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NECESSARY CONDITIONS FOR INFINITE HORIZON OPTIMAL CONTROL PROBLEMS WITH STATE CONSTRAINTS

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Abstract. Partial and full sensitivity relations are obtained for nonautonomous optimal control problems with infinite horizon subject to state constraints, assuming the associated value function to be locally Lipschitz in the state. Sufficient structural conditions are given to ensure such a Lipschitz regularity in presence of a positive discount factor, as it is typical of macroeconomics models.

1. **Introduction.** Consider the infinite horizon optimal control problem $\mathscr{B}_\infty$

$$\text{minimize} \int_{t_0}^{\infty} L(t, x(t), u(t))\, dt \tag{1}$$

over all the trajectory-control pairs subject to the state constrained control system

$$\begin{cases} x'(t) = f(t, x(t), u(t)) & \text{a.e. } t \in [t_0, \infty) \\ x(t_0) = x_0 & \\ u(t) \in U(t) & \text{a.e. } t \in [t_0, \infty) \\ x(t) \in A & t \in [t_0, \infty) \end{cases} \tag{2}$$

where $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $L : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are given, A is a nonempty closed subset of $\mathbb{R}^n$, $U : [0, \infty) \rightrightarrows \mathbb{R}^m$ is a Lebesgue measurable set valued map with closed nonempty images and $(t_0, x_0) \in [0, \infty) \times A$ is the initial datum. Every trajectory-control pair $(x(\cdot), u(\cdot))$ that satisfies the state constrained control system (2) is called feasible. We refer to such $x(\cdot)$ as a feasible trajectory. The infimum of the cost functional in (1) over all feasible trajectory-control pairs, with

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the initial datum (t_0, x_0) or if the integral in (1) is not well defined for every feasible trajectory-control pair $(x(\cdot), u(\cdot))$, is denoted by $V(t_0, x_0)$ (if no feasible trajectory-control pair exists at (t_0, x_0), or if the integral in (1) is not defined for every feasible pair, we set $V(t_0, x_0) = +\infty$). The function $V : [0, \infty) \times A \to \mathbb{R} \cup \{\pm\infty\}$ is called the value function of problem $\mathscr{B}_\infty$.

Infinite horizon problems have a very natural application in mathematical economics (see, for instance, the Ramsey model in [14]). In this case the planner seeks to find a solution to $\mathscr{B}_\infty$ (dealing with a maximization problem instead of a minimization one) with

$$L(t, x, u) = e^{-\lambda t} l(ug(x)) \quad \& \quad f(t, x, u) = \tilde{f}(x) - ug(x)$$

where $l(\cdot)$ is called the "utility" function, $\tilde{f}(\cdot)$ the "production" function, and $g(\cdot)$ the "consumption" function, while the variable x stands for the "capital" (in many applications one takes as constraint set $A = [0, \infty)$ with $U(\cdot) \equiv [-1, 1]$). The approach used by many authors to address this problem is to find necessary conditions of the first or second order (cfr. [1], [6], [9], [18]).

It happens quite often, in mathematical economics papers, that one considers as candidates for optimal solutions only trajectories satisfying simultaneously the unconstrained Pontryagin maximum principle and the state constraints. Such an approach, however, is incorrect as there are cases (see, e.g., Example 1) where no optimal trajectory exists in this class. There is, therefore, the need of a constrained maximum principle for infinite horizon problems with sufficiently general structure.

The literature dealing with necessary optimality conditions for unconstrained infinite horizon optimal control problems is quite rich (see, e.g., [3] and the reference therein), mostly under assumptions on f and L that guarantee the Lipschitz regularity of $V(\cdot, \cdot)$. On the contrary, recovering optimality conditions in the presence of state constraints appears quite a challenging issue for infinite horizon problems, despite all the available results for constrained Bolza problems with finite horizon (cfr. [19]).

As a matter of fact, necessary conditions in the form of the maximum principle and partial sensitivity relations have been obtained for infinite horizon convex problems under smooth functional constraints such as $h(t, x(t)) \geq 0$ (see, e.g., [16]). In this paper we prefer to deal with the constraint $h(t, x(t)) \leq 0$ (without loss of generality). For instance, suppose $(\bar{x}, \bar{u})$ is optimal at (t_0, x_0) for the problem

$$\begin{cases} \text{maximize } \int_{t_0}^{\infty} L(t, x(t), u(t))\, dt & \\ x'(t) = f(t, x(t), u(t)) & \text{a.e. } t \in [t_0, \infty) \\ x(t_0) = x_0 & \\ u(t) \in U & \text{a.e. } t \in [t_0, \infty) \\ h(t, x(t)) \leq 0 & t \in [t_0, \infty), \end{cases}$$

with U a closed convex subset of $\mathbb{R}^m$, $h \in C^2$, f and L continuous together with their partial derivatives with respect to x and u, and assume the inward pointing condition

$$\inf_{u \in U} \langle \nabla_x h(t, \bar{x}(t)), f(t, \bar{x}(t), u) - f(t, \bar{x}(t), \bar{u}(t)) \rangle < 0 \qquad \forall t \geqslant t_0.$$

If $h(t_0, x_0) < 0$, then one proves that there exist $q^0 \in \{0, 1\}$, a co-state $q(\cdot)$, and a nondecreasing function $\mu(\cdot)$, constant on any interval where $h(t, \bar{x}(t)) < 0$, such

that $(q^0, q(t_0)) \neq (0,0)$, $\mu(t_0) = 0$, and $q(\cdot)$ satisfies the adjoint equation

$$q(t) = q(t_0) - \int_{t_0}^{t} \nabla_x H(s, \bar{x}(s), q(s), \bar{u}(s))\, ds - \int_{[t_0,t]} \nabla_x h(s, \bar{x}(s))\, d\mu(s)$$

and the maximum principle

$$H(t, \bar{x}(t), q(t), \bar{u}(t)) = \max_{u \in U} H(t, \bar{x}(t), q(t), u) \qquad \text{a.e. } t \in [t_0, \infty),$$

where $H(t,x,p,u) := \langle p, f(t,x,u)\rangle + q^0 L(t,x,u)$. Furthermore in [7], using the language of the calculus of variations, the authors show that, under some very restrictive assumptions on f, if A is convex and $\operatorname{int} A \neq \emptyset$ then, for any optimal trajectory $\bar{x}(\cdot)$ of problem $\mathscr{B}_\infty$, there exists an absolutely continuous arc $q(\cdot)$ which satisfies the adjoint equation and the partial sensitivity relation $q(t) \in \partial_x V(t, \bar{x}(t))$ for all $t \in [t_0, \infty)$.

In the present work, for the first time we provide the normal maximum principle (i.e. $q_0 = 1$) together with partial and full sensitivity relations and a transversality condition at the initial time, under mild assumption on dynamics and constraints. To describe our results, assume for the sake of simplicity that $L(t,x,u) = e^{-\lambda t} l(x,u)$ is smooth, $U(\cdot) \equiv U$ is a closed subset of $\mathbb{R}^m$, $V(t,\cdot)$ is continuously differentiable, and denote by $N_A(y)$ the limiting normal cone to A at y. If $(\bar{x}, \bar{u})$ is optimal for $\mathscr{B}_\infty$ at $(t_0, x_0) \in [0,\infty) \times \operatorname{int} A$, then Theorem 4.2 below guarantees the existence of a locally absolutely continuous co-state $p(\cdot)$, a nonnegative Borel measure μ on $[t_0, \infty)$, and a Borel measurable selection $\nu(\cdot) \in \overline{\operatorname{co}}\, N_A(\bar{x}(\cdot)) \cap \mathbb{B}$ such that $p(\cdot)$ satisfies the adjoint equation

$$-p'(t) = \mathrm{d}_x f(t, \bar{x}(t), \bar{u}(t))^* \left(p(t) + \eta(t)\right) - e^{-\lambda t} \nabla_x l(\bar{x}(t), \bar{u}(t)) \qquad \text{a.e. } t \in [t_0, \infty),$$

the maximality condition

$$\begin{aligned} &\langle p(t) + \eta(t), f(t, \bar{x}(t), \bar{u}(t))\rangle - e^{-\lambda t} l(\bar{x}(t), \bar{u}(t)) \\ &= \max_{u \in U} \left\{ \langle p(t) + \eta(t), f(t, \bar{x}(t), u)\rangle - e^{-\lambda t} l(\bar{x}(t), u) \right\} \qquad \text{a.e. } t \in [t_0, \infty), \end{aligned}$$

and the transversality and sensitivity relations

$$-p(t_0) = \nabla_x V(t_0, \bar{x}(t_0)), \quad -\left(p(t) + \eta(t)\right) = \nabla_x V(t, \bar{x}(t)) \qquad \text{a.e. } t \in (t_0, \infty), \tag{3}$$

where $\eta(t_0) = 0$ and $\eta(t) = \int_{[t_0,t]} \nu(s)\, d\mu(s)$ for all $t \in (t_0, \infty)$. Observe that, if $\bar{x}(\cdot) \in \operatorname{int} A$, then $\nu(\cdot) \equiv 0$ and the usual maximum principle holds true. But if $\bar{x}(t) \in \partial A$ for some time t, then a measure multiplier factor, $\int_{[0,t]} \nu\, d\mu$, may arise modifying the adjoint equation.

Furthermore, the transversality condition and sensitivity relation in (3) lead to a significant economic interpretation (see [2], [17]): the co-state $p + \eta$ can be regarded as the "shadow price" or "marginal price", i.e., (3) describes the contribution to the value function (the optimal total utility) of a unit increase of capital x.

From the technical point of view, this paper relies on two main ideas. The first one consists in reformulating the infinite horizon problem as a Bolza problem on each finite time interval, which can be analyzed in detail by appealing to the existing theory for finite horizon problems. More precisely, fixing any $T > 0$, we have that

$$V(s,y) = \inf \left\{ V(T, x(T)) + \int_s^T L(t, x(t), u(t))\, dt \right\} \quad \forall (s,y) \in [0,T] \times A,$$

where the infimum is taken over all the feasible trajectory-control pairs (x,u) satisfying (2) with initial datum (s,y) (Lemma 4.1). Hence, problem $\mathscr{B}_\infty$ becomes

a Bolza problem on $[0,T]$ with the additional final cost $\phi^T(\cdot) = V(T,\cdot)$. Then, assuming the local Lipschitz regularity of $V(T,\cdot)$, we derive uniform bounds for the truncated co-states (Lemma 3.4) which in turn allow to pass to the limit as $t \to \infty$ in the necessary conditions (Theorem 4.2). The second key point is Therem 5.1 which provides structural assumptions on the data for V to be Lipschitz. A typical dynamic programming argument is used to obtain such a property for certain classes of Lagrangians, which include problems with a sufficiently large discount factor or a periodic dependence on time.

