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The Poincaré-Miranda theorem and Viability condition

Hélène Frankowska*

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Abstract

The aim of this note is to discuss the relation between the assumptions of the Poincaré-Miranda theorem and the viability condition, first used by Nagumo to prove existence of a solution to ODEs under state constraints (viable solutions). An interesting consequence of this observation is an extension of the Poincaré-Miranda theorem to arbitrary convex compact sets in locally convex Hausdorff vector spaces (instead of a parallelotope in an Euclidean space). We also recall a very short proof of this extension based on a Ky Fan inequality. This proof is not new, but seems to have passed unnoticed in the literature devoted to the Poincaré-Miranda theorem. In fact, recent variations of this theorem in ℓ^2 follow then in a simple straightforward way. The above extension also implies a generalization of the Lax intermediate value theorem to infinite dimensional spaces.

Keywords: Poincaré-Miranda theorem, intermediate value theorem, viability condition.

1 Introduction

In [20] H. Poincaré stated the following result. Let $K = \underbrace{[-L, L] \times .... \times [-L, L]}_{n} \subset \mathbb{R}^n$, where $L > 0$ is given and let $f = (f_1, ..., f_n) : K \to \mathbb{R}^n$ be a continuous function such that

$$\begin{aligned} f_i(x) \geq 0, \ \ \forall x \in \{(x_1, ..., x_n) \in K \ : x_i = -L\} \\ f_i(x) \leq 0, \ \ \forall x \in \{(x_1, ..., x_n) \in K \ : x_i = L\}. \end{aligned} \tag{1}$$

Then f has an equilibrium in K, in the sense that $f(\bar{x}) = 0$ for some $\bar{x} \in K$ (in [20] the above inequalities are strict).

This statement was accompanied by a comment that it follows from an important theorem of L. Kronecker and was rediscovered by S. Cinquini [5] with an incomplete proof. In [17] C. Miranda has shown its equivalence to the Brouwer fixed point theorem. It entered into the literature under the name of the Poincaré-Miranda theorem and sometimes has devoted sections in monographs on fixed point theory, see for instance [22]. We refer to J. Mawhin [15] for a passionate history of rediscovery of this result and also to [16] for its analytical proof using differential forms and for its extension to the Hilbert hypercube in ℓ^2. Further references can be found in [9, 16].

This note is not intended to be an exhaustive account on various generalizations and applications that appeared in the rich literature on this subject. Our aim is to link assumptions of this theorem to the so-called viability (tangential) condition and to show how some recent generalisations of the

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Poincaré-Miranda theorem can be deduced from it. Though [12] contains a similar observation (cf. Remarks 2.4 and 2.6 from [12]), it lacks details and refers to [10], while the proof we recall here is much simpler than the one given in [10].

The above mentioned tangential condition was used first in a paper by M. Nagumo [19] to investigate existence of solutions to ordinary differential equations under (not necessarily convex) state constraints and was extended later on to differential inclusions under state constraints. Viability theory was actively developing since 1978 with various applications to control theory, Hamilton-Jacobi equations, traffic control, etc. and the tangential condition was renamed the *viability condition*. As a bi-product of these studies, in the context of set-valued maps, a particularly simple proof based on a powerful Ky Fan inequality was found to deduce that a map satisfying the viability condition on a convex compact set has an equilibrium, cf. [2, Theorem 15.1.1 and Proposition 15.1.1]. Possibly, because of its set-valued character and assumptions involving the support functions, this proof passed unnoticed by those generalizing the Poincaré-Miranda theorem, even though [2, Corollary 15.1.4] corresponding to this theorem was deduced. The Ky Fan theorem [7], see also [8] for extensions and applications, is equivalent to a generalization of the KKM principle from [6] and, in particular, it implies the Brouwer and Kakutani fixed point theorems. Let us recall that the Brouwer fixed point theorem, the Sperner lemma, the KKM principle, and many results in topology and nonlinear analysis are mutually equivalent. A general result leading to a unified approach to the Poincaré-Miranda theorem, the Lax intermediate value theorem and fixed point theorems is particularly useful.

