HAL
open science

A Detailed Fluctuation Theorem for Heat Fluxes in Harmonic Networks out of Thermal Equilibrium

Mondher Damak, Mayssa Hammami, Claude-Alain Pillet

To cite this version:

Mondher Damak, Mayssa Hammami, Claude-Alain Pillet. A Detailed Fluctuation Theorem for Heat Fluxes in Harmonic Networks out of Thermal Equilibrium. Journal of Statistical Physics, 2020, 180 (1-6), pp.263-296. 10.1007/s10955-019-02398-x . hal-02124194v2

HAL Id: hal-02124194

https://hal.science/hal-02124194v2

Submitted on 26 Sep 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A Detailed Fluctuation Theorem for Heat Fluxes in Harmonic Networks out of Thermal Equilibrium

Mondher Damak[1] Mayssa Hammami[1,2] Claude-Alain Pillet[2]

[1]Departement of Mathematics, Faculty of Sciences of Sfax, University of Sfax, Tunisia.
[2]Aix Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France

Abstract. We continue the investigation, started in [JPS], of a network of harmonic oscillators driven out of thermal equilibrium by heat reservoirs. We study the statistics of the fluctuations of the heat fluxes flowing between the network and the reservoirs in the nonequilibrium steady state and in the large time limit. We prove a large deviation principle for these fluctuations and derive the fluctuation relation satisfied by the associated rate function.

Keywords. Large Deviations, Fluctuation Relations, Entropy Production, Heat Fluxes, Nonequilibrium Steady State.

1 Introduction

Fluctuation Relations (FRs for short) describe universal features of the statistical properties of physical systems. The first instance of such a relation goes back to 1905 and the celebrated work of Einstein on Brownian motion. Despite a few early occurrences in the literature[1], it is only after the works [ECM, ES, GC1, GC2] that FRs became a major research direction in nonequilibrium statistical mechanics, see, e.g., [Ja, Cr, LS, Ma, CG] on the theoretical side and [CCZ, CDF] on the experimental one. See also [Ga2, Ga3, Ru2] for more mathematically oriented introductions to the subject and the reviews [RM, Se, JPR, JOPP] for more exhaustive references. In this work, we shall adhere to the somewhat restrictive but mathematically precise perspective advocated by Gallavotti and Cohen in [GC1, GC2], and call FR a universal – i.e., model independent – symmetry property of the rate function describing the large deviations of some distinguished observable of a physical system in a steady state in the large time limit. As explained in [ECM, Ga1, LS, JPR], such FRs, sometimes coined *detailed FRs* in the physics literature, provide extensions of the well-known Green–Kubo and Onsager relations of linear response theory to the nonequilibrium regime.

While FRs in smooth chaotic dynamical systems on compact phase space are now pretty well understood, their status is more problematic in the presence of singularities and for general deterministic or stochastic dynamical system with non-compact phase space. As observed through the study of specific models, FRs may only have a limited domain of validity and/or acquire non-universal features in such circumstances, see [BGGZ, Fa, Vi, RT, RH].

- From the physical point of view, there are issues related to the proper choice of the relevant observable, as discussed in [JPS]. While most studies concern entropy production, work and heat transfer in time-dependent protocols involving a single heat reservoir, there is also some interests in investigating individual heat fluxes in multi-reservoir systems.

[1] see [BK], and also, in view of the now well understood connection with the thermodynamic formalism, [Ru1, Proposition 5.3.2, Equ. (3.9)].

- From the mathematical point of view, the problems are often related to the failure of standard approaches to the derivation of a large deviation principle (see the series [JNPS1]–[JNPS4] and [Ne]), or due to the technical difficulties met in applying the contraction principle, as in [BL, Section 3.4].

