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To cite this version:

P Cannarsa, Hélène Frankowska. VALUE FUNCTION, RELAXATION, AND TRANSVERSALITY CONDITIONS IN INFINITE HORIZON OPTIMAL CONTROL*. Journal of Mathematical Analysis and Applications, 2018, 457 (2), pp.1188-1217. hal-02126097

HAL Id: hal-02126097

https://hal.science/hal-02126097

Submitted on 10 May 2019

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VALUE FUNCTION, RELAXATION, AND TRANSVERSALITY CONDITIONS IN INFINITE HORIZON OPTIMAL CONTROL*

P. CANNARSA AND H. FRANKOWSKA

Abstract. We investigate the value function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ of the infinite horizon problem in optimal control for a general—not necessarily discounted—running cost and provide sufficient conditions for its lower semicontinuity, continuity, and local Lipschitz regularity. Then we use the continuity of $V(t,\cdot)$ to prove a relaxation theorem and to write the first order necessary optimality conditions in the form of a, possibly abnormal, maximum principle whose transversality condition uses limiting/horizontal supergradients of $V(0,\cdot)$ at the initial point. When $V(0,\cdot)$ is merely lower semicontinuous, then for a dense subset of initial conditions we obtain a normal maximum principle augmented by sensitivity relations involving the Fréchet subdifferentials of $V(t,\cdot)$. Finally, when V is locally Lipschitz, we prove a normal maximum principle together with sensitivity relations involving generalized gradients of V for arbitrary initial conditions. Such relations simplify drastically the investigation of the limiting behaviour at infinity of the adjoint state.

Keywords. Infinite horizon problem, value function, relaxation theorem, sensitivity relation, maximum principle.

1. Introduction

In some models of mathematical economics one encounters the following infinite horizon optimal control problem

$$W(x_0) = \inf \int_0^\infty e^{-\lambda t} \ell(x(t), u(t))\, dt$$

over all trajectory-control pairs (x, u), subject to the state equation

$$\begin{cases} x'(t) = f(x(t), u(t)), \quad u(t) \in U \quad \text{for a.e. } t \geq 0 \\ x(0) = x_0, \end{cases}$$

where controls $u(\cdot)$ are Lebesgue measurable functions and $\lambda > 0$. Its history goes back to Ramsey [18]. The term $e^{-\lambda t}$ is sometimes called a discount factor. The literature addressing this problem deals with traditional questions of existence of optimal solutions, regularity of W, necessary and sufficient optimality conditions. Usually assumptions are imposed to ensure the local Lipschitz continuity of W.

The question of necessary conditions is quite challenging, because unlike for classical finite horizon problems, transversality conditions are not immediate. Indeed, let $(\bar{x}, \bar{u})$ be a given optimal trajectory-control pair. It is well known that if ∞ in the above problem is replaced by some $T > t_0$, that is, the infinite horizon problem is reduced to the Bolza one

$$\text{minimize} \int_0^T e^{-\lambda t} \ell(x(t), u(t))\, dt \tag{1.1}$$

Date: March 9, 2017.

*Partially Supported by Bando Doppie Cattedre, U. Roma Tor Vergata.

over all trajectory-control pairs (x, u) of the system

$$\begin{cases} x'(t) = f(x(t), u(t)), \quad u(t) \in U \quad \text{for a.e. } t \in [0, T] \\ x(0) = x_0, \end{cases}$$

then the restriction of $(\bar{x}, \bar{u})$ to the time interval $[0, T]$ may loose optimality. For the Bolza problem, however, a necessary optimality condition is known and takes the form of the maximum principle: if $(\bar{x}, \bar{u})$ is an optimal trajectory-control pair for the above Bolza problem, then the solution $p_T := p$ of the adjoint system

$$-p'(t) = D_x f(\bar{x}(t), \bar{u}(t))^* p(t) - e^{-\lambda t} \ell_x(\bar{x}(t), \bar{u}(t)) \qquad \text{for a.e. } t \in [0, T], \quad p(T) = 0$$

satisfies the maximality condition

$$\langle p(t), f(\bar{x}(t), \bar{u}(t))\rangle - e^{-\lambda t}\ell(\bar{x}(t), \bar{u}(t)) = \max_{u \in U}(\langle p(t), f(\bar{x}(t), u)\rangle - e^{-\lambda t}\ell(\bar{x}(t), u)) \text{ a.e.} \tag{1.2}$$

We underline that the transversality condition $p(T) = 0$ is due the fact that there is no cost term depending on $x(T)$ in (1.1). If for any $i \geq 1$, the restriction of $(\bar{x}, \bar{u})$ to $[0, i]$, is optimal for the corresponding Bolza problem, then one is led to extract, whenever possible, a subsequence of $\{p_i\}$ that converges almost uniformly on $[0, +\infty[$ to a solution of the adjoint system

$$-p'(t) = D_x f(\bar{x}(t), \bar{u}(t))^* p(t) - e^{-\lambda t} \ell_x(\bar{x}(t), \bar{u}(t)) \qquad \text{for a.e. } t > 0, \tag{1.3}$$

satisfying the maximality condition (1.2) a.e. in $[0, \infty[$. This $p(\cdot)$, called sometimes a co-state, allows to test candidates for optimality. Though the transversality condition does disappear in this approach, some additional assumption on f, ℓ, λ alow to conclude that $\lim_{t\to+\infty} p(t) = 0$. In a way, zero may appear to be a candidate for the transversality condition at infinity.

To make the above rigourous, one can add the end point constraint $x(T) = \bar{x}(T)$. With such additional constraint the restriction of $(\bar{x}, \bar{u})$ to the time interval $[0, T]$, becomes optimal for the above Bolza problem. This results, however, in possibly abnormal maximum principles for finite horizon problems, and, *in fine*, leads to necessary optimality conditions not involving the cost function in (1.3), (1.2). Also in this approach the transversality condition at time T does disappear, becoming $-p(T) \in N_{\{\bar{x}(T)\}}(\bar{x}(T)) = \mathbb{R}^n$ (normal cone to $\{\bar{x}(T)\}$ at $\bar{x}(T)$).

Halkin was the first to observe in [15] that it may happen that the co-state is different from zero at infinity, i.e. $\lim_{t\to+\infty} p(t) \neq 0$ and that the maximum principles may be all abnormal. In his work, however he has changed the notion of optimal solution to a weaker one, to overcome the fact that restrictions of optimal solutions may become non-optimal, cf. [15, p. 269]. For a different modification of definition of optimal solution see also [22]. We refer to [1] for an extended overview of the literature devoted to transversality conditions, examples and counterexamples, and for important bibliographical comments and also to [2] for a further discussion.

Since then a major effort was made by many authors to get a normal maximum principle for the infinite horizon problem involving some transversality conditions. Also much more general, nonautonomous, infinite horizon problem, not necessarily containing the discount factor $e^{-\lambda t}$, started to be considered. In such case, W may become discontinuous and may take infinite values even when data are smooth and have sublinear growth.

Several ways were proposed in the literature to approach the infinite horizon problem. Let us mention just some of them.

1. Modifying the notion of (strong) optimal solution for the infinite horizon problems by replacing it by a family of finite horizon problems. This modification is done in such a way that a maximum principle can be associated to the restriction of $(\bar{x}, \bar{u})$ to the finite time interval $[0, T]$, Then, taking a limit in the obtained necessary optimality conditions when T increases to $+\infty$, leads to an adjoint system and a maximality condition on $[0, +\infty[$ but, in general, does not provide transversality conditions, cf. [15, 10].

2. Introducing the concept of locally weakly overtaking optimal (LWOO) solution for the infinite horizon problems and defining a co-state $p(\cdot)$ by integrating the adjoint system on $[t, \infty[$. Then, under mild assumptions, LWOO solutions satisfy the maximality condition for this particular $p(\cdot)$, cf. [2, 3]. There is no transversality condition in this approach.

3. Replacing the infinite horizon problem containing a discount factor $e^{-\lambda t}$ by a family of finite horizon problems on increasing time intervals involving additional state variables and cost functions in such a way that restrictions of $(\bar{x}, \bar{u})$ are locally optimal for these new problems. Then, writing the necessary conditions in the form of a normal maximum principle for these new problems with the transversality conditions specific to them and taking the limit, leads to a co-state $p(\cdot)$ defined on $[0, +\infty[$, cf. [16, 19, 23]. In addition, in this approach one gets an exact expression for the Hamiltonian of the auxiliary problem along any optimal trajectory-control pair and co-state. The presence of the discount factor allows to conclude that, under some boundedness assumptions, this Hamiltonian (along any optimal trajectory-control pair and co-state) and the co-state $p(\cdot)$ vanish at infinity. This approach is based on dynamic programming and uses the value function of the infinite horizon optimal control problem. Actually in [16] the value function is supposed to be C^1 to get these conclusions, while in [23] it is Lipschitz continuous.

4. Replacing the infinite horizon problem containing a discount factor $e^{-\lambda t}$ by a family of finite horizon problems on increasing time intervals involving a penalty term in such a way that optimal controls of finite horizon problems converge to $\bar{u}$. Then, using the finite horizon maximum principles for penalized problems, to get a co-state of the infinite horizon problem by taking appropriate limits, cf. [1]. Since for each finite horizon problem the co-state vanishes at the final time, some additional growth assumptions on f and ℓ allow to deduce that the co-state of the infinite horizon problem is equal to zero at infinity.

5. Developing the duality theory on weighted Sobolev spaces $L^p(0, +\infty; \mathbb{R}^n)$ with respect to the measure $e^{-\lambda t}dt$ (or more general measures) [4, 17, 20].

6. Using the sensitivity relations for the finite horizon problems that allow to write a transversality condition at the initial time instead of infinite time [4, 16, 19, 23]. This was done up to now for problems with the discount factor and for Lipschitz value functions.

Note that the transversality condition at infinity is a consequence of the adjoint equation and assumptions on data. This differs substantially from the finite horizon settings, where transversality conditions at final time is an independent requirement.

In the present work we consider a very general framework and use sensitivity relations for the Bolza problem obtained recently in [8, 6] as additional necessary optimality conditions. We neither change the notion of optimal solution, nor introduce a new cost function, but rely entirely on the dynamic programming principle. This allows us to state the maximum principle for the infinite horizon problem with a transversality condition at the initial time and also to deduce the behavior of the co-state p at infinity. We would like to underline here that introducing the value function into transversality conditions is an additional property of the co-state, usually absent in the maximum principles. Adding such sensitivity relations may even give necessary and sufficient optimality conditions for the finite horizon problems, cf. [7]. On the other hand, as we show below, they also allow to deduce the behavior of the co-state at infinity in a quite straightforward way. For this reason our approach simplifies the earlier investigations of the infinite horizon maximum principles. At the same time it does not apply to local minimizers.

The previous works [4, 19, 23] involving the value function in transversality conditions have addressed problems with a discount factor and locally Lipschitz value functions, while we are able to state some result also when W is merely lower semicontinuous or continuous and there

is no discount factor. More precisely, we consider the nonautonomous optimal control problem

$$V(t_0, x_0) = \inf \int_{t_0}^{\infty} L(t, x(t), u(t))\, dt \tag{1.4}$$

over all trajectory-control pairs (x, u), subject to the state equation

$$\begin{cases} x'(t) = f(t, x(t), u(t)), \quad u(t) \in U(t) \quad \text{for a.e. } t \in \mathbb{R}_+ \\ x(t_0) = x_0, \end{cases} \tag{1.5}$$

where $\mathbb{R}_+ = [0, +\infty[$, $t_0 \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$, $U : \mathbb{R}_+ \rightsquigarrow \mathbb{R}^m$ is a measurable set-valued map with closed images. Selections $u(t) \in U(t)$ are supposed to be measurable and $L : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$, $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are given mappings.

Remark 1.1. We assume that L takes nonnegative values for the sake of simplicity, but our approach applies also to L bounded from below by an integrable function, that is when $L(t, x, u) \geq \alpha(t)$ for a.e. $t \geq 0$ and all x, u, where $\alpha : \mathbb{R}_+ \to \mathbb{R}$ is integrable on $[0, +\infty)$. Thus the value function takes values in $(-\infty, +\infty]$. Without the bound from the below it may also take the value $-\infty$. It would be interesting to see how much our approach can be extended also to this more general case.

The above setting subsumes the classical infinite horizon optimal control problem when f and U are time independent, $L(t, x, u) = e^{-\lambda t}\ell(x, u)$ for some mapping $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ and $t_0 = 0$. Furthermore, in this particular case, for all $t \geq 0$,

$$V(t, x_0) = e^{-\lambda t} W(x_0).$$

Clearly if $(\bar{x}, \bar{u})$ is optimal at (t_0, x_0) then for every $T > t_0$,

$$V(t_0, x_0) = V(T, \bar{x}(T)) + \int_{t_0}^{T} L(t, \bar{x}(t), \bar{u}(t))\, dt.$$

Introducing $g_T(y) := V(T, y)$ we get the Bolza type problem

$$\text{minimize } g_T(x(T)) + \int_{t_0}^{T} L(t, x(t), u(t))\, dt$$

over all trajectory-control pairs (x, u), subject to the state equation

$$\begin{cases} x'(t) = f(t, x(t), u(t)), \quad u(t) \in U(t) \quad \text{for a.e. } t \in [t_0, T] \\ x(t_0) = x_0. \end{cases}$$

For this new problem, the restriction of any optimal trajectory control-pair $(\bar{x}, \bar{u})$ to $[t_0, T]$ is optimal. One finds then in the literature the corresponding necessary optimality conditions and Hamilton-Jacobi equations even if g_T is merely lower semicontinuous. Furthermore, when $V(t_0, \cdot)$ is locally Lipschitz, the sensitivity relation

$$-p(t_0) \in \partial_x V(t_0, x_0)$$

obtained in [12] and generalized in [6] can be considered as a transversality condition at the initial state (here $\partial_x V(t_0, x_0)$ denotes the generalized gradient of $V(t_0, \cdot)$ at x_0). In the present work we also obtain some results on the behavior of $p(\cdot)$ at infinity by considering similar sensitivity relations along optimal trajectories on the whole half-line.

