$ are linearly independent for all $x \in \mathbb{R}^n$.

If (H)′-(i) holds true, by the Chow-Rashevsky theorem (see [15, 23]), for any $x_0, y \in \mathbb{R}^n$ there exists an absolutely continuous arc $x : [0,1] \to \mathbb{R}^n$, with square integrable derivative, such that $x(0) = x_0$, $x(1) = y$, and

$$x'(t) \in \operatorname{span}\{f_1(x(t)), ..., f_m(x(t))\} \quad \text{for a.e. } t \in [0,1]. \tag{5}$$

An absolutely continuous arc on $[0,1]$ satisfying (5), with square integrable derivative, is said to be an *horizontal arc*.

Let us denote by $\mathscr{S}(x_0,x)$ the set of all horizontal arcs $\beta$ such that $\beta(0) = x_0$ and $\beta(1) = x$. Then, if (H)′ holds true, there exists a bijection between $\mathscr{S}(x_0,x)$ and

$L^2(0,1;\mathbb{R}^m)$ such that for any $\beta \in \mathscr{S}(x_0,x)$ there exists a unique $u_\beta \in L^2(0,1;\mathbb{R}^m)$ satisfying $\beta'(s) = \sum_{i=1}^m (u_\beta(s))_i f_i(\beta(s))$ for a.e. $s \in [0,1]$. We can associate to any horizontal arc $[0,1] \ni t \mapsto \beta(t)$ its length given by $l(\beta) = \int_0^1 |u_\beta(t)|\, dt$, and the *sub-Riemannian distance* between $x_0$ and $x$, written $d_{SR}(x_0,x)$, is $\inf\{l(\beta) \,|\, \beta \in \mathscr{S}(x_0,x)\}$. The following result is very useful (see [2]):

**Proposition 3.5.** *Assume* (H)′-(i). *For any* $x_0$, $x \in \mathbb{R}^n$

$$d_{SR}(x_0,x)^2 = e(x_0,x) := \inf\left\{\int_0^1 |u_\beta(t)|^2\, dt \,|\, \beta \in \mathscr{S}(x_0,x)\right\}.$$

The function $e(\cdot,\cdot)$ is said to be the *sub-Riemannian energy*, and an horizontal arc minimizing $e(x_0,x)$ is said to be a *geodesic* steering $x_0$ to $x$. A geodesic $\beta$ is called a *singular geodesic* (or *singular*) if the associated control $u_\beta$ is singular.

Consider the following minimization problem

$$\mathcal{E}_{x_0}(t,x) := \inf\left\{\int_0^t |u(s)|^2\, ds \,|\, u \in L^2(0,t;\mathbb{R}^m),\ x_{x_0,u}(t) = x\right\},$$

where $(t,x) \in [0,\infty) \times \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$. Then, by Proposition 3.5, it follows that $d_{SR}(x_0,x) = \sqrt{\mathcal{E}_{x_0}(1,x)}$ for all $x \in \mathbb{R}^n$, and, assuming that any geodesic connecting $x$ to $x_0 \neq x$ is not singular, by [11, Theorem 5] the function $d_{SR}(x_0,\cdot)$ is locally semiconcave on $\mathbb{R}^n \setminus \{x_0\}$. We would like to underline that the infimum of a family of semiconcave functions is not in general a semiconcave function. When each member of the family is semiconcave with same constant then the infimum is semiconcave too.

**Lemma 3.6** ([14])**.** *Let* $\{u \,|\, u \in \mathscr{Z}\}$ *be a family of* $C$*-semiconcave functions on* $\Gamma \subset \mathbb{R}^n$ *and put* $w(x) = \inf_{u\in\mathscr{Z}} u(x)$. *If* $w(x) \neq -\infty$ *for all* $x \in \Gamma$ *then* $w(\cdot)$ *is* $C$*-semiconcave on* $\Gamma$.

For any compact set $\Gamma \subset \mathbb{R}^n$, the sub-Riemannian distance between $\Gamma$ and $x$ is

$$d_{SR}(\Gamma,x) = \inf_{x_0\in\Gamma} d_{SR}(x_0,x).$$

**Definition 3.7.** We say that *there are no singular geodesics for* $\Gamma$ (associated to the distribution spanned by $f_1,...,f_m$) if any geodesics connecting $x$ to $y$ is not singular whenever $x \in \Gamma$ and $y \in \Gamma^c$.

Finally, in light of Proposition 3.5, Lemma 3.6, and Theorem 3.4, we get the following result:

**Corollary 3.8.** *Assume* (H)′. *Let* $\Gamma \subset \mathbb{R}^n$ *be a compact set and suppose that there are no singular geodesics for* $\Gamma$.

*Then* $d_{SR}(\Gamma,\cdot)$ *is locally semiconcave on* $\Gamma^c$.

## 4. Proof of the Main Result

We provide here the proof of our main result deferring technical details to the appendix. For any $k > 0$, $T > 0$, $x_0 \in \mathbb{R}^n$, and $\Gamma \subset [0,\infty) \times \mathbb{R}^n$, we introduce the following notation

$$\mathscr{U}_{x_0}(\Gamma) = \bigcup_{(s,y)\in\Gamma} U_{x_0}(s,y),$$

$$\mathscr{L}_k^T = \{u : [0,T] \to \mathbb{R}^m \mid \|u\|_\infty \leqslant k \text{ and } u \text{ is } k\text{-Lipschitz continuous}\}.$$

We equip the set $\mathscr{L}_k^T$ with the uniform norm.

**Lemma 4.1.** *Assume* (H) *and let $T > 0$. Then, for any $k' \geqslant 0$ there exists $K \geqslant 0$ such that for any $z \in G$ the map*

$$[0,T] \times \mathscr{L}_{k'}^T \ni (t,u) \mapsto \int_0^t L(x_{z,u}(s), u(s))\, ds \tag{6}$$

*is $K$-Lipschitz continuous and $K$-semiconcave.*

*Proof.* By Remark 6.1 from the Appendix, there exists $r = r(k', G) > 0$ such that $\|x_{z,u}\|_\infty \leqslant r$ for all $z \in G$ and all $u \in \mathscr{L}_{k'}^T$. Consider $\alpha > 0$, $\tilde{M} > 0$, depending on $k'$ and $G$, such that for all $x, y \in B_r(0)$ and all $u, w \in B_{k'}(0)$

$$|L(x,u) - L(y,w)| \leqslant \alpha\,(|x-y| + |u-w|) \qquad \& \qquad |L(x,u)| \leqslant \tilde{M}, \tag{7}$$

and for all $x, \eta$ with $[x+\eta, x-\eta] \subset B_{3r}(0)$ and every $u \in B_{k'}(0)$

$$L(x+\eta,u) + L(x-\eta,u) - 2L(x,u) \leqslant \alpha\,|\eta|^2. \tag{8}$$

Fix $z \in G$. Denote for simplicity the map in (6) by $C_t(u)$. Then, from Lemma 6.2 and (7), there exists $\sigma = \sigma(k', G) > 1$ such that for any $0 \leqslant s \leqslant t \leqslant T$, and any $u, w \in \mathscr{L}_{k'}^T$

$$\begin{aligned}
|C_t(u) - C_s(w)| &\leqslant \int_s^t |L(x_{z,w}(\xi), w(\xi))|\, d\xi \\
&\quad + \int_0^s |L(x_{z,u}(\xi), u(\xi)) - L(x_{z,w}(\xi), w(\xi))|\, d\xi \\
&\leqslant \tilde{M}\,|t-s| + \alpha \int_0^s (|x_{z,u}(\xi) - x_{z,w}(\xi)| + |u(\xi) - w(\xi)|)\, d\xi \\
&\leqslant \left(\tilde{M} + \alpha\sigma T\right)(|t-s| + \|u-w\|_\infty).
\end{aligned}$$

Now, let $t, h, u$, and $v$ be such that $[t-h, t+h] \subset [0,T]$ and $u - v, u + v \in \mathscr{L}_{k'}^T$. We have

$$\begin{aligned}
&C_{t+h}(u+v) + C_{t-h}(u-v) - 2C_t(u) \\
&= C_t(u+v) + C_t(u-v) - 2C_t(u) \\
&\quad + C_{t+h}(u+v) + C_{t-h}(u-v) - C_t(u+v) - C_t(u-v).
\end{aligned} \tag{9}$$

Then,

$$
\begin{aligned}
&C_t(u+v)+C_t(u-v)-2C_t(u)\\
&=\int_0^t \left(L(x_{z,u+v}(s),u(s)+v(s))+L(x_{z,u-v}(s),u(s)-v(s))-2L(x_{z,u}(s),u(s))\right)\,ds\\
&=\int_0^t \left(L(x_{z,u+v}(s),u(s))+L(2x_{z,u}(s)-x_{z,u+v}(s),u(s))-2L(x_{z,u}(s),u(s))\right)\,ds\\
&\quad+\int_0^t \left(L(x_{z,u+v}(s),u(s)+v(s))-L(x_{z,u+v}(s),u(s))\right)\,ds\\
&\quad+\int_0^t \left(L(x_{z,u-v}(s),u(s)-v(s))-L(x_{z,u-v}(s),u(s))\right)\,ds\\
&\quad+\int_0^t \left(L(x_{z,u-v}(s),u(s))-L(2x_{z,u}(s)-x_{z,u+v}(s),u(s))\right)\,ds.
\end{aligned}
\tag{10}
$$

From (8) and Lemma 6.2, there exists a constant $\sigma_0=\sigma_0(k',G)>0$ such that

$$
\int_0^t \left(L(x_{z,u+v}(s),u(s))+L(2x_{z,u}(s)-x_{z,u+v}(s),u(s))-2L(x_{z,u}(s),u(s))\right)\,ds\leqslant \sigma_0\left\|v\right\|_\infty^2.
$$

According to Remark 6.4 below, there exists $\sigma_1=\sigma_1(k',G)>0$ such that $|x_{z,u+v}(s)+x_{z,u-v}(s)-2x_{z,u}(s)|\leqslant\sigma_1\left\|v\right\|_{L^2}^2$ for all $s\in[0,T]$. Hence, by (7),

$$
\begin{aligned}
&\int_0^t \left(L(x_{z,u-v}(s),u(s))-L(2x_{z,u}(s)-x_{z,u+v}(s),u(s))\right)\,ds\\
&\leqslant \alpha\int_0^t |x_{z,u+v}(s)+x_{z,u-v}(s)-2x_{z,u}(s)|\,ds\\
&\leqslant \sigma_2\left\|v\right\|_\infty^2,
\end{aligned}
$$

where $\sigma_2=\sigma_2(k',G)>0$. For the second and third term in (10) we have, using the regularity of the Lagrangian in the second variable, Lemma 6.2, and the Cauchy-Schwarz inequality, that

$$
\begin{aligned}
&\int_0^t \left(L(x_{z,u+v}(s),u(s)+v(s))-L(x_{z,u+v}(s),u(s))\right)\,ds\\
&\qquad+\int_0^t \left(L(x_{z,u-v}(s),u(s)-v(s))-L(x_{z,u-v}(s),u(s))\right)\,ds\\
&=\int_0^t\int_0^1 \langle\,\nabla_u L(x_{z,u+v}(s),\xi v(s)+u(s)),v(s)\rangle d\xi\,ds\\
&\qquad-\int_0^t\int_0^1 \langle\,\nabla_u L(x_{z,u-v}(s),-\xi v(s)+u(s)),v(s)\rangle\,d\xi\,ds\\
&\leqslant \sigma_3\int_0^t \left(|x_{z,u+v}(s)-x_{z,u-v}(s)|+|v(s)|\right)|v(s)|\,ds\\
&\leqslant \sigma_4\left\|v\right\|_\infty^2,
\end{aligned}
$$

where $\sigma_i = \sigma_i(k', G) > 0$ for $i = 3, 4$. The above relations and (10) imply that

$$C_t(u+v) + C_t(u-v) - 2C_t(u) \leqslant \sigma_5 \|v\|_\infty^2 \tag{11}$$

for a suitable $\sigma_5 = \sigma_5(k', G) > 0$.