In order to reach a better understanding of FRs we need to further investigate simple models which allow for a clean mathematical treatment. Networks of oscillators [MN, EZ] are among the simplest candidates. While some progresses have been achieved in our understanding of the non-equilibrium dynamics of networks of anharmonic oscillators (see, e.g., the recent works [CE, CEHR]), a complete picture of FRs for these systems seems to be still out of reach of currently available techniques (to the best of our knowledge, the only partial results can be found in [RT] for chains of oscillators). The circumstances are much more favorable to networks of harmonic oscillators [CDF, KSD]. Despite being very special, the latter provide effective models for a wide range of systems and processes, from macroscopic electrical circuits [CCZ, GMR] to the microscopic dynamics of protein [HE, ADJ], including the motion of mesoscopic colloidal particles [Vi, JPC]. A novel control-theoretic approach to stochastically driven harmonic networks has been developed in [JPS]. There, FRs were obtained for various quantities related to the entropy produced by a general harmonic network driven out of equilibrium by thermal forcing (see Eq. (2.11) below for a typical result). The purpose of the present work is to continue these investigations, following the same control-theoretic strategy, and focusing on the individual energy currents flowing between the network and its environment.

Let us briefly describe the settings of [JPS] which will be used in this work. We focus on a collection, indexed by a finite set $\mathscr{I}$, of one-dimensional harmonic oscillators with position and momentum coordinates $q = (q_i)_{i\in\mathscr{I}}$ and $p = (p_i)_{i\in\mathscr{I}}$. The Hamiltonian of this system is the quadratic form

$$H(q,p) = \frac{1}{2}|p|^2 + \frac{1}{2}|\kappa q|^2,$$

where $|\cdot|$ denotes the Euclidean norm and κ an automorphism of $\mathbb{R}^{\mathscr{I}}$.

Besides the conservative harmonic forces deriving from this Hamiltonian, a subset of the oscillators, indexed by $\partial\mathscr{I} \subset \mathscr{I}$, is acted upon by thermal reservoirs. The latter are described by Langevin forces

$$f_i(p,q) = (2\gamma_i\vartheta_i)^{\frac{1}{2}}\,\dot{w}_i - \gamma_i p_i, \qquad (i \in \partial\mathscr{I}) \tag{1.1}$$

where $\vartheta_i > 0$ denotes the temperature of the ith–reservoir, $\gamma_i > 0$ the rate at which energy is dissipated in this reservoir and $\dot{w}_i$ is a standard white noise. We interpret the work[2]

$$\Phi_i(t) = \int_0^t \Big[(2\gamma_i\vartheta_i)^{\frac{1}{2}}\,p_i(s)\,\dot{w}_i(s) + \gamma_i(\vartheta_i - p_i(s)^2)\Big]\,\mathrm{d}s \tag{1.2}$$

performed by the Langevin force f_i during the time interval $[0,t]$ as the amount of heat injected in the network by the ith–reservoir during this period. We denote by $\Xi = \mathbb{R}^{\partial\mathscr{I}}$ the vector space where the heat currents $\Phi(t) = (\Phi_i(t))_{i\in\partial\mathscr{I}}$ take their values and write the associated Euclidean inner product as $\langle\xi,\Phi\rangle = \sum_{i\in\partial\mathscr{I}}\xi_i\Phi_i$.

The main results of the present work concern the statistics of the Ξ–valued process $\{\Phi(t)\}_{t\geq 0}$ induced by the *stationary* Markov process generated by the system of stochastic differential equations

$$\dot{q} = \nabla_p H(q,p), \qquad \dot{p} = -\nabla_q H(q,p) + f,$$

with appropriate initial conditions. More precisely, and under a controllability condition which ensures the existence and uniqueness of an invariant measure for this Markov process:

- We identify a subspace $\mathscr{L} \subset \Xi$ characterized by the fact that for $\xi \in \mathscr{L}$ one has

$$\langle\xi,\Phi(t)\rangle = \mathscr{Q}_\xi(q(t),p(t)) - \mathscr{Q}_\xi(q(0),p(0)) \tag{1.3}$$

[2] The integral there is to be taken in Itô's sense.

where $\mathscr{Q}_\xi$ is a quadratic form which is a first integral of the harmonic network. Applying a general result of [BJP] allows us to describe the asymptotics of (1.3) in the limit $t \to \infty$. Besides a large deviation principle (LDP for short) for the fluctuations of order t of this quantity, we also get the explicit form of its limiting distribution, which has full support. This is in sharp contrast with what would happen if $\mathscr{Q}_\xi$ was a bounded function on the phase space: the right-hand side of (1.3) – often coined a "boundary term" in the physics literature – would have no order t fluctuations and its limiting law would have compact support.

