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Propagation in a fractional reaction-diffusion equation in a periodically hostile environment

Alexis Léculier*, Sepideh Mirrahimi[†] and Jean-Michel Roquejoffre[‡]

May 10, 2019

Abstract

We provide an asymptotic analysis of a fractional Fisher-KPP type equation in periodic non-connected 1-dimensional media with Dirichlet conditions outside the domain. After demonstrating the existence and uniqueness of a non-trivial bounded stationary state n_+ , we prove that the stable state n_+ invades the unstable state 0 with a speed which is exponential in time.

Key-Words: Non-local fractional operator, Fisher KPP, asymptotic analysis, exponential speed of propagation, perturbed test function

AMS Class. No: 35K08, 35K57, 35B40, 35Q92.

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1 Introduction

1.1 Model and question

We focus on the following equation :

$$\begin{cases} \partial_t n + (-\Delta)^\alpha n = n - n^2, & \text{for } (x, t) \in \bigcup_{k \in \mathbb{Z}}]2kA, (2k+1)A[\times]0, \infty[, \\ n = 0, & \text{for } (x, t) \in \bigcup_{k \in \mathbb{Z}} [(2k+1)A, (2k+2)A] \times [0, \infty[, \\ n(x, 0) = n_0(x), \end{cases} \quad (1)$$

with A a constant and $(-\Delta)^\alpha$ the fractional Laplacian with $\alpha \in]0, 1[$ which is defined as follows :

$$(-\Delta)^\alpha n(x, t) = C_\alpha PV \int_{\mathbb{R}} \frac{n(x, t) - n(y, t)}{|x - y|^{1+2\alpha}} dy \quad \text{where } C_\alpha = \frac{4^\alpha \Gamma(\frac{1}{2} + \alpha)}{\pi^{\frac{\alpha}{2}} |\Gamma(-\alpha)|}.$$

The main aim of this paper is to describe the propagation front associated to (1). We show that the stable state invades the unstable state with an exponential speed.

1.2 Motivation

Equation (1) models the growth and the invasion of a species subject to a non-local dispersion in a periodically hostile environment. The function n stands for the density of the population. The fractional Laplacian describes the motions of individuals, it takes into account the possibility of "jump" (move rapidly) of individuals from one point to another, for instance because of human activities for animals or because of the wind for seeds. The logistic term $n - n^2$ represents the growth rate of the population. The originality of this model is the following, we forbid our species to invade some periodic patches and thus the reachable areas are disconnected. Here, we assume that the regions where the species can develop itself are homogeneous. Thanks to the non-local diffusion (which models the "jumps"), the species will invade all the "good" patches and the solution will converge to a non-null stable stationary state with a speed which grows exponentially fast.

Many works deal with the case of a standard diffusion ($\alpha = 1$) with homogenous or heterogeneous environment (see [13], [18], [1] and [15]). More close to this article, Guo and Hamel in [17] treat a Fisher-KPP equation with periodically hostile regions and a standard diffusion. The authors prove that the stable state will invade the unstable state in the connected component of the initial data. In our work, thanks to the non-local character of the fractional Laplacian, contrary to what happens in [17], we show that there exists a unique non-trivial positive bounded stationary state. Moreover this stationary state invades the unstable state 0 everywhere and not only on the connected component of the initial data.

1.3 Assumptions and results

Concerning the domain

$$\Omega := \bigcup_{k \in \mathbb{Z}}]2kA, (2k+1)A[$$

the constant A is such that for any $A > A_0$, the principale eigenvalue λ_A of the Dirichlet operator $(-\Delta)^\alpha - Id$ in $]0, A[$ is negative (the existence of such A_0 is provided by [3]). In other words, there

exists a map ϕ_A positive such that

$$\begin{cases} ((-\Delta)^\alpha - Id) \phi_A(x) = \lambda_A \phi_A(x) & \text{for } x \in]0, A[, \\ \phi_A(x) = 0 & \text{for } x \in]0, A[^c, \\ \phi_A \geq 0, \lambda_A < 0. \end{cases} \quad (\text{H1})$$

It is well known that the principal eigenvalue λ_0 of the Dirichlet operator $(-\Delta)^\alpha - Id$ in Ω is simple in the algebraic and geometric sense and moreover, the associated principal eigenfunction has a sign

$$i.e. \begin{cases} ((-\Delta)^\alpha - Id) \phi_0(x) = \lambda_0 \phi_0(x) & \text{for } x \in \Omega, \\ \phi_0(x) = 0 & \text{for } x \in \Omega^c, \\ \phi_0 \geq 0. \end{cases} \quad (2)$$

It is a direct application of the Krein-Rutman Theorem.

The first result of this paper is the following which ensures the existence and the uniqueness of a positive bounded stationary state of (1):

$$\begin{cases} (-\Delta)^\alpha n_+(x) = n_+(x) - n_+(x)^2 & \text{for } x \in \Omega, \\ n_+(x) = 0 & \text{for } x \in \Omega^c. \end{cases} \quad (3)$$

Theorem 1. *Under the assumption (H1), there exists a unique positive and bounded stationary state n_+ to (1). Moreover, we have $0 \leq n_+ \leq 1$ and n_+ is $2A$ -periodic.*

The existence is due to the negativity of the principal eigenvalue of the Dirichlet operator $(-\Delta)^\alpha - Id$ in $]0, A[$ which allows to construct by an iterative method a stationary state. To prove the uniqueness, we first prove that thanks to the non-local character of the fractional Laplacian, all the bounded stationary states behave like

$$\delta(x)^\alpha = \text{dist}(x, \partial\Omega)^\alpha 1_\Omega(x). \quad (4)$$

Hence, thanks to the maximum principle and the fractional Hopf Lemma (stated in [16] for instance), we get the result. We should underline that the uniqueness is clearly due to the non-local operator $(-\Delta)^\alpha$, and it does not hold in the case of a standard diffusion term ($\alpha = 1$). A direct consequence of the existence of a stationary solution of (1) is that λ_0 is strictly negative:

Corollary 1. *The principal eigenvalue of the Dirichlet operator $(-\Delta)^\alpha - Id$ in Ω is negative.*

To deal with (1), we first have to show that the solution at time $t = 1$ has algebraic tails in the interior of Ω :

$$\frac{c \times \delta(x)^\alpha}{1 + |x|^{1+2\alpha}} \leq n(x, t = 1) \leq \frac{C}{1 + |x|^{1+2\alpha}}. \quad (5)$$

The proof of (5) is an application of general results about the fractional Dirichlet heat kernel estimates given for instance in [8] or in [4]. Both of the two cited articles use a probabilistic approach. We propose in this work a deterministic proof of the lower bound of the fractional Dirichlet kernel estimates. Our proof is quite simple but the result is not as general than those presented in [8] and [4] because we work in finite time and with smooth domains. We do not provide the proof of the upper bound of the fractional Dirichlet kernel estimates since there is no difficulties to obtain such bound. The result is the following:

Theorem 2. Let Ω be a smooth domain of \mathbb{R}^d with $d \geq 1$. If we define p as the solution of the following equation

$$\begin{cases} \partial_t p + (-\Delta)^\alpha p = 0 & \text{for all } (x, t) \in \Omega \times]0, +\infty[, \\ p(x, t) = 0 & \text{for all } (x, t) \in \Omega^c \times [0, +\infty[, \\ p(x, t = 0) = n_0(x) \in C_0^\infty(\Omega, \mathbb{R}^+) \cap C_c^0(\mathbb{R}), \end{cases} \quad (6)$$

then there exists $c > 0$ and $C > 0$ such that for all $x \in \Omega$,

$$\frac{c \times \delta(x)^\alpha}{1 + |x|^{d+2\alpha}} \leq p(x, t = 1) \leq \frac{C}{1 + |x|^{d+2\alpha}}. \quad (7)$$

Once (5) is obtained we can focus on the propagation phenomena. The question of propagation in a Fisher KPP type equation involving a fractional Laplacian in a constant environment was first treated by Cabré and Roquejoffre in [7]. They proved that the front position is exponential in time (see also [9] for instance for some heuristic and numerical works predicting such behavior). Next, Cabré, Coulon and Roquejoffre proved in [6] the convergence to a stationary state with an exponential speed in a periodic heterogeneous environment. Thanks to a different approach introduced by Méléard and Mirrihami in [20], the authors proved the result of an exponential speed of propagation in a constant environment. More recently, [5], [24] and [19] extended this approach to derive the speed of propagation for different non-local operators in the case of a homogeneous environment for [5] and a periodic environment for [19] and [24]. The idea introduced in [20] is to use an asymptotic approach as known as "approximation of geometric optics". The main idea of this approach is to perform a long time-long range rescaling to catch the effective behavior of solution (see for instance [14] and [12] for the classical Laplacian case). We follow this general idea. Thus, we expect that in large time the propagation front is located in

$$\mathcal{B} = \{(x, t) \in \mathbb{R} \times \mathbb{R}^+ \mid (1 + 2\alpha) \log |x| < |\lambda_0| t\}.$$

where λ_0 is defined by (2). We perform the following change of variable

$$(x, t) \mapsto \left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}, \frac{t}{\varepsilon} \right). \quad (8)$$

A such scaling does not change the geometry of the set \mathcal{B} . Next, we rescale the solution of (1) as follows :

$$n_\varepsilon(x, t) = n\left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}, \frac{t}{\varepsilon}\right) \quad (9)$$

and a new steady state :

$$n_{+, \varepsilon}(x) = n_+\left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}\right). \quad (10)$$

For any set \mathcal{U} and any positive constant ν , we introduce the following new sets :

$$\mathcal{U}_\nu = \{x \in \mathcal{U} \mid \text{dist}(x, \partial\mathcal{U}) > \nu\}, \quad \mathcal{U}_{-\nu} = \{x \in \mathbb{R} \mid \text{dist}(x, \mathcal{U}) < \nu\} \quad \text{and} \quad \mathcal{U}^\varepsilon = \left\{x \in \mathbb{R} \mid |x|^{\frac{1}{\varepsilon}-1} x \in \mathcal{U}\right\}.$$

For reasons of brevity, we will always denote $(\mathcal{U}_\nu)^\varepsilon$ by $\mathcal{U}_\nu^\varepsilon$. We also introduce the following set :

$$\mathcal{B}^\delta = \{(x, t) \in \mathbb{R} \times [0, +\infty[\mid |x|^{1+2\alpha} \leq e^{|\lambda_0|t-\delta}\}.$$

Concerning the initial data, we assume that

$$n_0 \in C_0^\infty(\Omega, \mathbb{R}^+) \cap C_c^\infty(\mathbb{R}, \mathbb{R}^+), \quad n_0 \not\equiv 0. \quad (\text{H2})$$

Theorem 3. Assume (H1) and (H2). Then

(i) n_ε converges locally uniformly to 0 in \mathcal{B}^c ,

(ii) $\frac{n_\varepsilon}{n_{+,\varepsilon}}$ converges locally uniformly to 1 in $(\Omega^\varepsilon \times]0, +\infty[) \cap \mathcal{B}$.

A direct consequence of this result is the invasion of the unstable state by the stable state:

Corollary 2. Assume (H1) and (H2). Then for all $\nu > 0$:

(i) $\forall c < \frac{|\lambda_0|}{1+2\alpha}$, $\lim_{t \rightarrow +\infty} n(x, t) = n_+(x)$ uniformly on $x \in \{|x| < e^{ct}\}$,

(ii) $\forall C > \frac{|\lambda_0|}{1+2\alpha}$, $\lim_{t \rightarrow +\infty} n(x, t) = 0$ uniformly on $x \in \{|x| > e^{Ct}\}$.

1.4 Strategy and organization of the paper

One of the main arguments to prove Theorem 3, is that, using the rescaling (8), as $\varepsilon \rightarrow 0$, the term $((-\Delta)^\alpha(n)n^{-1})(|x|^{\frac{1}{\varepsilon}-1}x, \frac{t}{\varepsilon})$ vanishes. More precisely, one can provide a sub and a super-solution to the rescaled equation which are indeed a sub and a super-solution to a perturbation of an ordinary differential equation derived from (1) by omitting the term $(-\Delta)^\alpha$ and multiplying by the principal eigenfunction associated to the operator $(-\Delta)^\alpha - Id$ in $\Omega_{\pm\nu}$. These sub and super-solutions also have the property that when one applies the operator $f \mapsto (-\Delta)^\alpha(f)f^{-1}$ to such functions, the outcome is very small in the interior of Ω_ν and of order $\mathcal{O}(\varepsilon^2)$ as ε tends to 0. Moreover, these sub and super-solutions have algebraic decay at infinity. Obviously, we can not put the sub-solution below n_0 . However, using the heat kernel estimates given by Theorem 2, we are able to put the sub-solution below the solution n for $t \geq 1$. Finally, to prove the convergence of n_ε , we use the method of perturbed test functions from the theory of viscosity solutions and homogenization (introduced by Evans in [10] and [11]).

We close the introduction by noticing that all the presented results can be extended to the multi-dimensional case. Let Ω be a smooth non-connected periodic domain of \mathbb{R}^d

$$i.e. \Omega = \bigcup_{k \in \mathbb{Z}^d} \Omega_0 + a_k, \text{ with } \Omega_0 \text{ a smooth bounded domain of } \mathbb{R}^d \text{ and } a_k \in \mathbb{R}^d.$$

We assume that

$$(\Omega_0 + a_i) \cap (\Omega_0 + a_j) \neq \emptyset \text{ if and only if } i = j.$$

Moreover, if we denote e_i the i^{th} vector of the canonical basis of \mathbb{R}^d then $a_{k+e_i} - a_k = a_{e_i}$.

The assumption (H1) has to be adapted in the following way: we assume that the principal eigenvalue $\tilde{\lambda}_0$ of the Dirichlet operator $(-\Delta)^\alpha - Id$ in Ω_0 is negative. For the sake of simplicity, the study is done in one dimension. The only result presented in multi-dimension is the parabolic estimates of Theorem 2.

In section 2, we demonstrate Theorem 1. Next, section 3 is dedicated to the proof of Theorem 2. Section 4 introduces all the requirements to achieve the proof of Theorem 3. Finally, section 5 is devoted to the proof of Theorem 3. All along the article, the constant denoted by c or C may change from one line to another when there is no confusion possible and we drop the constant C_α and the Cauchy principal value $P.V.$ in front of the fractional Laplacian for better readability.

2 Uniqueness of the stationary state n_+

First, we state a proposition which gives the shape of any non-trivial bounded sub and super-solution to (3) near the boundary. Then, we use this result to prove the uniqueness result. Since

the proof of the existence is classical we do not provide it. It relies on an increasing sequences initialized by $\varepsilon\phi_A$ (see [23] for more details).