On the other hand,

$$\begin{aligned}
&C_{t+h}(u+v) + C_{t-h}(u-v) - C_t(u+v) - C_t(u-v)\\
&= \int_t^{t+h} (L(x_{z,u+v}(s), u(s)+v(s)) - L(x_{z,u+v}(t), u(s)+v(s)))\, ds\\
&\quad + \int_t^{t+h} (L(x_{z,u+v}(t), u(s)+v(s)) - L(x_{z,u+v}(t), u(s)-v(s)))\, ds\\
&\quad + \int_t^{t+h} (L(x_{z,u+v}(t), u(s)-v(s)) - L(x_{z,u-v}(t), u(s)-v(s)))\, ds\\
&\quad + \int_t^{t+h} L(x_{z,u-v}(t), u(s)-v(s))\, ds - \int_{t-h}^{t} L(x_{z,u-v}(s), u(s)-v(s))\, ds,
\end{aligned}$$

and, on account of (7) and the Lipschitz regularity of trajectories, the first three terms are bounded by $3C\left(h\|v\|_\infty + h^2\right)$, while, since $u - v \in \mathscr{L}^T_{2k'}$, there exists a constant $M = M(k', G) > 0$ satisfying

$$\begin{aligned}
&\int_t^{t+h} L(x_{z,u-v}(t), u(s)-v(s))\, ds - \int_{t-h}^{t} L(x_{z,u-v}(s), u(s)-v(s))\, ds\\
&= \int_t^{t+h} (L(x_{z,u-v}(t), (u-v)(s)) - L(x_{z,u-v}(s-h), (u-v)(s-h)))\, ds\\
&\leqslant Mh^2.
\end{aligned}$$

Then

$$C_{t+h}(u+v) + C_{t-h}(u-v) - C_t(u+v) - C_t(u-v) \leqslant 2(3C+M)\left(\|v\|_\infty^2 + h^2\right).$$

This and (9) and (11) complete the proof. □

**Remark 4.2.** Arguing in a similar way as in the first part of the proof of Lemma 4.1 and using Lemma 6.2, we have that, for any $T > 0$, the map

$$\mathbb{R}^+ \times C(0, T; \mathbb{R}^m) \times \mathbb{R}^n \ni (t, u, z) \mapsto \int_0^t L(x_{z,u}(s), u(s))\, ds$$

is continuous.

**Lemma 4.3.** *Assume* (H) *and suppose that the problem* (2)-(4) *does not admit singular minimizers.*

*Then the function* $(x_0, t, x) \mapsto V_{x_0}(t, x)$ *is continuous on* $G \times \mathscr{D}_G$.

*Proof.* Let $(\bar{x}_0, \bar{t}, \bar{x}) \in G \times \mathscr{D}_G$ and consider a sequence $(z_i, t_i, x_i) \to (\bar{x}_0, \bar{t}, \bar{x})$ in $G \times \mathscr{D}_G$ such that

$$\lim_{i\to\infty} V_{z_i}(t_i, x_i) = l \in \mathbb{R} \cup \{\pm\infty\}.$$

By [21, Theorem 8, Chap. 4]), for every $i \geqslant 1$ there exists a square integrable control $u_i(\cdot)$ such that $V_{z_i}(t_i, x_i) = \int_0^{t_i} L(x^i(s), u_i(s))\, ds$, where $x^i(\cdot)$ denotes the trajectory $x_{x_i,u_i}(\cdot)$ such that $x_{z_i,u_i}(t_i) = x_i$. Without loss of generality we can suppose that for all large $i$ the controls $\{u_i\}_i$ are defined on $[0, \bar{t}+1]$ putting $u_i \equiv 0$ on $[t_i, \bar{t}+1]$.

We first show that $V_{\bar{x}_0}(\bar{t}, \bar{x}) \geqslant l$. Notice that $V_{\bar{x}_0}(\bar{t}, \bar{x}) \neq +\infty$. Let $v(\cdot)$ be an optimal control steering $\bar{x}_0$ to $\bar{x}$ in time $\bar{t}$. From [11, Lemma 3], $v(\cdot)$ is continuous. Moreover, according to Lemma 6.6 below, there exists $\{v_j\}_{j=1}^n \subset C(0, \bar{t}; \mathbb{R}^m)$ such that the map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, defined by $\varphi(\beta) = \sum_{j=1}^n \beta_j \mathrm{d}E_{\bar{t}}(\bar{x}_0, v)(0, v_j)$, is an isomorphism. Then, from Lemma 6.2 and Corollary 6.5, we conclude that the map $\mathscr{E} : \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{R}^n$ defined by

$$\mathscr{E}(y, s, \beta) = \Big(y, s, E_s(y, v + \sum_{j=1}^n \beta_j v_j)\Big)$$

is $C^1$ in a neighborhood of $(\bar{x}_0, \bar{t}, 0)$ and $\det \mathrm{d}\mathscr{E}(\bar{x}_0, \bar{t}, 0) \neq 0$. So, applying the Inverse Function Theorem, the map $\mathscr{E}$ is open in a neighborhood of $(\bar{t}, \bar{x}_0, 0)$. It means that any point $(z_i, t_i, x_i)$, sufficiently close to $(\bar{x}_0, \bar{t}, \bar{x})$, admits a control $w^i = v + \sum_{j=1}^n \beta_j v_j$ close to $v$ in $C(0, \bar{t}+1; \mathbb{R}^m)$ such that $x_{z_i,w^i}(t_i) = x_i$. By Remark 4.2, we have that $\lim_i \int_0^{t_i} L(x_{z_i,w^i}(s), w^i(s))\, ds = \int_0^{\bar{t}} L(x_{\bar{x}_0,v}(s), v(s))\, ds$. So, since $V_{z_i}(t_i, x_i) \leqslant \int_0^{t_i} L(x_{z_i,w^i}(s), w^i(s))\, ds$ for all $i$, passing to the limit we deduce that $l \leqslant V_{\bar{x}_0}(\bar{t}, \bar{x})$. Consequently $l < +\infty$.

We next prove that $V_{\bar{x}_0}(\bar{t}, \bar{x}) \leqslant l$. By assumptions on $\phi(\cdot)$, there exists $\alpha$, $C > 0$ such that $\alpha r^2 \leqslant \phi(r)$ for all $r \geqslant C$. So

$$\begin{aligned}
\|u_i\|_{L^2}^2 &= \int_{[0,\bar{t}+1]\cap\{s:|u_i(s)|\geqslant C\}} |u_i(s)|^2\, ds + \int_{[0,\bar{t}+1]\cap\{s:|u_i(s)|\leqslant C\}} |u_i(s)|^2\, ds \\
&\leqslant \alpha^{-1} \int_{[0,\bar{t}+1]\cap\{s:|u_i(s)|\geqslant C\}} \phi(|u_i(s)|)\, ds + C^2(\bar{t}+1) \\
&\leqslant \alpha^{-1} \int_0^{\bar{t}+1} L(x^i(s), u_i(s))\, ds + (c + C^2)(\bar{t}+1).
\end{aligned}$$

Since $l < +\infty$, $\{\|u_i\|_{L^2}\}_i$ is bounded. By further extraction of a subsequence and from Gronwall's lemma and the Ascoli-Arzelà theorem, keeping the same notation, we have that $u_i \rightharpoonup \bar{u}$ in $L^2(0, \bar{t}+1; \mathbb{R}^m)$ and $x^i(\cdot)$ converges uniformly on $[0, \bar{t}+1]$ to an absolutely continuous trajectory $y(\cdot) := x_{\bar{x}_0,\bar{u}}(\cdot)$. Now, since $|y(t_i) - \bar{x}| \leqslant |y(t_i) - x^i(t_i)| + |x^i(t_i) - \bar{x}|$, we conclude that $\lim_i |y(t_i) - \bar{x}| = 0$. So $y(\bar{t}) = \bar{x}$. Then, from the convexity of $L$ with respect to the second variable (see [21, proof of Theorem 8 Chap. 3]), we deduce that $\lim_i \int_0^{t_i} L(x^i(s), u_i(s))\, ds \geqslant \int_0^{\bar{t}} L(y(s), \bar{u}(s))\, ds$. Hence $V_{\bar{x}_0}(\bar{t}, \bar{x}) \leqslant l$.

□

From the proof of [11, Lemma 3], Remark 4.2, and Lemma 4.3, we get the following compactness result.

**Lemma 4.4.** *Assume* (H) *and suppose that the problem* (2)-(4) *does not admit singular minimizers.*

*Then, for any nonempty compact subset* $\Gamma \subset \Omega_G$*, we have* $U_{x_0}(s,y) \neq \emptyset$ *for all* $(s,y) \in \Gamma$ *and* $x_0 \in G$*, and there exists* $k = k(G,\Gamma) > 0$ *such that*

$$\bigcup_{x_0 \in G} \mathscr{U}^{x_0}(\Gamma) \subset \mathscr{L}^T_k.$$

We give next an inverse mapping result for the end-point map.

