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INVARIANCE FOR QUASI-DISSIPATIVE SYSTEMS IN BANACH SPACES

P. CANNARSA, G. DA PRATO, AND H. FRANKOWSKA

Abstract. In a separable Banach space E, we study the invariance of a closed set K under the action of the evolution equation associated with a maximal dissipative linear operator A perturbed by a quasi-dissipative continuous term B. Using the distance to the closed set, we give a general necessary and sufficient condition for the invariance of K. Then, we apply our result to several examples of partial differential equations in Banach and Hilbert spaces.

1. Introduction

In a separable Banach space E, consider the Cauchy problem

$$\begin{cases} X'(t) = AX(t) + B(X(t)), & t \geq 0 \\ X(0) = x, \end{cases} \tag{1}$$

where $A : D(A) \subset E \to E$ is the infinitesimal generator of the strongly continuous semigroup of contractions e^{tA} on E and $B : E \to E$ is continuous and such that $B - MI$ is dissipative for some $M \geqslant 0$. As is well known, for every $x \in E$ problem (1) has a unique mild solution

$$X(t,x) = e^{tA}x + \int_0^t e^{(t-s)A} B(X(s,x))ds, \quad t \in [0,T]$$

which belongs to $C([0,T];E)$. Moreover, when $x \in D(A)$, the solution is strict (see [**?**] and Proposition 3.1 below).

The main object of this paper is the characterization of those nonempty closed sets K of E which are *invariant* for the flow $X(\cdot,x)$, that is, such that $x \in K$ implies $X(t,x) \in K$ for all $t \geq 0$. Such invariance properties are very useful, for instance, to obtain relaxation results for semilinear control systems under state constraints (see [**?**]).

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There is an extensive literature addressing domain invariance issues for partial differential equations (see, for instance, [**?**], [**?**, **?**], and [**?**]), but most of the existing results concern sufficient conditions for the invariance of K. Necessary and sufficient conditions for invariance can be found in the monograph by O. Carja, M. Necula and K. Vrabie [**?**]. Such results are, however, completely different from ours. Indeed, in the aforementioned monograph, the classical condition introduced by Nagumo [**?**] is extended to infinite dimensions using as a tool the set $\mathscr{F}_K^A(\xi)$, see [**?**, Definition 8.1.3], which reduces to the contingent cone at ξ when $A = 0$. We observe that the definition of such a set is given in terms of the semigroup e^{tA} rather than its infinitesimal generator A. Then, sufficient conditions reduce to asking that $B(\xi) \in \mathscr{F}_K^A(\xi)$ for all $\xi \in K$ (see [**?**, Theorem 8.1.2]), whereas necessary conditions require some additional compactness assumptions (see [**?**, Theorem 8.2.1]).

Our main result, Theorem 4.2 below, provides a necessary and sufficient condition for the invariance of K which is stated in terms of A and B. It essentially guarantees that K is invariant if and only if, for some $C \geqslant 0$ and $\delta > 0$,

$$D^- d_K(x)\,(Ax + B(x)) \leqslant C\, d_K(x) \qquad \forall x \in D(A) \cap K_\delta, \tag{2}$$

where $D^- d_K(x)$ is the lower Dini derivative of the distance $d_K(x)$ of x to K, defined in Section 2, and

$$K_\delta = \big\{x \in E \setminus K \ : \ d_K(x) < \delta\big\}.$$

This characterisation, though already quite simple, can be expressed in more geometric forms when K is a convex subset of a Hilbert space. Indeed, in this case, the lower Dini derivative of the distance can be computed using the orthogonal projection Π_K onto K. So, (2) becomes

$$\big\langle x - \Pi_K(x), Ax + B(x)\big\rangle \leqslant C d_K^2(x) \qquad \forall x \in D(A) \cap K_\delta \tag{3}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product on E. Moreover, if $D(A)$ is invariant for Π_K, then condition (3) further simplifies into

$$\big\langle p, Ax + B(x)\big\rangle \leqslant 0 \qquad \forall x \in \partial K \cap D(A)\,,\ \forall p \in N_K(x) \cap D(A), \tag{4}$$

where $N_K(x)$ denotes the normal cone to K at x.

Alltogether, the above results allow us to treat several examples of partial differential equations of evolution in Hilbert or Banach spaces, including both reaction-diffusion equations and equations modelling the dynamics of an age-structured population. For these examples we study, in spaces of continuous functions, the invariance of the set K of positive functions or, more generally, of functions satisfying unilateral constraints like in obstacle problems.

This paper is organized as follows. Section 2 is devoted to some preliminaries about dissipative operators and derivatives of the norm in a Banach space, following [**?**]. In Section 3, we gather several properties of solutions to (1) which

are needed to obtain our main results. Section 4 is devoted to the proof of necessary and sufficient conditions for the invariance of K. Finally, in Section 5, the invariance of several sets is discussed in Hilbert and Banach spaces.

2. Notation and preliminaries

In this section, we recall some well-known definitions and premilinary results that can be easily found in the literature (see, for instance, [**?**]).