The outline of the paper is as follows. In Section 2, we provide basic definitions, terminology, and facts from nonsmooth analysis. In Section 3, we give a bound on the total variation of measures associated to Mayer problems under state constraints. In Section 4, we focus on the main result, investigating problem $\mathscr{B}_\infty$ and stating sensitivity relations and transversality condition for the co-state. Finally, in the last Section, we prove the uniform Lipschitz continuity of a large class of value functions when A is compact.

2. **Preliminaries on nonsmooth analysis.** We denote by $\mathbb{B}$ the closed unit ball in $\mathbb{R}^n$ and by $|\cdot|$ the Euclidean norm. The interior of $C \subset \mathbb{R}^n$ is written as $\operatorname{int} C$. Given a nonempty subset C and a point x we denote the distance from x to C by $d_C(x) := \inf\{|x-y| : y \in C\}$, the convex hull of C by $\operatorname{co} C$, and its closure by $\overline{\operatorname{co}}\, C$. Take a family of sets $\{S(y) \subset \mathbb{R}^n : y \in D\}$ where $D \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. The sets[1]

$$\operatorname*{Lim\,inf}_{\substack{y\to x\\ D}} S(y) := \{\xi \in \mathbb{R}^n : \forall x_i \underset{D}{\longrightarrow} x,\ \exists \xi_i \to \xi \text{ s.t. } \xi_i \in S(x_i) \text{ for all } i\},$$
$$\operatorname*{Lim\,sup}_{\substack{y\to x\\ D}} S(y) := \{\xi \in \mathbb{R}^n : \exists x_i \underset{D}{\longrightarrow} x,\ \exists \xi_i \to \xi \text{ s.t. } \xi_i \in S(x_i) \text{ for all } i\}$$

are called, respectively, the *lower* and *upper limits* in the Kuratowski sense. Observe that these upper and lower limits are closed, possibly empty, and verify $\operatorname*{Lim\,inf}_{\substack{y\to x\\ D}} S(y) \subset \operatorname*{Lim\,sup}_{\substack{y\to x\\ D}} S(y)$.

We denote by $W^{1,1}(a,b;\mathbb{R}^n)$ the space of all absolutely continuous $\mathbb{R}^n$-valued functions $u : [a,b] \to \mathbb{R}^n$ endowed with the norm $\|u\|_{W^{1,1}(a,b)} = |u(a)| + \int_a^b |u'(t)|\, dt$. Let $u : [a,\infty) \to \mathbb{R}^n$, we write $u \in W^{1,1}_{\text{loc}}(a,\infty;\mathbb{R}^n)$ if $u|_{[a,b]} \in W^{1,1}(a,b;\mathbb{R}^n)$ for all $b > a$. Let I be a compact interval in $\mathbb{R}$. We denote by $C(I;\mathbb{R}^n)$ the set of all continuous $\mathbb{R}^n$-valued functions endowed with the *uniform norm* $\|u\|_{\infty,I} = \sup\{|u(t)| : t \in I\}$.

Let $G : [a,b] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a multifunction taking nonempty values. We say that $G(\cdot,x)$ is *absolutely continuous from the left*, uniformly on $R \subset \mathbb{R}^n$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite partition $a \leqslant t_1 < \tau_1 \leqslant t_2 < \tau_2 \leqslant \ldots \leqslant t_m < \tau_m \leqslant b$ of $[a,b]$ satisfying $\sum_1^m (\tau_i - t_i) < \delta$ and for any $x \in R$ we have $\sum_1^m d_{G(\tau_i,x)}(G(t_i,x)) < \varepsilon$, where for any $E, E' \subset \mathbb{R}^n$

$$d_E(E') := \inf\{\beta > 0 : E' \subset E + \beta\mathbb{B}\}.$$

Take a closed set $E \subset \mathbb{R}^n$ and $x \in E$. The *regular normal cone* $\hat{N}_E(x)$ to E at x and the *limiting normal cone* $N_E(x)$ to E at x are defined, respectively, by

[1] we write $y_i \underset{E}{\longrightarrow} x$ for $y_i \to x$ and $y_i \in E$ for any i.

$$\hat{N}_E(x) := \left\{ p \in \mathbb{R}^n \,:\, \limsup_{\substack{y \to x \\ E}} \frac{\langle p, y-x \rangle}{|y-x|} \leqslant 0 \right\}$$
$$N_E(x) := \operatorname*{Lim\,sup}_{\substack{y \to x \\ E}} \hat{N}_E(y).$$

We denote by $T_E^C(x) := (N_E(x))^-$ the *Clarke tangent cone* to E at x, where "$-$" stands for the negative polar of a set. It is well known that $\overline{\text{co}}\, N_E(x) = N_E^C(x)$ where $N_E^C(x) := (T_E^C(x))^-$ denotes the *Clarke normal cone* to E at x (cfr. [15, Chapter 6]). Take an extended-valued function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and define the effective domain of f by $\operatorname{dom} f := \{x \in \mathbb{R}^n \,:\, f(x) < +\infty\}$. We denote by $\operatorname{epi} f$ and $\operatorname{hypo} f$ the epigraph and hypograph of f respectively. The *subdifferential*, the *limiting subdifferential* and the *limiting superdifferential* of an extended real function f at $x \in \operatorname{dom} f$ are defined respectively by

$$\hat{\partial} f(x) := \left\{ \xi \in \mathbb{R}^n \,:\, (\xi, -1) \in \hat{N}_{\operatorname{epi} f}(x, f(x)) \right\}$$
$$\partial f(x) := \{ \xi \in \mathbb{R}^n \,:\, (\xi, -1) \in N_{\operatorname{epi} f}(x, f(x)) \}$$
$$\partial^+ f(x) := \{ \xi \in \mathbb{R}^n \,:\, (-\xi, 1) \in N_{\operatorname{hypo} f}(x, f(x)) \}.$$

If f is Lipschitz continuous on a neighborhood of $x \in \operatorname{dom} f$, then $\partial f(x)$ and $\partial^+ f(x)$ are nonempty. It is well known that $\hat{\partial} f(x) \neq \emptyset$ on a dense subset of $\operatorname{dom} f$, whenever f is lower semicontinuous.

3. **The value function.** Let $\tau > 0$ and $g^\tau : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function. Consider the problem $\mathscr{M}(g^\tau, \tau)$ on $[0, \tau]$

$$\text{minimize } g^\tau(x(\tau)) \tag{4}$$

over all the trajectories of the following differential inclusion under state constraints

$$\begin{cases} x'(t) \in F(t, x(t)) & \text{a.e. } t \in [t_0, \tau] \\ x \in W^{1,1}(t_0, \tau; \mathbb{R}^n) & \\ x(t_0) = x_0 & \\ x(t) \in \Omega & t \in [t_0, \tau] \end{cases} \tag{5}$$

with the initial datum $(t_0, x_0) \in [0, \tau] \times \Omega$, where $F : [0, \infty) \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a multifunction and $\Omega \subset \mathbb{R}^n$ a nonempty closed set. Every trajectory $x(\cdot)$ that satisfies the state constrained differential inclusion (5) is called *feasible*. The infimum of the cost in (4) over all feasible trajectories, with the initial datum (t_0, x_0), is denoted by $V^\tau(t_0, x_0)$ (if no feasible trajectory does exist, we define $V^\tau(t_0, x_0) = +\infty$). The function

$$V^\tau : [0, \tau] \times \Omega \to \mathbb{R} \cup \{\pm\infty\}$$

is called the *value function* of problem $\mathscr{M}(g^\tau, \tau)$. We say that $\bar{x}(\cdot)$ is a minimizer for problem $\mathscr{M}(g^\tau, \tau)$ at (t_0, x_0) if $\bar{x}$ is feasible, $\bar{x}(t_0) = x_0$ and $V(t_0, x_0) = g^\tau(\bar{x}(\tau))$.

We start with the main assumptions on $F(\cdot, \cdot)$ and Ω.

Hypothesis (H1):

- $F(\cdot, \cdot)$ takes closed nonempty values and $F(\cdot, x)$ is Lebesgue measurable for any $x \in \mathbb{R}^n$;
- there exists $k \in L^\infty([0, \infty); \mathbb{R}^+)$ such that $F(t, x) \subset k(t)(1 + |x|)\mathbb{B}$ for any $x \in \mathbb{R}^n$, a.e. $t \in [0, \infty)$;

- for all $R \geqslant 0$ there exists $\gamma_R \in L^1_{\rm loc}([0,\infty);\mathbb{R}^+)$ such that $F(t,x) \subset F(t,x') + \gamma_R(t)\,|x-x'|\,\mathbb{B}$ for any $x, x' \in B(0,R)$, a.e. $t \in [0,\infty)$;
- (Relaxed Inward Pointing Condition-IPC$'$) For any $(t,x) \in [0,\infty) \times \partial\Omega$ there exists a set $\Omega_{t,x} \subset [0,\infty)$ with null measure such that for any $v \in \mathbb{R}^n$ satisfying
$$v \in \operatorname*{Lim\,sup}_{\substack{(s,y)\to(t,x)\\ s\notin\Omega_{t,x}}} F(s,y) \quad \text{and} \quad \max_{n\in N_\Omega(x)\cap S^{n-1}} \langle\, n, v\rangle \geqslant 0$$
we can find $w \in \mathbb{R}^n$ such that
$$w \in \operatorname*{Lim\,inf}_{\substack{(s,y)\to(t,x)\\ s\notin\Omega_{t,x}}} \operatorname{co} F(s,y) \quad \text{and} \quad \max_{n\in N_\Omega(x)\cap S^{n-1}} \langle\, n, w-v\rangle < 0.$$

Let us denote by **(H2)** the hypothesis as in **(H1)** under an additional assumption

