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INVASIANCE FOR QUASI-DISSIPATIVE SYSTEMS
IN BANACH SPACES

P. CANNARSA, G. DA PRATO, AND H. FRANKOWSKA

ABSTRACT. In a separable Banach space $E$, we study the invariance of a closed set $K$ under the action of the evolution equation associated with a maximal dissipative linear operator $A$ perturbed by a quasi-dissipative continuous term $B$. Using the distance to the closed set, we give a general necessary and sufficient condition for the invariance of $K$. Then, we apply our result to several examples of partial differential equations in Banach and Hilbert spaces.

1. Introduction

In a separable Banach space $E$, consider the Cauchy problem

$$
\begin{align*}
X'(t) &= AX(t) + B(X(t)), \quad t \geq 0 \\
X(0) &= x,
\end{align*}
$$

(1)

where $A : D(A) \subset E \to E$ is the infinitesimal generator of the strongly continuous semigroup of contractions $e^{tA}$ on $E$ and $B : E \to E$ is continuous and such that $B - MI$ is dissipative for some $M \geq 0$. As is well known, for every $x \in E$ problem (1) has a unique mild solution

$$
X(t, x) = e^{tA}x + \int_0^t e^{(t-s)A}B(X(s, x))ds, \quad t \in [0, T]
$$

which belongs to $C([0, T]; E)$. Moreover, when $x \in D(A)$, the solution is strict (see [?] and Proposition 3.1 below).

The main object of this paper is the characterization of those nonempty closed sets $K$ of $E$ which are invariant for the flow $X(\cdot, x)$, that is, such that $x \in K$ implies $X(t, x) \in K$ for all $t \geq 0$. Such invariance properties are very useful, for instance, to obtain relaxation results for semilinear control systems under state constraints (see [?]).

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There is an extensive literature addressing domain invariance issues for partial differential equations (see, for instance, [?], [?, ?], and [?]), but most of the existing results concern sufficient conditions for the invariance of $K$. Necessary and sufficient conditions for invariance can be found in the monograph by O. Carja, M. Necula and K. Vrabie [?]. Such results are, however, completely different from ours. Indeed, in the aforementioned monograph, the classical condition introduced by Nagumo [?] is extended to infinite dimensions using as a tool the set $\mathcal{F}_K^A(\xi)$, see [?], Definition 8.1.3, which reduces to the contingent cone at $\xi$ when $A = 0$. We observe that the definition of such a set is given in terms of the semigroup $e^{tA}$ rather than its infinitesimal generator $A$. Then, sufficient conditions reduce to asking that $B(\xi) \in \mathcal{F}_K^A(\xi)$ for all $\xi \in K$ (see [?, Theorem 8.1.2]), whereas necessary conditions require some additional compactness assumptions (see [?, Theorem 8.2.1]).

Our main result, Theorem 4.2 below, provides a necessary and sufficient condition for the invariance of $K$ which is stated in terms of $A$ and $B$. It essentially guarantees that $K$ is invariant if and only if, for some $C \geq 0$ and $\delta > 0$,

$$D^-d_K(x)(Ax + B(x)) \leq Cd_K(x) \quad \forall x \in D(A) \cap K_\delta,$$

where $D^-d_K(x)$ is the lower Dini derivative of the distance $d_K(x)$ of $x$ to $K$, defined in Section 2, and

$$K_\delta = \{ x \in E \setminus K : d_K(x) < \delta \}.$$ 

This characterisation, though already quite simple, can be expressed in more geometric forms when $K$ is a convex subset of a Hilbert space. Indeed, in this case, the lower Dini derivative of the distance can be computed using the orthogonal projection $\Pi_K$ onto $K$. So, (2) becomes

$$\langle x - \Pi_K(x), Ax + B(x) \rangle \leq Cd_K^2(x) \quad \forall x \in D(A) \cap K_\delta$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $E$. Moreover, if $D(A)$ is invariant for $\Pi_K$, then condition (3) further simplifies into

$$\langle p, Ax + B(x) \rangle \leq 0 \quad \forall x \in \partial K \cap D(A), \forall p \in N_K(x) \cap D(A),$$

where $N_K(x)$ denotes the normal cone to $K$ at $x$.

Alltogether, the above results allow us to treat several examples of partial differential equations of evolution in Hilbert or Banach spaces, including both reaction-diffusion equations and equations modelling the dynamics of an age-structured population. For these examples we study, in spaces of continuous functions, the invariance of the set $K$ of positive functions or, more generally, of functions satisfying unilateral constraints like in obstacle problems.

This paper is organized as follows. Section 2 is devoted to some preliminaries about dissipative operators and derivatives of the norm in a Banach space, following [?]. In Section 3, we gather several properties of solutions to (1) which
are needed to obtain our main results. Section 4 is devoted to the proof of necessary and sufficient conditions for the invariance of $K$. Finally, in Section 5, the invariance of several sets is discussed in Hilbert and Banach spaces.

2. Notation and preliminaries

In this section, we recall some well-known definitions and preliminary results that can be easily found in the literature (see, for instance, [?]).