Proposition 1. (i) *If u is a smooth positive bounded function such that:*

1. $u(x) = 0$ for all $x \in \Omega^c$,
2. $(-\Delta)^\alpha u(x) \leq u(x) - u(x)^2$ for all $x \in \Omega$,

then there exists $C > 0$ such that

$$u(x) \leq C\delta(x)^\alpha.$$

(ii) *If v is a smooth positive bounded function such that:*

1. $v(x) = 0$ for all $x \in \Omega^c$,
2. $(-\Delta)^\alpha v(x) \geq v(x) - v(x)^2$ for all $x \in \Omega$,
3. $\exists x \in \Omega$ such that $v(x) > 0$,

then there exists $c > 0$ such that

$$c\delta(x)^\alpha \leq v(x).$$

Before giving the proof of Proposition 1, we recall a lemma proved in Annex B of [22] which states a useful barrier function.

Lemma 1 ([22]). *There exists $C > 0$ and a radial continuous function $\bar{\psi} \in H_{loc}^1(\mathbb{R})$ satisfying :*

$$\begin{cases} (-\Delta)^\alpha \bar{\psi} \geq 1 & \text{in }]-4, 4[\setminus]-1, 1[, \\ \bar{\psi} \equiv 0 & \text{in }]-1, 1[, \\ 0 \leq \bar{\psi} \leq C|x-1|^\alpha & \text{in }]-4, 4[\setminus]-1, 1[, \\ 1 \leq \bar{\psi} \leq C & \text{in } \mathbb{R} \setminus]-4, 4[. \end{cases} \quad (11)$$

Proof of Proposition 1. Proof of (i). Let u be a continuous positive bounded function such that $u = 0$ in Ω^c and $(-\Delta)^\alpha u \leq f(u)$ in Ω . To help the reading, we demonstrate the result near the boundary point 0. The proof works the same for each point of the boundary. We rescale the function given by Lemma 1:

$$\bar{\phi}(x) := \max\left(1, \left(\frac{A}{8}\right)^{2\alpha}, \max u\right) \bar{\psi}\left(\frac{8}{A}x + 1\right).$$

Then, according to Lemma 1, we have the following properties for $\bar{\phi}$:

$$\begin{cases} (-\Delta)^\alpha \bar{\phi} \geq 1 & \text{in }]0, \frac{3A}{8}[, \\ \bar{\phi} \equiv 0 & \text{in }]-\frac{A}{4}, 0[, \\ 0 \leq \bar{\phi} \leq C|x|^\alpha & \text{in }]0, \frac{3A}{8}[, \\ \max u \leq \bar{\phi} \leq C \max\left(1, \left(\frac{A}{8}\right)^{2\alpha}, \max u\right) & \text{in } \mathbb{R} \setminus \left[-\frac{5A}{8}, \frac{3A}{8}\right], \end{cases} \quad (12)$$

Next, we can deduce that in $[0, \frac{3A}{8}]^c$, $u \leq \bar{\phi}$. We want to prove by contradiction that $u \leq \bar{\phi}$ in $[0, \frac{3A}{8}]$. If we assume that there exists $x_0 \in [0, \frac{3A}{8}]$ such that $(\bar{\phi} - u)(x_0) < 0$. Then, there exists $x_1 \in]0, \frac{3A}{8}[$ such that $(\bar{\phi} - u)(x_1) = \min_{x \in \mathbb{R}}(\bar{\phi} - u)(x) < 0$. Thus,

$$(-\Delta)^\alpha(\bar{\phi} - u)(x_1) < 0, \text{ and } (-\Delta)^\alpha(\bar{\phi} - u)(x_1) \geq 1 - u(x_1) + u(x_1)^2 \geq 0,$$

a contradiction.

Proof of (ii). Let v be a continuous positive function bounded such that $v = 0$ in Ω^c and $(-\Delta)^\alpha v \geq f(v)$ in Ω . An easy but important remark is the following: thanks to the non-local character of the fractional Laplacian, since there exists $x \in \Omega$ such that $v(x) > 0$, we deduce that $v > 0$ in the whole domain Ω . Otherwise, we could conclude to the following contradiction :

$$\exists \underline{x} \in \Omega \text{ such that } v(\underline{x}) = 0 \text{ and } (-\Delta)^\alpha v(\underline{x}) - v(\underline{x}) + v(\underline{x})^2 = - \int_{\mathbb{R}} \frac{v(y)}{|x - y|^{1+2\alpha}} dy < 0.$$

Next, we define $\underline{w}_k : (x, t) \in \mathbb{R} \times [0, +\infty[\mapsto \underline{w}_k(x, t) \in \mathbb{R}$ as the solution of

$$\begin{cases} \partial_t \underline{w}_k + (-\Delta)^\alpha \underline{w}_k = \underline{w}_k - \underline{w}_k^2, & \text{in }]2kA, 2(k+1)A[\times]0, +\infty[, \\ \underline{w}_k(x, 0) = 0, & \text{in } \mathbb{R} \setminus]2kA, 2(k+1)A[\times]0, +\infty[, \\ \underline{w}_k(x, 0) = v(x), & \text{in }]2kA, 2(k+1)A[. \end{cases} \quad (13)$$

Thanks to the first remark, and recalling (H1), we deduce thanks to Theorem 5.1 in [3] that $\underline{w}_k(\cdot, t) \xrightarrow[t \rightarrow +\infty]{} \underline{w}_{stat}(\cdot)$. Thus, we conclude thanks to the maximum principle that

$$\underline{w}_{stat}(x) \leq v(x), \quad \forall x \in \mathbb{R}$$

with \underline{w}_{stat} the solution of

$$\begin{cases} (-\Delta)^\alpha \underline{w}_{stat} = \underline{w}_{stat} - \underline{w}_{stat}^2, & \text{in }]2kA, 2(k+1)A[, \\ \underline{w}_{stat} = 0 & \text{in } \mathbb{R} \setminus]2kA, 2(k+1)A[. \end{cases} \quad (14)$$

Note that the above \underline{w}_{stat} does not depend on the choice of k , i.e. $\underline{w}_k(\cdot, t)$ converges as t tends to $+\infty$ to the same \underline{w}_{stat} (up to a translation).

Since, $]2kA, 2(k+1)A[$ is bounded, we apply the results of [22] to find that there exists a constant $c > 0$ such that

$$c\delta(x)^\alpha 1_{]2kA, 2(k+1)A[}(x) \leq \underline{w}_{stat}(x) \leq v(x).$$

The previous analysis holds for every $k \in \mathbb{Z}$. We conclude that

$$c\delta(x)^\alpha \leq v(x). \quad (15)$$

□

Proof of Theorem 1. The argument relies on the fact that two steady solutions are comparable everywhere thanks to Proposition 1. This is in the spirit of [2] in a different context. Let u and v be two bounded steady solutions to (3). By the maximum principle, we easily have that for all $x \in \mathbb{R}$,

$$u(x) \leq 1 \text{ and } v(x) \leq 1.$$

We will assume that

$$v\left(\frac{A}{2}\right) \leq u\left(\frac{A}{2}\right). \quad (16)$$

Thanks to Proposition 1, we deduce the existence of two constants $0 < c \leq C$ such that:

$$c\delta(x)^\alpha \leq u(x) \leq C\delta(x)^\alpha \quad \text{and} \quad c\delta(x)^\alpha \leq v(x) \leq C\delta(x)^\alpha.$$

Thus there exists a constant $\lambda > 1$ such that for all $x \in \mathbb{R}$,

$$u(x) \leq \lambda v(x). \quad (17)$$

We set $l_0 := \inf \{\lambda \geq 1 \mid \forall x \in \mathbb{R}, u(x) \leq \lambda v(x)\}$. The point is to prove by contradiction that $l_0 = 1$. It implies that $\frac{A}{2}$ is a contact point. It will allow us to conclude thanks to the fractional maximum principle that $u = v$.

We assume by contradiction that $l_0 > 1$. Next, we define :

$$\tilde{w} = \inf_{x \in \Omega} \frac{(l_0 v - u)(x)}{\delta(x)^\alpha} \geq 0. \quad (18)$$

There are two cases to be considered.

Case 1: $\tilde{w} > 0$.

We show in this case that we can construct $l_1 < 1$ such that $u(x) \leq l_1 l_0 v(x)$ for all $x \in \mathbb{R}$ and it will be the contradiction. If $\tilde{w} > 0$, then there exists $\mu \in]0, 1[$ and $\nu > 0$ such that for all $x \in \Omega \setminus \Omega_\nu$,

$$\frac{(\mu l_0 v - u)(x)}{\delta(x)^\alpha} \geq \frac{\tilde{w}}{2}. \quad (19)$$

Actually, if there does not exist such couples (μ, ν) then we deduce that for all $n \in \mathbb{N}$, there exists $(x_n)_{n \in \mathbb{N}} \in \Omega$, such that $\delta(x_n) \leq \frac{1}{n}$ and

$$\frac{((1 - \frac{1}{n})l_0 v - u)(x_n)}{\delta(x_n)^\alpha} < \frac{\tilde{w}}{2}.$$

Passing to the liminf we get the following contradiction :

$$0 < \tilde{w} \leq \frac{\tilde{w}}{2}.$$

The existence of the couple (μ, ν) implies that

$$(\mu l_0 v - u)(x) \geq 0, \quad \forall x \in \Omega \setminus \Omega_\nu. \quad (20)$$

Next, we show that

$$\exists \rho > 0 \text{ such that } \rho \leq (l_0 v - u)(x) \quad \forall x \in \Omega_\nu. \quad (21)$$

Actually, if a such ρ does not exist then there exists a sequence $(x_n)_{n \in \mathbb{N}} \in \Omega$ such that $\delta(x_n) \geq \nu$ and $(l_0 v - u)(x_n) \xrightarrow{n \rightarrow +\infty} 0$. Then we obtain

$$\frac{(l_0 v - u)(x_n)}{\delta(x_n)^\alpha} \leq \frac{(l_0 v - u)(x_n)}{\nu^\alpha} \xrightarrow{n \rightarrow +\infty} 0$$

which is in contradiction with the fact that \tilde{w} is positive. The existence of a such ρ implies that

$$\left(\left(1 - \frac{\rho}{\max l_0 v}\right) l_0 v - u \right) (x) \geq 0, \quad \forall x \in \Omega_\nu. \quad (22)$$

Finally, if we define $l_1 = \max(\mu, 1 - \frac{\rho}{\max l_0 v + 1})$ then we obtain the desired contradiction. Therefore this case can not occur.

Case 2: $\tilde{w} = 0$.

We consider $(x_n)_{n \in \mathbb{N}}$ a minimizing sequence of \tilde{w} . There are 3 subcases : a subsequence of $(x_n)_{n \in \mathbb{N}}$ converges to $x_0 \in \Omega$, a subsequence of $(x_n)_{n \in \mathbb{N}}$ converges to $x_b \in \partial\Omega$ and any subsequence of $(x_n)_{n \in \mathbb{N}}$ diverges.

Subcase a: *There is $x_0 \in \Omega$, such that $\frac{(l_0 v - u)(x_0)}{\delta(x_0)^\alpha} = 0$.*

Since $x_0 \in \Omega$ we deduce that $(l_0 v - u)(x_0) = 0$. Hence, by the maximum principle, $u = l_0 v$. We deduce $l_0 v$ is a solution of (3) and we conclude that :

$$l_0(v - v^2) = l_0(-\Delta)^\alpha(v) = (-\Delta)^\alpha(l_0 v) = l_0 v - (l_0 v)^2. \quad (23)$$

This equation leads to $l_0 = 1$, a contradiction.

Subcase b: *There is $x_b \in \partial\Omega$, such that $\liminf_{\substack{x \rightarrow x_b, \\ x \in \Omega}} \frac{(l_0 v - u)(x)}{\delta(x)^\alpha} = 0$.*

Here is a summary of what we know:

- (i) $l_0 v - u \geq 0$,
- (ii) $(-\Delta)^\alpha(l_0 v - u) \geq -l_0(l_0 v - u)$,
- (iii) $(l_0 v - u)(x_b) = 0$.

According to the fractional Hopf Lemma [16], the previous assumptions leads to $\liminf_{\substack{x \rightarrow x_b, \\ x \in \Omega}} \frac{(l_0 v - u)(x)}{\delta(x)^\alpha} >$

0. However, we have assumed that $\liminf_{\substack{x \rightarrow x_b, \\ x \in \Omega}} \frac{(l_0 v - u)(x)}{\delta(x)^\alpha} = 0$, a contradiction.

Subcase c: *There exists a minimizing sequence $(x_n)_{n \in \mathbb{N}}$ such that $|x_n|$ tends to the infinity.*

We first set

$$\bar{x}_k = x_k - \lfloor \frac{x_k}{A} \rfloor,$$

where $\lfloor x \rfloor$ is the entire part of x . Since $\bar{x}_k \in [0, A]$, we deduce that up to a subsequence \bar{x}_k converges to $\bar{x}_\infty \in [0, A]$. Then we define:

$$u_k(x) = u(x + x_k) \text{ and } v_k(x) = v(x + x_k).$$

We also define the following set :

$$\Omega_\infty = \{x \in \mathbb{R} \mid x + \bar{x}_\infty \in \Omega\}.$$

For every compact set K of Ω_∞ , there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0, \forall x \in K, x + x_n \in \Omega_\infty.$$

Thus, for all $n \geq n_0$,

$$(-\Delta)^\alpha u_n(x) = u_n(x) - u_n(x)^2 \text{ and } (-\Delta)^\alpha v_n(x) = v_n(x) - v_n(x)^2 \quad \forall x \in K.$$

According to [21], we deduce that the sequences $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ converge up to a subsequence locally uniformly to u_∞ and v_∞ in $C^\beta(\Omega_\infty)$ with some $\beta > 2\alpha$. Hence we deduce that

$$(-\Delta)^\alpha u_\infty(x) = u_\infty(x) - u_\infty(x)^2 \text{ and } (-\Delta)^\alpha v_\infty(x) = v_\infty(x) - v_\infty(x)^2, \quad \forall x \in K.$$

Since it is true in every compact subset of Ω_∞ , it follows that

$$(-\Delta)^\alpha u_\infty(x) = u_\infty(x) - u_\infty(x)^2 \text{ and } (-\Delta)^\alpha v_\infty(x) = v_\infty(x) - v_\infty(x)^2, \forall x \in \Omega_\infty.$$

Remark that

$$l_0 v_\infty - u_\infty \geq 0 \text{ and } \liminf_{\substack{x \rightarrow 0 \\ x \in \Omega_\infty}} \frac{(l_0 v_\infty - u_\infty)(x)}{\text{dist}(x, \partial\Omega_\infty)^\alpha} = 0.$$

Hence, if $\bar{x}_\infty \in]0, A[$ then $0 \in \Omega_\infty$ and we fall in the subcase a).