- For any $R > 0$ there exists $r > 0$ such that $F(\cdot,x)$ is absolutely continuous from the left, uniformly over $x \in (\partial\Omega + r\mathbb{B}) \cap B(0,R)$,

and with the Relaxed Inward Pointing Condition (IPC$'$) replaced by

- (Relaxed Inward Pointing Condition-IPC) For any $(t,x) \in [0,\infty) \times \partial\Omega$
$$\operatorname*{Lim\,inf}_{\substack{(t',x')\to(t,x)\\ (t',x')\in[0,\infty)\times\Omega}} \operatorname{co} F(t',x') \bigcap \operatorname{int} T^C_\Omega(x) \neq \emptyset.$$

Remark 1. We note that, if F is continuous, then the IPC condition reduces to
$$\operatorname{co} F(t,x) \bigcap \operatorname{int} T^C_\Omega(x) \neq \emptyset \qquad \forall (t,x) \in [0,\infty) \times \partial\Omega. \tag{6}$$

Define the Hamiltonian
$$H(t,x,p) = \max_{v\in F(t,x)} \langle\, p, v\rangle \qquad \forall\, (t,x,p) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n.$$

Then, by the separation theorem, (6) is equivalent to
$$H(t,x,-p) > 0 \qquad \forall\, 0 \neq p \in N^C_\Omega(x).$$

Theorem 3.1 ([8])**.** *Assume* **(H1)***, let $g : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function and consider the problem $\mathscr{M}(g,\tau)$ with $\tau > 0$. Then $V^\tau(\cdot,\cdot)$ is locally Lipschitz continuous on $[0,\tau] \times \Omega$.*

Moreover, if $\bar{x}(\cdot)$ is a minimizer for $\mathscr{M}(g,\tau)$ with initial condition $(t_0,x_0) \in [0,\tau] \times \Omega$, then there exists $p \in W^{1,1}(t_0,\tau;\mathbb{R}^n)$, a different from zero nonnegative Borel measure μ on $[t_0,\tau]$ and a Borel measurable function $\nu : [t_0,\tau] \to \mathbb{R}^n$ such that, letting
$$q(t) = p(t) + \eta(t)$$
with
$$\eta(t) = \begin{cases} \int_{[t_0,t]} \nu(s)\, d\mu(s) & t \in (t_0,\tau] \\ 0 & t = t_0, \end{cases}$$
the following holds true:

(i) $\nu(t) \in \overline{\rm co} N_\Omega(\bar{x}(t)) \cap \mathbb{B}$ $\mu-$ *a.e.* $t \in [t_0,\tau]$;
(ii) $p'(t) \in \operatorname{co}\big\{r \,:\, (r,q(t)) \in N_{GrF(t,\cdot)}(\bar{x}(t),\bar{x}'(t))\big\}$ *for a.e.* $t \in [t_0,\tau]$;
(iii) $-q(\tau) \in \partial g(\bar{x}(\tau))$, $-q(t_0) \in \partial^+_x V^\tau(t_0,\bar{x}(t_0))$;
(iv) $\langle\, q(t), \bar{x}'(t)\rangle = \max\{\langle\, q(t), v\rangle \,:\, v \in F(t,\bar{x}(t))\}$ *for a.e.* $t \in [t_0,\tau]$;
(v) $-q(t) \in \partial^0_x V^\tau(t,\bar{x}(t))$ *for a.e.* $t \in (t_0,\tau]$;
(vi) $(H(t,\bar{x}(t),q(t)), -q(t)) \in \partial^0 V^\tau(t,\bar{x}(t))$ *for a.e.* $t \in (t_0,\tau]$,

where

$$\partial_x^0 V^\tau(t,x) := \operatorname*{Lim\,sup}_{\substack{x' \to x \\ \operatorname{int}\Omega}} \operatorname{co}\, \partial_x V^\tau(t,x'),$$

$$\partial^0 V^\tau(t,x) := \operatorname*{Lim\,sup}_{\substack{(t',x') \to (t,x) \\ [t_0,\tau]\times \operatorname{int}\Omega}} \operatorname{co}\, \partial V^\tau(t,x').$$

Remark 2. We would like to acknowledge here that the proof of the above result in [8] contains an erroneous claim which however does not have any impact neither on the rest of the proof nor on the final result. Namely on p. 373 the correct expression is $\partial h^>(\bar{x}(t), \bar{f}(t)) = (0,0,0,0,0,0,-1)$ whenever $\bar{x}(t) \in \operatorname{int} A$ and so the claim (27) is not correct. This does not influence however the rest of the arguments of the proof.

Theorem 3.2 ([8])**.** *The conclusion of Theorem 3.1 is also valid if we assume* **(H2)** *instead of* **(H1)**.

Definition 3.3. A family $\mathscr{G}$ of $\mathbb{R}$-valued functions defined on $E \subset \mathbb{R}^k$ is *uniformly locally Lipschitz continuous* on E if for all $R \geqslant 0$ there exists $L_R \geqslant 0$ such that

$$|\varphi(z) - \varphi(\tilde{z})| \leqslant L_R\, |z - \tilde{z}|$$

for all z, $\tilde{z} \in E \cap B(0,R)$ and $\varphi \in \mathscr{G}$.

Lemma 3.4. *Assume* **(H1)** *or* **(H2)**. *For all $j \in \mathbb{N}$ let $g^j : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function. Fix $(t_0, x_0) \in [0,\infty) \times \operatorname{int}\Omega$, $T > t_0$ and consider the problems $\mathscr{M}(g^j, j)$. Assume also that $\left\{V^j(\cdot,\cdot)\right\}_{j \geqslant T}$ are uniformly locally Lipschitz continuous on $[0,T] \times \Omega$. Let $\bar{x} \in W^{1,1}_{loc}(t_0,\infty;\Omega)$ be such that for any $j \geqslant T$ the restriction $\bar{x}|_{[t_0,j]}(\cdot)$ is a minimizer for problem $\mathscr{M}(g^j, j)$ with initial datum (t_0, x_0). Let, for every $j \in \mathbb{N}$, p_j, q_j, ν_j, and μ_j be as in the conclusion of Theorem 3.1 for the problem $\mathscr{M}(g^j, j)$.*

Then

(i) $\{p_j\}_{j \geqslant T}$ and $\{q_j\}_{j \geqslant T}$ are uniformly bounded on $[t_0, T]$;
(ii) the total variation of the measures $\{\tilde{\mu}_j\}_{j \geqslant T}$ on $[t_0, T]$ is uniformly bounded, where $\tilde{\mu}_j$ is defined by $\tilde{\mu}(dt) = |\nu_j(t)|\, \mu_j(dt)$.

The proof of the above lemma relies on the following proposition, which can be in turn justified following the same reasoning as in the proof of [11, Lemma 4.1].

Proposition 1. *Let $I \subset \mathbb{R}$ be an interval and $G : I \rightrightarrows \mathbb{R}^n$ be a lower semicontinuous set valued map such that $G(t)$ is a closed convex cone and $\operatorname{int} G(t) \neq \emptyset$ for all $t \in I$. Then for every $\varepsilon > 0$ there exists a continuous function $f : I \to \mathbb{R}^n$ such that for all $t \in \{s \in I \,:\, G(s) \neq \mathbb{R}^n\}$*

$$\sup_{n \in G(t)^- \cap S^{n-1}} \langle\, n, f(t) \rangle \leqslant -\varepsilon.$$

Proof of the Lemma 3.4. Since $\bar{x}(\cdot)$ is continuous, hence locally bounded, by the uniform local Lipschitz continuity of $\left\{V^j\right\}_j$ we deduce that

$$\sup \left\{ |\xi| \,:\, \xi \in \bigcup_{t \in [t_0,T]} \partial_x^0 V^j(t, \bar{x}(t)) \cup \partial_x^+ V^j(t_0, \bar{x}(t_0)),\, j \geqslant T \right\} < \infty. \tag{7}$$

By Theorem 3.1-(iii), (v) we know that

$$-q_j(T) \in \partial g^j(\bar{x}(T)),\ -q_j(t_0) = -p_j(t_0) \in \partial_x^+ V^j(t_0, \bar{x}(t_0))$$

and

$$-q_j(t) \in \partial_x^0 V^j(t, \bar{x}(t)) \qquad \text{a.e. } t \in (t_0, T]$$

for all $j \geqslant T$. Since q_j are right continuous on (t_0, T), from (7), it follows that

$$\left\{ \|q_j\|_{\infty,[t_0,T]} \right\}_{j \geqslant T} \text{ is bounded.} \tag{8}$$

Now, by a well-known property of Lipschitz multifunctions (cfr. [19, Proposition 5.4.2]), from (ii) of Theorem 3.1 and assumptions **(H1)** (respectively **(H2)**) it follows that there exists $\xi \in L^1_{\text{loc}}[t_0, \infty)$ such that $\left|p_j'(t)\right| \leqslant \xi(t)\,|q_j(t)|$ for a.e. $t \in [t_0, T]$ and all $j \geqslant T$. Hence, in view of (8),

$$\left\{ \|p_j\|_{\infty,[t_0,T]} \right\}_{j \geqslant T} \text{ is bounded.} \tag{9}$$

So, the conclusion (i) follows. Also, since $q_j(t) = p_j(t) + \eta_j(t)$, from (8) and (9) we deduce that

$$\left\{ \|\eta_j\|_{\infty,[t_0,T]} \right\}_{j \geqslant T} \text{ is bounded.} \tag{10}$$

Now let $\Gamma := \{s \in [t_0, T] : \bar{x}(s) \in \partial\Omega\}$. From the Relaxed Inward Pointing Condition, it follows that $\operatorname{int} T_\Omega^C(\bar{x}(t))$ is nonempty for all $t \in \Gamma$ and so $\operatorname{int} T_\Omega^C(\bar{x}(t))$ is nonempty for all $t \in [t_0, T]$. Furthermore, this implies that the set valued map $t \rightsquigarrow T_\Omega^C(\bar{x}(t))$ is lower semicontinuous on $[t_0, T]$. Since $\Gamma = \left\{s \in [t_0, T] : T_\Omega^C(\bar{x}(s)) \neq \mathbb{R}^n\right\}$, we can apply Proposition 1 with $\varepsilon = 2$ to conclude that there exists a continuous function $f : [t_0, T] \to \mathbb{R}^n$ such that

$$\sup_{n \in N_\Omega^C(\bar{x}(t)) \cap S^{n-1}} \langle\, f(t), n \rangle \leqslant -2 \qquad \forall t \in \Gamma. \tag{11}$$