Let $E$ be a real separable Banach space with norm $\| \cdot \|$ and let $f : E \rightarrow \mathbb{R}$ be a locally Lipschitz function. We define the lower Dini derivative of $f$ at a point $x \in E$ by

$$D^{-} f(x) v = \liminf_{\lambda \downarrow 0} \frac{f(x + \lambda v) - f(x)}{\lambda} \quad \forall v \in E.$$ 

The subdifferential of $\| \cdot \|$ at a point $x \in E$ is defined as

$$\partial \| x \| = \{ \phi \in E^* : \langle \phi, y \rangle \leq D^{-} \| x \| y, \forall y \in E \},$$

where $E^*$ denotes the dual of $E$, $\langle \cdot, \cdot \rangle$ the duality between $E^*$ and $E$. Observe that, since $x \mapsto \| x \|$ is a convex function on $E$, we have

$$D^{-} \| x \| y = \lim_{\lambda \downarrow 0} \frac{\| x + \lambda y \| - \| x \|}{\lambda}.$$ 

It is well known that $\partial \| x \|$ is a nonempty closed convex subset of $E^*$ for all $x \in E$. Such a set can be characterized as follows

$$\partial \| x \| = \{ \phi \in E^* : \langle \phi, x \rangle = \| x \|, \| \phi \|_* = 1 \} \quad \forall x \in E,$$

where $\| \cdot \|_*$ denotes the dual norm. Notably, for $x \neq 0$ we have that $\| \phi \|_* = 1$ for all $\phi \in \partial \| x \|$. Moreover, for all $x, y \in E$,

$$D^{-} \| x \| y = \max_{\phi \in \partial \| x \|} \langle \phi, y \rangle.$$

Example 2.1. Let $E = C([a, b])$ and let $x \in E \setminus \{0\}$. Define

$$\arg \max_{[0,1]} |x| = \{ \xi \in [a, b] : |x(\xi)| = \| x \|_{\infty} \}.$$ 

Then one can prove (see [?]) that $\phi \in \partial \| x \|$ if and only if $\phi$ is represented by a signed Radon measure $\mu_{\phi}$ on $[a, b]$, with total variation $\| \mu_{\phi} \| = 1$ and support in $\arg \max_{[0,1]} |x|$, such that, for all Borel sets $\Omega \subset [a, b]$,

$$\int_{\Omega} \text{sgn}(x(\xi)) \mu_{\phi}(d\xi) \geq 0$$

where

$$\text{sgn}(\lambda) = \begin{cases} 1 & \text{if } \lambda \geq 0 \\ 0 & \text{if } \lambda < 0. \end{cases}$$

The following lemma (see, e.g., [?, Appendix D]) will be used in what follows.
Lemma 2.2. Let $g : [0, T] \rightarrow E$ be differentiable at a point $t_0 \in [0, T]$. Then the function $\gamma(t) := \|g(t)\|$ is:

(a) right-differentiable at $t_0$ (provided that $t_0 < T$) and
\[
\frac{d\gamma}{dt}^+(t_0) := \lim_{t \downarrow t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0} = \max \{ \langle \phi, g'(t_0) \rangle : \phi \in \partial \|g(t_0)\| \},
\]
(b) left-differentiable at $t_0$ (provided that $t_0 > 0$) and
\[
\frac{d\gamma}{dt}^-(t_0) := \lim_{t \uparrow t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0} = \min \{ \langle \phi, g'(t_0) \rangle : \phi \in \partial \|g(t_0)\| \}.
\]

A mapping $F : D(F) \rightarrow E$, where $D(F) \subset E$, is called dissipative if
\[
\|x - y\| \leq \|x - y - \alpha(F(x) - F(y))\| \quad \forall \alpha > 0, \forall x, y \in D(F),
\]
and maximal dissipative if it is dissipative and $I - F$ is onto. Moreover, $F$ is called quasi-dissipative if $F - M I$ is dissipative for some constant $M \geq 0$. By using (5), it is easy to check that $F$ is quasi-dissipative if and only if
\[
\langle \phi, F(x) - F(y) \rangle \leq M \|x - y\|
\]
for any $x, y \in D(F)$ and some $\phi \in \partial \|x - y\|$.

Clearly, if $F$ is linear, then $F$ is dissipative if and only if
\[
\|x\| \leq \|x - \alpha F(x)\| \quad \forall \alpha > 0, \forall x \in D(F),
\]
or, equivalently,
\[
\forall x \in D(F) \exists \phi \in \partial \|x\| \text{ such that } \langle \phi, F(x) \rangle \leq 0.
\]

3. Quasi-dissipative Evolution Equations

We are here concerned with the Cauchy problem
\[
\begin{aligned}
X'(t) &= AX(t) + B(X(t)), \quad t \geq 0 \\
X(0) &= x
\end{aligned}
\]
under the following assumptions (H):

(H1) $A : D(A) \subset E \rightarrow E$ is a the infinitesimal generator of a strongly continuous semigroup of contractions on $E$, which will be denoted by $e^{tA}$,

(H2) $B : E \rightarrow E$ is continuous and quasi-dissipative.

As is well known, assumption (H1) is satisfied if and only if $A$ is maximal dissipative on $E$. The following result guarantees that (8) is well posed.

Proposition 3.1. Assume (H) and fix any $T > 0$. Then the following holds true.