If $\bar{x}_\infty \notin]0, A[$ then $0 \in \partial\Omega_\infty$ and we fall in the subcase b).

Both cases lead to a contradiction.

Thus, we conclude that $l_0 = 1$. □

Remark. Noticing that for all $x \in \Omega$, we have

$$(-\Delta)^\alpha (n_+(\cdot + 2A))(x) = \int_{\mathbb{R}} \frac{n_+(x + 2A) - n_+(y + 2A)}{|x + 2A - (y + 2A)|^{1+2\alpha}} dy = n_+(x + 2A) - n_+(x + 2A)^2,$$

we deduce by uniqueness of the solution of (3) that n_+ is $2A$ -periodic.

3 The fractional heat kernel and the preparation of the initial data

We first prove Theorem 2, then we apply it to estimate the position of n at time 1.

Proof of Theorem 2. We recall that for this proof Ω is a smooth domain of \mathbb{R}^d . First, let $\nu_0 > 0$ be such that for all $x_0 \in \partial\Omega_{\nu_0}$, the open ball $B(x_0, \nu_0) \subset \Omega$. Next, note that thanks to a translation and possibly a scaling, we can suppose the following hypothesis:

$$\exists \mu > 0 \text{ such that } \mu < n_0(x) \text{ for all } x \in B(0, 2). \quad (24)$$

Thanks to (24), we can focus our study on $x \in \Omega \setminus B(0, 2)$. Thus, it is sufficient to prove that there exists a constant $c > 0$ such that

$$\forall x \in \Omega \setminus B(0, 2), \text{ we have } \frac{c\delta(x)^\alpha}{(|x| + \nu)^{d+2\alpha}} \leq p(x, 1)$$

with $\nu = \min(\frac{1}{2}, \nu_0)$. To achieve the proof, there will be 3 steps.

First, we introduce a suitable decomposition of the fractional Laplacian to prove the existence of $c_1 > 0$ such that

$$\begin{cases} \frac{c_1}{|x|^{d+2\alpha}} \leq \partial_t p(x, t) + L^\alpha p(x, t) + \lambda p(x, t) & \text{for all } (x, t) \in (\Omega \setminus B(0, 2)) \times]0, 1], \\ p(x, t) = 0 & \text{for all } (x, t) \in \Omega^c \times [0, 1], \\ p(x, t = 0) = n_0(x) \in C_0^\infty(\Omega, \mathbb{R}^+) \end{cases} \quad (25)$$

where

$$L^\alpha p(x, t) = \int_{B(0, \nu)} \frac{p(x, t) - p(x + y, t)}{|y|^{d+2\alpha}} dy \text{ and } \lambda = \int_{\mathbb{R}^d \setminus B(0, \nu)} \frac{1}{|y|^{d+2\alpha}} dy.$$

In a second step, we will show that

$$\exists c_2 > 0 \text{ such that } \frac{c_2}{(|x| + \nu)^{d+2\alpha}} \leq p(x, t = 1) \text{ for all } x \in \Omega_\nu \setminus B(0, 2). \quad (26)$$

Finally, we prove the same kind of result near the boundary :

$$\exists c_3 > 0 \text{ such that } \frac{c_3 \delta(x)^\alpha}{(|x| + \nu)^{d+2\alpha}} \leq p(x, t = 1) \text{ for all } x \in (\Omega \setminus \Omega_\nu) \setminus B(0, 2). \quad (27)$$

Before starting the details of the proof, we will need two intermediate lemmas. Let ϕ_ν be the principal positive eigen function of the following equation:

$$\begin{cases} L^\alpha \phi_\nu = \mu_\nu \phi_\nu & \text{for all } x \in B(0, \nu) \\ \phi_\nu = 0 & \text{for all } x \in B(0, \nu)^c, \\ \phi_\nu \geq 0, \|\phi_\nu\|_\infty = 1. \end{cases}$$

The first lemma is the following.

Lemma 2. *Let w be the solution of the following problem:*

$$\begin{cases} \partial_t w + L^\alpha w = 1 & \text{for all } (x, t) \in B(0, \nu) \times]0, +\infty[\\ w(x, t) = 0 & \text{for all } (x, t) \in B(0, \nu)^c \times [0, +\infty[, \\ w(x, t = 0) = 0 & \text{for all } x \in B(0, \nu). \end{cases} \quad (28)$$

Then there exists a constant $c_\nu > 0$ such that

$$c_\nu \times \phi_\nu(x) \leq w(x, t = 1).$$

Proof. We define $\tau(t) = \frac{1}{\mu_\nu}(1 - e^{-\mu_\nu t})$ such that

$$\begin{cases} \tau'(t) + \mu_\nu \tau(t) = 1, \\ \tau(0) = 0. \end{cases}$$

Thanks to this choice of $\tau(t)$, the application $\underline{w}(x, t) := \tau(t) \times \phi_\nu(x)$ is a sub-solution to (28). Actually, we have

$$(\partial_t + L^\alpha)(\underline{w}) - 1 = \tau' \phi_\nu + \mu_\nu \tau \phi_\nu - 1 \leq \tau' \phi_\nu + \mu_\nu \tau \phi_\nu - \phi_\nu = \phi_\nu(\tau' + \mu_\nu \tau - 1) = 0.$$

Since $\underline{w}(t = 0) = 0 \leq w(t = 0)$, we can conclude thanks to the comparison principle that for all $t \geq 0$ and all $x \in \mathbb{R}^d$, we have

$$\underline{w}(x, t) \leq w(x, t) \Rightarrow \underline{w}(x, 1) = \frac{1}{\mu_\nu}(1 - e^{-\mu_\nu})\phi_\nu(x) = c_\nu \phi_\nu(x) \leq w(x, 1).$$

□

Next, we establish a barrier function for L^α in the spirit of those introduced in [22].

Lemma 3. *There exists a function $\underline{\psi}$ such that*

$$\begin{cases} L^\alpha \underline{\psi} \leq 0 & \text{in } B(0, \nu) \setminus B(0, \frac{\nu}{2}), \\ \underline{\psi} = 0 & \text{in } B(0, \nu)^c, \\ \underline{\psi} \leq 1 & \text{in } B(0, \frac{\nu}{2}), \\ \underline{c}(\nu - |x|)^\alpha \leq \underline{\psi} & \text{in } B(0, \nu), \\ \underline{\psi} \text{ is continuous in } B(0, \nu) \setminus B(0, \frac{\nu}{2}). \end{cases}$$

Proof. We just have to consider C large enough such that the first point and the third of the lemma holds with the following $\underline{\psi}$:

$$\underline{\psi}(x) := \left(\frac{1}{C}(\nu - |x|^2)^\alpha + \frac{1}{2}1_{B(0, \frac{\nu}{4})}(x) \right) 1_{B(0, \nu)}(x).$$

Indeed, defining $f(x) := (\nu - |x|^2)^\alpha$, we have for C large enough and $x \in B(0, \nu) \setminus B(0, \frac{\nu}{2})$

$$L^\alpha \underline{\psi}(x) \leq \frac{L^\alpha f(x)}{C} - \frac{1}{2} \int_{B(0, \frac{\nu}{4})} \frac{1}{|x-y|^{d+2\alpha}} dy \leq \frac{\sup_{B(0, \nu) \setminus B(0, \frac{\nu}{2})} |L^\alpha f|}{C} - \frac{m(B(0, 1))}{2 \times 5^{d+2\alpha}} \times \left(\frac{4}{\nu} \right)^{2\alpha} < 0.$$

The other conditions follow. □

Step 1. We first split the fractional Laplacian into 2 parts:

$$(-\Delta)^\alpha p(x, t) = \int_{\mathbb{R}^d \setminus B(0, \nu)} \frac{p(x, t) - p(x+y, t)}{|y|^{d+2\alpha}} dy + L^\alpha p(x, t) = I_1 + L^\alpha p(x, t).$$

For I_1 , we obtain :

$$I_1 = \int_{\mathbb{R}^d \setminus B(0, \nu)} \frac{p(x, t) - p(x+y, t)}{|y|^{d+2\alpha}} dy = \lambda p(x, t) - \int_{\mathbb{R}^d \setminus B(0, \nu)} \frac{p(x+y, t)}{|y|^{d+2\alpha}} dy.$$

Since $|x| > 2$, and since $p \geq 0$ (according to the comparison principle), we have

$$\inf_{\substack{t \in (0, 1) \\ z \in B(0, 1)}} p(z, t) \int_{B(0, 1)} \frac{1}{|z-x|^{d+2\alpha}} dz \leq \int_{B(-x, 1)} \frac{p(x+y, t)}{|y|^{d+2\alpha}} dy \leq \int_{\mathbb{R}^d \setminus B(0, \nu)} \frac{p(x+y, t)}{|y|^{d+2\alpha}} dy. \quad (29)$$

Let us notice that, $\inf_{\substack{t \in (0, 1) \\ z \in B(0, 1)}} p(z, t) > 0$.

Indeed, let ϕ_2 be the first positive eigen-function of the Dirichlet fractional Laplacian in $B(0, 2)$ and λ_2 the associated eigenvalue

$$i.e. \begin{cases} (-\Delta)^\alpha \phi_2 = \lambda_2 \phi_2 & \text{for } x \in B(0, 2), \\ \phi_2 = 0 & \text{for } x \in B(0, 2)^c, \\ \|\phi_2\|_\infty = 1. \end{cases}$$

Then the function

$$\underline{p}(x, t) := \mu \times \phi_2(x) \times e^{-\lambda_2 t}$$

is a sub-solution of (6) (where μ is defined by (24)). According to the maximum principle, we have

$$0 < \mu \times \min_{B(0,1)} \phi_2 \times e^{-|\lambda_2|} \leq p(x, t) \text{ for all } (x, t) \in B(0, 1) \times [0, 1].$$

We conclude to the existence of a positive constant $c_1 > 0$ such that:

$$\frac{c_1}{|x|^{d+2\alpha}} \leq \int_{\mathbb{R}^d \setminus [-\nu, \nu]} \frac{p(x+y, t)}{|y|^{d+2\alpha}} dy.$$

So we have shown (25). If we define $v(x, t) = e^{\lambda t} \times p(x, t)$, we find the following system:

$$\begin{cases} \frac{c_1}{|x|^{d+2\alpha}} \leq \partial_t v(x, t) + L^\alpha v(x, t) & \text{for } (x, t) \in (\Omega \setminus B(0, 2 - \nu)) \times]0, 1], \\ v(x, t) = 0 & \text{for } (x, t) \in \Omega^c \times [0, 1], \\ v(x, t = 0) = n_0(x) \in C_0^\infty(\Omega, \mathbb{R}^+). \end{cases} \quad (30)$$

Step 2. Pick $x_0 \in \Omega_\nu \setminus B(0, 2)$. Then, we have for all $(x, t) \in (\Omega \setminus B(0, 2 - \nu)) \times]0, 1]$:

$$\frac{c_1}{(|x_0| + \nu)^{d+2\alpha}} 1_{B(0, \nu)}(x - x_0) \leq \frac{c_1}{|x|^{d+2\alpha}} \leq \partial_t v(x, t) + L^\alpha v(x, t).$$

That leads to

$$1_{B(0, \nu)}(x - x_0) \leq \partial_t \left(\frac{(|x_0| + \nu)^{d+2\alpha}}{c_1} v \right) + L^\alpha \left(\frac{(|x_0| + \nu)^{d+2\alpha}}{c_1} v \right).$$

Then, according to the maximum principle and Lemma 2, we have

$$c_\nu \phi_\nu(x - x_0) \leq \frac{(|x_0| + \nu)^{d+2\alpha}}{c_1} v(x, t = 1). \quad (31)$$

If we evaluate (31) at x_0 , we obtain:

$$\frac{c_\nu c_1 e^{-\lambda} \phi_\nu(0)}{(|x_0| + \nu)^{d+2\alpha}} \leq p(x_0, t = 1).$$

Defining $c_2 = c_\nu c_1 e^{-\lambda} \phi_\nu(0)$, we have proved (26).

Step 3 : As in the proof of Proposition 1, we can show by contradiction that there exists a positive constant c_0 such that for all $x \in \mathbb{R}^d$,

$$c_0 \underline{\psi}(x) \leq \phi_\nu(x)$$

where $\underline{\psi}$ is defined in Lemma 3. Then we take $x_1 \in \Omega \setminus (\Omega_\nu \cup B(0, 2 - \nu))$. Since Ω is assumed to be smooth, there exists $x_0 \in \partial\Omega_\nu$ such that $x_1 \in B(x_0, \nu)$, $B(x_0, \nu) \subset \Omega$ and $\nu - |x_1 - x_0| = \delta(x_1)$. Thanks to (31) and the fourth point of Lemma 3, we deduce

$$c_\nu c_0 \underline{\delta}(x_1)^\alpha = c_\nu c_0 \underline{\delta}(\nu - |x_1 - x_0|)^\alpha \times 1_{B(0, \nu)}(x_1 - x_0) \leq c_\nu \phi_\nu(x_1 - x_0) \leq \frac{(|x_0| + \nu)^{d+2\alpha}}{c_1} v(x_1, t = 1).$$

We deduce that there exists $c_3 > 0$ such that (27) holds true.

Combining (24), (26) and (27) yields the conclusion of the Theorem. \square

We apply Theorem 2 to show that starting from $n(x, 0) \in C_0^\infty(\Omega) \cap C_c^\infty(\mathbb{R})$, the solution of (1) $n(\cdot, t = 1)$ has algebraic tails.

Proposition 2. *There exists two constants c_m and c_M depending on n_0 such that for all $x \in \Omega$, we have*

$$\frac{c_m \delta(x)^\alpha}{1 + |x|^{1+2\alpha}} \leq n(x, 1) \leq \frac{c_M}{1 + |x|^{1+2\alpha}}. \quad (32)$$

Proof. Defining $M := \max(\max n_0, 1)$, the solution n belongs to the set $[0, M]$ (0 is a sub-solution and M is a super-solution).

We begin with the proof that $\frac{c_m \delta(x)^\alpha}{1 + |x|^{1+2\alpha}} \leq n(x, 1)$.

