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On Fractional Cut Covers

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Abstract. Given an undirected graph, a minimum cut cover is a collection of cuts covering the whole set of edges and having minimum cardinality. This paper is dedicated to the fractional version of this problem where a fractional weight is computed for each cut such that, for each edge, the sum of the weights of all cuts containing it is no less than 1, while the sum of all weights is minimized. The fractional cover is computed for different graph classes among which the weakly bipartite graphs. Efficient algorithms are described to compute lower and upper bounds with worst-case performance guarantees. A general randomized approach is also presented giving new insights into Goemans and Williamson's algorithm for the maximum cut problem. Some numerical experiments are included to assess the quality of bounds.

Keywords: graph cuts, covering, approximation, randomized algorithm

1 Introduction

Let G denote an undirected simple graph with node set V and a non-empty edge set E. Given a node subset $S \subseteq V$, the *cut* defined by S, denoted by $\delta(S)$, is the set of all edges of E with exactly one endpoint in S: $\delta(S) = \{uv \in E\colon u \in S \text{ and } v \in V \setminus S\}$. A *cut cover* is a collection of cuts whose union is E. Let $\mathcal{C}$ denote the set of all cuts in G. Given a cut $C \in \mathcal{C}$, let $\chi^C \in \{0,1\}^E$ denote its incidence vector: $(\chi^C)_e = 1$ if and only if $e \in C$, for all $e \in E$. The all-ones vector indexed on the edge set is denoted by $\mathbf{1_E}$. The complete graph with order n is denoted by K_n. We also use $\mathbb{Z}_+$ (resp. $\mathbb{R}_+$) to denote the set of non-negative integer (resp. real) numbers.

The problem which consists in determining the minimum cardinality of a cut cover can be formulated as the following integer linear program.

$$(CC)\begin{cases} Z^*_{CC}(G) = \min \sum_{c\in\mathcal{C}} \lambda_C \\ \text{s.t.} \\ \quad \sum_{C\in\mathcal{C}} \lambda_C \chi^C \geq \mathbf{1_E}, \\ \quad \lambda \in \mathbb{Z}^{\mathcal{C}}_+. \end{cases}$$

Problem (CC) has applications in the testing for possible defects of printed circuit boards (PCB's) [15,28]. There the objective is to minimize the number of tests to be performed on a PCB in order to check it does not contain a short circuit. The situation can be modeled by means of an undirected simple graph G where each node corresponds to a connecting point of the PCB and each edge to one component of the PCB. A cut in G can then be interpreted as the testing of all the components of the PCB that are represented by the edges this cut contains.

Problem (CC) is generally $\mathcal{NP}$-hard, as it is now well-known that $Z^*_{CC}(G) = \lceil \log_2 \chi(G) \rceil$ for any graph G [21,22,28,30,33], where $\chi(G)$ stands for the chromatic number of G.

In this paper, we focus our attention on the *fractional cut cover problem* which corresponds to the linear relaxation of (CC), denoted by (FC).

$$(FC)\begin{cases} Z^*_{FC}(G) = \min \sum_{C\in\mathcal{C}} \lambda_C \\ \text{s.t.} \\ \quad \sum_{C\in\mathcal{C}} \lambda_C \chi^C \geq \mathbf{1_E}, \\ \quad \lambda \in \mathbb{R}^{\mathcal{C}}_+. \end{cases}$$

Problem (FC) was introduced by Samal in [37,38,39]. His main results will be reviewed in the next section.

In what follows, we may simply write Z^*_{FC} instead of $Z^*_{FC}(G)$ when G is clear from the context. Since (FC) corresponds to the linear relaxation of (CC), we have the upper bound: $Z^*_{FC} \leq \lceil \log_2 \chi(G) \rceil$. This inequality is sharp since it holds at equality, for instance, for bipartite graphs (in which case $Z^*_{FC} = \lceil \log_2 \chi(G) \rceil = 1$). Introducing the variable $\theta = \frac{1}{\sum_{C\in\mathcal{C}} \lambda_C}$ in formulation (FC), we can easily see that the latter is equivalent to the following linear program

$$(P1)\begin{cases} Z^*_{P1}(G) = \max \theta \\ \text{s.t.} \\ \quad \sum_{C\in\mathcal{C}} \lambda_C \chi^C \geq \theta \mathbf{1_E}, \\ \quad \sum_{C\in\mathcal{C}} \lambda_C \leq 1, \\ \quad \lambda \in \mathbb{R}^{\mathcal{C}}_+, \theta \in \mathbb{R}, \end{cases}$$

in particular, $Z^*_{FC(G)} = \frac{1}{Z^*_{P1}(G)}$. Here, we may interpret the variable λ_C as a weight assigned to the cut $C \in \mathcal{C}$. Define the weight of an edge $e \in E$ as the sum of the weights of the cuts containing it: $\sum_{C \in \mathcal{C}:\, e \in C} \lambda_C$. Then, problem $(P1)$ consists in determining non-negative weights to be assigned to all cuts such that the weights sum up to 1, and the minimum edge weight taken over all edges is maximized.

Let us mention a potential application of $(P1)$ for frequency assignment. Assume that there are only two frequencies to be assigned to the nodes of a wireless communication network. Let $G = (V, E)$ denote an undirected graph whose node set corresponds to the nodes of the network and there is an edge $uv \in E$ if and only if the assignment of the same frequency to the nodes u and v causes interferences. Note that determining a frequency assignment which minimizes the total number of interferences reduces to finding a maximum cardinality cut in G. Indeed, given a cut $\delta(S)$ with $S \subseteq V$, the corresponding assignment consists in setting the same frequency to all the nodes in S and the other available frequency to all the other nodes. Assume now that during one unit of time, we have the possibility of changing the frequency assignments. We wish to determine a sequence of assignments together with the duration of each (this is called a frequency hopping pattern in GSM networks), so as to maximize the minimum amount of time - taken over all edges $e \in E$ - that both endpoints of e are assigned with different frequencies. The portion of time of the assignment that is allocated to the cut $C \in \mathcal{C}$ is represented by the variable λ_C. Also, given the objective sense: "maximize", in any optimal solution of $(P1)$, the value of θ gives the minimum portion of time - taken over all edges $e \in E$ - that there are no interferences between both endpoints of e.

Contents of the paper

Section 2 is dedicated to the presentation of the main known results related to the fractional cut cover problem. In Section 3, we exhibit new bounds on Z^*_{FC} stemming from polynomial-time solvable approximations of (FC): one of them rests upon a cutting plane algorithm for solving (DC) using an approximate separation algorithm (Subsection 3.1), while another relies on an approximation of the cut polytope (Subsection 3.2). Polynomial-time randomized algorithms providing upper bounds on Z^*_{FC} and giving a new insight into Goemans and Willamson's algorithm [16] for the maximum cut problem are presented in Section 4. Computational experiments are reported in Section 5, before we conclude in Section 6.

2 Related work and some straightforward extensions

To the present authors' knowledge, investigations explicitly dedicated to the study of the value Z^*_{FC} seem to appear mainly in Samal's works [37,38,39] where the interest for this graph parameter

stems from investigations on graph homomorphisms; in particular, it could be used to prove the nonexistence of some graph homomorphisms. A *graph homomorphism* f from a graph $G = (V, E)$ to a graph $H = (V_H, E_H)$, written $f : G \to H$, is a function from V to V_H that maps endpoints of each edge in G to endpoints of an edge in H.

Samal [39] initially defined a n/k-cover as an n-tuple of cuts such that each edge belongs to at least k of them. He introduced the parameter $x(G) = \inf\{\frac{n}{k} : \text{a } n/k\text{-cover of } G \text{ exists}\}$, mentioning that it coincides with Z^*_{FC} as introduced before.

Let $Q_{n/k} = (V_Q, E_Q)$ denote the graph having $\{0, 1\}^n$ for set of vertices, and where xy forms an edge if and only if the Hamming distance between x and y is larger than or equal to k. Samal observed [39] that a graph has a n/k-cover if and only if it is homomorphic to $Q_{n/k}$. Indeed, starting from a n/k-cover $(X_1, ..., X_n)$ where $X_i = \delta(W_i)$, one can build a homomorphism $f = (f_1, ..., f_n) : V \to V_Q$ by setting $f_i(v) = 1$ if $v \in W_i$ and $f_i(v) = 0$ otherwise. Since each edge $uv \in E$ belongs to at least k cuts, the Hamming distance between $f(u)$ and $f(v)$ will be larger than or equal to k implying that $Q_{n/k}$ contains an edge $f(u)f(v)$. The other direction can be proved in a very similar way by building cuts starting from a homomorphism $f = (f_1, ..., f_n)$. Engström *et al.* [10] proved the next result which was conjectured earlier by Samal in [38].

Proposition 1. *[10] Let k, n be integers such that $k \leq n < 2k$. Then,*

$$x(Q_{n/k}) = \begin{cases} n/k \text{ if } k \text{ is even, and} \\ (n+1)/(k+1) \text{ otherwise.} \end{cases}$$

A straightforward lower bound related to the maximum cardinality of a cut is recalled in the next proposition. It can be derived by summing up the inequalities of the formulation (FC) and considering an optimal solution of it.

Proposition 2. *[39] The following inequality holds: $Z^*_{FC} \geq \frac{|E|}{mc(G)}$, where $mc(G)$ denotes the maximum cardinality of a cut in G. If G is edge-transitive, then equality holds.*

The *odd girth* of G, denoted by $\overline{g}$ is the length (≥ 3) of the smallest odd cycle in the graph. In case the graph G is bipartite, its odd girth is defined to be infinity. The following lower bound is easy to compute [32].