Let E be a real separable Banach space with norm $\|\cdot\|$ and let $f : E \to \mathbb{R}$ be a locally Lipschitz function. We define the lower Dini derivative of f at a point $x \in E$ by

$$D^- f(x)\, v = \liminf_{\lambda \downarrow 0} \frac{f(x+\lambda v) - f(x)}{\lambda} \qquad \forall v \in E.$$

The subdifferential of $\|\cdot\|$ at a point $x \in E$ is defined as

$$\partial\|x\| = \left\{\phi \in E^* \ : \ \langle \phi, y\rangle \leqslant D^-\|x\|\, y\,,\ \forall y \in E\right\},$$

where E^* denotes the dual of E, $\langle\cdot,\cdot\rangle$ the duality between E^* and E. Observe that, since $x \mapsto \|x\|$ is a convex function on E, we have

$$D^-\|x\|\, y = \lim_{\lambda \downarrow 0} \frac{\|x+\lambda y\| - \|x\|}{\lambda}\,.$$

It is well known that $\partial\|x\|$ is a nonempty closed convex subset of E^* for all $x \in E$. Such a set can be characterized as follows

$$\partial\|x\| = \left\{\phi \in E^* \ : \ \langle \phi, x\rangle = \|x\|\,,\ \|\phi\|_* \leqslant 1\right\} \qquad \forall x \in E\,,$$

where $\|\cdot\|_*$ denotes the dual norm. Notably, for $x \neq 0$ we have that $\|\phi\|_* = 1$ for all $\phi \in \partial\|x\|$. Moreover, for all $x, y \in E$,

$$D^-\|x\|\, y = \max_{\phi \in \partial\|x\|} \langle \phi, y\rangle. \tag{5}$$

Example 2.1. Let $E = C([a,b])$ and let $x \in E \setminus \{0\}$. Define

$$\arg\max_{[0,1]} |x| = \left\{\xi \in [a,b] \ : \ |x(\xi)| = \|x\|_\infty\right\}.$$

Then one can prove (see [**?**]) that $\phi \in \partial\|x\|$ if and only if ϕ is represented by a signed Radon measure μ_ϕ on $[a,b]$, with total variation $\|\mu_\phi\| = 1$ and support in $\arg\max_{[0,1]} |x|$, such that, for all Borel sets $\Omega \subset [a,b]$,

$$\int_\Omega \operatorname{sgn}\big(x(\xi)\big)\mu_\phi(d\xi) \geqslant 0$$

where

$$\operatorname{sgn}(\lambda) = \begin{cases} 1 & \text{if } \lambda \geqslant 0 \\ 0 & \text{if } \lambda < 0. \end{cases}$$

The following lemma (see, e.g., [**?**, Appendix D]) will be used in what follows.

Lemma 2.2. *Let $g : [0,T] \to E$ be differentiable at a point $t_0 \in [0,T]$. Then the function $\gamma(t) := \|g(t)\|$ is:*

(a) *right-differentiable at t_0 (provided that $t_0 < T$) and*

$$\frac{d\gamma}{dt^+}(t_0) := \lim_{t\downarrow t_0} \frac{\gamma(t)-\gamma(t_0)}{t-t_0} = \max\{\langle \phi, g'(t_0)\rangle : \quad \phi \in \partial\|g(t_0)\|\},$$

(b) *left-differentiable at t_0 (provided that $t_0 > 0$) and*

$$\frac{d\gamma}{dt^-}(t_0) := \lim_{t\uparrow t_0} \frac{\gamma(t)-\gamma(t_0)}{t-t_0} = \min\{\langle \phi, g'(t_0)\rangle : \quad \phi \in \partial\|g(t_0)\|\}.$$

A mapping $F : D(F) \to E$, where $D(F) \subset E$, is called *dissipative* if

$$\|x-y\| \leqslant \big\|x-y-\alpha(F(x)-F(y))\big\| \qquad \forall \alpha > 0\,,\ \forall\ x,y \in D(F),$$

and *maximal dissipative* if it is dissipative and $I-F$ is onto. Moreover, F is called *quasi-dissipative* if $F - MI$ is dissipative for some constant $M \geqslant 0$. By using (5), it is easy to check that F is quasi-dissipative if and only if

$$\langle \phi, F(x)-F(y)\rangle \leqslant M\|x-y\| \tag{6}$$

for any $x, y \in D(F)$ and some $\phi \in \partial\|x-y\|$.

Clearly, if F is *linear*, then F is dissipative if and only if

$$\|x\| \leqslant \big\|x-\alpha F(x)\big\| \qquad \forall \alpha > 0\,,\ \forall\ x \in D(F),$$

or, equivalently,

$$\forall\, x \in D(F)\ \exists\, \phi \in \partial\|x\| \ \text{ such that } \ \langle \phi, F(x)\rangle \leqslant 0. \tag{7}$$

3. Quasi-dissipative evolution equations

We are here concerned with the Cauchy problem

$$\begin{cases} X'(t) = AX(t) + B(X(t)), & t \geqslant 0 \\ X(0) = x \end{cases} \tag{8}$$

under the following assumptions **(H)**:

(H1) $A : D(A) \subset E \to E$ is a the infinitesimal generator of a strongly continuous semigroup of contractions on E, which will be denoted by e^{tA},

(H2) $B : E \to E$ is continuous and quasi-dissipative.

As is well known, assumption **(H1)** is satisfied if and only if A is maximal dissipative on E. The following result guarantees that (8) is well posed.