In the difference with the well investigated finite horizon problem, i.e. when ∞ in (1.4) is replaced by a real $T > t_0$, even when f, L, U are time independent and bounded and f, L are Lipschitz, it may happen that V is merely lower semicontinuous and takes infinite values. In this paper we also provide sufficient conditions for the continuity of V with respect to the state variable x that we exploit to prove a relaxation theorem for the infinite horizon problem.

Furthermore, we give sufficient conditions for the local Lipschitz continuity of V which allows the investigation of some sensitivity relations involving generalized gradients of the value function.

Under mild assumptions every optimal solution satisfies a, possibly abnormal, maximum principle, not involving transversality conditions, cf. Theorem 4.2 below. To investigate normality as well as transversality conditions, our main results exploit the value function. For instance, the maximum principle takes a normal form whenever the problem is calm with respect to the state variable at (t_0, x_0), cf. Theorem 5.4 below (the notion of calmness is recalled in Section 5 and uses the value function). Furthermore, for every upper semicontinuous function Φ "supporting" locally $V(t_0, \cdot)$ at x_0 a, possibly abnormal, maximum principle can be stated with the transversality conditions involving limiting (normal case) or horizontal limiting (abnormal case) supegradients of Φ at x_0, cf. Remark 5.2. In particular, if $V(t_0, \cdot)$ is continuous around x_0, then we get a maximum principle with transversality conditions of this type applied to $V(t_0, \cdot)$, cf. Theorem 5.1. Moreover, if f is differentiable with respect to x, then any element p_0 of the Fréchet subdifferential of $V(t_0, \cdot)$ at x_0 can be used to state a normal maximum principle with the transversality condition $p(t_0) = -p_0$, augmented by a sensitivity relation, cf. Theorem 4.1. In particular, if $V(t_0, \cdot)$ is locally Lipschitz around x_0, the maximum principle with the transversality condition involving the limiting superdifferential of $V(t_0, \cdot)$ at x_0 is normal. Furthermore, in this case additional sensitivity relations using the generalized gradients of $V(t, \cdot)$ hold true along any optimal trajectory, cf. Theorems 6.1 and 6.2.

The outline of the paper is as follows. Section 2 is devoted to some preliminaries and notations. In Section 3 we provide sufficient conditions for V to be lower semicontinuous, continuous or locally Lipschitz and discuss its relation with a finite horizon Bolza problem. We also give there a relaxation theorem. In Section 4 we state a maximum principle and a sensitivity relation involving the Fréchet subdifferential of the lower semicontinuous value function. Section 5 deals with continuous (around x_0) mapping $V(t_0, \cdot)$, where we obtain a maximum principle with a transversality condition involving the limiting superdifferential of $V(t_0, \cdot)$ at x_0. Finally Section 6 discusses maximum principles, sensitivity relations and transversality conditions at infinity for locally Lipschitz value functions.

2. Some Elements of Set-Valued and Non-Smooth Analysis

For a finite dimensional space X, denote by $B(x, R)$ the closed ball in X centered at $x \in X$ with radius $R > 0$. Let $K \subset \mathbb{R}^n$ and $x \in K$. The contingent cone to K at x consists of all vectors $v \in \mathbb{R}^n$ such that there exist sequences $h_i \to 0+,\ v_i \to v$ satisfying $x + h_i v_i \in K$. The limiting normal cone to a closed subset $K \subset \mathbb{R}^n$ at $x \in K$ is defined as the Peano-Kuratowski upper limit

$$N_K^L(x) = \mathrm{Limsup}_{y \to_K x} T_K(y)^-,$$

where $\to_K$ stands for the convergence in K and $T_K(y)^-$ is the negative polar of $T_K(y)$. It is well known that if x lies on the boundary of K, then $N_K^L(x)$ is not reduced to zero.

For $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ denote by $dom(\varphi)$ the domain of φ, that is the set of all $x \in \mathbb{R}^n$ such that $\varphi(x)$ is finite and by $epi(\varphi)$ and $hyp(\varphi)$ respectively its epigraph and hypograph. For any $x \in dom(\varphi)$ the lower directional derivative of φ at x in the direction $y \in \mathbb{R}^n$ is defined by

$$D_\uparrow \varphi(x) y = \liminf_{z \to y, h \to 0+} \frac{\varphi(x + hz) - \varphi(x)}{h}$$

and the Fréchet subdifferential of φ at x by

$$\partial^- \varphi(x) = \{p \mid \langle p, y \rangle \leq D_\uparrow \varphi(x) y,\ \forall\, y \in \mathbb{R}^n\}.$$

It is well known that $p \in \partial^- \varphi(x)$ if and only if $(p, -1) \in T_{epi(\varphi)}(x, \varphi(x))^-$ or, equivalently, if and only if

$$\liminf_{y \to x} \frac{\varphi(y) - \varphi(x) - \langle p, y - x \rangle}{|y - x|} \geq 0.$$

By the same arguments as those of [13, Proof of Proposition 1.1] we obtain the following result.

Proposition 2.1. *Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be Lebesgue measurable. For any $x \in dom(\varphi)$, a vector $p \in \partial^- \varphi(x)$ if and only if there exists a continuous mapping $\psi : \mathbb{R}^n \to \mathbb{R}$ such that $\psi(x) = \varphi(x)$, $\psi(y) < \varphi(y)$ for all $y \neq x$ and the Fréchet derivative of ψ at x is equal to p.*

Actually in [13, Proposition 1.1] φ is continuous and ψ is C^1. However for our purposes it is sufficient to have ψ continuous and differentiable only at x.

For all $p \in \mathbb{R}^n$ and $q \in \mathbb{R}$ satisfying $(p, q) \in N^L_{epi(\varphi)}(x, \varphi(x))$ we have $q \leq 0$. Furthermore, if $q < 0$, then $(p, q) \in N^L_{hyp(\varphi)}(x, \varphi(x))$ if an only if $(p/|q|, -1) \in N^L_{epi(\varphi)}(x, \varphi(x))$. Any $p \in \mathbb{R}^n$ satisfying $(p, -1) \in N^L_{epi(\varphi)}(x, \varphi(x))$ is called a limiting subgradient of φ at x, while if $(p, 0) \in N^L_{epi(\varphi)}(x, \varphi(x))$, then p is called a limiting horizontal subrgradient of φ at x. The sets of all limiting and limiting horizontal subgradients of φ at x are denoted by $\partial^{L,-}\varphi(x)$ and $\partial^{\infty,-}\varphi(x)$ respectively.

The limiting supergradients are defined in a somewhat similar way, but changing signs: every $p \in \mathbb{R}^n$ satisfying $(-p, +1) \in N^L_{hyp(\varphi)}(x, \varphi(x))$ is called a limiting supergradient of φ at x. If instead $(-p, 0) \in N^L_{hyp(\varphi)}(x, \varphi(x))$, then p is called a limiting horizontal supergradient of φ at x. The sets of all limiting and limiting horizontal supergradients of φ at x are denoted by $\partial^{L,+}\varphi(x)$ and $\partial^{\infty,+}\varphi(x)$ respectively.

Observe that if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable at x, then its sets of limiting sub and supergradients are reduced to the singleton $\{\nabla\varphi(x)\}$. This is the reason why it is natural to inverse the sign in the definition of the limiting supergradients. If φ is locally Lipschitz at x, then the sets of limiting horizontal subgradients and supergradients are empty and the set $co\, \partial^{L,-}\varphi(x)$ is the generalized gradient of φ at x, denoted by $\partial\varphi(x)$, where *co* stands for the convex hull. It is well known that for a locally Lipschitz φ,

$$\partial\varphi(x) = co\, \partial^{L,-}\varphi(x) = co\, \partial^{L,+}\varphi(x).$$

Finally, for any a, $b \in \mathbb{R}$, set $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

3. Value Function of the Infinite Horizon Problem

Consider the nonautonomous infinite horizon optimal control problem (1.4), (1.5) with data as described in the introduction. Every Lebesgue measurable $u : \mathbb{R}_+ \to \mathbb{R}^m$ satisfying $u(t) \in U(t)$ a.e. is called a control and the set of all controls is denoted by $\mathcal{U}$. Note that to state (1.4) we need only controls defined on $[t_0, +\infty[$. However, since throughout the paper the time interval varies, in order not to bother the reader with additional notations and without any loss of generality, we suppose that controls are defined on $[0, +\infty[$.

From now on and throughout the whole paper the following assumptions are imposed.

Assumptions **(H1)***:*

i) *There exist locally integrable functions $c, \theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that for a.e. $t \in \mathbb{R}_+$,*

$$|f(t, x, u)| \leq c(t)|x| + \theta(t), \quad \forall\, x \in \mathbb{R}^n,\ u \in U(t);$$

ii) *For every $R > 0$, there exists a locally integrable function $c_R : \mathbb{R}_+ \to \mathbb{R}_+$ and a modulus of continuity $\omega_R : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for a.e. $t \in \mathbb{R}_+$, $\omega_R(t, \cdot)$ is nondecreasing, $\lim_{r\to 0+} \omega_R(t, r) = 0$ and for all $x, y \in B(0, R)$, $u \in U(t)$,*

$$|f(t, x, u) - f(t, y, u)| \le c_R(t)|x - y|, \quad |L(t, x, u) - L(t, y, u)| \le \omega_R(t, |x - y|);$$

iii) *For all $x \in \mathbb{R}^n$, the mappings $f(\cdot, x, \cdot)$, $L(\cdot, x, \cdot)$ are Lebesgue-Borel measurable ;*

iv) *There exists a locally integrable function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ and a locally bounded nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for a.e. $t \in \mathbb{R}_+$,*

$$L(t, x, u) \le \beta(t)\phi(|x|), \quad \forall\, x \in \mathbb{R}^n,\ u \in U(t);$$

v) *$U(\cdot)$ is Lebesgue measurable and has closed nonempty images;*

vi) *For a.e. $t \in \mathbb{R}_+$, and for all $x \in \mathbb{R}^n$ the set*

$$F(t, x) := \big\{\big(f(t, x, u), L(t, x, u) + r\big) : u \in U(t) \text{ and } r \ge 0\big\}$$

is closed and convex.

The above assumptions imply, in particular, that to every control $u(\cdot)$ and $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ corresponds a solution $x(\cdot)$ of the system in (1.5) defined on $\mathbb{R}_+$. This solution is a locally absolutely continuous function and (x, u) is called a trajectory-control pair. When we need to specify, we denote by $x(\cdot; t_0, x_0, u)$ the trajectory of our control system corresponding to the control u and satisfying $x(t_0) = x_0$. By a standard application of Gronwall's lemma, from assumption i) above it follows that, for all $(t_0, x_0) \in [0, +\infty[\times\mathbb{R}^n$,

$$|x(t; t_0, x_0, u)| \le \Big(|x_0| + \int_{t_0}^{t} \theta(s)\, ds\Big) e^{\int_{t_0}^{t} c(s)\, ds} \quad \forall\, t \ge t_0. \tag{3.1}$$

Moreover, setting

$$M_t(T, R) = \Big(R + \int_{t}^{T} \theta(s)\, ds\Big) e^{\int_{t}^{T} c(s)\, ds} \quad \forall\, T \ge t \ge 0\,,\ R \ge 0, \tag{3.2}$$

we have that, for all $R \ge 0$,

$$|x_0| \le R \implies |x(t; t_0, x_0, u)| \le M_{t_0}(t, R) \quad \forall\, t \ge t_0. \tag{3.3}$$

The above bound, together with assumption ii) and the Gronwall lemma, yields the local Lipschitz dependence of trajectories on the initial conditions: for all R, $T > 0$, all $t_0 \in [0, T]$, and all x_0, $x_1 \in B(0, R)$ we have that

$$|x(t; t_0, x_1, u) - x(t; t_0, x_0, u)| \le |x_1 - x_0|\, e^{\int_{t_0}^{t} c_{M_{t_0}(T,R)}(s)\, ds} \qquad \forall\, t \in [t_0, T]. \tag{3.4}$$

Given a trajectory-control pair (x, u), set

$$\int_{t_0}^{\infty} L(t, x(t), u(t))\, dt = \lim_{T\to\infty} \int_{t_0}^{T} L(t, x(t), u(t))\, dt.$$

The extended function $V : [0, +\infty[\times\mathbb{R}^n \to \mathbb{R}_+\cup\{+\infty\}$ defined by (1.4), (1.5) is called the value function of the infinite horizon problem. For any $t_0 \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$ such that $V(t_0, x_0) < +\infty$, a trajectory-control pair $(\bar{x}, \bar{u})$ is called optimal for the infinite horizon problem at (t_0, x_0) if for every trajectory-control pair (x, u) satisfying $x(t_0) = x_0$ we have

$$\int_{t_0}^{\infty} L(t, \bar{x}(t), \bar{u}(t))\, dt \le \int_{t_0}^{\infty} L(t, x(t), u(t))\, dt.$$

The proof of the following Proposition is standard and is provided in the appendix for the sake of completeness.

Proposition 3.1. *Assume (H1). Then V is lower semicontinuous and for every $(t_0, x_0) \in dom(V)$, there exists a trajectory-control pair $(\bar{x}, \bar{u})$ satisfying $V(t_0, x_0) = \int_{t_0}^{\infty} L(t, \bar{x}(t), \bar{u}(t))\, dt$.*

Consider the relaxed infinite horizon problem

$$V^{rel}(t_0, x_0) = \inf \int_{t_0}^{\infty} \sum_{i=0}^{n} \lambda_i(t) L(t, x(t), u_i(t))\, dt \tag{3.5}$$

over all trajectory-control pairs of

$$\begin{cases} x'(t) = \sum_{i=0}^{n} \lambda_i(t) f(t, x(t), u_i(t)),\ u_i(t) \in U(t),\ \lambda_i(t) \geq 0,\ \sum_{i=0}^{n} \lambda_i(t) = 1 \\ x(t_0) = x_0, \end{cases} \tag{3.6}$$

where $u_i(\cdot),\ \lambda_i(\cdot)$ are Lebesgue measurable on $\mathbb{R}_+$ for $i = 0, ..., n$. Clearly $V^{rel} \leq V$.

