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DISTANCE ESTIMATES FOR STATE CONSTRAINED TRAJECTORIES OF INFINITE DIMENSIONAL DIFFERENTIAL INCLUSIONS

H. FRANKOWSKA, E.M. MARCHINI, AND M. MAZZOLA

ABSTRACT. This paper concerns estimates on the distance between a trajectory of a differential inclusion and the set of feasible trajectories of the same inclusion, feasible meaning confined to a given set of constraints. We apply these estimates to investigate Lipschitz continuity of the value functions arising in optimal control, and to variational inclusions, useful for proving non degenerate necessary optimality conditions. The main feature of our analysis is the infinite dimensional framework, which can be applied to models involving PDEs.

Keywords. Semilinear differential inclusion, state constraint, relaxation theorem, neighboring feasible trajectory theorem.

1. INTRODUCTION

Consider the differential inclusion

$$(1.1) \quad \dot{x}(t) \in F(t, x(t)), \quad \text{a.e. } t \in [t_0, 1], \quad x(t_0) = x_0$$

under the state constraint:

$$(1.2) \quad x(t) \in K, \quad \text{for } t \in [t_0, 1],$$

where $F : I \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a set-valued map, $I = [0, 1]$, $t_0 \in I$ and K is a closed subset of \mathbb{R}^n . Estimates on the distance between a trajectory of differential inclusion (1.1) and the set of all its feasible trajectories, that is satisfying in addition the state constraint (1.2), have been the object of an intense study, see [4, 5, 9, 18, 20, 21] and their bibliographies, to mention only a few. Such estimates are usually called neighbouring feasible trajectory theorems and their interest is due, on one hand, to constructive proofs, not requiring penalization and, on another hand, to the wide-spread applications in state constrained optimal control problems. Regularity properties of the value function, sensitivity relations, dynamic programming, non degenerate form of necessary optimality conditions, are some of the results obtained by using such distance estimates. Also the developed constructions of feasible trajectories did allow to obtain relaxation theorems under state constraints, i.e. to prove density of feasible trajectories in the set of relaxed feasible trajectories (that is when $F(t, x)$ are replaced by their convex envelopes).

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The infinite dimensional control systems and differential inclusions under state constraints, the natural framework for the description of many physical phenomena modeled by PDEs, as diffusion, vibration of strings or membranes, population dynamics, fluid dynamics, have been analyzed, by such direct approach, for the first time in [17], where, in particular, a relaxation theorem has been proved. The differential inclusion under investigation takes the following form

$$(1.3) \quad \dot{x}(t) \in \mathbb{A}x(t) + F(t, x(t)), \quad \text{a.e. } t \in [t_0, 1], \quad x(t_0) = x_0$$

subject to the similar, but infinite dimensional, state constraint:

$$(1.4) \quad x(t) \in K, \quad \text{for all } t \in [t_0, 1].$$

Here, the densely defined unbounded linear operator \mathbb{A} is the infinitesimal generator of a strongly continuous semigroup $S(t) : X \rightarrow X$, X is an infinite dimensional separable Banach space, $F : I \times X \rightsquigarrow X$ is a set-valued map with closed non-empty images and K is a given subset of X . The trajectories of (1.3) are understood in the mild sense (see [26]).

The main result in [17] states that, under a convenient inward pointing condition, any *feasible* trajectory, that is any mild solution x of (1.3) satisfying (1.4), can be approximated by a mild trajectory of (1.3) lying in the interior of K for $t > t_0$. As in the finite dimensional case, this kind of approximations plays a key role in many proofs dealing with constrained problems, see e.g. the references mentioned before. The aim of our paper is to make a further step forward and to extend results on distance estimates to the infinite dimensional setting. In particular, we prove a *neighboring feasible trajectory theorem* stating that under an inward pointing condition, there exists $c_0 > 0$ such that for any trajectory \hat{x} of (1.3) and for any $\varepsilon > 0$, one can construct a *feasible* trajectory x lying in the interior of K and satisfying

$$\|\hat{x} - x\|_{\mathcal{C}([t_0, 1], X)} \leq c_0 \left(\max_{t \in [t_0, 1]} \text{dist}_K(\hat{x}(t)) + \varepsilon \right),$$

where $\max_{t \in [t_0, 1]} \text{dist}_K(\hat{x}(t))$ is interpreted as a measure of the constraint violation by \hat{x} . As a consequence, we obtain two applications of these results to infinite dimensional control problems: sufficient conditions for local Lipschitz continuity of the value function associated to the Mayer optimal control problem under state constraints, and to variational inclusions, playing an important role in proving nondegenerate necessary conditions for optimality.

To obtain neighbouring feasible trajectory theorems in the finite dimensional framework under very general assumptions on F and K , in [18, 19] a new inward pointing condition has been introduced. When F is bounded and continuous near the boundary of K , it can be written in the following way:

$$(1.5) \quad \forall R > 0, \exists \rho > 0 \text{ such that } \forall \bar{x} \in \partial K \cap RB, \\ \text{if } \sigma(\bar{x}; v) \geq 0 \text{ for some } t \in I, v \in F(t, \bar{x}), \text{ then } \inf_{\bar{v} \in F(t, \bar{x})} \sigma(\bar{x}; \bar{v} - v) \leq -\rho.$$

In the above

$$\sigma(x; y) = \sup_{\xi \in \partial d_K(x)} \langle \xi, y \rangle, \quad \forall y \in X,$$

where $d_K(x)$ is the oriented distance from $x \in X$ to K defined by

$$d_K(x) = \begin{cases} \inf_{k \in K} \|x - k\|_X & \text{if } x \notin K \\ -\inf_{k \in (X \setminus K)} \|x - k\|_X & \text{otherwise,} \end{cases}$$

and $\partial d_K(x)$ denotes the Clarke generalized gradient of d_K at the point $x \in X$.

Below we will use also the classical notion of distance from $x \in X$ to K , defined by

$$\text{dist}_K(x) = \inf_{k \in K} \|x - k\|_X.$$

Condition (1.5) can be interpreted in the following way: for every $(t, \bar{x}) \in I \times \partial K$, any admissible velocity $v \in F(t, \bar{x})$ that is “pushing outward” in the sense that $\sigma(\bar{x}; v) \geq 0$, can be corrected by a “compensating velocity” $\bar{v} \in F(t, \bar{x})$, satisfying $\sigma(\bar{x}; \bar{v} - v) \leq -\rho$.

This kind of condition allows to treat constraints with nonsmooth boundary, arising naturally in applied sciences. Nevertheless, when K has a smooth boundary, then (1.5) reduces to the classical inward pointing condition

$$(1.6) \quad \forall R > 0, \exists \rho > 0 \text{ such that } \inf_{\bar{v} \in F(t, \bar{x})} \langle \nabla d_K(\bar{x}), \bar{v} \rangle \leq -\rho, \quad \forall (t, \bar{x}) \in I \times (\partial K \cap RB)$$

that is equivalent to the Soner condition [27] when K and F are bounded, see [19] for more details.

In Proposition 5.4 we discuss an analogous reduction of the inward pointing condition (1.5) when X is a reflexive Banach space.

We aim to work in great generality, to cover a large class of models: the infinite dimensional state space X is Banach, and the unbounded operator \mathbb{A} generates a semigroup $S(t)$ which is merely strongly continuous. For this reason, in our paper the semigroup $S(t)$ is involved in the formulation of the inward pointing conditions.

To simplify the notation, let us define for any positive η the set

$$\partial^\eta K = \{x \in K + \eta B : S(\tau)x \in \partial K + \eta B \text{ for some } \tau \in [0, \eta]\},$$

and for every $\eta > 0$ and $x \in X$, the set

$$\mathcal{A}^\eta(x) = \{(\tau, z) \in [0, \eta] \times X : S(\tau)x \in \partial K + \eta B, z \in B(S(\tau)x, \eta)\},$$

and the function $\Sigma_\eta(x; \cdot) : X \rightarrow [-\infty, +\infty)$ that associates to any $v \in X$ the value

$$\Sigma_\eta(x; v) = \sup_{(\tau, z) \in \mathcal{A}^\eta(x)} \sigma(z; S(\tau)v),$$

with the convention that the supremum over the empty set is equal to $-\infty$. The introduction of the sets $\partial^\eta K$ and $\mathcal{A}^\eta(x)$ is due to the fact that, because of the presence of the semigroup $S(\cdot)$ and since, in general, the boundary of K is not compact, we cannot consider anymore only the points on the boundary of K , but we are led to extend the inward pointing condition to points that are “sent” by the semigroup into a neighborhood of ∂K . Let us underline that $\partial K + \eta B \subset \partial^\eta K$ for any $\eta > 0$ and that this inclusion may be strict. The function $\Sigma_\eta(x; \cdot)$ allows to define “outward” and “compensating” velocities. Observe that it is equal to $-\infty$ whenever $S(\tau)x$ does not belong to $\partial K + \eta B$ for all $\tau \in [0, \eta]$.

Thus, in our very general context, the inward pointing condition takes the form

$$(1.7) \quad \forall R > 0, \exists \eta, \rho, M > 0, \exists J \subset I \text{ such that } \mu(J) = 1 \text{ and if } \Sigma_\eta(x; v) \geq 0 \\ \text{for some } t \in J, x \in RB \cap \partial^n K, v \in F(t, x), \text{ then } \exists \bar{v} \in F(t, x) \cap B(v, M) \text{ satisfying} \\ \max \{ \Sigma_\eta(x; \bar{v} - v); \Sigma_\eta(x; \bar{v}) \} \leq -\rho.$$

Because of the generality of setting, this inward pointing condition may seem complicated. Nevertheless, some simplifications of (1.7) hold: we will prove in Lemma 5.1 that, when working with locally bounded and convex valued F , condition (1.7) is reduced to

$$(1.8) \quad \forall R > 0, \exists \eta, \rho > 0, \exists J \subset I \text{ such that } \mu(J) = 1 \text{ and whenever } \Sigma_\eta(x; v) \geq 0 \\ \text{for some } t \in J, x \in RB \cap \partial^n K, v \in F(t, x), \text{ then } \exists \bar{v} \in F(t, x) \text{ satisfying} \\ \Sigma_\eta(x; \bar{v} - v) \leq -\rho.$$

Further, if some compactness is present, conditions (1.7), (1.8) can be drastically simplified and, in some cases, reduced to its finite dimensional counterpart (1.5) or (1.6), as we show in Section 5.

The paper is structured as follows: Section 2 contains notations, definitions and assumptions. In Section 3 we provide examples of physical models satisfying the main hypotheses in use. The new results are stated in Section 4, while in Section 5 we provide the conditions allowing to simplify (1.7) into (1.5) or (1.6), and applications to the concrete examples. Sections 6 and 7 deal with Lipschitz continuity of the value function and with variational inclusions, and the final Section 8 is devoted to the proofs of the main results.

2. PRELIMINARIES

In this section we list the notation and the main assumptions in use throughout the paper.

2.1. Notation.

- X is a separable Banach space;
- $B(x, r)$ denotes the closed ball of center $x \in X$ and radius $r > 0$; B is the closed unit ball in X centered at 0;
- given a Banach space Y , $\mathbb{L}(X, Y)$ denotes the Banach space of bounded linear operators from X into Y , $\mathcal{C}(I, X)$ the space of continuous functions from I to X , $L^p(I, X)$ the space of Bochner L^p integrable functions from I to X , and $L^\infty(I, X)$ the space of measurable essentially bounded functions from I to X ;
- $\langle \cdot, \cdot \rangle$ stands for the duality pairing on $X^* \times X$;
- μ is the Lebesgue measure on the real line;
- given a set-valued map $\mathcal{F} : X \rightsquigarrow X$, $x \in X$ and $y \in \mathcal{F}(x)$, the derivative $d\mathcal{F}(x, y) : X \rightsquigarrow X$ is defined by

$$v \in d\mathcal{F}(x, y)w \quad \Leftrightarrow \quad \lim_{h \rightarrow 0^+} \text{dist} \left(v, \frac{\mathcal{F}(x + hw) - y}{h} \right) = 0;$$

- the intermediate (or adjacent) cone to K at x is defined as

$$\begin{aligned} I_K(x) &= \left\{ v \in X : \lim_{h \rightarrow 0^+} \text{dist}\left(v, \frac{K - x}{h}\right) = 0 \right\} \\ &= \left\{ v \in X : \forall h_i \rightarrow 0^+, \exists w_i \rightarrow v \text{ such that } x + h_i w_i \in K \right\}. \end{aligned}$$

We will use the following notion of solution.