Proposition 3.1. *Assume* **(H)** *and fix any $T > 0$. Then the following holds true.*

(I): *For every $x \in E$ problem (8) has a unique mild solution $X(\cdot, x)$ which belongs to $C([0,T]; E)$ and satisfies*

$$X(t,x) = e^{tA}x + \int_0^t e^{(t-s)A} B(X(s,x))ds, \qquad \forall t \in [0,T]. \tag{9}$$

Moreover, for all $x, y \in E$

$$\|X(t,x) - X(t,y)\| \leqslant e^{Mt}\|x-y\| \qquad \forall t \in [0,T]. \tag{10}$$

(II): *If, in addition, $x \in D(A)$, then $X(\cdot,x) \in W^{1,\infty}(0,T;E) \cap L^\infty(0,T;D(A))$ and*

$$X'(t) = AX(t) + B(X(t)) \quad \text{a.e. on } [0,T]. \tag{11}$$

Furthermore, $X(\cdot,x)$ is differentiable at $t=0$ and

$$\frac{d}{dt}X(t,x)|_{t=0} = Ax + B(x). \tag{12}$$

Proof. The fact that (8) has a unique mild solution, given by (9), follows from assumption **(H)**, a result by Webb [**?**] which ensures that $A+B$ is maximal quasi-dissipative, and the classical Crandall-Liggett Theorem (see, e.g., [**?**]).

The fact that $X(\cdot,x) \in W^{1,\infty}(0,T;E) \cap L^\infty(0,T;D(A))$ when $x \in D(A)$ and (8) holds a.e. is also well-known, see, e.g., [**?**, Theorem 4.1.4].

The Lipschitz dependence of the flow with respect to initial data is a direct consequence of assumption **(H)** for $x, y \in D(A)$. Indeed, using (6) and Lemma 2.2 to compute the derivative of the function $t \mapsto \|X(t,x) - X(t,y)\|$ (which coincides with the left-derivative a.e.), we deduce that for a.e. $t \in [0,T]$

$$\begin{aligned}
&\frac{d}{dt}\|X(t,x) - X(t,y)\| \\
&= \min_{\phi \in \partial\|X(t,x)-X(t,y)\|} \big\langle \phi, A\big[X(t,x) - X(t,y)\big] + B(X(t,x)) - B(X(t,y))\big\rangle \\
&\leqslant M\|X(t,x) - X(t,y)\|.
\end{aligned}$$

Then, (10) follows by Gronwall's lemma for all $x, y \in D(A)$. Moreover, the same inequality can be recovered by density for all $x, y \in H$, because $X(\cdot,x_k) \to X(\cdot,x)$ uniformly on $[0,T]$ whenever $D(A) \ni x_k \to x$.

Finally, in order to prove (12), write

$$\begin{aligned}
\frac{X(t,x)-x}{t} &= \frac{e^{tA}x - x}{t} + \frac{1}{t}\int_0^t e^{(t-s)A}B(X(s,x))ds \\
&= \frac{e^{tA}x - x}{t} + \frac{1}{t}\int_0^t e^{(t-s)A}B(x)ds + \frac{1}{t}\int_0^t e^{(t-s)A}(B(X(s,x)) - B(x))ds \qquad (13)\\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Then, since $x \in D(A)$, we have that $I_1 \to Ax$, $I_2 \to B(x)$, and $I_3 \to 0$ as $t \to 0$. □

4. Invariance results

4.1. **Invariance of closed subsets of a Banach space.** Let $K \subset E$ be a nonempty closed set.

Definition 4.1. *We say that K is* invariant *for* (8) *if $X(t,x) \in K$ for all $x \in K$ and all $t \geq 0$.*

We denote by $d_K(x)$ the distance of x from K, that is,

$$d_K(x) = \inf_{y \in K} \|x - y\|, \quad \forall\, x \in E.$$

We recall that d_K is Lipschitz continuous (with Lipschitz constant 1) on E, and convex if K is convex.

We now give a necessary and sufficient condition for the invariance of K. For any $\delta > 0$ let

$$K_\delta = \big\{x \in E \setminus K \ : \ d_K(x) < \delta\big\}.$$

Theorem 4.2. *Assume* **(H)**. *Then K is* invariant *for* (8) *if and only if there exists $\delta > 0$ such that*

$$D^- d_K(x)\,(Ax + B(x)) \leqslant C\, d_K(x) \qquad \forall x \in D(A) \cap K_\delta \tag{14}$$

for some constant $C \geqslant 0$.

Proof. Suppose K is invariant for (8) and let $x \in D(A) \setminus K$. Then for all $t > 0$ there exists $x_t \in K$ such that

$$\|x - x_t\| \leqslant (1 + t^2) d_K(x). \tag{15}$$

Since $X(t, x_t) \in K$ for all $t \geqslant 0$, by (15) and (10) we deduce that

$$\begin{aligned}
\frac{1}{t}\left[d_K(X(t,x)) - d_K(x)\right] &= \frac{1}{t}\left[d_K(X(t,x)) - d_K(X(t,x_t)) - d_K(x)\right] \\
&\leqslant \frac{1}{t}\Big(\|X(t,x) - X(t,x_t)\| - \frac{\|x - x_t\|}{1+t^2}\Big) \\
&\leqslant \frac{1}{t}\Big(e^{Mt} - \frac{1}{1+t^2}\Big)\|x - x_t\| \qquad \forall t > 0
\end{aligned}$$

with M given by (6). Hence, again by (15), we obtain

$$\frac{d_K(X(t,x)) - d_K(x)}{t} \leqslant \Big(\frac{e^{Mt} - 1}{t} + \frac{t}{1+t^2}\Big)(1 + t^2) d_K(x) \qquad \forall t > 0.$$