**Proposition 4.5.** *Assume* (H) *and suppose that the problem* (2)-(4) *does not admit singular minimizers. Let* $\Gamma \subset \Omega_G$ *be a nonempty compact subset and define*

$$T = \sup\{t > 0 \,|\, \exists\, x \in \mathbb{R}^n,\, (t,x) \in \Gamma\},$$

$$\Lambda = \left\{(t,z,u) \in [0,\infty) \times G \times L^2(0,T;\mathbb{R}^m) \,|\, \exists\, x \in \mathbb{R}^n,\, u \in U_z(t,x), (t,x) \in \Gamma\right\}.$$

*If for some* $k > 0$

$$\bigcup_{x_0 \in G} \mathscr{U}^{x_0}(\Gamma) \subset \mathscr{L}^T_k, \tag{12}$$

*then there exist* $k' \geqslant k$*,* $r > 0$*, and* $\ell \geqslant 0$ *such that for any* $(t,z,u) \in \Lambda$ *we can find a map*

$$F_{t,z,u} : B_r(t) \times B_r(z) \times B_r(x_{z,u}(t)) \to \mathscr{L}^T_{k'},$$

*satisfying for all* $(t,z,u) \in \Lambda$*:*

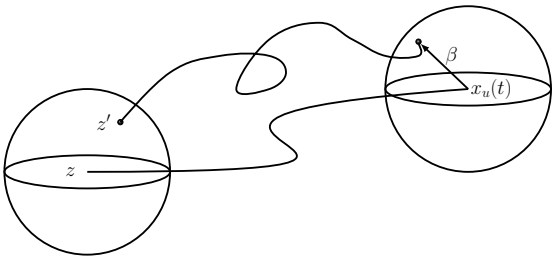
*(i)* $F_{t,z,u} \in C^{1,1}$*;*

*(ii)* $E_s(z', F_{t,z,u}(s,z',\beta)) = \beta$ *for all* $(s,z',\beta) \in B_r(t) \times B_r(z) \times B_r(x_{z,u}(t))$*;*

*(iii)* $\mathrm{d}F_{t,z,u}$ *is* $\ell$*-Lipschitz.*

*Proof.* Let $(t_0,z_0,u_0) \in \Lambda$. We know that $\mathrm{d}E_{t_0}(z_0,u_0)(0,\cdot)$ is surjective on $L^2(0,T;\mathbb{R}^m)$. Let $\mathscr{V} \subset C^1(0,T;\mathbb{R}^m)$ be a countable subset such that $\overline{\mathrm{span}\mathscr{V}} = L^2(0,T;\mathbb{R}^m)$. By Lemma 6.6, there exist $n$ linearly independent vectors $\{v^0_1,...,v^0_n\} \subset \mathscr{V}$ such that the map $A_0 : \mathbb{R}^n \to \mathbb{R}^n$, defined by $A_0(\alpha) = \sum_{i=1}^n \alpha_i \mathrm{d}E_{t_0}(z_0,u_0)(0,v^0_i)$, is an isomorphism. Define for any $(t,z,u) \in (0,\infty) \times \mathbb{R}^n \times L^2(0,T;\mathbb{R}^m)$ the map $\varphi^0_{t,z,u} : \mathbb{R}^n \to \mathbb{R}^n$ by $\varphi^0_{t,z,u}(\alpha) = \sum_{i=1}^n \alpha_i \mathrm{d}E_t(z,u)(0,v^0_i)$. By Lemma 6.2 and Corollary 6.5, there exist $\varrho_0 > 0$, $\mu_0 > 0$ such that for any $(t,z,u) \in \mathcal{J}_0 := B_{\varrho_0}(t_0) \times B_{\varrho_0}(z_0) \times B_{L^2}(u_0,\varrho_0)$ the map $\mathscr{E}^0_{t,z,u} : (0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to (0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n$, defined by $\mathscr{E}^0_{t,z,u}(s,y,\alpha) = (s,y,E_s(y,u+\sum_{i=1}^m \alpha_i v^0_i))$, satisfies for all $(t,z,u) \in \mathcal{J}_0$

$$|\det\left(\mathrm{d}\mathscr{E}^0_{t,z,u}(t,z,0)\right)| = \left|\det \varphi^0_{t,z,u}\right| \geqslant \mu_0.$$




Now, from (12) and the Ascoli-Arzelà theorem, the set $\bigcup_{x_0\in G}\mathscr{U}^{x_0}(\Gamma)$ is compact. Then there exists $N\in\mathbb{N}^+$ such that, for all $j=1,...,N$, we can find $\rho_j>0$, $\mu_j>0$, $(t_j,z_j,u_j)\in\Lambda$, and linearly independent $\left\{v_1^j,...,v_n^j\right\}\subset\mathscr{V}$, such that

$$\Lambda\subset\bigcup_{j=1,...,N}B_{\varrho_j}(t_j)\times B_{\varrho_j}(z_j)\times B_{L^2}(u_j,\varrho_j)=:\bigcup_{j=1,...,N}\mathcal{J}_j.$$

Defining for any $(t,z,u)\in\mathcal{J}_j$ the maps $\mathscr{E}^j_{t,z,u}:(0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n\to(0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n$ and $\varphi^j_{t,z,u}:\mathbb{R}^n\to\mathbb{R}^n$ by $\mathscr{E}^j_{t,z,u}(s,y,\alpha)=(s,y,E_s(y,u+\sum_{i=1}^m\alpha_iv_i^j))$ and $\varphi^j_{t,z,u}(\alpha)=\sum_{i=1}^n\alpha_i\mathrm{d}E_t(z,u)(0,v_i^j)$, we deduce that for all $(t,z,u)\in\mathcal{J}_j$ and $j=1,...,N$

$$|\det\left(\mathrm{d}\mathscr{E}^j_{t,z,u}(t,z,0)\right)|=\left|\det\varphi^j_{t,z,u}\right|\geqslant\mu_j\geqslant\min\left\{\mu_1,...,\mu_N\right\}>0. \tag{13}$$

Applying the Inverse Mapping Theorem to the map $\mathscr{E}^j_{t,z,u}$ and using a compactness argument, we conclude that for each $j$ there exists $r_j>0$ such that the set $\mathbb{V}_j(t,z,u):=(t-r_j,t+r_j)\times B_{r_j}(z)\times B_{r_j}(E_t(z,u))$ is isomorph to $\left(\mathscr{E}^j_{t,z,u}\right)^{-1}(\mathbb{V}_j(t,z,u))$ for any $(t,z,u)\in\mathcal{J}_j$. Put $r=\min\left\{r_1,...,r_N\right\}$ and define for any $(t,z,u)\in\mathcal{J}_j$,

$$F_{t,z,u}(t',z',\beta)=u+\sum_{i=1}^m\alpha_i(t',z',\beta)v_i^j\qquad\forall(t',z',\beta)\in\mathbb{V}_j(t,z,u),$$

where $\left(\mathscr{E}^j_{t,z,u}\right)^{-1}(t',z',\beta)=(t',z',\alpha(t',z',\beta))$. Notice that, since the coefficients $\alpha_i$ are bounded by a suitable constant $M\geqslant0$ and $v_i^j\in C^1(0,T;\mathbb{R}^m)$, there exists a constant $k'\geqslant k$ such that $F_{t,z,u}$ take values in $\mathscr{L}^T_{k'}$. Hence, $(i)$ and $(ii)$ follow. Moreover, from (13) and the $C^{1,1}_{\mathrm{loc}}$ regularity of the end-point map, there exists a constant $\ell\geqslant0$, depending only on $k$ and $G$, such that $\mathrm{d}F_{t,z,u}$ is $\ell$-Lipschitz for all $(t,z,u)\in\Lambda$. So, we get $(iii)$. □

*Proof of Theorem 3.4.* Define $\delta=\mathrm{dist}(\partial\Omega_G,\Gamma)$ and let $r>0$ be as in Proposition 4.5 (we can pick $r$ such that $r\leqslant\delta$). It is sufficient to prove the semiconcavity of $V_{x_0}$, uniformly in $x_0$, on the set $\left([t-r,t+r]\times\overline{B_r(x)}\right)\cap\Gamma$ whenever $(t,x)\in\Gamma$. So, fix $x_0\in G$ and let $u\in U_{x_0}(t,x)$ and $h$, $\eta\in\mathbb{R}^n$ be such that $[t-h,t+h]\times[x-\eta,x+\eta]\subset\left([t-r,t+r]\times\overline{B_r(x)}\right)\cap\Gamma$. Hence, denoting for simplicity $F_{t,x_0,u}$ by $F$ and using the same notation as in the proof of Lemma 4.1, from Lemma 4.4 and Proposition 4.5 we conclude that

$$V_{x_0}(t,x)=C_t(F(t,x_0,x)),$$

and for all $(t', x') \in \left([t-r, t+r] \times \overline{B_r(x)}\right) \cap \Gamma$

$$V_{x_0}(t', x') \leqslant C_{t'}(F(t', x_0, x')).$$

So, by Lemma 4.1 and Proposition 4.5, there exist $\tilde{C} = \tilde{C}(G, \Gamma) > 0$ and $C = C(G, \Gamma) > 0$ such that for all $x_0 \in G$

$$\begin{aligned}
&V_{x_0}(t+h, x+\eta) + V_{x_0}(t-h, x-\eta) - 2V_{x_0}(t, x) \\
&\leqslant C_{t+h}(F(t+h, x_0, x+\eta)) + C_{t-h}(F(t-h, x_0, x-\eta)) - 2C_t(F(t, x_0, x)) \\
&= C_{t+h}(F(t+h, x_0, x+\eta)) + C_{t-h}(F(t-h, x_0, x-\eta)) \\
&\qquad - 2C_t\left(\frac{F(t+h, x_0, x+\eta) + F(t-h, x_0, x-\eta)}{2}\right) \\
&\qquad + 2\left(C_t\left(\frac{F(t+h, x_0, x+\eta) + F(t-h, x_0, x-\eta)}{2}\right) - C_t(F(t, x_0, x))\right) \\
&\leqslant C|F(t+h, x_0, x+\eta) - F(t-h, x_0, x-\eta)|^2 \\
&\qquad + C\left|F(t+h, x_0, x+\eta) + F(t-h, x_0, x-\eta) - 2F(t, x_0, x)\right| \\
&\leqslant C\tilde{C}^2(h+\eta)^2 + C\tilde{C}\left(h^2+\eta^2\right).
\end{aligned}$$

Since all constants involved in the previous inequality depend only on $G$ and $\Gamma$, the conclusion follows. □

## 5. Sensitivity Relations

We investigate next sensitivity relations for the minimum time function. Let $\Gamma \subset \mathbb{R}^n$ be a compact subset.

**Remark 5.1.** It is known (see [20, Proposition 3.1]) that the sub-Riemannian distance between two points $y$ and $x$ is equal to the *minimum time* $\tau_{\{y\}}(x)$ to reach $y$ from $x$, associated to the control system

$$\begin{cases} y'(s) = \sum_{i=1}^m u_i(s)\, f_i(y(s)) \quad \text{for a.e. } s \geqslant 0 \\ y(0) = x \\ u \in \mathscr{B}_m, \end{cases} \tag{14}$$

where $\mathscr{B}_m$ denotes the set of all Lebesgue measurable controls $u : [0, \infty) \to \mathbb{R}^m$ such that $u(s) \in \overline{B_1(0)}$ for a.e. $s \geqslant 0$. So, the minimum time function $\tau_\Gamma(\cdot)$ to reach $\Gamma$ for the control system (14) satisfies $\tau_\Gamma(x) = \inf_{y\in\Gamma} \tau_{\{y\}}(x) = \inf_{y\in\Gamma} d_{SR}(y, x) = d_{SR}(\Gamma, x)$ for all $x \in \mathbb{R}^n$. A control $u \in \mathscr{B}_m$ is said to be *optimal* (for the minimum time function $\tau_\Gamma$) at $z$ if $\tau_\Gamma(z) = \theta_\Gamma(x_{z,u}(\cdot), u(\cdot))$.

