Phase retrieval for wide-band signals

Philippe Jaming, Karim Kellay, Rolando Perez Iii

PHASE RETRIEVAL FOR WIDE BAND SIGNALS

PHILIPPE JAMING, KARIM KELLAY & ROLANDO PEREZ III

Abstract. This study investigates the phase retrieval problem for wide-band signals. We solve the following problem: given $f \in L^2(\mathbb{R})$ with Fourier transform in $L^2(\mathbb{R}, e^{2c|x|}\,\mathrm{d}x)$, we find all functions $g \in L^2(\mathbb{R})$ with Fourier transform in $L^2(\mathbb{R}, e^{2c|x|}\,\mathrm{d}x)$, such that $|f(x)| = |g(x)|$ for all $x \in \mathbb{R}$. To do so, we first translate the problem to functions in the Hardy spaces on the disc via a conformal bijection, and take advantage of the inner-outer factorization. We also consider the same problem with additional constraints involving some transforms of f and g, and determine if these constraints force uniqueness of the solution.

1. Introduction

The phase retrieval problem refers to the recovery of the phase of a function f using given data on its magnitude $|f|$ and a priori assumptions on f. These problems are widely studied because of their physical applications in which the quantities involved are identified by their magnitude and phase, where the phase is difficult to measure while the magnitude is easily obtainable. Some physical applications of phase retrieval problems include works related to astronomy [11], lens design [12], x-ray crystallography [26], inverse scattering [27], and optics [28]. More physical examples were given in the survey article of Klibanov *et. al.* [22] and the book of Hurt [16]. A more recent overview of the phase retrieval problem was given by the article of Grohs *et. al.* [14], which discussed a more general formulation of the phase retrieval problem using Banach spaces and bounded linear functionals, as well as results related to the uniqueness and stability properties of the problem.

Phase retrieval problems have been given more interest because of progress in the discrete (finite-dimensional) case, starting with the work of Candès *et. al.* [7] and of Waldspurger *et. al.* [31], which formulated the phase retrieval problem as an optimization problem and used algorithms to determine the solutions. On the other hand, phase retrieval problems devoted to the continuous (infinite-dimensional) case have been solved in various settings, such as, for one-dimensional band-limited functions [1, 2, 32], for functions in the Hardy space on the disc without singular parts [5], for real-valued band-limited functions from the absolute values of their samples [29], for 2π-periodic time-limited signals from magnitude values on the unit circle [13], and for real-valued functions in the Sobolev spaces [15]. We refer the reader to [3] for more discussion and examples of continuous phase retrieval

2010 *Mathematics Subject Classification.* 30D05, 30H10, 42B10, 94A12.
Key words and phrases. phase retrieval, Hardy spaces.

problems, in particular on the stability of the problem and other useful references. Our aim in this paper is to investigate the phase retrieval problem for *wide-band* functions, namely functions with mildly decreasing Fourier transforms.

Before we summarize our results, let us give a quick overview of the phase retrieval problem in the band-limited and narrow band cases: given a band-limited function (i.e. a function with compactly supported Fourier transform) $f \in L^2(\mathbb{R})$, find all band-limited functions $g \in L^2(\mathbb{R})$ such that

$$|f(x)| = |g(x)| \text{ for all } x \in \mathbb{R}. \tag{1}$$

This problem in the class of compactly supported functions has been solved by Akutowicz [1, 2] in the mid-1950's, and independently by Walther [32] in 1963. To solve the problem, they first used the Paley-Wiener Theorem which states that f and g extend into holomorphic functions in the complex plane that are of exponential type, that is, f, g grow like $e^{a|z|}$. Next, they showed that (1) is then equivalent to

$$f(z)\overline{f(\bar{z})} = g(z)\overline{g(\bar{z})} \text{ for all } z \in \mathbb{C}. \tag{2}$$

Observe that (2) is a reformulation of (1) when z is *real* and is an equality between two holomorphic functions so that it is valid for all $z \in \mathbb{C}$. Finally, they used the Hadamard Factorization Theorem which states that holomorphic functions of exponential type are characterized by their zeros. Now, (2) implies that each zero of g is either a zero of f or a complex conjugate of a such a zero. Thus, it follows that g can be obtained by changing arbitrarily many zeros of f into their complex conjugates in the Hadamard factorization of g, and this is called *zero-flipping*.

McDonald [24] extended this proof to functions that have Fourier tranforms with very fast decrease at infinity. For instance, in the case of Gaussian decrease, if $|\widehat{f}(\xi)|, |\widehat{g}(\xi)| \lesssim e^{-a|\xi|^2}$, $a > 0$, then f, g extend to holomorphic functions of exponential type 2 so that Hadamard factorization can still be used. Thus, the solutions also can be obtained by zero-flipping. Furthermore, this proof extends to functions which satisfy an exponential decay condition of the form

$$|\widehat{f}(\xi)|, |\widehat{g}(\xi)| \lesssim e^{-a|\xi|^\alpha} \text{ for } a > 0 \text{ and } \alpha > 1, \tag{3}$$

but breaks down at $\alpha = 1$. Therefore, the main goal of this work is to investigate the phase retrieval problem for functions satisfying (3) but for $\alpha = 1$, i.e. $|\widehat{f}(\xi)|, |\widehat{g}(\xi)| \lesssim e^{-a|\xi|}$. Functions with this decay are sometimes called wide-band signals in the engineering community. Here, the functions f and g only extend holomorphically to an horizontal strip $\mathcal{S}_a = \{z \in \mathbb{C} : |\mathrm{Im} z| < a\}$ in the complex plane so that (2) only holds for $z \in \mathcal{S}_a$, which implies that Hadamard factorization cannot be used. To overcome this difficulty, we first reduce the problem to the Hardy space on the disc using a conformal bijection. We then exploit the inner-outer-Blashcke factorization in the Hardy space on the disc. The solution is now more evolved than the band-limited case as, aside from zero-flipping, the singular inner function and the outer function also take part in the solution. For our main results, we go back to the strip to solve our initial problem using an equivalent

inner-outer-Blaschke factorization on the strip, and provide analogs of the consequences shown on the phase retrieval problem on the disc.

Generally speaking, the solution set of a phase retrieval problem is very large. Thus, additional constraints are considered to reduce the solution set. For instance, Klibanov *et. al.* provided different examples of supplementary information to force uniqueness, or at least to reduce the solution set. We here follow the same aim of reducing the set of solutions by coupling two phase retrieval problems. For our first coupled problem, we add a condition involving a fixed reference signal h: $|g-h| = |f-h|$. We use its geometric interpretation to show that this problem has exactly two solutions. We will also look at the problem with an additional condition involving the Fourier transforms: $|\widehat{g}| = |\widehat{f}|$. This coupled problem is also known as the Pauli problem. Here, we do not get uniqueness for our case, in fact, we explicitly construct an uncountable family of solutions using Riesz products. Next, we look at the coupled problem with the condition $|Dg| = |Df|$ where D is a derivation operator. Using the special properties of D, f and g, we show that this coupled problem has exactly two solutions. Finally, for our last coupled problem, we add the condition $|g| = |f|$ on a segment on the strip. We show the uniqueness of the solution by using our main results.

This work is organized as follows. Section 2 is a quick review of definitions and results on analysis and Hardy spaces. Section 3 is devoted to the solution of the phase retrieval problem in the wide-band case. We will look at the phase retrieval problem on the unit disc and on the strip. Section 4 is devoted to the coupled phase retrieval problems.

2. Preliminaries

2.1. **Notation.** For a domain $\Omega \subset \mathbb{C}$, $\mathrm{Hol}(\Omega)$ is the set of holomorphic functions on Ω. For $F \in \mathrm{Hol}(\Omega)$ we denote by $Z(F)$ the set of zeros of F, counted with multiplicity. Write $\overline{\Omega} = \{\bar{z} : z \in \Omega\}$ and if $F \in \mathrm{Hol}(\Omega)$, we denote by F^* the function in $\mathrm{Hol}(\overline{\Omega})$ defined by $F^*(z) = \overline{F(\bar{z})}$. It will be convenient to denote the conjugation function by C, where $C(z) = \bar{z}$ for all $z \in \mathbb{C}$.