Definition 2.1. Let $t_0 \in I$ and $x_0 \in X$. A function $x \in \mathcal{C}([t_0, 1], X)$ is a (mild) solution of (1.3) with initial datum $x(t_0) = x_0$ if there exists a function $f^x \in L^1([t_0, 1], X)$ such that

$$(2.1) \quad f^x(t) \in F(t, x(t)), \quad \text{for a.e. } t \in (t_0, 1)$$

and

$$(2.2) \quad x(t) = S(t - t_0)x_0 + \int_{t_0}^t S(t - s)f^x(s) ds, \quad \text{for any } t \in [t_0, 1],$$

i.e. f^x is an integrable selection of the set valued map $t \rightsquigarrow F(t, x(t))$ and x is a mild solution (see [26]) of the initial value problem

$$\begin{cases} \dot{x}(t) = \mathbb{A}x(t) + f^x(t), & \text{for a.e. } t \in [t_0, 1] \\ x(t_0) = x_0. \end{cases}$$

We denote by $\mathcal{S}_{[t_0, 1]}^K(x_0)$ the set of feasible trajectories of (1.3) with initial datum x_0 . Since $S(t)$ is a strongly continuous semigroup, there exists $M_S \geq 1$ such that

$$(2.3) \quad \|S(t)\|_{\mathbb{L}(X, X)} \leq M_S, \quad \text{for any } t \in I.$$

The differential inclusion (1.3) is a convenient tool to investigate for example the semi-linear control system

$$(2.4) \quad \begin{cases} \dot{x}(t) = \mathbb{A}x(t) + f(t, x(t), u(t)), & u(t) \in U, \quad \text{a.e. } t \in [t_0, 1] \\ x(t_0) = x_0 \end{cases}$$

where U is a complete separable metric space. Setting $F(t, x) = f(t, x, U)$, we can reduce (2.4) to (1.3) by applying a measurable selection theorem.

2.2. Assumptions. The following conditions are assumed in the main results:

- positive invariance of K by the semigroup:

$$(2.5) \quad S(t)K \subset K, \quad \forall t \in I;$$

- for every $(t, x) \in I \times X$, $F(t, x)$ is closed, non empty, and, for any $x \in X$,

$$(2.6) \quad \text{the set-valued map } F(\cdot, x) \text{ is Lebesgue measurable;}$$

- for every $t \in I$, the set-valued map $F(t, \cdot)$ is locally Lipschitz in the following sense: for any $R > 0$, there exists $k_R \in L^1(I, \mathbb{R}^+)$ such that, for a.e. $t \in I$ and any $x, y \in RB$,

$$(2.7) \quad F(t, x) \subset F(t, y) + k_R(t)\|x - y\|_X B;$$

- there exists $\phi \in L^1(I, \mathbb{R}^+)$ such that, for a.e. $t \in I$ and any $x \in X$,
- $$(2.8) \quad F(t, x) \subset \phi(t)(1 + \|x\|_X)B.$$

3. EXAMPLES OF INVARIANT SETS

The great generality of the main assumptions allows to apply our results to a number of phenomena modeled by PDEs. The only restrictive condition regards the positive invariance of the state constraint K by the linear semigroup $S(t)$, namely assumption (2.5). A large literature has been devoted to ensure such a property. For instance, when K is closed and convex, necessary and sufficient conditions for (2.5) are well known. To mention only few contributions in this direction, see e.g. [7, 11, 16, 25] and the references therein. In particular, in the case when K is a cone inducing a partial ordering in the Banach space X , invariance properties have been deeply studied due to their importance in several applications. For instance in models describing populations phenomena this kind of constraint is a natural requirement. When \mathbb{A} is the realization of a second order elliptic operator, invariance of pointwise constraints (such as positivity of solutions) is provided by the maximum principle (and its generalizations). Among this variety of models, we have selected a family of problems satisfying (2.5), to which we can apply our main theorems, see Section 5 for further details.

3.1. The case of the contraction semigroups. A meaningful class of examples is provided by systems of type (1.3), where the operator \mathbb{A} generates a C_0 semigroup $S(t)$ of contraction. Here the constraint $K = RB$ fulfills the required invariance (2.5), allowing to impose an upper bound on the associated energy. In this framework, many systems arising from PDEs can be found in the literature.

Example 3.1 (*Boltzmann viscoelasticity*). We analyze at first a model for the isothermal viscoelasticity, see [17] and the references therein, where the same problem has been studied. This phenomenon is described by an integrodifferential equation. Namely, taking $\Omega \subset \mathbb{R}^3$, a bounded domain with smooth boundary $\partial\Omega$ representing the region occupied by an elastic body, and a *memory kernel* μ , accounting for the hereditary properties of the viscoelastic material, the displacement function $u = u(t, \mathbf{x})$, $t \in (0, 1)$ and $\mathbf{x} \in \Omega$, satisfies

$$(3.1) \quad \partial_{tt}u(t) - \Delta \left[u(t) - \int_0^\infty \mu(s)u(t-s) ds \right] \in F(t, u(t)),$$

(we omit the variable \mathbf{x}). Imposing Dirichlet boundary conditions, complies with the assumption that the body is kept fixed at the boundary of Ω . As detailed in [10, 17], by introducing an auxiliary variable in accord to the classical Dafermos' *history approach* [14], (3.1) can be rewritten as a differential inclusion of type (1.3), where, in a suitable functional Hilbert space X , a linear operator \mathbb{A} generates a strongly continuous semigroup of contractions $S(t) : X \rightarrow X$.

Example 3.2 (*The acoustic wave equation*). The second example arises in theoretical acoustics, when modeling transverse vibrations in a membrane whose boundary can vibrate. The differential inclusion ruled by the classical wave equation

$$(3.2) \quad \partial_{tt}u \in \Delta u + F(t, u),$$

for $u = u(t, \mathbf{x})$, $t \geq 0$, $\mathbf{x} \in \Omega$, where a smooth bounded domain Ω of \mathbb{R}^N , is endowed with the so called *acoustic boundary conditions*,

$$(3.3) \quad m\partial_{tt}\delta + d\partial_t\delta + k\delta + \rho u_t = 0$$

underlying the assumption that any $\mathbf{x} \in \partial\Omega$ reacts to the excess pressure of the wave as a resistive harmonic oscillator. Here $\delta = \delta(t, \mathbf{x})$, $t \geq 0$, $\mathbf{x} \in \partial\Omega$ is the normal displacement of the boundary, ρ is the density, $m, d, k \in \mathcal{C}(\partial\Omega)$ represent respectively the mass, the friction coefficient, and the spring constant. Assuming the boundary impenetrable, the following compatibility condition follows: $\partial_t\delta = \partial_\nu u$, where $\partial_\nu u$ is the directional derivative of u in the direction of the outward normal to Ω . These boundary conditions have been introduced by Moser and Ingard in [24], and formalized in a rigorous mathematical framework in [3], see also [13, 22]. As shown in [22], equation (3.2) can be rewritten in the form (1.3), with an operator \mathbb{A} that is a 4×4 matrix generating a strongly continuous semigroup of contractions in the Hilbert space

$$X = H^1(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega) \times L^2(\partial\Omega)$$

with norm

$$\|x\|_X^2 = \int_{\Omega} \left(\rho |\nabla u|^2 + \frac{\rho}{c^2} |\partial_t u|^2 \right) d\mathbf{x} + \int_{\partial\Omega} \left(k |\delta|^2 + m |\partial_t \delta|^2 \right) dS,$$

for $x = (u, \partial_t u, \delta, \partial_t \delta)$.

3.2. Conservative systems. We consider here the non convex set $K = RB \setminus rB$, with $0 < r < R$, complying with the request that the energy associated to the system should be constrained between two bounds. As example, we can take the acoustic wave equation (3.2)-(3.3). Defining as energy of the solution $E = \|x\|_X^2$, a simple estimate, see [22], shows that

$$\frac{dE}{dt} = -2 \int_{\partial\Omega} d |\partial_t \delta|^2 dS,$$

implying that (3.2)-(3.3) is a conservative system once the friction coefficient $d \equiv 0$. Hence the annulus $RB \setminus rB$, with $0 < r < R$ is positively invariant under the action of the associated semigroup, and (2.5) is satisfied for this type of constraint.

3.3. Pointwise constraints. A class of parabolic problems arising in heat diffusion is considered, where we impose a *pointwise* state constraint, such as the positivity of the solutions. With the exception of a one-dimensional problem, whose framework is the Hilbert space $H^1(0, 1)$ (embedded in $\mathcal{C}([0, 1])$), in higher dimension, in order to have $\text{Int } K \neq \emptyset$, we work in the Banach space of the continuous functions. At first the zero-obstacle problem is investigated, namely, we require the positivity of the solutions.

Example 3.3 (*A one-dimensional heat equation*). To model the heat flux in a cylindrical bar, with perfectly insulated lateral surface and whose length is much larger than its cross-section, a one-dimensional equation is introduced. For $u = u(t, s)$, $t, s \in [0, 1] \times [0, 1]$, let

$$(3.4) \quad \partial_t u \in \partial_{ss} u - u + F(t, u),$$

endowed with the Neumann boundary conditions, complying with the assumption that the heat flux at the two ends of the bar is zero. In order to rewrite (3.4) as (1.3), we introduce

the operator $\mathbb{A} = \partial_{ss} - I$ with domain $D(\mathbb{A}) = \{x \in H^2(0, 1) : x'(0) = x'(1) = 0\}$. \mathbb{A} is the infinitesimal generator of a strongly continuous semigroup $\tilde{S}(t)$ on $Y = L^2(0, 1)$. The space $X = H^1(0, 1)$ is an invariant subspace of Y , and the restriction of $\tilde{S}(t)$ to X (called $S(t)$) is a strongly continuous semigroup in X , see [28]. Then, we can apply our results to $S(t) : X \rightarrow X$ where the cone of nonnegative functions

$$K = \{x \in X : x(s) \geq 0, \text{ for } s \in [0, 1]\},$$

has non empty interior. Further, K is positively invariant under the action of $S(t)$. An interesting property of this constraint set, which will allow to simplify drastically our inward point condition, is that the set of the external normals to K is pre-compact, see [17] for the proof.

Example 3.4 (*Heat equation with dynamical boundary conditions*). In this example, we study a heat conduction process with dynamic (or Wentzell) boundary conditions

$$(3.5) \quad \begin{cases} \partial_t u \in \Delta u + F(t, u) & \text{in } (0, 1) \times \Omega \\ \partial_t u = -\partial_\nu u - \gamma u & \text{on } (0, 1) \times \partial\Omega. \end{cases}$$

Here $u = u(t, \mathbf{x})$, Ω is a bounded domain of \mathbb{R}^N with smooth boundary and $\gamma \in \mathcal{C}(\partial\Omega)$. This kind of systems has been the object of an intense research, due to its relevance from a modeling point of view, see e.g. [23]. As outlined in [2, 29], problem (3.5) can be seen as a heat equation with Wentzell boundary condition, and studied in the space $\mathcal{C}(\overline{\Omega})$, a natural space for such boundary conditions. We associate to (3.5) the operator

$$\mathbb{A} = \Delta, \quad \text{dom}(\mathbb{A}) = \{u \in \mathcal{C}_\nu^1(\overline{\Omega}) : \Delta u \in \mathcal{C}(\overline{\Omega}); \Delta u + \partial_\nu u + \gamma u = 0 \text{ on } \partial\Omega\},$$

with $\mathcal{C}_\nu^1(\overline{\Omega})$ being the space of functions $u \in \mathcal{C}(\overline{\Omega})$ for which the outer normal derivative $\partial_\nu u$ exists and is continuous on $\partial\Omega$, see [1, 15]. As proved e.g. in [15], this operator generates a positive compact C_0 semigroup in the space of continuous functions $\mathcal{C}(\overline{\Omega})$. Hence, the constraint set

$$K = \{u \in \mathcal{C}(\overline{\Omega}) : u(\mathbf{x}) \geq 0, \text{ for } \mathbf{x} \in \overline{\Omega}\}$$

has non-empty interior and (2.5) is satisfied.