Subsequently, to shorten notation, we write $f(x, u)$ in place of $\sum_{i=1}^m u_i f_i(x)$. Next we recall a result from [8, Theorem 3.1], stated under more general assumptions for the vector fields $f_1, ..., f_m$:

**Lemma 5.2** ([8]). *Assume* (H)-(i). *Let $A \subset \mathbb{R}^n$ be a closed set, $\bar{u}$ be an optimal control at $x_0 \in A^c$ for the minimum time function $\tau_A(\cdot)$, and put $\tau_0 = \tau_A(x_0)$.*

*Then for any $\xi \in T_{A^c}(x_{x_0,\bar{u}}(\tau_0))^-$ the solution $q : [0,\tau_0] \to \mathbb{R}^n$ of the adjoint system*

$$\begin{cases} -q'(t) = \mathrm{d}_x f(x_{x_0,\bar{u}}(t), \bar{u}(t))^* q(t) \quad \text{for a.e. } t \in [0,\tau_0] \\ q(\tau_A(x_0)) = -\xi \end{cases} \tag{15}$$

*satisfies the minimum principle*

$$\langle\, q(t), f(x_{x_0,\bar{u}}(t), \bar{u}(t))\rangle = \min_{u \in \overline{B_1(0)}} \langle\, q(t), f(x_{x_0,\bar{u}}(t), u)\rangle \quad \forall\, t \in [0,\tau_0].$$

We denote by $H$ the *Hamiltonian function* on $\mathbb{R}^n \times \mathbb{R}^n$, defined by

$$H(x,p) = \max_{u \in \overline{B_1(0)}} \langle\, p, f(x,u)\rangle.$$

**Proposition 5.3.** *Assume* (H)-(i). *Let $A \subset \mathbb{R}^n$ be a closed set and fix $x_0 \in A^c$. Let $\bar{u}$ be an optimal control at $x_0$ for the minimum time function and let $\xi \in N^P_{\overline{A^c}}(\bar{x})$, where $\bar{x} := x_{x_0,\bar{u}}(\tau_A(x_0))$.*

*The following statements hold true:*

*(i) if $H(\bar{x},\xi) \neq 0$, then*

$$-p(t) \in \partial^P \tau_A(x_{x_0,\bar{u}}(t)) \quad \forall\, t \in [0, \tau_A(x_0)),$$

*where $p(\cdot)$ solves (15) with final condition $p(\tau_A(x_0)) = \xi / H(\bar{x},\xi)$;*

*(ii) if $H(\bar{x},\xi) = 0$, then*

$$-p(t) \in \partial^{\infty,P} \tau_A(x_{x_0,\bar{u}}(t)) \quad \forall\, t \in [0, \tau_A(x_0)),$$

*where $p(\cdot)$ solves (15) with final condition $p(\tau_A(x_0)) = \xi$.*

*Proof.* Denote for simplicity by $\tau(\cdot)$ the minimum time function $\tau_A(\cdot)$. Let $q(\cdot)$ be the solution of (15) with final condition $-q(\tau(x_0)) = \xi \in N^P_{\overline{A^c}}(\bar{x})$, and put $\alpha := H(\bar{x},\xi)$ and $p(\cdot) := -q(\cdot)$. We only show the conclusions $(i)$ and $(ii)$ at $t = 0$, i.e.,

$$(p(0), \alpha) \in N^P_{\mathrm{hypo}\,\tau}(x_0, \tau(x_0)). \tag{16}$$

First of all we claim that $\alpha \geqslant 0$. Indeed, since $p(\tau(x_0)) \in N^P_{\overline{A^c}}(\bar{x})$, there exists $\sigma \geqslant 0$ such that $\langle\, \xi, \bar{y} - \bar{x}\rangle \leqslant \sigma\, |\bar{y} - \bar{x}|^2$ for all $\bar{y} \in \overline{A^c}$. So, for every $0 < t < \tau(x_0)$

$$\langle\, \xi, x_{x_0,\bar{u}}(t) - x_{x_0,\bar{u}}(\tau(x_0))\rangle \leqslant \sigma\, |x_{x_0,\bar{u}}(t) - x_{x_0,\bar{u}}(\tau(x_0))|^2,$$

and, dividing the previous inequality by $t - \tau(x_0)$, it follows

$$\begin{aligned} &\langle\, \xi, \frac{1}{t-\tau(x_0)} \int_{\tau(x_0)}^t f(x_{x_0,\bar{u}}(s), \bar{u}(s))\, ds\rangle \\ &\geqslant \sigma \frac{1}{t-\tau(x_0)}\, |x_{x_0,\bar{u}}(t) - x_{x_0,\bar{u}}(\tau(x_0))| \left| \int_{\tau(x_0)}^t f(x_{x_0,\bar{u}}(s), \bar{u}(s))\, ds \right|. \end{aligned} \tag{17}$$

Now,

$$\begin{aligned}&\frac{1}{t-\tau(x_0)}\int_{\tau(x_0)}^{t} f(x_{x_0,\bar u}(s),\bar u(s))\,ds\\&=\frac{1}{t-\tau(x_0)}\int_{\tau(x_0)}^{t}\left(f(x_{x_0,\bar u}(s),\bar u(s))-f(x_{x_0,\bar u}(\tau(x_0)),\bar u(s))\right)\,ds\\&\quad+\frac{1}{t-\tau(x_0)}\int_{\tau(x_0)}^{t} f(x_{x_0,\bar u}(\tau(x_0)),\bar u(s))\,ds.\end{aligned}\tag{18}$$

By Lemma 6.2, there exists a constant $C\geqslant 0$ such that

$$\left|\frac{1}{t-\tau(x_0)}\int_{\tau(x_0)}^{t}\left(f(x_{x_0,\bar u}(s),\bar u(s))-f(x_{x_0,\bar u}(\tau(x_0)),\bar u(s))\right)\,ds\right|\leqslant C\left|t-\tau(x_0)\right|.\tag{19}$$

Furthermore, since $f(x,\overline{B_1(0)})$ is compact and convex for all $x\in\mathbb{R}^n$,

$$\frac{1}{t-\tau(x_0)}\int_{\tau(x_0)}^{t} f(x_{x_0,\bar u}(\tau(x_0)),\bar u(s))\,ds\in f(x_{x_0,\bar u}(\tau(x_0)),\overline{B_1(0)}),$$

and there exist $t_i\to\tau(x_0)-$ and $u^*\in\overline{B_1(0)}$ satisfying

$$\frac{1}{t_i-\tau(x_0)}\int_{\tau(x_0)}^{t_i} f(x_{x_0,\bar u}(\tau(x_0)),\bar u(s))\,ds\to f(x_{x_0,\bar u}(\tau(x_0)),u^*).\tag{20}$$

So, using (18), (19), and (20), passing to the limit in (17) when $t=t_i$ and $t_i\to\tau(x_0)^-$ we get that

$$\langle\,\xi, f(x_{x_0,\bar u}(\tau(x_0)),u^*)\rangle\geqslant 0.$$

Hence, the claim holds true.

To prove (16), we have to show that there exists $\hat\sigma\geqslant 0$ such that for all $y\in\overline{A^c}$ and $\beta\leqslant\tau(y)$

$$\langle\,p(0),y-x_0\rangle+\alpha(\beta-\tau(x_0))\leqslant\hat\sigma\left(|y-x_0|^2+|\beta-\tau(x_0)|^2\right).\tag{21}$$

On account of [17, Proposition 1.5], we prove (21) for all $y\in\overline{A^c}$ and $\beta\leqslant\tau(y)$ with $|\tau(y)-\tau(x_0)|\leqslant 1$. Fix such $y$ and $\beta$, and let $\xi(\cdot)$ be the solution of the Cauchy problem

$$\begin{cases}\xi'(t)=f(\xi(t),\bar u(t)) & \text{for a.e. } t\geqslant 0\\ \xi(0)=y.\end{cases}\tag{22}$$

Case 1: $\tau(y)\leqslant\tau(x_0)$.

Put $y_1=\xi(\tau(y))\in\overline{A^c}$ and $x_1=x_{x_0,\bar u}(\tau(y))\in\overline{A^c}$. By Gronwall's lemma there exists $K\geqslant 0$ such that for any $s\in[0,\tau(x_0)]$

$$|\xi(s)-x_{x_0,\bar u}(s)|\leqslant e^{Ks}\,|y-x_0|\leqslant e^{K\tau(x_0)}\,|y-x_0|.\tag{23}$$

Furthermore,

$$\begin{aligned}
&D_s\langle p(s), \xi(s) - x_{x_0,\bar{u}}(s)\rangle \\
&= \langle p'(s), \xi(s) - x_{x_0,\bar{u}}(s)\rangle + \langle p(s), f(\xi(s), \bar{u}(s)) - f(x_{x_0,\bar{u}}(s), \bar{u}(s))\rangle \\
&= \langle -\mathrm{d}_x f(x_{x_0,\bar{u}}(s), \bar{u}(s))^* p(s), \xi(s) - x_{x_0,\bar{u}}(s)\rangle \\
&\quad + \langle p(s), f(\xi(s), \bar{u}(s)) - f(x_{x_0,\bar{u}}(s), \bar{u}(s))\rangle \\
&= \langle p(s), f(\xi(s), \bar{u}(s)) - f(x_{x_0,\bar{u}}(s), \bar{u}(s)) - \mathrm{d}_x f(x_{x_0,\bar{u}}(s), \bar{u}(s))(\xi(s) - x_{x_0,\bar{u}}(s))\rangle \\
&= \langle p(s), \int_0^1 (1-t)\mathrm{d}_x^2 f(t\xi(s) + (1-t)x_{x_0,\bar{u}}(s), \bar{u}(s))(\xi(s) - x_{x_0,\bar{u}}(s))^2\, dt\rangle,
\end{aligned}$$

and

$$\langle p(\tau(y)), y_1 - x_1\rangle = \langle p(0), y - x_0\rangle + \int_0^{\tau(y)} D_s\langle p(s), \xi(s) - x_{x_0,\bar{u}}(s)\rangle\, ds.$$

Applying (23) we deduce that there exists $\sigma_1 \geqslant 0$ (not depending on $y$) satisfying

$$\langle p(0), y - x_0\rangle \leqslant \langle p(\tau(y)), y_1 - x_1\rangle + \sigma_1 |y - x_0|^2. \tag{24}$$

Since $p(\cdot)$ is Lipschitz continuous, there exists $\sigma_2 \geqslant 0$ such that

$$\begin{aligned}
&\langle p(\tau(y)), y_1 - x_1\rangle = \langle p(\tau(x_0)), y_1 - x_1\rangle + \langle p(\tau(y)) - p(\tau(x_0)), y_1 - x_1\rangle \\
&\leqslant \langle p(\tau(x_0)), y_1 - x_1\rangle + \sigma_2 |\tau(y) - \tau(x_0)|\, |y_1 - x_1| \\
&\leqslant \langle p(\tau(x_0)), y_1 - x_1\rangle + \frac{\sigma_2}{2}\left(|\tau(y) - \tau(x_0)|^2 + |y_1 - x_1|^2\right),
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
&\langle p(\tau(x_0)), y_1 - x_1\rangle \\
&= \langle p(\tau(x_0)), x_{x_0,\bar{u}}(\tau(x_0)) - x_{x_0,\bar{u}}(\tau(y))\rangle + \langle p(\tau(x_0)), \xi(\tau(y)) - x_{x_0,\bar{u}}(\tau(x_0))\rangle \\
&= \int_{\tau(y)}^{\tau(x_0)} \langle p(\tau(x_0)), f(x_{x_0,\bar{u}}(s), \bar{u}(s))\rangle\, ds + \langle p(\tau(x_0)), \xi(\tau(y)) - x_{x_0,\bar{u}}(\tau(x_0))\rangle \\
&= \int_{\tau(y)}^{\tau(x_0)} (\langle p(\tau(x_0)), f(x_{x_0,\bar{u}}(s), \bar{u}(s)) - f(x_{x_0,\bar{u}}(\tau(x_0)), \bar{u}(\tau(x_0)))\rangle)\, ds \\
&\qquad + \alpha(\tau(x_0) - \tau(y)) + \langle p(\tau(x_0)), y_1 - \bar{x}\rangle.
\end{aligned} \tag{26}$$