Proposition 3. *[39] Let G be a nonbipartite graph with odd girth $\overline{g}$. Then, $Z^*_{FC} \geq \frac{\overline{g}}{\overline{g}-1}$. If G is an odd cycle, then $Z^*_{FC} = \frac{\overline{g}}{\overline{g}-1} = \frac{|E|}{|E|-1}$.*

The case of complete graphs is easy to handle. An optimal solution of (FC) is obtained by considering the same weight λ_C for all cuts $\delta(X)$ with $|X| = n/2$ (resp. $|X| = (n-1)/2$) when n is even (resp. odd) and $\lambda_C = 0$ otherwise.

Proposition 4. *[39]* $Z^*_{FC}(K_n) = 2\left(1 - \frac{1}{n+(n \mod 2)}\right)$.

Combining the propositions above and the fact that Z^*_{FC} is monotone non-decreasing with respect to edge inclusion, leads to the following corollary.

Corollary 1. *[39] Let $G = (V, E)$ denote an undirected simple graph with $|V| = n, |E| \geq 1$. Then, the following inequalities hold:* $1 \leq \frac{|E|}{mc(G)} \leq Z^*_{FC} \leq 2\left(1 - \frac{1}{n+(n \mod 2)}\right) < 2$.

Other bounds on Z^*_{FC} following from the work by Samal involve the chromatic number and the *vector chromatic number* [24]. The latter is the quantity $\chi_v = 1 - \frac{1}{\rho}$ where ρ denotes the minimum value such that there exists a mapping $f : V \to \mathbb{R}^V$ satisfying the conditions $\|f(v)\| = 1$, for all $v \in V$, and $(f(u))^T f(v) \leq \rho$, for all $uv \in E$, where the notation T stands for the transpose. Note that the vector chromatic number satisfies $\chi_v \geq 1$ and an approximation of χ_v can be computed efficiently using semidefinite programming (SDP) [24].

Proposition 5. *[39] The following inequalities hold:*

$$2\left(1 - \frac{1}{\chi_v}\right) \leq Z^*_{FC} \leq 2\left(1 - \frac{2}{\pi\chi_v}\right), \text{ and}$$
$$Z^*_{FC} \leq 2 - \frac{1}{\left\lceil \frac{\chi(G)}{2} \right\rceil}.$$

The last inequality involves the standard chromatic number. Note that if G is neither an odd-cycle nor a complete graph, then $\chi(G) \leq \Delta$ by Brook's theorem, where Δ denotes the maximum degree in G. Consequently, $Z^*_{FC} \leq 2 - \frac{1}{\left\lceil \frac{\Delta}{2} \right\rceil}$. This bound dominates the upper bound presented in Corollary 1.

Let $G(n, p)$ denote a random graph generated according to the Erdös-Rényi model [5,11]. For such graphs, Samal [39] proved the following.

Proposition 6. *[39]* $\frac{2}{2 - Z^*_{FC}(G(n,p))} = \Theta(\sqrt{pn})$ *a.a.s.*

McGuinness proved [31] that for any simple graph $G = (V, E)$ having k components and cogirth (i.e. the minimum cardinality of any nontrivial cut) $g^* \geq 3$, there is a family of at most $|E| - |V| + k$ cocycles (i.e. minimal cuts) which cover the edges of G at least twice. This implies the next result that is useful only if $|E| - |V| \leq 2$.

Proposition 7. *For any simple connected graph $G = (V, E)$ with cogirth at least three, we have* $Z^*_{FC} \leq \frac{|E|-|V|+1}{2}$.

Let us consider the dual program (DC) of (FC) that we mention hereafter for reference later.

$$(DC) \begin{cases} Z^*_{DC} = \max \sum_{e \in E} u_e \\ \text{s.t.} \\ \quad \sum_{e \in C} u_e \leq 1, \forall C \in \mathcal{C}, \\ \quad u \in \mathbb{R}^E_+. \end{cases}$$

It corresponds to the determination of nonnegative weights to be assigned to the edges such that the sum of the weights taken over all edges is maximized, and each cut has weight at most 1. Note that from linear programming (strong) duality we have $Z^*_{DC} = Z^*_{FC}$. Another special feature of Z^*_{FC} is its connection with the fractional chromatic number. Let $\mathcal{S}$ denote the set of all the stable sets in G. Given a stable set $S \in \mathcal{S}$, let $\chi^S \in \{0,1\}^V$ denote the incidence vector of S. The fractional chromatic number of the graph G, denoted by $\chi_f(G)$, is the optimal objective value of the linear program

$$(CHROM) \begin{cases} \min \sum_{S \in \mathcal{S}} \mu_S \\ \text{s.t.} \\ \quad \sum_{S \in \mathcal{S}} \mu_S \chi^S \geq \mathbf{1_V}, \\ \quad \mu \in \mathbb{R}^{\mathcal{S}}_+. \end{cases}$$

Given the relation $Z^*_{FC} \leq Z^*_{CC} = \lceil \log_2 \chi(G) \rceil$ mentioned above, we may naturally wonder whether there are connections between Z^*_{FC} and $\chi_f(G)$. A preliminary result is the following relation which was proved in [39] using graph homomorphisms. We provide an alternative proof based on linear programming and duality.

Proposition 8. *[39] The following inequality holds: $Z^*_{FC} \leq \frac{1}{2}\chi_f(G)$.*

Proof. The dual linear program of $(CHROM)$ may be formulated as follows.

$$(D_{CHROM}) \begin{cases} \max \sum_{v \in V} y_v \\ \text{s.t.} \\ \quad \sum_{v \in S} y_v \leq 1, \forall S \in \mathcal{S}, \\ \quad y \in \mathbb{R}^V_+. \end{cases}$$

Let $u^* \in \mathbb{R}^E_+$ denote an optimal solution of (DC). Then, we define a solution y for the linear program (D_{CHROM}) as follows: $y_v := \sum_{e \in \delta(v)} u^*_e$. Note that for any stable set $S \in \mathcal{S}$ we have that $\sum_{v \in S} y_v = \sum_{e \in \delta(S)} u^*_e \leq 1$, and thus y is a feasible solution for D_{CHROM}. Its objective value is a lower bound for $\chi_f(G)$ and is equal to $\sum_{v \in V} y_v = \sum_{v \in V} \sum_{e \in \delta(v)} u^*_e = 2Z^*_{FC}$. The result follows.

□

We terminate this section mentioning an upper bound on Z^*_{FC} based on Fajtlowicz'result [12] that, for any graph G with maximum clique size ω and maximum degree Δ, the inequality $\chi_f(G) \leq \frac{\omega+\Delta+1}{2}$ holds. Proposition 8 leads to another upper bound on Z^*_{FC} that is of potential interest for graphs such that $\omega + \Delta \leq 6$ (in view of Corollary 1). In particular, it is tight if G is a matching.

Corollary 2. *For any graph G with maximum clique size ω and maximum degree Δ the following inequality holds:*

$$Z^*_{FC} \leq \frac{\omega + \Delta + 1}{4}.$$

3 Bounds on Z^*_{FC} based on polynomial-time solvable approximations of (FC)

A SDP-based approximation algorithm has been proposed in [39] for the so called *cubical chromatic number* defined by $\chi_q = \frac{2}{2-Z^*_{FC}}$. It relies on relations between χ_q and the vector chromatic number χ_v (see Proposition 5), and it uses the results by Karger et al. [24] for approximating χ_v in polynomial time. The approximation ratio resulting from this approach is $\frac{\pi}{2}$. This algorithm can be used to get a $2(1-1/\pi)$ approximation of Z^*_{FC}. Indeed, using an approximation $\frac{2}{2-\overline{Z}_{FC}}$ of the cubical number χ_q that is satisfying $\frac{2}{2-\overline{Z}_{FC}} \leq \frac{\pi}{2}\chi_q = \frac{\pi}{2}\frac{2}{2-Z^*_{FC}} = \frac{\pi}{2-Z^*_{FC}}$, we get that $\overline{Z}_{FC} \leq (2-\frac{4}{\pi}) + \frac{2}{\pi}Z^*_{FC}$. Combining this with $1 \leq Z^*_{FC}$, we deduce that $\overline{Z}_{FC} \leq (2-\frac{2}{\pi})Z^*_{FC}$. A different approach leading to a better approximation is presented below. We will also provide a lower bound related to a relaxation of the cut polytope.

3.1 Approximation based on an approximate separation to solve (DC)

Since our approximation is using the celebrated algorithm of Goemans and Willamson [16] to solve the maximum cut problem, we recall the main ingredients of this algorithm. Given an undirected graph $G = (V, E)$ and edge weights $(w_e)_{e\in E}$ with $w_e \in \mathbb{R}$, for all $e \in E$, the maximum cut problem consists in finding a cut of G such that the sum of the weights of the edges it contains is maximized. This is a notorious $\mathcal{NP}$-hard problem (see, e.g., the survey [4] and the references therein). However, it can be approximated with the algorithm of [16] using semidefinite programming. The semidefinite relaxation below is solved, where $w_{ij} = 0$ if $ij \notin E$ and $w_{ij} = w_e$ for $e = ij$ otherwise, and the matrix Y is a $|V| \times |V|$ matrix that is required to be symmetric and positive semidefinite, which is denoted by $Y \succeq 0$.

$$(SDP_{GW}) \begin{cases} \max \frac{1}{4}\sum_{i\in V, j\in V} w_{ij}(1-Y_{ij}) \\ \text{s.t.} \\ \qquad Y_{ii} = 1, \forall i \in V, \\ \qquad Y \succeq 0, Y \in \mathbb{R}^{V\times V}. \end{cases}$$

An optimal solution Y^* of (SDP_{GW}) is a Gram matrix related to a set of $|V|$ vectors $(\sigma^i)_{i\in V}$ belonging to $\mathbb{R}^V$ (i.e. $Y^*_{ij} = (\sigma^i)^T\sigma^j$). Then, a randomly generated hyperplane in $\mathbb{R}^V$ containing the origin separates the set of vectors $(\sigma^i)_{i\in V}$ in two subsets, depending on their position with respect to the hyperplane. This naturally induces a cut. Under the assumption that all the edge weights are non-negative, Goemans and Willamson [16] show that the expectation of the weight of the cut produced in this way is less than or equal to α times the optimal weight, where $\alpha =$

$\min_{0\leq\theta\leq\pi} \frac{2}{\pi}\frac{\theta}{1-\cos\theta} > 0.87856$. A deterministic version of their algorithm is proposed by Mahajan and Ramesh [29].