Example 3.5 (*A further diffusion equation*). In this last example, we consider again a pointwise constraint: we want to control a temperature in a room by imposing that it is constrained between two fixed values. The model describing the heat process is governed by the heat equation with Neumann boundary conditions

$$(3.6) \quad \begin{cases} \partial_t u \in \Delta u + F(t, u) & \text{in } (0, 1) \times \Omega \\ \partial_\nu u = 0 & \text{on } (0, 1) \times \partial\Omega, \end{cases}$$

where $u = u(t, \mathbf{x})$ and Ω is a bounded domain of \mathbb{R}^N with smooth boundary. The operator associated to the system (3.6) generates a positive C_0 -semigroup on $Y = L^2(\Omega)$. As done in example 3.3, we can consider its restriction to the invariant subspace $X = \mathcal{C}(\overline{\Omega})$, see e.g. in [1], where it has been proved that the strongly continuous semigroup induced on X is positive and compact. Further, applying the parabolic maximum principle, see [6], we deduce that the state constraint

$$K = \{u \in \mathcal{C}(\overline{\Omega}) : -1 \leq u(\mathbf{x}) \leq 1, \mathbf{x} \in \overline{\Omega}\}$$

is positively invariant, yielding the validity of (2.5).

4. THE MAIN RESULTS

This section contains the main results of the paper. Their proofs are postponed to Section 8.

Consider the semilinear differential inclusion

$$(4.1) \quad \dot{x}(t) \in \mathbb{A}x(t) + F(t, x(t)), \quad \text{a.e. } t \in [t_0, 1],$$

subject to the state constraint:

$$(4.2) \quad x(t) \in K, \quad \text{for all } t \in [t_0, 1],$$

where K is a closed subset of X .

4.1. Neighbouring feasible trajectory theorems. Our first neighboring feasible trajectory theorem is stated in the greatest generality.

Theorem 4.1. *Assume (2.5)–(2.8) and (1.7). Then, for any $R_0 > 0$ there exists $c_0 > 0$ such that, for any $t_0 \in I$, any $\varepsilon > 0$, and any trajectory \hat{x} of (4.1) with $\hat{x}(t_0) \in K \cap R_0 B$, there exists $x \in \mathcal{S}_{[t_0, 1]}^K(\hat{x}(t_0))$ satisfying*

$$(4.3) \quad x(t) \in \text{Int } K, \quad \text{for any } t \in (t_0, 1]$$

and

$$(4.4) \quad \|\hat{x} - x\|_{C([t_0, 1], X)} \leq c_0 \left(\max_{t \in [t_0, 1]} \text{dist}_K(\hat{x}(t)) + \varepsilon \right).$$

The same conclusions are valid if (2.5)–(2.8) hold true with $\phi \in L^\infty(I, \mathbb{R}^+)$, F is convex valued, and (1.7) is replaced by (1.8).

Some applications (see Section 7) require weaker estimates than those expressed in Theorem 4.1 and hold true under weaker versions of conditions (1.7) and (1.8):

$$(4.5) \quad \forall D \subset X \text{ compact, } \exists \eta, \rho, M > 0, \exists J \subset I \text{ such that } \mu(J) = 1 \text{ and if} \\ \sup_{\tau \in [0, \eta], z \in B(x, \eta)} \sigma(z; S(\tau)v) \geq 0 \text{ for some } t \in J, x \in D \cap (\partial K + \eta B), v \in F(t, x), \text{ then} \\ \exists \bar{v} \in F(t, x) \cap B(v, M) \text{ with } \sup_{\tau \in [0, \eta], z \in B(x, \eta)} \{ \sigma(z; S(\tau)(\bar{v} - v)); \sigma(z; S(\tau)\bar{v}) \} \leq -\rho,$$

respectively

$$(4.6) \quad \forall D \subset X \text{ compact, } \exists \eta, \rho > 0, \exists J \subset I \text{ such that } \mu(J) = 1 \text{ and if} \\ \sup_{\tau \in [0, \eta], z \in B(x, \eta)} \sigma(z; S(\tau)v) \geq 0 \text{ for some } t \in J, x \in D \cap (\partial K + \eta B), v \in F(t, x), \text{ then} \\ \exists \bar{v} \in F(t, x) \text{ satisfying } \sup_{\tau \in [0, \eta], z \in B(x, \eta)} \sigma(z; S(\tau)(\bar{v} - v)) \leq -\rho.$$

Theorem 4.2. *Assume (2.5)–(2.8), (4.5) and that*

$$(4.7) \quad \text{for any compact } D_0 \subset X, \text{ there exists a compact } D \subset X \text{ such that} \\ \forall t_0 \in I, y([t_0, 1]) \subset D, \text{ for any trajectory } y \text{ of (4.1) satisfying } y(t_0) \in D_0.$$

Then, for every compact set $D_0 \subset X$ *, we can find* $c_0 = c(D_0) > 0$ *such that, for any* $t_0 \in I$ *, any* $\varepsilon > 0$ *, and any trajectory* \hat{x} *of (4.1) with* $\hat{x}(t_0) \in K \cap D_0$ *, there exists* $x \in \mathcal{S}_{[t_0, 1]}^K(\hat{x}(t_0))$ *satisfying (4.3) and (4.4).*

The same conclusions hold true if (4.5) is replaced by (4.6), $\phi \in L^\infty(I, \mathbb{R}^+)$ and F is convex valued.

The following proposition provides two sufficient conditions for (4.7) to be satisfied. The first of them holds true for instance in the model of Examples 3.4, 3.5.

Proposition 4.3. *Assume (2.8) and that either*

$$(4.8) \quad S(\cdot) \text{ is a compact semigroup}$$

or

$$(4.9) \quad \forall R > 0 \text{ there exists a compact } \mathcal{K}_R \subset X \text{ such that } F(t, x) \subset \mathcal{K}_R, \forall (t, x) \in I \times RB.$$

Then condition (4.7) is satisfied.

Proof. Let D_0 be a compact subset of X . Assume first (4.9) and consider a trajectory y of (4.1) with initial datum $y_0 = y(t_0) \in D_0$. By the representation formula (2.2),

$$y(t) = S(t - t_0)y_0 + \int_{t_0}^t S(t - s)f^y(s)ds,$$

with f^y as in (2.1) (and x replaced by y). By (2.8) and the Gronwall lemma there exists $R > 0$ only depending on D_0 and ϕ , such that $y(t) \in RB$, for any $t \in [t_0, 1]$. By the continuity properties of the semigroup $S(t)$ and by the compactness assumption (4.9), there exists a compact set $\tilde{D} \subset X$, depending on D_0 and R satisfying

$$S(t - t_0)y_0 \in \tilde{D} \text{ and } S(t - s)f^y(s) \in \tilde{D}, \quad \text{for any } t \text{ and a.e. } s \text{ in } [t_0, 1]$$

implying the claimed

$$y([t_0, 1]) \subset D = 2\overline{c_0\tilde{D}}.$$

In case assumption (4.8) holds, our proof is based on [8, Lemma 5.4]. Consider the relaxed system

$$(4.10) \quad \dot{x}(t) \in \mathbb{A}x(t) + \overline{c_0}F(t, x(t)), \quad \text{a.e. } t \in [t_0, 1].$$

Adapting to the differential inclusions setting the assumptions of Lemma 5.4 in [8], we deduce that the set

$$\mathcal{S}(D_0) = \{\tilde{x} \text{ solution to (4.10) with } \tilde{x}(t_0) \in D_0\}$$

is compact in $\mathcal{C}([t_0, 1], X)$. This implies that the set

$$D = \{\tilde{x}(t) : \tilde{x} \in \mathcal{S}(D_0), t \in [t_0, 1]\}$$

is compact. Indeed, let $x_i(t_i) \in D$, taking a subsequence and keeping the same notation we have that $t_i \rightarrow \tilde{t}$, for some $\tilde{t} \in [t_0, 1]$ and, for some trajectory $\tilde{x} \in \mathcal{S}(D_0)$,

$$\sup_{t \in [t_0, 1]} \|\tilde{x}_i(t) - \tilde{x}(t)\|_X \rightarrow 0.$$

This yields

$$\begin{aligned} \|x_i(t_i) - \tilde{x}(\tilde{t})\|_X &\leq \|x_i(t_i) - \tilde{x}(t_i)\|_X + \|\tilde{x}(t_i) - \tilde{x}(\tilde{t})\|_X \\ &\leq \sup_{t \in [t_0, 1]} \|x_i(t) - \tilde{x}(t)\|_X + \|\tilde{x}(t_i) - \tilde{x}(\tilde{t})\|_X \rightarrow 0, \end{aligned}$$

proving the compactness. Finally, since any solution of (1.3) is also a solution of (4.10), D is as claimed. \square

4.2. Approximation and relaxation theorems. Results obtained in [17] can be generalized by using a less restrictive inward pointing condition stated in terms of the mapping $\Sigma_\eta(x; \cdot)$. We do not provide here the proofs, since they follow easily by slight modifications of the proofs contained in [17].

Theorem 4.4. *Assume (2.5)–(2.8) and that*

$$(4.11) \quad \forall \bar{x} \in \partial K, \exists \eta, \rho, M > 0, \exists J \subset I \text{ such that } \mu(J) = 1 \text{ and if} \\ \sup_{\tau \in [0, \eta], z \in B(x, \eta)} \sigma(z; S(\tau)v) \geq 0 \text{ for some } t \in J, x \in K \cap B(\bar{x}, \eta), v \in F(t, x), \text{ then} \\ \exists \bar{v} \in F(t, x) \cap B(v, M) \text{ satisfying } \Sigma_\eta(x; \bar{v} - v) \leq -\rho.$$

Then, for any $\varepsilon > 0$ and any trajectory \hat{x} of (4.1), (4.2), there exists $x \in \mathcal{S}_{[t_0, 1]}^K(\hat{x}(t_0))$ satisfying (4.3) and

$$(4.12) \quad \|\hat{x} - x\|_{C([t_0, 1], X)} \leq \varepsilon.$$

A consequence of the previous result is the following relaxation theorem:

Theorem 4.5. *Assume (2.5)–(2.8) and that*

$$\forall \bar{x} \in \partial K, \exists \eta, \rho, M > 0, \exists J \subset I \text{ such that } \mu(J) = 1 \text{ and if} \\ \sup_{\tau \in [0, \eta], z \in B(x, \eta)} \sigma(z; S(\tau)v) \geq 0 \text{ for some } t \in J, x \in K \cap B(\bar{x}, \eta), v \in \overline{\text{co}}F(t, x), \text{ then} \\ \exists \bar{v} \in \overline{\text{co}}F(t, x) \cap B(v, M) \text{ satisfying } \Sigma_\eta(x; \bar{v} - v) \leq -\rho.$$

Then, for any $\varepsilon > 0$ and any trajectory \hat{x} of

$$\dot{x}(t) \in \mathbb{A}x(t) + \overline{\text{co}}F(t, x(t)), \quad \text{a.e. } t \in [t_0, 1]$$

and (4.2), there exists $x \in \mathcal{S}_{[t_0, 1]}^K(\hat{x}(t_0))$ satisfying (4.3) and (4.12).