- We then focus on the component $\Phi(t)^\perp$ of the heat flux orthogonal to $\mathscr{L}$. We show that it has fluctuations of order t satisfying a local LDP whose rate function I is the (partial) Legendre transform of a real analytic function g for which we provide several explicit representations. In particular, we connect g to the spectral properties of a finite dimensional matrix and the domain of validity of the LDP to some associated algebraic Riccati equation. Both functions g and I satisfy a FR.

- We derive a simple sufficient condition, in terms of the solutions to the above mentioned Riccati equation, which ensures that our LDP and the associated FR hold globally. We also provide several examples where our condition is fulfilled. This shows that there is a regime where the components of the heat flux along the subspace $\mathscr{L}$ are responsible for the failure of the global FR for the entropy production observed in [JPS]. Our examples show, however, that our sufficient condition does not survive strong thermal forcing.

- In cases where the LDP for $\Phi(t)^\perp$ holds globally without our sufficient condition being satisfied, we show that the rate function only satisfies the universal FR on a proper subset of Ξ which is again described in terms of the solutions to the Riccati equation.

The remaining parts of the paper are organized as follows. In Section 2, we introduce a general class of harmonic networks driven out of thermal equilibrium by heat reservoirs. We describe the stochastic processes generated by their nonequilibrium dynamics and, within this probabilistic framework, we identify the fluxes of energy flowing between the network and the heat reservoirs. Then, we briefly recall some results of [JPS] on the fluctuations of the entropy produced by the networks in a nonequilibrium steady state: a large deviation principle and the associated FRs. Finally, we sketch a naive argument which will motivate the approach followed in this work.

In Section 3, we formulate our main results on the fluctuations of the heat fluxes in a nonequilibrium steady state of the network. Under a natural controllability assumption, we provide an explicit formula for the large time asymptotics of the cumulant generating function of these fluxes. Then, we describe the resulting *local* LDP for the fluctuations of the heat fluxes and the associated FRs. Finally, under an additional assumption on the network, we provide a *global* LDP with its associated FRs.

In Section 4, we provide some specific examples to which our results apply. The final Section 5 collects all the proofs of our results.

Acknowledgements. This research was supported by the Agence Nationale de la Recherche (ANR) through the grant *NONSTOPS* (ANR-17-CE40-0006) and the CNRS collaboration grant *Fluctuation theorems in stochastic systems*. The work of C.-A.P. has been carried out in the framework of the Labex *Archimède* (ANR-11-LABX-0033) and of the *A*MIDEX* project (ANR-11-IDEX-0001-02), funded by the "Investissements d'Avenir" French Government program managed by the ANR.

Parts of this work were performed during the visits of M.D. and M.H. at the University of Toulon and of C.-A.P. at the University of Sfax. We thank the CPT, the University of Toulon and the Mathematics Department of Sfax for their hospitality and support. M.H. and C.-A.P. are also grateful to the Centre de Recherches Mathématiques de l'Université de Montréal for its hospitality and the Simons foundation and CNRS for their support during their stay in Montréal in the fall 2018.

2 The model

2.1 Setup

In order to set up the notation to be used in the sequel, we briefly recall the general framework of [JPS], referring the reader to this paper for more details.

Notations and conventions. Let E and F be real or complex Hilbert spaces. $L(E,F)$ denotes the set of (continuous) linear operators $A : E \to F$ and $L(E) = L(E,E)$. For $A \in L(E,F)$, $A^* \in L(F,E)$ denotes the adjoint of A, $\|A\|$ its operator norm, $\operatorname{Ran} A \subset F$ its range and $\operatorname{Ker} A \subset E$ its kernel. We denote the spectrum of $A \in L(E)$ by $\operatorname{sp}(A)$. A is non-negative (resp. positive), written $A \geq 0$ (resp. $A > 0$), if it is self-adjoint and $\operatorname{sp}(A) \in [0,\infty[$ (resp. $\operatorname{sp}(A) \subset]0,\infty[$). We write $A \geq B$ whenever $A - B \geq 0$. A pair $(A,Q) \in L(E) \times L(F,E)$ is said to be controllable if the smallest A-invariant subspace of E containing $\operatorname{Ran} Q$ is E itself. Denoting by $\mathbb{C}_{\mp}$ the open left/right half-plane, $A \in L(E)$ is said to be stable/anti-stable whenever $\operatorname{sp}(A) \subset \mathbb{C}_{\mp}$.