Since d_K is Lipschitz and $X(\cdot, x)$ is differentiable at $t = 0$ by Proposition 3.1, the lower limit of the left-hand side as $t \downarrow 0$ coincides with $D^- d_K(x)(Ax + B(x))$. Therefore

$$D^- d_K(x)\,(Ax + B(x)) \leqslant M\, d_K(x)$$

which is (14) with $C = M$.

Conversely, assume (14) and fix any $x \in K,\ T > 0$ so that $X([0,T], x) \subset K \cup K_\delta$. Suppose for a moment that $X(T, x) \in K_\delta$ and let

$$t_0 = \max\{t \in [0,T] \ : \ X(t,x) \in K\}.$$

Without loss of generality, we can assume $t_0 = 0$. Let $\{x_j\} \subset D(A)$ be any sequence converging to x. Then, on account of (10),

$$X(\cdot, x_j) \to X(\cdot, x) \text{ uniformly on } [0,T]. \tag{16}$$

Next, fix $\varepsilon \in (0,T)$ and observe that, in view of (16), there exist $\delta_\varepsilon > 0$ and $j_\varepsilon \in \mathbb{N}$ such that

$$X([\varepsilon,T], x_j) \subset K_\delta, \quad d_K(X(t,x_j)) \geqslant \delta_\varepsilon \qquad \forall t \in [\varepsilon,T],\ \forall j \geqslant j_\varepsilon. \tag{17}$$

Owing to Proposition 3.1, for all such j and a.e. $t \in [0,T]$ we have that the derivative $X'(t,x_j)$ does exist, $X(t,x_j) \in D(A)$, and

$$X'(t,x_j) = AX(t,x_j) + B(X(t,x_j)) \quad \text{for a.e. } t \in [0,T]. \tag{18}$$

Thus, we obtain by (14)

$$D^- d_K(X(t,x_j))\,(AX(t,x_j)+B(X(t,x_j))) \leqslant C\, d_K(X(t,x_j)) \quad \text{a.e. } t \in [0,T]. \tag{19}$$

Now, let $j \geqslant j_\varepsilon$ and consider the Lipschitz function

$$\phi(t) = d_K(X(t,x_j)) \qquad (t \in [\varepsilon,T]).$$

Since, ϕ is a.e. differentiable, for a.e. $t \in (\varepsilon,T)$ such that (18) holds true, the derivative $\phi'(t)$ does exist. Fix such t and consider $h_i \to 0+$ such that

$$\begin{aligned} &D^- d_K(X(t,x_j))(AX(t,x_j) + B(X(t,x_j))) \\ &\quad = \lim_{i\to\infty} \tfrac{1}{h_i}\Big\{ d_K\big(X(t,x_j) + h_i\big[AX(t,x_j) + B(X(t,x_j))\big]\big) - d_K(X(t,x_j))\Big\}. \end{aligned}$$

Then

$$\begin{aligned} \phi'(t) &= \lim_{h\downarrow 0} \frac{d_K(X(t+h,x_j)) - d_K(X(t,x_j))}{h} \\ &= \lim_{i\to\infty} \frac{1}{h_i}\Big\{ d_K\big(X(t,x_j) + h_i\big[AX(t,x_j) + B(X(t,x_j))\big]\big) - d_K(X(t,x_j))\Big\} \\ &= D^- d_K(X(t,x_j))\,\big(AX(t,x_j) + b(X(t,x_j))\big). \end{aligned}$$

Therefore, in view of (19), $\phi'(t) \leqslant C\,\phi(t)$ for a.e. $t \in [\varepsilon,T]$. We can then apply the Gronwall lemma to deduce that

$$d_K(X(t,x_j)) \leqslant e^{C(t-\varepsilon)}\, d_K(X(\varepsilon,x_j)) \qquad \forall t \in [\varepsilon,T].$$

Since the above estimate holds for every $j \geqslant j_\varepsilon$ we can pass to the limit as $j \to \infty$ to obtain

$$d_K(X(t,x)) \leqslant e^{C(t-\varepsilon)}\, d_K(X(\varepsilon,x)) \qquad \forall t \in [\varepsilon,T].$$

So, taking $t = T$,

$$d_K(X(T,x)) \leqslant \lim_{\varepsilon\downarrow 0} e^{C(T-\varepsilon)} d_K(X(\varepsilon,x)) = e^{CT} d_K(x) = 0,$$

which contradicts $X(T,x) \notin K$. This shows that $X(t,x) \in K$ for all $t \geqslant 0$. $\square$

Remark 4.3. The above proof shows that, in order to deduce the invariance of K, it suffices to assume that condition (14) is satisfied for the points $x \in D(A) \setminus K$ which belong to some open neighborhood of K, say V. Consequently, the quasi-dissipativity of B could also be required just on $(D(A) \setminus K) \cap V$.