Proposition 9. *A value $\overline{Z}$ satisfying $Z^*_{FC} - \epsilon \leq \overline{Z} \leq (\frac{1}{\alpha} + \epsilon)Z^*_{FC}$ can be computed in polynomial time, for any $\epsilon > 0$.*

Proof. The proof we give is inspired by [7,13,23] where an approximate separation algorithm is used within the framework of the ellipsoid algorithm.

From Corollary 1, we have: $1 \leq Z^*_{FC} = Z^*_{DC} \leq 2$. We now apply binary search in the interval $[1, 2]$ to determine an approximation of Z^*_{DC} with precision ϵ.

Given a value $\tau \in [1, 2]$, the following linear program is solved with a cutting plane algorithm.

$$(D_\tau) \begin{cases} \max \sum_{e\in E} u_e \\ \text{s.t.} \\ \quad \sum_{e\in E} u_e \geq \tau, \\ \quad \sum_{e\in C} u_e \leq 1, \forall C \in \mathcal{C}, \\ \quad u \in \mathbb{R}^E_+. \end{cases}$$

The formulation (D_τ) is obtained from (DC) by adding a constraint enforcing a lower bound on the optimal objective value. The cutting plane algorithm to solve (D_τ) proceeds as follows.

- Step 1. A first relaxation of (D_τ) is built. Let $\widehat{\mathcal{C}} \subseteq \mathcal{C}$ denote a set of at most $|E|$ different cuts and covering all the edges, that is, for each $e \in E$, there exists some cut $C \in \widehat{\mathcal{C}}$, such that $e \in C$. The relaxation (RD_τ) of (D_τ) is initialized with

$$(RD_\tau) \begin{cases} \max \sum_{e\in E} u_e \\ \text{s.t.} \\ \quad \sum_{e\in E} u_e \geq \tau, \\ \quad \sum_{e\in C} u_e \leq 1, \forall C \in \widehat{\mathcal{C}}, \\ \quad u \in \mathbb{R}^E_+. \end{cases}$$

- Step 2. The current relaxation (RD_τ) is solved by the ellipsoid algorithm. Then, an approximate separation algorithm is used to determine if some inequality of the form "$\sum_{e\in C} u_e \leq 1$" is violated by the solution found, for some $C \in \mathcal{C}$.
- Step 3. In case a violated inequality is found, the set $\widehat{\mathcal{C}}$ is updated by adding this inequality to the current relaxation (RD_τ) and Step 2 is repeated. Otherwise the algorithm stops.

Let us now describe the approximate separation algorithm that is used. Given an optimal solution u^* of the current relaxation (RD_τ) of (D_τ) with objective value $Z_D(\tau)$, we solve the SDP relaxation

of the maximum cut problem with edge cost function u^* [16]. Using the fact that all the entries of u^* are non-negative, for any $\epsilon' > 0$, it is possible to compute in polynomial time (in the input size and $\log(\frac{1}{\epsilon'})$) [16,29] a cut with objective value Z_{cut}, greater than or equal to $\alpha(Z^*_{sdp}(u^*) - \epsilon')$, where $Z^*_{sdp}(u^*)$ denotes the optimal objective value of the SDP relaxation (SDP_{GW}) with the edge costs u^*. In what follows, we take $\epsilon' = \epsilon$. In case the cut found has an objective value strictly larger than 1, the corresponding inequality in the formulation of (DC) is added to the current relaxation (RD_τ). Otherwise (i.e. the cut found has weight at most 1), we stop.

So, for some given value τ, with the separation algorithm described before, the ellipsoid algorithm stops either when infeasibility is detected (i.e. D_τ has no feasible solution) or because no violated constraint is found by the approximate separation algorithm. At each iteration, τ belongs to an interval $[lb, ub]$ (initially we have $lb = 1$ and $ub = 2$). This interval is updated by binary search in a standard way. If infeasibility is detected for the current value of τ, then we set $ub = \tau$ and we take $\tau = \frac{lb+ub}{2}$. On the other hand, if no violated constraint is detected, then we set $lb = \tau$ and we update τ in the same way as before. The binary search stops when $ub - lb$ falls below ϵ.

Let $\overline{\tau}$ denote the largest value of τ for which the ellipsoid algorithm stopped because no violated inequality has been found. Then, given the precision of the binary search, we know that $Z^*_{DC} \leq \overline{\tau} + \epsilon$ since by taking $\tau = \overline{\tau} + \epsilon$, we get an unfeasible problem D_τ. In addition, let $\overline{u}$ denote the approximate solution obtained when approximately solving $D_{\overline{\tau}}$ as mentioned before. The maximum weight of any cut with respect to the edge costs $(\overline{u}_e)_{e \in E}$ is upper bounded by $\frac{1}{\alpha} + \epsilon$. This implies $\frac{\alpha}{1+\alpha\epsilon}\overline{u}$ is feasible for (DC), and thus $Z^*_{DC} \geq \frac{\alpha}{1+\alpha\epsilon}\overline{\tau}$.

By taking $\overline{Z} = \overline{\tau}$ we get that $Z^*_{FC} - \epsilon \leq \overline{Z} \leq (\frac{1}{\alpha} + \epsilon) Z^*_{FC}$.

To finish the proof, let us prove that the whole algorithm is polynomial-time. Since the number of iterations of the binary search is $\mathcal{O}(\ln(1/\epsilon))$, we only have to show that (D_τ) can be solved in polynomial time. If a strong separation algorithm (see [18] for an accurate definition) were used, then the results by Grötschel *et al.* [18, Chapter 6] show that for solving (D_τ) the number of calls to such a separation algorithm would be polynomial. In our case, since the procedure we described above corresponds to the application of such a method with a premature stop, the number of calls to the approximate separation algorithm is polynomial. This implies that the number of inequalities added to (RD_τ) is polynomially bounded. Consequently, each relaxation (RD_τ) can be solved in polynomial time, and since the separation takes polynomial time, it follows that the whole procedure to (approximately) solve D_τ is polynomial-time. □

The cutting-plane approach used in this section clearly implies that if the graph G belongs to a class of graphs for which the maximum cut problem can be solved in polynomial time, then by

equivalence of separation and optimization, Z^*_{DC} can also be computed in polynomial time. Several such classes are known (see, e.g., [4,9]).

3.2 Approximations by modifying the feasible region of (DC)

Given a polytope Q, let $ext(Q)$ denote the set of its extreme points. Consider the linear program, denoted (D_Λ), whose formulation follows and where Λ stands for a polytope contained in $\mathbb{R}^E$.

$$(D_\Lambda) \begin{cases} Z^*_D(\Lambda) = \max \sum_{e\in E} u_e \\ \text{s.t.} \\ \quad \sum_{e\in E} \alpha_e u_e \leq 1, \forall \alpha \in ext(\Lambda), \\ \quad u \in \mathbb{R}^E_+. \end{cases}$$

Taking for Λ the cut polytope of the graph G: $CUT(G)$, we get formulation (DC). Furthermore, it is well-known that $CUT(G) \subseteq MET(G)$, where $MET(G) \subseteq \mathbb{R}^E$ denotes the metric polytope of the graph $G = (V, E)$ [1,2], that is defined by the following set of constraints:

$$\begin{cases} \sum_{e\in F} x_e - \sum_{e\in C\setminus F} x_e \leq |F| - 1, \forall \text{ cycle } C, F \subseteq C, |F| \text{ odd}, \\ 0 \leq x_e \leq 1, \forall e \in E. \end{cases}$$

Recall that, for a graph G that is not contractible to K_5, we have $CUT(G) = MET(G)$ [2]. Since optimizing a linear function over $MET(G)$ can be done in polynomial time (due to the equivalence of optimization and separation [18], and the fact that there exists a polynomial time separation oracle for $MET(G)$ [2]), it follows that for a graph not contractible to K_5, $Z^*_{FC} = Z^*_D(MET(G))$ can be computed in polynomial time. For graphs not satisfying this restriction, we have the next general result.

Proposition 10. *Let G denote an undirected graph. Then, we have*

$$Z^*_D(MET(G)) \leq Z^*_{FC} = Z^*_{DC}, \tag{1}$$

and the quantity $Z^*_D(MET(G))$ can be computed in polynomial time.

The next proposition states that the value of $Z^*_D(MET(G))$ is related to the odd girth.

Proposition 11. *Let $G = (V, E)$ denote a nonbipartite undirected graph with odd girth $\bar{g}$. Then $Z^*_D(MET(G)) = \frac{\bar{g}}{\bar{g}-1}$.*

Proof. Let C be a cycle of length $\bar{g}$ and define a vector u with $u_e = \frac{1}{\bar{g}-1}$ for $e \in C$ and $u_e = 0$ otherwise. Then, for any vector $\alpha \in MET(G)$, we have $\sum_{e\in C} \alpha_e \leq |C| - 1$, implying that $u^T\alpha \leq 1$. Hence u is feasible for $\left(D_{MET(G)}\right)$ and thus $Z^*_D(MET(G)) \geq \sum_{e\in E} u_e = \frac{\bar{g}}{\bar{g}-1}$.