The unit disc $\mathbb{D}$ is defined as $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and its boundary $\mathbb{T}$ is defined by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Let $c > 0$ and $\mathcal{S}_c$ be the strip defined as $\mathcal{S}_c := \{z \in \mathbb{C} : |\mathrm{Im} z| < c\}$, and $\mathcal{S} := \mathcal{S}_1$.

For a nonnegative and locally integrable function ω on $\mathbb{R}$, the weighted L^2 space on $\mathbb{R}$ is given by

$$L^2_\omega(\mathbb{R}) = L^2(\mathbb{R}, \omega \mathrm{d}t) = \left\{ f \text{ is measurable} : ||f||^2_{L^2_\omega(\mathbb{R})} = \int_{\mathbb{R}} |f(t)|^2 \omega(t)\, \mathrm{d}t < +\infty \right\}.$$

Finally, consider a measure space $(X_1, \mathcal{A}_1, \mu)$, a measurable space $(X_2, \mathcal{A}_2)$, and a measurable map $\psi : X_1 \to X_2$. Recall that the pullback measure of X_1 by ψ is given by

$$\psi_* \mu(A) = \mu(\psi(A))$$

for $A \in \mathcal{A}_1$. Equivalently, if h is a function such that $f \circ \psi$ is integrable on X_1 with respect to μ, then we have the change of variables formula

$$\int_{X_2} h \, \mathrm{d}(\psi_* \mu) = \int_{X_1} h \circ \psi \, \mathrm{d}\mu.$$

2.2. **Hardy Spaces on the Disc.** Recall that the Hardy space on the disc $\mathbb{D}$ is defined as

$$H^2(\mathbb{D}) = \left\{ F \in \mathrm{Hol}(\mathbb{D}) : ||F||^2_{H^2(\mathbb{D})} = \sup_{0 \le r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{i\theta})|^2 \, \mathrm{d}\theta < +\infty \right\},$$

and

$$H^\infty(\mathbb{D}) = \left\{ F \in \mathrm{Hol}(\mathbb{D}) : ||F||_{H^\infty(\mathbb{D})} = \sup_{w \in \mathbb{D}} |F(w)| < +\infty \right\}.$$

We will need the following key facts. First, every $F \in H^2(\mathbb{D})$ admits a radial limit $F(e^{i\theta}) = \lim_{r \to 1} F(re^{i\theta})$ for almost every $e^{i\theta} \in \mathbb{T}$ (see e.g [23, Lemma 3.10]) with $F \in L^2(\mathbb{T})$, $\widehat{F}(n) = 0$ for $n = -1, -2, ...$, and $\log |F| \in L^1(\mathbb{T})$. Furthermore [23, Section 7.6], every function $F \in H^2(\mathbb{D})$ can be uniquely decomposed as

$$F = e^{i\gamma} B_F S_F O_F$$

where $e^{i\gamma} \in \mathbb{T}$, B_F is the Blaschke product formed from the zeros of F, S_F is a singular inner function, and O_F is the outer part of F. More precisely, the Blaschke product is defined for all $w \in \mathbb{D}$ as

$$B_F(w) = \prod_{\alpha \in Z(F)} b_\alpha(w), \tag{4}$$

where

$$b_\alpha(w) = \begin{cases} w & \text{if } \alpha = 0 \\ \dfrac{\alpha}{|\alpha|} \dfrac{\alpha - w}{1 - \bar{\alpha} w} & \text{if } \alpha \neq 0 \end{cases}.$$

The singular part is given by

$$S_F(w) = \exp\left(\int_{\mathbb{T}} \frac{w + e^{i\theta}}{w - e^{i\theta}} \, \mathrm{d}\nu_F \left(e^{i\theta} \right) \right), \tag{5}$$

where ν_F is a finite positive singular measure (with respect to the Lebesgue measure). Finally, the outer part is determined by the modulus of the radial limit of F

$$O_F(w) = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{w + e^{i\theta}}{w - e^{i\theta}} \log |F\left(e^{i\theta} \right)| \, \mathrm{d}\theta \right). \tag{6}$$

2.3. **Hardy Spaces on the Strip.** There are essentially two ways of defining the Hardy space on the strip $\mathcal{S}$. To start, let us define the conformal bijection $\phi : \mathcal{S} \longrightarrow \mathbb{D}$ given by

$$\phi(z) := \tanh\left(\frac{\pi}{4} z \right).$$

Observe that ϕ has the following properties: $\phi^* = \phi$, $\phi(\mathbb{R}) = [-1, 1]$, and $\phi(\partial\mathcal{S} \cap \mathbb{C}_\pm) = \mathbb{T}_\pm$, where $\mathbb{C}_+, \mathbb{C}_-$ denote the upper and lower halves of $\mathbb{C}$

respectively, and $\mathbb{T}_+, \mathbb{T}_-$ denote the upper and lower halves of $\mathbb{T}$ respectively. On one hand we shall consider the following Hardy spaces defined

$$H^2(\mathcal{S}) = \left\{ f \in \mathrm{Hol}(\mathcal{S}) : f \circ \phi^{-1} \in H^2(\mathbb{D}) \right\},$$

and $||f||_{H^2(\mathcal{S})} = ||f \circ \phi^{-1}||_{H^2(\mathbb{D})}$. It can then be shown [4, Theorem 2.2] that $H^2(\mathcal{S}) = H^2_W(\mathcal{S})$ isometrically where

$$H^2_W(\mathcal{S}) = \left\{ f \in \mathrm{Hol}(\mathcal{S}) : ||f||^2_{H^2_W(\mathcal{S})} = \sup_{|y|<1} \int_{\mathbb{R}} \frac{|f(t+iy)|^2}{|W(t+iy)|}\, \mathrm{d}t < +\infty \right\},$$

and $W(z) = \dfrac{1}{4\cosh^2(\frac{\pi}{4}z)} = \pi\phi'(z)$ for all $z \in \mathcal{S}$.

Now this last space can be identified to the natural analogue of the Hardy space on the disc:

$$H^2_\tau(\mathcal{S}) = \left\{ f \in \mathrm{Hol}(\mathcal{S}) : ||f||^2_{H^2_\tau(\mathcal{S})} = \sup_{|y|<1} \int_{\mathbb{R}} |f(t+iy)|^2\, \mathrm{d}t < +\infty \right\}.$$

More precisely $f \in H^2_\tau(\mathcal{S})$ if and only if $W^{1/2} f \in H^2_W(\mathcal{S})$ if and only if $\widehat{f} \in L^2(\mathbb{R}, e^{2|\xi|} d\xi)$.

Finally by using [4, Theorem 2.1], we obtain the factorization on $H^2_\tau(\mathcal{S})$.

Lemma 2.1. *Let $f \in H^2_\tau(\mathcal{S})$. Then the unique inner-outer factorization of f is given by*

$$f(z) = \frac{e^{i\gamma} B_F(\phi(z)) S_F(\phi(z)) O_F(\phi(z))}{W(z)^{1/2}}$$

for all $z \in \mathcal{S}$ and for some $\gamma \in \mathbb{R}$. For all $z \in \mathcal{S}$, the Blaschke product B_f is given by

$$B_f(z) = \prod_{\beta \in Z(f)} b_{\phi(\beta)}(\phi(z)), \tag{7}$$

while the singular inner function S_f is given by

$$S_f(z) = \exp\left(\int_{\partial\mathcal{S}} \frac{\phi(z)+\phi(\zeta)}{\phi(z)-\phi(\zeta)}\, \mathrm{d}\mu_f(\zeta) \right) \tag{8}$$

where $\mu_f = {\phi^{-1}}_* \nu_F$ is the pullback measure of ν_F on $\partial\mathcal{S}$, and the outer function O_f is given by

$$\begin{aligned} O_f(z) = \exp\Bigg(& \frac{-1}{2\pi i} \int_{\mathbb{R}} \frac{\phi(z)+\phi(x+i)}{\phi(z)-\phi(x+i)} \frac{\phi'(x+i)}{\phi(x+i)} \log|W(x+i)^{1/2} f(x+i)|\, \mathrm{d}x \\ & + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\phi(z)+\phi(x-i)}{\phi(z)-\phi(x-i)} \frac{\phi'(x-i)}{\phi(x-i)} \log|W(x-i)^{1/2} f(x-i)|\, \mathrm{d}x \Bigg). \end{aligned} \tag{9}$$

Proof. For $F \in H^2(\mathbb{D})$ and $z \in \mathcal{S}$, by Theorem 2.1 from [4] we have $F(\phi(z)) = W^{1/2}(z) f(z)$ and equivalently,

$$f(z) = \frac{F(\phi(z))}{W(z)^{1/2}} = \frac{e^{i\gamma} B_F(\phi(z)) S_F(\phi(z)) O_F(\phi(z))}{W(z)^{1/2}}.$$

Note that this is well-defined on $\mathcal{S}$ since $W(z) = \pi\phi'(z) \neq 0$ for any $z \in \mathcal{S}$.