There are various proofs in the literature of the Poincaré-Miranda theorem, cf. [1, 9, 13, 16, 22, 23, 24], based on the degree theory, the Sperner lemma and its variations, the Steinhaus chessboard theorem, monotone operators theory, differential forms in $\mathbb{R}^n$, topological methods. The interest of applying instead the Ky Fan inequality lies in a very short argument from [2], simpler and much more general than those being used to prove the Poincaré-Miranda theorem. We included it here because it is particularly straightforward for single-valued maps. In many situations the viability condition is easy to verify. To illustrate it we provide very short proofs of some known results.

2 An extension of the Poincaré-Miranda theorem

In this note X is a real locally convex Hausdorff vector space, X^* is its continuous dual space and $\langle \cdot, \cdot \rangle$ is the duality pairing. For a set $C \subset X$, ∂C denotes its boundary, $\overline{C}$ its closure and $\operatorname{Int} C$ its interior. $\mathbb{R}_+$ and $\mathbb{R}_-$ stand, respectively, for all the non-negative and non-positive reals and B for the closed unit ball in $\mathbb{R}^n$. Let K be a nonempty convex subset of X. Recall that for any $x \in K$ the tangent cone to K at x is defined by

$$T_K(x) = \overline{\bigcup_{\lambda \geq 0} \lambda(K - x)}.$$

Then $T_K(x)$ is a closed convex cone and $T_K(x) = X$ for every $x \in \operatorname{Int} K$. Note that $\lambda(K-x) \subset \mu(K-x)$ for all $0 \leq \lambda \leq \mu$. The normal cone to K at x is defined by $N_K(x) := \{q \in X^* \mid \langle q, y-x \rangle \leq 0 \ \ \forall\, y \in K\}$.

A mapping $f : K \to X$ satisfies the *viability condition* if $f(x) \in T_K(x)$ for every $x \in K$ or, equivalently, if $\langle p, f(x) \rangle \leq 0$ for all $x \in \partial K$ and $p \in N_K(x)$.

Example 2.1 (Paralelotope) *Consider $z = (z_1, ..., z_n) \in \mathbb{R}^n$, reals $L_i \geq 0$, $i = 1, ..., n$, the convex compact set $K = [z_1 - L_1, z_1 + L_1] \times .... \times [z_n - L_n, z_n + L_n] \subset \mathbb{R}^n$ and let $x = (x_1, ..., x_n) \in K$. For*

every $i \in \{1, ..., n\}$ define the cone in $\mathbb{R}$

$$M_i = \begin{cases} \mathbb{R}_+ & \text{if} \quad x_i = z_i - L_i, \text{ and } \ L_i \neq 0 \\ \mathbb{R}_- & \text{if} \quad x_i = z_i + L_i, \text{ and } \ L_i \neq 0 \\ \{0\} & \text{if} \quad L_i = 0 \\ \mathbb{R} & \text{if} \quad x_i \in]z_i - L_i, z_i + L_i[\end{cases}$$

Then, by the very definition of the tangent cone, $T_K(x) = M_1 \times .... \times M_n$.

In particular, if $z = 0$ and for some $L \geq 0$ we have $L_i = L$ for all i, then any $f : K \to \mathbb{R}^n$ as in (1) satisfies the viability condition.

Example 2.2 (Hilbert cube in ℓ^2) *Let $X = \ell^2$ and $K = \{(x_1, x_2, ...) \in \ell^2 \, : \, |x_i| \leq 1/i\}$. Then K is convex and compact. Fix any $x = (x_1, x_2, ...) \in K$. For every integer $i \geq 1$ define the set M_i as in Example 2.1 with $z_i = 0$ and $L_i = 1/i$.*

Then $M_1 \times .... \times M_n \times \{0\} \times \{0\} \times .... \subset T_K(x)$ for every $n \geq 1$. Consider any $v = (v_1, v_2, ...) \in \ell^2$ such that $v_i \in M_i$. Since $T_K(x)$ is closed, taking the limit of $w_n := (v_1, ..., v_n, 0, 0, ...) \in \ell^2$ when $n \to \infty$ we deduce that $v \in T_K(x)$. Conversely, for any $\lambda \geq 0$ and $v \in \lambda(K - x)$ we have $v \in M_1 \times M_2 \times ...$. Hence $T_K(x) \subset \{v \in \ell^2 \, : \, v_i \in M_i\}$ and, therefore, $T_K(x) = \{v \in \ell^2 \, : \, v_i \in M_i\}$.