Since $p(\tau(x_0)) \in N^P_{\overline{A^c}}(\bar{x})$, we have

$$\begin{aligned}
\langle p(\tau(x_0)), y_1 - \bar{x}\rangle &\leqslant \sigma |y_1 - \bar{x}|^2 \\
&\leqslant \sigma\left(|\tau(x_0) - \tau(y)|^2 + |y_1 - \bar{x}|^2\right),
\end{aligned} \tag{27}$$

and there exists $\sigma_3 \geqslant 0$ such that for all $s \in [0, \tau(x_0)]$

$$\begin{aligned}
&\langle p(\tau(x_0)), f(x_{x_0,\bar{u}}(s), \bar{u}(s)) - f(x_{x_0,\bar{u}}(\tau(x_0)), \bar{u}(\tau(x_0)))\rangle \\
&= \langle p(\tau(x_0)), f(x_{x_0,\bar{u}}(s), \bar{u}(s)) - f(x_{x_0,\bar{u}}(\tau(x_0)), \bar{u}(s))\rangle \\
&\quad + \langle p(\tau(x_0)), f(x_{x_0,\bar{u}}(\tau(x_0)), \bar{u}(s)) - f(x_{x_0,\bar{u}}(\tau(x_0)), \bar{u}(\tau(x_0)))\rangle \\
&\leqslant \sigma_3 |\tau(x_0) - s|.
\end{aligned} \tag{28}$$

So, using (26) and inequalities (27) and (28), it follows that

$$\langle p(\tau(x_0)), y_1 - x_1\rangle \leqslant \alpha\left(\tau(x_0) - \tau(y)\right) + \max\left(\sigma, \sigma_3\right)\left(|\tau(x_0) - \tau(y)|^2 + |y_1 - \bar{x}|^2\right). \tag{29}$$

On account of (23), by (29) there exists $\sigma_4 \geqslant 0$ satisfying

$$\langle p(\tau(x_0)), y_1 - x_1\rangle \leqslant \alpha\left(\tau(x_0) - \tau(y)\right) + \sigma_4\left(|y - x_0|^2 + |\tau(x_0) - \tau(y)|^2\right).$$

Finally, from (23), (24), and (25), we deduce (21) (we assumed $\beta \leqslant \tau(y)$).

Case 2: $\tau(y) > \tau(x_0)$.

We claim that

$$(p(\tau(x_0)), \alpha) \in N^P_{\mathrm{hypo}\,\tau}(\bar{x}, 0). \tag{30}$$

The inclusion (30) means that there exists $\sigma_5 \geqslant 0$ satisfying for all $\tilde{y} \in \overline{A^c}$ and $\tilde{\beta} \leqslant \tau(\tilde{y})$

$$\langle p(\tau(x_0)), \tilde{y} - \bar{x}\rangle + \alpha\tilde{\beta} \leqslant \sigma_5\left(|\tilde{y} - \bar{x}|^2 + \tilde{\beta}^2\right). \tag{31}$$

If $\tilde{\beta} \leqslant 0$, then (31) follows from the condition $p(\tau(x_0)) \in N^P_{\overline{A^c}}(\bar{x})$. On the other hand, suppose that $0 < \tilde{\beta} \leqslant \tau(\tilde{y}) \leqslant 1$ and let $z(\cdot)$ be the solution of the problem

$$\begin{cases} z'(t) = f(z(t), \bar{u}(\tau(x_0))) & \text{for a.e. } t \geqslant 0 \\ z(0) = \tilde{y}. \end{cases}$$

Define $\bar{y}_1 = z(\tilde{\beta})$ and observe that $\bar{y}_1 \in \overline{A^c}$. So, letting $K \geqslant 0$ to be the Lipschitz constant of $f$ with respect to the space variable on the compact set $\{y \in \mathbb{R}^n \,|\, \tau(y) \leqslant 1\}$, we deduce that for all $0 < t \leqslant \tilde{\beta}$

$$\begin{aligned} |z(t) - \tilde{y}| &\leqslant \int_0^t |f(z(s), \bar{u}(\tau(x_0)))|\, ds \\ &\leqslant \int_0^t |f(z(s), \bar{u}(\tau(x_0))) - f(\tilde{y}, \bar{u}(\tau(x_0)))|\, ds + t\,|f(\tilde{y}, \bar{u}(\tau(x_0)))| \\ &\leqslant K\int_0^t |z(s) - \tilde{y}|\, ds + t\,|f(\tilde{y}, \bar{u}(\tau(x_0)))| \\ &\leqslant K\int_0^t |z(s) - \tilde{y}|\, ds + \tilde{\beta}\,|f(\tilde{y}, \bar{u}(\tau(x_0)))|. \end{aligned}$$

From Gronwall's lemma, it follows that for all $0 < t \leqslant \tilde{\beta} \leqslant 1$

$$\begin{aligned} |z(t) - \tilde{y}| &\leqslant \tilde{\beta}e^{Kt}\,|f(\tilde{y}, \bar{u}(\tau(x_0)))| \\ &\leqslant \tilde{\beta}e^{K}\left(|f(\tilde{y}, \bar{u}(\tau(x_0))) - f(\bar{x}, \bar{u}(\tau(x_0)))| + |f(\bar{x}, \bar{u}(\tau(x_0)))|\right) \\ &\leqslant \tilde{\beta}Ke^{K}\,|\tilde{y} - \bar{x}| + \tilde{\beta}e^{K}\,|f(\bar{x}, \bar{u}(\tau(x_0)))| \\ &\leqslant Ke^{K}\,|\tilde{y} - \bar{x}| + \tilde{\beta}e^{K}\,|f(\bar{x}, \bar{u}(\tau(x_0)))| \\ &\leqslant \tilde{K}e^{K}\left(|\tilde{y} - \bar{x}| + \tilde{\beta}\,|f(\bar{x}, \bar{u}(\tau(x_0)))|\right), \end{aligned} \tag{32}$$

where $\tilde{K} = K + 1$. Now

$$\langle p(\tau(x_0)), \tilde{y} - \bar{x} \rangle = \langle p(\tau(x_0)), \tilde{y} - \bar{y}_1 \rangle + \langle p(\tau(x_0)), \bar{y}_1 - \bar{x} \rangle, \tag{33}$$

and, combining the inclusion $p(\tau(x_0)) \in N^P_{\overline{A^c}}(\bar{x})$ with (32), it follows that

$$\begin{aligned} \langle p(\tau(x_0)), \bar{y}_1 - \bar{x} \rangle &\leqslant \sigma \left| \bar{y}_1 - \bar{x} \right|^2 \\ &\leqslant 2\sigma \left( \left| \bar{y}_1 - \tilde{y} \right|^2 + \left| \tilde{y} - \bar{x} \right|^2 \right) \\ &\leqslant 2\sigma \tilde{K}^2 e^{2\tilde{K}} \left( 2 \left| \tilde{y} - \bar{x} \right|^2 + 2\tilde{\beta}^2 \left| f(\bar{x}, \bar{u}(\tau(x_0))) \right|^2 + \left| \tilde{y} - \bar{x} \right|^2 \right) \\ &\leqslant \tilde{\sigma}_5 \left( \tilde{\beta}^2 + \left| \tilde{y} - \bar{x} \right|^2 \right), \end{aligned} \tag{34}$$

for a suitable constant $\tilde{\sigma}_5 \geqslant 0$. On the other hand,

$$\begin{aligned} &\langle p(\tau(x_0)), \tilde{y} - \bar{y}_1 \rangle \\ &= - \int_0^{\tilde{\beta}} \langle p(\tau(x_0)), f(z(s), \bar{u}(\tau(x_0))) \rangle \, ds \\ &= - \int_0^{\tilde{\beta}} \langle p(\tau(x_0)), f(\bar{x}, \bar{u}(\tau(x_0))) \rangle \, ds \\ &\quad - \int_0^{\tilde{\beta}} \langle p(\tau(x_0)), f(z(s), \bar{u}(\tau(x_0))) - f(\bar{x}, \bar{u}(\tau(x_0))) \rangle \, ds \\ &= -\alpha\tilde{\beta} - \int_0^{\tilde{\beta}} \langle p(\tau(x_0)), f(z(s), \bar{u}(\tau(x_0))) - f(\bar{x}, \bar{u}(\tau(x_0))) \rangle \, ds. \end{aligned} \tag{35}$$

Furthermore, putting $\tilde{\sigma} = |p(\tau(x_0))| \, K$,

$$\begin{aligned} &- \int_0^{\tilde{\beta}} \langle p(\tau(x_0)), f(z(s), \bar{u}(\tau(x_0))) - f(\bar{x}, \bar{u}(\tau(x_0))) \rangle \, ds \\ &\leqslant |p(\tau(x_0))| \int_0^{\tilde{\beta}} \left| f(z(s), \bar{u}(\tau(x_0))) - f(\bar{x}, \bar{u}(\tau(x_0))) \right| \, ds \\ &\leqslant \tilde{\sigma} \int_0^{\tilde{\beta}} \left| z(s) - \bar{x} \right| \, ds \\ &\leqslant \tilde{\sigma} \left( \int_0^{\tilde{\beta}} \left| z(s) - \bar{x} \right| \, ds + \tilde{\beta} \left| \tilde{y} - \bar{x} \right| \right) \\ &\leqslant \tilde{\sigma} \left( \int_0^{\tilde{\beta}} (e^{\tilde{K}} \left| \tilde{y} - \bar{x} \right| + s \left| f(\bar{x}, \bar{u}(\tau(x_0))) \right|) \, ds + \frac{1}{2}\tilde{\beta}^2 + \frac{1}{2} \left| \tilde{y} - \bar{x} \right|^2 \right) \\ &\leqslant \tilde{\sigma} \left( e^{\tilde{K}} \tilde{\beta} \left| \tilde{y} - \bar{x} \right| + \frac{1}{2}\tilde{\beta}^2 \left| f(\bar{x}, \bar{u}(\tau(x_0))) \right| + \frac{1}{2}\tilde{\beta}^2 + \frac{1}{2} \left| \tilde{y} - \bar{x} \right|^2 \right) \\ &\leqslant \hat{\sigma}_5 \left( \tilde{\beta}^2 + \left| \tilde{y} - \bar{x} \right|^2 \right), \end{aligned} \tag{36}$$

for a suitable constant $\hat{\sigma}_5 \geqslant 0$. Now, from (35) and (36) it follows that

$$\langle p(\tau(x_0)), \tilde{y} - \bar{y}_1 \rangle + \alpha\tilde{\beta} \leqslant \hat{\sigma}_5 \left( \tilde{\beta}^2 + |\tilde{y} - \bar{x}|^2 \right). \tag{37}$$

Hence (31) follows from (33), (34), and (37).
Now, consider the solution $\xi(\cdot)$ of (22) and put $\bar{y} = \xi(\tau(x_0)) \in \overline{A^c}$. In a similar fashion as in the previous step, there exists $\sigma_6 \geqslant 0$ such that

$$\langle p(0), y - x_0 \rangle \leqslant \langle p(\tau(x_0)), \bar{y} - \bar{x} \rangle + \sigma_6 |y - x_0|^2. \tag{38}$$

From the dynamic programming principle it follows that if $\beta \leqslant \tau(y)$ then $\beta - \tau(x_0) \leqslant \tau(y) - \tau(x_0) \leqslant \tau(\bar{y})$, so by (31) we have

$$\langle p(\tau(x_0)), \bar{y} - \bar{x} \rangle + \alpha (\beta - \tau(x_0)) \leqslant \sigma_5 \left( |\bar{y} - \bar{x}|^2 + |\beta - \tau(x_0)|^2 \right),$$

and, using (23), we deduce that there exists $\sigma_7 \geqslant 0$ not depending on $y$ such that

$$\langle p(\tau(x_0)), \bar{y} - \bar{x} \rangle + \alpha (\beta - \tau(x_0)) \leqslant \sigma_7 \left( |y - x_0|^2 + |\beta - \tau(x_0)|^2 \right). \tag{39}$$

So, combining (38) and (39), we deduce (21).