4.3. The case of Hilbert spaces. When X is a Hilbert space, the inward pointing conditions can be reformulated by using normal vectors to the boundary of K , instead of generalized gradients of the oriented distance function. This formulation is useful in the applications, as in Examples 3.1, 3.2, 3.3.

Let $\langle \cdot, \cdot \rangle_X$ denote the scalar product in X and let K be a proper closed subset of X such that $K = \overline{\text{Int } K}$. Denote by Z the set of points $z \in X \setminus \partial K$ admitting a unique projection $P_{\partial K}(z)$ on ∂K . This set is dense in X . For every $z \in Z$, set

$$n_z = \frac{z - P_{\partial K}(z)}{\|z - P_{\partial K}(z)\|_X} \text{sgn}(d_K(z)).$$

Let us define for any $\eta > 0$ and $x \in X$, the set

$$\mathcal{B}^\eta(x) = \{(\tau, z) \in [0, \eta] \times Z : S(\tau)x \in \partial K + \eta B, z \in B(S(\tau)x, \eta)\} \subset \mathcal{A}^\eta(x),$$

and the extended real valued function $\Pi_\eta(x; \cdot) : X \rightarrow [-\infty, +\infty)$ that associates to any $v \in X$ the value

$$\Pi_\eta(x; v) = \sup_{(\tau, z) \in \mathcal{B}^\eta(x)} \langle n_z, S(\tau)v \rangle,$$

again with the convention that the supremum over the empty set takes the value $-\infty$. Then an inward pointing condition can be expressed in this setting by replacing (1.7) by

$$(4.13) \quad \forall R > 0, \exists \eta, \rho, M > 0, \exists J \subset I \text{ such that } \mu(J) = 1 \text{ and whenever } \Pi_\eta(x; v) \geq 0$$

for some $t \in J$, $x \in RB \cap \partial^n K$, $v \in F(t, x)$, then $\exists \bar{v} \in F(t, x) \cap B(v, M)$ satisfying

$$\max \{ \Pi_\eta(x; \bar{v} - v); \Pi_\eta(x; \bar{v}) \} \leq -\rho$$

to get the same conclusions as those in Theorems 4.1, 4.2, 4.4, 4.5. To prove it, it is sufficient to follow arguments from [17] using the new conditions.

5. REMARKS ON THE INWARD POINTING CONDITIONS

In this section we analyze the inward pointing conditions. In the presence of compactness assumptions some simplifications hold. The first result below deals with (1.8) and (4.6), useful when working with relaxed differential inclusions, hence important in many applications. In the other three propositions of this section, we provide conditions which guarantee the validity of the neighboring feasible trajectory theorem under the classical inward pointing conditions (1.5) and (1.6).

Lemma 5.1. *Let $F : I \times X \rightsquigarrow X$ be a set-valued map with convex nonempty images, satisfying (2.8) with $\phi \in L^\infty(I, \mathbb{R}^+)$. Then condition (1.8) implies (1.7) and (4.6) implies (4.5).*

Proof. We only prove that (1.8) implies (1.7), the proof of the second implication being analogous.

Assume (1.8), fix $R > 0$, and let η , ρ and J be as in (1.8). Set $\alpha = \|\phi\|_{L^\infty(I; \mathbb{R}^+)}$, $M = 2\alpha(1 + R)$ and $\tilde{J} \subset J$ such that $\mu(\tilde{J}) = 1$, and

$$F(t, x) \subset \alpha(1 + \|x\|_X)B \quad \forall (t, x) \in \tilde{J} \times X.$$

Let $t \in \tilde{J}$, $x \in RB \cap \partial^n K$ and $v \in F(t, x)$ satisfy

$$\Sigma_\eta(x; v) \geq 0.$$

Then

$$\mathcal{D} = \left\{ \bar{v} \in F(t, x) : \Sigma_\eta(x; \bar{v} - v) \leq -\rho \right\}$$

is nonempty. It is clear that $\mathcal{D} \subset B(v, M)$. The proof is performed in two steps.

Step 1. We claim that, there exists $\bar{v} \in \mathcal{D}$ such that

$$(5.1) \quad \Sigma_\eta(x; \bar{v}) < 0.$$

Indeed, arguing by contradiction, assume that any $\bar{v} \in \mathcal{D}$ satisfies

$$(5.2) \quad \Sigma_\eta(x; \bar{v}) \geq 0.$$

Let $\bar{v}_0 \in \mathcal{D}$, then by (1.8) and (5.2), there exists $\bar{v}_1 \in F(t, x)$ such that

$$\Sigma_\eta(x; \bar{v}_1 - \bar{v}_0) \leq -\rho.$$

Notice that

$$\Sigma_\eta(x; \bar{v}_1 - v) \leq \Sigma_\eta(x; \bar{v}_1 - \bar{v}_0) + \Sigma_\eta(x; \bar{v}_0 - v) \leq -2\rho,$$

implying $\bar{v}_1 \in \mathcal{D}$. Iterating the procedure we obtain a sequence $\{\bar{v}_n\}_{n \in \mathbb{N}} \subset \mathcal{D}$ satisfying, for any $n \in \mathbb{N}$,

$$\Sigma_\eta(x; \bar{v}_n - \bar{v}_{n-1}) \leq -\rho.$$

Hence, for any $n \in \mathbb{N}$,

$$\Sigma_\eta(x; \bar{v}_n - \bar{v}_0) \leq \sum_{i=1}^n \Sigma_\eta(x; \bar{v}_i - \bar{v}_{i-1}) \leq -n\rho,$$

leading to a contradiction since by (2.3) and (2.8)

$$|\Sigma_\eta(x; \bar{v} - \bar{v}_0)| \leq M_S |\bar{v} - \bar{v}_0| \leq M_S M,$$

for any $\bar{v} \in F(t, x)$. The claim of Step 1 follows.

Step 2. We prove that there exists $\bar{v} \in \mathcal{D}$ such that

$$(5.3) \quad \Sigma_\eta(x; \bar{v}) \leq -\rho.$$

Let $\bar{v} \in \mathcal{D}$ satisfy (5.1) and consider the continuous function in $[0, 1]$

$$\varphi(\lambda) = \Sigma_\eta(x; \lambda \bar{v} + (1 - \lambda)v).$$

Since $\varphi(0) \geq 0$ and $\varphi(1) < 0$, we get $\varphi(\bar{\lambda}) = 0$ for some $\bar{\lambda} \in [0, 1)$. Further, as $v, \bar{v} \in F(t, x)$ and F has convex images, $\lambda \bar{v} + (1 - \lambda)v \in F(t, x)$, for any $\lambda \in [0, 1]$. Then, by (1.8), for some $\bar{v} \in F(t, x)$,

$$\Sigma_\eta(x; \bar{v} - \bar{\lambda} \bar{v} - (1 - \bar{\lambda})v) \leq -\rho,$$

implying

$$\Sigma_\eta(x; \bar{v} - v) \leq \Sigma_\eta(x; \bar{v} - \bar{\lambda} \bar{v} - (1 - \bar{\lambda})v) + \Sigma_\eta(x; \bar{\lambda}(\bar{v} - v)) \leq -(1 + \bar{\lambda})\rho.$$

Hence $\bar{v} \in \mathcal{D}$. Further,

$$\Sigma_\eta(x; \bar{v}) \leq \Sigma_\eta(x; \bar{v} - \bar{\lambda} \bar{v} - (1 - \bar{\lambda})v) + \varphi(\bar{\lambda}) \leq -\rho,$$

yielding (5.3) and ending the proof. \square

In case some compactness is assumed, we can use (1.5) as the inward pointing condition.

Proposition 5.2. *Let $F : I \times X \rightsquigarrow X$ be a set-valued map satisfying (2.7)–(2.8) with $k_R, \phi \in L^\infty(I, \mathbb{R}^+)$. If*

$$(5.4) \quad F(\cdot, \bar{x}) \text{ is continuous for any } \bar{x} \in \partial K,$$

and

$$(5.5) \quad F(t, \bar{x}) \text{ is compact, for any } t \in I \text{ and any } \bar{x} \in \partial K$$

then, assumption (1.5) implies (4.5) (for a possibly smaller ρ).

Proof. Let D be a compact set and $R > 0$ be such that $D \subset RB$. Set $\alpha = \|\phi\|_{L^\infty(I; \mathbb{R}^+)}$, $\beta = \|k_{2R}\|_{L^\infty(I; \mathbb{R}^+)}$ and let $J \subset I$ be such that $\mu(J) = 1$ and

$$(5.6) \quad F(t, x) \subset \alpha(1 + \|x\|_X)B \quad \forall (t, x) \in J \times X,$$

$$(5.7) \quad F(t, x) \subset F(t, y) + \beta\|x - y\|_X B \quad \forall (t, x, y) \in J \times RB \times RB.$$

By (5.6), for any $t \in J$ and any $x \in D$,

$$(5.8) \quad F(t, x) \subset \alpha(1 + R)B$$

implying, in particular, that

$$\|v - w\|_X \leq M := 2\alpha(1 + R),$$

for any $v, w \in F(t, x)$. Observe also that, by (5.4), the inclusion (5.8) is valid for every $(t, x) \in I \times (\partial K \cap D)$.

Fix $\bar{x} \in \partial K \cap D$ and let $\rho > 0$ be as in (1.5). As in the first step in the proof of Lemma 5.1, for any $t \in I$ and $v \in F(t, \bar{x})$ satisfying $\sigma(\bar{x}; v) \geq 0$, define

$$\mathcal{D}_{t,v} = \left\{ \bar{v} \in F(t, \bar{x}) : \sigma(\bar{x}; \bar{v} - v) \leq -\rho \right\}.$$

By (1.5) we know that $\mathcal{D}_{t,v} \neq \emptyset$. The proof proceeds in three steps.

Step 1. We claim that there exists $\bar{v} \in \mathcal{D}_{t,v}$ satisfying

$$(5.9) \quad \sigma(\bar{x}; \bar{v}) < 0.$$

Arguing by contradiction, we assume that $\sigma(\bar{x}; \bar{v}) \geq 0$, for any $\bar{v} \in \mathcal{D}_{t,v}$ and fix $\bar{v}_0 \in \mathcal{D}_{t,v}$. Then by (1.5), there exists \bar{v}_1 such that

$$\sigma(\bar{x}; \bar{v}_1 - \bar{v}_0) \leq -\rho.$$

Further, $\bar{v}_1 \in \mathcal{D}_{t,v}$, indeed

$$\sigma(\bar{x}; \bar{v}_1 - v) \leq \sigma(\bar{x}; \bar{v}_1 - \bar{v}_0) + \sigma(\bar{x}; \bar{v}_0 - v) \leq -2\rho.$$

Iterating the procedure we obtain a sequence $\{\bar{v}_n\}_{n \in \mathbb{N}} \subset \mathcal{D}_{t,v}$ satisfying, for any $n \in \mathbb{N}$,

$$\sigma(\bar{x}; \bar{v}_n - \bar{v}_{n-1}) \leq -\rho \quad \text{hence} \quad \sigma(\bar{x}; \bar{v}_n - \bar{v}_0) \leq \sum_{i=1}^n \sigma(\bar{x}; \bar{v}_i - \bar{v}_{i-1}) \leq -n\rho.$$

A contradiction, since by (2.3) and (2.8), for any $\bar{v} \in \mathcal{D}_{t,v}$,

$$|\sigma(\bar{x}; \bar{v}_n - \bar{v}_0)| \leq M,$$

implying the first claim.