Theorem 3.2. *Assume (H1) i)-v) with $\omega_R(t, r) = \bar{c}_R(t) r$, for a locally integrable $\bar{c}_R : \mathbb{R}_+ \to \mathbb{R}_+$, and that, for a.e. $t \in \mathbb{R}_+$ and all $x \in \mathbb{R}^n$, the set*

$$\big\{ \big(f(t, x, u), L(t, x, u) \big) : u \in U(t) \big\}$$

is compact. If for every $t \geq 0$, $V^{rel}(t, \cdot) : \mathbb{R}^n \to \mathbb{R}$ is continuous, then $V^{rel} = V$ on $\mathbb{R}_+ \times \mathbb{R}^n$. In particular, if a trajectory-control pair $(\bar{x}, \bar{u})$ is optimal for (1.4), (1.5), then it is also optimal for the relaxed problem (3.5)-(3.6).

Remark 3.3. a) Notice that if $U(t)$ is compact for a.e. $t \geq 0$ and f, L are continuous with respect to u, then the above compactness assumption holds true.

b) Sufficient conditions for continuity of $V^{rel}(t, \cdot)$ are given in Theorem 3.5 below.

c) Theorem 3.2 allows to avoid convexity requirement in assumption (H1) *vi*).

Proof. For $v = (u_0, ..., u_n),\ \Lambda = (\lambda_0, ..., \lambda_n)$ define

$$\hat{f}(t, x, v, \Lambda) = \sum_{i=0}^{n} \lambda_i f(t, x, u_i), \quad \hat{L}(t, x, v, \Lambda) = \sum_{i=0}^{n} \lambda_i L(t, x, u_i).$$

Thus the relaxed problem is of type (1.4)-(1.5) with $f,\ L$ replaced by $\hat{f},\ \hat{L}$ and $U(t)$ replaced by

$$\underbrace{U(t) \times ... \times U(t)}_{n+1} \times \{(\lambda_0, ..., \lambda_n) \,|\, \lambda_i \geq 0\ \forall\, i,\ \Sigma_{i=0}^{n} \lambda_i = 1\}. \tag{3.7}$$

Let $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$. If $V^{rel}(t_0, x_0) = \infty$, then $V(t_0, x_0) = \infty$. Assume next that $V^{rel}(t_0, x_0) < \infty$. By Proposition 3.1 applied to $\hat{f}$, $\hat{L}$ and the control map (3.7), there exists $(\bar{x}(\cdot), \bar{v}(\cdot), \bar{\Lambda}(\cdot))$ satisfying (3.6) such that

$$V^{rel}(t_0, x_0) = \int_{t_0}^{\infty} \hat{L}(t, \bar{x}(t), \bar{v}(t), \bar{\Lambda}(t))\, dt.$$

Fix $\varepsilon > 0$ and set $x^0(t_0) = x_0,\ u^0(t_0) = \bar{u}(t_0)$. By the dynamic programming principle,

$$V^{rel}(t_0, x_0) = V^{rel}(t_0 + 1, \bar{x}(t_0 + 1)) + \int_{t_0}^{t_0+1} \hat{L}(t, \bar{x}(t), \bar{v}(t), \Lambda(t)) dt.$$

By the relaxation theorem in the finite horizon context, for every $\delta > 0$ there exists a measurable selection $u^1(t) \in U(t)$ for $t \in]t_0, t_0 + 1]$ such that the solution x^1 of the system

$$x' = f(t, x, u^1(t)),\ x^1(t_0) = x_0$$

satisfies

$$|\bar{x}(t_0+1)-x^1(t_0+1)|<\delta, \quad \left|\int_{t_0}^{t_0+1}\hat{L}(t,\bar{x}(t),\bar{v}(t),\bar{\Lambda}(t))dt-\int_{t_0}^{t_0+1}L(t,x^1(t),u^1(t))dt\right|<\delta.$$

Taking $\delta\in(0,\varepsilon/4)$ sufficiently small and using continuity of $V^{rel}(t_0+1,\cdot)$ we may assume that (x^1,u^1) is so that

$$V^{rel}(t_0+1,\bar{x}(t_0+1))\geq V^{rel}(t_0+1,x^1(t_0+1))-\frac{\varepsilon}{4}.$$

We proceed by induction. Assume that for some integer $k\geq 1$ and every $j\leq k$ we have constructed a trajectory-control pair (x^j,u^j) on the time interval $[t_0,t_0+j]$ satisfying for all $1\leq j\leq k$ the following two conditions,

(i) the restriction of (x^j,u^j) to $[t_0,t_0+j-1]$ is equal to (x^{j-1},u^{j-1}) ;

(ii) $V^{rel}(t_0+j-1,x^{j-1}(t_0+j-1))\geq V^{rel}(t_0+j,x^j(t_0+j))+\int_{t_0+j-1}^{t_0+j}L(t,x^j(t),u^j(t))dt-\frac{\varepsilon}{2^j}$.

Then

$$V^{rel}(t_0,x_0)\geq V^{rel}(t_0+k,x^k(t_0+k))+\int_{t_0}^{t_0+k}L(t,x^j(t),u^j(t))dt-\varepsilon(\frac{1}{2}+...+\frac{1}{2^k}).$$

We extend (x^k,u^k) on the time interval $[t_0+k,t_0+k+1]$ by applying a relaxation theorem. Indeed, by the same arguments as above, there exists a trajectory-control pair (x,u) of

$$x'(t)=f(t,x(t),u(t)) \quad \text{a.e. } t\in]t_0+k,t_0+k+1]$$

such that $x^{k+1}(t_0+k)=x^k(t_0+k)$ and

$$V^{rel}(t_0+k,x^k(t_0+k))\geq V^{rel}(t_0+k+1,x(t_0+k+1))+\int_{t_0+k}^{t_0+k+1}L(t,x(t),u(t))dt-\frac{\varepsilon}{2^{k+1}}.$$

Set $(x^{k+1},u^{k+1})=(x^k,u^k)$ on $[t_0,t_0+k]$ and $(x^{k+1},u^{k+1})=(x,u)$ on $]t_0+k,t_0+k+1]$.

In this way we obtain the sequence (x^j,u^j) satisfying (i), (ii) for all integer $j\geq 1$. Define a trajectory-control pair (x,u) of (1.5) by setting $(x(t),u(t))=(x^j(t),u^j(t))$ if $t\in[t_0+j-1,t_0+j]$. Since $V^{rel}\geq 0$ we deduce that for every $j\geq 1$,

$$V^{rel}(t_0,x_0)\geq\int_{t_0}^{t_0+j}L(t,x(t),u(t))dt-\varepsilon.$$

Finally, because the sequence of integrals $\int_{t_0}^{t_0+j}L(t,x(t),u(t))dt,\ j=1,...$ is increasing and bounded, it converges to $\int_{t_0}^{\infty}L(t,x(t),u(t))dt$. Consequently, by the last inequality,

$$V^{rel}(t_0,x_0)\geq V(t_0,x_0)-\varepsilon.$$

This yields $V^{rel}\geq V$ and ends the proof. □

We indicate next a simple link between the infinite horizon problem and the Bolza problem. Let $t_0\geq 0$ and $T>t_0$. Define $g(x)=V(T,x)$ for all $x\in\mathbb{R}^n$ and consider the Bolza problem

$$\text{minimize}\, g(x(T))+\int_{t_0}^{T}L(t,x(t),u(t))\,dt$$

over all trajectory-control pairs (x,u) of the system

$$\begin{cases} x'(t)=f(t,x(t),u(t)), & u(t)\in U(t) \text{ for a.e. } t\in[t_0,T] \\ x(t_0)=x_0. \end{cases}$$

If (H1) holds true, then, by Proposition 3.1, $V(T,\cdot)$ is lower semicontinuous and, by the well known existence theorems, for every $x_0 \in \mathbb{R}^n$ satisfying $V(t_0,x_0) < +\infty$, the above Bolza problem has an optimal solution. Define its value function $V^B : [t_0,T]\times\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ by

$$V^B(s_0,y_0) = \inf\{g(x(T)) + \int_{s_0}^T L(t,x(t;s_0,y_0,u),u(t))\,dt \mid \ u\in\mathcal{U}\}.$$

Proposition 3.4. *Under assumption (H1), $V^B(s_0,y_0) = V(s_0,y_0)$ for all $s_0\in[t_0,T]$, $y_0\in\mathbb{R}^n$. Furthermore, if $(\bar x,\bar u)$ is optimal for the infinite horizon problem at (t_0,x_0) then the restriction of $(\bar x,\bar u)$ to $[t_0,T]$ is optimal for the above Bolza problem.*

Proof. Fix $(s_0,y_0)\in[t_0,T]\times\mathbb{R}^n$. If $V^B(s_0,y_0)=+\infty$, then $V^B(s_0,y_0)\ge V(s_0,y_0)$. If it is finite, then consider an optimal trajectory-control pair $(\bar x,\bar u)$ for the Bolza problem at (s_0,y_0) defined on $[s_0,T]$. Let a trajectory control pair (x,u) be such that $x(T)=\bar x(T)$ and $V(T,\bar x(T)) = \int_T^\infty L(s,x(s),u(s))ds$. Then

$$V^B(s_0,y_0) = \int_{s_0}^T L(s,\bar x(s),\bar u(s))ds + \int_T^\infty L(s,x(s),u(s))ds \ge V(s_0,y_0).$$

Thus $V^B\ge V$. Conversely, if $V(s_0,y_0)=+\infty$, then $V(s_0,y_0)\ge V^B(s_0,y_0)$. If it is finite, then consider an optimal trajectory-control pair $(\bar x,\bar u)$ for the infinite horizon problem at (s_0,y_0) defined on $[s_0,+\infty[$. By the dynamic programming principle,

$$V(s_0,y_0) = \int_{s_0}^T L(s,\bar x(s),\bar u(s))ds + \int_T^\infty L(s,\bar x(s),\bar u(s))ds \ge V(T,\bar x(T)) + \int_{s_0}^T L(s,\bar x(s),\bar u(s))ds.$$

This yields, $V(s_0,y_0)\ge V^B(s_0,y_0)$. Hence $V(s_0,y_0)=V^B(s_0,y_0)$. The point $(s_0,y_0)\in[t_0,T]\times\mathbb{R}^n$ being arbitrary, the proof follows. □

Under some additional assumptions, the value function is continuous with respect to x.

Theorem 3.5. *Let (H1) hold with time independent $c(t)\equiv c\ge 0$ and $\theta(t)\equiv\theta\ge 0$. Assume also that $c_R(t)\equiv\delta\ge 0$ for all $R>0$ and that, for all x, $y\in\mathbb{R}^n$, $u\in U(t)$ and a.e. $t\ge 0$,*

$$|L(t,y,u)-L(t,x,u)| \le \bar\omega(t,|x-y|)\big[L(t,x,u)\wedge L(t,y,u) + h(t,|x|\vee|y|)\big], \tag{3.8}$$

where functions $\bar\omega:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$, $h:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ enjoy the following properties: $\bar\omega$, h are Lebesgue-Borel measurable, $\bar\omega$ is bounded and $\lim_{r\to 0+}\bar\omega(t,r)=0$ for a.e. $t>0$, $\bar\omega(t,\cdot)$ and $h(t,\cdot)$ are nondecreasing, and

$$\int_0^\infty h(t,(R+\theta t)e^{ct})dt < +\infty \quad \forall\, R\ge 0. \tag{3.9}$$

If $dom(V)\ne\varnothing$, then $dom(V)=\mathbb{R}_+\times\mathbb{R}^n$ and $V(t,\cdot)$ is continuous for any $t\ge 0$.

Proof. Let $(t_0,x_0)\in dom(V)$ and let $(\bar x_0,\bar u_0)$ be a trajectory-control pair at (t_0,x_0) satisfying $V(t_0,x_0)=\int_{t_0}^\infty L(t,\bar x_0(t),\bar u_0(t))\,dt$. For any fixed $x\in\mathbb{R}^n$, let $\bar x(\cdot)=x(\cdot;t_0,x,\bar u_0)$. Then, in view of (3.8), we have that

$$\begin{aligned} V(t_0,x)-V(t_0,x_0) &\le \int_{t_0}^\infty \big[L(t,\bar x(t),\bar u_0(t)) - L(t,\bar x_0(t),\bar u_0(t))\big]\,dt \\ &\le \int_{t_0}^\infty \bar\omega(t,|\bar x(t)-\bar x_0(t)|)\big[L(t,\bar x_0(t),\bar u_0(t)) + h(t,|\bar x(t)|\vee|\bar x_0(t)|)\big]\,dt. \end{aligned} \tag{3.10}$$

Moreover, by (3.1), we obtain

$$|\bar x(t)|\vee|\bar x_0(t)| \le \big[|x|+|x_0|+\theta(t-t_0)\big]e^{c(t-t_0)} \quad \forall\, t\ge t_0.$$

So, (3.9) yields

$$\int_{t_0}^{\infty} h(t, |\bar{x}(t)| \vee |\bar{x}_0(t)|)\, dt \leq \int_{t_0}^{\infty} h\big(t, \big[|x| + |x_0| + \theta(t-t_0)\big] e^{c(t-t_0)}\big) dt < +\infty \tag{3.11}$$

Since $\bar{\omega}$ is bounded, by combining (3.10) and (3.11) we conclude that $V(t_0, x) < \infty$. So, $dom(V(t_0, \cdot)) = \mathbb{R}^n$. Moreover, by the dynamic programming principle, $V(t, \bar{x}_0(t)) < +\infty$ for all $t \geq t_0$. Therefore, $dom(V) \supset [t_0, \infty[\times\mathbb{R}^n$. Now, fix any $(t, x) \in [0, t_0[\times\mathbb{R}^n$ and let $(\bar{x}, \bar{u})$ be any trajectory-control pair satisfying $\bar{x}(t) = x$. By the dynamic programming principle,

$$V(t, x) \leq \int_t^{t_0} L(s, \bar{x}(s), \bar{u}(s))\, ds + V(t_0, \bar{x}(t)).$$

Since $\int_t^{t_0} L(s, \bar{x}(s), \bar{u}(s))\, ds < +\infty$ in view of (3.1) and assumption **(H1)**-iv), we conclude that $(t, x) \in dom(V)$. So, $dom(V) = \mathbb{R}_+ \times \mathbb{R}^n$.