4.2. **Invariance of convex subsets of a Hilbert space.** The above invariance result holds, in particular, when E is a Hilbert space. In this case, we can use the scalar product on E, denoted by $\langle \cdot, \cdot \rangle$, to characterize dissipative operators. Indeed, condition (6) reduces to

$$\langle x - y, B(x) - B(y) \rangle \leqslant M \|x - y\|^2 \qquad \forall x, y \in E.$$

Similarly, when K is convex, our necessary and sufficient condition for invariance can be given in terms of the orthogonal projection onto K, denoted by $\Pi_K(\cdot)$. We have that $d_K(x) = \|x - \Pi_K(x)\|$ for all $x \in E$. Moreover, as is well known, $d_K \in C^1(E \setminus K)$ and the gradient of d_K at x is given by

$$\nabla d_K(x) = \frac{x - \Pi_K(x)}{d_K(x)} \qquad \forall x \in E \setminus K. \tag{20}$$

Consequently,

$$D^- d_K(x)\, v = \langle \nabla d_K(x), v \rangle \qquad \forall x \in E \setminus K\,,\ \forall v \in E.$$

So, by applying Theorem 4.2 we derive the following necessary and sufficient condition for the invariance of K.

Corollary 4.4. *Assume* **(H)**. *Then a closed convex set $K \subset E$ is* invariant *for* (8) *if and only if there exist constants $C \geqslant 0$, $\delta > 0$ such that*

$$\big\langle x - \Pi_K(x), Ax + B(x) \big\rangle \leqslant C d_K^2(x) \qquad \forall x \in D(A) \cap K_\delta. \tag{21}$$

When $D(A)$ is invariant for the projection onto K, the above condition for invariance can be reduced to the one given by the following proposition, where

$$N_K(x) = \big\{ p \in E \ :\ \langle p, y - x \rangle \leqslant 0\,,\ \forall y \in K \big\}$$

is the normal cone to K at a point $x \in K$.

Proposition 4.5. *Assume* **(H)** *and suppose that $\Pi_K(D(A)) \subset D(A)$. Then K is* invariant *for* (8) *if and only if*

$$\big\langle p, Ax + B(x) \big\rangle \leqslant 0 \qquad \forall x \in \partial K \cap D(A)\,,\ \forall p \in N_K(x) \cap D(A). \tag{22}$$

Proof. If K is invariant for (8), then (21) holds true. Let $x \in \partial K \cap D(A)$ and $p \in N_K(x) \cap D(A)$. Then $x_\lambda := x + \lambda p \in D(A) \setminus K$ and $\Pi_K(x_\lambda) = x$ for all $\lambda > 0$. Thus (21) yields $\langle x_\lambda - x, Ax_\lambda + B(x_\lambda) \rangle \leqslant C\lambda^2 \|p\|^2$ for all small $\lambda > 0$. So, dividing by λ,

$$\langle p, Ax + \lambda Ap + B(x_\lambda) \rangle \leqslant C\lambda \|p\|^2.$$

Passing to the limit as $\lambda \downarrow 0$ and recalling that B is continuous we obtain (22).

Next, suppose (22) holds true, let $x \in D(A) \setminus K$, and set $\bar{x} = \Pi_K(x)$. Since $x - \bar{x} \in N_K(\bar{x}) \cap D(A)$, by (22) we have that

$$\langle x - \bar{x}, A\bar{x} + B(\bar{x})\rangle \leqslant 0.$$

The above inquality, together with assumptions **(H)** and (7), yields

$$\begin{aligned}
&\langle x - \bar{x}, Ax + B(x)\rangle \\
&\quad\leqslant \langle x - \bar{x}, A\bar{x} + B(\bar{x})\rangle + \langle x - \bar{x}, A(x - \bar{x})\rangle + \langle x - \bar{x}, B(x) - B(\bar{x})\rangle \\
&\quad\leqslant M\,\|x - \bar{x}\|^2 = M\, d_K^2(x)\,.
\end{aligned}$$

We have thus obtained (21), which in turn guarantees the invariance of K. □

5. Examples

5.1. **The unit ball in a Hilbert space.** Let $(E, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let K be the closed unit ball centered at 0. Then

$$d_K(x) = \max\{0, \|x\| - 1\} \quad (x \in E), \tag{23}$$

and

$$\Pi_K(x) = \begin{cases} \frac{x}{\|x\|}, & \text{if } \|x\| \geqslant 1 \\ \\ x, & \text{if } \|x\| \leqslant 1. \end{cases} \tag{24}$$

The necessary and sufficient condition (21) becomes

$$\frac{\|x\| - 1}{\|x\|}\,\langle x, Ax + B(x)\rangle \leqslant C(\|x\| - 1)^2, \quad \forall\, x \in D(A) \cap K_\delta,$$

which is equivalent to

$$\langle Ax + B(x), x\rangle \leqslant C\|x\|(\|x\| - 1) \;\; \forall\, x \in D(A) \;\text{ such that }\; 1 + \delta > \|x\| > 1. \tag{25}$$

The "boundary" necessary and sufficient condition for the invariance (22) becomes

$$\big\langle x, Ax + B(x)\big\rangle \leqslant 0 \qquad \forall\, x \in D(A) \text{ such that } \|x\| = 1. \tag{26}$$