(I): For every $x \in E$ problem (8) has a unique mild solution $X(\cdot, x)$ which belongs to $C([0, T]; E)$ and satisfies
\[
X(t, x) = e^{tA}x + \int_0^t e^{(t-s)A}B(X(s, x))ds, \quad \forall t \in [0, T].
\]
Moreover, for all $x, y \in E$

$$\|X(t, x) - X(t, y)\| \leq e^{Mt}\|x - y\| \quad \forall t \in [0, T].$$

\textbf{(II):} If, in addition, $x \in D(A)$, then $X(\cdot, x) \in W^{1, \infty}(0, T; E) \cap L^{\infty}(0, T; D(A))$ and

$$X'(t) = AX(t) + B(X(t)) \quad \text{a.e. on } [0, T].$$

Furthermore, $X(\cdot, x)$ is differentiable at $t = 0$ and

$$\frac{d}{dt} X(t, x)|_{t=0} = Ax + B(x).$$

\textbf{Proof.} The fact that (8) has a unique mild solution, given by (9), follows from assumption (H), a result by Webb [?] which ensures that $A + B$ is maximal quasi-dissipative, and the classical Crandall-Liggett Theorem (see, e.g., [?, Theorem 4.1.4]).

The fact that $X(\cdot, x) \in W^{1, \infty}(0, T; E) \cap L^{\infty}(0, T; D(A))$ when $x \in D(A)$ and (8) holds a.e. is also well-known, see, e.g., [?, Theorem 4.1.4].

The Lipschitz dependence of the flow with respect to initial data is a direct consequence of assumption (H) for $x, y \in D(A)$. Indeed, using (6) and Lemma 2.2 to compute the derivative of the function $t \mapsto \|X(t, x) - X(t, y)\|$ (which coincides with the left-derivative a.e.), we deduce that for a.e. $t \in [0, T]$

$$
\frac{d}{dt} \|X(t, x) - X(t, y)\| = \min_{\phi \in \partial \|X(t, x) - X(t, y)\|} \left\langle \phi, A\left[X(t, x) - X(t, y)\right] + B(X(t, x)) - B(X(t, y)) \right\rangle \\
\leq M\|X(t, x) - X(t, y)\|.
$$

Then, (10) follows by Gronwall’s lemma for all $x, y \in D(A)$. Moreover, the same inequality can be recovered by density for all $x, y \in H$, because $X(\cdot, x_k) \to X(\cdot, x)$ uniformly on $[0, T]$ whenever $D(A) \ni x_k \to x$.

Finally, in order to prove (12), write

$$
\frac{X(t, x) - x}{t} = \frac{e^{tA}x - x}{t} + \frac{1}{t} \int_0^t e^{(t-s)A}B(X(s, x))ds \\
= \frac{e^{tA}x - x}{t} + \frac{1}{t} \int_0^t e^{(t-s)A}B(x)ds + \frac{1}{t} \int_0^t e^{(t-s)A}(B(X(s, x)) - B(x))ds \\
= I_1 + I_2 + I_3.
$$

Then, since $x \in D(A)$, we have that $I_1 \to Ax$, $I_2 \to B(x)$, and $I_3 \to 0$ as $t \to 0$. □

4. Invariance results

4.1. Invariance of closed subsets of a Banach space. Let $K \subset E$ be a nonempty closed set.
Definition 4.1. We say that \( K \) is invariant for (8) if \( X(t, x) \in K \) for all \( x \in K \) and all \( t \geq 0 \).

We denote by \( d_K(x) \) the distance of \( x \) from \( K \), that is,

\[
d_K(x) = \inf_{y \in K} \| x - y \|, \quad \forall \ x \in E.
\]

We recall that \( d_K \) is Lipschitz continuous (with Lipschitz constant 1) on \( E \), and convex if \( K \) is convex.

We now give a necessary and sufficient condition for the invariance of \( K \). For any \( \delta > 0 \) let

\[
K_\delta = \{ x \in E \ ; \ d_K(x) < \delta \}.
\]

Theorem 4.2. Assume (H). Then \( K \) is invariant for (8) if and only if there exists \( \delta > 0 \) such that

\[
D^- d_K(x) (Ax + B(x)) \leq C d_K(x) \quad \forall x \in D(A) \cap K_\delta
\]

for some constant \( C \geq 0 \).

Proof. Suppose \( K \) is invariant for (8) and let \( x \in D(A) \setminus K \). Then for all \( t > 0 \) there exists \( x_t \in K \) such that

\[
\| x - x_t \| \leq (1 + t^2) d_K(x).
\]

Since \( X(t, x) \in K \) for all \( t \geq 0 \), by (15) and (10) we deduce that

\[
\frac{1}{t} \left[ d_K(X(t, x)) - d_K(x) \right] = \frac{1}{t} \left[ d_K(X(t, x)) - d_K(X(t, x_t)) - d_K(x) \right]
\]

\[
\leq \frac{1}{t} \left( \| X(t, x) - X(t, x_t) \| - \frac{\| x - x_t \|}{1 + t^2} \right)
\]

\[
\leq \frac{1}{t} \left( e^{Mt} - \frac{1}{1 + t^2} \right) \| x - x_t \| \quad \forall t > 0
\]

with \( M \) given by (6). Hence, again by (15), we obtain

\[
\frac{d_K(X(t, x)) - d_K(x)}{t} \leq \left( \frac{e^{Mt} - 1}{t} + \frac{t}{1 + t^2} \right) (1 + t^2) d_K(x) \quad \forall t > 0.
\]

Since \( d_K \) is Lipschitz and \( X(\cdot, x) \) is differentiable at \( t = 0 \) by Proposition 3.1, the lower limit of the left-hand side as \( t \downarrow 0 \) coincides with \( D^- d_K(x)(Ax + B(x)) \).