We remark that the function f does not depend on j but only on $\bar{x}(\cdot)$ and T. Now, consider $\tilde{f} \in C^\infty([t_0, T]; \mathbb{R}^n)$ such that $\left\|f - \tilde{f}\right\|_{\infty,[t_0,T]} \leqslant 1$. We obtain from (11)

$$\sup_{n \in N_\Omega^C(\bar{x}(t)) \cap S^{n-1}} \langle\, \tilde{f}(t), n \rangle \leqslant -1. \tag{12}$$

Then, from (12) we deduce that for all $j \geqslant T$,

$$\begin{aligned}
&\int_{[t_0,T]} \langle\, \tilde{f}(s), \nu_j(s) \rangle \, d\mu_j(s) \\
&= \int_{[t_0,T] \cap \{s\,:\,\nu_j(s) \neq 0\}} \langle\, \tilde{f}(s), \nu_j(s) \rangle \, d\mu_j(s) \\
&= \int_{[t_0,T] \cap \{s\,:\,\nu_j(s) \neq 0\}} \langle\, \tilde{f}(s), \frac{\nu_j(s)}{|\nu_j(s)|} \rangle \, |\nu_j(s)| \, d\mu_j(s) \\
&\leqslant -\int_{[t_0,T] \cap \{s\,:\,\nu_j(s) \neq 0\}} |\nu_j(s)| \, d\mu_j(s) \\
&= -\int_{[t_0,T]} |\nu_j(s)| \, d\mu_j(s).
\end{aligned}$$

So,

$$\int_{[t_0,T]} |\nu_j(s)| \, d\mu_j(s) \leqslant \int_{[t_0,T]} \langle\, -\tilde{f}(s), \nu_j(s) \rangle \, d\mu_j(s). \tag{13}$$

Furthermore, from (10), integrating by parts, we obtain that, for some constant $C \geqslant 0$ and all $j \geqslant T$,

$$\begin{aligned}
&\int_{[t_0,T]} \langle -\tilde{f}(s), \nu_j(s)\rangle \, d\mu_j(s) \\
&= \int_{[t_0,T]} -\tilde{f}(s) \, d\eta_j(s) \\
&= -\eta_j(T)\tilde{f}(T) + \int_{[t_0,T]} \eta_j(s)\tilde{f}'(s) \, ds \\
&\leqslant C \left(\left\| \tilde{f} \right\|_\infty + (T-t_0) \left\| \tilde{f}' \right\|_\infty \right).
\end{aligned} \tag{14}$$

Now, since $\tilde{f}$ does not depend on j, from (13) and (14) we deduce (ii). □

4. **The infinite horizon optimal control problem.** Consider the infinite horizon optimal control problem with state constraints $\mathscr{B}_\infty$ as in (1)-(2). We define the Hamiltonian function on $[0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n$ by

$$\mathcal{H}(t,x,p) = \sup \left\{ \langle p, f(t,x,u)\rangle - L(t,x,u) \, : \, u \in U(t) \right\}.$$

Let us denote by **(h)** the following assumptions:

- there exist two locally essentially bounded functions $b, \theta : \mathbb{R}^+ \to \mathbb{R}^+$ and a nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for a.e. $t \in \mathbb{R}^+$ and for all $x \in \mathbb{R}^n$, $u \in U(t)$
$$\begin{aligned} |f(t,x,u)| &\leqslant b(t)\,(1+|x|)\,, \\ |L(t,x,u)| &\leqslant \theta(t)\Psi(|x|); \end{aligned}$$
- for any $R > 0$ there exist two locally integrable functions c_R, $\alpha_R, : \mathbb{R}^+ \to \mathbb{R}^+$ such that for a.e. $t \in \mathbb{R}^+$ and for all x, $y \in B(0,R)$, $u \in U(t)$,
$$\begin{aligned} |f(t,x,u) - f(t,y,u)| &\leqslant c_R(t)|x-y|, \\ |L(t,x,u) - L(t,y,u)| &\leqslant \alpha_R(t)\,|x-y|\,; \end{aligned}$$
- for all $x \in \mathbb{R}^n$ the mappings $f(\cdot,x,\cdot)$, $L(\cdot,x,\cdot)$ are Lebesgue-Borel measurable;
- For a.e. $t \in \mathbb{R}^+$, and for all $x \in \mathbb{R}^n$ the set
$$\{(f(t,x,u), L(t,x,u)) \, : \, u \in U(t)\}$$
is closed;
- the Relaxed Inward Pointing Condition-IPC$'$ is satisfied;
- for all $(t_0,x_0) \in [0,\infty) \times A$ the limit $\lim_{T\to\infty} \int_{t_0}^T L(t,x(t),u(t))\,dt$ exists for all trajectory-control pairs (x,u) satisfying (2) with initial datum (t_0,x_0);
- $V(t_0,x_0) \neq -\infty$ for all $(t_0,x_0) \in [0,\infty) \times A$.

Remark 3. A sufficient condition to guarantee that the last two hypothesis in **(h)** are satisfied is to assume that L takes nonnegative values. Alternatively, we may assume that for any initial datum (t_0,x_0) there exists a function $\phi_{t_0,x_0} \in L^1(0,\infty)$ such that $L(t,x(t),u(t)) \geqslant \phi_{t_0,x_0}(t)$ a.e. $t \in [t_0,\infty)$ for all trajectory-control pairs (x,u) satisfying (2).

The above hypotheses guarantee the existence and uniqueness of the solution to the differential equation in (2) for every initial datum x_0 and every control. So,

denoting by $x_{x_0,u_0}(\cdot)$ such solution starting from x_0 at time t_0, associated with a control $u_0(\cdot)$, by Gronwall's lemma and our growth assumptions

$$|x_{x_0,u_0}(t)| \leqslant \Big(|x_0| + (t-t_0)\,\|b\|_{\infty,[t_0,t]}\Big)\, e^{(t-t_0)\|b\|_{\infty,[t_0,t]}} \qquad \forall t \geqslant t_0. \tag{15}$$

In particular, feasible trajectories are uniformly bounded on every compact time interval. Moreover, setting

$$M_{t_0,R}(t) = \Big(R + (t-t_0)\,\|b\|_{\infty,[t_0,t]}\Big)\, e^{(t-t_0)\|b\|_{\infty,[t_0,t]}},$$

by (15), Gronwall's lemma, and our assumptions we have that for all $R, t > 0$, all $t_0 \in [0,t]$, and all $x_0, x_1 \in B(0,R)$

$$|x_{x_1,u}(s) - x_{x_0,u}(s)| \leqslant |x_1 - x_0|\, e^{\int_{t_0}^{s} c_{M_{t_0,R(t)}}(\xi)\,d\xi} \qquad \forall s \in [t_0,t]. \tag{16}$$

Define the extended value function $V : [0,\infty) \times A \to \mathbb{R} \cup \{+\infty\}$ of problem $\mathscr{B}_\infty$ by

$$V(t_0,x_0) := \inf \int_{t_0}^{\infty} L(t,x(t),u(t))\,dt,$$

where the infimum is taken over all trajectory-control pairs (x,u) that satisfy (2) with the initial datum $(t_0,x_0) \in [0,\infty) \times A$.

We denote by $\operatorname{dom} V$ the set $\{(t_0,x_0) \in [0,\infty) \times A : V(t_0,x_0) < +\infty\}$, and we say that a pair $(\bar{x},\bar{u})$ is optimal for $\mathscr{B}_\infty$ at $(t_0,x_0) \in \operatorname{dom} V$ if

$$\int_{t_0}^{\infty} L(t,\bar{x}(t),\bar{u}(t))\,dt \leqslant \int_{t_0}^{\infty} L(t,x(t),u(t))\,dt$$

for any feasible trajectory-control pair (x,u) starting from x_0 at time t_0.

Lemma 4.1. *Let $T \geqslant 0$ and assume* **(h)**. *Consider the Bolza problem $\mathscr{B}_T$*

$$\text{minimize } \left\{ V(T,x(T)) + \int_{t_0}^{T} L(t,x(t),u(t))\,dt \right\}$$

over all the trajectory-control pairs satisfying the state constrained equation

$$\begin{cases} x'(t) = f(t,x(t),u(t)) & \text{a.e. } t \in [t_0,T] \\ x(t_0) = x_0 & \\ u(t) \in U(t) & \text{a.e. } t \in [t_0,T] \\ x(t) \in A & t \in [t_0,T]. \end{cases}$$

Denote by $V_{\mathscr{B}_T} : [0,T] \times A \to \mathbb{R} \cup \{+\infty\}$ the value function of the above problem. Then

$$V_{\mathscr{B}_T}(\cdot,\cdot) = V(\cdot,\cdot) \qquad \text{on } [0,T] \times A. \tag{17}$$

Furthermore, if $(\bar{x},\bar{u})$ is optimal at $(t_0,x_0) \in [0,T] \times A$ for $\mathscr{B}_\infty$, then the restriction of $(\bar{x},\bar{u})$ to the time interval $[t_0,T]$ is optimal for the Bolza problem $\mathscr{B}_T$ too.

Proof. Let $(t_0,x_0) \in [0,T] \times A$ and $\varepsilon > 0$. If $V(t_0,x_0) = +\infty$, then $V(t_0,x_0) \geqslant V_{\mathscr{B}_T}(t_0,x_0)$. Otherwise, there exists a feasible trajectory-control pair $(x_\varepsilon,u_\varepsilon)$ for

problem $\mathscr{B}_\infty$ at (t_0, x_0) such that

$$\begin{aligned} V(t_0,x_0) &\geqslant \int_{t_0}^{T} L(s, x_\varepsilon(s), u_\varepsilon(s))\, ds + \int_{T}^{\infty} L(s, x_\varepsilon(s), u_\varepsilon(s))\, ds - \varepsilon \\ &\geqslant \int_{t_0}^{T} L(s, x_\varepsilon(s), u_\varepsilon(s))\, ds + V(T, x_\varepsilon(T)) - \varepsilon \\ &\geqslant V_{\mathscr{B}_T}(t_0,x_0) - \varepsilon. \end{aligned} \tag{18}$$

Since ε is arbitrary, we obtain $V(t_0,x_0) \geqslant V_{\mathscr{B}_T}(t_0,x_0)$.