Example 2.3 *Let $X = \ell^2$, $z = (z_1, z_2, ...) \in \ell^2$, $(L_1, L_2, ...) \in \ell^2$. Consider the convex compact set $K = \{(x_1, x_2, ...) \in \ell^2 \, : \, |x_i - z_i| \leq L_i\} \subset \ell^2$. If $z = 0$ and $L_i = 1/i$ we get the Hilbert hypercube. Fix any $(x_1, x_2, ...) \in K$ and define M_i as in Example 2.1. Exactly as before we obtain $T_K(x) = \{(v_1, v_2, ...) \in \ell^2 \, : \, v_i \in M_i\}$.*

The next theorem can be seen as an extension of the Poincaré-Miranda theorem to arbitrary convex compact sets. It is a counterpart of the fixed point theorem from [10].

Theorem 2.4 *Consider a nonempty convex compact subset $K \subset X$ and a continuous mapping $f : K \to X$ satisfying the viability condition. Then $f(\bar{x}) = 0$ for some $\bar{x} \in K$.*

By Example 2.1, Theorem 2.4 implies the Poincaré-Miranda theorem. From Theorem 2.4 and Examples 2.2, 2.3 we deduce the following result.

Corollary 2.5 ([16, 21]) *Let $K \subset \ell^2$ be as in Example 2.3 and $f = (f_1, f_2, ...) : K \to \ell^2$ be continuous and satisfy the following condition: for every integer $i \geq 1$, $f_i(x) \geq 0$ if $x_i = z_i - L_i$ and $f_i(x) \leq 0$ if $x_i = z_i + L_i$. Then there exists $\bar{x} \in K$ such that $f(\bar{x}) = 0$.*

Remark 2.6 *(i) Assume that for every integer $i \geq 1$: either (a) $f_i(x) \geq 0$ if $x_i = z_i - L_i$ and $f_i(x) \leq 0$ if $x_i = z_i + L_i$ or (b) $f_i(x) \leq 0$ if $x_i = z_i - L_i$ and $f_i(x) \geq 0$ if $x_i = z_i + L_i$. Define for every i the function $\bar{f}_i = f_i$ if (a) holds true and $\bar{f}_i = -f_i$ in the case of (b). By Corollary 2.5, $\bar{f}$ has an equilibrium in K. Since any equilibrium of $\bar{f}$ is also an equilibrium of f we deduce [21, Theorem 2.1] and [16, Theorem 5.1].*

(ii) In the same vein, [11, Theorem 2] follows from Theorem 2.4.

To prove Theorem 2.4 we recall the celebrated Ky Fan inequality. A real valued function defined on a convex set K is called quasi-concave if for every $r \in \mathbb{R}$ the set $\{x \in K \,|\, f(x) > r\}$ is convex.

Theorem 2.7 ([7]) *Let $K \subset X$ be a nonempty convex compact set and $g : K \times K \to \mathbb{R}$ be such that $g(\cdot, y)$ is lower semicontinuous for each $y \in K$ and $g(x, \cdot)$ is quasi-concave for each $x \in K$. Then there exists $\bar{x} \in K$ satisfying $\sup_{y \in K} g(\bar{x}, y) \leq \sup_{y \in K} g(y, y)$.*

Proof of Theorem 2.4. The proof follows the lines of [2, Proof of Theorem 15.1.1] that we simplify thanks to the single-valuedness of f.