Therefore

\[
D^- d_K(x) (Ax + B(x)) \leq M d_K(x)
\]

which is (14) with \( C = M \).

Conversely, assume (14) and fix any \( x \in K \), \( T > 0 \) so that \( X([0, T], x) \subset K \cup K_\delta \). Suppose for a moment that \( X(T, x) \in K_\delta \) and let

\[
t_0 = \max\{ t \in [0, T] : X(t, x) \in K \}.
\]
Without loss of generality, we can assume \( t_0 = 0 \). Let \( \{x_j\} \subset D(A) \) be any sequence converging to \( x \). Then, on account of (10),

\[
(16) \quad X(\cdot, x_j) \to X(\cdot, x) \text{ uniformly on } [0, T].
\]

Next, fix \( \varepsilon \in (0, T) \) and observe that, in view of (16), there exist \( \delta_\varepsilon > 0 \) and \( j_\varepsilon \in \mathbb{N} \) such that

\[
(17) \quad X([\varepsilon, T], x_j) \subset K_\delta, \quad d_K(\varepsilon, x_j) \geq \delta \quad \forall t \in [\varepsilon, T], \forall j \geq j_\varepsilon.
\]

Owing to Proposition 3.1, for all such \( j \) and a.e. \( t \in [0, T] \) we have that the derivative \( X'(t, x_j) \) does exist, \( X(t, x_j) \in D(A) \), and

\[
(18) \quad X'(t, x_j) = AX(t, x_j) + B(\varepsilon, x_j) \quad \text{for a.e. } t \in [0, T].
\]

Thus, we obtain by (14)

\[
(19) \quad D^-d_K(X(t, x_j))(AX(t, x_j) + B(\varepsilon, x_j)) \leq C d_K(X(t, x_j)) \quad \text{a.e. } t \in [0, T].
\]

Now, let \( j \geq j_\varepsilon \) and consider the Lipschitz function

\[
\phi(t) = d_K(X(t, x_j)) \quad (t \in [\varepsilon, T]).
\]

Since, \( \phi \) is a.e. differentiable, for a.e. \( t \in (\varepsilon, T) \) such that (18) holds true, the derivative \( \phi'(t) \) does exist. Fix such \( t \) and consider \( h_i \to 0+ \) such that

\[
D^-d_K(X(t, x_j))(AX(t, x_j) + B(\varepsilon, x_j)) = \lim_{h_i \to \infty} \frac{1}{h_i} \left\{ d_K(X(t, x_j) + h_i[AX(t, x_j) + B(\varepsilon, x_j)]) - d_K(X(t, x_j)) \right\}.
\]

Then

\[
\phi'(t) = \lim_{h_i \to 0} \frac{d_K(X(t + h, x_j)) - d_K(X(t, x_j))}{h} = \lim_{h_i \to \infty} \frac{1}{h_i} \left\{ d_K(X(t, x_j) + h_i[AX(t, x_j) + B(\varepsilon, x_j)]) - d_K(X(t, x_j)) \right\} = D^-d_K(X(t, x_j))(AX(t, x_j) + b(\varepsilon, x_j)).
\]

Therefore, in view of (19), \( \phi'(t) \leq C \phi(t) \) for a.e. \( t \in [\varepsilon, T] \). We can then apply the Gronwall lemma to deduce that

\[
d_K(X(t, x_j)) \leq e^{C(t-\varepsilon)} d_K(X(\varepsilon, x_j)) \quad \forall t \in [\varepsilon, T].
\]

Since the above estimate holds for every \( j \geq j_\varepsilon \) we can pass to the limit as \( j \to \infty \) to obtain

\[
d_K(X(t, x)) \leq e^{C(t-\varepsilon)} d_K(X(\varepsilon, x)) \quad \forall t \in [\varepsilon, T].
\]

So, taking \( t = T \),

\[
d_K(X(T, x)) \leq \lim_{\varepsilon \to 0} e^{C(T-\varepsilon)} d_K(X(\varepsilon, x)) = e^{CT} d_K(x) = 0,
\]

which contradicts \( X(T, x) \notin K \). This shows that \( X(t, x) \in K \) for all \( t \geq 0 \). \( \square \)
Passing to the limit as \( \lambda \) by \( \lambda \) Thus (21) yields
\[
\langle x - y, B(x) - B(y) \rangle \leq M\|x - y\|^2 \quad \forall x, y \in E.
\]
Similarly, when \( K \) is convex, our necessary and sufficient condition for invariance can be given in terms of the orthogonal projection onto \( K \), denoted by \( \Pi_K(\cdot) \). We have that \( d_K(x) = \|x - \Pi_K(x)\| \) for all \( x \in E \). Moreover, as is well known, \( d_K \in C^1(E \setminus K) \) and the gradient of \( d_K \) at \( x \) is given by
\[
(20) \quad \nabla d_K(x) = \frac{x - \Pi_K(x)}{d_K(x)} \quad \forall x \in E \setminus K.
\]
Consequently,
\[
D^-d_K(x)v = \langle \nabla d_K(x), v \rangle \quad \forall x \in E \setminus K, \ \forall v \in E.
\]
So, by applying Theorem 4.2 we derive the following necessary and sufficient condition for the invariance of \( K \).