On the other hand, if $V_{\mathscr{B}_T}(t_0,x_0) = +\infty$, then $V_{\mathscr{B}_T}(t_0,x_0) \geqslant V(t_0,x_0)$. Otherwise, there exists a feasible trajectory-control pair $(\tilde{x}_\varepsilon, \tilde{u}_\varepsilon)$ for problem $\mathscr{B}_T$ at (t_0,x_0) such that

$$V_{\mathscr{B}_T}(t_0,x_0) \geqslant \int_{t_0}^{T} L(s, \tilde{x}_\varepsilon(s), \tilde{u}_\varepsilon(s))\, ds + V(T, \tilde{x}_\varepsilon(T)) - \varepsilon.$$

By (15) and our assumptions on L, $\int_{t_0}^{T} |L(s, \tilde{x}_\varepsilon(s), \tilde{u}_\varepsilon(s))|\, ds < \infty$. Hence $(T, \tilde{x}_\varepsilon(T)) \in \operatorname{dom} V$. So, there exists a feasible trajectory-control pair $(\hat{x}_\varepsilon, \hat{u}_\varepsilon)$ for problem $\mathscr{B}_\infty$ at $(T, \tilde{x}_\varepsilon(T))$ such that

$$\begin{aligned} V_{\mathscr{B}_T}(t_0,x_0) &\geqslant \int_{t_0}^{T} L(s, \tilde{x}_\varepsilon(s), \tilde{u}_\varepsilon(s))\, ds + \int_{T}^{\infty} L(s, \hat{x}_\varepsilon(s), \hat{u}_\varepsilon(s))\, ds - 2\varepsilon \\ &= \int_{t_0}^{\infty} L(s, x(s), u(s))\, ds - 2\varepsilon, \end{aligned} \tag{19}$$

where $x(\cdot)$ is the trajectory starting from x_0 at time t_0 satisfying the ordinary differential equation in (2) with the control u given by

$$u(s) := \begin{cases} \tilde{u}_\varepsilon(s) & s \in [t_0, T] \\ \hat{u}_\varepsilon(s) & s \in (T, \infty). \end{cases}$$

Since $u(\cdot) \in U(\cdot)$ and $x([t_0,\infty)) \subset A$, (x,u) is feasible for problem $\mathscr{B}_\infty$ at (t_0,x_0). Then, by (19), $V_{\mathscr{B}_T}(t_0,x_0) \geqslant V(t_0,x_0) - 2\varepsilon$ and, since ε is arbitrary, $V_{\mathscr{B}_T}(t_0,x_0) \geqslant V(t_0,x_0)$.

The last part of the conclusion follows from (18), by setting $\varepsilon = 0$, $(x_\varepsilon, u_\varepsilon) = (\bar{x}, \bar{u})$, and using that $V_{\mathscr{B}_T}(t_0,x_0) = V(t_0,x_0)$. □

Theorem 4.2. *Assume* **(h)** *and suppose that $V(i,\cdot)$ is locally Lipschitz continuous on A for all large $i \in \mathbb{N}$. Then V is locally Lipschitz continuous on $[0,\infty) \times A$.*

Moreover, if $(\bar{x}, \bar{u})$ is optimal for $\mathscr{B}_\infty$ at $(t_0,x_0) \in [0,\infty) \times \operatorname{int}\Omega$, then there exist $p \in W^{1,1}_{\mathrm{loc}}(t_0,\infty;\mathbb{R}^n)$, a nonnegative Borel measure μ on $[t_0,\infty)$, and a Borel measurable function $\nu : [t_0,\infty) \to \mathbb{R}^n$ such that, setting

$$q(t) = p(t) + \eta(t)$$

with

$$\eta(t) = \begin{cases} \int_{[t_0,t]} \nu(s)\, d\mu(s) & t \in (t_0,\infty) \\ 0 & t = t_0, \end{cases}$$

the following holds true:

(i) $\nu(t) \in \overline{\mathrm{co}}\, N_A(\bar{x}(t)) \cap \mathbb{B}$ $\mu-$a.e. $t \in [t_0,\infty)$;

(ii) $p'(t) \in \mathrm{co}\, \big\{ r : (r, q(t), -1) \in N_{\mathrm{Gr}\, F(t,\cdot)}(\bar{x}(t), \bar{x}'(t), L(t, \bar{x}(t), \bar{u}(t))) \big\}$ *for a.e.* $t \in [t_0,\infty)$ *where* $F(t,x) = \{(f(t,x,u), L(t,x,u)) : u \in U(t)\}$;

(iii) $-p(t_0) \in \partial_x^+ V(t_0, \bar{x}(t_0))$, $-q(t) \in \partial_x^0 V(t, \bar{x}(t))$ *for a.e.* $t \in (t_0, \infty)$;
(iv) $\langle q(t), f(t, \bar{x}(t), \bar{u}(t))\rangle - L(t, \bar{x}(t), \bar{u}(t)) = \max_{u \in U(t)} \langle q(t), f(t, \bar{x}(t), u)\rangle - L(t, \bar{x}(t), u)$ *for a.e.* $t \in [t_0, \infty)$;
(v) $(\mathcal{H}(t, \bar{x}(t), q(t)), -q(t)) \in \partial^0 V(t, \bar{x}(t))$ *for a.e.* $t \in (t_0, \infty)$.

Remark 4. (a) Define $G(t, x) = \{(f(t, x, u), L(t, x, u)) : u \in U(t)\}$ and assume that for all $R > 0$ there exists $r > 0$ such that $G(\cdot, x)$ is absolutely continuous from the left uniformly over $(\partial A + r\mathbb{B}) \cap B(0, R)$. Then the conclusion of Theorem 4.2 holds if IPC$'$ is replaced by IPC;
(b) Theorem 4.2 implies a weaker hamiltonian inclusion

$$(-p'(t), \bar{x}'(t)) \in co\, \partial_{(x,p)} \mathcal{H}(t, \bar{x}(t), q(t)) \qquad \text{a.e. } t \in [t_0, \infty)$$

(cfr. comment (e)-[8, p. 362]);
(c) If $V(i, \cdot)$ is locally Lipschitz continuous on A for all large i, then, under assumptions of Theorem 4.2, $V(t, \cdot)$ is locally Lipschitz on A for every $t \geq 0$;
(d) See Section 5 for sufficient conditions for Lipschitz continuity of $V(t, \cdot)$ in the autonomous case when A is compact. Also, sufficient conditions for the Lipschitz continuity of $V(t, \cdot)$ in the nonautonomous case for unbounded A were recently investigated in [5].

Proof of Theorem 4.2. For any $j \in \mathbb{N}$ such that $j \geqslant t_0$ consider the Bolza problem $\mathscr{B}_j$. We can rewrite the problem as a Mayer one on $\mathbb{R}^{n+1}$: keeping the same notation as in Section 3, consider the Mayer problems $\mathscr{M}(g^j, j)$ on $\mathbb{R}^{n+1}$ with

$$g^j(\xi, z) := V(j, \xi) + z,$$

$\tilde{F}(t, x, z) := \{(f(t, x, u), L(t, x, u)) : u \in U(t)\}$ and $\Omega = A \times \mathbb{R}$.

Denoting by V^j the extended value function on $[0, j] \times \Omega$ for problem $\mathscr{M}(g^j, j)$ it follows, by standard arguments (cfr. [10, Chapter 7]), that

$$V^j(t, x, z) = V_{\mathscr{B}_j}(t, x) + z \tag{20}$$

for all $(t, x, z) \in [0, j] \times A \times \mathbb{R}$. Since, for all large j, $V(j, \cdot)$ is locally Lipschitz continuous on A, also g^j is locally Lipschitz on $A \times \mathbb{R}$. For every j consider a locally Lipschitz function $\bar{g}^j : \mathbb{R}^{n+1} \to \mathbb{R}$ that coincides with g^j on $A \times \mathbb{R}$. Note that replacing g^j by $\bar{g}^j$ does not change the value function of the Bolza problem $\mathscr{B}_j$. So, applying Theorem 3.1, it follows that V^j is locally Lipschitz on $[0, j] \times A \times \mathbb{R}$ for all large j. Then $V_{\mathscr{B}_j}$ is locally Lipschitz on $[0, j] \times A$ and so, by Lemma 4.1, the value function V is locally Lipschitz on $[0, j] \times A$. By the arbitrariness of j, V is locally Lipschitz continuous on $[0, \infty) \times A$. Hence, if $T > 0$, from (20) and (17) it follows that V^j's are uniformly locally Lipschitz continuous on $[0, T] \times A \times \mathbb{R}$ for all $j \geqslant T$.

Since the restriction of $(\bar{x}, \bar{u})$ to $[t_0, j]$ is optimal for $V_{\mathscr{B}_j}$ at (t_0, x_0), setting

$$\bar{z}(t) = \int_{t_0}^{t} L(s, \bar{x}(s), \bar{u}(s))\, ds,$$

we have that the restriction of $(\bar{X} := (\bar{x}, \bar{z}), \bar{u})$ to $[t_0, j]$ is optimal for V^j at $(t_0, (x_0, 0))$ too. So, we may apply Theorem 3.1 with $\bar{g}^j$ instead of g^j on each time interval $[t_0, j]$ with $j \in \mathbb{N} \cap [t_0, \infty)$. Denoting by X the pair (x, z) in $\mathbb{R}^{n+1}$, we obtain that there exist absolutely continuous arcs $\{P_j\}_j$ and functions $\{\Phi_j\}_j$ of bounded variation defined on $[t_0, j]$, and nonnegative measures $\{\mu_j\}_j$ on $[t_0, j]$ such that $\{\Phi_j\}_j$ are continuous from the right on (t_0, j) and

(a) $Q_j(t) = P_j(t) + \Phi_j(t)$, where $\Phi_j(t_0) = 0$, $\Phi_j(t) = \int_{[t_0,t]} \Pi_j(s)\, d\mu_j(s)$ for all $t \in (t_0, j]$ for some Borel measurable selections $\Pi_j(s) \in \overline{\mathrm{co}}\, N_\Omega(\bar{X}(s)) \cap B$ μ_j-a.e. $s \in [t_0, j]$;

(b) $P_j'(t) \in \mathrm{co}\, \Big\{R \,:\, (R, Q_j(t)) \in N_{\mathrm{Gr}\, \tilde{F}(t,\cdot)}(\bar{X}(t), \bar{X}'(t))\Big\}$ for a.e. $t \in [t_0, j]$;

(c) $-Q_j(t_0) \in \partial_X^+ V^j(t_0, \bar{X}(t_0))$;

(d) $\langle\, Q_j(t), \bar{X}'(t)\rangle = \max\Big\{\langle\, Q_j(t), v\rangle \,:\, v \in \tilde{F}(t, \bar{X}(t))\Big\}$ for a.e. $t \in [t_0, j]$;

(e) $-Q_j(t) \in \partial_X^0 V^j(t, \bar{X}(t))$ for a.e. $t \in (t_0, j]$;

(f) $(\tilde{H}(t, \bar{X}(t), Q_j(t)), -Q_j(t)) \in \partial^0 V^j(t, \bar{X}(t))$ for a.e. $t \in (t_0, j]$,

where $\tilde{H}(t, X, P) = \max_{v \in \tilde{F}(t,X)} \langle\, P, v\rangle$.