Finally, from the dynamic programming principle and with the same technique as those used for the case $t = 0$, we show that the conclusion holds on the whole time interval $[0, \tau(x_0))$. □

**Proposition 5.4.** *Let $u$ be a $C$-semiconcave function on an open set $\mathscr{O} \subset \mathbb{R}^n$, with $C > 0$. Suppose that, for some $\lambda \in \mathbb{R}$,*

$$\begin{aligned} &(i) \quad U_\lambda := \{x \in \mathscr{O} \,|\, u(x) \leqslant \lambda\} \neq \emptyset \\ &(ii) \quad \overline{\partial U_\lambda \cap \mathscr{O}} \subset \mathscr{O} \text{ and } \overline{\partial U_\lambda \cap \mathscr{O}} \text{ is compact} \qquad (40) \\ &(iii) \quad \exists\, \alpha > 0 \text{ such that } D^+u(x) \cap B_\alpha(0)^c \neq \emptyset \quad \forall x \in \mathscr{O}, \qquad (41) \end{aligned}$$

*then there exists $r > 0$ such that for all $x \in \partial U_\lambda \cap \mathscr{O}$ we can find $\hat{v}_x \in S^1$ satisfying*

$$\overline{B_r(x + r\hat{v}_x)} \subset U_\lambda.$$

*Proof.* We claim the following: if $x \in \partial U_\lambda \cap \mathscr{O}$, $p \in D^+u(x)$, and $R' > 0$ are such that

$$\overline{B_{R'}(x)} \subset \mathscr{O} \quad \& \quad p \neq 0, \tag{42}$$

then for $\hat{v} := -p/|p|$ we have $\overline{B_{r'}(x + r'\hat{v})} \subset U_\lambda$, where $r' = \min\{R'/2, |p|/2C\}$. Indeed, for such $r'$, by (42), we have $[x, x - r'\dfrac{p}{|p|} + r'v] \subset \mathscr{O}$ for all $v \in S^1$, and, applying [14, Proposition 3.3.1], we get

$$\begin{aligned} u(x - r'\frac{p}{|p|} + r'v) &\leqslant u(x) + \langle p, r'v - r'\frac{p}{|p|} \rangle + Cr'^2 \left| v - \frac{p}{|p|} \right|^2 \\ &\leqslant \lambda + r'\langle p, v \rangle - r'|p| + 2Cr'^2 \left( 1 - \frac{\langle p, v \rangle}{|p|} \right) \\ &= \lambda + r' (\langle p, v \rangle - |p|) \left( 1 - \frac{2Cr'}{|p|} \right) \\ &\leqslant \lambda. \end{aligned}$$

So, the claim holds true for $\hat{v} = -p/|p|$.

Now, denote $R_x = \sup\left\{r > 0 \,|\, \overline{B_r(x)} \subset \mathscr{O}\right\}$ for all $x \in \partial U_\lambda \cap \mathscr{O}$. If $R_x = +\infty$ for some $x \in \partial U_\lambda \cap \mathscr{O}$, then $\mathscr{O} = \mathbb{R}^n$. Otherwise, we claim that there exists $R > 0$ such that $R_x \geqslant R$ for all $x \in \partial U_\lambda \cap \mathscr{O}$. Indeed, otherwise there exists a sequence $\{x_i\}_i \subset \partial U_\lambda \cap \mathscr{O}$ such that $\overline{B_{R_{x_i}+\varepsilon}(x_i)} \cap \mathscr{O}^c \neq \emptyset$ for all $i \geqslant 1$ and $R_{x_i} \to 0$ for any $\varepsilon > 0$. Using (40), by further subsequence extraction, we can suppose that $x_i \to \bar{x} \in \overline{\partial U_\lambda \cap \mathscr{O}}$ (then $\bar{x} \in \overline{\mathscr{O}}$), and since $d(x_i, \mathscr{O}^c) \leqslant R_{x_i} + \varepsilon$, passing to the limit we obtain $d(\bar{x}, \mathscr{O}^c) \leqslant \varepsilon$. By arbitrariness of $\varepsilon$ it follows that $d(\bar{x}, \mathscr{O}^c) = 0$ and so $\bar{x} \in \mathscr{O}^c$. Hence $\bar{x} \in \overline{\mathscr{O}} \cap \mathscr{O}^c = \partial\mathscr{O}$, in contradiction with (40). So, the claim holds true. We can conclude that for some $R > 0$, $\overline{B_R(x)} \subset \mathscr{O}$ for all $x \in \partial U_\lambda \cap \mathscr{O}$. From the first claim, we deduce that for any $x \in \partial U_\lambda \cap \mathscr{O}$ and any $p_x \in D^+u(x) \cap B_\alpha(0)^c$, $\hat{v}_x := -p_x/|p_x|$ satisfies $\overline{B_{r_x}(x + r_x\hat{v}_x)} \subset U_\lambda$, where $r_x = \min\{R_x/2, |p_x|/2C\}$. Finally, using (41), we have $r_x \geqslant \min\{R/2, \alpha/2C\}$, and the conclusion follows with $r = \min\{R/2, \alpha/2C\}$. □

We state next the main result of this section.

**Theorem 5.5.** *Assume* (H)′. *Let $\Gamma \subset \mathbb{R}^n$ be a compact set and suppose that there are no singular geodesics for $\Gamma$. Let $x_0 \in \Gamma^c$ and $\bar{u}$ be an optimal control for the minimum time function at $x_0$. Denote $\Gamma_t = \{y \in \mathbb{R}^n \,|\, \tau_\Gamma(y) \leqslant t\}$.*

*Then the solution of the adjoint equation*

$$-p'(t) = \mathrm{d}_x f(x_{x_0,\bar{u}}(t), \bar{u}(t))^* p(t) \quad \text{for a.e. } t \in [0, \tau_\Gamma(x_0)]$$

*satisfy the sensitivity relation*

$$-p(t) \in \partial^P \tau_\Gamma(x_{x_0,\bar{u}}(t)) \quad \forall t \in [0, \tau_\Gamma(x_0)), \tag{43}$$

*and the transversality condition*

$$p(\tau_\Gamma(x_0)) \in \operatorname*{Limsup}_{t \to \tau_\Gamma(x_0)-} N^P_{\Gamma_t^c}(x_{x_0,\bar{u}}(t)). \tag{44}$$

*Proof.* First of all we notice that by Corollary 3.8 and Remark 5.1 the minimum time function $\tau_\Gamma$ is locally semiconcave on $\Gamma^c$. Let $\{\lambda_i\}_i \subset (0, \infty)$ with $\lambda_i \to 0+$ and write for simplicity $\Gamma_i$ in place of $\Gamma_{\lambda_i}$. It is easy to see that the level sets $\Gamma_i$ are compact. Hence, for all $i \geqslant 1$ there exist open sets $A_i, B_i, D \subset \mathbb{R}^n$ such that $D$ is bounded and

$$\Gamma \subset \overline{B_i} \subset \operatorname{int}\Gamma_i \quad \& \quad \Gamma_i \subset A_i \subset D. \tag{45}$$

Putting $\mathscr{O}_i = A_i \backslash \overline{B_i}$, it follows from (45) that $\partial\Gamma_i \cap \mathscr{O}_i$ is bounded and $\partial\Gamma_i = \{y \in \mathbb{R}^n \,|\, \tau_\Gamma(y) = \lambda_i\} \subset \mathscr{O}_i$. So, $\overline{\partial\Gamma_i \cap \mathscr{O}_i} \subset \mathscr{O}_i$. Consider now the eikonal equation $|F(y)^*\nabla\tau_\Gamma(y)| - 1 = 0$ on $\Gamma^c$, where $F(y)$ is the matrix whose columns are the vectors $f_1(y), ..., f_m(y)$. Since $\tau_\Gamma$ is a viscosity solution of such an equation, we have that $|F(y)^*p| - 1 = 0$ for all $y \in \Gamma^c$ and all $p \in D^*\tau(y)$ (see [11, Section 5.3]). So, putting $M > \max\left\{\|F(y)\| \,|\, y \in \overline{D}\right\}$, from (1) we deduce that for all $i \geqslant 1$ and all $y \in \mathscr{O}_i$ there exists $p_i \in D^+\tau_\Gamma(y)$ such that $|p_i| \geqslant M^{-1}$. Hence, applying Proposition

5.4, for all $i \geqslant 1$ there exists $r_i > 0$ such that for any $y \in \partial\Gamma_i$ we can find a unit vector $\hat{v}_i(y)$ satisfying

$$\overline{B_{r_i}(y + r_i\hat{v}_i(y))} \subset \Gamma_i. \tag{46}$$

We note that (46) implies that the set $N^P_{\overline{\Gamma_i^c}}(y)$ contains a nonnull vector for all $y \in \partial\Gamma_i$. Furthermore, since $\tau_\Gamma$ is locally Lipschitz continuous on $\Gamma^c$, applying [14, Theorem 8.2.3], we have for all $i \geqslant 1$ and all $y \in \partial\Gamma_i$ that $H(y,\xi) \neq 0$ for any $\xi \in N^P_{\overline{\Gamma_i^c}}(y) \cap S^1$. We next construct a solution $p(\cdot)$, solving the adjoint equation, associated to the sequence $\{\lambda_i\}_i$ as follows. For all $i \geqslant 1$ pick $\xi_i \in N^P_{\overline{\Gamma_i^c}}(y_i) \cap S^1$ where $y_i = x_{x_0,\bar{u}}(\tau_{\partial\Gamma_i}(x_0))$. We denote by $p_i(\cdot)$ the solution of (15) on $[0, \tau_{\partial\Gamma_i}(x_0)]$, with final condition $\xi_i H(y_i,\xi_i)^{-1}$, and we extend such functions as solutions of the adjoint equation in (15) to the whole interval $[0,\tau_\Gamma(x_0)]$ (we continue to denote by $p_i(\cdot)$ such extended functions). Notice that, since $\tau_{\partial\Gamma_i}(\cdot) = \tau_\Gamma(\cdot) - \lambda_i$ on $\Gamma_i^c$ for all $i \geqslant 1$, $\partial^P\tau_{\partial\Gamma_i}(y) = \partial^P\tau_\Gamma(y)$ for all $y \in \Gamma_i^c$. From Proposition 5.3 it follows that

$$-p_i(t) \in \partial^P\tau_\Gamma(x_{x_0,\bar{u}}(t)) \qquad \forall t \in [0,\tau_{\partial\Gamma_i}(x_0)]. \tag{47}$$