Let \underline{n} be the solution of :

$$\begin{cases} \partial_t \underline{n}(x, t) + (-\Delta)^\alpha \underline{n}(x, t) = -M \underline{n}(x, t) & \text{for all } (x, t) \in \Omega \times]0, +\infty[, \\ \underline{n}(x, t) = 0 & \text{for all } (x, t) \in \Omega^c \times [0, +\infty[, \\ \underline{n}(x, 0) = n_0(x) & \text{for all } x \in \mathbb{R}, \end{cases} \quad (33)$$

Thanks to the maximum principle, we deduce that for all $(x, t) \in \mathbb{R} \times [0, +\infty[$, we have

$$\underline{n}(x, t) \leq n(x, t).$$

Moreover, if we define $\underline{p}(x, t) = e^{Mt} \underline{n}(x, t)$, we find that \underline{p} is solution of (6). According to Theorem 2, we deduce that there exists $c_m > 0$ such that

$$\frac{c_m \delta(x)^\alpha}{(1 + |x|^{1+2\alpha})} \leq \underline{n}(x, t = 1) \leq n(x, t = 1). \quad (34)$$

The proof works the same for the other bound. \square

In what follows, we make a translation in time to keep $n(x, 1)$ as our initial data. In other words, we will suppose that there exists c_m and c_M such that:

$$\forall x \in \mathbb{R}, \frac{c_m \times \delta(x)^\alpha}{1 + |x|^{1+2\alpha}} \leq n(x, 0) \leq \frac{c_M}{1 + |x|^{1+2\alpha}}. \quad (\text{H2}')$$

4 Rescaling and preparation for the proof of Theorem 3

In this section, we introduce all the requirements for the proof of Theorem 3. We begin by performing the rescaling (8). Next, we obtain a sub-solution and a super-solution for the rescaled problem with ε small enough. Finally, we prove some convergence results on the principal eigenvalue λ_ν of $(-\Delta)^\alpha - Id$ in Ω_ν and on $n_{+, \nu}$ the stationary state of (3) in Ω_ν when ν is small.

We perform the scaling (8) on equation (1) with the new initial condition satisfying (H2'). The equation becomes

$$\begin{cases} \varepsilon \partial_t n_\varepsilon + (-\Delta)_\varepsilon^\alpha n_\varepsilon = n_\varepsilon(1 - n_\varepsilon) & \text{for } (x, t) \in \Omega^\varepsilon \times]0, +\infty[, \\ n_\varepsilon(x, t) = 0 & \text{for } (x, t) \in \Omega^{\varepsilon^c} \times [0, +\infty[, \\ n_\varepsilon(x, 0) = n_{0, \varepsilon}(x) & \in C_0^\infty(\Omega^\varepsilon, \mathbb{R}^+). \end{cases} \quad (1_\varepsilon)$$

where $(-\Delta)_\varepsilon^\alpha n_\varepsilon(x, t) = (-\Delta)^\alpha n \left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}, \frac{t}{\varepsilon} \right)$.

Set

$$g(x) := \frac{1}{1 + |x|^{1+2\alpha}}.$$

We first state in the spirit of [6] the behavior of g under the fractional Laplacian.

Lemma 4. *Let γ be a positive constant in $]0, \alpha[$ such that $2\alpha - \gamma < 1$. Let $\chi \in C^\alpha(\mathbb{R})$ be a periodic positive function and $g(x) := \frac{1}{1+|x|^{1+2\alpha}}$. Then there exists a positive constant C , such that we have for all $x \in \mathbb{R}$:*

(i) for all $a > 0$,

$$|(-\Delta)^\alpha g(ax)| \leq a^{2\alpha} C g(ax),$$

(ii) for all $a \in]0, 1]$,

$$|\tilde{K}(g(a \cdot), \chi)(x)| \leq \frac{Ca^{2\alpha-\gamma}}{1 + (a|x|)^{1+2\alpha}} = Ca^{2\alpha-\gamma} g(a|x|),$$

where $\tilde{K}(u, v)(x) = \int_{\mathbb{R}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+2\alpha}} dy$ is such that

$$(-\Delta)^\alpha(u \times v) = (-\Delta)^\alpha(u) \times v + u \times (-\Delta)^\alpha(v) - \tilde{K}(u, v).$$

Since, the same kind of result can be found in the appendix A of [19], we do not provide the proof of this lemma. Note that here, the lemma is stated with less regularity on χ such than in [19]. Nevertheless, there is no difficulty to adapt the proof.

Notation. As we have introduced $(-\Delta)_\varepsilon^\alpha n_\varepsilon(x, t) = (-\Delta)^\alpha n \left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}, \frac{t}{\varepsilon} \right)$, we introduce

$$\tilde{K}_\varepsilon(u, v)(x, t) = \tilde{K}(u, v)(|x|^{\frac{1}{\varepsilon}-1}, \frac{t}{\varepsilon}).$$

For any application $h : \mathbb{R} \mapsto \mathbb{R}$, we define

$$\begin{aligned} h_\varepsilon : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto h(|x|^{\frac{1}{\varepsilon}-1}x). \end{aligned}$$

According to (H1), taking $\nu \in]0, \frac{A-A_0}{2}[$, the principale eigenvalue λ_ν of the Dirichlet operator $(-\Delta)^\alpha - Id$ in Ω_ν is negative. We will denote the associated periodic and positive eigenfunction by ϕ_ν .

Theorem 4. *We assume (H1) and (H2'). Let ν be a positive constant such that $\nu < \frac{A-A_0}{2}$.*

Let C_m belongs to $]0, \min(\frac{|\lambda_\nu|}{2(\max \phi_\nu + 1)}, \frac{c_m \nu^\alpha}{\max \phi_\nu + 1})[$, C_M to $] \max(\frac{2|\lambda_\nu|}{\min \phi_\nu}, \frac{c_M}{\min \phi_\nu}), +\infty[$ and δ to $]0, \min(\sqrt{|\lambda_\nu| - C_m \max \phi_\nu}, \sqrt{C_M \min \phi_\nu - |\lambda_\nu|})[$. Then there exists $\varepsilon_\nu > 0$ such that for all $\varepsilon < \varepsilon_\nu$, the following is true.

1. If f_ε^m is defined as

$$f_\varepsilon^m(x, t) = \frac{C_m e^{-\frac{\delta}{\varepsilon}}}{1 + e^{-\frac{t}{\varepsilon}(|\lambda_\nu| - \varepsilon^2) - \frac{\delta}{\varepsilon}} |x|^{\frac{1+2\alpha}{\varepsilon}}} \times (\phi_{\nu, \varepsilon}(x) + \varepsilon),$$

then it is a sub-solution of (1_ε) in $\Omega_\nu^\varepsilon \times]0, +\infty[$.

2. If f_ε^M is defined as

$$\frac{C_M}{1 + e^{-\frac{t}{\varepsilon}(|\lambda_\nu| + \varepsilon^2) - \frac{\delta}{\varepsilon}|x|^{\frac{1+2\alpha}{\varepsilon}}}} \times \phi_{-\nu, \varepsilon}(x),$$

then it is a super-solution of (1 $_\varepsilon$).

3. For all $(x, t) \in \mathbb{R} \times [0, +\infty[$, $f_\varepsilon^m(x, t) \leq n_\varepsilon(x, t) + \varepsilon$ and $n_\varepsilon(x, t) \leq f_\varepsilon^M(x, t)$.

Proof. We begin by proving (1). Let (x, t) be in $\Omega_\nu^\varepsilon \times]0, +\infty[$. We define:

$$\psi_\varepsilon(x, t) = \frac{C_m e^{-\frac{\delta}{\varepsilon}}}{1 + e^{-\frac{t}{\varepsilon}(|\lambda_\nu| - \varepsilon^2) - \frac{\delta}{\varepsilon}|x|^{\frac{1+2\alpha}{\varepsilon}}}}, \text{ thus } f_\varepsilon^m(x, t) = \psi_\varepsilon(x, t) \times (\phi_{\nu, \varepsilon}(x) + \varepsilon).$$

We bound $\varepsilon \partial_t \psi_\varepsilon$ from above:

$$\begin{aligned} \varepsilon \partial_t \psi_\varepsilon(x, t) &= \varepsilon \frac{C_m \frac{(|\lambda_\nu| - \varepsilon^2)}{\varepsilon} e^{-\frac{t}{\varepsilon}(|\lambda_\nu| - \varepsilon^2) - \frac{\delta}{\varepsilon}|x|^{\frac{1+2\alpha}{\varepsilon}}} e^{-\frac{\delta}{\varepsilon}}}{(1 + e^{-\frac{t}{\varepsilon}(|\lambda_\nu| - \varepsilon^2) - \frac{\delta}{\varepsilon}|x|^{\frac{1+2\alpha}{\varepsilon}}})^2} \\ &= \psi_\varepsilon(x, t) \left[(|\lambda_\nu| - \varepsilon^2) \frac{e^{-\frac{t}{\varepsilon}(|\lambda_\nu| - \varepsilon^2) - \frac{\delta}{\varepsilon}|x|^{\frac{1+2\alpha}{\varepsilon}}}}{1 + e^{-\frac{t}{\varepsilon}(|\lambda_\nu| - \varepsilon^2) - \frac{\delta}{\varepsilon}|x|^{\frac{1+2\alpha}{\varepsilon}}}} \right] \\ &\leq \psi_\varepsilon(x, t) [|\lambda_\nu| - \varepsilon^2 - \psi_\varepsilon(x, t) (\phi_{\nu, \varepsilon}(x) + \varepsilon)]. \end{aligned} \quad (35)$$

The last inequality holds because by denoting by $D = e^{-\frac{t}{\varepsilon}(|\lambda_\nu| - \varepsilon^2) - \frac{\delta}{\varepsilon}|x|^{\frac{1+2\alpha}{\varepsilon}}}$ and using the definition of C_m , we obtain for all $\varepsilon < \min(\sqrt{\frac{|\lambda_\nu|}{2}}, 1)$

$$\begin{aligned} |\lambda_\nu| - \varepsilon^2 - \psi_\varepsilon(\phi_{\nu, \varepsilon} + \varepsilon) - (|\lambda_\nu| - \varepsilon^2) \frac{D}{1 + D} &= \frac{|\lambda_\nu| - \varepsilon^2 - C_m(\phi_{\nu, \varepsilon} + \varepsilon) e^{-\frac{\delta}{\varepsilon}}}{1 + D} \\ &\geq \frac{|\lambda_\nu| - \varepsilon^2 - \frac{|\lambda_\nu|}{2}}{1 + D} \\ &\geq 0. \end{aligned}$$

Next, we compute $(-\Delta)_\varepsilon^\alpha f_\varepsilon^m(x, t)$

$$(-\Delta)_\varepsilon^\alpha f_\varepsilon^m(x, t) = (\phi_{\nu, \varepsilon} + \varepsilon)(x) (-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t) + \psi_\varepsilon(x, t) (-\Delta)_\varepsilon^\alpha \phi_{\nu, \varepsilon}(x) - \tilde{K}_\varepsilon(\psi, (\phi_\nu + \varepsilon))(x, t).$$

Combining (35) and the above equality we find:

$$\begin{aligned} &\varepsilon \partial_t f_\varepsilon^m(x, t) + (-\Delta)_\varepsilon^\alpha f_\varepsilon^m(x, t) - f_\varepsilon^m(x, t) (1 - f_\varepsilon^m(x, t)) \\ &\leq f_\varepsilon^m(x, t) (|\lambda_\nu| - \varepsilon^2 - f_\varepsilon^m(x, t)) + (\phi_{\nu, \varepsilon}(x) + \varepsilon) (-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t) + \psi_\varepsilon(x, t) (-\Delta)_\varepsilon^\alpha \phi_{\nu, \varepsilon}(x) \\ &\quad - \tilde{K}_\varepsilon(\psi, \phi_\nu + \varepsilon)(x, t) - f_\varepsilon^m(x, t) (1 - f_\varepsilon^m(x, t)) \\ &= f_\varepsilon^m(x, t) (|\lambda_\nu| - \varepsilon^2) + (\phi_{\nu, \varepsilon}(x) + \varepsilon) (-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t) \\ &\quad + (\lambda_\nu + 1) \psi_\varepsilon(x, t) \phi_{\nu, \varepsilon}(x) - f_\varepsilon^m(x, t) - \tilde{K}_\varepsilon(\psi, \phi_\nu + \varepsilon)(x, t) \\ &= f_\varepsilon^m(x, t) (|\lambda_\nu| - \varepsilon^2) + (\phi_{\nu, \varepsilon}(x) + \varepsilon) (-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t) \\ &\quad + (\lambda_\nu + 1) \psi_\varepsilon(x, t) (\phi_{\nu, \varepsilon}(x) + \varepsilon)(x, t) - \varepsilon (\lambda_\nu + 1) \psi_\varepsilon(x, t) - f_\varepsilon^m(x, t) - \tilde{K}_\varepsilon(\psi, \phi_\nu + \varepsilon)(x, t) \\ &= -\varepsilon^2 f_\varepsilon^m(x, t) + (\phi_{\nu, \varepsilon}(x) + \varepsilon) (-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t) - \varepsilon (\lambda_\nu + 1) \psi_\varepsilon(x, t) - \tilde{K}_\varepsilon(\psi, \phi_\nu + \varepsilon)(x, t). \end{aligned} \quad (36)$$

Thanks to Lemma 4, we obtain

$$\begin{aligned} |(-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t)| &= |C_m e^{-\frac{\delta}{\varepsilon}} (-\Delta)_\varepsilon^\alpha (g_\varepsilon(e^{\frac{-[t(|\lambda_\nu| - \varepsilon^2) + \delta]}{1+2\alpha}})) (x)| \\ &\leq |C_m e^{-\frac{2\alpha[t(|\lambda_\nu| - \varepsilon^2) + \delta]}{\varepsilon(1+2\alpha)} - \frac{\delta}{\varepsilon}} (g_\varepsilon)(e^{\frac{-[t(|\lambda_\nu| - \varepsilon^2) + \delta]}{1+2\alpha}} x)|. \end{aligned}$$

We deduce that there exists $\varepsilon_2 > 0$ such that for all $\varepsilon < \varepsilon_2$:

$$|(-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t)| \leq \frac{\varepsilon^2}{3} \psi_\varepsilon(x, t). \quad (37)$$

Since $(\phi_\nu + \varepsilon)$ is periodic, positive and C^α according to [22] (Proposition 1.1), we conclude from Lemma 4 that there exists $\gamma \in]0, \alpha[$ and a constant C such that

$$|\tilde{K}_\varepsilon(\psi, \phi_\nu + \mu)(x, t)| \leq C e^{-\frac{[t(|\lambda_\nu| - \varepsilon^2) + \delta](2\alpha - \gamma)}{\varepsilon(1+2\alpha)}} \psi_\varepsilon(x, t).$$

We deduce the existence of $\varepsilon_3 > 0$ such that for all $\varepsilon < \varepsilon_3$, we have

$$|\tilde{K}_\varepsilon(\psi, \phi_\nu + \mu)(x, t)| \leq \frac{\varepsilon^3}{3} \psi_\varepsilon(x, t). \quad (38)$$

Noticing that $\lambda_\nu + 1 > 0$, inserting (37) and (38) into (36), we conclude that for all $\varepsilon < \varepsilon_\nu := \min(\varepsilon_2, \varepsilon_3, \sqrt{\frac{|\lambda_\nu|}{2}}, 1)$ and $(x, t) \in \Omega_\nu^\varepsilon \times [0, +\infty[$ we have:

$$\begin{aligned} &\varepsilon \partial_t f_\varepsilon^m(x, t) + (-\Delta)_\varepsilon^\alpha f_\varepsilon^m(x, t) - f_\varepsilon^m(x, t) + f_\varepsilon^m(x, t)^2 \\ &\leq -\varepsilon^2 f_\varepsilon^m(x, t) + (\phi_{\nu, \varepsilon} + \mu)(x) (-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t) - \varepsilon(\lambda_\nu + 1) \psi_\varepsilon(x, t) - \tilde{K}_\varepsilon(\psi, \phi_\nu + \varepsilon)(x, t) \\ &\leq -\varepsilon^2 f_\varepsilon^m(x, t) + \frac{\varepsilon^2}{3} f_\varepsilon(x, t) + \frac{\varepsilon^2}{3} f_\varepsilon(x, t) \\ &\leq -\frac{\varepsilon^2}{3} f_\varepsilon^m(x, t) \\ &\leq 0. \end{aligned}$$

Therefore, f_ε^m is a sub-solution of (1_ε) .