**Remark 4.3.** The above proof shows that, in order to deduce the invariance of \( K \), it suffices to assume that condition (14) is satisfied for the points \( x \in D(A) \setminus K \) which belong to some open neighborhood of \( K \), say \( V \). Consequently, the quasi-dissipativity of \( B \) could also be required just on \( (D(A) \setminus K) \cap V \).

### 4.2. Invariance of convex subsets of a Hilbert space

The above invariance result holds, in particular, when \( E \) is a Hilbert space. In this case, we can use the scalar product on \( E \), denoted by \( \langle \cdot, \cdot \rangle \), to characterize dissipative operators. Indeed, condition (6) reduces to
\[
\langle x - y, B(x) - B(y) \rangle \leq M\|x - y\|^2 \quad \forall x, y \in E.
\]

**Proposition 4.5.** Invariance of convex subsets of a Hilbert space. Assume (H). Then a closed convex set \( K \subset E \) is invariant for (8) if and only if there exist constants \( C \geq 0, \delta > 0 \) such that
\[
(21) \quad \langle x - \Pi_K(x), Ax + B(x) \rangle \leq Cd_K^2(x) \quad \forall x \in D(A) \cap K_\delta.
\]

When \( D(A) \) is invariant for the projection onto \( K \), the above condition for invariance can be reduced to the one given by the following proposition, where
\[
N_K(x) = \{ p \in E : \langle p, y - x \rangle \leq 0, \ \forall y \in K \}
\]
is the normal cone to \( K \) at a point \( x \in K \).

**Corollary 4.4.** Assume (H). Then a closed convex set \( K \subset E \) is invariant for (8) if and only if there exist constants \( C \geq 0, \delta > 0 \) such that
\[
(21) \quad \langle x - \Pi_K(x), Ax + B(x) \rangle \leq Cd_K^2(x) \quad \forall x \in D(A) \cap K_\delta.
\]

When \( D(A) \) is invariant for the projection onto \( K \), the above condition for invariance can be reduced to the one given by the following proposition, where
\[
N_K(x) = \{ p \in E : \langle p, y - x \rangle \leq 0, \ \forall y \in K \}
\]
is the normal cone to \( K \) at a point \( x \in K \).

**Proposition 4.5.** Assume (H) and suppose that \( \Pi_K(D(A)) \subset D(A) \). Then \( K \) is invariant for (8) if and only if
\[
(22) \quad \langle p, Ax + B(x) \rangle \leq 0 \quad \forall x \in \partial K \cap D(A), \ \forall p \in N_K(x) \cap D(A).
\]

**Proof.** If \( K \) is invariant for (8), then (21) holds true. Let \( x \in \partial K \cap D(A) \) and \( p \in N_K(x) \cap D(A) \). Then \( x_\lambda := x + \lambda p \in D(A) \setminus K \) and \( \Pi_K(x_\lambda) = x \) for all \( \lambda > 0 \). Thus (21) yields
\[
\langle x_\lambda - x, Ax_\lambda + B(x_\lambda) \rangle \leq Cd_{d_K^2}(x_\lambda) \quad \forall \lambda > 0.
\]

Thus (21) yields
\[
\langle p, Ax + \lambda Ap + B(x_\lambda) \rangle \leq C\lambda\|p\|^2.
\]
Passing to the limit as \( \lambda \downarrow 0 \) and recalling that \( B \) is continuous we obtain (22).
Next, suppose (22) holds true, let \( x \in D(A) \setminus K \), and set \( \bar{x} = \Pi_K(x) \). Since \( x - \bar{x} \in N_K(\bar{x}) \cap D(A) \), by (22) we have that
\[
\langle x - \bar{x}, A\bar{x} + B(\bar{x}) \rangle \leq 0.
\]
The above inequality, together with assumptions (H) and (7), yields
\[
\langle x - \bar{x}, A\bar{x} + B(x) \rangle \\
\leq \langle x - \bar{x}, A\bar{x} + B(\bar{x}) \rangle + \langle x - \bar{x}, A(x - \bar{x}) \rangle + \langle x - \bar{x}, B(x) - B(\bar{x}) \rangle \\
\leq M \| x - \bar{x} \|^2 = M d_K^2(x).
\]
We have thus obtained (21), which in turn guarantees the invariance of \( K \). \( \Box \)

5. Examples

5.1. The unit ball in a Hilbert space. Let \( (E, \langle \cdot, \cdot \rangle) \) be a Hilbert space and let \( K \) be the closed unit ball centered at 0. Then
\[
d_K(x) = \max\{0, \|x\| - 1\} \quad (x \in E),
\]
and
\[
\Pi_K(x) = \begin{cases} 
\frac{x}{\|x\|}, & \text{if } \|x\| \geq 1 \\
x, & \text{if } \|x\| \leq 1.
\end{cases}
\]
The necessary and sufficient condition (21) becomes
\[
\frac{\|x\| - 1}{\|x\|} \langle x, Ax + B(x) \rangle \leq C(\|x\| - 1)^2, \quad \forall x \in D(A) \cap K_\delta,
\]
which is equivalent to
\[
\langle Ax + B(x), x \rangle \leq C\|x\| (\|x\| - 1) \quad \forall x \in D(A) \text{ such that } 1 + \delta > \|x\| > 1.
\]
The ”boundary” necessary and sufficient condition for the invariance (22) becomes
\[
\langle x, Ax + B(\bar{x}) \rangle \leq 0 \quad \forall x \in D(A) \text{ such that } \|x\| = 1.
\]