Letting $L = \max\{\|\mathrm{d}_x f(x_{x_0,\bar{u}}(t),\bar{u}(t))\| \mid t \in [0,\tau_\Gamma(x_0)]\}$, by Gronwall's lemma we get $|p_i(t)| \leqslant e^{tL}|p_i(0)|$ and $|p_i'(t)| \leqslant L|p_i(t)|$ for all $t \in [0,\tau_\Gamma(x_0)]$. Since $\partial^P\tau_\Gamma(x_0)$ is bounded, the sequence $\{|p_i(0)|\}_i$ is bounded. So, applying the Ascoli-Arzelà and the Dunford-Pettis theorems, taking a subsequence and keeping the same notations, there exists an absolutely continuous function $p(\cdot)$ on $[0,\tau_\Gamma(x_0)]$ such that $p_i \to p$ uniformly on $[0,\tau_\Gamma(x_0)]$ and $p_i' \rightharpoonup p'$ in $L^1(0,\tau_\Gamma(x_0))$. Such $p(\cdot)$ satisfies the adjoint equation on $[0,\tau_\Gamma(x_0)]$.

Now, using the closedness of $\partial^P\tau_\Gamma(y)$ for any $y \in \Gamma^c$, we get (43) by passing to the limit in (47), and from the definition of upper limit we get (44). □

## 6. Appendix

Below we assume that $T > 0$.

**Remark 6.1.** If (H)-(i) holds true and $B \subset \mathbb{R}^n$, $\mathscr{W} \subset L^2(0,T;\mathbb{R}^m)$ are bounded subsets then, by Gronwall's lemma, it follows that there exists $r = r(\mathscr{W}, B) > 0$ such that

$$\|x_{x_0,u}\|_\infty \leqslant r \quad \forall u \in \mathscr{W},\ \forall x_0 \in B. \tag{48}$$

If $\mathscr{W} \subset B_{L^\infty}(0,R)$ for $R \geqslant 0$, then all trajectories $x_{x_0,u}(\cdot)$ with $(x_0,u) \in B \times \mathscr{W}$ are uniformly Lipschitz continuous on $[0,T]$.

**Lemma 6.2.** *Assume* (H)-(i). *Let $B \subset \mathbb{R}^n$ and $\mathscr{W} \subset L^2(0,T;\mathbb{R}^m)$ be bounded subsets.*

*Then there exists $C = C(\mathscr{W}, B) \geqslant 0$ such that for all $t \in [0,T]$, $z, y \in B$, and $v, w \in \mathscr{W}$*

$$|x_{z,v}(t) - x_{y,w}(t)| \leqslant C\left(\|v-w\|_{L^2} + |z-y|\right). \tag{49}$$

*In particular, if $\mathscr{W} \subset B_{L^\infty}(0,R)$, then there exists $\tilde{C} = \tilde{C}(R,B) \geqslant 0$ such that for all $t, s \in [0,T]$, $z, y \in B$, and $v, w \in \mathscr{W}$*

$$|x_{z,v}(t) - x_{y,w}(s)| \leqslant \tilde{C}\left(\|v-w\|_{L^2} + |z-y| + |t-s|\right).$$

*Proof.* Let $B \subset \mathbb{R}^n$ and $\mathscr{W} \subset L^2(0,T;\mathbb{R}^m)$ be bounded subsets. We have

$$\begin{aligned}
&|x_{z,v}(t) - x_{y,w}(t)| \\
&= \left| \int_0^t \sum_{i=1}^m v_i(s)\, f_i(x_{z,v}(s))\, ds - \int_0^t \sum_{i=1}^m w_i(s)\, f_i(x_{y,w}(s))\, ds + z - y \right| \\
&\leqslant \left| \int_0^t \sum_{i=1}^m (v_i(s) - w_i(s))\, f_i(x_{z,v}(s))\, ds - \int_0^t \sum_{i=1}^m w_i(s)\, \left(f_i(x_{y,w}(s)) - f_i(x_{z,v}(s))\right)\, ds \right| \\
&\quad + |z-y| \\
&\leqslant \sum_{i=1}^m \left( \int_0^t |(v_i(s) - w_i(s))|\, |f_i(x_{z,v}(s))|\, ds + \int_0^t |w_i(s)|\, |f_i(x_{y,w}(s)) - f_i(x_{z,v}(s))|\, ds \right) \\
&\quad + |z-y|.
\end{aligned} \tag{50}$$

From (48) it follows that $x_{z,v}(\cdot)$ takes values in a compact set of $\mathbb{R}^n$ for all $v \in \mathscr{W}$ and $z \in B$. Then there exists $M = M(\mathscr{W}, B) \geqslant 0$ such that $\mathscr{W} \subset B_{L^2}(0,M)$ and for all $z, y \in B$, $v, w \in \mathscr{W}$, $s \in [0,T]$, and $i = 1, ..., m$ holds

$$|f_i(x_{z,v}(s))| \leqslant M \quad \& \quad |f_i(x_{z,v}(s)) - f_i(x_{y,w}(s))| \leqslant M\, |x_{z,v}(s) - x_{y,w}(s)|. \tag{51}$$

Now, from the Cauchy-Schwarz inequality and since $\sum_{i=1}^m |v_i| \leqslant \sqrt{m}\,|v|$, we have

$$\begin{aligned}
&\sum_{i=1}^m \int_0^t |v_i(s) - w_i(s)|\, |f_i(x_{z,v}(s))|\, ds \\
&\leqslant M \sum_{i=1}^m \int_0^t |v_i(s) - w_i(s)|\, ds \\
&\leqslant M\sqrt{m} \int_0^T |v(s) - w(s)|\, ds \\
&\leqslant M\sqrt{mT}\, \|v-w\|_{L^2}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^m \int_0^t |w_i(s)|\ |f_i(x_{y,w}(s)) - f_i(x_{z,v}(s))|\ ds \\
&\leqslant M \sum_{i=1}^m \int_0^t |w_i(s)|\, |x_{y,w}(s) - x_{z,v}(s)|\ ds \\
&= M \int_0^t \left( \sum_{i=1}^m |w_i(s)| \right) |x_{y,w}(s) - x_{z,v}(s)|\ ds.
\end{aligned}$$

On account of the above inequalities, (50) becomes

$$\begin{aligned}
|x_{z,v}(t) - x_{y,w}(t)| \leqslant M\sqrt{mT}\|v - w\|_{L^2} + |z - y| \\
+ M \int_0^t \left( \sum_{i=1}^m |w_i(s)| \right) |x_{z,v}(s) - x_{y,w}(s)|\ ds.
\end{aligned}$$

Hence, applying Gronwall's lemma, for some $C = C(\mathscr{W}, B) \geqslant 0$ we get (49).

Finally, if $\mathscr{W} \subset B_{L^\infty}(0, R)$, the last conclusion follows from (49) and Remark 6.1. □

**Proposition 6.3.** *Assume* (H)-(i). *Then* $E_T \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n \times L^2(0, T; \mathbb{R}^m))$.

*Proof.* Let $(y, u) \in \mathbb{R}^n \times L^2(0, T; \mathbb{R}^m)$ and consider a bounded neighbourhood of $(y, u)$ in $\mathbb{R}^n \times L^2(0, T; \mathbb{R}^m)$ of the form $B_\delta(y) \times B_{L^2}(u, \delta)$, with $\delta > 0$. Set for simplicity $\Delta x(\cdot) = x_{y+h,u+v}(\cdot)$ and $x(\cdot) = x_{y,u}(\cdot)$. So

$$\begin{aligned}
&x'_{y+h,u+v}(t) - x'_{y,u}(t) \\
&= \sum_{i=1}^m (u_i(t) + v_i(t)) f_i(\Delta x(t)) - \sum_{i=1}^m u_i(t) f_i(x(t)) \\
&= \sum_{i=1}^m v_i(t) f_i(\Delta x(t)) + \sum_{i=1}^m u_i(t) \left( f_i(\Delta x(t)) - f_i(x(t)) \right).
\end{aligned} \tag{52}$$

Observe that

$$\begin{aligned}
&f_i(\Delta x(t)) - f_i(x(t)) \\
&= \mathrm{d}f_i(x(t))(\Delta x(t) - x(t)) + \int_0^1 (1 - s)\mathrm{d}^2 f_i((1 - s)x(t) + s\Delta x(t))(\Delta x(t) - x(t))^2\, ds.
\end{aligned}$$

Hence we can rewrite (52) as

$$\begin{aligned} &x'_{y+h,u+v}(t) - x'_{y,u}(t) \\ &= \sum_{i=1}^{m} v_i(t) f_i(\Delta x(t)) \\ &\quad + \sum_{i=1}^{m} u_i(t) \mathrm{d}f_i(x(t))(\Delta x(t) - x(t)) \\ &\quad + \sum_{i=1}^{m} u_i(t) \int_0^1 (1-s) \mathrm{d}^2 f_i((1-s)x(t) + s\Delta x(t))(\Delta x(t) - x(t))^2 \, ds. \end{aligned} \tag{53}$$

Since $\sum_{i=1}^m v_i(t) f_i(\Delta x(t)) = \sum_{i=1}^m v_i(t) f_i(x(t)) + \sum_{i=1}^m v_i(t) \left(f_i(\Delta x(t)) - f_i(x(t))\right)$, letting

$$\begin{cases} \xi(t) = \Delta x(t) - x(t) \\ A(t) = \sum_{i=1}^m u_i(t) \mathrm{d}f_i(x(t)) \\ B(t) = (f_1(x(t))|...|f_m(x(t))), \end{cases}$$

the equation (53) becomes

$$\xi' = A\xi + Bv + R, \tag{54}$$

where

$$\begin{aligned} R(t) = &\sum_{i=1}^{m} v_i(t) \left(f_i(\Delta x(t)) - f_i(x(t))\right) \\ &+ \sum_{i=1}^{m} u_i(t) \int_0^1 (1-s) \mathrm{d}^2 f_i((1-s)x(t) + s\Delta x(t))(\Delta x(t) - x(t))^2 \, ds. \end{aligned}$$

We remark that $\Delta x$ and $x$ depend on starting points $y+h$, $y$ and on controls $u+v$, $u$ respectively, while the matrices $A$ and $B$ depend only on $y$ and $u$.