Assume for a moment that $f(x) \neq 0$ for every $x \in K$. Then, by the separation theorem for every $x \in K$ there exists $p_x \in X^*$ such that $\langle p_x, f(x)\rangle > 0$. We associate with every $p \in X^*$ the set $U(p) = \{y \in K : \langle p, f(y)\rangle > 0\}$. By the continuity of f it is open. Then $K \subset \bigcup_{p\in X^*} U(p)$. Consider $p_1, ..., p_k \in X^*$ such that $V_i := U(p_i)$ $i = 1, ..., k$ form a finite open subcovering of K. Let ψ_i be a partition of unity subordinated to $\{V_i\}_{i=1}^k$ and define $g : K \times K \to \mathbb{R}$ by

$$g(x, y) = \sum_{i=1}^{k} \psi_i(x)\langle p_i, y - x\rangle.$$

Continuity of ψ_i and p_i imply that g is continuous with respect to x. Clearly it is also concave in y and $g(y, y) = 0$. By Theorem 2.7 there exists $\bar{x} \in K$ such that $\sum_{i=1}^k \psi_i(\bar{x})\langle p_i, y - \bar{x}\rangle \leq 0$ for every $y \in K$. Thus $\bar{p} := \sum_{i=1}^k \psi_i(\bar{x})p_i$ satisfies $\langle \bar{p}, v\rangle \leq 0$ for every $v \in T_K(\bar{x})$. In particular $\langle \bar{p}, f(\bar{x})\rangle \leq 0$.

On the other hand if i is so that $\bar{x} \notin V_i$, then $\psi_i(\bar{x}) = 0$ and if i is so that $\bar{x} \in V_i$, then $\langle p_i, f(\bar{x})\rangle > 0$. Since $\sum_{i=1}^k \psi_i(\bar{x}) = 1$ and $\psi_i \geq 0$, we get $\sum_{i=1}^k \psi_i(\bar{x})\langle p_i, f(\bar{x})\rangle > 0$, leading to a contradiction. □

Corollary 2.8 ([10]) *Consider a nonempty convex compact subset $K \subset X$ and a continuous mapping $f : K \to X$ such that $f(x) - x \in T_K(x)$ for all $x \in K$ (or $x - f(x) \in T_K(x)$ for all $x \in K$). Then $f(\bar{x}) = \bar{x}$ for some $\bar{x} \in K$. In particular, if $f(K) \subset K$, then f has a fixed point in K.*

Remark 2.9 *The proof of Theorem 2.4 uses a partition of unity similarly to [4], where assumptions like in Corollary 2.8 involve $\bigcup_{\lambda\geq 0} \lambda(K - x)$ in the place of $T_K(x)$ (but in [4] the Brouwer fixed point theorem is applied instead of the Ky Fan inequality). Taking $T_K(x)$ strengthens results of [4]. Indeed, if $f : K \to X$ satisfies assumptions of [4, Theorem 1], then $\bar{f}(x) := f(x) - x$ is as in Theorem 2.4. Thus $\bar{f}$ has an equilibrium in K and therefore f has a fixed point in K. Similarly, if $f : K \to X$ satisfies assumptions of [4, Theorem 2], then, by Theorem 2.4, f has a fixed point in K. On the other hand, let K be the closed unit ball in $\mathbb{R}^n$, and $f : K \to \mathbb{R}^n$ be continuous, not identically zero on ∂K and such that $\langle x, f(x)\rangle = 0$ for all $x \in \partial K$. By Theorem 2.4, f has an equilibrium at some $\bar{x} \in K$. Thus $\bar{x}$ is a fixed point of the mapping $K \ni x \mapsto \bar{f}(x) := x - f(x)$. Such $\bar{f}$ does not satisfy assumptions of [4, Theorems 1, 2]. In [10], the results of [4] were improved using the tangent cones $T_K(x)$ with more complicated proofs than the one recalled above.*

3 Some consequences of Theorem 2.4

Corollary 2.8 yields the Schauder fixed point theorem. The following covering property proved in [10] is very important in many applications. We provide below a different proof of it.