The formulas for the Blaschke product and singular inner function easily follow from $B_f(z) = B_F(\phi(z))$ and $S_f(z) = S_F(\phi(z))$. For the outer function, we need to split the integral since $\phi(\partial\mathcal{S} \cap \mathbb{C}_\pm) = \mathbb{T}_\pm$. Hence, the outer function is given by

$$O_f(z) = O_F(\phi(z)) = \exp\left(\frac{1}{2\pi}\int_0^\pi \frac{\phi(z)+e^{i\theta}}{\phi(z)-e^{i\theta}} \log|F\left(e^{i\theta}\right)|\,\mathrm{d}\theta + \frac{1}{2\pi}\int_{-\pi}^0 \frac{\phi(z)+e^{i\theta}}{\phi(z)-e^{i\theta}} \log|F\left(e^{i\theta}\right)|\,\mathrm{d}\theta\right)$$

for all $z \in \mathcal{S}$. By applying the substitutions $e^{i\theta} = \phi(x+i),\ \theta \in [0,\pi]$ on $\mathbb{T}^+$ and $e^{i\theta} = \phi(x-i),\ \theta \in [-\pi, 0]$ on $\mathbb{T}^-$, we get (9). □

3. Phase Retrieval in $H^2_\tau(\mathcal{S})$

3.1. **Reduction of the Problem.** In this section, we consider $f, g \in L^2(\mathbb{R})$ with $\widehat{f}, \widehat{g} \in L^2(\mathbb{R}, e^{2c|\xi|}\mathrm{d}\xi)$ such that $|f(x)| = |g(x)|$ for every $x \in \mathbb{R}$. Our goal is to determine, for a given f, all possible g's.

To do so, let us write $f_c(x) = f(cx)$ and $g_c(x) = g(cx)$ so that $f_c, g_c \in L^2(\mathbb{R})$ with $\widehat{f_c}, \widehat{g_c} \in L^2(\mathbb{R}, e^{2|\xi|}d\xi)$ and $|f_c(x)| = |g_c(x)|$ for every $x \in \mathbb{R}$ so that it is enough to consider the case $c = 1$.

Note that $\widehat{f}, \widehat{g} \in L^2(\mathbb{R}, e^{2|\xi|}\mathrm{d}\xi)$ if and only if $f, g \in H^2_\tau(\mathcal{S})$. Thus, f and g extend holomorphically to $\mathcal{S}$ and $|f(x)| = |g(x)|$ for every $x \in \mathbb{R}$ can be written as

$$f(x)\overline{f(\bar{x})} = g(x)\overline{g(\bar{x})} \text{ for all } x \in \mathbb{R}. \tag{10}$$

But now, (10) is an equality between two holomorphic functions on $\mathbb{R}$ so that it is valid also for all $x \in \mathcal{S}$. In other words, we are now trying to solve the following problem: given $f \in H^2_\tau(\mathcal{S})$, find all $g \in H^2_\tau(\mathcal{S})$ such that

$$f(z)f^*(z) = g(z)g^*(z) \text{ for all } z \in \mathcal{S}. \tag{11}$$

It turns out that this problem is easier to solve when transfering the problem to the disc. Multiplying $W^{1/2}(z)$, $\overline{W^{1/2}(\bar{z})}$ to both sides of (11), we obtain

$$(W^{1/2}f)(z)\overline{(W^{1/2}f)(\bar{z})} = (W^{1/2}g)(z)\overline{(W^{1/2}g)(\bar{z})}$$

for all $z \in \mathcal{S}$. According to [4], the functions $F = W^{1/2}f \circ \phi^{-1}$ and $G = W^{1/2}g \circ \phi^{-1}$ are in $H^2(\mathbb{D})$. Hence, by applying the substitution $z = \phi^{-1}(w)$ and $\bar{z} = \phi^{-1}(\bar{w})$ to the previous equation, we get

$$F(w)F^*(w) = G(w)G^*(w) \text{ for all } w \in \mathbb{D}. \tag{12}$$

Therefore, we have translated the equality on the strip to an equivalent equality on the disk. Finally, we are now trying to solve the following problem on the disc: given $F \in H^2(\mathbb{D})$, find all $G \in H^2(\mathbb{D})$ such that (12) holds for all $w \in \mathbb{D}$. Note that (12) is equivalent to $|F(w)|^2 = |G(w)|^2$ for $w \in (-1, 1)$.

3.2. **The Phase Retrieval Problem on the Disc.** In this section, we look at the equivalent phase retrieval problem on the disc.

Let $F \in H^2(\mathbb{D})$ and write $F = B_F S_F O_F$ with B_F, S_F, O_F given in equations (4), (5) and (6), respectively. The factorization of F^* is given by

$$F^* = e^{i\lambda} B_{F^*} S_{F^*} O_{F^*} = e^{i\lambda} B_F^* S_F^* O_F^*.$$

Since the factorization in $H^2(\mathbb{D})$ is unique, we have $B_{F^*} = B_F^*$, $S_{F^*} = S_F^*$, and $O_{F^*} = O_F^*$. Hence, for all $w \in \mathbb{D}$, the Blaschke product formed from the zeros of F^* is given by

$$B_{F^*}(w) = B_F^*(w) = \prod_{\alpha \in Z(F)} b_{\bar{\alpha}}(w) = \prod_{\alpha \in \overline{Z(F)}} b_\alpha(w). \tag{13}$$

The singular part of F^* is given by

$$S_{F^*}(w) = S_F^*(w) = \exp\left(\int_{\mathbb{T}} \frac{w + e^{i\theta}}{w - e^{i\theta}} \, \mathrm{d}(C_* \nu_F)\left(e^{i\theta}\right) \right), \tag{14}$$

for all $w \in \mathbb{D}$, where $C_* \nu_F$ is the pullback measure of $\mathbb{T}$ by the conjugation function C. Finally, for all $w \in \mathbb{D}$, the outer part of F^* is given by

$$O_{F^*}(w) = O_F^*(w) = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{w + e^{i\theta}}{w - e^{i\theta}} \log |F\left(e^{-i\theta}\right)| \, \mathrm{d}\theta \right). \tag{15}$$

We use all of the facts above to prove the following lemma.

Lemma 3.1. *Let $F, G \in H^2(\mathbb{D})$. Then*

$$|F(w)|^2 = |G(w)|^2 \text{ for all } w \in (-1, 1)$$

if and only if

(i) *the zero sets of F and G satisfy*

$$Z(F) \cup \overline{Z(F)} = Z(G) \cup \overline{Z(G)};$$

(ii) *the singular measures ν_F and ν_G, associated with F and G respectively, satisfy*

$$\nu_F + C_* \nu_F = \nu_G + C_* \nu_G$$

on $\mathbb{T}$; and

(iii) *the radial limits satisfy*

$$|F\left(e^{i\theta}\right) F\left(e^{-i\theta}\right)| = |G\left(e^{i\theta}\right) G\left(e^{-i\theta}\right)|$$

almost everywhere on $\mathbb{T}$.