We consider the harmonic network described in the Introduction. The configuration space $\mathbb{R}^{\mathscr{I}}$ is endowed with its Euclidean structure and the phase space $\Gamma = \mathbb{R}^{\mathscr{I}} \oplus \mathbb{R}^{\mathscr{I}}$ is equipped with its canonical symplectic structure. On these spaces, $|\ |$ and $\cdot$ denote the Euclidean norm and inner product, respectively. Recall that the Euclidean inner product of the space $\Xi = \mathbb{R}^{\partial\mathscr{I}}$ is written $\langle \xi, \Phi \rangle = \sum_{i\in\partial\mathscr{I}} \xi_i \Phi_i$.

Convention. We identify $\xi \in \Xi$ with the element of $L(\Xi)$ defined by

$$\xi : (u_i)_{i\in\partial\mathscr{I}} \mapsto (\xi_i u_i)_{i\in\partial\mathscr{I}}.$$

In particular, whenever we write inequalities involving such ξ, they are always to be interpreted as operator inequalities.

Introducing the linear map $\iota \in L(\mathbb{R}^{\partial\mathscr{I}}, \mathbb{R}^{\mathscr{I}})$ defined by

$$\iota : (u_i)_{i\in\partial\mathscr{I}} \mapsto (\sqrt{2\gamma_i}\, u_i)_{i\in\partial\mathscr{I}} \oplus 0_{\mathbb{R}^{\mathscr{I}\setminus\partial\mathscr{I}}},$$

we set

$$x = \begin{bmatrix} p \\ \kappa q \end{bmatrix}, \qquad A = \begin{bmatrix} -\frac{1}{2}u^* & -\kappa^* \\ \kappa & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} \iota \\ 0 \end{bmatrix} \vartheta^{\frac{1}{2}}. \tag{2.1}$$

The internal energy of the network then writes $h(x) = \frac{1}{2}|x|^2$, and its dynamics is described by the following system of Itô stochastic differential equations

$$\mathrm{d}x(t) = Ax(t)\mathrm{d}t + Q\mathrm{d}w(t), \tag{2.2}$$

w denoting a standard Ξ–valued Wiener process. The solution of the Cauchy problem associated to (2.2), with initial condition $x(0) = x_0$, can be written explicitly as

$$x(t) = \mathrm{e}^{tA}x_0 + \int_0^t \mathrm{e}^{(t-s)A} Q\mathrm{d}w(s). \tag{2.3}$$

This relation defines a family of Γ–valued Markov processes indexed by the initial condition $x_0 \in \Gamma$. The generator of this degenerate diffusion process is given by

$$L = \frac{1}{2}\nabla \cdot B\nabla + Ax \cdot \nabla, \tag{2.4}$$

where $B = QQ^*$. We shall denote $\mathbb{P}_{x_0}$ the probability measure induced on the path space $C([0,\infty[,\Gamma)$ and by $\mathbb{E}_{x_0}$ the corresponding expectation functional. Given a probability measure ν on Γ, we further set $\mathbb{P}_\nu = \int \mathbb{P}_x \nu(\mathrm{d}x)$ and define similarly $\mathbb{E}_\nu$.

For later references, we note the following structural relations

$$\mathrm{Ker}(A-A^*)=\{0\},\quad A+A^*=-Q\vartheta^{-1}Q^*,\quad Q^*Q>0,\quad [\vartheta,Q^*Q]=0. \tag{2.5}$$

Moreover, denoting by θ the time-reversal involution $(p,q)\mapsto(-p,q)$ of Γ,

$$\theta=\theta^*=\theta^{-1},\quad \theta Q=-Q,\quad \theta A\theta=A^*. \tag{2.6}$$

Setting

$$\Omega=\frac{1}{2}(A-A^*),$$

we also have

$$\theta B\theta=B^*=B,\quad \theta\Omega\theta=\Omega^*=-\Omega. \tag{2.7}$$

In the sequel, we shall further assume the Kalman condition

(C) The pair (A,Q) is controllable.