Step 2. Now, we prove that

$$(5.10) \quad \begin{aligned} &\exists \tilde{\rho} > 0 \text{ such that if } \sigma(\bar{x}; v) \geq 0 \text{ for some } t \in I, v \in F(t, \bar{x}), \\ &\text{then } \exists \bar{v} \in F(t, \bar{x}) \text{ satisfying } \max \{ \sigma(\bar{x}; \bar{v} - v), \sigma(\bar{x}; \bar{v}) \} \leq -\tilde{\rho}. \end{aligned}$$

Notice first that the set

$$\mathcal{E} = \left\{ (t, v) \in I \times X : v \in F(t, \bar{x}) \text{ and } \sigma(\bar{x}; v) \geq 0 \right\}$$

is compact. Indeed, let $(t_n, v_n) \in \mathcal{E}$, then up to a subsequence $t_n \rightarrow t \in I$. By the continuity assumptions on F , there exists $\bar{v}_n \in F(t, \bar{x})$ such that

$$\|v_n - \bar{v}_n\|_X \leq \varepsilon_n, \quad \text{with } \varepsilon_n \rightarrow 0.$$

The compactness assumption (5.5) allows to deduce that, up to a subsequence, $\bar{v}_n \rightarrow \bar{v} \in F(t, \bar{x})$ in X , implying that also $v_n \rightarrow \bar{v}$ in X . Since $\sigma(\bar{x}; v_n) \geq 0$ for any $n \in \mathbb{N}$, also $\sigma(\bar{x}; \bar{v}) \geq 0$, yielding that $(t, \bar{v}) \in \mathcal{E}$. Now, define for every $(t, v) \in \mathcal{E}$, the set

$$\widehat{\mathcal{D}}_{t,v} = \left\{ \bar{v} \in F(t, \bar{x}) : \sigma(\bar{x}; \bar{v} - v) \leq -\frac{\rho}{2} \right\}.$$

As in the previous step we show that

$$\inf_{\bar{v} \in \widehat{\mathcal{D}}_{t,v}} \sigma(\bar{x}; \bar{v}) = -\rho_{t,v} < 0$$

for some $\rho_{t,v} > 0$. In order to prove the claim of Step 2, it remains to show that

$$\inf \{ \rho_{t,v} : (t, v) \in \mathcal{E} \} = \bar{\rho} > 0.$$

Assume for a moment that $\bar{\rho} = 0$. Then we can find a sequence $(t_n, v_n) \in \mathcal{E}$ such that, for any $\bar{v}_n \in \widehat{\mathcal{D}}_{t_n, v_n}$, there exist $\xi_n \in \partial d_K(\bar{x})$ satisfying

$$\langle \xi_n, \bar{v}_n \rangle \geq -\frac{1}{n}.$$

Since \mathcal{E} is compact, up to a subsequence $(t_n, v_n) \rightarrow (t, v) \in \mathcal{E}$. By Step 1 there exists $\bar{v} \in F(t, \bar{x})$ such that

$$(5.11) \quad \langle \xi, \bar{v} - v \rangle \leq -\rho, \quad \langle \xi, \bar{v} \rangle < 0, \quad \text{for every } \xi \in \partial d_K(\bar{x}).$$

By continuity, there exist $\bar{v}_n \in F(t_n, \bar{x})$ converging to \bar{v} when $n \rightarrow \infty$. Then for all large n ,

$$\sup_{\xi \in \partial d_K(\bar{x})} \langle \xi, \bar{v}_n - v_n \rangle \leq -\frac{\rho}{2}.$$

Therefore $\bar{v}_n \in \widehat{\mathcal{D}}_{t_n, v_n}$. Let ξ_n be as above and consider its subsequence converging weakly-star to some $\bar{\xi} \in \partial d_K(\bar{x})$. Thus $\langle \bar{\xi}, \bar{v} \rangle \geq 0$ leading to a contradiction with (5.11).

Hence, taking $\tilde{\rho} = \min \{ \rho/2, \bar{\rho} \}$, we obtain the second claim.

Step 3. We will show that there exists $\eta, \tilde{\rho} > 0$ such that, for any $t \in J, x \in D \cap (\partial K + \eta B)$, and $v \in F(t, x)$ satisfying

$$\sup_{\tau \in [0, \eta]} \sigma(z; S(\tau)v) \geq 0, \quad \text{for some } z \in B(x, \eta),$$

we have

$$\left\{ \bar{v} \in F(t, x) : \sup_{\tau \in [0, \eta], z \in B(x, \eta)} \{ \sigma(z; S(\tau)(\bar{v} - v)), \sigma(z; S(\tau)\bar{v}) \} \leq -\tilde{\rho} \right\} \neq \emptyset.$$

Assume by contradiction that we can find sequences $\delta_i \rightarrow 0+$,

$$t_i \in J, \quad x_i \in D \cap \left(\partial K + \frac{1}{i} B \right), \quad z_i \in B \left(x_i, \frac{1}{i} \right), \quad \xi_i \in \partial d_K(z_i), \quad 0 \leq \tau_i \leq \frac{1}{i}, \quad v_i \in F(t_i, x_i)$$

such that

$$(5.12) \quad \langle \xi_i, S(\tau_i)v_i \rangle \geq -\frac{1}{i},$$

and, for any $\bar{v}_i \in F(t_i, x_i)$ there exist

$$0 \leq s_i^1, s_i^2 \leq \frac{1}{i}, \quad y_i^1 \in B \left(x_i, \frac{1}{i} \right), \quad y_i^2 \in B \left(x_i, \frac{1}{i} \right), \quad \zeta_i^1 \in \partial d_K(y_i^1), \quad \zeta_i^2 \in \partial d_K(y_i^2)$$

satisfying

$$\max \{ \langle \zeta_i^1, S(s_i^1)(\bar{v}_i - v_i) \rangle, \langle \zeta_i^2, S(s_i^2)\bar{v}_i \rangle \} > -\delta_i.$$

Passing to the limit $i \rightarrow \infty$ we obtain that, up to subsequences,

$$(5.13)$$

$$t_i \rightarrow t \in I, \quad x_i, z_i \rightarrow \bar{x} \in \partial K \cap D \text{ in } X, \quad \xi_i \xrightarrow{*} \xi_0 \text{ weakly-star in } X^*, \quad v_i \rightarrow v \in F(t, \bar{x}) \text{ in } X.$$

Concerning the last limit, notice that by (5.4) and (5.7), for any $k \in \mathbb{N}$ there exists i_k such that

$$v_i \in F(t_i, x_i) \subset F(t, \bar{x}) + \frac{1}{k} B, \quad \text{for any } i \geq i_k$$

implying that $\|v_i - w_i\|_X \leq \frac{1}{k}$, for some $w_i \in F(t, \bar{x})$ and, as $F(t, \bar{x})$ is a compact set, up to a subsequence, $w_i \rightarrow v \in F(t, \bar{x})$ yielding $v_i \rightarrow v$. Further, by [12, Proposition 2.1.5], $\xi_0 \in \partial d_K(\bar{x})$. Finally, as $\tau_i \rightarrow 0$, from (5.12) we deduce that

$$0 \leq \lim_{i \rightarrow \infty} \langle \xi_i, S(\tau_i)v_i \rangle = \langle \xi_0, v \rangle.$$

Then, from (5.10) there exists $\bar{v} \in F(t, \bar{x})$ such that

$$(5.14) \quad \max \{ \sigma(\bar{x}; \bar{v} - v), \sigma(\bar{x}; \bar{v}) \} \leq -\tilde{\rho}.$$

Let $\bar{v}_i \in F(t_i, x_i)$ such that $\bar{v}_i \rightarrow \bar{v}$ in X . Taking subsequences and keeping the same notation we obtain

$$(5.15) \quad y_i^1, y_i^2 \rightarrow \bar{x} \text{ in } X, \quad \zeta_i^1 \xrightarrow{*} \zeta^1, \quad \zeta_i^2 \xrightarrow{*} \zeta^2 \text{ weakly-star in } X^*,$$

and

$$(5.16) \quad 0 \leq \liminf_{i \rightarrow \infty} \max \{ \langle \zeta_i^1, \bar{v}_i - v_i \rangle, \langle \zeta_i^2, \bar{v}_i \rangle \} = \max \{ \langle \zeta^1, \bar{v} - v \rangle, \langle \zeta^2, \bar{v} \rangle \}.$$

As $\zeta^1, \zeta^2 \in \partial d_K(\bar{x})$, (5.16) contradicts (5.14), ending the proof. \square

Proposition 5.3. *Let X be reflexive, $F : I \times X \rightsquigarrow X$ be a set-valued map with nonempty images, satisfying (5.4), (2.7)–(2.8) with $k_R, \phi \in L^\infty(I, \mathbb{R}^+)$, and that, for any $\bar{x} \in \partial K$, $F(t, \bar{x})$ is convex, and*

$$(5.17)$$

the set-valued map $\partial d_K(\cdot)$ is upper semicontinuous at \bar{x} and $\partial d_K(\bar{x})$ is compact.

Then, assumption (1.5) implies (4.5).

If X is Hilbert, then (1.7) can be expressed in terms of normal vectors to ∂K as in (4.13). The model in Example 3.3 provides a concrete situation in which this simplified version of the inward pointing condition can be used.

Proof of Proposition 5.3. The proof is similar to the one of Proposition 5.2, modifying suitably limits in (5.13) and (5.15), as outlined in the proof of Proposition 2 in [17]. \square

Proposition 5.4. *Assume X is reflexive, (5.4) and that $k_R, \phi \in L^\infty(I, \mathbb{R}^+)$. If for any $\bar{x} \in \partial K$, $F(t, \bar{x})$ is convex, and the function d_K is continuously differentiable on ∂K , then, assumption (1.6) implies (4.5).*

Note that if X is a Hilbert space and the state constraint K is the ball RB or the annulus $RB \setminus rB$, as in Sections 3.1 and 3.2, then $d_K \in C^1$ on a neighborhood of ∂K .

Proof of Proposition 5.4. Note that (1.6) implies immediately (1.5). Once (1.5) is proved, we verify (4.5), as in the proof of Proposition 5.3. \square

6. LIPSCHITZ CONTINUITY OF THE VALUE FUNCTION

The first application of our approximation theorems concerns the local Lipschitz continuity of the value function associated to the Mayer problem of optimal control theory. Namely, given a cost function $g : X \rightarrow \mathbb{R}$, and $x_0 \in K$, consider the problem

$$\min \{g(x(1)) : x \in \mathcal{S}_{[0,1]}^K(x_0)\}.$$

The corresponding value function $V : I \times X \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$V(t_0, y_0) = \inf \{g(x(1)) : x \in \mathcal{S}_{[t_0,1]}^K(y_0)\}, \quad \text{for any } t_0 \in I, y_0 \in X,$$

with the convention that $\inf \emptyset = +\infty$.