Proof. Let $F, G \in H^2(\mathbb{D})$. Note that FF^* and GG^* have decompositions given by

$$FF^* = B_F B_{F^*} S_F S_{F^*} O_F O_{F^*} \text{ and } GG^* = B_G B_{G^*} S_G S_{G^*} O_G O_{G^*}.$$

Notice that $B_F B_{F^*}$ is again a Blaschke product, $S_F S_{F^*}$ is again a singular inner function, and $O_F O_{F^*}$ is again an outer function. Indeed, for all $w \in \mathbb{D}$, (13) implies that

$$B_F(w) B_{F^*}(w) = \prod_{\alpha \in Z(F) \cup \overline{Z(F)}} b_\alpha(w),$$

while (14) implies that

$$S_F(w)S_{F^*}(w) = \exp\left(\frac{1}{2\pi}\int_{\mathbb{T}} \frac{w+e^{i\theta}}{w-e^{i\theta}}\, \mathrm{d}\left(\nu_F + C_*\nu_F\right)(e^{i\theta})\right),$$

and finally, (15) implies that

$$O_F(w)O_{F^*}(w) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{w+e^{i\theta}}{w-e^{i\theta}} \log|F(e^{i\theta})F(e^{-i\theta})|\, \mathrm{d}\theta\right).$$

Thus, writing the same for GG^* and using the uniqueness of the decomposition, $FF^* = GG^*$ implies that $B_F B_{F^*} = B_G B_{G^*}$, which in turn implies that

$$Z(F) \cup \overline{Z(F)} = Z(G) \cup \overline{Z(G)}.$$

Furthermore, $FF^* = GG^*$ also implies that

$$S_F S_{F^*} = S_G S_{G^*} \text{ and } O_F O_{F^*} = O_G O_{G^*}.$$

Thus,

$$\nu_F + C_*\nu_F = \nu_G + C_*\nu_G$$

on $\mathbb{T}$, and by Fatou's theorem [23, Lemma 3.10], we have for almost every $\theta \in \mathbb{R}$

$$\lim_{r\to 1}(O_F(re^{i\theta})O_{F^*}(re^{i\theta})) = \lim_{r\to 1}(O_G(re^{i\theta})O_{G^*}(re^{i\theta})),$$

which in turn implies that

$$|F\left(e^{i\theta}\right)F\left(e^{-i\theta}\right)| = |G\left(e^{i\theta}\right)G\left(e^{-i\theta}\right)|$$

almost everywhere on $\mathbb{T}$. □

We can now construct such G's to solve the equivalent phase retrieval problem on the disc. Let $\mathcal{N}^+$ denote the Smirnov class, namely those functions holomorphic on $\mathbb{D}$ of the form $f = g/h$, where g and h are bounded and holomorphic on $\mathbb{D}$ and h is outer function. g is outer, then f is called outer function. Note that if $f \in N^+$ then by Fatou's Theorem [10, Theorem 1.3], the radial limit f_* almost everywhere on $\mathbb{T}$ and $\log|f^*| \in L^1(\mathbb{T})$. The following corollary immediately follows from Lemma 3.1.

Corollary 3.2. *Let $F, G \in H^2(\mathbb{D})$. Then $|F| = |G|$ on $(-1,1)$ if and only if the inner-outer decomposition of F and G are given by*

$$F = e^{i\gamma}B_F S_F O_F \text{ and } G = {}^{i\gamma'} B_G S_G O_G$$

where

- $\gamma, \gamma' \in \mathbb{R}$;
- B_F, S_F, O_F *are given by* (4), (5), (6) *respectively;*
- B_G *is the Blaschke product associated with the set* $A \cup (\overline{Z(F)\backslash A})$ *for some* $A \subset Z(F)$;
- S_G *is the singular inner function associated with the positive singular measure* $\nu_G = \nu_F + \rho$, *where* ρ *is an odd real singular measure; and*
- $O_G = UO_F$ *where* $U \in N^+$ *is an outer function and* $U = 1/U^*$ *on* $\mathbb{D}$.

Remark 3.3. *We write $\rho = \rho_+ - \rho_-$, where ρ_+ is the positive part while ρ_- is the negative part. Note that the positive part and the negative part have disjoint supports. Since ρ is an odd measure, $C_*\rho = -\rho$ and given $E \subset \mathbb{T}$ such that $E \cap \overline{E} = \emptyset$, we have* $\mathrm{supp}\rho_+ \subset E$. *Thus, we take $\rho_- = C_*\rho_+$. Furthermore, for ν_G to be positive, we need the condition $C_*\rho_+ \leq \nu_F$, or equivalently, $\rho_+ \leq C_*\nu_F$.*

Proof of Corollary 3.2. Let $F, G \in H^2(\mathbb{D})$ with inner-outer decompositions as defined on Corollary 3.2. Observe that the properties of the Blaschke product B_G and the singular inner function S_G immediately follow from Lemma 3.1. For the outer function, by Lemma 3.1, we have

$$|O_F(e^{i\theta})O_F(e^{-i\theta})| = |O_G(e^{i\theta})O_G(e^{-i\theta})| \tag{16}$$

almost everywhere on $\mathbb{T}$. Hence,

$$\log|O_G(e^{i\theta})| = \log|O_F(e^{i\theta})| + \log|U(e^{i\theta})|$$

almost everywhere on $\mathbb{T}$, where $\log|U(e^{i\theta})|$ is an odd real-valued function of θ and $\log|U| \in L^1(\mathbb{T})$. Since $|O_G(e^{i\theta})| = |O_F(e^{i\theta})U(e^{i\theta})|$ almost everywhere on $\mathbb{T}$ and O_G and O_F are outer functions, we get

$$O_G(z) = O_F(z)O_U(z), \qquad z \in \mathbb{D}.$$

Hence $O_U = O_G/O_F \in N^+$. Moreover, (16) implies that $|O_U(e^{i\theta})O_U(e^{-i\theta})| = 1$ almost everywhere on $\mathbb{T}$, and so $O_U(z)O_U^*(z) = 1$ on $\mathbb{D}$. □

We can actually identify the solutions of the phase retrieval problem on the disc in terms of a factorization. Let us consider an analog of the result of McDonald [24, Proposition 1].

Corollary 3.4. *Let $F, G \in H^2(\mathbb{D})$. Then $|F| = |G|$ on $(-1, 1)$ if and only if there exist $u, v \in \mathrm{Hol}(\mathbb{D})$ such that $F = uv$ and $G = uv^*$.*

Proof. Let $F, G \in H^2(\mathbb{D})$. By Corollary 3.2, we have the factorizations

$$F = B_F S_F O_F \text{ and } G = B_F S_F O_F.$$

First, observe that we can write the Blaschke products B_F and B_G as

$$B_F = B_1 B_2 \text{ and } B_G = B_1 B_2^*$$

where B_1 is the Blaschke product associated with $A \subset Z(F)$ and B_2 is the Blaschke product associated with $Z(F)\backslash A$. On the other hand, we can write the singular measures ν_F and ν_G as

$$\nu_F = \nu_1 + \nu_2 \text{ and } \nu_G = \nu_1 + C_*\nu_2$$

where

$$\nu_1 = \nu_f + \frac{\rho_+ - C_*\rho_+}{2} \text{ and } \nu_2 = \frac{C_*\rho_+ - \rho_+}{2},$$

so that $S_F = S_{\nu_1}S_{\nu_2}$ and $S_G = S_{\nu_1}S_{\nu_2}^*$.