Next, we prove that $V(t_0, \cdot)$ is upper semicontinuous at any $x_0 \in \mathbb{R}^n$ for any fixed $t_0 \geq 0$. Let $\{x_k\}_{k\in\mathbb{N}}$ be any sequence converging to x_0 and let $\bar{x}_k(t) = x(t; t_0, x_k, \bar{u}_0)$. Then (3.10) becomes, for $x = x_k$,

$$V(t_0, x_k) - V(t_0, x_0) \leq \int_{t_0}^{\infty} \bar{\omega}(t, |\bar{x}_k(t) - \bar{x}_0(t)|)\big[L(t, \bar{x}_0(t), \bar{u}_0(t)) + h(t, |\bar{x}_k(t)| \vee |\bar{x}_0(t)|)\big]\, dt.$$

Without loss of generality assume $|x_k| \leq 1 + |x_0|$, so that

$$h(t, |\bar{x}_k(t)| \vee |\bar{x}_0(t)|) \leq h\big(t, [1 + |x_0| + \theta(t-t_0)] e^{c(t-t_0)}\big) \qquad \forall\, t \geq t_0.$$

Moreover, appealing to (3.4) we also have that

$$|\bar{x}_k(t) - \bar{x}_0(t)| \leq |x_k - x_0|\, e^{\delta(t-t_0)} \qquad \forall\, t \geq t_0.$$

Therefore, $\bar{\omega}(t, |\bar{x}_k(t) - \bar{x}_0(t)|) \to 0$ for every $t \geq t_0$ as $k \to \infty$. So, by the Dominated Convergence Theorem we conclude that $\limsup_{k\to\infty} V(t_0, x_k) \leq V(t_0, x_0)$. □

Under further restrictions, V turns out to be locally Lipschitz. We begin by proving Lipschitz continuity with respect to x.

Lemma 3.6. *Assume (H1) with time independent $c(t) \equiv c \geq 0$, $\theta(t) \equiv \theta \geq 0$, $c_R(t) \equiv \delta \geq 0$ for all $R > 0$ and suppose that, for all x, $y \in \mathbb{R}^n$, $u \in U(t)$ and a.e. $t \geq 0$,*

$$|L(t, y, u) - L(t, x, u)| \leq k(t, |x| \vee |y|)|x - y|,$$

where $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is Lebesgue-Borel measurable, $k(t, \cdot)$ is nondecreasing, and

$$\int_0^{\infty} e^{\delta t} k(t, (R + \theta t)e^{ct}) dt < +\infty \quad \forall\, R \geq 0. \tag{3.12}$$

If $dom(V) \neq \varnothing$, then $V(t, \cdot)$ is locally Lipschitz continuous on $\mathbb{R}^n$ for all $t \geq 0$ and

$$|V(t, x_2) - V(t, x_1)| \leq e^{-\delta t}\, K_t(R)\, |x_2 - x_1| \qquad \forall\, x_1, x_2 \in B(0, R) \tag{3.13}$$

where

$$K_t(R) := \int_t^{\infty} e^{\delta\tau} k\big(\tau, M_t(\tau, R)\big) d\tau < +\infty \qquad \forall\, t \geq 0 \tag{3.14}$$

and $M_t(\tau, R) = [R + \theta(\tau - t)]e^{c(\tau-t)}$ is a special case of the function defined in (3.2).

Proof. It suffices to prove (3.13). Fix any $t_0 \geq 0$, $R > 0$ and $x_1, x_2 \in B(0,R)$. Let $(\bar{x}_1, \bar{u})$ be a trajectory-control pair satisfying $\bar{x}_1(t_0) = x_1$, $V(t_0, x_1) = \int_{t_0}^{\infty} L(t, \bar{x}_1(t), \bar{u}(t))\, dt$ and define $\bar{x}_2(\cdot) = x(\cdot; t_0, x_2, \bar{u})$. Owing to (3.3) and (3.4), we know that

$$|\bar{x}_1(t)| \vee |\bar{x}_2(t)| \leq [R + \theta(t - t_0)]e^{c(t-t_0)} \ \ \& \ \ |\bar{x}_2(t) - \bar{x}_1(t)| \leq |x_2 - x_1|\, e^{\delta(t-t_0)} \qquad \forall\, t \geq t_0.$$

Therefore

$$V(t_0, x_2) - V(t_0, x_1) \leq e^{-\delta t_0}|x_2 - x_1| \int_{t_0}^{\infty} e^{\delta t} k\big(t, [R + \theta(t - t_0)]e^{c(t-t_0)}\big) dt = e^{-\delta t_0}\, K_{t_0}(R)\, |x_2 - x_1|.$$

By exchanging the role of x_1 and x_2 we obtain (3.13). □

Remark 3.7. We observe that the function K_t above has the following property:

$$K_t(M_s(t,R)) \leq K_s(R) \qquad \forall\, t \geq s \geq 0\,. \tag{3.15}$$

Indeed

$$\begin{aligned} K_t(M_s(t,R)) &= \int_t^{\infty} e^{\delta\tau} k\big(\tau, [R + \theta(t-s)]e^{c(\tau-s)} + \theta(\tau - t)]e^{c(\tau-t)}\big) d\tau \\ &= \int_t^{\infty} e^{\delta\tau} k\big(\tau, Re^{c(\tau-s)} + \theta[(t-s)e^{c(\tau-s)} + (\tau - t)e^{c(\tau-t)}]\big) d\tau \\ &\leq \int_t^{\infty} e^{\delta\tau} k\big(\tau, [R + \theta(\tau - s)]e^{c(\tau-s)}]\big) d\tau \leq K_s(R). \end{aligned}$$

Corollary 3.8. *Under the assumptions of Lemma 3.6, fix any $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ and let $(\bar{x}_0, \bar{u})$ be any trajectory-control pair satisfying $\bar{x}_0(t_0) = x_0$. Then for all $t \geq t_0$ we have*

$$|V(t, x_2) - V(t, x_1)| \leq e^{-\delta t}\, K_{t_0}(1 + |x_0|)\, |x_2 - x_1| \qquad \forall\, x_1, x_2 \in B(\bar{x}_0(t), 1) \tag{3.16}$$

where $K_t(\cdot)$ is defined in (3.14).

Proof. Appealing to (3.13) we have that for all $t \geq t_0$

$$|V(t, x_2) - V(t, x_1)| \leq e^{-\delta t}\, K_t\big(1 + |\bar{x}_0(t)|\big)\, |x_2 - x_1| \qquad \forall\, x_1, x_2 \in B(\bar{x}_0(t), 1)$$

Moreover, $|\bar{x}_0(t)| \leq M_{t_0}(t, |x_0|)$ for all $t \geq t_0$. So, since $K_t(\cdot)$ is nondecreasing, by (3.15) we obtain

$$K_t\big(1 + |\bar{x}_0(t)|\big) \leq K_t\big(1 + M_{t_0}(t, |x_0|)\big) \leq K_t\big(M_{t_0}(t, 1 + |x_0|)\big) \leq K_{t_0}(1 + |x_0|)$$

The conclusion follows. □

We now complete the analysis by proving the joint Lipschitz continuity of V in (t, x).

Theorem 3.9. *Assume (H1) with time independent $c(t) \equiv c \geq 0$, $\theta(t) \equiv \theta \geq 0$ and nonincreasing $\beta(\cdot)$. Assume also $c_R(t) \equiv \delta \geq 0$ for all $R > 0$ and suppose that, for all x, $y \in \mathbb{R}^n$, $u \in U(t)$ and a.e. $t \geq 0$,*

$$|L(t, x, u) - L(t, y, u)| \leq k(t, |x| \vee |y|)|x - y|, \tag{3.17}$$

where $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is as in Lemma 3.6. If dom$(V) \neq \varnothing$, then dom$(V) = \mathbb{R}_+ \times \mathbb{R}^n$, V is locally Lipschitz continuous, and for every $T \geq t \geq 0$ and $R > 0$ we have that

$$|V(t_2, x_2) - V(t_1, x_1)| \leq e^{-\delta t}\, K_t(R)\, |x_2 - x_1| + N_t(T, R)|t_2 - t_1|$$

for all $t_1, t_2 \in [t, T]$ and $x_1, x_2 \in B(0, R)$, where $K_t(R)$ is given by (3.14) *and*

$$N_t(T, R) = e^{-\delta t} K_t(R)\big[\, \theta + c\, M_t(T, R) + \beta(t)\phi\big(M_t(T, R)\big)\, \big]. \tag{3.18}$$

Proof. Fix any $R > 0$ and $T \geq t \geq 0$, and let $t_1, t_2 \in [t, T]$ and $x_1, x_2 \in B(0, R)$. We distinguish two cases.

$\boxed{t_2 \geq t_1}$ Let $(\bar{x}_1, \bar{u}_1)$ be an optimal trajectory-control pair satisfying $\bar{x}_1(t_1) = x_1$. By (3.3) we have

$$|\bar{x}_1(s)| \leq M_{t_1}(s, R) \quad \forall\, s \geq t_1 \tag{3.19}$$

where M_{t_1} is defined in (3.2). Now, the dynamic programming principle, (3.13) and (3.15) yield

$$\begin{aligned} V(t_2, x_2) - V(t_1, x_1) &\leq V(t_2, x_2) - V(t_2, \bar{x}_1(t_2)) - \int_{t_1}^{t_2} L(s, \bar{x}_1(s), \bar{u}_1(s))\, ds \\ &\leq e^{-\delta t_2} K_{t_2}\big(M_{t_1}(t_2, R)\big)\, |x_2 - \bar{x}_1(t_2)| \\ &\leq e^{-\delta t_2} K_{t_1}(R)\big(|x_2 - x_1| + |x_1 - \bar{x}_1(t_2)|\big) \end{aligned} \tag{3.20}$$

Next, on account of **(H1)**-i) and (3.19),

$$\begin{aligned} |x_1 - \bar{x}_1(t_2)| &= \Big| \int_{t_1}^{t_2} f(s, \bar{x}_1(s), \bar{u}_1(s))\, ds \Big| \\ &\leq \int_{t_1}^{t_2} (\theta + c\,|\bar{x}_1(s)|)\, ds \leq \big(\theta + c\, M_{t_1}(t_2, R)\big)|t_2 - t_1|. \end{aligned}$$

By combining the last estimate with (3.20), we conclude that

$$V(t_2, x_2) - V(t_1, x_1) \leq e^{-\delta t_2} K_{t_1}(R)\Big\{|x_2 - x_1| + \big(\theta + c\, M_{t_1}(t_2, R)\big)|t_2 - t_1|\Big\}.$$

Since $M_{t_1}(t_2, R) \leq M_t(T, R)$ and $K_{t_1}(R) \leq K_t(R)$, we thus obtain

$$\begin{aligned} V(t_2, x_2) - V(t_1, x_1) &\leq e^{-\delta t} K_t(R) \Big\{|x_2 - x_1| + \big(\theta + c\, M_t(T, R)\big)|t_2 - t_1|\Big\} \\ &\leq e^{-\delta t} K_t(R)\, |x_2 - x_1| + N_t(T, R)|t_2 - t_1| \end{aligned}$$

where N_t is given by (3.18).

$\boxed{t_2 < t_1}$ The reasoning is similar to the one above. Let $(\bar{x}_2, u_2)$ be any trajectory-control pair satisfying $\bar{x}_2(t_2) = x_2$. Invoke dynamic programming, (3.13), and **(H1)**-iv) to derive

$$\begin{aligned} V(t_2, x_2) - V(t_1, x_1) &\leq V(t_1, \bar{x}_2(t_1)) - V(t_1, x_1) + \int_{t_2}^{t_1} L(s, \bar{x}_2(s), u_2(s))\, ds \\ &\leq e^{-\delta t_1} K_{t_1}\big(M_{t_2}(t_1, R)\big)\, |\bar{x}_2(t_1)) - x_1| + \int_{t_2}^{t_1} \beta(s)\, \phi\big(|\bar{x}_2(s)|\big) ds \\ &\leq e^{-\delta t_1} K_{t_2}(R)\big(|x_2 - x_1| + |\bar{x}_2(t_1) - x_2|\big) + \beta(t_2)\, \phi\big(M_{t_2}(t_1, R)\big)\, |t_2 - t_1|. \end{aligned}$$

Since we have, as above,

$$|\bar{x}_2(t_1) - x_2| \leq \int_{t_2}^{t_1} (\theta + c\,|\bar{x}_2(s)|)\, ds \leq \big(\theta + c\, M_{t_2}(t_1, R)\big)|t_2 - t_1|,$$

once again we conclude that

$$V(t_2, x_2) - V(t_1, x_1) \leq e^{-\delta t} K_t(R)\, |x_2 - x_1| + N_t(T, R)|t_2 - t_1|.$$

Therefore, the above inequality holds true for all $t_1, t_2 \in [t, T]$. The conclusion follows. □

Remark 3.10. Even the Lipschitz constant of V with respect to time and space can be estimated along any trajectory-control pair as in Corollary 3.8. More precisely, for any given trajectory-control pair $(\bar{x}_0, \bar{u})$ with $\bar{x}_0(t_0) = x_0$ and $t \geq t_0$, for all $(t_i, x_i) \in \mathbb{R}_+ \times \mathbb{R}^n$ $(i = 1, 2)$ satisfying $t \leq t_i \leq t + 1$ and $|x_i - \bar{x}_0(t)| \leq 1$ we have that

$$\begin{aligned} &|V(t_2, x_2) - V(t_1, x_1)| \\ &\quad \leq \quad e^{-\delta t} K_{t_0}(1 + |x_0|)\Big\{|x_2 - x_1| + \big[\,\theta + c\, M_t(t+1, R_t) + \beta(t)\phi\big(M_t(t+1, R_t)\big)\,\big]|t_2 - t_1|\Big\} \end{aligned}$$

where $R_t = 1 + |\bar{x}_0(t)| \leq 1 + M_{t_0}(t, |x_0|)$. Since

$$M_t(t+1, R_t) \leq \big[\,1 + \theta + M_{t_0}(t, |x_0|)\,\big]e^c \leq \big[\,1 + \theta\,\big]e^c + \big[\,|x_0| + \theta(t - t_0)\,\big]\, e^{c(t-t_0+1)},$$

it is easy to realise that if $\delta > c$, then the Lipschitz constant of V at $(t, \bar{x}_0(t))$ can be estimated from the above when $t \to \infty$ upon the behaviour at infinity of the function

$$t \mapsto e^{-\delta t}\, \beta(t)\, \phi\big((R + \theta t)\, e^{ct}\big)$$

for $R > 0$ sufficiently large.