The proof of (2) follows the same arguments as the proof of (1).

For the proof of (3), we only have to check that the initial data are ordered in the right way. According to $(H2')$ and the definition of C_m , we have that

$$\forall x \in \Omega_\nu^\varepsilon, f_\varepsilon^m(x, 0) \leq \frac{C_m(\max \phi_\nu + \varepsilon)}{1 + |x|^{\frac{1+2\alpha}{\varepsilon}}} \leq \frac{c_m \nu^\alpha + \varepsilon}{1 + |x|^{\frac{1+2\alpha}{\varepsilon}}} \leq \frac{c_m \delta(x)^\alpha}{1 + |x|^{\frac{1+2\alpha}{\varepsilon}}} + \varepsilon \leq n_\varepsilon(x, 0) + \varepsilon.$$

Furthermore, for all $(x, t) \in (\Omega_\nu^\varepsilon)^c \times [0, +\infty[$, we know that $\varepsilon = f_\varepsilon^m(x, t) \leq n_\varepsilon(x, t) + \varepsilon$. Thus we conclude from the maximum principle that for all $(x, t) \in \mathbb{R} \times [0, +\infty[$, we have

$$f_\varepsilon^m(x, t) \leq n_\varepsilon(x, t) + \varepsilon. \quad (39)$$

The other inequality can be obtained following similar arguments. \square

Before dealing with the proof of Theorem 3, we need intermediate results about λ_0 and n_+ . We approach λ_0 by λ_ν where λ_ν is the principal eigenvalue of the Dirichlet Laplacian $(-\Delta)^\alpha - Id$ in Ω_ν . The second proposition approaches the stationary state n_+ by the stationary state of $(-\Delta)^\alpha n_{\nu,+} = n_{\nu,+} - n_{\nu,+}^2$ in Ω_ν and 0 outside Ω_ν .

Proposition 3. *The map $(\nu \in]-\frac{A-A_0}{2}, \frac{A-A_0}{2}[\mapsto \lambda_\nu)$ is increasing and continuous.*

We let the proof of this proposition to the reader since there is no difficulty. The proof relies on the Rayleigh quotient for the monotony and on the uniqueness of the principal eigenvalue for the continuity.

Proposition 4. *For all $\nu \in [0, A - A_0[$, there exists a unique steady state $n_{\nu,+}$ to the following equation:*

$$\begin{cases} (-\Delta)^\alpha n_{\nu,+} = n_{\nu,+}(1 - n_{\nu,+}) & \text{for } x \in \Omega_\nu, \\ n_{\nu,+}(x) = 0 & \text{for } x \in \Omega_\nu^c. \end{cases} \quad (40)$$

Proof. The proof of the existence and uniqueness follows the proof of Theorem 1 since the hypothesis (H1) is verified. \square

Then, we continue by proving that the dependance of $n_{\nu,+}$ on the parameter ν is continuous. As Proposition 4, the proof is classical. Nevertheless, it uses Theorem 1 so we provide it.

Proposition 5. *The steady state $n_{\nu,+}$ converges uniformly to n_+ as $\nu \rightarrow 0$.*

Proof. The convergence outside Ω is trivial since $n_{\nu,+} = n_+ = 0$.

Let μ be a positive constant in $]0, \frac{A-A_0}{2}[$. We show the convergence inside Ω_μ . We first show that for all $0 < \nu_1 < \nu_2$ we have

$$n_{\nu_2,+} \leq n_{\nu_1,+} \leq n_+. \quad (41)$$

Actually, take $w_0 \in C_0^\infty(\Omega_{\nu_2}, \mathbb{R}^+) \cap C_c(\mathbb{R}, \mathbb{R}^+)$. Then we consider the three systems:

$$\begin{cases} \partial_t w_1 + (-\Delta)^\alpha w_1 = w_1(1 - w_1) & \text{for } (x, t) \in \Omega_{\nu_1} \times]0, +\infty[, \\ w_1(x, t) = 0 & \text{for } (x, t) \in \Omega_{\nu_1}^c \times [0, +\infty[, \\ w_1(x, 0) = w_0(x), \end{cases}$$

$$\begin{cases} \partial_t w_2 + (-\Delta)^\alpha w_2 = w_2(1 - w_2) & \text{for } (x, t) \in \Omega_{\nu_2} \times]0, +\infty[, \\ w_2(x, t) = 0 & \text{for } (x, t) \in \Omega_{\nu_2}^c \times [0, +\infty[, \\ w_2(x, 0) = w_0(x), \end{cases}$$

and

$$\begin{cases} \partial_t w + (-\Delta)^\alpha w = w(1 - w) & \text{for } (x, t) \in \Omega_\times \times]0, +\infty[, \\ w(x, t) = 0 & \text{for } (x, t) \in \Omega^c \times [0, +\infty[, \\ w(x, 0) = w_0(x). \end{cases}$$

According to the maximum principle, we have that for all $(x, t) \in \mathbb{R} \times [0, +\infty[$

$$w_2(x, t) \leq w_1(x, t) \leq w(x, t).$$

If we take the limit in time in the above inequalities, we find (41). Thus, the map $(\nu \mapsto n_{\nu,+})$ is decreasing. Since $n_{\nu,+}$ is bounded, we deduce that $n_{\nu,+} \xrightarrow[\nu \rightarrow 0]{} n_{0,+}$ pointwise. Moreover, $n_{0,+}$ is a steady solution to the equation (1). By uniqueness of n_+ (induced by Theorem 1), we deduce that

$$n_{0,+} = n_+.$$

Since the sequence $(n_{\nu,+})_{\nu \geq 0}$ is decreasing and the limit n_+ is continuous, we deduce from the Dini Theorem that the convergence is uniform in $[\mu, A - \mu]$. Moreover, since $n_{\nu,+}$ and n_+ are periodic, it is enough to prove that the convergence is uniform in the set $[\mu, A - \mu]$ to conclude to the uniform convergence in the whole domain Ω_μ . Thus, $n_{\nu,+}$ converges uniformly to n_+ in Ω_μ .

We deduce that for all $\tau > 0$ and for all $\mu > 0$, there exists $\nu_0 > 0$ such that for all $\nu \in]0, \nu_0[$,

$$|n_+(x) - n_{\nu,+}(x)| \leq \tau, \quad \forall x \in \Omega_\mu \cup \Omega^c.$$

It remains to prove the same relation in $\Omega \setminus \Omega_\mu$. Thanks to Proposition 1, we know that there exists $C > 0$ such that for all $x \in \mathbb{R}$

$$n_+(x) \leq C\delta(x)^\alpha.$$

We deduce that for all $\tau > 0$, there exists $\mu_0 > 0$ such that for all $\mu \in]0, \mu_0[$, we have

$$n_+(x) \leq \frac{\tau}{2}, \quad \forall x \in \Omega \text{ such that } \delta(x) < \mu.$$

Thanks to (41), we deduce that for all $\mu \in]0, \mu_0[$ and for all $\nu > 0$, we have

$$|n_+(x) - n_{\nu,+}(x)| \leq 2n_+(x) \leq \tau, \quad \forall x \in \Omega \setminus \Omega_\mu.$$

We conclude that $n_{\nu,+}$ converges uniformly to n_+ as $\nu \rightarrow 0$. □

5 The proof of Theorem 3

We first provide the proof of the convergence of n_ε to 0 in \mathcal{B}^c . Next, we prove of the convergence of n_ε to $n_{+,\varepsilon}$ in $(\Omega^\varepsilon \times \mathbb{R}^+) \cap \mathcal{B}$. This proof is the difficult part of this section.

Proof of the convergence of n_ε to 0 in \mathcal{B}^c . Let K be a compact set of \mathcal{B}^c . According to Theorem 4, we know that for all $(x, t) \in K$,

$$0 \leq n_\varepsilon(x, t) \leq f_\varepsilon^M(x, t) \leq \frac{C \max \phi_{-\nu}}{1 + e^{\frac{-t(|\lambda_{-\nu}| + \varepsilon^2) - \delta + (1+2\alpha) \log(|x|)}{\varepsilon}}},$$

where δ can be taken as small as we want. For δ small enough we have that for all $(x, t) \in K$

$$2\delta \leq (1 + 2\alpha) \log(|x|) - |\lambda_0|t.$$

Thanks to Proposition 3, we deduce the existence of $\nu_1 > 0$ and ε_1 such that for all $(x, t) \in K$ we have for all $\nu < \nu_1$ and $\varepsilon < \varepsilon_1$,

$$\delta \leq (1 + 2\alpha) \log(|x|) - t|\lambda_0| + t(|\lambda_0| - |\lambda_{-\nu}| - \varepsilon^2) - \delta = -t(|\lambda_{-\nu}| + \varepsilon^2) - \delta + (1 + 2\alpha) \log(|x|).$$

The conclusion follows. □

Proof of the convergence of n_ε to $n_{+, \varepsilon}$ in $(\Omega^\varepsilon \times \mathbb{R}^+) \cap \mathcal{B}$. It is sufficient to prove the uniform convergence in the set $(\Omega_{2\tilde{\delta}}^\varepsilon \times B(\underline{t}, r)) \cap \mathcal{B}^{2\tilde{\delta}}$ with $B(\underline{t}, r)$ a neighborhood of some positive time \underline{t} and for all positive constants $\tilde{\delta}$ arbitrarily small. More precisely, let \underline{t} be a positive time and $\tilde{\delta}$ be a positive constant. We define

$$V(\underline{t}) = B(\underline{t}, r') \quad \text{with } r' := \min \left(\frac{\underline{t}}{2}, \frac{\tilde{\delta}}{8|\lambda_0|}, \sqrt{\frac{\tilde{\delta}}{16}} \right) \quad (42)$$

as a neighborhood of \underline{t} . The aim is to show that for all $\mu > 0$, there exists $\varepsilon_0 > 0$ (depending on $\tilde{\delta}$ and μ) such that we have for all $\varepsilon < \varepsilon_0$ and all $(x, t) \in (\Omega_{2\tilde{\delta}}^\varepsilon \times B(\underline{t}, \frac{r'}{2})) \cap \mathcal{B}^{2\tilde{\delta}}$

$$\left| \frac{n_\varepsilon(x, t)}{n_{+, \varepsilon}(x)} - 1 \right| \leq \mu. \quad (43)$$

The proof is split into two parts :

Part A) For all $\varepsilon > 0$ small enough, we have for all $(x, t) \in (\Omega_{2\tilde{\delta}}^\varepsilon \times B(\underline{t}, \frac{r'}{2})) \cap \mathcal{B}^{2\tilde{\delta}}$,

$$\frac{n_\varepsilon(x, t)}{n_{+, \varepsilon}(x)} \leq 1 + \mu.$$

Part B) For all $\varepsilon > 0$ small enough we have for all $(x, t) \in (\Omega_{2\tilde{\delta}}^\varepsilon \times B(\underline{t}, \frac{r'}{2})) \cap \mathcal{B}^{2\tilde{\delta}}$,

$$1 - \mu \leq \frac{n_\varepsilon(x, t)}{n_{+, \varepsilon}(x)},$$

Proof of Part A. We start with the easiest part. A super-solution to (1) is obtained by taking sup n_0 as an initial datum. The conclusion follows.

The non trivial part is the lower bound. The main difficulty wich comes from the boundary will be encountered in this second part. We will overcome it by studying a modification of $\frac{n_\varepsilon(x, t)}{n_{+, \varepsilon}(x)}$ which localizes the minimum of $\frac{n_\varepsilon(x, t)}{n_{+, \varepsilon}(x)}$. The modification is defined as follows:

$$\Psi_\varepsilon(x, t) := \begin{cases} \frac{(n_\varepsilon(x, t) + \varepsilon)(1 + e^{\frac{-|\lambda_0|t_0 + \tilde{\delta}}{\varepsilon}} |x|^{\frac{1+2\alpha}{\varepsilon}}) e^{\frac{(t-t_0)^2}{\varepsilon}}}{n_{\tilde{\nu}, +, \varepsilon}(x)} & \text{if } x \in \Omega_{\tilde{\nu}}^\varepsilon \\ +\infty & \text{otherwise,} \end{cases} \quad (44)$$

where $\tilde{\nu}$ will be chosen later. This is very much in the spirit of the comparison techniques for viscosity solutions.

Proof of Part B. Let $t_0 \in V(\underline{t})$. In order to achieve this part, we split the study into 5 main steps:

(i) We show that for $\tilde{\nu} \in [0, \tilde{\delta}]$, to be specified later, the minimum of Ψ_ε is attained at some point $(x_\varepsilon, t_\varepsilon) \in (\Omega_{\tilde{\nu}} \times V(\underline{t})) \cap \mathcal{B}$.