Since $O_G = UO_F$ where U is an outer function, $U \in N^+$ and $UU^* = 1$ on $\mathbb{D}$. We write

$$O_F = O_F U^{1/2} U^{-1/2}$$

and

$$O_G = O_F U^{1/2} U^{1/2} = O_F U^{1/2} (U^{-1/2})^*$$

Therefore, we take

$$u = B_1 S_{\nu_1} O_F U^{1/2} \text{ and } v = B_2 S_{\nu_2} U^{-1/2}.$$

□

3.3. **Back to the Strip.** In this section, we go back to the phase retrieval problem on the strip. Using Corollary 2.1, we see that Lemma 3.1 translates to functions on $H^2_\tau(\mathcal{S})$. By a change of variable and by applying the inner-outer factorization on $H^2_\tau(\mathcal{S})$, we have:

Lemma 3.5. *Let $f, g \in H^2_\tau(\mathcal{S})$. Then*

$$|f(z)|^2 = |g(z)|^2 \text{ for all } z \in \mathbb{R}$$

if and only if

(i) *the zero sets of f and g satisfy*

$$Z(f) \cup \overline{Z(f)} = Z(g) \cup \overline{Z(g)};$$

(ii) *the singular measures μ_f and μ_g, associated with f and g respectively, satisfy*

$$\mu_f + C_* \mu_f = \mu_g + C_* \mu_g$$

on $\partial\mathcal{S}$; and

(iii) *the boundary values satisfy*

$$|f(x+i)f(x-i)| = |g(x+i)g(x-i)|$$

almost everywhere on $\mathbb{R}$.

We now construct the solutions of the problem on the strip. Let $N^+_\tau(\mathcal{S})$ the Smirnov class of holomorphic functions in the $\mathcal{S}$ such that $f = F(\phi(z))/W^{1/2}(z)$ where $F \in N^+$. The following result immediately follows from Lemma 3.5.

Theorem 3.6. *Let $f, g \in H^2_\tau(\mathcal{S})$. Then $|f| = |g|$ on $\mathbb{R}$ if and only if the inner-outer decomposition of f and g are given by*

$$f =^{i\gamma} W^{-1/2} B_f S_f O_f \text{ and } g = e^{i\gamma'} W^{-1/2} B_g S_g O_g$$

where

- $\gamma, \gamma' \in \mathbb{R}$;
- B_f, S_f, O_f *are given by* (7), (8), (9) *respectively;*
- B_g *is the Blaschke product associated with the set* $A \cup (\overline{Z(f) \backslash A})$ *with* $A \subset Z(f)$;
- S_g *is the singular inner function associated with the positive singular measure* $\mu_g = \mu_f + \sigma$, *where* σ *is an odd real singular measure, given by* $\sigma = \sigma_+ - C_*\sigma_+$, *satisfying* $C_*\sigma = -\sigma$ *and* $\sigma_+ \leq C_*\mu_f$; *and*
- O_g *is the outer part of* uO_f *where* $u \in N^+_\tau(\mathcal{S})$ *is an outer function and* $u = 1/u^*$ *on* $\mathcal{S}$.

Remark 3.7. *Observe that possible trivial solutions to the problem on the strip are given by:*

$$(1) \quad g(z) = ce^{i\eta z} f(z) \qquad \text{and} \qquad (2) \quad g(z) = ce^{i\eta z} f^*(z)$$

with $|c| = 1$ and $\eta \in \mathbb{R}$. Those trivial solutions are retrieved as follows

– *the factor c is $c = e^{i(\gamma - \gamma')}$*

– *the factor $e^{i\eta z}$ is the factor u of the outer part as $e^{i\eta z}(e^{i\eta z})^* = 1$*

– the replacement of f by f^* is obtained by taking $A = \emptyset$ for the Blaschke part, $\sigma = C_*\mu_f - \mu_f$ so that $\mu_g = C_*\mu_f$ for the inner part and finally $u = O_{f^*}/O_f$ so that the outer part of g is $Og = uO_f = O_{f^*}$.

Corollary 3.4 also translates to a result on the strip with a simple change of variable.

Corollary 3.8. *Let $f, g \in H^2_\tau(\mathcal{S})$. Then $|f| = |g|$ on $\mathbb{R}$ if and only if there exist $u, v \in \mathrm{Hol}(\mathcal{S})$ such that $f = uv$ and $g = uv^*$.*

Finally, we go back to our initial phase retrieval problem. The following result directly follows from Theorem 3.6.

Corollary 3.9. *Let $f \in L^2(\mathbb{R})$ and $\widehat{f} \in L^2(\mathbb{R}, e^{2c|\xi|}d\xi)$. Then there exists $g \in L^2(\mathbb{R})$ such that $\widehat{g} \in L^2(\mathbb{R}, e^{2c|\xi|}d\xi)$, $|f(x)| = |g(x)|$ for all $x \in \mathbb{R}$, and*

$$g = e^{i\kappa}W^{-1/2}B_gS_gO_g$$

where

- *$e^{i\kappa} \in \mathbb{T}$;*
- *B_g is the Blaschke product associated with the set $A \cup (\overline{Z(f)\backslash A})$ with $A \subset Z(f)$;*
- *S_g is the singular inner function associated with the positive singular measure $\mu_g = \mu_f + \sigma$, where σ is an odd real singular measure, given by $\sigma = \sigma_+ - C_*\sigma_+$, satisfying $C_*\sigma = -\sigma$ and $\sigma_+ \leq C_*\mu_f$; and*
- *O_g is the outer part of uO_f where $u \in N^+_\tau(\mathcal{S})$ is an outer function and $u = 1/u^*$ on $\mathcal{S}$.*

4. Coupled Phase Retrieval Problems

In this section, we are investigating coupled phase retieval problems, i.e. problems of the form $|u| = |v|$, $|Tu| = |Tv|$ where T is some transform. This additional assumption involving T may either lead to uniqueness or at least to the reduction of the set of solutions.

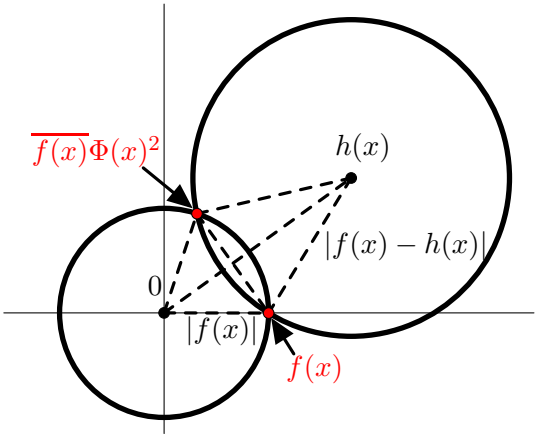
4.1. **Adding a Fixed Reference.** Klibanov *et. al.* [22] considered the following constrained problem:

$$|g| = |f| \text{ and } |g - h| = |f - h| \tag{17}$$

where h is a fixed reference signal. They were able to show that there are at most two solutions of this problem. For the following result, we look at a similar problem. It turns out that for the wide-band case, we also obtain two solutions.

Theorem 4.1. *Let $f, g \in H^2_\tau(\mathcal{S})$ and h be a nonzero complex-valued function such that $\Phi = e^{i\arg h}$ is bounded and analytic on $\mathbb{R}$. Suppose that $|g(x)| = |f(x)|$ and $|g(x) - h(x)| = |f(x) - h(x)|$ for (a.e.) $x \in \mathbb{R}$. Then there exists two solutions of this problem, namely $g(x) = f(x)$ or $g(x) = \overline{f(x)}\Phi(x)^2$, for $x \in \mathbb{R}$.*

Proof. Consider the two circles on $\mathbb{C}$: $\mathcal{C}(0, |f(x)|)$ and $\mathcal{C}(h(x), |f(x) - h(x)|)$. These two circles have two intersection points, one being $f(x)$, the other being $\overline{f(x)}\Phi(x)^2$ (eventually being the same as the first one).



The circles $\mathcal{C}(0,|f(x)|)$ and $\mathcal{C}(h(x),|f(x)-h(x)|)$.