Theorem 3.1 *Consider a nonempty convex compact set $K \subset X$ and a continuous mapping $\varphi : K \to X$ satisfying $x - \varphi(x) \in T_K(x)$. Then $K \subset \varphi(K)$.*

Proof. Fix any $z \in K$ and define $f(x) = z - \varphi(x)$ for all $x \in K$. Then f is continuous from K into X. Moreover $z - x \in T_K(x)$ for any $x \in K$. Since the set $T_K(x)$ is a convex cone we have $f(x) = (z - x) + (x - \varphi(x)) \in T_K(x)$. Hence f satisfies the viability condition and therefore there exists $\bar{x} \in K$ such that $f(\bar{x}) = 0$. Consequently, $\varphi(\bar{x}) = z$. □

The second statement of the result below is due to Bohl [3].

Corollary 3.2 *Let $\Omega \subset \mathbb{R}^n$ be convex, open and bounded, and $\varphi : \overline{\Omega} \to \mathbb{R}^n$ be continuous and satisfy $\varphi(x) = x$ for every $x \in \partial\Omega$. Then $\overline{\Omega} \subset \varphi(\overline{\Omega})$ and $\Omega \subset \varphi(\Omega)$.*

In particular, if $K = [-L_1, L_1] \times .... \times [-L_n, L_n]$ for some $L_i \geq 0$, $i = 1, ..., n$, and $\varphi : K \to \mathbb{R}^n$ is continuous and satisfies $\varphi(x) = x$ for every $x \in \partial K$, then $K \subset \varphi(K)$ and $\mathrm{Int}\, K \subset \varphi(\mathrm{Int}\,(K))$.

Proof. Since $0 = x - \varphi(x) \in T_{\overline{\Omega}}(x)$ for every $x \in \partial\Omega$, Theorem 3.1 yields the first inclusion. Fix any $z \in \Omega$ and consider $x \in \overline{\Omega}$ such that $z = \varphi(x)$. If $x \in \partial\Omega$, then $z = x \in \partial\Omega$ leading to a contradiction. Hence $z \in \varphi(\Omega)$. □

We next extend a result of Lax [14].

Corollary 3.3 (Intermediate Value Theorem) *Let φ be a continuous map from the closed unit ball $B \subset \mathbb{R}^n$ into $\mathbb{R}^n$ such that $\langle x, \varphi(x)\rangle \geq 1$ for all $x \in \partial B$. Then $B \subset \varphi(B)$.*

In [14] it is assumed that $\varphi(x) = x$ for all $x \in \partial B$, that naturally satisfies our assumption. **Proof.** For any $x \in \partial B$, $T_B(x) = \{v \in \mathbb{R}^n : \langle x, v\rangle \leq 0\}$. By the assumption on φ, $\langle x, x - \varphi(x)\rangle = |x|^2 - \langle x, \varphi(x)\rangle \leq 0$ for every $x \in \partial B$. Hence φ satisfies assumptions of Theorem 3.1. □

The second statement of the result below is due to Bohl, [3].

Corollary 3.4 *Let K be as in Corollary 3.2. There is no continuous $\varphi : K \to \mathbb{R}^n$ that does not vanish on K and satisfies $-\varphi(x) \in T_K(x)$ for every $x \in \partial K$.*

In particular, there is no continuous $\varphi : K \to \mathbb{R}^n$ that does not vanish on K and satisfies $\varphi(x) = x$ for every $x \in \partial K$.

Proof. Observe that $f := -\varphi$ satisfies assumptions of Theorem 2.4. Thus f vanishes at some point in K and so does φ. □

Corollary 3.5 ([18]) *Let $B(0, r)$ be the closed ball in $\mathbb{R}^n$ of center zero and radius $r > 0$ and $f : B(0, r) \to \mathbb{R}^n$ be a continuous mapping satisfying the following condition: $\langle x, f(x)\rangle \leq 0$ for all $x \in \partial B(0, r)$. Then there exists $\bar{x} \in B(0, r)$ such that $f(\bar{x}) = 0$.*

Proof. Set $K = B(0, r)$. Since $T_K(x) = \{v \in \mathbb{R}^n : \langle x, v\rangle \leq 0\}$ for every $x \in \partial B(0, r)$, f satisfies the viability condition. Theorem 2.4 completes the proof. □

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