(ii) We show that for $\tilde{\nu}$ small enough $t_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} t_0$.

(iii) We use this minimum to prove the existence of $\varepsilon(\tilde{\delta}, \mu, t_0)$ such that for all $\varepsilon \leq \varepsilon(\tilde{\delta}, \mu, t_0)$

$$1 - \frac{\mu}{8} \leq \frac{n_\varepsilon(x_\varepsilon, t_\varepsilon)}{n_{+,\varepsilon}(x_\varepsilon)}. \quad (45)$$

(iv) We bring back (45) to (x_0, t_0) where x_0 is any point of $\Omega_{2\tilde{\delta}} \cap \left\{ x \in \mathbb{R} \mid |x| \leq e^{\frac{|\lambda_0|t_0 - 2\tilde{\delta}}{1+2\alpha}} \right\}$.

(v) We show the existence of $\varepsilon(\tilde{\delta}, \mu)$ such that we have

$$\forall \varepsilon \leq \varepsilon(\tilde{\delta}, \mu) \quad \text{and} \quad \forall (x, t) \in \left(\Omega_{2\tilde{\delta}}^\varepsilon \times B(\underline{t}, \frac{r'}{2}) \right) \cap \mathcal{B}^{2\tilde{\delta}}, \quad 1 - \mu \leq \frac{n_\varepsilon(x, t)}{n_{+,\varepsilon}(x)}.$$

Step (i). We focus on the localisation of the minima of Ψ_ε defined by (44).

We recall that according to Theorem 4, for all $\tilde{\nu} \in]0, \frac{A-A_0}{2}[$ there exists $\varepsilon_{\tilde{\nu}} > 0$ such that for all $\varepsilon \leq \varepsilon_{\tilde{\nu}}$ we have

$$(\phi_{\frac{\tilde{\nu}}{2}, \varepsilon}(x) + \varepsilon) \times \frac{C_m e^{-\frac{\delta}{\varepsilon}}}{1 + e^{-\frac{t}{\varepsilon}(|\lambda_{\frac{\tilde{\nu}}{2}}| - \varepsilon^2) - \frac{\delta}{\varepsilon}} |x|^{\frac{1+2\alpha}{\varepsilon}}} \leq n_\varepsilon(x, t) + \varepsilon. \quad (46)$$

Remark that the constant δ in (46) can be taken as small as we want, thus we will take it such that

$$\delta \leq \frac{\tilde{\delta}}{4}. \quad (47)$$

We want to prove that the minimum of Ψ_ε is taken in $(\Omega_{\tilde{\nu}}^\varepsilon \times V(\underline{t})) \cap \mathcal{B}$. More precisely, the goal now is to prove that in $(\Omega_{\tilde{\nu}}^\varepsilon \times V(\underline{t})) \cap \mathcal{B}^c$, $\Psi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} +\infty$ whereas $\Psi_\varepsilon(x, t_0)$ is bounded for all

$$x \in \Omega_{2\tilde{\nu}}^\varepsilon \cap \left\{ x \in \mathbb{R} \mid |x| < e^{\frac{|\lambda_0|t_0 - 2\tilde{\delta}}{1+2\alpha}} \right\}.$$

Before dealing with the details, we provide one last inequality given by (46). For all $(x, t) \in (\Omega_{\tilde{\nu}}^\varepsilon \times V(\underline{t})) \cap \mathcal{B}^c$ we have :

$$\begin{aligned} \Psi_\varepsilon(x, t) &\geq (\phi_{\frac{\tilde{\nu}}{2}, \varepsilon}(x) + \varepsilon) \times \frac{C_m e^{-\frac{\delta}{\varepsilon}}}{1 + e^{-\frac{t(|\lambda_{\frac{\tilde{\nu}}{2}}| - \varepsilon^2) + \delta}{\varepsilon}} |x|^{\frac{1+2\alpha}{\varepsilon}}} \times \frac{(1 + e^{\frac{-|\lambda_0|t_0 + \tilde{\delta}}{\varepsilon}} |x|^{\frac{1+2\alpha}{\varepsilon}}) e^{\frac{(t-t_0)^2}{\varepsilon}}}{n_{\tilde{\nu}, +, \varepsilon}(x)} \\ &\geq C_m \times \frac{\min_{\Omega_{\tilde{\nu}}} \phi_{\frac{\tilde{\nu}}{2}}}{\max n_{\tilde{\nu}, +}} \times \frac{e^{-\frac{\delta}{\varepsilon}} (1 + e^{\frac{-|\lambda_0|t_0 + \tilde{\delta}}{\varepsilon}} |x|^{\frac{1+2\alpha}{\varepsilon}}) e^{\frac{(t-t_0)^2}{\varepsilon}}}{1 + e^{-\frac{t(|\lambda_{\frac{\tilde{\nu}}{2}}| - \varepsilon^2) + \delta}{\varepsilon}} |x|^{\frac{1+2\alpha}{\varepsilon}}} \\ &\geq C \times \frac{e^{\frac{-\delta - |\lambda_0|t_0 + \tilde{\delta} + (1+2\alpha) \log |x| + (t-t_0)^2}{\varepsilon}}}{1 + e^{-\frac{t(|\lambda_{\frac{\tilde{\nu}}{2}}| - \varepsilon^2) + \delta}{\varepsilon}} |x|^{\frac{1+2\alpha}{\varepsilon}}}. \end{aligned} \quad (48)$$

Proof that Ψ_ε diverges in $(\Omega_{\tilde{\nu}}^\varepsilon \times V(\underline{t})) \cap \mathcal{B}^c$. Let $(x, t) \in (\Omega_{\tilde{\nu}}^\varepsilon \times V(\underline{t})) \cap \mathcal{B}^c$. We have to consider two cases :

Case 1: $-t(|\lambda_{\frac{\tilde{\nu}}{2}}| - \varepsilon^2) - \delta + (1+2\alpha) \log |x| \leq 0$.

In this case,

$$1 + e^{-\frac{t}{\varepsilon}(|\lambda_{\frac{\tilde{\nu}}{2}}| - \varepsilon^2) - \frac{\delta}{\varepsilon}} |x|^{\frac{1+2\alpha}{\varepsilon}} \leq 2.$$

According to (42), (47) and since $(x, t) \in \mathcal{B}^c$, we have

$$\frac{\tilde{\delta}}{4} \leq \tilde{\delta} - \delta + |\lambda_0|(t - t_0) - |\lambda_0|t + (1 + 2\alpha) \log |x| + (t - t_0)^2 = -\delta - |\lambda_0|t_0 + \tilde{\delta} + (1 + 2\alpha) \log |x| + (t - t_0)^2.$$

Inserting these two above inequalities in (48) gives

$$\Psi_\varepsilon(x, t) \geq C \times \frac{e^{\frac{-\delta + \tilde{\delta} - |\lambda_0|t_0 + (1 + 2\alpha) \log |x| + (t - t_0)^2}{\varepsilon}}}{2} \geq \frac{C}{2} e^{\frac{\tilde{\delta}}{4\varepsilon}} \xrightarrow{\varepsilon \rightarrow 0} +\infty. \quad (49)$$

Case 2: $-t(|\lambda_{\frac{\tilde{\nu}}{2}}| - \varepsilon^2) - \delta + (1 + 2\alpha) \log |x| > 0$.

In this case note that

$$1 + e^{\frac{-t(|\lambda_{\frac{\tilde{\nu}}{2}}| - \varepsilon^2) - \delta + (1 + 2\alpha) \log |x|}{\varepsilon}} \leq 2e^{\frac{-t(|\lambda_{\frac{\tilde{\nu}}{2}}| - \varepsilon^2) - \delta + (1 + 2\alpha) \log |x|}{\varepsilon}}.$$

Thus we deduce that in (48) we have

$$\Psi_\varepsilon(x, t) \geq C \times \frac{e^{\frac{-\delta - |\lambda_0|t_0 + \tilde{\delta} + (1 + 2\alpha) \log |x| + (t - t_0)^2}{\varepsilon}}}{2e^{-\frac{t(|\lambda_{\frac{\tilde{\nu}}{2}}| - \varepsilon^2) - \delta + (1 + 2\alpha) \log |x|}{\varepsilon}}} = \frac{C}{2} \times e^{\frac{-|\lambda_0|t_0 + \tilde{\delta} + (t - t_0)^2 + t(|\lambda_{\frac{\tilde{\nu}}{2}}| - \varepsilon^2)}{\varepsilon}}.$$

Thanks to Proposition 3 and (42), there exists two positive constants $\tilde{\nu}_1$ and ε_1 such that for all $\tilde{\nu} < \tilde{\nu}_1$ and $\varepsilon < \varepsilon_1$, we have

$$|t(|\lambda_{\frac{\tilde{\nu}}{2}}| - \varepsilon^2) - |\lambda_0|t_0 + (t - t_0)^2| \leq \frac{\tilde{\delta}}{2}, \quad \forall t \in V(\underline{t}). \quad (50)$$

We deduce that

$$\Psi_\varepsilon(x, t) \geq C e^{\frac{\tilde{\delta}}{2\varepsilon}} \xrightarrow{\varepsilon \rightarrow 0} +\infty. \quad (51)$$

Proof that $\Psi_\varepsilon(x, t_0)$ is bounded for $x \in \Omega_{2\tilde{\delta}}^\varepsilon \cap \left\{ x \in \mathbb{R} \mid |x| < e^{\frac{|\lambda_0|t_0 - 2\tilde{\delta}}{1 + 2\alpha}} \right\}$. Take $x \in \Omega_{2\tilde{\delta}}^\varepsilon \cap \left\{ x \in \mathbb{R} \mid |x| < e^{\frac{|\lambda_0|t_0 - 2\tilde{\delta}}{1 + 2\alpha}} \right\}$, it follows that

$$-|\lambda_0|t_0 + (1 + 2\alpha) \log(|x|) + \tilde{\delta} \leq -\tilde{\delta}.$$

According the maximum principle, we have for $\varepsilon < 1$

$$n_\varepsilon(x) + \varepsilon \leq 2.$$

We find that

$$\Psi_\varepsilon(x, t_0) \leq \frac{2(1 + e^{-\frac{\tilde{\delta}}{\varepsilon}})}{\min_{\Omega_{2\tilde{\nu}}} n_{\tilde{\nu},+}} \leq \frac{4}{\min_{\Omega_{2\tilde{\nu}}} n_{\tilde{\nu},+}}.$$

We deduce that for all $\tilde{\nu} < \min(\frac{A-A_0}{2}, \tilde{\nu}_1, \tilde{\delta})$ and all $\varepsilon < \min(\varepsilon_1, \varepsilon_{\tilde{\nu}}, 1)$, there exists $(x_\varepsilon, t_\varepsilon) \in (\Omega_{\tilde{\nu}}^\varepsilon \times V(\underline{t})) \cap \mathcal{B}$ such that $\Psi_\varepsilon(x_\varepsilon, t_\varepsilon) = \min_{(x,t) \in \mathbb{R} \times V(\underline{t})} \Psi_\varepsilon(x, t)$. Remark that this minimum is global in space and local in time.

Step (ii). We show by contradiction that

$$\exists \tilde{\nu}_2 > 0, \text{ such that } \forall \tilde{\nu} \leq \tilde{\nu}_2 \text{ we have } t_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} t_0.$$

If it does not hold, there exists a constant $\tau > 0$ such that $|t_{\varepsilon_k} - t_0| > \tau$ with $\varepsilon_k \xrightarrow{k \rightarrow +\infty} 0$. According to Proposition 3, there exists $\tilde{\nu}_2 > 0$ and $\varepsilon_2 > 0$ such that for all $\tilde{\nu} < \tilde{\nu}_2$ and for all $\varepsilon \leq \varepsilon_2$ we have

$$t||\lambda_0| + \varepsilon^2 - |\lambda_{\frac{\tilde{\nu}}{2}}|| \leq \frac{\tau^2}{2}, \quad \forall t \in V(\underline{t}). \quad (52)$$

We are going to show that $\Psi_{\varepsilon_k}(x_{\varepsilon_k}, t_{\varepsilon_k}) \xrightarrow{k \rightarrow +\infty} +\infty$ which is a contradiction. Thanks to (46), we obtain

$$\begin{aligned} \Psi_{\varepsilon_k}(x_{\varepsilon_k}, t_{\varepsilon_k}) &\geq C_m \times \frac{\min_{\Omega_{\tilde{\nu}}} \phi_{\frac{\tilde{\nu}}{2}}}{\max n_{\tilde{\nu},+}} \times \frac{e^{-\frac{\delta}{\varepsilon_k}} (1 + e^{\frac{-|\lambda_0|t_0 + \tilde{\delta}}{\varepsilon_k}} |x_{\varepsilon_k}|^{\frac{1+2\alpha}{\varepsilon_k}}) e^{\frac{(t_{\varepsilon_k} - t_0)^2}{\varepsilon_k}}}{1 + e^{-\frac{t_{\varepsilon_k}}{\varepsilon_k} (|\lambda_{\frac{\tilde{\nu}}{2}}| - \varepsilon_k^2) - \frac{\delta}{\varepsilon_k}} |x_{\varepsilon_k}|^{\frac{1+2\alpha}{\varepsilon_k}}} \\ &\geq C_m \times \frac{\min_{\Omega_{\tilde{\nu}}} \phi_{\frac{\tilde{\nu}}{2}}}{\max n_{\tilde{\nu},+}} \times \frac{e^{\frac{-|\lambda_0|t_0 + (1+2\alpha) \log |x_{\varepsilon_k}| + \tilde{\delta} - \delta + (t_{\varepsilon_k} - t_0)^2}{\varepsilon_k}}}{1 + e^{-\frac{t_{\varepsilon_k}}{\varepsilon_k} (|\lambda_{\frac{\tilde{\nu}}{2}}| - \varepsilon_k^2) - \frac{\delta}{\varepsilon_k}} |x_{\varepsilon_k}|^{\frac{1+2\alpha}{\varepsilon_k}}}. \end{aligned}$$

Recalling that $(x_{\varepsilon_k}, t_{\varepsilon_k}) \in \mathcal{B}$, we deduce that $(1 + 2\alpha) \log |x_{\varepsilon_k}| - |\lambda_0|t_{\varepsilon_k} < 0$ and then

$$\Psi_{\varepsilon_k}(x_{\varepsilon_k}, t_{\varepsilon_k}) \geq C \times \frac{e^{\frac{\tilde{\delta} - \delta + |\lambda_0|(t_{\varepsilon_k} - t_0) + (t_{\varepsilon_k} - t_0)^2}{\varepsilon_k}}}{1 + e^{-\frac{t_{\varepsilon_k} (|\lambda_{\frac{\tilde{\nu}}{2}}| - \varepsilon_k^2) + \delta}{\varepsilon_k}} |x_{\varepsilon_k}|^{\frac{1+2\alpha}{\varepsilon_k}}}.$$