Therefore, for each $x\in\mathbb{R}$, either $g(x)=f(x)$ or $g(x)=\overline{f(x)}\Phi(x)^2$. By the pigeonhole principle, one of these two alternatives is valid on a set of positive measure. But f,g and $\overline{f}\Phi^2$ are all analytic so that if $g=f$ on a set of positive measure, then $g=f$ everywhere, otherwise if $g=\overline{f}\Phi^2$ on a set of positive measure, then $g=\overline{f}\Phi^2$ everywhere as well. □

Remark 4.2. *If we do not assume Φ to be analytic, then $\bar{b}\Phi^2$ may not be analytic and would therefore not be a solution.*

4.2. **Pauli's Problem.** For our next result, we add a constraint involving the Fourier transforms:

$$|g|=|f| \text{ and } |\widehat{g}|=|\widehat{f}| \tag{18}$$

This problem is due to Pauli, who speculated that (18) would imply $g=cf$ for some $c\in\mathbb{T}$. However, one may construct many pairs (f,g) satisfying (18) for which this is not the case (*see e.g.*Vogt [30], Corbett and Hurst [8, 9]). Such pairs are now called *Pauli partners*. In the band-limited case, Ismagilov [17] and the first author [18] have independently shown that the set of the Pauli partners may be arbitrarily large. However, altough this is not explicitly stated in [17, 18], for a given band-limited f only finitely band-limited partners (up to trivial solutions) are constructed. The following result shows that the solution set of the Pauli problem in the wide-band case may be arbitrarily large as well and even uncountable.

Theorem 4.3. *There exists $f\in H^2_\tau(\mathcal{S})$ which has a nondenumerable infinity of Pauli partners which are not constant multiples of one another.*

Proof. The proof is a direct adaptation of [17, 18].

Let $\{\alpha_n\}_{n=0}^\infty$ be a sequence of non-zero real numbers such that $\sum_{n=1}^{+\infty}|\alpha_n|^2<\infty$ and consider the associated Riesz product

$$R_\alpha(x)=\prod_{n=1}^{\infty}\big(1+2i\alpha_n\sin(2\pi 3^n x)\big).$$

For properties of Riesz products, we refer the reader to the book of Katznelson [21]. We may write this Riesz product as a Fourier series

$$R_\alpha(x)=\sum_{k\in\mathbb{Z}}a_k e^{2\pi ikx}. \tag{19}$$

Next, let $\varphi \in L^2(\mathbb{R})$ be such that $\widehat{\varphi}$ is supported on $[0,1]$ and note that $\widehat{\varphi}$ is bounded. For all $x \in \mathbb{R}$, take $f = R_\alpha \varphi$. As

$$f(x) = \left(\sum_{k \in \mathbb{Z}} a_k e^{2\pi i k x} \right) \varphi(x),$$

we get

$$\widehat{f}(\xi) = \sum_{k \in \mathbb{Z}} a_k \widehat{\varphi}(\xi - k).$$

Now, observe that $a_k = 0$ unless there exists an integer N and $\eta_1, \ldots, \eta_N \in \{-1, 0, 1\}$ with $\eta_N \neq 0$ such that $k = \sum_{j=1}^{N} \eta_j 3^j$. Further, N and the η_j's are uniquely determined by k. In this case, a simple computation shows that $3^{N-1} \leq |k| \leq 3^{N+1}$ and that

$$|a_k| = \prod_{j=1}^{N} |\alpha_j|. \tag{20}$$

Therefore, if we chose $0 < |\alpha_j| \leq e^{-2 \cdot 3^{j+1}}$, we get

$$|a_k| \leq |\alpha_N| \leq e^{-2 \cdot 3^{N+1}} \leq e^{-2|k|}.$$

As a consequence, for $k \leq |\xi| \leq k+1$,

$$|\widehat{f}(\xi)| = |a_k| |\widehat{\varphi}(\xi - k)| \leq e^{-2|k|} \|\widehat{\varphi}\|_\infty \leq C e^{-2|\xi|}.$$

It follows that $f \in H^2_\tau(\mathcal{S})$.

Next, let $\varepsilon = \{\varepsilon_n\}_{n=1}^\infty \in \{-1,1\}^{\mathbb{N}}$ and $\alpha(\varepsilon) = \{\alpha_n \varepsilon_n\}_{n=1}^\infty$. In particular, for $\varepsilon = \mathbf{1} = (1, 1, \ldots)$, $\alpha(\mathbf{1}) = \alpha$. Observe that the associated Riesz product

$$R_{\alpha(\varepsilon)}(x) = \prod_{n=1}^{\infty} \left(1 + 2i\alpha_n \varepsilon_n \sin(2\pi 3^n x)\right) = \sum_{k \in \mathbb{Z}} a_k(\varepsilon) e^{2\pi i k x}$$

has the following properties:

- for every $x \in \mathbb{R}$, $|R_{\alpha(\varepsilon)}(x)| = |R_\alpha(x)|$;
- for every $k \in \mathbb{Z}$, $|a_k(\varepsilon)| = |a_k|$.

This last property follows directly from (20). Note also that $R_{\alpha(\varepsilon)}$ is not a constant multiple of $R_{\alpha(\varepsilon')}$ if $\varepsilon \neq \varepsilon'$.

It remains to define $f_\varepsilon = R_{\alpha(\varepsilon)} \varphi$. Then f_ε has the following properties:

- $f_\varepsilon \in H^2_\tau(\mathcal{S})$ and f_ε is not a constant multiple of $f_{\varepsilon'}$ if $\varepsilon \neq \varepsilon'$;
- $|f_\varepsilon(x)| = |f_{\varepsilon'}(x)|$ for all $x \in \mathbb{R}$;
- $|\widehat{f_\varepsilon}(\xi)| = |\widehat{f_{\varepsilon'}}(\xi)|$ since for $k \leq |\xi| \leq k+1$, $k \in \mathbb{Z}$,

$$|\widehat{f_\varepsilon}(\xi)| = |a_k(\varepsilon)| |\widehat{\varphi}(\xi - k)| = |a_k(\varepsilon')| |\widehat{\varphi}(\xi - k)| = |\widehat{f_{\varepsilon'}}(\xi)|.$$

□

4.3. **Derivation Operator.** We first look at a direct consequence of Corollary 3.8. Let $b, q \in \mathbb{R}$ with $|q| < 1$. For all $z \in \mathcal{S}$ and $f \in H^2_\tau(\mathcal{S})$, consider the operator $\dfrac{\partial}{\partial z}$ which gives the derivative of f, the operator δ given by

$$\delta(f)(z) = f(z+b) - f(z),$$

and the operator γ given by

$$\gamma(f)(z) = f(qz) - f(z).$$

The key property is that if D is one of $\dfrac{\partial}{\partial z}, \delta$ or γ and $\varphi, \psi \in H^2_\tau(\mathcal{S})$, then

$$D(\varphi \cdot \psi) = D\varphi \cdot \psi + \varphi \cdot D\psi.$$

McDonald [24, Theorem 1] considered the coupled phase retrieval problem: f, g entire, $|g(x)| = |f(x)|$ with the additional constraint $|Dg(x)| = |Df(x)|$ for $x \in \mathbb{R}$. McDonald showed that if $f = uv$ and $g = uv^*$, then $|Dg| = |Df|$ is equivalent to

$$\left(\frac{Dv}{v} - \frac{Dv^*}{v^*}\right)\left(\frac{Du^*}{u^*} - \frac{Du}{u}\right) = \frac{DfDf^* - DgDg^*}{ff^*} = 0,$$

which imposes strong restrictions on either u or v. With these, McDonald was able to significantly reduce the solution set into two solutions. As a consequence of Corollary 3.8, McDonald's result directly extends to the wide-band case. We omit the proof as it is *mutadis mutandis* the one provided by McDonald.

Corollary 4.4. *Let $f, g \in H^2_\tau(\mathcal{S})$ and D be one of the operators $\dfrac{\mathrm{d}}{\mathrm{d}x}, \delta$ or γ. Suppose that $|g(x)| = |f(x)|$ and $|Dg(x)| = |Df(x)|$ for (a.e.) $x \in \mathbb{R}$. Then:*

(i) *For the cases $D = \dfrac{\mathrm{d}}{\mathrm{d}x}$ and $D = \gamma$, either $g = \beta f$ or $g = \beta f^*$ for some constant $\beta \in \mathbb{R}$.*
(ii) *For the case $D = \delta$, either $g = Vf$ or $g = Vf^*$ where V is a meromorphic function that has period b and continuous and unimodular on $\mathbb{R}$.*

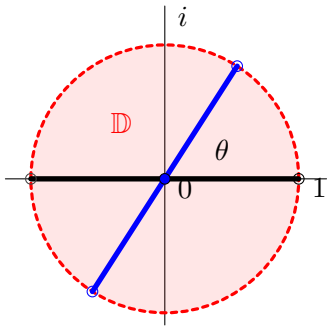
4.4. **Modulus on a Segment on $\mathcal{S}$.** In the spirit of what was done by Boche *et. al.* [5], we now consider that $|g(z)| = |f(z)|$ for z in a curve on $\mathcal{S}$. A similar idea can also be found in [19]. For this part, we add the fact that $|g(z)| = |f(z)|$ for every z on a segment lying on the strip $\mathcal{S}$. We first look at this additional constraint on the phase retrieval problem on the disc.