On the other hand, consider the vector α defined by $\alpha_e = \frac{\bar{g}-1}{\bar{g}}$ for all $e \in E$. Let us show that $\alpha \in MET(G)$. We clearly have $0 \leq \alpha_e \leq 1$ for all $e \in E$. For any cycle C' and any subset of edges $F \subseteq C'$ of odd cardinality, we have

$$\sum_{e \in F} \alpha_e - \sum_{e \in C' \setminus F} \alpha_e = \frac{\bar{g}-1}{\bar{g}}(|F| - |C' \setminus F|).$$

If $F = C'$, then we see that $\frac{\bar{g}-1}{\bar{g}}|C'| \leq |C'| - 1$, since $|C'| \geq \bar{g}$. Otherwise, F is a proper subset of C' and we have

$$\frac{\bar{g}-1}{\bar{g}}(|F| - |C' \setminus F|) \leq \frac{\bar{g}-1}{\bar{g}}(|F| - 1) \leq |F| - 1.$$

Hence, for any feasible $u \in D_{MET(G)}$, we have $u^T \alpha \leq 1$ leading to the reverse inequality $\sum_{e \in E} u_e \leq \frac{\bar{g}}{\bar{g}-1}$. □

We already observed that for graphs not contractible to K_5, Z^*_{FC} can be computed in polynomial time. From Proposition 11, we get the exact value of Z^*_{FC} for this kind of graphs: $Z^*_{FC} = \frac{\bar{g}}{\bar{g}-1}$. In fact, as will be shown below, the result holds for the more general class of weakly bipartite graphs defined in [19]. It is shown in [14] that each graph not contractible to K_5 is weakly bipartite. Weakly bipartite graphs have been characterized by Guenin in [20] proving a conjecture of Seymour [40].

Proposition 12. *Let G denote a weakly bipartite graph with odd girth $\bar{g} < +\infty$. Then, $Z^*_{FC} = \frac{\bar{g}}{\bar{g}-1}$.*

Proof. Consider the polytope $ODD(G)$ given by constraints

$$\begin{cases} \sum_{e \in C} x_e \leq |C| - 1, \forall \text{ cycle } C, |C| \text{ odd}, \\ 0 \leq x_e \leq 1, \forall e \in E. \end{cases}$$

This polytope clearly contains $MET(G)$, leading to $Z^*_D(ODD(G)) \leq Z^*_D(MET(G))$. Moreover, the vector u introduced in the proof of Proposition 11 is also feasible for $D_{ODD(G)}$ implying that $Z^*_D(ODD(G)) \geq \frac{\bar{g}}{\bar{g}-1}$. Combining the two inequalities we deduce that $Z^*_D(ODD(G)) = Z^*_D(MET(G)) = \frac{\bar{g}}{\bar{g}-1}$. Let us now consider the polytope $P_B(G)$ defined as the convex hull of the incidence vectors of all bipartite subgraphs of G $\left(P_B(G) = Conv\left\{\chi^F, F \subset E, (V,F) \text{ is bipartite}\right\}\right)$. Observe that $CUT(G) \subset P_B(G)$, and when edge weights are non-negative, a maximum weight bipartite subgraph is obtained by computing a maximum weight cut. Thus, imposing that $u^T \alpha \leq 1$ for any $\alpha \in CUT(G)$ is equivalent to the same requirement for any $\alpha \in P_B(G)$. This leads to $Z^*_{FC} = Z^*_D(CUT(G)) = Z^*_D(P_B(G))$. Finally, since weakly bipartite graphs are those for which $P_B(G) = ODD(G)$, we have $Z^*_D(ODD(G)) = Z^*_D(P_B(G))$, leading to $Z^*_{FC} = \frac{\bar{g}}{\bar{g}-1}$.

Note that by combining Propositions 11 and 10, we deduce that $Z^*_{FC} \geq \frac{\bar{g}}{\bar{g}-1}$ if $\bar{g} < +\infty$, which is already known from Proposition 3. Observe also that the odd cycle case mentioned in Proposition 3 is a special case of Proposition 12.

Other lower bounds of better quality might be obtained using a better relaxation of $P_B(G)$ (see [3]).

4 Randomized algorithms

4.1 General framework

Given an undirected simple graph $G = (V, E)$, let $\mathcal{E}$ denote some abstract set of elements (particular definitions will be given and investigated later). For each $i \in V$, let $\sigma^i \in \mathbb{R}^{\mathcal{E}}$ denote a vector of real values indexed on $\mathcal{E}$. For each $e \in \mathcal{E}$, let u_e denote a normally-distributed random variable $u_e \sim N(0, 1)$. The random variables $(u_e)_{e\in\mathcal{E}}$ are supposed to be independent. We then define, for each $i \in V$, the random variable

$$Y_i = \sum_{e\in\mathcal{E}} \sigma_e^i u_e.$$

Assuming the vectors $\left(\sigma^i\right)_{i\in V}$ are given, the proposed randomized scheme generates a cut $\delta(S)$ induced by the sign of the random variables Y_i, $i \in V$. An edge ij will then appear in a cut with a probability $Pr(Y_iY_j \leq 0)$, where $Pr(.)$ denotes the probability of an event.

This randomized scheme will be used to provide good feasible solutions of problem $(P1)$. For each cut $C \in \mathcal{C}$ in G, define λ_C as the probability that the randomized algorithm outputs the cut C. Then, the probability for any edge $ij \in E$ to belong to the cut output by the algorithm is given by $\sum_{C\in\mathcal{C}:\ ij\in C} \lambda_C$. This namely implies that $\left(\lambda, \theta = \min\limits_{ij\in E} \sum\limits_{C\in\mathcal{C}:\ ij\in C} \lambda_C\right)$ is a feasible solution for $(P1)$ leading to a lower bound for Z^*_{P1} and thus an upper bound for Z^*_{FC} for the case when $\theta \neq 0$.

Let us show that the probability for an edge to belong to a cut can be computed exactly. Our randomized scheme is intimately related to the rounding technique of Goemans and Williamson [16]. As already said in Section 3.1, after solving (SDP_{GW}) and obtaining unit vectors $(\sigma^i)_{i\in V}$, a random hyperplane is generated and used to partition V. Let u be the normal vector corresponding to this hyperplane. u is then a random vector belonging to the hypersphere. A basic geometric fact observed by Goemans and Williamson is that the probability to have i and j in the cut is given by $Pr(u^T\sigma^i.u^T\sigma^j \leq 0) = \frac{1}{\pi}\arccos\left((\sigma^i)^T\sigma^j\right)$. This is also related to earlier work of Grothendieck (see, e.g., [25]). One way to generate a random uniformly distributed vector u on the hypersphere is to generate each component u_i according to a normal distribution $N(0, 1)$, then normalize it [34]. Since we are only interested in the sign of the terms $u^T\sigma^i$, the normalization step of u can be skipped. Observe that $u^T\sigma^i$ is nothing but the random variable Y_i. As presented now, our randomized scheme can consequently be seen as another way of expressing the rounding technique

of [16]. This immediately leads to the following expression of the probability for an edge to be in the random cut. We are using here the standard notation $\|.\|$ to denote the Euclidean norm of a vector.

Lemma 1. *The probability for an edge ij to belong to a cut produced by the randomized scheme is given by* $Pr(Y_iY_j \leq 0) = \frac{1}{\pi}\arccos\left(\frac{(\sigma^i)^T\sigma^j}{\|\sigma^i\|\|\sigma^j\|}\right)$.

Let us consider the parameter $\hat{\theta}(\sigma)$ representing the smallest probability for an edge to appear in a cut:

$$\hat{\theta}(\sigma) = \min_{ij\in E} \frac{1}{\pi}\arccos\left(\frac{(\sigma^i)^T\sigma^j}{\|\sigma^i\|\|\sigma^j\|}\right).$$

Using Lemma 1 and the discussion above, one can build a feasible solution to $(P1)$ by taking λ_C equal to the probability to obtain the cut C (by the randomized scheme), and setting θ to $\hat{\theta}(\sigma)$. We thus obtain the following bounds.

Proposition 13. $Z^*_{P1} \geq \hat{\theta}(\sigma)$ *and thus* $Z^*_{FC} \leq \frac{1}{\hat{\theta}(\sigma)}$ *when* $\hat{\theta}(\sigma) \neq 0$.

The rest of this section will be devoted to proposing several choices for the vectors σ^i leading to tractable uppers bounds.

4.2 Tripartition based method

Let us consider the case where each element $e \in \mathcal{E}$ corresponds to a partition of V into three sets: (V^0_e, V^1_e, V^2_e). Some node subsets defining the partition may be empty. For each $e \in \mathcal{E}$, let $(\sigma_{e,1}, \sigma_{e,2}) \in \mathbb{R}^2$ and define the vectors $(\sigma^i)_{i\in V}$ as follows:

$$\sigma^i_e = \begin{cases} \sigma_{e,1} \text{ if } i \in V^1_e, \\ \sigma_{e,2} \text{ if } i \in V^2_e, \\ 0 \text{ otherwise.} \end{cases} \tag{2}$$

By Lemma 1, the probability $Pr(Y_iY_j \leq 0)$ that an edge $ij \in E$ belongs to the cut output by the algorithm is given by

$$\frac{1}{\pi}\arccos\left(\frac{\sum\limits_{e\in\mathcal{E}:\ (i,j)\in(V^1_e\times V^2_e)\cup(V^2_e\times V^1_e)} \sigma_{e,1}\sigma_{e,2} + \sum\limits_{e\in\mathcal{E}:\ (i,j)\in(V^1_e\times V^1_e)} (\sigma_{e,1})^2 + \sum\limits_{e\in\mathcal{E}:\ (i,j)\in(V^2_e\times V^2_e)} (\sigma_{e,2})^2}{\sqrt{\sum\limits_{e\in\mathcal{E}:\ i\in V^1_e} (\sigma_{e,1})^2 + \sum\limits_{e\in\mathcal{E}:\ i\in V^2_e} (\sigma_{e,2})^2}\sqrt{\sum\limits_{e\in\mathcal{E}:\ j\in V^1_e} (\sigma_{e,1})^2 + \sum\limits_{e\in\mathcal{E}:\ j\in V^2_e} (\sigma_{e,2})^2}}\right).$$