Example 5.1. Let us now consider the special case \( H = L^2(\Omega) \), where \( \Omega \) is a bounded domain of \( \mathbb{R}^n \) with a smooth boundary \( \partial \Omega \). Let \( A \) be given by
\[
Ax(\xi) = \Delta x(\xi), \quad \forall x \in D(A) = H^2(\Omega) \cap H_0^1(\Omega).
\]
Then the iff condition (21) reduces to
\[
\langle B(x), x \rangle \leq C\|x\| (\|x\| - 1) + \|\nabla x\|^2 \quad \forall x \in D(A) \text{ such that } 1 + \delta > \|x\| > 1,
\]
where we have denoted by \( \|\nabla x\| \) the \( L^2 \)-norm of the gradient of \( x \). The necessary and sufficient condition (26) becomes
\[
\langle B(x), x \rangle \leq \|\nabla x\|^2, \quad \forall x \in D(A) \text{ such that } \|x\| = 1.
\]
When $B$ is given by the composition operator $B(x)(\xi) = \beta(x(\xi))$, where $\beta \in C^1(\mathbb{R})$ and $\beta(0) = 0$, it is easy to see that (28) holds true if
\[ \beta'(s) \leq \lambda_0 \quad \forall s \in \mathbb{R} \]
where $\lambda_0$ is the first eigenvalue of $-A$.

5.2. Invariance of a half-space in a Hilbert space. Let $(E, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let
\[ K = \{ x \in E : \langle x, a \rangle \leq 0 \}, \]
where $a \in E$ and $\|a\| = 1$. Then we have
\[ d_K(x) = \max \{ 0, \langle x, a \rangle \} \quad (x \in E) \]
and
\[ \Pi_K(x) = \begin{cases} x - \langle x, a \rangle a, & \forall x \notin K \\ x, & \forall x \in K. \end{cases} \]
The necessary and sufficient condition (21) becomes
\[ \langle x, a \rangle \langle Ax + B(x), a \rangle \leq C \langle x, a \rangle^2, \quad \forall x \in D(A), \delta > \langle x, a \rangle > 0, \]
which is equivalent to
\[ \langle Ax + B(x), a \rangle \leq C \langle x, a \rangle, \quad \forall x \in D(A), \delta > \langle x, a \rangle > 0. \]

5.3. Further examples. In this section, we study three examples: the cone of nonnegative functions, the convex constraint associated with an obstacle problem for a parabolic equation, and a first order equation in population dynamics.

Example 5.2 (The positive cone). Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^n$. In the Hilbert space $E = L^2(\mathcal{O})$ consider the cone (with empty interior)
\[ K = \{ x \in H : x(\xi) \geq 0 \ \text{a.e.} \ \xi \in \mathcal{O} \}. \]
Set
\[ x^+(\xi) = \max \{ x(\xi), 0 \}, \quad x^-(\xi) = \max \{ -x(\xi), 0 \}, \quad \forall \xi \in \mathcal{O}. \]
Then
\[ d_K(x) = \| x^- \|, \quad \Pi_K(x) = x^+, \quad x - \Pi_K(x) = -x^-. \]
The necessary and sufficient condition (21) becomes
\[ -\langle x^-, Ax + B(x) \rangle \leq C \| x^- \|^2 \quad \forall x \in D(A) \cap K_\delta. \]
Assume that $A$ is given by (27). Then we have
\[ \langle Ax^+, x^- \rangle = -\int_\mathcal{O} \nabla x^+(\xi) \cdot \nabla x^-(\xi) d\xi = 0 \]
and (33) can be recast as follows:

\[(34) -\langle x^-, B(x) \rangle \leq C\|x^-\|^2 + \int_\partial \|\nabla x^- (\xi)\|^2 d\xi \quad \forall x \in D(A) \text{ with } 0 < \|x^-\| < \delta.\]

Suppose now \(B : E \to E\) is given by

\[(35) B(x)(\xi) = \beta(x(\xi)) \quad (\xi \in [0,1])\]

where \(\beta : \mathbb{R} \to \mathbb{R}\) is a continuous function such that \(\beta(0) = 0\) and

\[(36) (\beta(s) - \beta(r))(s - r) \leq M(s - r)^2 \quad \forall r, s \in \mathbb{R}\]

for some constant \(M \geq 0\). Then we have that

\[-\langle x^-, B(x) \rangle = -\int_\partial x^-(\xi) \beta(- x^-(\xi)) \, d\xi \leq M\|x^-\|^2.\]

So, condition (34) is satisfied with \(C = M\). Consequently, the cone of positive functions is invariant for the parabolic problem

\[(37) \begin{cases} \frac{\partial X}{\partial t} - \Delta X - \beta(X) = 0 & \text{in } (0, +\infty) \times \partial \mathcal{O} \\ X = 0 & \text{on } (0, +\infty) \times \partial \mathcal{O} \\ X(0, \xi) = x(\xi) & \xi \in \mathcal{O}. \end{cases}\]

This result also follows from the parabolic maximum principle (see, e.g., [?]).