Lemma 4.5. *Let $f, g \in H^2(\mathbb{D})$ such that $|g(x)| = |f(x)|$ for $x \in (-1, 1)$ and*

$$|g(z)| = |f(z)| \text{ for } z \in e^{i\theta}(-1, 1) \tag{21}$$

where $\theta \notin \pi\mathbb{Q}$. Then $g = cf$ for some $c \in \mathbb{T}$.

Proof. Let $f, g \in H^2(\mathbb{D})$ and $\mathscr{Z} = Z(f) \triangle Z(g)$. Since $|g(x)| = |f(x)|$ for all $x \in (-1, 1)$, we have $\mathscr{Z} = \overline{\mathscr{Z}}$. It clearly follows that $\mathscr{Z} \cap \mathbb{R} = \emptyset$.



The disc $\mathbb{D}$ and the segment $e^{i\theta}(-1,1)$.

Since $|g(x)| = |f(x)|$ for all $x \in e^{i\theta}(-1,1)$, we have $\mathscr{Z} = \mathrm{Ref}_\theta \mathscr{Z}$ where Ref_θ refers to a reflection with respect to the segment $e^{i\theta}(-1,1)$. Hence, by composing $\mathscr{Z} = \overline{\mathscr{Z}}$ and $\mathscr{Z} = \mathrm{Ref}_\theta \mathscr{Z}$, we get that $\mathscr{Z} = \mathrm{Rot}_{2\theta}\mathscr{Z}$, where $\mathrm{Rot}_{2\theta}$ refers to a counterclockwise 2θ-rotation with respect to 0. Now, since $\theta \notin \pi\mathbb{Q}$, either $\mathscr{Z} = \emptyset$ or $\mathscr{Z}$ is uncountable. Since the zero set is discrete, $\mathscr{Z}$ cannot be uncountable, and so $\mathscr{Z} = \emptyset$. Hence, $Z(f) = Z(g)$, which implies that the Blaschke products formed by the zeros of f and g given by B_f and B_g respectively, are equal.

Now, observe that since $|g(x)| = |f(x)|$ for all $x \in (-1,1)$, Lemma 3.1 implies that for $e^{i\zeta} \in \mathbb{T}$,

$$\nu_f(e^{i\zeta}) + \nu_f(e^{-i\zeta}) = \nu_g(e^{i\zeta}) + \nu_g(e^{-i\zeta}).$$

Using this equation, the Fourier coefficients of ν_f and ν_g satisfy

$$\widehat{\nu_f}(n) + \widehat{\nu_f}(-n) = \widehat{\nu_g}(n) + \widehat{\nu_g}(-n), \text{ for all } n \in \mathbb{N}. \tag{22}$$

On the other hand, $|g(x)| = |f(x)|$ for all $x \in e^{i\theta}(-1,1)$ implies that $|f(e^{i\theta}x)| = |g(e^{i\theta}x)|$ for all $x \in (-1,1)$. For $z \in \mathbb{D}$, we now write $F(z) = f(e^{i\theta}z)$ and $G(z) = g(e^{i\theta}z)$ so that $F, G \in H^2(\mathbb{D})$ and $|F(w)| = |G(w)|$ for all $w \in (-1,1)$. Note that for $w, z \in \mathbb{D}$, we have

$$\begin{aligned} S_F(w) &= \exp\left(\int_{\mathbb{T}} \frac{w + e^{i\zeta}}{w - e^{i\zeta}} d\nu_F(e^{i\zeta})\right) \\ &= \exp\left(\int_{\mathbb{T}} \frac{ze^{i\theta} + e^{i\zeta}}{ze^{i\theta} - e^{i\zeta}} d\nu_F(e^{i\zeta})\right), \end{aligned}$$

and so by letting $u = \zeta - \theta$, we get

$$S_F(w) = \exp\left(\int_{\mathbb{T}} \frac{z + e^{iu}}{z - e^{iu}} d\nu_f(e^{i(u+\theta)})\right).$$

Thus by Lemma 3.1, we have for $e^{i\zeta} \in \mathbb{T}$,

$$\nu_f(e^{i(\theta+\zeta)}) + \nu_f(e^{i(\theta-\zeta)}) = \nu_g(e^{i(\theta+\zeta)}) + \nu_g(e^{i(\theta-\zeta)}). \tag{23}$$

Next, define the measure μ on $\mathbb{T}$ by $\mu(e^{i\zeta}) = \nu_f(e^{i(\zeta+\theta)})$ for $e^{i\zeta} \in \mathbb{T}$, with Fourier coefficients given by

$$\widehat{\mu}(n) = \int_{\mathbb{T}} e^{-in\theta} d\nu_f(e^{i(\zeta+\theta)}) = e^{in\theta}\widehat{\nu_f}(n)$$

for $n \in \mathbb{N}$. Hence, the previous equation and (23) imply that for $n \in \mathbb{N}$,

$$e^{in\theta}\widehat{\nu_f}(n) + e^{-in\theta}\widehat{\nu_f}(-n) = e^{in\theta}\widehat{\nu_g}(n) + e^{-in\theta}\widehat{\nu_g}(-n).$$

Now this equation together with (22) imply that

$$\widehat{\nu_g}(n) = \frac{e^{-in\theta}\widehat{\nu_f}(n) - e^{in\theta}\widehat{\nu_f}(n)}{e^{-in\theta} - e^{in\theta}} = \widehat{\nu_f}(n)$$

and $\widehat{\nu_g}(-n) = \widehat{\nu_f}(-n)$, for all $n \in \mathbb{N}$. It follows that $\nu_f = \nu_g$ and so $S_f = S_g$.

We now prove the same for the outer part. Since $|g(x)| = |f(x)|$ for all $x \in (-1, 1)$, Lemma 3.1 again implies that for a.e. $e^{i\zeta} \in \mathbb{T}$,

$$\log|f(e^{i\zeta})| + \log|f(e^{-i\zeta})| = \log|g(e^{i\zeta})| + \log|g(e^{-i\zeta})|.$$

For $e^{i\zeta} \in \mathbb{T}$, letting $h_f(e^{i\zeta}) = \log|f(e^{i\zeta})|$ implies that the Fourier coefficients of h_f and h_g satisfy

$$\widehat{h_f}(n) + \widehat{h_f}(-n) = \widehat{h_g}(n) + \widehat{h_g}(-n), \text{ for all } n \in \mathbb{N}. \tag{24}$$

On the other hand, by definition of F and G, we have for a.e. $e^{i\zeta} \in \mathbb{T}$,

$$\log|f(e^{i(\theta+\zeta)})| + \log|f(e^{i(\theta-\zeta)})| = \log|g(e^{i(\theta+\zeta)})| + \log|g(e^{i(\theta-\zeta)})|.$$

Using this equation and a similar argument to te one for the Fourier coefficients of the singular measures, we get that for $n \in \mathbb{N}$,

$$e^{in\theta}\widehat{h_f}(n) + e^{-in\theta}\widehat{h_f}(-n) = e^{in\theta}\widehat{h_g}(n) + e^{-in\theta}\widehat{h_g}(-n).$$

Hence, by this equation and (24) we get that $\widehat{h_g}(n) = \widehat{h_f}(n)$ for all $n \in \mathbb{Z}$. Therefore $h_f = h_g$, and so $O_f = O_g$.