A relaxation of the problem which consists in determining vectors $(\sigma^i)_{i\in V}$ as defined above and maximizing the minimum probability $Pr(Y_iY_j < 0)$ among all $ij \in E$ is given by the following SOCP

(Second Order Cone Programming) formulation, where we introduced the variables $z_e^{1,1}, z_e^{2,2}, z_e^{1,2}$ representing the quantities $(\sigma_{e,1})^2, (\sigma_{e,2})^2$ and $\sigma_{e,1}\sigma_{e,2}$, respectively.

$$(SOCP)\begin{cases} \min \gamma \\ \text{s.t.} \\ \sum\limits_{e\in\mathcal{E}:\ (i,j)\in(V_e^1\times V_e^2)\cup(V_e^2\times V_e^1)} z_e^{1,2} + \sum\limits_{e\in\mathcal{E}:\ (i,j)\in(V_e^1\times V_e^1)} z_e^{1,1} + \sum\limits_{e\in\mathcal{E}:\ (i,j)\in(V_e^2\times V_e^2)} z_e^{2,2} \leq \gamma, \forall ij \in E, \\ \sum\limits_{e\in\mathcal{E}:\ i\in V_e^1} z_e^{1,1} + \sum\limits_{e\in\mathcal{E}:\ i\in V_e^2} z_e^{2,2} = 1, \forall i \in V, \\ (z_e^{1,2})^2 \leq z_e^{1,1} z_e^{2,2}, \forall e \in \mathcal{E}, \\ \gamma \in \mathbb{R}, (z_e^{1,1}, z_e^{2,2}) \in \mathbb{R}_+^2, z_e^{1,2} \in \mathbb{R}, \forall e \in \mathcal{E}. \end{cases}$$

Notice that constraints $(z_e^{1,2})^2 \leq z_e^{1,1} z_e^{2,2}$ are hyperbolic constraints that can be expressed through SOCP constraints. Since we are minimizing γ, we do not lose optimality by assuming that $z_e^{1,2} = -\sqrt{z_e^{1,1}.z_e^{2,2}}$. One can then take $\sigma_{e,1} = \sqrt{z_e^{1,1}}$, $\sigma_{e,2} = -\sqrt{z_e^{2,2}}$ and $z_e^{1,2} = -\sqrt{z_e^{1,1} z_e^{2,2}} = \sigma_{e,1}\sigma_{e,2}$.

Using Proposition 13, we get the following upper bound.

Proposition 14. *Let γ^* be the optimal objective value of $(SOCP)$. Then $Z_{FC}^* \leq \frac{\pi}{\arccos(\gamma^*)}$.*

On a case when $\mathcal{E}$ is defined by a set of cuts

Consider now the more particular case when each element $e \in \mathcal{E}$ is associated with a cut of G: it is of the form $(V_e^0, V_e^1, V_e^2) = (\emptyset, S_e, V \setminus S_e)$ for some node subset $S_e \subseteq V$. For our purposes, we shall assume $\cup_{e\in\mathcal{E}}\delta(S_e) = E$. Consider, in addition, the following restriction set on the coefficients $(\sigma_{e,1}, \sigma_{e,2})$, for each $e \in \mathcal{E}$: $\sigma_{e,1} = -\sigma_{e,2}$. Then, introducing for each $e \in \mathcal{E}$ a variable λ_e representing the quantity $|\sigma_{e,1}|^2 = |\sigma_{e,2}|^2$, the former SOCP formulation reduces to the following LP.

$$(P2)\begin{cases} \min \gamma \\ \text{s.t.} \\ -\sum\limits_{e\in\mathcal{E}:\ ij\in\delta(S_e)} \lambda_e + \sum\limits_{e\in\mathcal{E}:\ ij\in E\setminus\delta(S_e)} \lambda_e \leq \gamma, \forall ij \in E, \\ \sum\limits_{e\in\mathcal{E}} \lambda_e = 1, \\ \gamma \in \mathbb{R}, \lambda \in \mathbb{R}_+^{\mathcal{E}}. \end{cases}$$

Using the equation of the formulation $(P2)$ and rewriting the first inequality, we obtain that the last optimization problem is equivalent to the following one.

$$(P3)\begin{cases} \max \theta \ \ \left(= \frac{1-\gamma}{2}\right) \\ \text{s.t.} \\ \qquad \sum_{e\in\mathcal{E}\,:\, ij\in\delta(S_e)} \lambda_e \geq \theta, \forall ij \in E, \\ \qquad \sum_{e\in\mathcal{E}} \lambda_e = 1, \\ \qquad \theta \in \mathbb{R}, \lambda \in \mathbb{R}_+^{\mathcal{E}}. \end{cases}$$

Let γ^* denote the optimal objective value of $(P2)$. Let (λ^*, θ^*) denote an optimal solution of $(P3)$, and consider the following parameterization for the coefficients $(\sigma_{e,1}, \sigma_{e,2})_{e\in\mathcal{E}}$: $\sigma_{e,1} = \sqrt{\lambda_e^*} = -\sigma_{e,2}$, for all $e \in \mathcal{E}$. From the assumption that $\cup_{e\in\mathcal{E}}\delta(S_e) = E$, we get that $\theta^* > 0$ and $\arccos(\gamma^*) > 0$. Using Proposition 13, we deduce that $Z_{FC}^* \leq \frac{\pi}{\arccos(\gamma^*)}$. Observe that $(P3)$ is a restriction of $(P1)$, implying that $Z_{FC}^* \leq \frac{1}{\theta^*} = \frac{2}{1-\gamma^*}$. One can compare the two upper bounds by studying the function $\gamma \to \frac{\pi(1-\gamma)}{2\arccos(\gamma)}$. It is easy to see that the upper bound $\frac{2}{1-\gamma^*}$ is lower than $\frac{\pi}{\arccos(\gamma^*)}$ when $\gamma^* < 0$ (i.e., $\theta^* > \frac{1}{2}$) while the situation is reversed if $\gamma^* > 0$.

Moreover, observe that if all the cuts are considered in $\mathcal{E}$, then $(P3)$ is nothing else than $(P1)$. We consequently have $\theta^* = Z_{P1}^* = \frac{1}{Z_{FC}^*}$. In this case, the ratio $\frac{\frac{\pi}{\arccos(1-2\theta^*)}}{Z_{FC}^*}$ is equal to $\frac{\pi\theta^*}{\arccos(1-2\theta^*)}$. This ratio is upper bounded by $\frac{1}{\alpha}$, where α is recalled in Section 3.1. In other words, we can achieve the same approximation ratio as the one obtained in Section 3.1 by considering this randomized procedure. Observe however that this result is only of theoretical interest since to get this ratio we had to include all the cuts in $\mathcal{E}$.

On a case when $\mathcal{E}$ is defined by the set of edges

Let us now consider the case when there is a bijection between the sets $\mathcal{E}$ and E: each element $e \in \mathcal{E}$ is associated with some edge $ij \in E$ and is of the form (V_e^0, V_e^1, V_e^2) with $V_e^0 = V \setminus \{i, j\}$, $V_e^1 = \{i\}$ and $V_e^2 = \{j\}$. We will make an abuse of notation by writing that $\mathcal{E} = E$ and identifying an edge $ij \in E$ with the element $e \in \mathcal{E}$ mapped to ij. Using the notation introduced at the beginning of the

section, formulation $(SOCP)$ specialized to this case becomes:

$$(SOCP_{edge}) \begin{cases} \min \gamma \\ s.t. \\ \quad z_e^{1,2} \leq \gamma, \forall e \in E, \\ \quad \sum\limits_{e \in \delta(i):\ i \in V_e^1} z_e^{1,1} + \sum\limits_{e \in \delta(i):\ i \in V_e^2} z_e^{2,2} = 1, \forall i \in V, \\ \quad (z_e^{1,2})^2 \leq z_e^{1,1} z_e^{2,2}, \forall e \in \mathcal{E}, \\ \quad \gamma \in \mathbb{R}, (z_e^{1,1}, z_e^{2,2}) \in \mathbb{R}_+^2, z_e^{1,2} \in \mathbb{R}, \forall e \in \mathcal{E}. \end{cases}$$

Observe that for $e = ij$ we have $z_e^{1,1} = (\sigma_{e,1})^2 = (\sigma_e^i)^2$, $z_e^{2,2} = (\sigma_{e,2})^2 = (\sigma_e^j)^2$ and $z_e^{1,2} = \sigma_{e,1}\sigma_{e,2} = \sigma_e^i \sigma_e^j$. Solving the SOCP problem above is then equivalent to solve to the following problem:

$$(P_{edge}) \begin{cases} \min \gamma \\ s.t. \\ \quad \sigma_e^i \sigma_e^j \leq \gamma, \forall e = ij \in E, \\ \quad \sum\limits_{e \in \delta(i)} (\sigma_e^i)^2 = 1, \forall i \in V, \\ \quad \gamma \in \mathbb{R}, \sigma^i \in \mathbb{R}^E, \forall i \in V. \end{cases}$$

Let $\tilde{\gamma}$ denote the optimal objective value of $(SOCP_{edge})$ (or (P_{edge})). From Proposition 13, we obtain the upper bound described below.

Proposition 15.

$$Z_{FC}^* \leq \frac{\pi}{\arccos(\tilde{\gamma})}. \tag{3}$$

Also note that a feasible solution of (P_{edge}) can be obtained by setting $(\sigma_e^i)^2 = \frac{1}{d_i}$ for each $e \in \delta(i)$ and $\hat{\gamma} = -\min\limits_{ij \in E} \frac{1}{\sqrt{d_i d_j}}$ where d_i denotes the degree of node i. This leads to the following upper bound on Z_{FC}^*.