Let $P_j(t) = (p_j(t), p_j^0(t))$, $Q_j(t) = (q_j(t), q_j^0(t))$, $\Phi_j(t) = (\eta_j(t), \eta_j^0(t))$, and $\Pi_j(t) = (\nu_j(t), \nu_j^0(t))$. Using the definition of limiting normal vectors as limits of strict normal vectors, relations (a)-(c), and the fact that $N_\Omega(\bar{X}(\cdot)) = N_A(\bar{x}(\cdot)) \times \{0\}$ we obtain

$$p_j'(t) \in \mathrm{co}\, \big\{r \,:\, (r, q_j(t), q_j^0(t)) \in N_{\mathrm{Gr}\, F(t,\cdot)}(\bar{x}(t), \bar{x}'(t), L(t, \bar{x}(t), \bar{u}(t)))\big\}$$

$$\text{a.e. } t \in [t_0, j],$$

$$(p_j^0)' \equiv 0, \quad p_j^0(t_0) = -1, \quad \nu_j^0 \equiv 0, \quad \eta_j^0 \equiv 0, \quad q_j^0 \equiv -1.$$

Thus, on account of (d)-(f), for a.e. $t \in [t_0, j]$ we derive the extended Euler-Lagrange condition

$$p_j'(t) \in \mathrm{co}\, \big\{r \,:\, (r, q_j(t), -1) \in N_{\mathrm{Gr}\, F(t,\cdot)}(\bar{x}(t), \bar{x}'(t), L(t, \bar{x}(t), \bar{u}(t)))\big\} \tag{21}$$

where $q_j(t) = p_j(t) + \eta_j(t)$, with

$$\eta_j(t) = \begin{cases} \int_{[t_0,t]} \nu_j(s)\, d\mu_j(s) & t \in (t_0, j] \\ 0 & t = t_0, \end{cases} \tag{22}$$

and $\nu_j(t) \in \overline{\mathrm{co}}\, N_A(\bar{x}(t)) \cap \mathbb{B}$ μ_j-a.e. on $[t_0, j]$, satisfy the maximum principle

$$\begin{aligned} &\langle\, q_j(t), f(t, \bar{x}(t), \bar{u}(t))\rangle - L(t, \bar{x}(t), \bar{u}(t)) \\ &= \max_{u \in U(t)} \langle\, q_j(t), f(t, \bar{x}(t), u)\rangle - L(t, \bar{x}(t), u) \qquad \text{a.e. } t \in [t_0, j], \end{aligned} \tag{23}$$

the transversality condition in terms of limiting superdifferential

$$-p_j(t_0) \in \partial_x^+ V(t_0, x_0), \tag{24}$$

and the sensitivity relations

$$-q_j(t) \in \partial_x^0 V(t, \bar{x}(t)) \qquad \text{a.e. } t \in (t_0, j], \tag{25}$$

$$(\mathcal{H}(t, \bar{x}(t), q_j(t)), -q_j(t)) \in \partial^0 V(t, \bar{x}(t)) \qquad \text{a.e. } t \in (t_0, j]. \tag{26}$$

We extend the functions p_j and η_j to whole interval (j, ∞) as the constants $p_j(j)$ and $\eta_j(j)$, respectively. We denote again by p_j and η_j such extensions.

We divide the proof into three steps. Let k be an integer such that $k > t_0$.

Step 1. Applying Lemma 3.4 to problems $\mathscr{M}(g^j, j)$, we known that $\{p_j\}_{j \geqslant k}$ and $\{q_j\}_{j \geqslant k}$ are uniformly bounded on $[t_0, k]$. Furthermore, for some $\xi \in L^1_{\mathrm{loc}}([0, \infty); \mathbb{R}^+)$ and a.e. $t \in [t_0, j]$, we have $\left|p_j'(t)\right| \leqslant \xi(t)\, |q_j(t)|$ for all j. So, by the Ascoli-Arzelà and Dunford-Pettis theorems we have, taking a subsequence and keeping the

same notation, that there exists an absolutely continuous function $p^k : [t_0, k] \to \mathbb{R}^n$ such that

$$\begin{aligned} &p_j \to p^k \text{ uniformly on } [t_0, k] \\ &p_j' \rightharpoonup (p^k)' \text{ in } L^1(t_0, k). \end{aligned}$$

Furthermore, from Lemma 3.4 again, we known that $\{\eta_j\}_{j \geqslant k}$ is uniformly bounded on $[t_0, k]$ and the total variation of such functions is uniformly bounded on $[t_0, k]$. So, applying Helly's selection theorem, taking a subsequence and keeping the same notation, we deduce that there exists a function of bounded variation η^k on $[t_0, k]$ such that $\eta_j \to \eta^k$ pointwise on $[t_0, k]$ (notice that since $\eta_j(t_0) = 0$ for all j then $\eta^k(t_0) = 0$). Furthermore, from Lemma 3.4-(ii) we deduce that there exists a nonnegative measure μ^k on $[t_0, k]$ such that, by further extraction of a subsequence, $\tilde{\mu}_j \rightharpoonup^* \mu^k$ in $C([t_0, k]; \mathbb{R})^*$, where $\tilde{\mu}_j(dt) = |\nu_j(t)|\, \mu_j(dt)$. Let

$$\gamma_j(t) := \begin{cases} \dfrac{\nu_j(t)}{|\nu_j(t)|} & \nu_j(t) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\gamma_j(t) \in \overline{\text{co}} N_A(\bar{x}(t)) \cap \mathbb{B}$ $\tilde{\mu}_j-$a.e. $t \in [t_0, k]$ is a Borel measurable selection, applying [19, Proposition 9.2.1], we deduce that, for a subsequence j_i, there exists a Borel measurable function ν^k such that

$$\nu^k(\cdot) \in \overline{\text{co}} N_A(\bar{x}(\cdot)) \cap \mathbb{B} \qquad \mu^k\text{-a.e. on } [t_0, k]$$

and for all $\phi \in C([t_0, k]; \mathbb{R}^n)$

$$\int_{[t_0,k]} \langle\, \phi(s), \gamma_{j_i}(s) \rangle\, d\tilde{\mu}_{j_i}(s) \to \int_{[t_0,k]} \langle\, \phi(s), \nu^k(s) \rangle\, d\mu^k(s) \qquad \text{as } i \to \infty. \tag{27}$$

Now since for $t \in (t_0, k]$

$$\begin{aligned} \eta_{j_i}(t) = \int_{[t_0,t]} \nu_{j_i}(s)\, d\mu_{j_i}(s) &= \int_{[t_0,t] \cap \{s\,:\, \nu_{j_i}(s) \neq 0\}} \nu_{j_i}(s)\, d\mu_{j_i}(s) \\ &= \int_{[t_0,t]} \gamma_{j_i}(s)\, d\tilde{\mu}_{j_i}(s), \end{aligned}$$

from (27) it follows that for all $t \in (t_0, k]$

$$\eta^k(t) = \int_{[t_0,t]} \nu^k(s)\, d\mu^k(s).$$

By Mazur's theorem, as in [4, Theorem 7.2.2], using the closedness of $\partial_x^+ V(t_0, x_0)$, $\partial_x^0 V(t, \bar{x}(t))$ and convexity in (21), passing to the limit in (24), (21), and (23) on $[t_0, k]$, and in (25) and (26) on $(t_0, k]$, we obtain condition (iv) on $[t_0, k]$, inclusions (ii) on $[t_0, k]$, (iii) and (v) at t_0 and on $(t_0, k]$.

Step 2. Consider now the interval $[t_0, k+1]$. By the same argument as in the first step, taking suitable subsequences $\{p_{j_{i_l}}\}_l \subset \{p_{j_i}\}_i$ and $\{\eta_{j_{i_l}}\}_l \subset \{\eta_{j_i}\}_i$, we deduce that there exist an absolutely continuous function p^{k+1}, a function of bounded variation η^{k+1}, and a nonnegative measure μ^{k+1} which satisfy condition (iv) on

$[t_0, k+1]$, inclusions (ii) on $[t_0, k+1]$, (iii) and (v) at t_0 and on $(t_0, k+1]$. Moreover

$$\begin{aligned} &p_{j_{i_l}} \to p^{k+1} \text{ uniformly on } [t_0, k+1] \\ &p'_{j_{i_l}} \rightharpoonup (p^{k+1})' \text{ in } L^1(t_0, k+1) \\ &p^{k+1}|_{[t_0,k]} = p^k, \end{aligned}$$

and for all $t \in [t_0, k+1]$

$$\eta_{j_{i_l}}(t) \to \eta^{k+1}(t) = \begin{cases} \int_{[t_0,t]} \nu^{k+1}(s)\, d\mu^{k+1}(s) & t \in (t_0, k+1] \\ 0 & t = t_0, \end{cases}$$

where $\nu^{k+1}(\cdot) \in \overline{\text{co}} N_A(\bar{x}(\cdot)) \cap \mathbb{B}$ $\quad \mu^{k+1}$-a.e. on $[t_0, k+1]$ is a Borel measurable selection. Furthermore, since $\eta^{k+1}|_{[t_0,k]} = \eta^k$ and $\mu^{k+1}|_{[t_0,k]} = \mu^k$, we have that

$$\nu^{k+1}|_{[t_0,k]} = \nu^k \qquad \mu^k\text{-a.e. on } [t_0, k].$$

We see that the functions p^{k+1}, η^{k+1}, and ν^{k+1} extend the functions p^k, η^k, and ν^k respectively, and measure μ^{k+1} extends measure μ^k.