Finally, since $B_f = B_g$, $S_f = S_g$ and $O_f = O_g$, we have $g = cf$ for some $c \in \mathbb{T}$. □

We now consider the coupled phase retrieval problem on the strip that includes a more general form of the constraint given in (21). Using the previous lemma, we establish the uniqueness of the solution of the following problem.

Theorem 4.6. *Let $f, g \in H^2_\tau(\mathcal{S})$ such that $|g(x)| = |f(x)|$ for $x \in \mathbb{R}$ and*

$$|g(z)| = |f(z)| \text{ for } z \in \left(-e^{i\theta} + a, e^{i\theta} + a\right)$$

where $a \in \mathbb{R}$ and $\theta \notin \pi\mathbb{Q}$. Then $g = cf$ for some $c \in \mathbb{T}$.

Proof. Without loss of generality, we let $a = 0$ so that the segment intersects the real line at the origin. Consider $f_{1/2}(z) = f(\frac{1}{2}z)$, $g_{1/2}(z) = g(\frac{1}{2}z)$ for all $z \in \mathbb{D}$. Observe that $f_{1/2}, g_{1/2} \in H^2(\mathbb{D})$, and $|g_{1/2}| = |f_{1/2}|$ on $(-1, 1)$ and on $e^{i\theta}(-1, 1)$. Hence, $g_{1/2} = cf_{1/2}$ on $\mathbb{D}$ for some $c \in \mathbb{T}$ by the Lemma 4.5, and so $g = cf$ on $\frac{1}{2}\mathbb{D}$. Therefore, since $f, g \in \mathrm{Hol}(\mathcal{S})$ and $g = cf$ on $\frac{1}{2}\mathbb{D}$ so we have $g = cf$ on $\mathcal{S}$. □

Acknowledgements. The third author is supported by a scholarship from Campus France and CHED, Philippines.

This study has been carried out with financial support from the French State, managed by the French National Research Agency (ANR) in the frame of the Investments for the future Programme IdEx Bordeaux - CPU (ANR-10-IDEX-03-02).

This paper was completed during the first author's visit at the Shrödinger Institute, Vienna, during the workshop "Operator Related Function Theory". We kindly acknoledge ESI's hospitality.

Some results in this paper have been announced in [20]. We thank the referees of that announcement for their helpfull comments that also lead to improvements here.

References

[1] E. Akutowicz, *On the determination of the phase of Fourier integral I*, Trans. Amer. Math. Soc., **83** (1956), 179–192.

[2] E. Akutowicz, *On the determination of the phase of Fourier integral II*, Proc. Amer. Math. Soc., **8** (1957), 234–238.

[3] R. Alaifari, I. Daubechies, P. Grohs and R. Yin, *Stable Phase Retrieval in Infinite Dimensions*, Found. Comput. Math., (2018). https://doi.org/10.1007/s10208-018-9399-7

[4] A. Bakan and S. Kaijser, *Hardy spaces for the strip*, J. Math. Anal. Appl., **333** (2007), 347–364.

[5] H. Boche, N. Li and V. Pohl, 'Phase retrieval in spaces of analytic functions on the unit disk' in *2017 International Conference on Sampling Theory and Applications (SampTA)*, Tallinn, Estonia, 2017.

[6] B. Bodmann and N. Hamme, *Stable phase retrieval with low-redundancy frames*, Adv. Compu. Math. **41** (2015) 317–331.

[7] E. Candès, X. Li and M. Soltanolkotabi, *Phase retrieval via Wirtinger flow: theory and algorithms*, IEEE Trans. Inform. Theory, **61** (2014), 1985–2007.

[8] J. Corbett and C. Hurst, *What is needed to determine a state*, manuscript.

[9] J. Corbett and C. Hurst, *Are wave functions uniquely determined by their position and momentum distributions?*, J. Austral. Math. Soc., **20** (1978), 182–201.

[10] P. L . Duren, *The Theory of H^p spaces*, Academic Press, New York, 1970.

[11] J. Dainty and J. Fienup, *Phase retrieval and image reconstruction for astronomy, in: H. Stark (Ed.), Image Recovery: Theory and Application*, Academic Press, 1987, 231–275.

[12] D. Dobson, *Phase reconstruction via nonlinear least squares*, Inverse Problems, **8** (1992), 541–548.

[13] Y. Fu and L. Li, *Phase retrieval of time-limited signals*, Acta Math. Sci. Ser. B (Engl. Ed.), **30** (2010), 39-46.

[14] P. Grohs, S. Koppensteiner and M. Rathmair, *The mathematics of phase retrieval*, arXiv:1901.07911.

[15] D. Han and Y. Li, *Phase retrieval of real-valued functions in Sobolev space*, Acta Math. Sin. (Engl. Ser.), **34** (2018), 1778–1794.

[16] N. Hurt, *Phase Retrieval and Zero Crossing (Mathematical Methods in Image Reconstruction)*, Math. And Its Appl., Kluwer Academic Publisher, 1989.

[17] R. Ismagilov, *On the Pauli problem*, Funksional Anal. i Prilozhen, **30** (1986), 82–84. In Russian, translation in Funct. Anal. Appl., **30** (1996), 138–140.

[18] P. Jaming, *Phase retrieval techniques for radar ambiguity problems*, J. Fourier Anal. Appl., **5** (1999), 309–329.

[19] P. Jaming, *Uniqueness results in an extension of Pauli's phase retrieval*, Appl. Comp. Harm. Anal., **37** (2014) 413-441.

[20] P. Jaming, K. Kellay & R. Perez III, *Phase Retrieval for Wide Band Signals*, SAMPTA 2019, Bordeaux, France.
[21] Y. Katznelson, *An Introduction to Harmonic Analysis*, Dover, 1976.
[22] M. Klibanov, P. Sacks and A. Tikhonravov, *The phase retrieval problem*, Inverse Prob., **11** (1995), 1–28.
[23] J. Mashreghi, *Representation Theorems in Hardy Spaces*, Cambridge University Press, 2009.
[24] J. McDonald, *Phase retrieval and magnitude retrieval of entire functions*, J. Fourier Anal. Appl., **10** (2004), 259–267.
[25] J. McDonald, *Phase retrieval of H^2-functions*, J. Math. Anal. Appl., **314** (2006), 162–173.
[26] R. Millane, *Phase retrieval in crystallography and optics*, J. Opt. Soc. Amer. A, **7** (1990), 394–411.
[27] P. Sacks, *Reconstruction of steplike potentials*, Wave Motion, **18** (1993), 21–30.
[28] B. Seifert, H. Stolz and M. Tasche, *Nontrivial ambiguities for blind frequency-resolved optical gating and the problem of uniqueness*, J. Opt. Soc. Amer. B, **21** (2004), 1089–1097.
[29] G. Thakur, *Reconstruction of bandlimited functions from unsigned samples*, J. Fourier Anal. Appl., **17** (2011), 720–732.
[30] A. Vogt, 'Position and momentum distributions do not determine the quantum mechanical state' in *Mathematical Foundations of Quantum Theory* (A. Marlow, editor), Academic Press (1978), 365–372.
[31] I. Waldspurger, A. d'Aspremont and S. Mallat, *Phase recovery, MaxCut and complex semidefinite programming*, Math. Prog., **149** (2015), 47–81.
[32] A. Walther, *The question of phase retrieval in optics*, Opt. Acta, **10** (1963), 41–49.

Ph. Jaming, Univ. Bordeaux, CNRS, Bordeaux INP, IMB, UMR 5251, F-33400, Talence, France
E-mail address: Philippe.Jaming@math.u-bordeaux.fr

K. Kellay, Univ. Bordeaux, CNRS, Bordeaux INP, IMB, UMR 5251, F-33400, Talence, France
E-mail address: kkellay@math.u-bordeaux.fr

R. Perez III , Univ. Bordeaux, CNRS, Bordeaux INP, IMB, UMR 5251, F-33400, Talence, France

Institute of Mathematics, University of the Philippines Diliman, 1101 Quezon City, Philippines
E-mail address: rolando-iii.perez@u-bordeaux.fr