Corollary 3. *Let* $\hat{\gamma} = -\min\limits_{ij \in E} \frac{1}{\sqrt{d_i d_j}}$, *then*

$$Z_{FC}^* \leq \frac{\pi}{\arccos(\hat{\gamma})}. \tag{4}$$

Though it may be naturally expected that the bound of Corollary 3 is of poorer quality than the one of Proposition 15, the former has the advantage of not requiring solving any SOCP. In addition, it has the nice feature of corresponding to an optimal solution of (P_{edge}) for the case when the graph $G = (V, E)$ is regular, as we show next.

Proposition 16. *If the graph* $G = (V, E)$ *is regular, then* $\tilde{\gamma} = \hat{\gamma}$.

Proof. From the constraints of (P_{edge}), we should have $\gamma \geq \sigma_e^i \sigma_e^j$ for each edge $ij = e$. This leads to $-2\gamma \leq (\sigma_e^i)^2 + (\sigma_e^j)^2$. Summing up the latter inequalities over all $ij \in E$ and using the equations of (P_{edge}) leads to $-2|E|\gamma \leq |V|$. Thus $-\frac{|V|}{2|E|}$ is a lower bound on the optimal objective value of (P_{edge}). Since the graph is here regular, this lower bound coincides with the objective value of the feasible solution defined by $(\sigma_e^i)^2 = \frac{1}{d_i}$ and $\hat{\gamma} = -\min_{ij \in E} \frac{1}{\sqrt{d_i d_j}}$, we deduce that $\tilde{\gamma} = \hat{\gamma}$. □

4.3 Another approximation algorithm

We already mentioned bounds on Z_{FC}^* (see Proposition 5) which directly follow from Samal's work [39] and that can be computed efficiently using SDP. In this subsection, we study a variant of the general scheme from Section 4.1 providing a better upper bound and leading to a $\frac{1}{\alpha}$-approximation algorithm, still using SDP.

Consider the following parameterization for the general scheme (Section 4.1): $\mathcal{E} = V$. Each vector σ^i belongs to $\mathbb{R}^V$.

We know from Proposition 13 that $Z_{FC}^* \leq \frac{1}{\tilde{\theta}(\sigma)} = \max_{ij \in E} \frac{\pi}{\arccos\left(\frac{(\sigma^i)^T \sigma^j}{\|\sigma^i\| \|\sigma^j\|}\right)}$.

Looking for the best bound of this form reduces to determining normalized vectors $(\overline{\sigma}^i)_{i \in V}$ in $\mathbb{R}^V$ minimizing the quantity $\max_{ij \in E} \left(\frac{\pi}{\arccos\left(\left(\overline{\sigma}^i\right)^T \overline{\sigma}^j\right)} \right)$. Since the function $x \mapsto \arccos(x)$ is strictly decreasing on $[-1, 1]$, the latter comes down to finding normalized vectors minimizing the quantity $\max_{ij \in E} \left(\left(\overline{\sigma}^i\right)^T \overline{\sigma}^j \right)$. Such vectors can be obtained by solving the following semidefinite program which coincides with the one introduced by Karger et al. [24] to compute the vector chromatic number χ_v. (Remember that $\chi_v = 1 - \frac{1}{Z_{SDP1}^*}$ was already mentioned before Proposition 5.)

$$(SDP1) \begin{cases} Z_{SDP1}^* = \min t \\ s.t. \\ \quad X_{ij} \leq t, \forall ij \in E, \\ \quad X_{ii} = 1, \forall i \in V, \\ \quad X \succcurlyeq 0, X \in \mathbb{R}^{V \times V}. \end{cases}$$

A feasible solution of $(SDP1)$ with $t = \frac{-1}{k-1}$ corresponds to Karger et al.'s vector k-coloring [24]. From our discussion above the next result follows.

Proposition 17. *The following inequality holds.*

$$Z_{FC}^* \leq \frac{\pi}{\arccos(Z_{SDP1}^*)}.$$

Corollary 4. *A $\frac{1}{\alpha} + \epsilon$ factor approximation of Z_{FC}^* can be computed in polynomial time, for any $\epsilon > 0$.*

Proof. The lower bound on Z^*_{FC} from Proposition 5 (due to Samal [39]) and the upper bound from Proposition 17 lead to

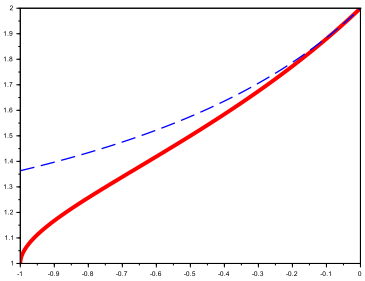
$$\frac{2}{1-Z^*_{SDP1}} \leq Z^*_{FC} \leq \frac{\pi}{\arccos(Z^*_{SDP1})}.$$

Then, the result follows from the definition of α and the fact that a feasible solution of $(SDP1)$ with objective value at most $Z^*_{SDP1}+\epsilon$ can be computed in polynomial time in n and $\log(\frac{1}{\epsilon})$ [17] for any $\epsilon > 0$. Observe that we use here the fact $\frac{\pi}{\arccos(Z^*_{SDP1})} / \frac{2}{1-Z^*_{SDP1}} \leq \max_{x\in[-1,1]} \frac{\pi(1-x)}{2\arccos(x)} = \frac{1}{\min_{0\leq\theta\leq\pi} \frac{2}{\pi}\frac{\theta}{1-\cos\theta}} = \frac{1}{\alpha}$.

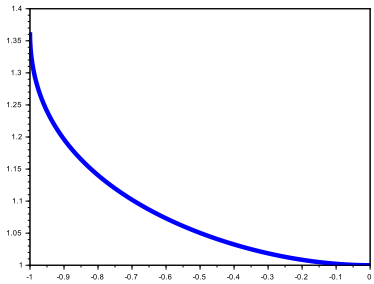
For sake of completeness, we give here a proof of the lower bound. Given an optimal solution $((\lambda_C)_{C\in\mathcal{C}}, \theta)$ of $(P1)$, consider the matrix $X = \sum_{C\in\mathcal{C}} \lambda_C \mathcal{I}_C{}^T \mathcal{I}_C$, where $\mathcal{I}_C$ is the vector of size $|V|$ induced by the cut C (i.e. the component $\mathcal{I}_C$ related to vertex v is either 1 or -1 depending on which side of the partition induced by C contains the vertex v). The matrix X is obviously positive semidefinite with all the diagonal terms equal to 1. For each edge $ij \in E$, we have $X_{ij} = \sum_{C\in\mathcal{C}:ij\notin C} \lambda_C - \sum_{C\in\mathcal{C}:ij\in C} \lambda_C = 1 - 2\sum_{C\in\mathcal{C}:ij\in C} \lambda_C$. Since $\sum_{C\in\mathcal{C}:ij\in C} \lambda_C \geq \theta$, we get that $X_{ij} \leq 1-2\theta$ for each edge ij. Moreover X is a feasible solution of $(SDP1)$. This leads to $Z^*_{SDP1} \leq 1-2\theta = 1 - \frac{2}{Z^*_{FC}}$ showing the validity of the lower bound. □

Remark 1. To illustrate the improvement the upper bound of Proposition 17 represents over the one from Proposition 5, we represent in Figure 1 the functions $f_1 : x \to \frac{\pi}{\arccos(x)}$, $f_2 : x \to 2\left(1 - \frac{2x}{\pi(x-1)}\right)$ and the ratio $\frac{f_2(x)}{f_1(x)}$ for $x \in [-1, 0]$.

Fig. 1. A comparison of the upper bounds on Z^*_{FC} from Propositions 5 and 17



(a) f_1 (thick line) and f_2 (dashed line) on the interval [-1,0]



(b) The ratio f_2/f_1 on the interval [-1,0]

5 Computational experiments

In this section we report computational experiments to compare different bounds on Z^*_{FC} and to illustrate their respective quality for different graph families.

5.1 Graph instances

The graphs used in our computational experiments fall into one of the following cases, where n and k denote positive integer values; d, p stand for real values in $[0, 1]$.

- C_n: cycle with order n.
- K_n: complete graph with order n.
- $P(n, d)$: planar graph randomly generated, with density parameter $d \in [0, 1]$ (so that the number of edges is about $3(|V| - 2)d$. Recall that the maximum number of edges of a planar graph with order greater than 2 is $3(|V| - 2)$).
- $G(n, p)$: random graph (Erdös-Rényi model [11]) with edge probability p.
- Pet: Petersen graph.
- $\mathcal{M}_k$: Mycielski graph with parameter k (see hereafter).
- Z_k: Zykov graph with parameter k (see hereafter).

Used graphs of the type $P(n, d)$ and $G(n, p)$ have been generated using *rudy* [36].

Mycielski graphs Consider the following graph transformation introduced by Mycielski [35]. Given a graph $G = (V, E)$ with $V = \{v_1, ..., v_n\}$, let $M(G)$ denote the graph with vertex set $V_M = V \cup \{z_1, \ldots, z_n, w\}$ and edge set $E_M = E \cup E' \cup E''$, with $E' = \{z_i w \colon i \in \{1, 2, \ldots, n\}\}$, and $E'' = \{v_i z_j \colon v_i v_j \in E\}$.

We have $\chi(M(G)) = \chi(G) + 1$. Let $\mathcal{M}_1$ denote the graph K_2, and define $\mathcal{M}_{k+1} = M(\mathcal{M}_k)$, for each integer $k \geq 2$. So we have $\chi(\mathcal{M}_k) = k + 1$. Regarding the fractional chromatic number, Larsen et al. [27] proved the following equation holds.

$$\chi_f(\mathcal{M}_k) = \chi_f(\mathcal{M}_{k-1}) + \frac{1}{\chi_f(\mathcal{M}_{k-1})}. \tag{5}$$

The gaps $\chi(\mathcal{M}_k) - \lceil \chi_f(\mathcal{M}_k) \rceil$ and $\chi_f(\mathcal{M}_k) - \omega(\mathcal{M}_k)$ become arbitrarily large as k increases, where for a given graph G, $\omega(G)$ denotes the maximum cardinality of a clique in G. In fact, even the ratios $\frac{\chi(\mathcal{M}_k)}{\chi_f(\mathcal{M}_k)}$ and $\frac{\chi_f(\mathcal{M}_k)}{\omega(\mathcal{M}_k)}$ approach infinity as k increases.