**Example 5.3 (An obstacle problem).** Let \(E\) be the Banach space

\[E = \{ x \in C([0,1]) : x(0) = 0 = x(1) \}\]

with the uniform norm

\[\|x\|_\infty = \max_{\xi \in [0,1]} |x(\xi)| \quad \forall x \in E.\]

Let \(f \in C^2([0,1])\) be a function such that \(f(0) \leq 0\) and \(f(1) \leq 0\) and consider the closed convex set

\[K^f = \{ x \in E : x(\xi) \geq f(\xi), \forall \xi \in [0,1] \}.\]

We observe that the interior of \(K^f\) is nonempty if \(f(0), f(1) < 0\).

Define \(A : D(A) \subset E \to E\) by

\[\begin{cases} D(A) = \{ x \in E \cap C^2([0,1]) : \frac{d^2 x}{d \xi^2}(0) = 0 = \frac{d^2 x}{d \xi^2}(1) \} \\ A x(\xi) = \frac{d^2 x}{d \xi^2}(\xi) \quad \forall \xi \in [0,1] \end{cases}\]

and let \(B : E \to E\) be given by (35), where \(\beta : \mathbb{R} \to \mathbb{R}\) is a continuous function satisfying (36) such that \(\beta(0) = 0\). It is well known that \(A\) and \(B\) satisfy assumptions (H) on \(E\). So, we can use condition (14) to study the invariance of \(K^f\) for the parabolic problem (37) or, equivalently, the solvability of the obstacle problem

\[x(\xi) \geq f(\xi), \forall \xi \in [0,1] \implies X(t, \xi) \geq f(\xi), \forall (t, \xi) \in [0, \infty) \times [0,1].\]
Noting that
\[ d_{K}(x) = \| (x - f)^- \|_{\infty} \quad \forall x \in E, \]
for any \( x \in D(A) \) and \( \delta > 0 \) we have that
\[ x \in K_{\delta}^I \iff 0 < \| (x - f)^- \|_{\infty} < \delta. \]
Fix any \( x \in D(A) \cap K_{\delta}^I, y \in E, \) and let \( h_j \downarrow 0 \) be a sequence of positive numbers such that
\[ (38) \quad D^{-} d_{K}(x) y = \lim_{j \to \infty} \frac{\| (x + h_j y - f)^- \|_{\infty} - \| (x - f)^- \|_{\infty}}{h_j}. \]
Let \( \xi_j \in [0, 1] \) be such that \( \| (x + h_j y - f)^- \|_{\infty} = (x + h_j y - f)^-(\xi_j). \) Since \( (x - f)^- \neq 0, \) for \( j \) sufficiently large we have that
\[ (39) \quad \| (x + h_j y - f)^- \|_{\infty} = (x - f)^-(\xi_j) - h_j y(\xi_j) \leq \| (x - f)^- \|_{\infty} - h_j y(\xi_j). \]
By a compactness argument, we can assume that \( \xi_j \xrightarrow{j \to \infty} \xi_0 \in [0, 1]. \) Moreover, the continuity of \( x \) ensures that
\[ \xi_0 \in \arg \max_{[0, 1]} (x - f)^- = \{ \xi \in [0, 1] : (x - f)^-(\xi) = \| (x - f)^- \|_{\infty} \}. \]
Combining (38) and (39) we conclude that
\[ D^{-} d_{K}(x) y \leq -y(\xi_0) \leq \max \{ -y(\xi) : \xi \in \arg \max_{[0, 1]} (x - f)^- \}. \]
Thus, appealing to (14), we recover the following sufficient condition for the invariance of \( K: \) for all \( x \in D(A) \) such that \( 0 < \| (x - f)^- \|_{\infty} < \delta \)
\[ (40) \quad \max \left\{ -\frac{d^2 x}{d \xi^2}(\xi) - \beta(x(\xi)) : \xi \in \arg \max_{[0, 1]} (x - f)^- \right\} \leq C\| (x - f)^- \|_{\infty}. \]
Now, observe that \( \arg \max_{[0, 1]} (x - f)^- \subset (0, 1) \) because \( 0 < \| (x - f)^- \|_{\infty} \) and \( (x - f)^-(0) = 0 = (x - f)^-(1). \) Therefore,
\[ \frac{d^2 x}{d \xi^2}(\xi) \geq \frac{d^2 f}{d \xi^2}(\xi) \quad \forall \xi \in \arg \max_{[0, 1]} (x - f)^-. \]
Moreover, owing to (36)
\[ -\left( \beta(x(\xi)) - \beta(f(\xi)) \right) \leq M\| (x - f)^- \|_{\infty} \quad \forall \xi \in \arg \max_{[0, 1]} (x - f)^-. \]
Thus,
\[ -\frac{d^2 x}{d \xi^2}(\xi) - \beta(x(\xi)) \leq -\frac{d^2 f}{d \xi^2}(\xi) - \beta(f(\xi)) + M\| (x - f)^- \|_{\infty} \quad \forall \xi \in \arg \max_{[0, 1]} (x - f)^-, \]
and we conclude that a sufficient condition for (40) to be satisfied is that
\[ (41) \quad \frac{d^2 f}{d \xi^2} + \beta(f) \geq 0 \quad \text{in} \quad [0, 1]. \]
In particular, since (41) is satisfied for $f \equiv 0$, we obtain that the cone of nonnegative functions is invariant for system (37) also when studied in $C([0, 1])$.