Theorem 6.1. *Assume (1.7), (2.5)–(2.8), and that g is locally Lipschitz continuous on K , that is, for any $R > 0$, there exists $g_R > 0$ such that, for any $x, y \in RB \cap K$,*

$$|g(x) - g(y)| \leq g_R \|x - y\|_X.$$

Then V is continuous on $I \times K$ and the map $y_0 \mapsto V(t_0, y_0)$ is locally Lipschitz continuous on K uniformly in time, that is, for any $R > 0$, there exist $l_R > 0$ such that, for any $t_0 \in I, y_1, y_2 \in RB \cap K$,

$$(6.1) \quad |V(t_0, y_1) - V(t_0, y_2)| \leq l_R \|y_1 - y_2\|_X.$$

Proof. The key ingredient is the neighboring feasible trajectory Theorem 4.1. Indeed, let $R > 0$ be fixed, c_0 be as in Theorem 4.1 and take $y_1 \neq y_2 \in K \cap RB$. Let \tilde{R} be such that any solution x to (4.1) with the initial datum in RB satisfies $x(t) \in \tilde{R}B$, for any t . By the definition of value function, we know that, for any $t_0 \in I$, there exists $x_1 \in \mathcal{S}_{[t_0,1]}^K(y_1)$ such that

$$g(x_1(1)) \leq V(t_0, y_1) + \|y_1 - y_2\|_X.$$

Taking the function

$$w_2(t) = S(t - t_0)y_2 + \int_{t_0}^t S(t - s) f^{x_1}(s) ds,$$

with f^{x_1} the selection corresponding to x_1 in (2.2), by (2.3) we have, for any $t \in [t_0, 1]$,

$$\|x_1(t) - w_2(t)\|_X \leq M_S \|y_1 - y_2\|_X.$$

Since for a.e. $s \in [t_0, 1]$,

$$\begin{aligned} f^{x_1}(s) &\in F(s, x_1(s)) \subset F(s, w_2(s)) + k_{\bar{R}}(s) \|x_1(s) - w_2(s)\|_X \\ &\subset F(s, w_2(s)) + k_{\bar{R}}(s) M_S \|y_1 - y_2\|_X, \end{aligned}$$

by an infinite dimensional version of the Filippov theorem, see [17, Lemma A.1], there exists z_2 solution to (4.1) such that $z_2(t_0) = y_2$ and for any $t \in [t_0, 1]$,

$$\|w_2(t) - z_2(t)\|_X \leq M_S e^{M_S \|k_{\bar{R}}\|_{L^1}} \int_{t_0}^t k_{\bar{R}}(s) M_S \|y_1 - y_2\|_X ds \leq C_1 \|y_1 - y_2\|_X$$

with $C_1 = M_S e^{M_S \|k_{\bar{R}}\|_{L^1}} \|k_{\bar{R}}\|_{L^1} M_S$. From Theorem 4.1, we obtain the existence of $x_2 \in \mathcal{S}_{[t_0, 1]}^K(y_2)$ such that

$$\|x_2 - z_2\|_{C([t_0, 1], X)} \leq c_0 \left(\max_{t \in [t_0, 1]} \text{dist}_K(z_2(t)) + \|y_1 - y_2\|_X \right) \leq c_0 ((C_1 + M_S) + 1) \|y_1 - y_2\|_X,$$

so that, for a positive constant C_2 depending only on R , and for any $t \in [t_0, 1]$,

$$\|x_2(t) - x_1(t)\|_X \leq C_2 \|y_1 - y_2\|_X.$$

This estimate jointly with the definition of the value function imply that

$$V(t_0, y_2) \leq g(x_2(1)) \leq g(x_1(1)) + g_{\bar{R}} C_2 \|y_1 - y_2\|_X \leq V(t_0, y_1) + l_R \|y_1 - y_2\|_X,$$

where $l_R = g_{\bar{R}} C_2 + 1$. By interchanging y_1 and y_2 in the previous calculation, we obtain (6.1).

In order to prove the continuity of V on $I \times K$, consider $t_1, t_2 \in I$ and $y_0 \in K$. Suppose $t_1 < t_2$ and let $x \in \mathcal{S}_{[t_1, 1]}^K(y_0)$ such that

$$g(x(1)) \leq V(t_1, y_0) + t_2 - t_1.$$

The properties of the value function ensure that

$$(6.2) \quad V(t_1, y_0) \leq V(t_2, x(t_2)) \leq g(x(1)) \leq V(t_1, y_0) + t_2 - t_1.$$

Let $R > 0$ be such that $x(t) \in RB$ for every $t \in [t_1, 1]$. Then we can conclude the proof by remarking that inequality (6.2), the Lipschitz regularity (6.1), and (2.8), imply

$$\begin{aligned} |V(t_1, y_0) - V(t_2, y_0)| &\leq V(t_2, x(t_2)) - V(t_1, y_0) + |V(t_2, x(t_2)) - V(t_2, y_0)| \\ &\leq t_2 - t_1 + l_R \|x(t_2) - y_0\|_X \\ &\leq t_2 - t_1 + l_R \|(S(t_2 - t_1) - Id)y_0\|_X + l_R \left\| \int_{t_1}^{t_2} S(t_2 - t) f^x(t) dt \right\|_X \\ &\leq t_2 - t_1 + l_R \|(S(t_2 - t_1) - Id)y_0\|_X + l_R M_S (1 + R) \int_{t_1}^{t_2} \phi(t) dt. \end{aligned}$$

□

7. VARIATIONAL INCLUSIONS

In this section, as application of the approximation Theorem 4.2, we prove a theorem on variational inclusions which is the infinite dimensional version of Theorem 4.1 in [18].

Theorem 7.1. *Under assumptions of Theorem 4.1 or Theorem 4.2, let $x_0 \in K$, $\bar{x} \in \mathcal{S}_{[t_0,1]}^K(x_0)$, $h_i \rightarrow 0+$, $w_i \rightarrow w_0 \in I_K(x_0)$ be such that $x_0 + h_i w_i \in K$, for all i . Consider a solution w to the linearized inclusion*

$$(7.1) \quad \dot{w}(t) \in \mathbb{A}w(t) + d_x F(t, \bar{x}(t), f^{\bar{x}}(t))w(t), \quad \text{for a.e. } t \in [t_0, 1]$$

with $w(t_0) = w_0$ and subject to the constraint

$$w(t) \in I_K(\bar{x}(t)), \quad \text{for a.e. } t \in [t_0, 1].$$

If

$$(7.2) \quad \lim_{h \rightarrow 0+} \frac{1}{h} \max_{t \in [t_0, 1]} \text{dist}_K(\bar{x}(t) + hw(t)) = 0,$$

then for any i there exists $x_i \in \mathcal{S}_{[t_0,1]}^K(x_0 + h_i w_i)$ such that

$$\frac{x_i - \bar{x}}{h_i} \rightarrow w \quad \text{in } \mathcal{C}([t_0, 1], X).$$

Proof. Fix a solution w to (7.1) and let $\pi(s) \in d_x F(s, \bar{x}(s), f^{\bar{x}}(s))w(s)$ a.e. be an integrable selection such that

$$w(t) = S(t)w_0 + \int_0^t S(t-s)\pi(s)ds \quad \forall t \in [t_0, 1].$$

By the definition of derivative, we obtain that

$$(7.3) \quad \lim_{h \rightarrow 0+} \text{dist}\left(\pi(t), \frac{F(t, \bar{x}(t) + hw(t)) - f^{\bar{x}}(t)}{h}\right) = 0, \quad \text{for a.e. } t \in [t_0, 1].$$

Set

$$y_i(t) = \bar{x}(t) + h_i(w(t) + w_i - w_0) \quad \text{and} \quad \gamma_i(t) = \text{dist}(f^{\bar{x}}(t) + h_i \pi(t), F(t, y_i(t))).$$

From (2.7) and (7.3), we obtain that $\int_{t_0}^1 \gamma_i(s)ds = o(h_i)$. Hence applying Lemma A.1 from [17] there exists \tilde{x}_i solving (4.1) with initial datum $\tilde{x}_i(t_0) = x_0 + h_i w_i$ such that

$$(7.4) \quad \frac{\tilde{x}_i - y_i}{h_i} \rightarrow 0 \quad \text{in } \mathcal{C}([t_0, 1], X), \quad \text{implying} \quad \frac{\tilde{x}_i - \bar{x}}{h_i} \rightarrow w \quad \text{in } \mathcal{C}([t_0, 1], X).$$

Now consider the compact set $D_0 = \cup_{i \in \mathbb{N}} \{x_0 + h_i w_i\} \cup \{x_0\}$. In light of Theorem 4.2, there exists c_{D_0} such that for any i , we can find $x_i \in \mathcal{S}_{[t_0,1]}^K(x_0 + h_i w_i)$ satisfying

$$(7.5) \quad \|x_i - \tilde{x}_i\|_{\mathcal{C}([t_0,1], X)} \leq c_{D_0} \left(\max_{t \in [t_0, 1]} \text{dist}_K(\tilde{x}_i(t)) + \frac{h_i}{i} \right).$$

Further,

$$\begin{aligned} \text{dist}_K(\tilde{x}_i(t)) &\leq \|\tilde{x}_i(t) - y_i(t)\|_X + \text{dist}_K(\bar{x}(t) + h_i(w(t) + w_i - w_0)) \\ &\leq \|\tilde{x}_i(t) - y_i(t)\|_X + h_i \|w_i - w_0\|_X + \text{dist}_K(\bar{x}(t) + h_i w(t)). \end{aligned}$$

Hence, assumption (7.2) allows to conclude. Indeed,

$$\begin{aligned} \left\| \frac{x_i - \bar{x}}{h_i} - w \right\|_{\mathcal{C}([t_0, 1], X)} &\leq \frac{1}{h_i} \|\tilde{x}_i - x_i\|_{\mathcal{C}([t_0, 1], X)} + \left\| \frac{\tilde{x}_i - \bar{x}}{h_i} - w \right\|_{\mathcal{C}([t_0, 1], X)} \\ &\leq c_{D_0} \frac{1}{h_i} \left(\max_{t \in [t_0, 1]} \text{dist}_K(\tilde{x}_i(t)) + \frac{h_i}{i} \right) + o(1) \\ &\leq \frac{c_{D_0}}{h_i} \|\tilde{x}_i - y_i\|_{\mathcal{C}([t_0, 1], X)} + \frac{c_{D_0}}{h_i} \max_{t \in [t_0, 1]} \text{dist}_K(\bar{x}(t) + h_i w(t)) + c_{D_0} \|w_i - w_0\|_X + o(1) \rightarrow 0, \end{aligned}$$

in light of (7.4), (7.5), ending the proof. \square

8. PROOFS OF THEOREMS 4.1 AND 4.2

To prove the main theorems of Section 4, we need the following lemma.

Lemma 8.1. *Under the assumptions of Theorem 4.1, for every $R_0 > 0$, there exist $c, \delta > 0$ such that, for any $\varepsilon' > 0$, any $\bar{t} \in [t_0, 1]$, and any solution y of (4.1) with $y(t_0) \in K \cap R_0 B$ and $y(\bar{t}) \in K$, we can find a solution $x_{\varepsilon'}$ of (4.1) satisfying*

$$\begin{aligned} x_{\varepsilon'}(\bar{t}) &= y(\bar{t}), \quad x_{\varepsilon'}(t) \in \text{Int } K, \quad \text{for any } t \in (\bar{t}, (\bar{t} + \delta) \wedge 1], \\ \text{and } \|x_{\varepsilon'} - y\|_{\mathcal{C}([\bar{t}, 1], X)} &\leq c \cdot \max_{t \in [\bar{t}, 1]} \text{dist}_K(y(t)) + \varepsilon'. \end{aligned}$$

The same conclusions are valid if (2.5)–(2.8) hold true with $\phi \in L^\infty(I, \mathbb{R}^+)$, F is convex valued, and (1.7) is replaced by (1.8).