Zykov graphs The Zykov product $\mathcal{Z}(G_1, G_2, \ldots, G_n)$ of simple graphs $G_1, G_2, \ldots, G_n$ is formed by taking the disjoint union of all the graphs $G_i = (V_i, E_i)$ and, for each possible choice of $(x_1, x_2, \ldots, x_n) \in V_1 \times V_2 \times \ldots \times V_n$, adding a new vertex adjacent to each node in the set $\{x_1, x_2, \ldots, x_n\}$. Zykov graphs are the graphs $(Z_i)_{i=1}^{+\infty}$, where Z_1 is the graph with a single vertex and $Z_{n+1} = \mathcal{Z}(Z_1, Z_2, \ldots, Z_n)$ (see [41, p.215], [42]). Charbit and Sereni [8] proved that equation (5) also holds for Zykov graphs.

5.2 Computational results

Hereafter, we describe the notation used for the evaluated lower ($LB0, \ldots, LB3$) and upper ($UB0, \ldots, UB5$) bounds on Z^*_{FC}, and give some implementation details.

- $LB0 = \frac{\overline{g}}{\overline{g}-1}$ (see Proposition 3).
- $LB1 = \frac{|E|}{\tilde{mc}(G)}$ where $\tilde{mc}(G)$ denotes the optimal objective value of the SDP relaxation of the maximum cut problem given by Goemans and Williamson [16] recalled in Section 3.1. The fact that this quantity is a lower bound for Z^*_{FC} is a corollary of Proposition 2.
- $LB2 = 2\left(1 - \frac{1}{\chi_v}\right)$ (see Proposition 5). The vector chromatic number χ_v is obtained solving a semidefinite program [24].
- $LB3 = \alpha \frac{\pi}{\arccos(Z^*_{SDP1})}$ (see proof of Corollary 4).
- $UB0 = 2 - 1/\left\lceil \frac{\Delta+1}{2} \right\rceil$ if the graph G is a complete graph or an odd cycle and $UB0 = 2 - 1/\left\lceil \frac{\Delta}{2} \right\rceil$ otherwise, where Δ denotes the maximum degree in G. This bound follows from Proposition 5 and Brooks' theorem.
- $UB1$: value obtained by solving (DC) using an approximate separation algorithm (Section 3.1). More precisely, for each call, our separation procedure generates 100 cuts: they result from the application of Goemans and Williamson's algorithm [16] (using 100 random vectors, each one leading to one cut). For each cut generated, we apply a local improvement heuristic: it basically consists in checking whether changing the position of one node in the partition improves the objective value of the resulting cut, and this is iterated as long as improvements are obtained. Finally, we use the cut with maximum weight among all the ones obtained in this manner to determine whether an inequality will be added to the current relaxation. Note that we are not using the ellipsoid algorithm in our implementation (only CPLEX is used to solve the linear programs).
- $UB2 = 2\left(1 - \frac{2}{\pi\chi_v}\right)$ (see Proposition 5).
- $UB3 = \frac{1}{\alpha} LB3 = \frac{\pi}{\arccos(Z^*_{SDP1})}$ (see Proposition 17).

– $UB4 = \frac{\chi_f(G)}{2}$ (see Proposition 8). The value $\chi_f(G)$ is obtained solving (D_{CHROM}) by constraint generation and using an exact separation procedure. Here the separation reduces to an integer linear program that is solved by CPLEX.
– $UB5$: two bounds are reported in this field of the table. The value on the left corresponds to the bound given by Proposition 15. The value on the right corresponds to the bound of Corollary 3.

Computations have been performed using a processor Intel Core i7-2640M CPU @ 2.80GHz x 4, 7.7 Gio RAM. All linear programs have been solved using the CPLEX solver. The semidefinite programs were solved by CSDP [6]. The CPU time in seconds is mentioned in brackets. They are omitted for some bounds when they are negligible for all the evaluated instances (this is notably the case for $LB0$, $UB0$ and the second bound reported in $UB5$) or when the bound is of poor quality (such as the first bound reported in $UB5$). A computation time of '0.00' for some instances means less than 5 ms. As the time required to compute the bounds $LB2$, $LB3$, $UB2$ and $UB3$ is almost the same, it is only mentioned once: in the field $LB2$. The time limit has been set to one hour (3600s). An asterisk is reported for the case when computation times exceed this limit. For the case of bounds which could not be computed within this time limit and which are obtained by applying a constraint generation algorithm, i.e. $UB1$ and $UB4$, the reported value corresponds to the optimal objective of the last relaxation solved. For the bounds relying on the resolution of a single semidefinite program ($LB1$, $LB2$, $LB3$, $UB2$, $UB3$, $UB5$) we let the solver terminate the optimization process even in the cases when the time limit was exceeded.

On random graphs the best lower bounds have been obtained with $LB2$. Regarding computation times, $LB1$ is also interesting for the largest instances among the random graphs (Table 2). $LB0$ tends to dominate the other lower bounds on particularly structured graphs (such as Petersen, Mycielski and Zykov graphs; also recall that, by Proposition 12, $LB0$ coincides with Z^*_{FC} for odd cycles and planar graphs).

With respect to the upper bounds, $UB1$ gives the best results whenever it could be computed within the time limits. Otherwise $UB3$ appears also to be interesting in terms of computation times and quality of the bound.

6 Conclusion

Several approaches have been proposed to get many lower and upper bounds for the fractional cut cover problem. Numerical experiments allowed us to evaluate the quality of these bounds and their cost in terms of computing time. We first proposed a polynomial-time $1/\alpha$ approximation

Table 1. Computational results

Instance	$\|V\|$	$\|E\|$	$LB0$	$LB1$	$LB2$	$LB3$	$UB0$	$UB1$	$UB2$	$UB3$	$UB4$	$UB5$
C_7	7	7	1.16	1.05 (0.00)	1.05 (0.00)	1.02	1.50	1.17 (0.03)	1.40	1.17	1.17 (0.03)	1.50 / 1.50
C_9	9	9	1.12	1.03 (0.00)	1.03 (0.00)	0.98	1.50	1.13 (0.03)	1.39	1.13	1.13 (0.03)	1.50 / 1.50
C_{11}	11	11	1.10	1.02 (0.00)	1.02 (0.00)	0.96	1.50	1.10 (0.04)	1.38	1.10	1.10 (0.04)	1.50 / 1.50
C_{13}	13	13	1.08	1.01 (0.00)	1.01 (0.00)	0.95	1.50	1.09 (0.08)	1.38	1.09	1.09 (0.05)	1.50 / 1.50
C_{15}	15	15	1.07	1.01 (0.00)	1.01 (0.00)	0.94	1.50	1.08 (0.11)	1.38	1.08	1.08 (0.06)	1.50 / 1.50
K_{10}	10	45	1.50	1.80 (0.00)	1.80 (0.01)	1.64	1.80	1.80 (0.19)	1.88	1.87	5.00 (0.00)	1.87 / 1.87
K_{11}	11	55	1.50	1.81 (0.00)	1.81 (0.01)	1.65	1.84	1.84 (0.16)	1.88	1.88	5.50 (0.00)	1.89 / 1.89
K_{12}	12	66	1.50	1.83 (0.00)	1.83 (0.01)	1.66	1.84	1.84 (0.31)	1.90	1.90	6.00 (0.00)	1.90 / 1.90
K_{13}	13	78	1.50	1.84 (0.00)	1.84 (0.01)	1.66	1.86	1.86 (0.32)	1.91	1.90	6.50 (0.00)	1.90 / 1.90
K_{14}	14	91	1.50	1.85 (0.00)	1.85 (0.01)	1.67	1.86	1.86 (0.51)	1.91	1.91	7.00 (0.00)	1.91 / 1.91
K_{15}	15	105	1.50	1.86 (0.00)	1.86 (0.02)	1.68	1.88	1.88 (0.49)	1.92	1.92	7.50 (0.00)	1.92 / 1.92
Pet	10	15	1.25	1.06 (0.00)	1.10 (0.00)	1.09	1.5	1.25 (0.05)	1.43	1.25	1.25 (0.07)	1.65 / 1.65
$\mathcal{M}_2$ (C_5)	5	5	1.25	1.10 (0.00)	1.11 (0.00)	1.09	1.50	1.25 (0.01)	1.44	1.25	1.25 (0.02)	1.50 /1.50
$\mathcal{M}_3$ (Grötzsch)	11	20	1.25	1.16 (0.00)	1.16 (0.00)	1.16	1.67	1.25 (0.07)	1.47	1.33	1.45 (0.17)	1.71 / 1.73
$\mathcal{M}_4$	23	71	1.25	1.20 (0.00)	1.20 (0.01)	1.20	1.84	1.34 (1.47)	1.50	1.38	1.63 (1.97)	1.83 / 1.86
Z_4	18	36	1.25	1.13 (0.00)	1.17 (0.01)	1.17	1.80	1.30 (0.35)	1.48	1.34	1.45 (0.78)	1.75 / 1.81
$P(25, 0.7)$	25	48	1.50	1.26 (0.01)	1.50 (0.01)	1.44	1.80	1.50 (0.52)	1.68	1.64	2.00 (0.04)	1.80 / 1.87
$P(25, 0.7)$	25	48	1.50	1.24 (0.01)	1.50 (0.02)	1.44	1.84	1.50 (0.54)	1.68	1.64	2.00 (0.02)	1.79 / 1.88
$P(25, 0.9)$	25	62	1.50	1.34 (0.01)	1.50 (0.02)	1.44	1.84	1.50 (1.51)	1.68	1.64	2.00 (0.16)	1.82 / 1.89
$P(25, 0.9)$	25	62	1.50	1.37 (0.01)	1.50 (0.01)	1.44	1.86	1.50 (1.27)	1.68	1.64	2.00 (0.26)	1.83 / 1.91
$P(50, 0.7)$	50	100	1.50	1.24 (0.02)	1.50 (0.04)	1.44	1.89	1.50 (7.28)	1.68	1.64	2.00 (0.26)	1.83 / 1.93
$P(50, 0.7)$	50	100	1.50	1.26 (0.02)	1.50 (0.04)	1.44	1.91	1.50 (6.33)	1.68	1.64	2.00 (0.28)	1.83 / 1.93
$P(50, 0.9)$	50	129	1.50	1.35 (0.02)	1.50 (0.05)	1.44	1.90	1.50 (31.58)	1.68	1.64	2.00 (1.11)	1.84 / 1.94
$P(50, 0.9)$	50	129	1.50	1.35 (0.02)	1.50 (0.05)	1.44	1.88	1.50 (32.68)	1.68	1.64	2.00 (0.97)	1.83 / 1.92
$P(100, 0.7)$	100	205	1.50	1.26 (0.08)	1.50 (0.17)	1.44	1.89	1.50 (346.03)	1.68	1.64	2.00 (9.57)	1.84 / 1.93
$P(100, 0.7)$	100	205	1.50	1.25 (0.09)	1.50 (0.17)	1.44	1.92	1.50 (237.59)	1.68	1.64	2.00 (5.57)	1.84 / 1.94
$P(100, 0.9)$	100	264	1.50	1.36 (0.09)	1.50 (0.24)	1.44	1.92	1.51* (> 3600)	1.68	1.64	2.00 (78.41)	1.87 / 1.94
$P(100, 0.9)$	100	264	1.50	1.34 (0.09)	1.50 (0.23)	1.44	1.93	1.51* (> 3600)	1.68	1.64	2.00 (83.27)	1.87 / 1.95
$P(200, 0.7)$	200	415	1.50	1.24 (0.39)	1.50 (1.09)	1.44	1.95	1.52* (> 3600)	1.68	1.64	2.00 (49.32)	1.86 / 1.97
$P(200, 0.7)$	200	415	1.50	1.24 (0.37)	1.50 (0.99)	1.44	1.94	1.52* (> 3600)	1.68	1.64	2.00 (53.52)	1.86 / 1.96
$P(200, 0.9)$	200	534	1.50	1.35 (0.35)	1.50 (1.50)	1.44	1.95	1.60* (> 3600)	1.68	1.64	2.00 (773.79)	1.89 / 1.97
$P(200, 0.9)$	200	534	1.50	1.36 (0.34)	1.50 (1.39)	1.44	1.95	1.60* (> 3600)	1.68	1.64	2.00 (760.34)	1.88 / 1.96