4. Maximum Principle and Sensitivity for LSC Value Function

If $f(t, \cdot, u)$ and $L(t, \cdot, u)$, are differentiable, denote $D_x f$ and L_x their (partial) Jacobian and gradient with respect to x.

We shall need the following assumption :

(H2) For every $R > 0$, there exists a locally integrable function $\alpha_R : \mathbb{R}_+ \to \mathbb{R}_+$ such that for a.e. $t \in \mathbb{R}_+$,

$$|L(t, x, u) - L(t, y, u)| \leq \alpha_R(t)|x - y|, \quad \forall\, x, y \in B(0, R),\ \forall\, u \in U(t);$$

We introduce next the Hamiltonian $H : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$H(t, x, p) := \sup_{u \in U(t)} \big(\langle p, f(t, x, u)\rangle - L(t, x, u)\big).$$

Then $H(t, x, \cdot)$ is convex. Under assumptions (H1), (H2) for a.e. $t \geq 0$, the supremum is attained for all $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$ and $H(t, \cdot, p)$ is Lipschitz on $B(0, R)$ with the Lipschitz constant $c_R(t)|p| + \alpha_R(t)$.

Recall that if $V(t, \cdot)$ is lower semicontinuous, then $\partial_x^- V(t, x_0) \neq \varnothing$ on a dense subset of points x_0 of the domain of $V(t_0, \cdot)$, where $\partial_x^- V(t, x_0)$ denotes the partial Fréchet subdifferential of $V(t, \cdot)$ at x_0.

Theorem 4.1. *Assume (H1), (H2) and let $(\bar{x}, \bar{u})$ be optimal at $(t_0, x_0) \in dom(V)$ with $\partial_x^- V(t_0, x_0) \neq \varnothing$. If $f(t, \cdot, u)$ and $L(t, \cdot, u)$ are differentiable for all $t \in \mathbb{R}_+, u \in U(t)$, then for every $p_0 \in \partial_x^- V(t_0, x_0)$ the solution $p(\cdot)$ of the adjoint system*

$$-p'(t) = D_x f(t, \bar{x}(t), \bar{u}(t))^* p(t) - L_x(t, \bar{x}(t), \bar{u}(t)) \qquad a.e.\ \ t \geq t_0, \quad p(t_0) = -p_0$$

satisfies the maximality condition

$$\langle p(t), f(t, \bar{x}(t), \bar{u}(t))\rangle - L(t, \bar{x}(t), \bar{u}(t)) = H(t, \bar{x}(t), p(t)) \tag{4.1}$$

a.e. in $[t_0, +\infty[$ and the sensitivity relation

$$-p(t) \in \partial_x^- V(t, \bar{x}(t)) \quad \forall\, t \geq t_0.$$

Furthermore, if assumptions of Lemma 3.6 hold true with $\delta > 0$, then $p(t)$ converges exponentially to zero when $t \to \infty$.

Proof. Let $p_0 \in \partial_x^- V(t_0, x_0)$ By Proposition 2.1 there exists a continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$, differentiable at x_0, such that $\varphi(x_0) = V(t_0, x_0)$, $\nabla\varphi(x_0) = p_0$ and $\varphi(\cdot) \leq V(t_0, \cdot)$. For any $T > t_0$, consider the functional

$$J_T(u) = \int_{t_0}^{T} L(t, x(t; T, \bar{x}(T), u(t))\, dt - \varphi(x(t_0))$$

that we wish to minimize over all trajectory-control pairs (x, u) of the following control system

$$\begin{cases} x'(t) = f(t, x(t), u(t)),\ u(t) \in U(t) & \text{for a.e. } t \in [t_0, T] \\ x(T) = \bar{x}(T). \end{cases} \tag{4.2}$$

We claim that $(\bar{x}, \bar{u})$ is optimal for this new problem. Indeed, assume for a moment that for some control u and a trajectory $x(\cdot)$ of (4.2) we have

$$\int_{t_0}^{T} L(t, \bar{x}(t), \bar{u}(t))\, dt - V(t_0, x_0) > \int_{t_0}^{T} L(t, x(t), u(t))\, dt - \varphi(x(t_0)).$$

Define $\tilde{x} : \mathbb{R}_+ \to \mathbb{R}^n$ by $\tilde{x}(\cdot) = x(\cdot)$ on $[t_0, T]$ and $\tilde{x}(\cdot) = \bar{x}(\cdot)$ on $[T, +\infty[$. The control $\tilde{u}$ is defined in a similar way. Adding $\int_T^\infty L(t, \bar{x}(t), \bar{u}(t))\, dt$ to both sides of the last inequality we obtain

$$\int_{t_0}^{\infty} L(t, \bar{x}(t), \bar{u}(t))\, dt - V(t_0, x_0) > \int_{t_0}^{\infty} L(t, \tilde{x}(t), \tilde{u}(t))\, dt - \varphi(x(t_0)).$$

Therefore $V(t_0, x(t_0)) < \varphi(x(t_0))$. This contradicts the choice of φ and proves our claim.

For this new finite horizon problem the classical maximum principle implies that the solution $p(\cdot)$ of the adjoint system

$$-p'(t) = D_x f(t, \bar{x}(t), \bar{u}(t))^* p(t) - L_x(t, \bar{x}(t), \bar{u}(t)) \quad \text{a.e. } t \in [t_0, T], \quad p(t_0) = -p_0$$

satisfies the maximality condition

$$\langle p(t), f(t, \bar{x}(t), \bar{u}(t))\rangle - L(t, \bar{x}(t), \bar{u}(t)) = H(t, \bar{x}(t), p(t))$$

a.e. in $[t_0, T]$. Since $p(\cdot)$ is uniquely defined and $T > t_0$ is arbitrary, the above equality holds true a.e. in $[t_0, +\infty[$.

For all $r \geq t_0$ let $\Psi(\cdot, r) \in \mathbb{R}^{n\times n}$ be the matrix-valued solution of

$$\Psi_s'(s, r) = D_x f\big(s, \bar{x}(s), \bar{u}(s)\big)\Psi(s, r) \ \text{ a.e. } s \geq t_0,\ \Psi(r, r) = I\,.$$

Fix $t > t_0$ and $y \in \mathbb{R}^n$ and consider $h_i \to 0+,\ y_i \to y$ such that

$$D_{\uparrow x} V(t, \bar{x}(t))y = \lim_{i\to\infty} \frac{V(t, \bar{x}(t) + h_i y_i) - V(t, \bar{x}(t))}{h_i} \in [-\infty + \infty],$$

where $D_{\uparrow x} V(t, \bar{x}(t))y$ denotes the lower directional derivative of $V(t, \cdot)$ at $\bar{x}(t)$ in the direction y. For any i, let x_i be the solution of the system

$$x' = f(s, x, \bar{u}(s)), \quad x(t) = \bar{x}(t) + h_i y_i$$

on $[t_0, t]$. Then the difference quotients $\frac{x_i(s) - \bar{x}(s)}{h_i}$ converge uniformly on $[t_0, t]$ to the solution w of the linear system

$$w'(s) = D_x f\big(s, \bar{x}(s), \bar{u}(s)\big)w(s) \ \text{ a.e. in } [t_0, t],\ \ w(t) = y$$

that can be represented by $w(s) = \Psi(s, t)y$. By the dynamic programming principle,

$$V(t_0, x_i(t_0)) \leq V(t, \bar{x}(t) + h_i y_i) + \int_{t_0}^{t} L(s, x_i(s), \bar{u}(s))ds.$$

Subtracting $V(t_0,x_0) = V(t,\bar{x}(t)) + \int_{t_0}^t L(s,\bar{x}(s),\bar{u}(s))ds$ from the both sides of the above inequality, dividing by h_i and passing to the limit we obtain

$$D_{\uparrow x}V(t_0,x_0)(w(t_0)) \le D_{\uparrow x}V(t,\bar{x}(t))y + \int_{t_0}^t L_x(s,\bar{x}(s),\bar{u}(s))w(s)ds$$

For $p(\cdot)$ as in our theorem we have $\frac{d}{ds}\langle p(s),w(s)\rangle = \big\langle L_x\big(s,\bar{x}(s),\bar{u}(s)\big),w(s)\big\rangle$. Consequently,

$$D_{\uparrow x}V(t_0,x_0)(w(t_0)) \le D_{\uparrow x}V(t,\bar{x}(t))y + \langle p(t),y\rangle - \langle p_0,w(t_0)\rangle.$$

Hence

$$0 \le D_{\uparrow x}V(t_0,x_0)(w(t_0)) - \langle -p_0,w(t_0)\rangle \le D_{\uparrow x}V(t,\bar{x}(t))y - \langle -p(t),y\rangle$$

and therefore $-p(t) \in \partial_x^- V(t,\bar{x}(t))$. The last statement follows from Corollary 3.8.

□

The above necessary optimality condition holds true only when the subdifferential $\partial_x^- V(t_0,x_0)$ is nonempty and involves elements of this subdifferential in the transversality condition. An alternative way consists in using limiting supergradients. This is done in the next section, but before, for the sake of completeness, we state one more result that follows directly from nonsmooth maximum principles for the Bolza problem, but does not include any transversality condition.

Below, let $\partial_x f$ denote the partial generalized Jacobian of f with respect to x and $\partial_x L$ the partial generalized gradient of L with respect to x.

Theorem 4.2. *Assume (H1), (H2) and let $(\bar{x},\bar{u})$ be optimal at some $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$. Then*
i) either there exists a solution $p(\cdot)$ of the adjoint inclusion

$$-p'(t) \in \partial_x f(t,\bar{x}(t),\bar{u}(t))^* p(t) - \partial_x L(t,\bar{x}(t),\bar{u}(t)) \qquad \text{for a.e. } t \ge t_0 \tag{4.3}$$

satisfying the maximality condition (4.1) a.e. in $[t_0,+\infty[$
ii) or there exists a nonvanishing solution $p(\cdot)$ of the adjoint inclusion

$$-p'(t) \in \partial_x f(t,\bar{x}(t),\bar{u}(t))^* p(t) \quad \text{for a.e. } t \ge t_0, \tag{4.4}$$

satisfying a.e. in $[t_0,+\infty[$ the abnormal maximum principle

$$\langle p(t), f(t,\bar{x}(t),\bar{u}(t))\rangle = \max_{u\in U(t)} \langle p(t), f(t,\bar{x}(t),u)\rangle. \tag{4.5}$$

Moreover, if for all large $t > t_0$, the value function $V(t,\cdot)$ is Lipschitz at $\bar{x}(t)$ with a Lipschitz constant independent from t, then the normal maximum principle i) holds true.

Furthermore, if there exist integrable mappings $C_f : \mathbb{R}_+ \to \mathbb{R}_+$ and $C_L : \mathbb{R}_+ \to \mathbb{R}_+$ such that for a.e. $t \ge 0$ and all $u \in U(t)$, $f(t,\cdot,u)$ is $C_f(t)$-Lipschitz and $L(t,\cdot,u)$ is $C_L(t)$-Lipschitz, then $\lim_{t\to\infty} p(t)$ does exist.

The above theorem lacks the transversality condition and also it does not exclude the abnormality of the maximum principle. For this reason its conclusion is less informative that the one of Theorem 4.1. On the other hand it holds true for all the initial conditions.

Proof. For any integer $i \ge t_0$, consider the problem of minimizing the functional

$$J_i(u) = \int_{t_0}^i L(t,x(t),u(t))\,dt + V(i,x(i))$$

over all trajectory-control pairs (x,u) of the control system

$$\begin{cases} x'(t) = f(t,x(t),u(t)), \quad u(t) \in U(t) \quad \text{for a.e. } t \in [t_0,i] \\ x(t_0) = x_0. \end{cases}$$

Then $(\bar{x}, \bar{u})$ is optimal for this new problem that we can rewrite as

$$\text{minimize} \int_{t_0}^{i} L(t, x(t), u(t))\, dt + z(i)$$

over trajectory-control pairs of

$$\begin{cases} x'(t) = f(t, x(t), u(t)), \quad u(t) \in U(t) & \text{for a.e. } t \in [t_0, i] \\ z' = 0 & \text{for a.e. } t \in [t_0, i] \\ (x(i), z(i)) \in epi(V(i, \cdot)),\ x(t_0) = x_0. \end{cases}$$

Then, setting $z(t) \equiv V(i, \bar{x}(i))$ we deduce that $((\bar{x}, z), \bar{u})$ is a minimizer of this problem. We apply the maximum principle [21, Theorem 6.2.1] stated for the Mayer problem that we adapt to the above Bolza problem in the usual way. Then there exist $\lambda_i \in \{0, 1\}$, and absolutely continuous functions $p_i : [t_0, i] \to \mathbb{R}^n,\ q_i : [t_0, i] \to \mathbb{R}$, not vanishing simultaneously, satisfying the adjoint inclusion

$$-p_i'(t) \in \partial_x f(t, \bar{x}(t), \bar{u}(t))^* p_i(t) - \lambda_i \partial_x L(t, \bar{x}(t), \bar{u}(t)),\ q_i'(t) = 0 \ \text{ for a.e. } \ t \in [t_0, i], \tag{4.6}$$

the maximality condition

$$\langle p_i(t), f(t, \bar{x}(t), \bar{u}(t))\rangle - \lambda_i L(t, \bar{x}(t), \bar{u}(t)) = \max_{u \in U(t)} (\langle p_i(t), f(t, \bar{x}(t), u)\rangle - \lambda_i L(t, \bar{x}(t), u))$$

a.e. in $[t_0, i]$ and transversality conditions :

$$(p_i(t_0), q_i(t_0)) \in \mathbb{R}^n \times \{0\}, \quad -(p_i(i), q_i(i)) \in \lambda_i(0, 1) + N^L_{epi(V(i,\cdot))}(\bar{x}(i), V(i, \bar{x}(i))).$$

Then $q_i \equiv 0$ and therefore

$$-(p_i(i), \lambda_i) \in N^L_{epi(V(i,\cdot))}(\bar{x}(i), V(i, \bar{x}(i))).$$

We extend p_i on the interval $]i, +\infty[$ as a solution of (4.6) equal to $p_i(i)$ at time i.