By Lemma 6.2, there exists $C = C(\delta) > 0$ such that for all $t \in [0,T]$

$$|\Delta x(t) - x(t)| \leqslant C(|h| + \|v\|_{L^2}) \quad \forall (h,v) \in B_\delta(y) \times B_{L^2}(u,\delta). \tag{55}$$

Observe that there exists $\tilde{M} \geqslant 0$ such that, by (55) and (51),

$$\begin{aligned} |R(t)| &\leqslant \sum_{i=1}^{m} |v_i(t)| \, |f_i(\Delta x(t)) - f_i(x(t))| + \tilde{M} C \left(|h| + \|v\|_{L^2}\right)^2 \sum_{i=1}^{m} |u_i(t)| \\ &\leqslant MC \left(|h| + \|v\|_{L^2}\right) \sum_{i=1}^{m} |v_i(t)| + \tilde{M} C \left(|h| + \|v\|_{L^2}\right)^2 \sum_{i=1}^{m} |u_i(t)|. \end{aligned} \tag{56}$$

Solving the system (54) with initial condition $\xi(0) = h$ we have that

$$\begin{aligned}\xi(t) = X(t)h + \int_0^t X(t)X(s)^{-1}B(s)v(s)\,ds \\ + \int_0^t X(t)X(s)^{-1}R(s)\,ds\end{aligned} \tag{57}$$

where $X(\cdot)$ is the fundamental solution, i.e.,

$$\begin{cases} X'(t) = A(t)X(t) \quad \text{for a.e. } t \in [0,T] \\ X(0) = I. \end{cases}$$

Furthermore, letting $C_1 = \max\left\{\|X(T)X(s)^{-1}\| \mid s \in [0,T]\right\}$ and $C_2 = \max\left\{MC, \tilde{M}C\right\}$, we have that

$$\begin{aligned}&\left|\int_0^T X(T)X(s)^{-1}R(s)\,ds\right| \\ &\leqslant \int_0^T \left\|X(T)X(s)^{-1}\right\| |R(s)|\,ds \\ &\leqslant C_1 \int_0^T |R(s)|\,ds \\ &\leqslant C_1C_2\left((|h| + \|v\|_{L^2})\sum_{i=1}^m \int_0^T |v_i(t)|\,dt + (|h| + \|v\|_{L^2})^2 \sum_{i=1}^m \int_0^T |u_i(t)|\,dt\right) \\ &\leqslant L\left(|h| + \|v\|_{L^2}\right)^2,\end{aligned}$$

where $L \geqslant 0$ is a suitable constant depending only on $\delta$. Finally, from (57) it follows that

$$\left|\Delta x(T) - x(T) - \left(X(T)h + \int_0^T X(T)X(s)^{-1}B(s)v(s)\,ds\right)\right| \leqslant L\left(|h| + \|v\|_{L^2}\right)^2$$

and since

$$(h,v) \mapsto X(T)h + \int_0^T X(T)X(s)^{-1}B(s)v(s)\,ds$$

is linear and continuous on $\mathbb{R}^n \times L^2(0,T;\mathbb{R}^m)$, we get

$$\mathrm{d}E_T(y,u)(h,v) = X(T)h + \int_0^T X(T)X(s)^{-1}B(s)v(s)\,ds. \tag{58}$$

Now, from (58) and regularity of $f_i$'s it follows that there exists $\tilde{C} = \tilde{C}(\delta) > 0$ such that

$$\begin{aligned}&|\mathrm{d}E_T(y_1,u_1)(\hat{y},\hat{u}) - \mathrm{d}E_T(y_2,u_2)(\hat{y},\hat{u})| \\ &\leqslant \tilde{C}\left(|y_1 - y_2| + \|u_1 - u_2\|_{L^2}\right)\left(|\hat{y}| + \|\hat{u}\|_{L^2}\right),\end{aligned}$$

and so

$$\|\mathrm{d}E_T(y_1,u_1)-\mathrm{d}E_T(y_2,u_2)\|\leqslant\tilde{C}\left(|y_1-y_2|+\|u_1-u_2\|_{L^2}\right). \tag{59}$$

□

**Remark 6.4.** *From the last part of the proof of Proposition 6.3 we deduce that for any $\tau>0$, $r>0$, and $R>0$ we can find $C=C(\tau,r,R)>0$ satisfying $\|\mathrm{d}E_s(y_1,u_1)-\mathrm{d}E_s(y_2,u_2)\|\leqslant C\left(|y_1-y_2|+\|u_1-u_2\|_{L^2}\right)$ for all $s\in[0,\tau]$, $u_1,u_2\in B_{L^2}(0,R)$, and $y_1,y_2\in B(0,r)$. Therefore $|x_{z,u+v}(s)+x_{z,u-v}(s)-2x_{z,u}(s)|\leqslant C\|v\|^2_{L^2}$ for all $s\in[0,\tau]$, $u,v\in B_{L^2}(0,R)$, and $z\in B(0,r)$.*

**Corollary 6.5.** *Assume* (H)-(i). *Then the map*

$$[0,\tau]\times\mathbb{R}^n\times L^2(0,\tau;\mathbb{R}^m)\ni(s,y,u)\mapsto\mathrm{d}E_s(y,u)$$

*is continuous.*

*Proof.* Notice that, from continuous dependence on parameters of solutions to ODE's and from the expression of the differential in (58), for any $\tau>0$ and every bounded subset $\mathcal{A}\subset\mathbb{R}^n\times L^2(0,\tau;\mathbb{R}^m)$, the maps $\{s\mapsto\mathrm{d}E_s(y,u)\,|\,s\in[0,\tau]\}$ are equicontinuous for $(y,u)\in\mathcal{A}$ and the constant $\tilde{C}$ that appears in (59) may be taken the same for $0\leqslant T\leqslant\tau$, $(y_1,u_1),(y_2,u_2)\in\mathcal{A}$. □

The following result is well known.

**Lemma 6.6.** *Let $X$ be a separable normed space and $\Phi:X\to\mathbb{R}^n$ be a linear, continuous, and surjective operator. Consider $\{x_i\}_i$ dense in $X$.*

*Then there exist linearly independent vectors $x_1,...,x_n$ such that $\Phi:W\to\mathbb{R}^n$ is an isomorphism, where $W=\mathrm{span}\,\{x_1,...,x_n\}$.*

*Proof.* Let $\{x_i\}_i$ be dense in $X$. Then there exists a countable increasing family of finite dimensional subspaces $W_k=\mathrm{span}\,\{x_i\}_{i=1}^k$ such that $\overline{\cup_k W_k}=X$. So, $\Phi(W_k)$ is a finite dimensional subspace and $\Phi(W_k)$ is increasing. Hence there exists $k_0$ such that $\Phi(W_{k_0})=\mathbb{R}^n$. So we can choose $n$ linearly independent vectors $x_1,...,x_n$ in $W_{k_0}$ such that $\Phi$ is onto and injective on $W=\mathrm{span}\,\{x_1,...,x_n\}$. □

## References


[1] A. Agrachev. Compactness for sub-riemannian length-minimizers and subanalyticity, in: Control theory and its applications (Grado, 1998). *Rend. Sem. Mat. Univ. Politec. Torino 56 (1998)*, 4:1–12, 2001.

[2] A. Bellaiche and J.-J. Risler. *Sub-Riemannian geometry*, volume 144. Birkhäuser Verlag, Basel, 1996.

[3] P. Cannarsa and H. Frankowska. Some characterizations of optimal trajectories in control theory. *SIAM Journal on Control and Optimization*, 29(6):1322–1347, 1991.

[4] P. Cannarsa and H. Frankowska. Interior sphere property of attainable sets and time optimal control problems. *ESAIM: Control, Optimisation and Calculus of Variations*, 12(2):350–370, 2006.

[5] P. Cannarsa and H. Frankowska. Local regularity of the value function in optimal control. *Systems & Control Letters*, 62(9):791–794, 2013.

[6] P. Cannarsa and H. Frankowska. From pointwise to local regularity for solutions of Hamilton–Jacobi equations. *Calculus of Variations and Partial Differential Equations*, 49(3-4):1061–1074, 2014.

[7] P. Cannarsa, H. Frankowska, and T. Scarinci. Second-order sensitivity relations and regularity of the value function for Mayer's problem in optimal control. *SIAM Journal on Control and Optimization*, 53(6):3642–3672, 2015.

[8] P. Cannarsa, H. Frankowska, and C. Sinestrari. Optimality conditions and synthesis for the minimum time problem. *Set-valued analysis*, 8(1-2):127–148, 2000.

[9] P. Cannarsa, A. Marigonda, and K.-T. Nguyen. Optimality conditions and regularity results for time optimal control problems with differential inclusions. *Journal of Mathematical Analysis and Applications*, 427(1):202–228, 2015.

[10] P. Cannarsa, C. Pignotti, and C. Sinestrari. Semiconcavity for optimal control problems with exit time. *Discrete and Continuous Dynamical Systems*, 6(4):975–997, 2000.

[11] P. Cannarsa and L. Rifford. Semiconcavity results for optimal control problems admitting no singular minimizing controls. *Annales de l'IHP Analyse non linéaire*, 25(4):773–802, 2008.

[12] P. Cannarsa and C. Sinestrari. Convexity properties of the minimum time function. *Calculus of Variations and Partial Differential Equations*, 3(3):273–298, 1995.

[13] P. Cannarsa and C. Sinestrari. On a class of nonlinear time optimal control problems. *Discrete Contin. Dynam. Systems*, 1(2):285–300, 1995.

[14] P. Cannarsa and C. Sinestrari. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, volume 58. Springer Science & Business Media, 2004.

[15] W.-L. Chow. Über systeme von linearen partiellen differentialgleichungen erster ordnung. *Mathematische Annalen*, 117(1):98–105, 1940.

[16] F.-H. Clarke. *Optimization and nonsmooth analysis*, volume 5. SIAM, 1990.

[17] F.-H Clarke, Y.-S. Ledyaev, R.-J. Stern, and P.-R. Wolenski. *Nonsmooth analysis and control theory*, volume 178. Springer Science & Business Media, 2008.

[18] G. Colombo and K.-T. Nguyen. On the structure of the minimum time function. *SIAM Journal on Control and Optimization*, 48(7):4776–4814, 2010.

[19] L. Hörmander. Hypoelliptic second order differential equations. *Acta Mathematica*, 119(1):147–171, 1967.

[20] D. Jerison and A. Sánchez-Calle. Subelliptic, second order differential operators. *Complex analysis, III (College Park, Md., 1985–86)*, 1277:46–77, 1987.

[21] E.-B. Lee and L. Markus. *Foundations of optimal control theory*. Krieger Publishing Company, 1967.

[22] R. Montgomery. *A tour of subriemannian geometries, their geodesics and applications*. Number 91. American Mathematical Soc., 2006.

[23] P.-K. Rashevsky. About connecting two points of a completely nonholonomic space by admissible curve. *Uch. Zapiski Ped. Inst. Libknechta*, 2:83–94, 1938.

[24] L. Rifford. *Sub-Riemannian geometry and optimal transport*. Springer Science & Business Media, 2014.

[25] E. Trélat. Some properties of the value function and its level sets for affine control systems with quadratic cost. *Journal of Dynamical and Control Systems*, 6(4):511–541, 2000.

[26] R. Vinter. *Optimal control*. Birkhäuser Boston Inc., Boston, MA, 2010.