By definition of τ and thanks to (42), we have

$$\tau^2 < \tilde{\delta} \left(1 - \frac{1}{4} - \frac{1}{4}\right) + \tau^2 \leq \tilde{\delta} - \delta + |\lambda_0|(t_{\varepsilon_k} - t_0) + (t_{\varepsilon_k} - t_0)^2.$$

We deduce that

$$\Psi_{\varepsilon_k}(x_{\varepsilon_k}, t_{\varepsilon_k}) \geq C \times \frac{e^{\frac{\tau^2}{\varepsilon_k}}}{1 + e^{\frac{-t_{\varepsilon_k} (|\lambda_{\frac{\tilde{\nu}}{2}}| - \varepsilon_k^2) + (1+2\alpha) \log |x_{\varepsilon_k}| - \delta}{\varepsilon_k}}}.$$

Moreover, since $(x_{\varepsilon_k}, t_{\varepsilon_k}) \in \mathcal{B}$, we have

$$\begin{aligned} -t_{\varepsilon_k} (|\lambda_{\frac{\tilde{\nu}}{2}}| - \varepsilon_k^2) + (1 + 2\alpha) \log |x_{\varepsilon_k}| - \delta &= -t_{\varepsilon_k} (|\lambda_{\frac{\tilde{\nu}}{2}}| - |\lambda_0| - \varepsilon_k^2) - |\lambda_0|t_{\varepsilon_k} + (1 + 2\alpha) \log |x_{\varepsilon_k}| - \delta \\ &\leq |t_{\varepsilon_k} (|\lambda_{\frac{\tilde{\nu}}{2}}| - |\lambda_0| - \varepsilon_k^2)|. \end{aligned}$$

Since $t_{\varepsilon_k} \in V(\underline{t})$ and thanks to (52), we obtain that

$$-t_{\varepsilon_k} (|\lambda_{\frac{\tilde{\nu}}{2}}| - \varepsilon_k^2) + (1 + 2\alpha) \log |x_{\varepsilon_k}| - \delta \leq |t_{\varepsilon_k} (|\lambda_{\frac{\tilde{\nu}}{2}}| - |\lambda_0| - \varepsilon_k^2)| \leq \frac{\tau^2}{2}.$$

We deduce the following contradiction

$$\Psi_{\varepsilon_k}(x_{\varepsilon_k}, t_{\varepsilon_k}) \geq \frac{C}{2} \times e^{\frac{\tau^2}{2\varepsilon_k}} \xrightarrow{k \rightarrow +\infty} +\infty.$$

Step (iii). We go back to (1_ε) . We want to demonstrate that

$$1 - \frac{\mu}{8} \leq \frac{n_\varepsilon(x_\varepsilon, t_\varepsilon)}{n_{+, \varepsilon}(x_\varepsilon)}. \quad (53)$$

In order to achieve it, we first fix $\tilde{\nu}$ as follows. According to Proposition 3, there exists $\tilde{\nu}_3$ such that for all $\tilde{\nu} < \tilde{\nu}_3$,

$$\forall x \in \Omega_{\tilde{\nu}}, \quad 1 - \frac{\mu}{16} \leq \frac{n_{\tilde{\nu},+}(x)}{n_+(x)}. \quad (54)$$

For the following, we fix $\tilde{\nu} = \min(\tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3, \tilde{\delta})$.

Next, we evaluate (1_ε) at $(x_\varepsilon, t_\varepsilon)$ and we divide by $(n_\varepsilon(x_\varepsilon, t_\varepsilon) + \varepsilon)$ to find :

$$\begin{aligned} (n_\varepsilon \times n_\varepsilon (n_\varepsilon + \varepsilon)^{-1})(x_\varepsilon, t_\varepsilon) &= -\varepsilon (\partial_t(n_\varepsilon + \varepsilon)(n_\varepsilon + \varepsilon)^{-1})(x_\varepsilon, t_\varepsilon) \\ &\quad - ((-\Delta)_\varepsilon^\alpha((n_\varepsilon + \varepsilon)(n_\varepsilon + \varepsilon)^{-1} - n_\varepsilon(n_\varepsilon + \varepsilon)^{-1}))(x_\varepsilon, t_\varepsilon). \end{aligned} \quad (55)$$

The term $(n_\varepsilon \times n_\varepsilon (n_\varepsilon + \varepsilon)^{-1})(x_\varepsilon, t_\varepsilon)$ has two terms that we are going to control separately.

Control of $\varepsilon (\partial_t(n_\varepsilon + \varepsilon)(n_\varepsilon + \varepsilon)^{-1})(x_\varepsilon, t_\varepsilon)$. We prove that

$$-\frac{\mu \min_{\Omega_{\tilde{\nu}}} n_+}{32} \leq \varepsilon \partial_t(n_\varepsilon + \varepsilon)(n_\varepsilon + \varepsilon)^{-1}(x_\varepsilon, t_\varepsilon). \quad (56)$$

Since $(x_\varepsilon, t_\varepsilon)$ is a local minimum in time of Ψ_ε , we obtain

$$\begin{aligned} \varepsilon \partial_t \Psi_\varepsilon(x_\varepsilon, t_\varepsilon) &= 0 \\ \Rightarrow \varepsilon \partial_t \left(\frac{(n_\varepsilon(x_\varepsilon, t_\varepsilon) + \varepsilon)(1 + e^{-\frac{|\lambda_0|t_0 + \tilde{\delta}}{\varepsilon}} |x_\varepsilon|^{\frac{1+2\alpha}{\varepsilon}}) e^{\frac{(t_\varepsilon - t_0)^2}{\varepsilon}}}{n_{\tilde{\nu},+, \varepsilon}(x_\varepsilon)} \right) &= 0 \\ \Rightarrow \varepsilon (\partial_t(n_\varepsilon + \varepsilon)(n_\varepsilon + \varepsilon)^{-1})(x_\varepsilon, t_\varepsilon) &= -2(t_\varepsilon - t_0). \end{aligned}$$

Since $t_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} t_0$ (thanks to *Step (ii)*), we conclude to the existence of $\varepsilon_{t_0} > 0$ such that for all $\varepsilon < \varepsilon_{t_0}$, (56) holds true.

Control of $(-\Delta)_\varepsilon^\alpha(n_\varepsilon + \varepsilon)(n_\varepsilon + \varepsilon)^{-1}(x_\varepsilon, t_\varepsilon)$. We first introduce a new notation :

$$h_\varepsilon(x, t) = \frac{e^{-\frac{(t-t_0)^2}{\varepsilon}}}{1 + e^{-\frac{|\lambda_0|t_0 + \tilde{\delta}}{\varepsilon}} |x|^{\frac{1+2\alpha}{\varepsilon}}} = e^{-\frac{(t-t_0)^2}{\varepsilon}} \times g_\varepsilon(e^{-\frac{|\lambda_0|t_0 + \tilde{\delta}}{1+2\alpha}} x)$$

such that :

$$\Psi_\varepsilon(x, t) = \begin{cases} \frac{(n_\varepsilon(x, t) + \varepsilon)}{h_\varepsilon(x, t) \times n_{\tilde{\nu},+, \varepsilon}(x)} & \text{if } x \in \Omega_{\tilde{\nu}}^\varepsilon \\ +\infty & \text{otherwise.} \end{cases}$$

Then, since $\Psi_\varepsilon(x_\varepsilon, t_\varepsilon) = \min_{(x,t) \in \mathbb{R} \times V(\underline{t})} \Psi_\varepsilon(x, t)$, we deduce that for all $y \in \Omega_{\tilde{\nu}}^\varepsilon$

$$\Psi_\varepsilon(x_\varepsilon, t_\varepsilon) \leq \Psi_\varepsilon(y, t_\varepsilon) \Rightarrow \frac{n_{\tilde{\nu},+, \varepsilon}(y) h_\varepsilon(y, t_\varepsilon)}{n_{\tilde{\nu},+, \varepsilon}(x_\varepsilon) h_\varepsilon(x_\varepsilon, t_\varepsilon)} - 1 \leq \frac{n_\varepsilon(y, t_\varepsilon) + \varepsilon}{n_\varepsilon(x_\varepsilon, t_\varepsilon) + \varepsilon} - 1. \quad (57)$$

Moreover, for all $y \in \mathbb{R} \setminus \Omega_{\tilde{\nu}}^\varepsilon$,

$$-1 = \frac{n_{\tilde{\nu},+, \varepsilon}(y) h_\varepsilon(y, t_\varepsilon)}{n_{\tilde{\nu},+, \varepsilon}(x_\varepsilon) h_\varepsilon(x_\varepsilon, t_\varepsilon)} - 1 \leq \frac{n_\varepsilon(y, t_\varepsilon) + \varepsilon}{n_\varepsilon(x_\varepsilon, t_\varepsilon) + \varepsilon} - 1. \quad (58)$$

The combination of (57) and (58) gives :

$$-(-\Delta)_\varepsilon^\alpha (n_{\tilde{\nu},+,\varepsilon} h_\varepsilon) (n_{\tilde{\nu},+,\varepsilon} h_\varepsilon)^{-1} (x_\varepsilon, t_\varepsilon) \leq -(-\Delta)_\varepsilon^\alpha ((n_\varepsilon + \varepsilon)(n_\varepsilon + \varepsilon)^{-1}) (x_\varepsilon, t_\varepsilon). \quad (59)$$

Conclusion : Back to (1_ε). Inserting (56) and (59) in (55) and remarking that $n_\varepsilon \times n_\varepsilon (n_\varepsilon + \varepsilon)^{-1} \leq n_\varepsilon$, we find that

$$n_{\tilde{\nu},+,\varepsilon}(x_\varepsilon) - \frac{\mu \min n_+}{32} - ((-\Delta)_\varepsilon^\alpha (n_{\tilde{\nu},+,\varepsilon} h_\varepsilon) (n_{\tilde{\nu},+,\varepsilon} h_\varepsilon)^{-1} - n_\varepsilon (n_\varepsilon + \varepsilon)^{-1} + n_{\tilde{\nu},+,\varepsilon}) (x_\varepsilon, t_\varepsilon) \leq n_\varepsilon (x_\varepsilon, t_\varepsilon). \quad (60)$$

Next, we show that for ε small enough we have

$$-\frac{\mu \min n_+}{32} \leq ((-\Delta)_\varepsilon^\alpha (n_{\tilde{\nu},+,\varepsilon} h_\varepsilon) (n_{\tilde{\nu},+,\varepsilon} h_\varepsilon)^{-1} - n_\varepsilon (n_\varepsilon + \varepsilon)^{-1} + n_{\tilde{\nu},+,\varepsilon}) (x_\varepsilon, t_\varepsilon). \quad (61)$$

We perform the computation of $(-\Delta)_\varepsilon^\alpha (n_{\tilde{\nu},+,\varepsilon} h_\varepsilon) (n_{\tilde{\nu},+,\varepsilon} h_\varepsilon)^{-1} (x_\varepsilon, t_\varepsilon)$:

$$\begin{aligned} & (-\Delta)_\varepsilon^\alpha (n_{\tilde{\nu},+,\varepsilon} h_\varepsilon) (n_{\tilde{\nu},+,\varepsilon} h_\varepsilon)^{-1} (x_\varepsilon, t_\varepsilon) \\ &= (-\Delta)_\varepsilon^\alpha (n_{\tilde{\nu},+,\varepsilon}) (n_{\tilde{\nu},+,\varepsilon})^{-1} (x_\varepsilon) + (-\Delta)_\varepsilon^\alpha \left(g_\varepsilon \left(e^{-\frac{(\lambda_0 |t_0 + \tilde{\delta})}{1+2\alpha}} \cdot \right) \right) \left(g_\varepsilon \left(e^{-\frac{(\lambda_0 |t_0 + \tilde{\delta})}{1+2\alpha}} \cdot \right) \right)^{-1} (x_\varepsilon, t_\varepsilon) \\ &+ \tilde{K}_\varepsilon \left(n_{\tilde{\nu},+,\varepsilon}, g \left(e^{-\frac{(\lambda_0 |t_0 + \tilde{\delta})}{1+2\alpha}} \cdot \right) \right) \left(n_{\tilde{\nu},+,\varepsilon} g_\varepsilon \left(e^{-\frac{(\lambda_0 |t_0 + \tilde{\delta})}{1+2\alpha}} \cdot \right) \right)^{-1} (x_\varepsilon, t_\varepsilon). \end{aligned}$$

According to Lemma 4, there exists $\varepsilon_3 > 0$ such that for all $\varepsilon < \varepsilon_3$ we have

$$\begin{aligned} & (-\Delta)_\varepsilon^\alpha (n_{\tilde{\nu},+,\varepsilon} h_\varepsilon) (n_{\tilde{\nu},+,\varepsilon} h_\varepsilon)^{-1} (x_\varepsilon, t_\varepsilon) \\ &= (-\Delta)_\varepsilon^\alpha (n_{\tilde{\nu},+,\varepsilon}) (n_{\tilde{\nu},+,\varepsilon})^{-1} (x_\varepsilon) + o_\varepsilon(1) \\ &\geq 1 - n_{\tilde{\nu},+,\varepsilon}(x_\varepsilon) - \frac{\mu \min n_+}{64} \end{aligned}$$

and

$$1 - \frac{\mu \min n_+}{64} \leq (n_\varepsilon (n_\varepsilon + \varepsilon)^{-1}) (x_\varepsilon, t_\varepsilon).$$

We deduce that (61) holds true.

Inserting (61) in (60), we find that for all $\varepsilon \leq \min(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_{\tilde{\nu}}, \varepsilon_{t_0})$

$$n_{\tilde{\nu},+,\varepsilon}(x_\varepsilon) - \frac{\mu \min n_+}{16} \leq n_\varepsilon(x_\varepsilon, t_\varepsilon).$$

Dividing by $n_{+,\varepsilon}(x_\varepsilon)$ in the previous inequality, we find thanks to (54) that (53) holds true.