Example 5.1. Let us now consider the special case $H = L^2(\mathscr{O})$, where $\mathscr{O}$ is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\mathscr{O}$. Let A be given by

$$Ax(\xi) = \Delta x(\xi), \quad \forall\, x \in D(A) = H^2(\mathscr{O}) \cap H_0^1(\mathscr{O}). \tag{27}$$

Then the iff condition (21) reduces to

$$\langle B(x), x\rangle \leqslant C\|x\|(\|x\| - 1) + \|\nabla x\|^2 \quad \forall\, x \in D(A) \text{ such that } 1 + \delta > \|x\| > 1,$$

where we have denoted by $\|\nabla x\|$ the L^2-norm of the gradient of x. The necessary and sufficient condition (26) becomes

$$\langle B(x), x\rangle \leqslant \|\nabla x\|^2, \quad \forall\, x \in D(A) \text{ such that } \|x\| = 1. \tag{28}$$

When B is given by the composition operator $B(x)(\xi) = \beta(x(\xi))$, where $\beta \in C^1(\mathbb{R})$ and $\beta(0) = 0$, it is easy to see that (28) holds true if

$$\beta'(s) \leqslant \lambda_{\mathscr{O}} \qquad \forall s \in \mathbb{R}$$

where $\lambda_{\mathscr{O}}$ is the first eigenvalue of $-A$.

5.2. **Invariance of a half-space in a Hilbert space.** Let $(E, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let

$$K = \{x \in E : \ \langle x, a \rangle \leqslant 0\},$$

where $a \in E$ and $\|a\| = 1$. Then we have

$$d_K(x) = \max\{0, \langle x, a \rangle\} \quad (x \in E) \tag{29}$$

and

$$\Pi_K(x) = \begin{cases} x - \langle x, a \rangle \, a, & \forall \, x \notin K \\ \\ x, & \forall \, x \in K. \end{cases} \tag{30}$$

The necessary and sufficient condition (21) becomes

$$\langle x, a \rangle \, \langle Ax + B(x), a \rangle \leqslant C \langle x, a \rangle^2, \quad \forall \, x \in D(A), \, \delta > \langle x, a \rangle > 0,$$

which is equivalent to

$$\langle Ax + B(x), a \rangle \leqslant C \langle x, a \rangle, \quad \forall \, x \in D(A), \, \delta > \langle x, a \rangle > 0. \tag{31}$$

5.3. **Further examples.** In this section, we study three examples: the cone of nonnegative functions, the convex constraint associated with an obstacle problem for a parabolic equation, and a first order equation in population dynamics.

Example 5.2 (*The positive cone*)**.** Let $\mathscr{O}$ be a bounded domain in $\mathbb{R}^n$. In the Hilbert space $E = L^2(\mathscr{O})$ consider the cone (with empty interior)

$$K = \{x \in H : \ x(\xi) \geqslant 0 \ \text{ a.e. } \xi \in \mathscr{O}\}.$$

Set

$$x^+(\xi) = \max\{x(\xi), 0\}, \quad x^-(\xi) = \max\{-x(\xi), 0\}, \quad \forall \xi \in \mathscr{O}. \tag{32}$$

Then

$$d_K(x) = \|x^-\|, \quad \Pi_K(x) = x^+, \quad x - \Pi_K(x) = -x^-.$$

The necessary and sufficient condition (21) becomes

$$-\langle x^-, Ax + B(x) \rangle \leqslant C \|x^-\|^2 \quad \forall x \in D(A) \cap K_\delta. \tag{33}$$

Assume that A is given by (27). Then we have

$$\langle Ax^+, x^- \rangle = -\int_{\mathscr{O}} \nabla x^+(\xi) \cdot \nabla x^-(\xi) d\xi = 0$$

and (33) can be recast as follows:

$$-\langle x^-, B(x)\rangle \leqslant C\|x^-\|^2 + \int_{\mathscr{O}} \|\nabla x^-(\xi)\|^2 d\xi \quad \forall x \in D(A) \text{ with } 0 < \|x^-\| < \delta. \tag{34}$$

Suppose now $B : E \to E$ is given by

$$B(x)(\xi) = \beta(x(\xi)) \qquad (\xi \in [0,1]) \tag{35}$$

where $\beta : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\beta(0) = 0$ and

$$\big(\beta(s) - \beta(r)\big)(s - r) \leqslant M(s - r)^2 \qquad \forall r, s \in \mathbb{R} \tag{36}$$

for some constant $M \geqslant 0$. Then we have that

$$-\langle x^-, B(x)\rangle = -\int_{\mathscr{O}} x^-(\xi)\beta\big(-x^-(\xi)\big)\, d\xi \leqslant M\|x^-\|^2.$$

So, condition (34) is satisfied with $C = M$. Consequently, the cone of positive functions is invariant for the parabolic problem

$$\begin{cases} \frac{\partial X}{\partial t} - \Delta X - \beta(X) = 0 & \text{in } (0,+\infty) \times \mathscr{O} \\ X = 0 & \text{on } (0,+\infty) \times \partial\mathscr{O} \\ X(0,\xi) = x(\xi) & \xi \in \mathscr{O}. \end{cases} \tag{37}$$

This result also follows from the parabolic maximum principle (see, e.g., [**?**]).