We first investigate the case when there exists an infinite subsequence $\{i_k\}_{k\geq 1}$ such that for every k, $\lambda_{i_k} = 1$ and $\{p_{i_k}(t_0)\}_{k\geq 1}$ is bounded. Then for every $T > t_0$, the restrictions of p_{i_k} to $[t_0, T]$ are equibounded and the restrictions of p'_{i_k} to $[t_0, T]$ are integrably bounded. Applying the same classical arguments as those recalled in the proof of Proposition 3.1 and using the upper semicontinuity of set-valued mappings $\partial_x f(t, \cdot, \bar{u}(t))$, $\partial_x L(t, \cdot, \bar{u}(t))$ having convex values, we show that a subsequence of $\{p_{i_k}\}$ converges almost uniformly on $[t_0, +\infty[$ to a solution $p(\cdot)$ of the adjoint inclusion (4.3) such that for a.e. $t \geq t_0$, (4.1) holds true.

Consider next the case when for an infinite subsequence $\{i_k\}_{k\geq 1}$ we have $\lambda_{i_k} = 1$ and $\lim_{k\to\infty} |p_{i_k}(t_0)| = \infty$. Then set

$$\gamma_k(t) = \frac{p_{i_k}(t)}{|p_{i_k}(t_0)|}$$

and observe that

$$-\gamma_k'(t) \in \partial_x f(t, \bar{x}(t), \bar{u}(t))^* \gamma_k(t) - \partial_x L(t, \bar{x}(t), \bar{u}(t))/|p_{i_k}(t_0)| \qquad \text{for a.e. } \ t \geq t_0,$$

and for a.e. $t \in [t_0, i_k]$ and all $u \in U(t)$,

$$\langle \gamma_k(t), f(t, \bar{x}(t), \bar{u}(t))\rangle - L(t, \bar{x}(t), \bar{u}(t))/|p_{i_k}(t_0)| \geq \langle \gamma_k(t), f(t, \bar{x}(t), u)\rangle - L(t, \bar{x}(t), u)/|p_{i_k}(t_0).$$

As before we can find a subsequence of $\{\gamma_k\}$ converging almost uniformly to a solution p of (4.4) satisfying (4.5) a.e. in $[t_0, +\infty[$. Moreover $p(t_0) \neq 0$.

The remaining case is $\lambda_i = 0$ for all large i. That is for all large i,

$$-p_i'(t) \in \partial_x f(t, \bar{x}(t), \bar{u}(t))^* p_i(t) \qquad \text{for a.e. } \ t \geq t_0,$$

$$\langle p_i(t), f(t, \bar{x}(t), \bar{u}(t))\rangle = \max_{u \in U(t)} (\langle p_i(t), f(t, \bar{x}(t), u)\rangle \qquad \text{for a.e. } \ t \in [t_0, i].$$

Then $p_i(t_0) \neq 0$ for all large i. Setting $\gamma_i(t) = p_i(t)/|p_i(t_0)|$, we extract a subsequence of $\{\gamma_i\}_{i\geq 1}$ converging almost uniformly on $[t_0, +\infty[$ to a solution $p(\cdot)$ of (4.4) satisfying (4.5) a.e. in $[t_0, +\infty[$. Clearly $p(t_0) \neq 0$.

Assume next that for all large $t > t_0$ the value function $V(t, \cdot)$ is Lipschitz at $\bar{x}(t)$ with a Lipschitz constant independent from t. We deduce that for all large i, the relation $-(p_i(i), \lambda_i) \in N^L_{epi(V(i,\cdot))}(\bar{x}(i), V(i, \bar{x}(i)))$ implies that $\lambda_i = 1$ and $\{p_i(i)\}$ are bounded by the common Lipschitz constant of $V(t, \cdot)$ at $\bar{x}(t)$ (for large t).

To prove the last statement, we first show that $p(\cdot)$ is bounded. Indeed, let $\lambda = 1$ if we are in Case $i)$ and $\lambda = 0$ if we are in Case $ii)$. Consider the constant function $y(t) = p(s_0)$ for all $t \geq s_0$. Then

$$|y'(t) - D_x f(t, \bar{x}(t), \bar{u}(t))^* y(t) - \lambda L_x(t, \bar{x}(t), \bar{u}(t))| \leq C_f(t)|p(s_0)| + C_L(t)$$

a.e. in $[s_0, \infty[$. By the Filippov theorem, for every $t \geq s_0$

$$|p(t) - p(s_0)| \leq \int_{s_0}^{t} e^{\int_s^t C_f(\tau)d\tau}(C_f(s)|p(s_0)| + C_L(s))ds,$$

implying the boundedness of p. Set $M = \sup_{t\geq t_0} |p(t)|$

Consider any sequence $t_i \geq 0$ converging to $+\infty$ and for every i define the mapping $y_i \equiv p(t_i)$. As before, for all large i and $t \geq t_i$

$$|p(t) - p(t_i)| \leq \int_{t_i}^{t} e^{\int_s^t C_f(\tau)d\tau}(C_f(s)M + C_L(s))ds.$$

This and our assumption imply that $p(t_i)$ is Cauchy and therefore it converges to some $p_\infty \in \mathbb{R}^n$. The last inequality yields $\lim_{t\to\infty} p(t) = p_\infty$. □

5. Transversality Condition for Continuous Value Function

We have derived a maximum principle with a transversality condition at the initial time on a dense subset of the domain of $V(t_0, \cdot)$. Actually, a maximum principle holds true at all points of the domain of $V(t_0, \cdot)$ with a less precise transversality condition and under an addition assumption of continuity of $V(t_0, \cdot)$ on a neighborhood of x_0. This will be the topic of this section.

Theorem 5.1. *Let (H1) $i) - v)$, (H2) hold and $(t_0, x_0) \in dom(V)$. Assume that $V(t_0, \cdot)$ is continuous on a neighborhood of x_0 and that $(\bar{x}, \bar{u})$ is optimal at (t_0, x_0). Then*

$i)$ either there exists a solution $p(\cdot)$ of the adjoint inclusion (4.3) satisfying the maximality condition (4.1) a.e. in $[t_0, +\infty[$ and the transversality condition

$$-p(t_0) \in \partial_x^{L,+} V(t_0, x_0);$$

$ii)$ or there exists a nonvanishing solution $p(\cdot)$ of the adjoint inclusion (4.4) satisfying the abnormal maximality condition (4.5) a.e. in $[t_0, +\infty[$ and the transversality condition

$$-p(t_0) \in \partial_x^{\infty,+} V(t_0, x_0).$$

Proof. Let $r > 0$ be such that $V(t_0, \cdot)$ is continuous on $B(x_0, r)$ and $W : \mathbb{R}^n \to \mathbb{R}$ be a continuous function that coincides with $V(t_0, \cdot)$ on $B(x_0, r)$. For any integer $i \geq t_0$, consider the functional

$$J_i(u) = \int_{t_0}^{i} L(t, x(t), u(t))\, dt - W(x(t_0))$$

that we wish to minimize over all controls $u(\cdot) \in \mathcal{U}$ and corresponding trajectories x of

$$\begin{cases} x'(t) = f(t, x(t), u(t)) & \text{for a.e. } t \in [t_0, i] \\ x(i) = \bar{x}(i). \end{cases}$$

Using a contradiction argument we show that $(\bar{x}, \bar{u})$ is strongly locally optimal for this new problem of Bolza type, in the sense that for any trajectory-control pair of the above system satisfying $\sup_{t\in[t_0,i]} |x(t) - \bar{x}(t)| < r$ we have $\int_{t_0}^{i} L(t, \bar{x}(t), \bar{u}(t))\, dt \le \int_{t_0}^{i} L(t, x(t), u(t))\, dt$. Let us rewrite this Bolza problem in the following way:

$$\text{minimize} \int_{t_0}^{i} L(t, x(t), u(t))\, dt - z(t_0)$$

over trajectory-control pairs of

$$\begin{cases} x'(t) = f(t, x(t), u(t)), \quad u(t) \in U(t) & \text{for a.e. } t \in [t_0, i] \\ z' = 0 & \text{for a.e. } t \in [t_0, i] \\ x(i) = \bar{x}(i), \quad (x(t_0), z(t_0)) \in hyp(W). \end{cases}$$

Then, setting $z(t) \equiv W(x_0)$ we deduce that $((\bar{x}, z), \bar{u})$ is a strong local minimizer of this problem. Applying the maximum principle from [21, Theorem 6.2.1] adapted to the Bolza problem, we deduce that there exist $\lambda_i \in \{0, 1\}$ and absolutely continuous functions $p_i : [t_0, i] \to \mathbb{R}^n$, $q_i : [t_0, i] \to \mathbb{R}$, not vanishing simultaneously, satisfying the adjoint system

$$-p_i'(t) \in \partial_x f(t, \bar{x}(t), \bar{u}(t))^* p_i(t) - \lambda_i \partial_x L(t, \bar{x}(t), \bar{u}(t)),\ q_i'(t) = 0 \quad \text{for a.e. } t \in [t_0, i], \tag{5.1}$$

the maximality condition

$$\langle p_i(t), f(t, \bar{x}(t), \bar{u}(t))\rangle - \lambda_i L(t, \bar{x}(t), \bar{u}(t)) = \max_{u\in U(t)} (\langle p_i(t), f(t, \bar{x}(t), u)\rangle - \lambda_i L(t, \bar{x}(t), u))$$

a.e. in $[t_0, i]$ and the transversality conditions

$$(p_i(t_0), q_i(t_0)) \in N^L_{hyp(W)}(x_0, W(x_0)), \quad -(p(i), q(i)) \in \lambda_i(0, -1) + \mathbb{R}^n \times \{0\}.$$

Thus $q(\cdot) \equiv \lambda_i$. Extend p_i on the interval $]i, +\infty[$ as a solution of (5.1) equal to $p_i(i)$ at time i.

If there exists an infinite subsequence $\{i_k\}_{k\ge1}$ such that for every k, $\lambda_{i_k} = 1$ and $\{p_{i_k}(t_0)\}_{k\ge1}$ is bounded, then $q_{i_k}(\cdot) = 1$ and $(p_{i_k}(t_0), 1) \in N^L_{hyp(W)}(x_0, W(x_0))$. Then a subsequence of $\{p_{i_k}(t_0)\}_{k\ge1}$ converges to some $-p_0$. Hence $p_0 \in \partial_x^{L,+} V(t_0, x_0)$.

Fix any $T > t_0$. Observe next that for some $R > 0$ and for all k and a.e. $t \in [t_0, T]$,

$$|p_{i_k}'(t)| \le c_R(t)|p_{i_k}(t)| + \alpha_R(t).$$

Thus, a subsequence of $\{p_{i_k}\}_{k\ge1}$ converges almost uniformly on $[t_0, +\infty[$ to a solution $p(\cdot)$ of the adjoint inclusion (4.3) such that $p(t_0) = -p_0$ and (4.1) holds for a.e. in $[t_0, +\infty[$.

Consider next the case when there exists an infinite subsequence $\{i_k\}_{k\ge1}$ such that for every k, $\lambda_{i_k} = 1$ and $\lim_{k\to\infty} |p_{i_k}(t_0)| = \infty$. Then define

$$\gamma_k(t) = \frac{p_{i_k}(t)}{|p_{i_k}(t_0)|}$$

and observe that $(\gamma_k(t_0), \frac{1}{|p_{i_k}(t_0)|}) \in N^L_{hyp(W)}(x_0, W(x_0))$. Then a subsequence of $\{\gamma_k(t_0)\}$ converges to some $-p_0$ and $p_0 \in \partial_x^{\infty,+} V(t_0, x_0)$ with its norm equal to one. Then

$$-\gamma_k'(t) \in \partial_x f(t, \bar{x}(t), \bar{u}(t))^* \gamma_k(t) - \partial_x L(t, \bar{x}(t), \bar{u}(t))/|p_{i_k}(t_0)| \qquad \text{for a.e. } t \ge t_0,$$

and for a.e. $t \in [t_0, i_k]$ and all $u \in U(t)$,

$$\langle \gamma_k(t), f(t, \bar{x}(t), \bar{u}(t))\rangle - L(t, \bar{x}(t), \bar{u}(t))/|p_{i_k}(t_0)| \ge \langle \gamma_k(t), f(t, \bar{x}(t), u)\rangle - L(t, \bar{x}(t), u)/|p_{i_k}(t_0)|.$$

As before we show that there exists a subsequence of $\{\gamma_k\}$ converging almost uniformly on $[t_0, +\infty[$ to a solution p of (4.4) such that $p(t_0) = -p_0$ and (4.5) holds for a.e. in $[t_0, +\infty[$.

The remaining case is $\lambda_i = 0$ for all large i. Then for all large i, $0 \neq (p_i(t_0), 0) \in N^L_{hyp(W)}(x_0, W(x_0))$,

$$-p_i'(t) \in \partial_x f(t, \bar{x}(t), \bar{u}(t))^* p_i(t) \qquad \text{for a.e. } \ t \geq t_0,$$

$$\langle p_i(t), f(t, \bar{x}(t), \bar{u}(t))\rangle = H(t, \bar{x}(t), p_i(t)) \qquad \text{for a.e. } \ t \in [t_0, i].$$

Set $\gamma_i(t) = p_i(t)/|p_i(t_0)|$ and extract a subsequence $\{\gamma_{i_k}\}_{k_geq1}$ such that $\lim_{k\to\infty} \gamma_{i_k}(t_0) = -p_0$ for some $p_0 \in \partial_x^{\infty,+} V(t_0, x_0)$.

In the same way as before we show that a subsequence of γ_{i_k} converges almost uniformly on $[t_0, +\infty[$ to a solution $p(\cdot)$ of (4.4) such that $p(t_0) = -p_0$ and (4.5) holds a.e. in $[t_0, +\infty[$. □

Remark 5.2. Observe that if in Theorem 5.1 instead of assuming continuity of $V(t_0, \cdot)$ on a neighborhood of x_0 we consider an upper semicontinuous function $\Phi : \mathbb{R}^n \to \mathbb{R}$ satisfying $\Phi(\cdot) \leq V(t_0, \cdot)$ on $B(x_0, r)$ for some $r > 0$ and $\Phi(x_0) = V(t_0, x_0)$, then the very same result can be stated with the transversality conditions involving limiting and horizontal limiting superdifferentials of Φ at x_0 instead of $V(t_0, \cdot)$. Indeed, it is sufficient to replace W by Φ in the above proof.