Proof. Fix $R_0 > 0$ and $\varepsilon' > 0$. Let y be a solution of (4.1) satisfying the assumptions of the lemma. By the Gronwall lemma, in light of (2.2), (2.3), and (2.8), there exists $R > 0$ depending only on R_0 such that

$$(8.1) \quad y(t) \in \frac{R}{2} B \quad \text{and} \quad S(\tau) y(t) \in R B, \quad \text{for any } \tau \in I, t \in [t_0, 1].$$

We can assume, without loss of generality, that $\varepsilon' < R$. Taking $\eta, \rho, M, J, k_R, \phi$ as in (1.7), (2.7), (2.8), we define

$$(8.2) \quad C = M_S M \left(2M_S \|k_R\|_{L^1} e^{M_S \|k_R\|_{L^1}} + 1 \right), \quad c = \frac{2C}{\rho},$$

and choose

$$(8.3) \quad 0 < \delta \leq \frac{1}{2C} \min\{R, \eta\}$$

such that, for any measurable $E \subset I$ with $\mu(E) \leq \delta$,

$$(8.4) \quad \int_E k_R(s) ds < \frac{\rho \|k_R\|_{L^1}}{\rho + 2C} \quad \text{and} \quad \int_E \phi(s) ds < \frac{\eta}{2M_S(1 + R)}.$$

Finally, we set

$$\Gamma = \left\{ s \in [\bar{t}, 1] \cap J : \text{for some } \tau_0 \leq \eta, S(\tau_0) y(s) \in \partial K + \eta B \quad \text{and} \right. \\ \left. \sup_{z \in B(S(\tau_0) y(s), \eta)} \sigma(z; S(\tau_0) f^y(s)) \geq 0 \right\}$$

and

$$(8.5) \quad T = \begin{cases} (\bar{t} + \delta) \wedge 1, & \text{if } \mu(\Gamma \cap [\bar{t}, (\bar{t} + \delta) \wedge 1]) < \frac{1}{C} \left[c \cdot \max_{t \in [\bar{t}, 1]} \text{dist}_K(y(t)) + \varepsilon' \right] \\ \min \left\{ s \geq \bar{t} : \mu(\Gamma \cap [\bar{t}, s]) = \frac{1}{C} \left[c \cdot \max_{t \in [\bar{t}, 1]} \text{dist}_K(y(t)) + \varepsilon' \right] \right\}, & \text{otherwise.} \end{cases}$$

By assumption (1.7) and the measurable selection theorem, there exists a measurable selection $\bar{v}(s) \in F(s, y(s))$ such that, for any $s \in \Gamma \cap [\bar{t}, T]$

$$(8.6) \quad \|\bar{v}(s) - f^y(s)\|_X \leq M$$

and

$$(8.7) \quad \sup_{(\tau, z) \in \mathcal{A}^\eta(y(s))} \left\{ \sigma(z; S(\tau)[\bar{v}(s) - f^y(s)]), \sigma(z; S(\tau)\bar{v}(s)) \right\} \leq -\rho.$$

Define

$$(8.8) \quad f_{\varepsilon'}(s) = \begin{cases} \bar{v}(s), & \text{for every } s \in \Gamma \cap [\bar{t}, T] \\ f^y(s), & \text{otherwise.} \end{cases}$$

Consider the trajectory

$$\begin{aligned} y_{\varepsilon'}(t) &= S(t - \bar{t})y(\bar{t}) + \int_{\bar{t}}^t S(t - s)f_{\varepsilon'}(s)ds \\ &= S(t - \bar{t})y(\bar{t}) + \int_{\Gamma \cap [\bar{t}, t \wedge T]} S(t - s)\bar{v}(s)ds + \int_{[\bar{t}, t] \setminus (\Gamma \cap [\bar{t}, t \wedge T])} S(t - s)f^y(s)ds. \end{aligned}$$

Applying (2.2), (2.3), (8.2), (8.5), (8.6) and (8.8), we obtain that on $[\bar{t}, 1]$

$$\begin{aligned} \|y_{\varepsilon'}(t) - y(t)\|_X &\leq \int_{\bar{t}}^t \|S(t - s)[f_{\varepsilon'}(s) - f^y(s)]\|_X ds \leq M_S \int_{\Gamma \cap [\bar{t}, t \wedge T]} \|\bar{v}(s) - f^y(s)\|_X ds \\ &\leq M_S M \mu(\Gamma \cap [\bar{t}, t \wedge T]) \leq c \cdot \max_{t \in [\bar{t}, 1]} \text{dist}_K(y(t)) + \varepsilon'. \end{aligned}$$

Hence, by (8.2), (8.3)

$$\|y_{\varepsilon'}(t) - y(t)\|_X \leq M_S M \delta \leq \frac{R}{2}.$$

We can apply an infinite dimensional version of the Filippov Theorem, see [17, Lemma A.1], to deduce that there exists a solution $x_{\varepsilon'}$ of (4.1) with $x_{\varepsilon'}(\bar{t}) = y(\bar{t})$ satisfying for any $t \in [\bar{t}, 1]$

$$\|x_{\varepsilon'}(t) - y_{\varepsilon'}(t)\|_X \leq 2M_S e^{M_S \|k_R\|_{L^1}} M_S M \mu(\Gamma \cap [\bar{t}, t \wedge T]) \int_{\bar{t}}^t k_R(s)ds,$$

$$\|f^{x_{\varepsilon'}}(t) - f_{\varepsilon'}(t)\|_X \leq k_R(t) \mu(\Gamma \cap [\bar{t}, t \wedge T]) \left(2M_S e^{M_S \|k_R\|_{L^1}} M_S M \int_{\bar{t}}^t k_R(s)ds + 2M_S M \right).$$

Therefore

$$\begin{aligned} \|x_{\varepsilon'}(t) - y(t)\|_X &\leq \|x_{\varepsilon'}(t) - y_{\varepsilon'}(t)\|_X + \|y_{\varepsilon'}(t) - y(t)\|_X \\ &\leq 2M_S \|k_R\|_{L^1} e^{M_S \|k_R\|_{L^1}} M_S M \mu(\Gamma \cap [\bar{t}, t \wedge T]) + M_S M \mu(\Gamma \cap [\bar{t}, t \wedge T]) \\ &= C \mu(\Gamma \cap [\bar{t}, t \wedge T]) \leq c \cdot \max_{t \in [\bar{t}, 1]} d_K(y(t)) + \varepsilon'. \end{aligned}$$

Further, by the definition of δ in (8.3), for any $t \in [\bar{t}, (\bar{t} + \delta) \wedge 1]$, we obtain

$$(8.9) \quad \|x_{\varepsilon'}(t) - y(t)\|_X \leq \min \left\{ \frac{R}{2}, \frac{\eta}{2} \right\},$$

$$(8.10) \quad \|x_{\varepsilon'}(t) - y_{\varepsilon'}(t)\|_X \leq \frac{C}{\|k_R\|_{L^1}} \mu(\Gamma \cap [\bar{t}, t \wedge T]) \int_{\bar{t}}^t k_R(s) ds \leq \frac{\rho}{2} \mu(\Gamma \cap [\bar{t}, t \wedge T])$$

and

$$\|f^{x_{\varepsilon'}}(t) - f_{\varepsilon'}(t)\|_X \leq k_R(t) \mu(\Gamma \cap [\bar{t}, t \wedge T]) \left(\frac{\rho}{2} + 2M_S M \right).$$

We claim that $x_{\varepsilon'}(t) \in \text{Int } K$ for any $t \in (\bar{t}, (\bar{t} + \delta) \wedge 1]$. Assume first that $t \in (\bar{t}, (\bar{t} + \delta) \wedge 1]$ satisfies $\mu(\Gamma \cap [\bar{t}, t \wedge T]) = 0$. This implies that $t \leq T$, since if $t > T$, $\mu(\Gamma \cap [\bar{t}, t \wedge T]) = \mu(\Gamma \cap [\bar{t}, T]) = 0$ yielding by the definition of T , the contradictory $t > (\bar{t} + \delta) \wedge 1 = T$. Hence, we obtain that $\mu(\Gamma \cap [\bar{t}, t]) = 0$ and $x_{\varepsilon'} \equiv y$ in $[\bar{t}, t]$, so our claim becomes $y(t) \in \text{Int } K$. From the definition of Γ , we have that for a.e. $s \in [\bar{t}, t]$ and every $\tau \in [0, \eta]$ either $S(\tau)y(s) \notin \partial K + \eta B$ or $S(\tau)y(s) \in \partial K + \eta B$ and

$$\sup_{\substack{\xi \in \partial d_K(z) \\ z \in B(S(\tau)y(s), \eta)}} \langle \xi, S(\tau)f^y(s) \rangle < 0.$$

Observe that, by (2.8) and (8.4), for every $s \in [\bar{t}, t]$ we have

$$(8.11) \quad \int_{\bar{t}}^s \|S(t-r)f^y(r)\|_X dr \leq \frac{\eta}{2}$$

implying together with (2.5),

$$(8.12) \quad d_K(S(t-s)y(s)) \leq d_K(S(t-\bar{t})y(\bar{t})) + \left\| \int_{\bar{t}}^s S(t-r)f^y(r) dr \right\|_X \leq \frac{\eta}{2}.$$

Consequently, if there exists $\bar{s} \in [\bar{t}, t]$ such that $S(t-\bar{s})y(\bar{s}) \notin \partial K + \eta B$, we obtain by (8.11)

$$d_K(y(t)) \leq d_K(S(t-\bar{s})y(\bar{s})) + \left\| \int_{\bar{s}}^t S(t-r)f^y(r) dr \right\|_X \leq -\eta + \frac{\eta}{2} < 0,$$

yielding $y(t) \in \text{Int } K$. Otherwise, for every $s \in [\bar{t}, t]$, $S(t-s)y(s) \in \partial K + \eta B$ and

$$(8.13) \quad \sup_{\substack{\xi \in \partial d_K(z) \\ z \in B(S(t-s)y(s), \eta)}} \langle \xi, S(t-s)f^y(s) \rangle < 0 \text{ for a.e. } s \in [\bar{t}, t].$$

Applying the mean value theorem (see [12]) and (2.2), there exist $z \in [y(t), S(t - \bar{t})y(\bar{t})]$ and $\xi \in \partial d_K(z)$ such that

$$(8.14) \quad \begin{aligned} d_K(y(t)) &= d_K(S(t - \bar{t})y(\bar{t})) + \langle \xi, y(t) - S(t - \bar{t})y(\bar{t}) \rangle \\ &= d_K(S(t - \bar{t})y(\bar{t})) + \int_{\bar{t}}^t \langle \xi, S(t - s)f^y(s) \rangle ds. \end{aligned}$$

Since, for any $s \in [\bar{t}, t]$,

$$\begin{aligned} \|z - S(t - s)y(s)\|_X &\leq \|z - y(t)\|_X + \|y(t) - S(t - s)y(s)\|_X \\ &\leq \|S(t - \bar{t})y(\bar{t}) - y(t)\|_X + \|y(t) - S(t - s)y(s)\|_X \leq \eta, \end{aligned}$$

by (2.5), (8.13) and (8.14) we have that $d_K(y(t)) < 0$, implying the claim.

Now we consider the case $\mu(\Gamma \cap [\bar{t}, t \wedge T]) > 0$. Notice that, by (8.1) and (8.9), $x_{\varepsilon'}(t) \in RB$, for any $t \in [\bar{t}, (\bar{t} + \delta) \wedge 1]$. Now, if there exists $\bar{s} \in [\bar{t}, t]$ such that $d_K(S(t - \bar{s})y(\bar{s})) < -\eta$, then using (2.2), (2.8), and (8.4), we obtain

$$\begin{aligned} d_K(x_{\varepsilon'}(t)) &\leq d_K(S(t - \bar{s})y(\bar{s})) + \|x_{\varepsilon'}(t) - S(t - \bar{t})y(\bar{t})\|_X + \|S(t - \bar{t})y(\bar{t}) - S(t - \bar{s})y(\bar{s})\|_X \\ &< -\eta + 2M_S(1 + R) \int_{\bar{t}}^t \phi(s) ds < 0. \end{aligned}$$

Therefore, recalling also (8.1) and (8.12), we can restrict our attention to the case where

$$S(t - s)y(s) \in \partial K + \eta B \quad \forall s \in [\bar{t}, t],$$

that is, $(t - s, z) \in \mathcal{A}^\eta(y(s))$ for every $s \in [\bar{t}, t]$ and $z \in B(S(t - s)y(s), \eta)$. By (8.7), we deduce that for all $s \in [\bar{t}, t] \cap \Gamma$

$$(8.15) \quad \sup_{\substack{\xi \in \partial d_K(z) \\ z \in B(S(t - s)y(s), \eta)}} \left\{ \langle \xi, S(t - s)[\bar{v}(s) - f^y(s)] \rangle, \langle \xi, S(t - s)\bar{v}(s) \rangle \right\} \leq -\rho.$$

On the other hand, for a.e. $s \in [\bar{t}, t] \setminus \Gamma$ we have

$$(8.16) \quad \sup_{\substack{\xi \in \partial d_K(z) \\ z \in B(S(t - s)y(s), \eta)}} \langle \xi, S(t - s)f^y(s) \rangle < 0.$$