We recall (see [JPS, Theorem 3.2] and references therein) that under this assumption the process (2.3) admits a unique invariant measure μ, the centered Gaussian measure on Γ with covariance

$$M=\int_0^\infty \mathrm{e}^{sA}B\mathrm{e}^{sA^*}\,\mathrm{d}s, \tag{2.8}$$

which is also characterized as the unique solution of the Lyapunov equation $AM+MA^*+B=0$.

Remark. If the environment is in thermal equilibrium at temperature $T_0>0$, i.e., if $\vartheta=T_0I_\Xi$, then $M=T_0I_\Gamma$ and μ is the Gibbs measure $\mu(\mathrm{d}x)\propto \mathrm{e}^{-h(x)/T_0}\mathrm{d}x$.

2.2 Heat fluxes and entropy production

Following [JPS], we shall interpret the work (1.2) performed by the Langevin force during the time interval $[0,t]$ as the amount of heat injected in the network by the ith–reservoir during this period. In [JPS], a LDP and extended fluctuation relations were proven for the total amount of entropy dissipated in the reservoirs, i.e., the entropy produced by the network

$$\mathfrak{S}(t)=-\langle\vartheta^{-1},\Phi(t)\rangle. \tag{2.9}$$

Let us briefly recall these results.

Assuming the Kalman condition **(C)**, the family $\{\mathfrak{S}(t)\}_{t\geq 0}$ satisfies a global LDP with a good rate function $I:\mathbb{R}\to[0,+\infty]$, i.e., for any Borel set $S\subset\mathbb{R}$, one has

$$-\inf_{s\in\dot{S}}I(s)\leq\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{P}_\mu[t^{-1}\mathfrak{S}(t)\in S]\leq\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}_\mu[t^{-1}\mathfrak{S}(t)\in S]\leq-\inf_{s\in\bar{S}}I(s)$$

where $\dot{S}$ and $\bar{S}$ denote respectively the interior and the closure of S. The mean entropy production rate

$$\mathrm{ep}=\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_\mu[\mathfrak{S}(t)] \tag{2.10}$$

exists and is non-negative. Whenever $\mathrm{ep}>0$, the rate function satisfies the FR

$$I(-s)-I(s)=s \tag{2.11}$$

for $|s|\leq\mathrm{ep}$. However, this universal relation fails for $|s|>\mathrm{ep}$ where, instead, a model dependent *extended fluctuation relation* holds (see [JPS, Section 3.7]).

Our aim here is to derive a LDP and FRs for the individual heat fluxes $\Phi(t)=(\Phi_i(t))_{i\in\partial\mathscr{I}}$. To motivate our approach, let us sketch a naive argument leading to the desired results.

The Gärtner-Ellis theorem (see, e.g., [DZ] or [dH]) is a well-known route to the LDP. To follow it, one has to show the existence and some regularity properties of the large time limit

$$e(\xi) = \lim_{t\to\infty} \frac{1}{t} g_t(\xi) \tag{2.12}$$

of the cumulant generating function

$$g_t(\xi) = \log \mathbb{E}_\mu \left[\mathrm{e}^{\langle \xi, \Phi(t)\rangle} \right]. \tag{2.13}$$

A simple calculation yields the expression

$$\langle \xi, \Phi(t)\rangle = \frac{t}{2}\operatorname{tr}(Q\xi Q^*) + \int_0^t \left[x(s)\cdot Q\xi \mathrm{d}w(s) - \frac{1}{2} x(s)\cdot Q\vartheta^{-1/2}\xi\vartheta^{-1/2}Q^* x(s)\mathrm{d}s \right]. \tag{2.14}$$

By Itô calculus, one has

$$\mathrm{d}(\mathrm{e}^{\langle \xi,\Phi(t)\rangle} f(x(t))) = \mathrm{e}^{\langle \xi,\Phi(t)\rangle} \left[(L_\xi f)(x(t))\mathrm{d}t + \langle \xi Q^* x(t) f(x(t)) + Q^*\nabla f(x(t)), \mathrm{d}w(t)\rangle \right],$$

where

$$L_\xi = \frac{1}{2}\nabla\cdot B\nabla + A_\xi x\cdot\nabla - \frac{1}{2} x\cdot C_\xi x + \frac{1}{2}\operatorname{tr}(Q\xi Q^*),$$

is a deformation of the Markov generator (2.4), the matrices A_ξ and C_ξ being given by

$$A_\xi = A + Q\xi Q^*, \qquad C_\xi = Q\xi(\vartheta^{-1} - \xi)Q^*. \tag{2.15}$$