We recall the following definition.

Definition 5.3. The infinite horizon problem is called calm with respect to the state variable at $(t_0, x_0) \in dom(V)$ if

$$\liminf_{y\to x_0} \frac{V(t_0, y) - V(t_0, x_0)}{|y - x_0|} > -\infty.$$

Theorem 5.4. *Assume (H1) $i) - v)$, (H2) and that the infinite horizon problem is calm with respect to the state variable at $(t_0, x_0) \in dom(V)$. If a trajectory-control pair $(\bar{x}, \bar{u})$ is optimal at (t_0, x_0), then there exists a solution $p(\cdot)$ of the adjoint inclusion (4.3) satisfying the maximality condition (4.1) a.e. in $[t_0, +\infty[$.*

Proof. The calmness assumption implies that for some $r > 0$ and $c \geq 0$ we have

$$V(t_0, y) \geq V(t_0, x_0) - c|y - x_0| \ \ \forall\, y \in B(x_0, r).$$

Consider any Lipschitz function $\Phi : \mathbb{R}^n \to \mathbb{R}$ such that $\Phi(y) = V(t_0, x_0) - c|y - x_0|$ on $B(x_0, r)$. Using Remark 5.2 we complete the proof. □

The next result states that with some optimal trajectories at (t_0, x_0) we can associate a maximum principle whose transversality condition involves the limiting subdifferential $\partial_x^{L,-} V(t_0, x_0)$.

Theorem 5.5. *Let $(t_0, x_0) \in dom(V)$. Assume (H1), (H2), that $f(t, \cdot, u)$ and $L(t, \cdot, u)$ are differentiable for all $t \in \mathbb{R}_+$, $u \in U(t)$ and that $V(t_0, \cdot)$ is continuous on a neighborhood of x_0. Then for every $p_0 \in \partial_x^{L,-} V(t_0, x_0)$ there exists a solution $(x, p)(\cdot)$ of the Hamiltonian inclusion*

$$(-p, x)' \in \partial_{(x,p)} H(t, x, p), \quad (x(t_0), p(t_0)) = (x_0, -p_0) \tag{5.2}$$

such that for some control u the pair (x, u) is optimal at (t_0, x_0) and $-p(t) \in \partial_x^{L,-} V(t, \bar{x}(t))$ for all $t \geq t_0$.

Remark 5.6. Recall that for any solution $(p, x)(\cdot)$ of the Hamiltonian system (5.2) there exists a control $u(\cdot)$ corresponding to $x(\cdot)$. Furthermore, for any such control, the triple $(x, u, p)(\cdot)$ satisfied the maximality condition (4.1) for a.e. $t \geq t_0$.

Proof. Consider a sequence (x_0^i, p_0^i) converging to (x_0, p_0) such that $p_0^i \in \partial_x^- V(t_0, x_0)$. Let $(\bar{x}_i, \bar{u}_i)$ be any optimal trajectory-control pair at (t_0, x_0^i). By Theorem 4.1 for every i the solution p_i of the adjoint system

$$-p_i'(t) = D_x f(t, \bar{x}_i(t), \bar{u}_i(t))^* p_i(t) - L_x(t, \bar{x}_i(t), \bar{u}_i(t)) \quad \text{for a.e. } t \geq t_0, \quad p_i(t_0) = -p_0^i$$

satisfies $-p_i(t) \in \partial_x^- V(t, \bar{x}_i(t))$ for all $t \geq t_0$ and the maximality condition

$$\langle p_i(t), f(t, \bar{x}_i(t), \bar{u}_i(t))\rangle - L(t, \bar{x}_i(t), \bar{u}_i(t)) = H(t, \bar{x}_i(t), p_i(t))$$

a.e. in $[t_0, +\infty[$. It is well known that this yields

$$(-p_i, \bar{x}_i)'(t) \in \partial_{(x,p)} H(t, \bar{x}_i(t), p_i(t)) \quad \text{for a.e. } t \geq t_0. \tag{5.3}$$

Consider a subsequence of $\{(\bar{x}_i, p_i)\}_i$ converging almost uniformly to a locally absolutely continuous function $(\bar{x}, p)$ and such that for every $T > t_0$, $(\bar{x}_i', p_i')_{|[t_0,T]}$ converge weakly in $L^1(t_0, T; \mathbb{R}^n)$ to $(\bar{x}', p')_{|[t_0,T]}$. Then $-p(t) \in \partial_x^{L,-} V(t, \bar{x}(t))$ for all $t \geq t_0$. By the proof of Proposition 3.1, for some $\bar{u} \in \mathcal{U}$ the pair $(\bar{x}, \bar{u})$ is a trajectory-control pair of system (1.5) and

$$V(t_0, x_0) = \lim_{i \to \infty} V(t_0, x_0^i) \geq \int_{t_0}^{\infty} L(t, \bar{x}(t), \bar{u}(t)) dt.$$

Hence $(\bar{x}, \bar{u})$ is optimal for (1.4) at (t_0, x_0). Passing to the limit in (5.3) and using the Mazur theorem and the upper semicontinuity of $\partial_{(x,p)} H(t, \cdot, \cdot)$ we deduce that

$$(-p, \bar{x})'(t) \in \partial_{(x,p)} H(t, \bar{x}(t), p(t)) \quad \text{for a.e. } t \geq t_0.$$

□

We say that $H(t, x, \cdot)$ is strictly convex if for all $q_i \in \partial_p H(t, x, y_i)$, where $i = 1, 2$ and $y_1 \neq y_2$ we have $\langle q_1 - q_2, y_1 - y_2\rangle > 0$.

Corollary 5.7. *Let $(t_0, x_0) \in dom(V)$. Assume (H1), (H2), that $f(t, \cdot, u)$ and $L(t, \cdot, u)$ are differentiable for all $t \in \mathbb{R}_+, u \in U(t)$ and that $V(t_0, \cdot)$ is continuous on a neighborhood of x_0. If H is strictly convex with respect to the last variable, then there exist at least as many optimal solutions to the infinite horizon problem as elements in $\partial_x^{L,-} V(t_0, x_0)$.*

Proof. By Theorem 5.5 with every $p_0 \in \partial_x^{L,-} V(t_0, x_0)$ we can associate a solution (x, p) of the Hamiltonian system

$$(-p, x)'(t) \in \partial_{(x,p)} H(t, x(t), p(t)) \quad \text{for a.e. } t \geq t_0, \quad x(t_0) = x_0,\ p(t_0) = -p_0$$

such that for some control u the pair (x, u) is optimal at (t_0, x_0). It is not difficult to deduce from this inclusion that

$$x'(t) \in \partial_p H(t, x(t), p(t)) \quad \text{for a.e. } t \geq t_0.$$

Consider $p_0^1,\ p_0^2 \in \partial_x^{L,-} V(t_0, x_0)$ and let $(x_i, p_i),\ i = 1, 2$ be some solutions of the above Hamiltonian inclusion with p_0 replaced by p_0^i for $i = 1, 2$. If $x_1 = x_2$ on $[t_0, +\infty[$, then for a.e. $t \geq t_0$ we have

$$0 = \langle x_1'(t) - x_2'(t), p_1(t) - p_2(t)\rangle.$$

By the strict convexity of $H(t, x_1(t), \cdot)$ this implies that $p_1(t) = p_2(t)$ a.e. and from the continuity of p_i we deduce that $p_1 = p_2$. Thus $p_0^1 = p_0^2$. □

6. Maximum Principle and Sensitivity Relations for Locally Lipschitz Value Function

In this section we show that if the value function is locally Lipschitz with respect to the second variable, then *every* optimal trajectory-control pair of the infinite horizon problem satisfies the maximum principle and a sensitivity relation involving the partial generalized gradient of the value function. In this respect conclusions of theorems in this section are stronger than those of Sections 4, 5, provided stronger assumptions are imposed on the value function and the generalized gradients are used instead of the Fréchet/limiting subgradients.

Theorem 6.1. *Assume (H1) i) − v), (H2) and that for all large $T > 0$, the mapping $V(T, \cdot)$ is locally Lipschitz. Then for every $t \geq 0$, $V(t, \cdot)$ is locally Lipschitz with the local Lipschitz constant depending only on the magnitude of t.*

Moreover, if $(\bar{x}, \bar{u})$ is optimal at some $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, then there exists a solution $p(\cdot)$ of the adjoint inclusion (4.3), satisfying the maximality condition (4.1) a.e. in $[t_0, +\infty[$ and the sensitivity relations

$$-p(t_0) \in \partial_x V(t_0, x_0), \quad -p(t) \in \partial_x V(t, \bar{x}(t)) \quad \text{for a.e. } t > t_0.$$

Furthermore, if assumptions of Lemma 3.6 hold true with $\delta > 0$, then $p(t)$ converges exponentially to zero when $t \to \infty$.

Proof. For any sufficiently large integer i consider the Bolza problem

$$\text{minimize } V(i, x(i)) + \int_0^i L(t, x(t), u(t))\, dt$$

over all trajectory-control pairs (x, u) of

$$\begin{cases} x'(t) = f(t, x(t), u(t)), \quad u(t) \in U(t) \qquad \text{for a.e. } t \in [0, i] \\ x(0) = x_0, \end{cases}$$

where $x_0 \in \mathbb{R}^n$. Denote by V^B its value function. It is well known that, under our assumptions, if i is large, then the mapping $V^B(t, \cdot)$ is locally Lipschitz with the local Lipschitz constants independent from $t \in [0, i]$. This and Proposition 3.4 imply the first claim of our theorem.

Let $(\bar{x}, \bar{u})$ be optimal at some $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$. Define the pseudo-Hamiltonian $\mathcal{H} : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ by

$$\mathcal{H}(t, x, p, u) := \langle p, f(t, x, u) \rangle - L(t, x, u).$$

By [11], for any fixed (t, p, u) its generalized gradient with respect to x at $y \in \mathbb{R}^n$ is contained in

$$p\partial_x f(t, y, u) - \partial_x L(t, y, u).$$

From Proposition 3.4 and [12, Theorem 3.1] we deduce the existence of absolutely continuous functions $p_i : [t_0, i] \to \mathbb{R}^n$ satisfying the adjoint inclusion

$$-p_i'(t) \in \partial_x \mathcal{H}(t, \bar{x}(t), p_i(t), \bar{u}(t)) \subset \partial_x f(t, \bar{x}(t), \bar{u}(t))^* p_i(t) - \partial_x L(t, \bar{x}(t), \bar{u}(t)) \text{ for a.e. } t \in [t_0, i],$$

the maximality condition

$$\langle p_i(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle - L(t, \bar{x}(t), \bar{u}(t)) = H(t, \bar{x}(t), p_i(t))$$

a.e. in $[t_0, i]$ and the sensitivity relations $-p_i(t_0) \in \partial_x V(t_0, x_0)$,

$$-p_i(t) \in \partial_x V(t, \bar{x}(t)) \quad \text{for a.e. } t \in [t_0, i].$$

We extend each p_i on $[i, +\infty[$ as a solution of the adjoint inclusion

$$-p_i'(t) \in \partial_x f(t, \bar{x}(t), \bar{u}(t))^* p_i(t) - \partial_x L(t, \bar{x}(t), \bar{u}(t)) \quad \text{for a.e. } t > i$$

starting at $p_i(i)$ at time i.

Using the diagonalization process, extract a subsequence p_{i_k} converging almost uniformly to a locally absolutely continuous function $p : [t_0, +\infty[\to \mathbb{R}^n$ such that for every $T > t_0$ the restrictions of p'_{i_k} to $[t_0, T]$ converge weakly in $L^1(t_0, T; \mathbb{R}^n)$ to p' restricted to $[t_0, T]$. Then $p(\cdot)$ is as required, because the sets $\partial_x f(t, \bar{x}(t), \bar{u}(t))$, $\partial_x L(t, \bar{x}(t), \bar{u}(t))$, are convex, compact and integrably bounded, the sets $\partial_x V(t, \bar{x}(t))$ are compact and $H(t, \bar{x}(t), \cdot)$ is continuous.

To prove the last statement consider $A \subset \mathbb{R}_+$ of null Lebesgue measure such that $-p(t) \in \partial_x V(t, \bar{x}(t))$ for all positive $t \notin A$. By Corollary 3.8 there exists a constant $\gamma > 0$ such that for all $t \notin A$ sufficiently large, $|p(t)| \leq \gamma e^{-\delta t}$. Continuity of p ends the proof. □

The next result uses a different adjoint inclusion and provides one more sensitivity relation. The assumptions are even stronger, implying that the value function is locally Lipschitz in both variables. This sensitivity relation helps to study the limit at infinity of $H(t, \bar{x}(t), p(t))$, whenever $\bar{x}$ is an optimal trajectory and p is a corresponding co-state.

Theorem 6.2. *Assume (H1) i)-v), (H2) with bounded $c(\cdot)$, $\theta(\cdot)$, $\beta(\cdot)$ and that for every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ the set $\{(f(t, x, u), L(t, x, u)) : u \in U(t)\}$ is closed. If for all large $T > 0$, $V(T, \cdot)$ is locally Lipschitz, then V is locally Lipschitz on $[0, \infty[\times \mathbb{R}^n$.*

Let $(\bar{x}, \bar{u})$ be optimal at some $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$. Then there exists a locally absolutely continuous function $p : [t_0, +\infty[\to \mathbb{R}^n$ such that

$$(-p, \bar{x})'(t) \in \partial_{(x,p)} H(t, \bar{x}(t), p(t)), \quad \text{for a.e. } t \geq t_0 \tag{6.1}$$

satisfying the transversality condition

$$-p(t_0) \in \partial_x^{L,+} V(t_0, x_0) \tag{6.2}$$

and the two sensitivity relations

$$-p(t) \in \partial_x V(t, \bar{x}(t)) \quad \text{for a.e. } t > t_0 \ ; \tag{6.3}$$

$$(H(t, \bar{x}(t), p(t)), -p(t)) \in \partial V(t, \bar{x}(t)) \quad \text{for a.e. } t > t_0. \tag{6.4}$$

Furthermore, if assumptions of Lemma 3.6 hold true with $\delta > 0$, then $p(t)$ converges exponentially to zero when $t \to \infty$.