Example 5.3 (*An obstacle problem)***.** Let E be the Banach space

$$E = \{x \in C([0,1]) \ : \ x(0) = 0 = x(1)\}$$

with the uniform norm

$$\|x\|_\infty = \max_{\xi\in[0,1]} |x(\xi)| \qquad \forall x \in E.$$

Let $f \in C^2([0,1])$ be a function such that $f(0) \leqslant 0$ and $f(1) \leqslant 0$ and consider the closed convex set

$$K^f = \big\{x \in E \ : \ x(\xi) \geqslant f(\xi)\,,\ \forall \xi \in [0,1]\big\}.$$

We observe that the interior of K^f is nonempty if $f(0), f(1) < 0$.

Define $A : D(A) \subset E \to E$ by

$$\begin{cases} D(A) = \{x \in E \cap C^2([0,1]) \ : \ \frac{d^2x}{d\xi^2}(0) = 0 = \frac{d^2x}{d\xi^2}(1)\} \\ Ax(\xi) = \frac{d^2x}{d\xi^2}(\xi) \qquad \forall \xi \in [0,1] \end{cases}$$

and let $B : E \to E$ by given by (35), where $\beta : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying (36) such that $\beta(0) = 0$. It is well known that A and B satisfy assumptions **(H)** on E. So, we can use condition (14) to study the invariance of K^f for the parabolic problem (37) or, equivalently, the solvability of the obstacle problem

$$x(\xi) \geqslant f(\xi)\,,\ \forall \xi \in [0,1] \quad \Longrightarrow \quad X(t,\xi) \geqslant f(\xi)\,,\ \forall (t,\xi) \in [0,\infty) \times [0,1].$$

Noting that

$$d_{K^f}(x) = \|(x-f)^-\|_\infty \qquad \forall x \in E,$$

for any $x \in D(A)$ and $\delta > 0$ we have that

$$x \in K^f_\delta \quad \Longleftrightarrow \quad 0 < \|(x-f)^-\|_\infty < \delta.$$

Fix any $x \in D(A) \cap K^f_\delta$, $y \in E$, and let $h_j \downarrow 0$ be a sequence of positive numbers such that

$$D^- d_{K^f}(x)\, y = \lim_{j\to\infty} \frac{\|(x+h_j y - f)^-\|_\infty - \|(x-f)^-\|_\infty}{h_j}. \tag{38}$$

Let $\xi_j \in [0,1]$ be such that $\|(x+h_j y - f)^-\|_\infty = (x+h_j y - f)^-(\xi_j)$. Since $(x-f)^- \not\equiv 0$, for j sufficiently large we have that

$$\|(x+h_j y - f)^-\|_\infty = (x-f)^-(\xi_j) - h_j y(\xi_j) \leqslant \|(x-f)^-\|_\infty - h_j y(\xi_j). \tag{39}$$

By a compactness argument, we can assume that $\xi_j \overset{j\to\infty}{\to} \xi_0 \in [0,1]$. Moreover, the continuity of x ensures that

$$\xi_0 \in \arg\max_{[0,1]} (x-f)^- = \big\{\xi \in [0,1] \ : \ (x-f)^-(\xi) = \|(x-f)^-\|_\infty\big\}.$$

Combining (38) and (39) we conclude that

$$D^- d_{K^f}(x)\, y \leqslant -y(\xi_0) \leqslant \max\big\{ -y(\xi) \ : \ \xi \in \arg\max_{[0,1]} (x-f)^-\big\}.$$

Thus, appealing to (14), we recover the following sufficient condition for the invariance of K: for all $x \in D(A)$ such that $0 < \|(x-f)^-\|_\infty < \delta$

$$\max\Big\{ -\frac{d^2x}{d\xi^2}(\xi) - \beta(x(\xi)) \ : \ \xi \in \arg\max_{[0,1]} (x-f)^-\Big\} \leqslant C\|(x-f)^-\|_\infty. \tag{40}$$

Now, observe that $\operatorname{argmax}_{[0,1]} (x-f)^- \subset (0,1)$ because $0 < \|(x-f)^-\|_\infty$ and $(x-f)^-(0) = 0 = (x-f)^-(1)$. Therefore,

$$\frac{d^2x}{d\xi^2}(\xi) \geqslant \frac{d^2 f}{d\xi^2}(\xi) \qquad \forall \xi \in \arg\max_{[0,1]} (x-f)^-.$$

Moreover, owing to (36)

$$-\big(\beta(x(\xi)) - \beta(f(\xi))\big) \leqslant M\|(x-f)^-\|_\infty \qquad \forall \xi \in \arg\max_{[0,1]} (x-f)^-.$$

Thus,

$$-\frac{d^2x}{d\xi^2}(\xi) - \beta(x(\xi)) \leqslant -\frac{d^2 f}{d\xi^2}(\xi) - \beta(f(\xi)) + M\|(x-f)^-\|_\infty \quad \forall \xi \in \arg\max_{[0,1]} (x-f)^-,$$

and we conclude that a sufficient condition for (40) to be satisfied is that

$$\frac{d^2 f}{d\xi^2} + \beta(f) \geqslant 0 \quad \text{in} \quad [0,1]. \tag{41}$$

In particular, since (41) is satisfied for $f \equiv 0$, we obtain that the cone of nonnegative functions is invariant for system (37) also when studied in $C([0,1])$.