Step 3. Repeating the argument of the second step for any interval $[t_0, k+s]$ with $s \in \mathbb{N}$, we can extend p^k, η^k, ν^k and μ^k to the whole interval $[t_0, \infty)$, extracting every time a subsequence of the previously constructed subsequence. Finally, we conclude that there exists a locally absolutely continuous function $p : [t_0, \infty) \to \mathbb{R}^n$, a function of locally bounded variation $\eta : [t_0, \infty) \to \mathbb{R}$, a nonnegative measure μ on $[t_0, \infty)$, and a Borel measurable selection $\nu(t) \in \overline{\text{co}}\, N_A(\bar{x}(t)) \cap \mathbb{B}$ $\quad \mu-$a.e. $t \in [t_0, \infty)$ satisfying the conclusion of the therorem. □

5. **Uniform Lipschitz continuity of a class of value functions.** We now investigate the uniform Lipschitz continuity of a class of value functions. In this section, we assume that f is time independent, i.e., $f(t, x, u) = f(x, u)$, $U(\cdot) \equiv U$ is closed, A is compact, and assumptions **(h)** hold true. Then, thanks to Remark 4 (a) and to (6), (IPC′) can be replaced by the simpler condition

$$\max_{u \in U} \langle -p, f(x, u) \rangle > 0 \qquad \forall\, 0 \neq p \in N_A^C(x) \quad \forall\, x \in \partial A.$$

Theorem 5.1. *Assume that*

$$L(t, x, u) = e^{-\lambda t} l(x, u).$$

Then the function $v(\cdot) := V(0, \cdot)$ is Lipschitz continuous on A for all large $\lambda > 0$.

Consequently, the value function $V(t, x)$ of problem $\mathscr{B}_\infty$, which is equal to $e^{-\lambda t} v(x)$, is Lipschitz continuous on A uniformly in $t \geqslant 0$ for all large $\lambda > 0$.

Proof. By our assumptions, $\operatorname{dom} v = A$ and v is bounded. For any $\tilde{x} \in A$ let us denote by $\mathscr{U}_{\tilde{x}}$ the set of all Lebesgue measurable functions $u : [0, 1] \to \mathbb{R}^m$ such that $u(t) \in U$ a.e. $t \geq 0$ and $x_{\tilde{x},u}(s) \in A$ for all $s \in [0, 1]$. By the dynamic programming principle it follows that for any distinct $x_1, x_0 \in A$ there exists a control u_0 feasible at x_0 for problem $\mathscr{B}_\infty$, such that

$$v(x_0) + |x_1 - x_0| > \int_0^1 e^{-\lambda s} l(x_{x_0,u_0}(s), u_0(s))\, ds + e^{-\lambda} v(x_{x_0,u_0}(1)).$$

Thus, applying again the dynamic programming principle, it follows that for any $u_1 \in \mathscr{U}_{x_1}$

$$\begin{aligned} v(x_1) - v(x_0) &\leqslant |x_1 - x_0| + \left| \int_0^1 e^{-\lambda s} \left[l(x_{x_1,u_1}(s), u_1(s)) - l(x_{x_0,u_0}(s), u_0(s)) \right] ds \right| \\ &\quad + e^{-\lambda} \left| v(x_{x_1,u_1}(1)) - v(x_{x_0,u_0}(1)) \right| \\ &\leqslant |x_1 - x_0| + \left| \int_0^1 e^{-\lambda s} \left[l(x_{x_1,u_1}(s), u_1(s)) - l(x_{x_1,u_0}(s), u_0(s)) \right] ds \right| \\ &\quad + \left| \int_0^1 e^{-\lambda s} \left[l(x_{x_1,u_0}(s), u_0(s)) - l(x_{x_0,u_0}(s), u_0(s)) \right] ds \right| \\ &\quad + e^{-\lambda} \left| v(x_{x_1,u_1}(1)) - v(x_{x_0,u_0}(1)) \right| . \end{aligned} \tag{28}$$

By (15), there exists a constant $M \geqslant 0$ such that for all $x \in A$ and all Lebesgue measurable $u : [0,1] \to \mathbb{R}^m$ with $u(t) \in U$ a.e., the trajectories $x_{x,u}(\cdot)$ take values in $B(0,M)$ on the time interval $[0,1]$. Let $C' > 0$ be a Lipschitz constant for l on $B(0,M)$, with respect to the space variable. Then, by (16), there exists $c > 1$ such that for all $x_1, x_0 \in A$

$$\begin{aligned} &\int_0^1 e^{-\lambda s} \left| l(x_{x_1,u_0}(s), u_0(s)) - l(x_{x_0,u_0}(s), u_0(s)) \right| ds \\ &\leqslant C' \int_0^1 e^{-\lambda s} \left| x_{x_1,u_0}(s) - x_{x_0,u_0}(s) \right| ds \\ &\leqslant C' \cdot c \left| x_1 - x_0 \right| . \end{aligned} \tag{29}$$

So, putting $C = C' \cdot c + 1$, from (28) it follows that

$$\begin{aligned} v(x_1) - v(x_0) \leqslant C \left| x_1 - x_0 \right| &+ \left| \int_0^1 e^{-\lambda s} \left[l(x_{x_1,u_1}(s), u_1(s)) - l(x_{x_1,u_0}(s), u_0(s)) \right] ds \right| \\ &+ e^{-\lambda} \left| v(x_{x_1,u_1}(1)) - v(x_{x_0,u_0}(1)) \right| . \end{aligned} \tag{30}$$

Now we claim that there exist a constant $\beta = \beta(f,l) \geqslant 1$ and a control $u_1 \in \mathscr{U}_{x_1}$ such that

$$\begin{aligned} \left| \int_0^1 e^{-\lambda s} \left[l(x_{x_1,u_1}(s), u_1(s)) - l(x_{x_1,u_0}(s), u_0(s)) \right] ds \right| &\leqslant \beta \left| x_1 - x_0 \right| , \\ \left| x_{x_1,u_1}(1) - x_{x_0,u_0}(1) \right| &\leqslant \beta \left| x_1 - x_0 \right| . \end{aligned} \tag{31}$$

Indeed, if $\max_{s\in[0,1]} d_A(x_{x_1,u_0}(s)) = 0$ then $u_0 \in \mathscr{U}_{x_1}$. So, (31) follows taking $u_1 = u_0$. Otherwise, suppose $\max_{s\in[0,1]} d_A(x_{x_1,u_0}(s)) > 0$ and consider the following control system in $\mathbb{R}^{n+1}$

$$\begin{cases} x'(s) = f(x(s), u(s)) & \text{a.e. } s \in [0,1] \\ z'(s) = e^{-\lambda s} l(x(s), u(s)) & \text{a.e. } s \in [0,1] \\ x(0) = \tilde{x},\ z(0) = 0 & \\ u(\cdot) \text{ is Lebesgue measurable} & \\ u(s) \in U & \text{a.e. } s \in [0,1]. \end{cases} \tag{32}$$

Let us denote by $(X_{\tilde{x},u}, u)$ the trajectory-control pair that satisfies (32) where $X_{\tilde{x},u}(\cdot) := (x_{\tilde{x},u}(\cdot), z_{0,u}(\cdot))$. Set $\Omega := A \times \mathbb{R}$. By the neighbouring feasible trajectory theorem [12, Theorem 3.3], there exists a constant $\beta \geqslant 1$ (depending only on f and l) and a control $u_1 \in \mathscr{U}_{x_1}$ such that

$$\|X_{x_1,u_1} - X_{x_1,u_0}\|_{\infty,[0,1]} \leqslant \beta \left(\max_{s\in[0,1]} d_\Omega(X_{x_1,u_0}(s)) \right). \tag{33}$$

Since $d_\Omega(X_{x_1,u_0}(\cdot)) = d_A(x_{x_1,u_0}(\cdot))$ and $x_{x_0,u_0}(\cdot) \in A$ we have

$$\begin{aligned}
&\|X_{x_1,u_1} - X_{x_1,u_0}\|_{\infty,[0,1]} \\
&\leqslant \beta \max_{s\in[0,1]} \left\{ \inf_{\gamma\in\Omega} |X_{x_1,u_0}(s) - \gamma| \right\} \\
&\leqslant \beta \max_{s\in[0,1]} \{|x_{x_1,u_0}(s) - x_{x_0,u_0}(s)|\} \\
&\leqslant \beta \cdot c\,|x_1 - x_0|\,.
\end{aligned}$$

Furthermore

$$\left| \int_0^1 e^{-\lambda s} \left[l(x_{x_1,u_1}(s), u_1(s)) - l(x_{x_1,u_0}(s), u_0(s)) \right] ds \right| \leqslant \|X_{x_1,u_1} - X_{x_1,u_0}\|_{\infty,[0,1]}$$

$$|x_{x_1,u_1}(1) - x_{x_0,u_0}(1)| \leqslant \|X_{x_1,u_1} - X_{x_1,u_0}\|_{\infty,[0,1]} + c\,|x_1 - x_0|\,.$$

So, replacing β with $2\beta \cdot c$, (31) follows.

Now, let $0 \leqslant r \leqslant 1$. Combining the inequalities in (30) and (31) we obtain that for all $x_1, x_0 \in A$ with $|x_1 - x_0| \leqslant r$

$$v(x_1) - v(x_0) \leqslant (C + \beta)\, r + e^{-\lambda}\omega(\beta r)$$

where

$$\omega(r) := \sup_{\substack{|h-h'|\leqslant r \\ h,h'\in A}} |v(h) - v(h')|\,.$$

By the symmetry of the previous inequality with respect to x_1 and x_0 we have that

$$|v(x_1) - v(x_0)| \leqslant (C + \beta)\, r + e^{-\lambda}\omega(\beta r). \tag{34}$$

Letting $\theta := e^{-\lambda}$ and $\alpha := C + \beta$, we deduce from (34) that for all $0 \leqslant r \leqslant 1$

$$\omega(r) \leqslant \alpha r + \theta\, \omega(\beta r). \tag{35}$$

So, Lemma 5.2 below yields the Lipschitz continuity of v for $\lambda > \log \beta$.

The last part of the conclusion follows observing that $V(t, \cdot) = e^{-\lambda t} v(\cdot)$. □

The next lemma (proved in the Appendix) extends [13, Lemma 2.1].