Moreover if assumptions of Theorem 3.9 are satisfied, $\delta > c$ and

$$ess - lim_{t\to\infty} \beta(t)\phi((R + \theta t)e^{ct}) = 0, \quad \forall\, R > 0,$$

then also ess-$lim_{t\to\infty} H(t, \bar{x}(t), p(t)) = 0$, where ess-lim denotes the essential limit.

Proof. Our proof uses [6, Theorem 2.2] stated for the Mayer problem under state-constraints with $c_R(\cdot)$ independent from R. However the proof given there can be easily adapted to the case when $c_R(\cdot)$ depends on R. For any $x_0 \in \mathbb{R}^n$ and positive integer i consider the Bolza problem

$$\text{minimize } \left(V(i, x(i)) + \int_{t_0}^{i} L(t, x(t), u(t))\, dt \right)$$

over all controls $u(\cdot) \in \mathcal{U}$ and trajectories x of

$$\begin{cases} x'(t) = f(t, x(t), u(t)), & \text{for a.e. } t \in [0, i] \\ x(0) = x_0. \end{cases}$$

Denote by V^B its value function. We rewrite it as the Mayer problem

$$\text{minimize } V(i, x(i)) + z(i)$$

over all controls $u(\cdot) \in \mathcal{U}$ and trajectories (x, z) of

$$\begin{cases} x'(t) = f(t, x(t), u(t)) & \text{for a.e. } t \in [0, i], \\ z'(t) = L(t, x(t), u(t)) & \text{for a.e. } t \in [0, i], \\ x(0) = x_0, \ z(0) = 0. \end{cases}$$

From [6, Theorem 2.2] it follows that for all large i, V^B is locally Lipschitz on $[0, i] \times \mathbb{R}^n$. Hence Proposition 3.4 implies that V is locally Lipschitz on $[0, i] \times \mathbb{R}^n$. By the arbitrariness of $i > 0$, V is locally Lipschitz on $[0, \infty[\times\mathbb{R}^n$.

Let $(\bar{x}, \bar{u})$ be optimal at some $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$. Setting $\bar{z}(t) = \int_{t_0}^t L(s, \bar{x}(s), \bar{u}(s))ds$, we deduce that $(\bar{x}, \bar{z})$ is optimal for this new problem. The relations between the maximum principle for the Bolza and Mayer problems are well known and so below we skip the usual calculations leading to an extra co-state equal to the constant -1.

To obtain the maximum principle and the sensitivity relations, it is enough to apply Proposition 3.4, [6, Theorem 2.2] and the comment e) following immediately this theorem to get absolutely continuous mappings $p_i : [t_0, i] \to \mathbb{R}^n$ satisfying

$$(-p_i, \bar{x})'(t) \in \partial_{(x,p)} H(t, \bar{x}(t), p_i(t)), \quad \text{for a.e. } t \in]t_0, i], \tag{6.5}$$

the transversality condition

$$-p_i(t_0) \in \partial_x^{L,+} V(t_0, x_0) \tag{6.6}$$

and the sensitivity relations

$$-p_i(t) \in \partial_x V(t, \bar{x}(t)) \quad \text{for a.e. } t \in]t_0, i] \ ; \tag{6.7}$$

$$(H(t, \bar{x}(t), p_i(t)), -p_i(t)) \in \partial V(t, \bar{x}(t)) \quad \text{for a.e. } t \in]t_0, i]. \tag{6.8}$$

We extend p_i on $]i, +\infty[$ by taking a solution of (6.5) on $[i, +\infty[$ starting at $p_i(i)$ at time i. It remains to find a subsequence p_{i_k} converging almost uniformly to a locally absolutely continuous function $p : [t_0, +\infty[\to \mathbb{R}^n$ such that for every $T > t_0$ the sequence $(p'_{i_k})_{|[t_0,T]}$ converges weakly in $L^1(t_0, T; \mathbb{R}^n)$ to p' restricted to $[t_0, T]$. Passing to the limit in (6.5) - (6.8) and using the closedness of $\partial_x^{L,+} V(t_0, x_0)$, $\partial_x V(t, \bar{x}(t))$, $\partial V(t, \bar{x}(t))$ and the convexity and upper semicontinuity of generalized gradients $\partial_{(x,p)} H(t, \cdot, \cdot)$ we prove (6.1) - (6.4).

The fact that $p(t)$ converges exponentially to zero follows by the same arguments as in the proof of Theorem 6.1. Remark 3.10 implies the last statement.

□

Example 6.3. Given $\lambda > 0$, a closed nonempty set $U \subset \mathbb{R}^m$ and a Borel measurable $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$, consider the classical infinite horizon problem:

$$\text{minimize} \int_0^\infty e^{-\lambda t} \ell(x(t), u(t))\, dt$$

over all trajectory-control pairs (x, u), subject to the state equation

$$\begin{cases} x'(t) = f(x(t), u(t)), \quad u(t) \in U \quad \text{for a.e. } t \geq 0 \\ x(0) = x_0. \end{cases}$$

Assume that f satisfies assumptions **(H1)** with time independent $c(t) \equiv c$, $\theta(t) = \theta$ and $c_R(t) \equiv \delta$ for all $R > 0$. Suppose in addition that $\ell(0, u) \leq M$ for some $M \geq 0$ and all $u \in U$ and

$$|\ell(x, u) - \ell(y, u)| \leq C\big[1 + (|x| \vee |y|)^r\big]|x - y| \quad \forall\, x, y \in \mathbb{R}^n\,,\ u \in U$$

for some constants $C, r \geq 0$. Then, taking

$$\beta(t) = e^{-\lambda t} \quad \text{and} \quad \phi(s) = M + C\big[1 + s^r\big]s,$$

it is easy to check that Theorem 3.9 can be applied provided that

$$\lambda > \delta + c\,r. \tag{6.9}$$

Indeed, assumption (3.17) holds true taking

$$k(t,s) = Ce^{-\lambda t}(1+s^r) \qquad t,s \geq 0,$$

which in turn satisfies (3.12) because

$$\int_0^\infty e^{\delta t} k(t,(R+\theta t)e^{ct})dt = C\int_0^\infty e^{(\delta-\lambda)t}\Big\{1+\big[(R+\theta t)e^{ct}\big]^r\Big\}dt < +\infty \quad \forall\, R \geq 0$$

owing to (6.9). The above assumptions were used in [4] to study analogous problems for linear f and convex compact U.

In particular, if

$$\theta := \sup_{u\in U} \ell(0,u) + \sup_{u\in U} |f(0,u)| < \infty$$

and there exists $0 \leq \delta < \lambda$ such that $(f(\cdot,u)),\ \ell(\cdot,u))$ is $\delta-$Lipschitz for every $u \in U$, then, taking $k(t,s) = e^{-\lambda t}\delta s,\ c = \delta,\ \phi(s) = \theta + \delta s$, we can apply Corollary 3.8 to any fixed optimal trajectory $\bar{x}(\cdot)$ starting from a point $x_0 \in \mathbb{R}^n$. We deduce that the adjoint state $p(\cdot)$ in Theorem 6.1 satisfies

$$|p(t)| \leq e^{-\delta t}\, K_0(1+|x_0|) \qquad \forall\, t \geq 0, \tag{6.10}$$

where

$$K_0(1+|x_0|) = \delta\int_0^\infty e^{(\delta-\lambda)\tau}(1+|x_0|+\theta(\tau-t))d\tau < \infty.$$

It is immediate to check that (6.10) yields the integrability properties of $p(\cdot)$ obtained in [4, Eq. (1.7)].

7. Appendix

7.1. **Proof of Proposition 3.1.**

Proof. Let $(t_0,x_0) \in dom(V)$. Consider a minimising sequence of trajectory-control pairs (x_i,u_i) satisfying $x_i(t_0) = x_0$. By (H1) i) and the Gronwall lemma, for every $T > 0$, the restrictions of x_i to $[0,T]$ are equibounded. Using again (H1) i) and the Ascoli-Arzela theorem we verify that there exists a subsequence x_{i_k} converging almost uniformly to a continuous function $\bar{x} : \mathbb{R}_+ \to \mathbb{R}^n$. By the Dunford-Pettis theorem, taking a subsequence and keeping the same notation, we may assume that for some locally integrable functions $y : \mathbb{R}_+ \to \mathbb{R}^m$, $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ and for every $T > 0$, the restrictions of $(x'_{i_k}(\cdot), L(\cdot, x_{i_k}(\cdot), u_{i_k}(\cdot))$ to $[0,T]$ converge weakly in $L^1(0,T;\mathbb{R}^n\times\mathbb{R}_+)$ to $(y,\gamma)|_{[0,T]}$. Taking the limit yields

$$\bar{x}(t) = x_0 + \int_{t_0}^t y(s)ds \quad \forall\, t \geq 0.$$

Hence $\bar{x}$ is locally absolutely continuous and, by the Lebesgue theorem, $\bar{x}'(t) = y(t)$ a.e. in $\mathbb{R}_+$.

Furthermore, for every $T > t_0$,

$$\int_{t_0}^\infty L(t, x_{i_k}(t), u_{i_k}(t))\, dt \geq \int_{t_0}^T L(t, x_{i_k}(t), u_{i_k}(t))\, dt$$

and, taking the limit, we obtain $V(t_0,x_0) \geq \int_{t_0}^T \gamma(t)dt$.

Since $T > 0$ is arbitrary, we conclude that

$$V(t_0,x_0) \geq \int_{t_0}^\infty \gamma(t)dt. \tag{7.1}$$

To prove that $\bar{x}$ corresponds to an optimal trajectory, fix any $T > 0$ and observe that, by (H1), for some $R > 0$ depending on T and for a.e. $t \in [0, T]$,

$$(x'_{i_k}(t), L(t, x_{i_k}(t), u_{i_k}(t))) \in F(t, x_{i_k}(t)) \subset F(t, \bar{x}(t)) + (c_R(t)|x_{i_k}(t) - \bar{x}(t)| + \omega(t, |x_{i_k}(t) - \bar{x}(t)|))B.$$

Fix any $\varepsilon > 0$. Then for all large k,

$$(x'_{i_k}(t), L(t, x_{i_k}(t), u_{i_k}(t))) \in F(t, \bar{x}(t)) + (c_R(t)\varepsilon + \omega(t, \varepsilon))B \quad \text{for a.e. } t \in [0, T].$$

Finally, observe that the sets $F(t, \bar{x}(t)) + (c_R(t)\varepsilon + \omega(t, \varepsilon))B$ are closed and convex. Since the restrictions of $(x'_{i_k}(\cdot), L(\cdot, x_{i_k}(\cdot), u_{i_k}(\cdot)))$ to $[0, T]$ converge weakly in $L^1(0, T; \mathbb{R}^n \times \mathbb{R}_+)$ to $(\bar{x}', \gamma)|_{[0,T]}$ from the Mazur theorem we deduce that

$$(\bar{x}'(t), \gamma(t)) \in F(t, \bar{x}(t)) + (c_R(t)\varepsilon + \omega(t, \varepsilon))B \quad \text{for a.e. } t \in [0, T].$$

By the arbitrariness of $\varepsilon > 0$, $(\bar{x}'(t), \gamma(t)) \in F(t, \bar{x}(t))$ for a.e. $t \in [0, T]$ and therefore $(\bar{x}'(t), \gamma(t)) \in F(t, \bar{x}(t))$ for a.e. $t \in \mathbb{R}_+$. By the measurable selection theorem there exist a control $\bar{u}(\cdot)$ and a measurable function $r : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\bar{x}'(t) = f(t, \bar{x}(t), \bar{u}(t)), \quad \gamma(t) = L(t, \bar{x}(t), \bar{u}(t)) + r(t).$$

This and (7.1) yield $V(t_0, x_0) \geq \int_{t_0}^{\infty} L(t, \bar{x}(t), \bar{u}(t))dt$ and therefore $(\bar{x}, \bar{u})$ is optimal.

To prove lower semicontinuity consider a sequence (t_0^i, x_0^i) converging to some (t_0, x_0) when $i \to \infty$. We have to show that $\liminf_{i\to\infty} V(t_0^i, x_0^i) \geq V(t_0, x_0)$. It is enough to consider the case when $\{V(t_0^i, x_0^i)\}_{i\geq 1}$ is bounded. Let (x_i, u_i) be the corresponding optimal trajectory-control pairs. We extend $(x_i(\cdot), L(\cdot, x_i(\cdot), u_i(\cdot))$ on $[0, t_0^i[$ by setting $x_i(s) = x_i(t_0^i)$, $L(s, x_i(s), u_i(s)) = 0$ on $[0, t_0^i[$. Using exactly the same arguments as before we extract a subsequence x_{i_k} converging almost uniformly to a locally absolutely continuous function $\bar{x} : \mathbb{R}_+ \to \mathbb{R}^n$, satisfying $\bar{x}(t_0) = x_0$ and such that for every $T > 0$ the restrictions of $(x'_{i_k}(\cdot), L(\cdot, x_{i_k}(\cdot), u_{i_k}(\cdot)))$ to $[0, T]$ converge weakly in $L^1(0, T; \mathbb{R}^m \times \mathbb{R}_+)$ to $(\bar{x}, \gamma)|_{[0,T]}$, where $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a locally integrable function. The proof ends reasoning similarly to the first part.

□

Remark 7.1. Instead of the sublinear growth assumption (H1) i) on f we could alternatively assume that there exists a locally integrable function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\langle f(t, x, u), x\rangle \leq c(t)(1 + |x|^2) \quad \forall\, x \in \mathbb{R}^n,\ u \in U(t) \tag{7.2}$$

and that for every $t_0 \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$ we can find $r > 0$ and an integrable function $c_0 \in L^1(t_0, t_0 + r; \mathbb{R}_+)$ satisfying $\sup_{u\in U(t)} |f(t, x_0, u)| \leq c_0(t)$ a.e. in $[t_0, t_0 + r]$. Then, under all the other assumptions of (H1), to every control $u(\cdot)$ and initial data $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ corresponds an unique locally absolutely continuous solution of

$$\begin{cases} x'(t) &= f(t, x(t), u(t)) \quad \text{for a.e. } t \in \mathbb{R}_+ \ , \\ x(t_0) &= x_0. \end{cases}$$

Then (7.2) implies that for every $T > 0$, the restrictions of x_i to $[0, T]$ are equibounded and the rest of the proof remains the same as the one given above.

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