Since (41) implies that $f$ is a subsolution of the boundary value problem in (37), the invariance result we have just proved could have also been derived by maximum principle arguments.

**Example 5.4 (An age-structured population model).** We now discuss the positivity of solutions to the age-structured population model

$$
\begin{aligned}
\frac{\partial X}{\partial t}(t,a) + \frac{\partial X}{\partial a}(t,a) + \mu(a)X(t,a) &= \phi(a, X(t,a)), \quad a \in [0, a_1], t \geq 0 \\
X(t,0) &= \int_0^{a_1} \beta(a)X(t, a) \, da, \quad t \geq 0 \\
X(0,a) &= x_0(a), \quad a \in [0, a_1].
\end{aligned}
$$

which was analysed in Lebesgue spaces in [?] (see also [?]). Here, $X(t,a)$ is the population density of age $a$ at time $t$, $\mu$ is the mortality rate, $\beta$ the birth rate, and $a_1 > 0$ is the maximal age. Moreover, $\phi$ is a real-valued continuous function modelling an exterior source, such as (inward or outward) migration. Observe that, since $\phi$ has no prescribed sign, in principle solutions might be forced to assume negative values, even for positive $x_0$.

We study the above problem in the space $E = C([0,a_1])$, where the positive cone

$$
K = \{x \in E : x(a) \geq 0, \forall a \in [0, a_1]\}
$$

has nonempty interior. Our assumptions (A) on the data are as follows:

(A1) $\mu, \beta \in K$ and

$$
\int_0^{a_1} \beta(a) \, da < 1
$$

(A2) $\phi : [0, a_1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\phi(0) = 0$ and

$$
(\phi(a, s) - \phi(a, r))(s - r) \leq M(s - r)^2 \quad \forall a \in [0, a_1], \forall r, s \in \mathbb{R}
$$

for some constant $M \geq 0$.

In order to recast problem (42) as a semilinear evolution equation in $E$, we define

$$
\begin{aligned}
D(A) &= \{x \in C^1([0, a_1]) : x(0) = \int_0^{a_1} \beta(a)x(a) \, da\} \\
Ax(a) &= -\frac{d}{da}(a) - \mu(a)x(a) \quad \forall x \in D(A), \forall a \in [0, a_1]
\end{aligned}
$$

and

$$
f(x)(a) = \phi(a, x(a)) \quad \forall x \in E, \forall a \in [0, a_1].
$$

Then, one can show that $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $E$. We omit the proof of this result that can be easily deduced arguing as in [?], but for the dissipativity of $A$. 
For this, let \( x \in D(A) \setminus \{0\} \) and let \( \bar{a} \in \arg \max_{[0,a_1]} |x| \). Observe that \( \bar{a} \neq 0 \) for otherwise the age relation satisfied by all elements of \( D(A) \) would yield

\[
\|x\|_\infty = |x(0)| = \left| \int_0^{a_1} \beta(a) x(a) \, da \right| \leq \int_0^{a_1} \beta(a) |x(a)| \, da \leq |x(0)| \int_0^{a_1} \beta(a) \, da
\]

and (43) would imply that \( x \equiv 0 \). So, taking the Dirac delta centered at \( \bar{a} \), \( \delta_{\bar{a}} \), by Example 2.1 we have that \( \mu_x := \text{sgn}(x(\bar{a})) \cdot \delta_{\bar{a}} \in \partial \|x\| \). Now, we claim that

\[
\langle \mu_x, Ax \rangle = -\text{sgn}(x(\bar{a})) \frac{dx}{da}(\bar{a}) - \mu(\bar{a}) |x(\bar{a})| \leq 0.
\]

Indeed, since \( \bar{a} \neq 0 \), it must be either \( \bar{a} \in (0,a_1) \), hence \( \frac{dx}{da}(\bar{a}) = 0 \), or \( \bar{a} = a_1 \) and

\[
\text{sgn}(x(\bar{a})) \frac{dx}{da}(\bar{a}) \geq 0.
\]

Thus, we can use condition (14) to study the invariance of \( K \). Since

\[
d_K(x) = \|x^-\|_\infty \quad \forall x \in E,
\]

arguing as in Example 5.3 we conclude a sufficient condition for the invariance of \( K \) is that, for some \( \rho > 0 \) and all \( x \in D(A) \) such that \( 0 < \|x^-\|_\infty < \rho \),

\[
\max \left\{ \frac{dx}{da}(a) + \mu(a)x(a) - \phi(a,x(a)) : a \in \arg \max_{[0,a_1]} x \right\} \leq C\|x^-\|_\infty.
\]

The above condition holds true because, as above, \( 0 \notin \arg \max_{[0,a_1]} x^- \) and so

\[
\frac{dx}{da}(a) + \mu(a)x(a) \leq 0 \quad \forall a \in \arg \max_{[0,a_1]} x^-,
\]

while

\[
-\phi(a,x(a)) \leq M\|x^-\|_\infty \quad \forall a \in \arg \max_{[0,a_1]} x^-\]

in view of (44). Therefore, \( K \) is invariant for problem (42).

References

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