Table 2. Computational results on random graphs

Instance	$\|V\|$	$\|E\|$	$LB0$	$LB1$	$LB2$	$LB3$	$UB0$	$UB1$	$UB2$	$UB3$	$UB4$	$UB5$
$G(25, 0.25)$	25	75	1.50	1.29 (0.01)	1.50 (0.03)	1.44	1.84	1.50 (1.40)	1.69	1.65	2.00 (0.70)	1.83 / 1.89
$G(25, 0.25)$	25	75	1.50	1.29 (0.01)	1.50 (0.02)	1.44	1.80	1.50 (1.53)	1.69	1.65	2.00 (0.21)	1.83 / 1.88
$G(50, 0.25)$	50	306	1.50	1.42 (0.01)	1.60 (0.182)	1.51	1.91	1.67 (16.31)	1.75	1.73	2.54 (25.79)	1.91 / 1.94
$G(50, 0.25)$	50	306	1.50	1.41 (0.01)	1.60 (0.256)	1.51	1.90	1.67 (22.44)	1.75	1.73	2.63 (19.62)	1.91 / 1.94
$G(100, 0.25)$	100	1238	1.50	1.52 (0.05)	1.66 (7.11)	1.55	1.95	1.77* (>3600)	1.79	1.78	3.65 (1096.08)	1.96 / 1.97
$G(100, 0.25)$	100	1238	1.50	1.52 (0.08)	1.64 (5.44)	1.54	1.95	1.77* (>3600)	1.78	1.76	3.61 (1222.07)	1.96 / 1.97
$G(200, 0.25)$	200	4975	1.50	1.63 (0.32)	1.73 (247.81)	1.60	1.98	1.93* (>3600)	1.84	1.83	6.47* (> 3600)	1.98 / 1.99
$G(200, 0.25)$	200	4975	1.50	1.62 (0.27)	1.73 (247.58)	1.60	1.98	1.93* (>3600)	1.84	1.83	6.35* (> 3600)	1.98 / 1.99
$G(25, 0.5)$	25	150	1.50	1.45 (0.01)	1.66 (0.06)	1.55	1.88	1.68 (9.17)	1.79	1.78	3.09 (0.84)	1.91 / 1.92
$G(25, 0.5)$	25	150	1.50	1.48 (0.01)	1.67 (0.04)	1.55	1.89	1.67 (5.30)	1.79	1.77	3.00 (0.3)	1.91 / 1.93
$G(50, 0.5)$	50	613	1.50	1.59 (0.02)	1.73 (0.92)	1.60	1.95	1.75* (>3600)	1.84	1.83	4.32 (27.56)	1.96 / 1.96
$G(50, 0.5)$	50	613	1.50	1.57 (0.02)	1.73 (0.83)	1.60	1.95	1.76* (>3600)	1.84	1.83	4.41 (31.57)	1.96 / 1.97
$G(100, 0.5)$	100	2475	1.50	1.69 (0.08)	1.80 (44.67)	1.64	1.97	1.91* (>3600)	1.88	1.88	6.60 (830.44)	1.98 / 1.98
$G(100, 0.5)$	100	2475	1.50	1.68 (0.06)	1.80 (33.07)	1.64	1.97	1.91* (>3600)	1.88	1.88	6.74 (827.46)	1.98 / 1.98
$G(200, 0.5)$	200	9950	1.50	1.76 (0.27)	1.86 (1941.85)	1.67	1.99	2.00* (>3600)	1.92	1.91	11.71* (> 3600)	1.99 / 1.99
$G(200, 0.5)$	200	9950	1.50	1.76 (0.29)	1.86 (1871.78)	1.67	1.99	2.00* (>3600)	1.92	1.91	11.55* (> 3600)	1.99 / 1.99
$G(25, 0.75)$	25	225	1.50	1.63 (0.01)	1.76 (0.09)	1.62	1.91	1.78 (52.30)	1.86	1.85	4.50 (0.34)	1.94 / 1.94
$G(25, 0.75)$	25	225	1.50	1.62 (0.01)	1.77 (0.09)	1.62	1.91	1.80 (8.59)	1.86	1.86	4.50 (0.19)	1.94 / 1.94
$G(50, 0.75)$	50	919	1.50	1.73 (0.02)	1.85 (2.54)	1.67	1.96	1.88* (>3600)	1.91	1.91	7.54 (9.36)	1.97 / 1.98
$G(50, 0.75)$	50	919	1.50	1.73 (0.02)	1.85 (2.47)	1.67	1.96	1.88* (>3600)	1.91	1.91	7.42 (8.41)	1.97 / 1.98
$G(100, 0.75)$	100	3713	1.50	1.80 (0.06)	1.89 (107.01)	1.69	1.98	1.97* (>3600)	1.94	1.94	11.91 (221.48)	1.99 / 1.99
$G(100, 0.75)$	100	3713	1.50	1.80 (0.06)	1.89 (107.11)	1.69	1.98	1.97* (>3600)	1.94	1.94	11.49 (221.67)	1.99 / 1.99
$G(200, 0.75)$	200	14925	1.50	1.85 (0.26)	1.92* (> 3600)	1.71*	1.99	2.04* (>3600)	1.96*	1.96*	19.75* (> 3600)	2.00 / 2.00
$G(200, 0.75)$	200	14925	1.50	1.85 (0.26)	1.92*(> 3600)	1.71*	1.99	2.04* (>3600)	1.96*	1.96*	19.71* (> 3600)	2.00 / 2.00

algorithm based on a cutting-plane algorithm where an approximate separation oracle is used. Then, approximating the cut polytope by the metric polytope we obtained some lower bounds that are tight for some graph classes such as those not contractible to K_5. Another relaxation based on odd cycle inequalities implied that the bound is also tight for the more general class of weakly bipartite graphs.

A general framework based on the generation of a set of independent normally distributed random variables (one variable u_e for each $e \in \mathcal{E}$) and the construction, for each vertex i, of a random variable Y_i linearly depending on the variables $(u_e)_{e\in\mathcal{E}}$ was presented. We showed that upper bounds can be obtained by solving an SOCP when the set $\mathcal{E}$ corresponds to tripartitions of V. When the tripartitions are induced by the set of edges, we get simpler bounds. The case where tripartitions are induced by cuts leads to upper bounds that are within a ratio of $1/\alpha$ from optimum.

If alternately to tripartitions, we consider $\mathcal{E} = V$ and we solve an appropriate SDP instead of an SOCP, we obtain another polynomial-time $1/\alpha$ approximation algorithm. As a consequence, we got new insights into the random hyperplane procedure of Goemans and Williamson's algorithm for the maximum cut problem.

To conclude, let us mention some of the numerous questions deserving investigation. We assumed in the presented framework that the variables u_e are normally distributed. One might consider other distributions for which the analysis can be conducted and the obtained optimization problems still remain easy to solve. Another natural extension that can be considered is the fractional multi-cut cover problem.

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