Lemma 5.2. *Let $R > 0$ and $\omega : [0, R] \to [0, \infty)$ be a nondecreasing function. Suppose that there exists $0 < \theta < 1$, $\alpha > 0$, $\beta \geqslant 1$ such that*

$$\omega(r) \leqslant \alpha r + \theta\omega(\beta r) \qquad \forall\, 0 \leqslant r \leqslant R/\beta. \tag{36}$$

Let $m \geqslant 1$ be a real number such that $\theta^m \beta < 1$. Then there exists a constant $C \geqslant 0$ such that

$$\omega(r) \leqslant C r^{1/m} \qquad \forall\, 0 \leqslant r \leqslant R.$$

Remark 5. (a) From Theorem 5.1 and Theorem (4.2)-(*iii*), since $V(t, \cdot) = e^{-\lambda t} v(\cdot)$, it follows that

$$\lim_{t\to\infty} q(t) = 0.$$

(b) From (35) and Lemma 5.2 it follows that, given any $\lambda > 0$, $\{V(t,\cdot)\}_{t\geqslant 0}$ are uniformly Hölder continuous on A of exponent $1/m$ for all $m \geq 1$ such that $m > (\log\beta)/\lambda$, where β is as in the above proof.

Corollary 1. *Assume that $L(t,x,u) = e^{-\lambda t} l(t,x,u)$ and there exists $T > 0$ such that l is time independent for all $t \geqslant T$. Then $\{V(t,\cdot)\}_{t\geqslant 0}$ are uniformly Lipschitz continuous on A for all large $\lambda > 0$.*

Corollary 2. *Assume that $L(t,x,u) = e^{-\lambda t} l(t,x,u)$ with the further assumption: $l(\cdot,x,u)$ is T-periodic, i.e. there exists $T > 0$ such that $l(t+T,x,u) = l(t,x,u)$ for all $t \geqslant 0$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Then $\{V(t,\cdot)\}_{t\geqslant 0}$ are uniformly Lipschitz continuous on A for all large $\lambda > 0$.*

Proof. Fix $t \in [0,\infty)$. Then, by the dynamic programming principle, for any $x, x_0 \in A$ there exists u_0 feasible for $\mathscr{B}_\infty$ at x_0 such that

$$\begin{aligned} &V(t,x_1) - V(t,x_0) \\ &\leqslant |x_1 - x_0| + \left| \int_t^{t+T} e^{-\lambda s} \left[l(s, x_{x_1,u_1}(s), u_1(s)) - l(s, x_{x_0,u_0}(s), u_0(s)) \right] ds \right| \\ &\qquad + |V(t+T, x_{x_1,u_1}(t+T)) - V(t+T, x_{x_0,u_0}(t+T))| \end{aligned}$$

for any u_1 feasible for $\mathscr{B}_\infty$ at x_1. Now, the periodicity of l in the time variable implies that $V(s+T,x) = e^{-\lambda T} V(s,x)$. From the previous inequality it follows that

$$\begin{aligned} &V(t,x_1) - V(t,x_0) \\ &\leqslant |x_1 - x_0| + \left| \int_t^{t+T} e^{-\lambda s} \left[l(s, x_{x_1,u_1}(s), u_1(s)) - l(s, x_{x_0,u_0}(s), u_0(s)) \right] ds \right| \\ &\qquad + e^{-\lambda T} \left| V(t, x_{x_1,u_1}(t+T)) - V(t, x_{x_0,u_0}(t+T)) \right|. \end{aligned}$$

Proceeding as in the proof of Theorem 5.1, by the neighbouring feasible trajectory theorem [12, Theorem 3.3] there exist two constants $\beta \geqslant 1$ and $C \geqslant 0$ (depending only on f, l, and T) such that, for all $|x_1 - x_0| \leqslant r \leqslant 1$, we have that

$$|V(t,x_1) - V(t,x_0)| \leqslant (C+\beta)\, r + e^{-\lambda T} \sup_{\substack{|h-h'| \leqslant \beta r \\ h,h' \in A}} |V(t,h) - V(t,h')|.$$

The conclusion follows applying Lemma 5.2 for $\lambda > (\log\beta)/T$. □

Example 1. In this example we will show the fallacy of applying the unconstrained Pontryagin maximum principle to $\mathscr{B}_\infty$ in order to obtain candidates for optimality that satisfy some given state constraints.

Consider the following infinite horizon optimal control problem:

$$\text{maximize } J(u) = \int_0^\infty e^{-\lambda t} (x(t) + u(t))\, dt$$

over all trajectory-control pairs (x,u) satisfying

$$\begin{cases} x'(t) = -a u(t) & \text{a.e. } t \geqslant 0 \\ x(0) = 1 \\ u(t) \in [-1,1] & \text{a.e. } t \geqslant 0 \\ x(t) \in (-\infty, 1] & t \geqslant 0, \end{cases} \tag{37}$$

with $a > \lambda > 0$.

Applying the Pontryagin maximum principle for unconstrained problems, it follows that any optimal trajectory-control pair satisfies one of the following three relations:

(i) $x^-(t) = 1 + at$ associated with $u^-(t) \equiv -1$;
(ii) $x^+(t) = 1 - at$ associated with $u^+(t) \equiv +1$;
(iii) $x^\pm(t) = (1 - at)\chi_{[0,\bar{t}]}(t) + (1 - a\bar{t} + a(t - \bar{t}))\chi_{(\bar{t},\infty)}(t)$ associated with $u^\pm(t) = \chi_{[0,\bar{t}]}(t) - \chi_{(\bar{t},\infty)}(t)$, for some $\bar{t} > 0$.

Excluding now the trajectories x^- and $x^\pm$, since they are not feasible, this analysis leads to the conclusion that x^+ is the only candidate for optimality. But one can easily see that the feasible trajectory $\bar{x}(t) \equiv 1$, associated with the control $u(t) \equiv 0$, verifies

$$J(\bar{u}) > J(u^+).$$

Appendix.

Proof of Lemma 5.2. Suppose first that $m = 1$. Let $\theta < \tau < 1$ be such that $\tau\beta \leqslant 1$. Then $\tau R \leqslant \dfrac{R}{\beta}$ and by the growth assumption in (36) and the monotonicity of ω, we have that

$$\begin{aligned} \omega(\tau R) &\leqslant \alpha\tau R + \theta\omega(\beta\tau R) \\ &\leqslant \alpha\tau R + \theta\omega(R). \end{aligned} \tag{38}$$

Applying again (36), the monotonicity of ω, and (38) we obtain

$$\begin{aligned} \omega(\tau^2 R) &\leqslant \alpha\tau^2 R + \theta\omega(\tau R) \\ &\leqslant \alpha\tau^2 R + \theta\left[\alpha\tau R + \theta\omega(R)\right] \\ &= \alpha\tau R(\tau + \theta) + \theta^2\omega(R). \end{aligned}$$

So, by induction on $k \in \mathbb{N}$ it is straightforward to show that

$$\begin{aligned} \omega(\tau^k R) &\leqslant \alpha\tau R(\tau^{k-1} + \theta\tau^{k-2} + ... + \theta^{k-1}) + \theta^k\omega(R) \\ &= \alpha R\tau^k\left[1 + \frac{\theta}{\tau} + ... + \left(\frac{\theta}{\tau}\right)^k\right] + \theta^k\omega(R) \\ &< \alpha R\tau^k\frac{1}{1 - \theta/\tau} + \theta^k\omega(R) \\ &= \frac{\alpha R}{\tau - \theta}\tau^{k+1} + \theta^k\omega(R). \end{aligned}$$

Now let $r \in [0, R]$. Then there exists $k \in \mathbb{N}$ such that $\tau^{k+1}R < r \leqslant \tau^k R$. Finally

$$\begin{aligned} \omega(r) &\leqslant \frac{\alpha R}{\tau - \theta}\tau^{k+1} + \theta^k\omega(R) \\ &\leqslant \frac{\alpha}{\tau - \theta}\tau^{k+1}R + \tau^{k+1}R\frac{\omega(R)}{\tau R} \\ &\leqslant \left(\frac{\alpha}{\tau - \theta} + \frac{\omega(R)}{\tau R}\right)r. \end{aligned}$$

The conclusion holds true with $C = \dfrac{\alpha}{\tau - \theta} + \dfrac{\omega(R)}{\tau R}$.

If $m > 1$, by the growth assumption in (36) and the monotonicity of ω we have that

$$\begin{aligned} \omega(\theta^m R) &\leqslant \alpha\theta^m R + \theta\omega(\beta\theta^m R) \\ &\leqslant \alpha\theta^m R + \theta\omega(R). \end{aligned} \tag{39}$$

Applying again (36), monotonicity, and (39) we obtain

$$\begin{aligned} \omega(\theta^{2m} R) &\leqslant \alpha\theta^{2m} R + \theta\omega(\theta^m R) \\ &\leqslant \alpha\theta^{2m} R + \theta\left[\alpha\theta^m R + \theta\omega(R)\right] \\ &= \alpha\theta^{m+1} R(1+\theta^{m-1}) + \theta^2\omega(R). \end{aligned}$$

So, by induction on $k \in \mathbb{N}$ it is straightforward to show that

$$\begin{aligned} \omega(\theta^{km} R) &\leqslant \alpha\theta^{m+k-1} R(1+\theta^{m-1}+\ldots+\theta^{(k-1)(m-1)}) + \theta^k\omega(R) \\ &< \alpha R\theta^{m+k-1}\frac{1}{1-\theta^{m-1}} + \theta^k\omega(R) \\ &= \left(\frac{\alpha R\theta^{m-1}}{1-\theta^{m-1}} + \omega(R)\right)\theta^k. \end{aligned}$$

Now let $r \in [0, R]$. Then there exists $k \in \mathbb{N}$ such that $\theta^{(k+1)m} R < r \leqslant \theta^{km} R$. Thus,

$$\omega(r) \leqslant \omega(\theta^{km} R) \leqslant \tilde{C}\theta^k < \frac{\tilde{C}}{\theta}\left(\frac{r}{R}\right)^{1/m} = \left(\frac{\tilde{C}}{\theta R^{1/m}}\right) r^{1/m}$$

where $\tilde{C} = \dfrac{\alpha R\theta^{m-1}}{1-\theta^{m-1}} + \omega(R)$. The conclusion follows with $C = \tilde{C}/\theta R^{1/m}$. □

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