If $t \in (T, (\bar{t} + \delta) \wedge 1]$, then

$$(8.17) \quad \mu(\Gamma \cap [\bar{t}, T]) = \frac{1}{C} \left[c \cdot \max_{t \in [\bar{t}, 1]} \text{dist}_K(y(t)) + \varepsilon' \right].$$

From the mean value theorem, as above, for some $z \in [y(t), x_{\varepsilon'}(t)]$ and $\xi \in \partial d_K(z)$, we have

$$(8.18) \quad d_K(x_{\varepsilon'}(t)) = d_K(y(t)) + \langle \xi, x_{\varepsilon'}(t) - y(t) \rangle.$$

Further, since by (8.9) and (8.11), for any $s \in [\bar{t}, t]$,

$$\|z - S(t - s)y(s)\|_X \leq \|z - y(t)\|_X + \left\| \int_s^t S(t - r)f^y(r) dr \right\|_X \leq \|x_{\varepsilon'}(t) - y(t)\|_X + \frac{\eta}{2} \leq \eta,$$

then $z \in B(S(t-s)y(s), \eta)$. Hence, from (8.2), (8.8), (8.10), (8.15), (8.17), and (8.18), we obtain

$$\begin{aligned} d_K(x_{\varepsilon'}(t)) &= d_K(y(t)) + \langle \xi, x_{\varepsilon'}(t) - y_{\varepsilon'}(t) \rangle + \langle \xi, y_{\varepsilon'}(t) - y(t) \rangle \\ &\leq d_K(y(t)) + \frac{\rho}{2} \mu(\Gamma \cap [\bar{t}, T]) + \int_{\Gamma \cap [\bar{t}, T]} \langle \xi, S(t-s)[\bar{v}(s) - f^y(s)] \rangle ds \\ &\leq d_K(y(t)) - \frac{\rho}{2} \mu(\Gamma \cap [\bar{t}, T]) = d_K(y(t)) - \frac{\rho}{2C} c \cdot \max_{s \in [\bar{t}, 1]} d_K(y(s)) - \frac{\rho}{2C} \varepsilon' < 0. \end{aligned}$$

Then, $x_{\varepsilon'}(t) \in \text{Int } K$. Finally, let $t \in (\bar{t}, T]$. By the mean value theorem there exist $z \in [S(t-\bar{t})y(\bar{t}), y_{\varepsilon'}(t)]$ and $\xi \in \partial d_K(z)$ such that

$$(8.19) \quad d_K(y_{\varepsilon'}(t)) = d_K(S(t-\bar{t})y(\bar{t})) + \langle \xi, y_{\varepsilon'}(t) - S(t-\bar{t})y(\bar{t}) \rangle.$$

Applying again (2.2), (2.8), and (8.4), we have for every $s \in [\bar{t}, t]$

$$\begin{aligned} \|z - S(t-s)y(s)\|_X &\leq \|z - S(t-\bar{t})y(\bar{t})\|_X + \|S(t-\bar{t})y(\bar{t}) - S(t-s)y(s)\|_X \\ &\leq \|y_{\varepsilon'}(t) - S(t-\bar{t})y(\bar{t})\|_X + \|S(t-\bar{t})y(\bar{t}) - S(t-s)y(s)\|_X \\ &\leq 2M_S(1+R) \int_{\bar{t}}^t \phi(r) dr \leq \eta. \end{aligned}$$

Then from (8.8), (8.10), (8.15), (8.16) and (8.19), we obtain

$$\begin{aligned} d_K(x_{\varepsilon'}(t)) &\leq \|x_{\varepsilon'}(t) - y_{\varepsilon'}(t)\|_X + d_K(y_{\varepsilon'}(t)) \\ &\leq \frac{\rho}{2} \mu(\Gamma \cap [\bar{t}, t]) + d_K(S(t-\bar{t})y(\bar{t})) + \int_{\bar{t}}^t \langle \xi, S(t-s)f_{\varepsilon'}(s) \rangle ds \\ &\leq \frac{\rho}{2} \mu(\Gamma \cap [\bar{t}, t]) + \int_{\Gamma \cap [\bar{t}, t]} \langle \xi, S(t-s)\bar{v}(s) \rangle ds + \int_{[\bar{t}, t] \setminus \Gamma} \langle \xi, S(t-s)f^y(s) \rangle ds \\ &\leq \left(\frac{\rho}{2} - \rho\right) \mu(\Gamma \cap [\bar{t}, t]) < 0. \end{aligned}$$

Also in this case $x_{\varepsilon'}(t) \in \text{Int } K$.

If (2.5)–(2.8) are valid with $\phi \in L^\infty(I, \mathbb{R}^+)$, convex valued F , and (1.7) replaced by (1.8), the same claim follows from Lemma 5.1 and the proof above. \square

Proof of Theorem 4.1. The construction of the trajectory x , based on Lemma 8.1, proceeds exactly as in [18, p.745]. \square

Proof of Theorem 4.2. Notice that, by the uniform continuity of the semigroup $S(t)$ over compact sets, we have that for any compact D and any $\eta > 0$, there exists $0 < \tilde{\eta} \leq \eta/2$ such that, for any $\tau \leq \tilde{\eta}$ and any $x \in D$, $\|S(\tau)x - x\|_X \leq \eta/2$. Then, it is possible to prove that assumption (4.5) implies

$$(8.20) \quad \forall D \subset X \text{ compact, } \exists \tilde{\eta}, \rho, M > 0, \exists J \subset I \text{ such that } \mu(J) = 1 \text{ and whenever}$$

$$\Sigma_{\tilde{\eta}}(x; v) \geq 0 \text{ for some } t \in I, x \in D \cap \partial^{\tilde{\eta}} K, v \in F(t, x),$$

$$\text{then } \exists \bar{v} \in F(t, x) \cap B(v, M) \text{ satisfying } \max \{ \Sigma_{\tilde{\eta}}(x; \bar{v} - v); \Sigma_{\tilde{\eta}}(x; \bar{v}) \} \leq -\rho.$$

Then, following the same proof as the one of Lemma 8.1 we obtain that for every compact $D_0 \subset X$ there exist $c, \delta > 0$ such that, for any $\varepsilon' > 0$, any $\bar{t} \in [t_0, 1]$, and any solution y of (1.3) with $y(t_0) \in K \cap D_0$ and $y(\bar{t}) \in K$, we can find a solution $x_{\varepsilon'}$ of (1.3) satisfying

$$x_{\varepsilon'}(\bar{t}) = y(\bar{t}), \quad x_{\varepsilon'}(t) \in \text{Int } K, \text{ for any } t \in (\bar{t}, (\bar{t} + \delta) \wedge 1],$$

$$\text{and } \|x_{\varepsilon'} - y\|_{L^\infty([\bar{t}, 1], X)} \leq c \cdot \max_{t \in [\bar{t}, 1]} \text{dist}_K(y(t)) + \varepsilon'.$$

This allows to conclude as in Theorem 4.1.

With the help of Lemma 5.1, the same conclusions hold true if (2.5)–(2.8) are valid with $\phi \in L^\infty(I, \mathbb{R}^+)$, F is convex valued, and (1.7) is replaced by (1.8). □

REFERENCES

- [1] W. Arendt *Semigroups and evolution equations: functional calculus, regularity and kernel estimates. Evolutionary equations. Vol. I, 1-85, Handb. Differ. Equ.* North-Holland, Amsterdam, 2004.
- [2] W. Arendt, G. Metafuno, D. Pallara, S. Romanelli, *The Laplacian with Wentzell-Robin boundary conditions on spaces of continuous functions*, Semigroup Forum **67** (2003), 247–261.
- [3] J.Y Beale, S.I. Rosencrans, *Acoustic boundary conditions*, Bull. Amer. Math. Soc. **80** (1974), 1276–1278.
- [4] P. Bettiol, A. Bressan, R. B. Vinter, *On trajectories satisfying a state constraint: $W^{1,1}$ estimates and counter-examples*, SIAM J. Control Optim. **48** (2010), 4664–4679.
- [5] P. Bettiol, H. Frankowska, R. B. Vinter, *L^∞ estimates on trajectories confined to a closed subset*, J. Differential Equations **252** (2012), 1912–1933.
- [6] H. Brézis, *Analyse fonctionnelle: théorie et applications*, Masson, Paris, 1983.
- [7] H. Brezis, A. Pazy, *Semigroups of nonlinear contractions on convex sets*, J. Funct. Anal. **6** (1970), 237–281.
- [8] P. Cannarsa, H. Frankowska, *Value function and optimality conditions for semilinear control problems*, Appl. Math. Optim. **26** (1992), 139–169.
- [9] P. Cannarsa, H. Frankowska, E. M. Marchini, *On Bolza optimal control problems with constraints*, Discrete Contin. Dyn. Syst. **11** (2009), 629–653.
- [10] P. Cannarsa, H. Frankowska, E.M. Marchini, *Optimal control for evolution equations with memory*, J. Evol. Equ. **13** (2013), 197–227.
- [11] K.N. Chueh, C.C. Conley, J.A. Smoller, *Positively invariant regions for systems of nonlinear equation*, Indiana Univ. Math. J. **26** (1977), 373–392.
- [12] F. H. Clarke, *Optimization and nonsmooth analysis*, SIAM, Philadelphia, 1990.
- [13] G.M. Coclite, G.R. Goldstein, J.A. Goldstein, *Stability estimates for nonlinear hyperbolic problems with nonlinear Wentzell boundary conditions*, Z. Angew. Math. Phys. **64** (2013), 733–753.
- [14] C. M. Dafermos: *Asymptotic stability in viscoelasticity*, Arch. Ration. Mech. Anal. **37** (1970), 297–308.
- [15] K.-J. Engel, *The Laplacian on $C(\bar{\Omega})$ with generalized Wentzell boundary conditions*, Arch. Math. **81** (2003), 548–558.
- [16] K.-J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, New York, 2000.
- [17] H. Frankowska, E.M. Marchini, M. Mazzola, *A relaxation result for state constrained inclusions in infinite dimension*, Mathematical Control and Related Fields (MCRF) **6** (2016), 113–141.
- [18] H. Frankowska, M. Mazzola, *Discontinuous solutions of Hamilton-Jacobi-Bellman equation under state constraints*, Calc. Var. Partial Differential Equations **46** (2013), 725–747.
- [19] H. Frankowska, M. Mazzola, *On relations of the adjoint state to the value function for optimal control problems with state constraints*, Nonlinear Differ. Equ. Appl. **20** (2013), 361–383.

- [20] H. Frankowska, F. Rampazzo, *Filippov's and Filippov-Wazewski's theorems on closed domains*, J. Differential Equations **161** (2000), 449–478.
- [21] H. Frankowska, R.B. Vinter, *Existence of neighbouring feasible trajectories: Applications to dynamic programming for state constrained optimal control problems*, J. Optim. Theory Appl. **104** (2000), 21–40.
- [22] C.G. Gal, G.R. Goldstein, J.A. Goldstein, *Oscillatory boundary conditions for acoustic wave equations*, J. Evol. Equ. **3** (2003), 623–635.
- [23] G.R. Goldstein. *Derivation and physical interpretation of general boundary conditions*, Adv. Diff. Eqns. **11** (2006), 457–480.
- [24] P.M. Moser, K.U. Ingard, *Theoretical acoustics*, McGraw Hill, New York, 1968.
- [25] E.M. Ouhabaz, *Invariance of closed convex sets and domination criteria for semigroups*, Potential Analysis **5** (1996), 611–625.
- [26] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.
- [27] H. M. Soner, *Optimal control with state-space constraints*, SIAM J. Control Optim. **24** (1986), 552–561.
- [28] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer, New York, 1997.
- [29] J.L. Vazquez, E. Vitillaro, *Heat equation with dynamical boundary conditions of locally reactive type*, Semigroup Forum **74** (2007), 1–40.

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