Step (iv). Take $x_0 \in \Omega_{2\tilde{\delta}}^\varepsilon \cap \left\{ x \mid (1+2\alpha) \log |x| \leq |\lambda_0| t_0 - 2\tilde{\delta} \right\}$. Now we want to bring back (53) at the point (x_0, t_0) . Remark that since $\tilde{\nu} \leq \tilde{\delta}$ we have $\Omega_{2\tilde{\delta}}^\varepsilon \subset \Omega_{\tilde{\nu}}^\varepsilon$. We also recall that thanks to Proposition 3,

$$\forall x \in \mathbb{R}, n_{\tilde{\nu},+}(x) \leq n_+(x). \quad (62)$$

Then there are two cases :

- 1) $h_\varepsilon(x_\varepsilon, t_\varepsilon) \leq h_\varepsilon(x_0, t_0)$,
- 2) $h_\varepsilon(x_\varepsilon, t_\varepsilon) > h_\varepsilon(x_0, t_0)$.

Case 1 : $h_\varepsilon(x_\varepsilon, t_\varepsilon) \leq h_\varepsilon(x_0, t_0)$.

Since $\Psi(x_\varepsilon, t_\varepsilon) = \min_{(x,t) \in \mathbb{R} \times V(\underline{t})} \Psi_\varepsilon(x, t)$ and according to (53) and (62) we deduce that

$$\begin{aligned} 1 - \frac{\mu}{4} &\leq 1 - \frac{\mu}{8} \leq \frac{n_\varepsilon(x_\varepsilon, t_\varepsilon)}{n_{+, \varepsilon}(x_\varepsilon)} \\ &\leq \frac{n_\varepsilon(x_\varepsilon, t_\varepsilon) + \varepsilon}{n_{+, \varepsilon}(x_\varepsilon)} \\ &\leq \frac{(n_\varepsilon(x_\varepsilon, t_\varepsilon) + \varepsilon)}{n_{\tilde{\nu}, +, \varepsilon}(x_\varepsilon)} \times \frac{h_\varepsilon(x_\varepsilon, t_\varepsilon)}{h_\varepsilon(x_\varepsilon, t_\varepsilon)} \\ &\leq \frac{(n_\varepsilon(x_0, t_0) + \varepsilon)}{n_{\tilde{\nu}, +, \varepsilon}(x_0)} \times \frac{h_\varepsilon(x_\varepsilon, t_\varepsilon)}{h_\varepsilon(x_0, t_0)} \leq \frac{(n_\varepsilon(x_0, t_0) + \varepsilon)}{n_{\tilde{\nu}, +, \varepsilon}(x_0)}. \end{aligned}$$

Case 2 : $h_\varepsilon(x_\varepsilon, t_\varepsilon) > h_\varepsilon(x_0, t_0)$.

Since $(x_0, t_0) \in \mathcal{B}_{2\tilde{\delta}}$ we have

$$-|\lambda_0|t_0 + (1 + 2\alpha) \log(|x|) + \tilde{\delta} < -\tilde{\delta}.$$

We deduce the existence of ε_4 such that for all $\varepsilon < \varepsilon_4$, we have

$$1 - \frac{\mu}{8} \leq \frac{1}{1 + e^{-\frac{\tilde{\delta}}{\varepsilon}}} \leq \frac{1}{1 + e^{-\frac{(|\lambda_0|t_0 + \tilde{\delta})}{\varepsilon}} |x_0|^{\frac{1+2\alpha}{\varepsilon}}} = h_\varepsilon(x_0, t_0). \quad (63)$$

Moreover, thanks to the definition of h_ε , we have

$$1 \leq e^{\frac{(t_\varepsilon - t_0)^2}{\varepsilon}} \times (1 + e^{\frac{-|\lambda_0|t_0 + \tilde{\delta}}{\varepsilon}} |x|^{\frac{1+2\alpha}{\varepsilon}}) = h_\varepsilon(x_\varepsilon, t_\varepsilon)^{-1}. \quad (64)$$

Recalling that $\Psi_\varepsilon(x_\varepsilon, t_\varepsilon) = \min_{(x,t) \in \mathbb{R} \times V(\underline{t})} \Psi_\varepsilon(x, t)$ and the inequalities (62), (63) and (64), we find that

$$\begin{aligned} 1 - \frac{\mu}{4} &\leq (1 - \frac{\mu}{8})^2 \leq \frac{n_\varepsilon(x_\varepsilon, t_\varepsilon)}{n_{+, \varepsilon}(x_\varepsilon)} \\ &\leq \frac{n_\varepsilon(x_\varepsilon, t_\varepsilon) + \varepsilon}{n_{+, \varepsilon}(x_\varepsilon)} \\ &\leq \frac{(n_\varepsilon(x_\varepsilon, t_\varepsilon) + \varepsilon)}{n_{\tilde{\nu}, +, \varepsilon}(x_\varepsilon)} \times \frac{h_\varepsilon(x_0, t_0)}{h_\varepsilon(x_\varepsilon, t_\varepsilon)} \\ &= \Psi_\varepsilon(x_\varepsilon, t_\varepsilon) \times h_\varepsilon(x_0, t_0) \\ &\leq \Psi_\varepsilon(x_0, t_0) \times h_\varepsilon(x_0, t_0) = \frac{n_\varepsilon(x_0, t_0) + \varepsilon}{n_{\tilde{\nu}, +, \varepsilon}(x_0)}. \end{aligned}$$

We deduce that for all $\varepsilon < \varepsilon(\tilde{\delta}, \mu, t_0) := \min_{i \in \{1, \dots, 4\}} (\varepsilon_i, \varepsilon_{\tilde{\nu}}, \varepsilon_{t_0}, 1, \frac{\mu}{4 \min_{\Omega_{2\tilde{\delta}}} n_{\tilde{\nu}, +}})$ and all $x_0 \in \Omega_{2\tilde{\delta}}^\varepsilon \cap$

$\left\{ x \in \mathbb{R} \mid |x| \leq e^{\frac{|\lambda_0|t_0 - 2\tilde{\delta}}{1+2\alpha}} \right\}$ we have

$$1 - \frac{\mu}{2} \leq \frac{n_\varepsilon(x_0, t_0) + \varepsilon}{n_{\tilde{\nu}, +, \varepsilon}(x_0)} - \frac{\varepsilon}{n_{\tilde{\nu}, +, \varepsilon}(x_0)} = \frac{n_\varepsilon(x_0, t_0)}{n_{\tilde{\nu}, +, \varepsilon}(x_0)}.$$

Since $x_0 \in \Omega_{2\tilde{\delta}}^\varepsilon \subset \Omega_{\tilde{\nu}}^\varepsilon$, we deduce that (54) holds true then we conclude thanks to the previous inequality that

$$1 - \mu \leq \left(1 - \frac{\mu}{2}\right) \times \left(1 - \frac{\mu}{8}\right) \leq \frac{n_\varepsilon(x_0, t_0)}{n_{\tilde{\nu}, +, \varepsilon}(x_0)} \times \frac{n_{\tilde{\nu}, +, \varepsilon}(x_0)}{n_{+, \varepsilon}(x_0)} = \frac{n_\varepsilon(x_0, t_0)}{n_{+, \varepsilon}(x_0)}.$$

Step (v). We show by contradiction that there exists $\varepsilon(\tilde{\delta}, \mu) > 0$ such that for all $\varepsilon \leq \varepsilon(\tilde{\delta}, \mu)$ and all $(x, t) \in \left(\Omega_{2\tilde{\delta}}^\varepsilon \times B(\underline{t}, \frac{r'}{2})\right) \cap \mathcal{B}^{2\tilde{\delta}}$ we have

$$1 - \mu \leq \frac{n_\varepsilon(x, t)}{n_{+, \varepsilon}(x)}.$$

Assume that such a non null $\varepsilon(\tilde{\delta}, \mu)$ does not exist. Next, for all $t \in V(\underline{t})$ we define

$$\bar{\varepsilon}(\tilde{\delta}, \mu, t) = \sup \left\{ \varepsilon_0 > 0 \mid 1 - \mu \leq \frac{n_\varepsilon(x, t)}{n_{+, \varepsilon}(x)} \text{ for all } \varepsilon < \varepsilon_0, x \in \Omega_{2\tilde{\delta}}^\varepsilon \cap \left\{ x \in \mathbb{R} \mid |x| \leq e^{\frac{|\lambda_0|t - 2\tilde{\delta}}{1+2\alpha}} \right\} \right\}.$$

Thanks to *Step (iv)*, we know that for all $t \in V(\underline{t})$

$$\bar{\varepsilon}(\tilde{\delta}, \mu, t) > 0.$$

Since we assume that $\varepsilon(\tilde{\delta}, \mu)$ does not exist, we deduce the existence of $(t_k)_{k \in \mathbb{N}} \in B(\underline{t}, \frac{r'}{2})$, such that

$$\bar{\varepsilon}(\tilde{\delta}, \mu, t_k) \rightarrow 0.$$

Up to an extraction, we assume that $t_k \xrightarrow[k \rightarrow +\infty]{} t_\infty$ with $t_\infty \in \overline{B}(\underline{t}, \frac{r'}{2})$. Since $t_\infty \in V(\underline{t})$, we deduce that $\bar{\varepsilon}(\frac{\tilde{\delta}}{2}, \frac{\mu}{2}, t_\infty) > 0$. We are going to exhibit a contradiction based on this fact.

Since we assume that $\bar{\varepsilon}(\tilde{\delta}, \mu, t_k) \rightarrow 0$, we deduce that

$$\exists \underline{N} \in \mathbb{N} \text{ such that } \forall k \geq \underline{N}, \bar{\varepsilon}(\tilde{\delta}, \mu, t_k) < \frac{\bar{\varepsilon}(\frac{\tilde{\delta}}{2}, \frac{\mu}{2}, t_\infty)}{2}.$$

We deduce that for all $k > \underline{N}$, there exists $\varepsilon_k \in]\frac{\bar{\varepsilon}(\frac{\tilde{\delta}}{2}, \frac{\mu}{2}, t_\infty)}{2}, \bar{\varepsilon}(\frac{\tilde{\delta}}{2}, \frac{\mu}{2}, t_\infty)[$ such that the following holds true :

$$\exists x_{k, \varepsilon_k} \in \Omega_{2\tilde{\delta}}^{\varepsilon_k} \cap \left\{ x \in \mathbb{R} \mid |x| \leq e^{\frac{|\lambda_0|t_k - 2\tilde{\delta}}{1+2\alpha}} \right\} \text{ such that } \frac{n_{\varepsilon_k}(x_{k, \varepsilon_k}, t_k)}{n_{\varepsilon_k, +}(x_{k, \varepsilon_k})} < 1 - \mu.$$

Moreover, since $t_k \rightarrow t_\infty$, we deduce the existence of $N_0 \in \mathbb{N}$ such that for all $k \geq N_0$ we have

$$\Omega_{2\tilde{\delta}}^{\varepsilon_k} \cap \left\{ x \in \mathbb{R} \mid |x| \leq e^{\frac{|\lambda_0|t_k - 2\tilde{\delta}}{1+2\alpha}} \right\} \subset \Omega_{\tilde{\delta}}^{\varepsilon_k} \cap \left\{ x \in \mathbb{R} \mid |x| \leq e^{\frac{|\lambda_0|t_\infty - \tilde{\delta}}{1+2\alpha}} \right\}. \quad (65)$$

Furthermore, the set $K := \left(\left(\Omega_{\tilde{\delta}}^{\varepsilon_k} \times V(\underline{t}) \right) \cap \mathcal{B}^{\tilde{\delta}} \right) \times]\frac{\bar{\varepsilon}(\frac{\tilde{\delta}}{2}, \frac{\mu}{2}, t_\infty)}{2}, \bar{\varepsilon}(\frac{\tilde{\delta}}{2}, \frac{\mu}{2}, t_\infty)[$ is compact. Since the map $((x, t, \varepsilon) \mapsto n_\varepsilon(x, t))$ is continuous, we deduce that $((x, t, \varepsilon) \mapsto n_\varepsilon(x, t))$ is uniformly continuous in K . We conclude to the existence of $N_1 \geq N_0$ such that for all $k \geq N_1$, all $\varepsilon \in]\frac{\bar{\varepsilon}(\frac{\tilde{\delta}}{2}, \frac{\mu}{2}, t_\infty)}{2}, \bar{\varepsilon}(\frac{\tilde{\delta}}{2}, \frac{\mu}{2}, t_\infty)[$ and all $x \in \Omega_{\tilde{\delta}}^{\varepsilon_k} \cap \left\{ x \in \mathbb{R} \mid |x| < e^{\frac{|\lambda_0|t_\infty - \tilde{\delta}}{1+2\alpha}} \right\}$ we have

$$1 - \frac{\mu}{2} < \frac{n_\varepsilon(x, t_k)}{n_\varepsilon(x, t_\infty)}. \quad (66)$$

By definition of $\bar{\varepsilon}(\frac{\tilde{\delta}}{2}, \frac{\mu}{2}, t_\infty)$, we deduce thanks to (65) and (66) that for all $\varepsilon \in]\frac{\bar{\varepsilon}(\frac{\tilde{\delta}}{2}, \frac{\mu}{2}, t_\infty)}{2}, \bar{\varepsilon}(\frac{\tilde{\delta}}{2}, \frac{\mu}{2}, t_\infty)[$ and all $x \in \Omega_{2\tilde{\delta}}^\varepsilon \cap \left\{ x \in \mathbb{R} \mid |x| < e^{\frac{|\lambda_0|t_k - 2\tilde{\delta}}{1+2\alpha}} \right\}$ we have

$$1 - \mu \leq \left(1 - \frac{\mu}{2}\right)^2 \leq \frac{n_\varepsilon(x, t_k)}{n_\varepsilon(x, t_\infty)} \times \frac{n_\varepsilon(x, t_\infty)}{n_{+, \varepsilon}(x)} = \frac{n_\varepsilon(x, t_k)}{n_{+, \varepsilon}(x)}.$$

Since the previous inequality holds true for $\varepsilon = \varepsilon_k$ and $x = x_{k, \varepsilon_k}$ for $k \geq \max(\underline{N}, N_1)$, the contradiction follows.

We conclude that for all $\mu > 0$, $\tilde{\delta} > 0$ and $\underline{t} > 0$, there exists $\varepsilon(\tilde{\delta}, \mu) > 0$ and $r > 0$ such that for all $\varepsilon < \varepsilon(\tilde{\delta}, \mu)$ and for all $(x, t) \in \left(\Omega_{2\tilde{\delta}}^\varepsilon \times B(\underline{t}, r)\right) \cap \mathcal{B}^{2\tilde{\delta}}$ we have

$$1 - \mu \leq \frac{n_\varepsilon(x, t)}{n_{+, \varepsilon}(x)}.$$

□

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