Since (41) implies that f is a subsolution of the boundary value problem in (37), the invariance result we have just proved could have also been derived by maximum principle arguments.

Example 5.4 (*An age-structured population model*)**.** We now discuss the positivity of solutions to the age-structured population model

$$\begin{cases} \dfrac{\partial X}{\partial t}(t,a) + \dfrac{\partial X}{\partial a}(t,a) + \mu(a)X(t,a) = \phi\big(a, X(t,a)\big), & a \in [0,a_1], t \geqslant 0 \\ X(t,0) = \displaystyle\int_0^{a_1} \beta(a)X(t,a)\,da, & t \geqslant 0 \\ X(0,a) = x_0(a). & a \in [0,a_1]\,. \end{cases} \tag{42}$$

which was analysed in Lebesgue spaces in [**?**] (see also [**?**]). Here, $X(t,a)$ is the population density of age a at time t, μ is the mortality rate, β the birth rate, and $a_1 > 0$ is the maximal age. Moreover, ϕ is a real-valued continuous function modelling an exterior source, such as (inward or outward) migration. Observe that, since ϕ has no prescribed sign, in principle solutions might be forced to assume negative values, even for positive x_0.

We study the above problem in the space $E = C([0,a_1])$, where the positive cone

$$K = \big\{x \in E \ : \ x(a) \geqslant 0\,,\ \forall a \in [0,a_1]\big\}$$

has nonempty interior. Our assumptions **(A)** on the data are as follows:

(A1) $\mu, \beta \in K$ and

$$\int_0^{a_1} \beta(a)\,da < 1 \tag{43}$$

(A2) $\phi : [0,a_1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\phi(0) = 0$ and

$$\big(\phi(a,s) - \phi(a,r)\big)(s-r) \leqslant M(s-r)^2 \qquad \forall a \in [0,a_1],\ \forall r,s \in \mathbb{R} \tag{44}$$

for some constant $M \geqslant 0$.

In order to recast problem (42) as a semilinear evolution equation in E, we define

$$\begin{cases} D(A) = \Big\{x \in C^1([0,a_1]) : \ x(0) = \displaystyle\int_0^{a_1} \beta(a)x(a)\,da\Big\} \\ Ax(a) = -\frac{dx}{da}(a) - \mu(a)x(a) \quad \forall x \in D(A),\ \forall a \in [0,a_1] \end{cases}$$

and

$$f(x)(a) = \phi\big(a, x(a)\big) \quad \forall x \in E,\ \forall a \in [0,a_1].$$

Then, one can show that $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup of contractions on E. We omit the proof of this result that can be easily deduced arguing as in [**?**], but for the dissipativity of A.

For this, let $x \in D(A) \setminus \{0\}$ and let $\bar{a} \in \arg\max_{[0,a_1]} |x|$. Observe that $\bar{a} \neq 0$ for otherwise the age relation satisfied by all elements of $D(A)$ would yield

$$\|x\|_\infty = |x(0)| = \Big| \int_0^{a_1} \beta(a)x(a)\,da \Big| \leqslant \int_0^{a_1} \beta(a)|x(a)|\,da \leqslant |x(0)| \int_0^{a_1} \beta(a)\,da$$

and (43) would imply that $x \equiv 0$. So, taking the Dirac delta centered at $\bar{a}$, $\delta_{\bar{a}}$, by Example 2.1 we have that $\mu_x := \operatorname{sgn}(x(\bar{a})) \cdot \delta_{\bar{a}} \in \partial\|x\|$. Now, we claim that

$$\langle \mu_x, Ax \rangle = -\operatorname{sgn}(x(\bar{a}))\,\frac{dx}{da}\,(\bar{a}) - \mu(\bar{a})\,|x(\bar{a})| \leqslant 0\,.$$

Indeed, since $\bar{a} \neq 0$, it must be either $\bar{a} \in (0, a_1)$, hence $\frac{dx}{da}\,(\bar{a}) = 0$, or $\bar{a} = a_1$ and

$$\operatorname{sgn}(x(\bar{a}))\,\frac{dx}{da}\,(\bar{a}) \geqslant 0\,.$$

Thus, we can use condition (14) to study the invariance of K. Since

$$d_K(x) = \|x^-\|_\infty \quad \forall x \in E,$$

arguing as in Example 5.3 we conclude a sufficient condition for the invariance of K is that, for some $\rho > 0$ and all $x \in D(A)$ such that $0 < \|x^-\|_\infty < \rho$,

$$\max\Big\{ \frac{dx}{da}\,(a) + \mu(a)x(a) - \phi(a, x(a)) \;:\; a \in \arg\max_{[0,a_1]} x^- \Big\} \leqslant C\|x^-\|_\infty. \tag{45}$$

The above condition holds true because, as above, $0 \notin \arg\max_{[0,a_1]} x^-$ and so

$$\frac{dx}{da}\,(a) + \mu(a)x(a) \leqslant 0 \qquad \forall a \in \arg\max_{[0,a_1]} x^-,$$

while

$$-\phi(a, x(a)) \leqslant M\|x^-\|_\infty \qquad \forall a \in \arg\max_{[0,a_1]} x^-$$

in view of (44). Therefore, K is invariant for problem (42).

References

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