A naive application of Girsanov formula yields

$$\mathbb{E}_x \left[\mathrm{e}^{\langle \xi,\Phi(t)\rangle} f(x(t)) \right] = \left(\mathrm{e}^{tL_\xi} f \right)(x),$$

and in particular

$$g_t(\xi) = \log \int (\mathrm{e}^{tL_\xi} 1)(x)\mu(\mathrm{d}x).$$

Given the specific form of L_ξ and the fact that it generates a positivity preserving semigroup, it is natural to seek an eigenvector Ψ_ξ to its dominant eigenvalue $\lambda_\xi = \max\{\operatorname{Re}\lambda \mid \lambda \in \operatorname{sp}(L_\xi)\}$ in the Gaussian form

$$\Psi_\xi(x) = \mathrm{e}^{-\frac{1}{2}x\cdot X_\xi x}.$$

A simple calculation shows that the eigenvalue problem splits into the following algebraic Riccati equation for a symmetric matrix X,

$$\mathscr{R}_\xi(X) \equiv XBX - XA_\xi - A_\xi^* X - C_\xi = 0, \tag{2.16}$$

and the relation

$$\lambda_\xi = \frac{1}{2}\operatorname{tr}(Q\xi Q^* - BX_\xi)$$

where X_ξ denotes the *maximal*[3] solution of (2.16). From the structural relations (2.5)–(2.7) and the fact that $[\xi, \vartheta] = 0$ one easily deduces that the formal adjoint L_ξ^* of L_ξ is given by

$$L_\xi^* = \Theta L_{\vartheta^{-1}-\xi}\Theta, \tag{2.17}$$

where the map Θ is defined by $\Theta f = f\circ\theta$.

Assuming L_ξ to have a non-vanishing spectral gap, we obtain

$$\int (\mathrm{e}^{tL_\xi} 1)(x)\mu(\mathrm{d}x) = \mathrm{e}^{t\lambda_\xi}\left(d_\xi + o(1)\right)$$

[3] X_ξ is maximal whenever $X \le X_\xi$ for all self-adjoint X such that $\mathscr{R}_\xi(X) = 0$.

as $t \to \infty$, and hence

$$\lim_{t\to\infty} \frac{1}{t} g_t(\xi) = \lambda_\xi,$$

provided the prefactor

$$d_\xi = \frac{\int \Psi_{\vartheta^{-1}}(\theta x)\Psi_{\vartheta^{-1}-\xi}(\theta y)\Psi_\xi(x)\mathrm{d}x\mathrm{d}y}{\int \Psi_{\vartheta^{-1}}(\theta x)\Psi_{\vartheta^{-1}-\xi}(\theta y)\Psi_\xi(y)\mathrm{d}x\mathrm{d}y} = \frac{\det(X_{\vartheta^{-1}})^{1/2}\det(X_\xi + \theta X_{\vartheta^{-1}-\xi}\theta)^{1/2}}{\det(X_\xi + \theta X_{\vartheta^{-1}}\theta)^{1/2}\det(X_{\vartheta^{-1}-\xi})^{1/2}} \tag{2.18}$$

is finite and positive.[4] The fluctuation relation

$$\lambda_{\vartheta^{-1}-\xi} = \lambda_\xi \tag{2.19}$$

then follows from (2.17).

Assuming the limiting cumulant generating function $\xi \mapsto \lambda_\xi$ to be everywhere differentiable on Ξ, the Gärtner-Ellis theorem yields the LDP

$$-\inf_{\varphi\in\dot{F}} I(\varphi) \le \liminf_{t\to\infty} \frac{1}{t}\log\mathbb{P}_\mu[t^{-1}\Phi(t)\in F] \le \limsup_{t\to\infty} \frac{1}{t}\log\mathbb{P}_\mu[t^{-1}\Phi(t)\in F] \le -\inf_{\varphi\in\bar{F}} I(\varphi),$$

where $\dot{F}/\bar{F}$ denotes the interior/closure of the Borel set $F \subset \Xi$, the rate function being given by the Legendre transform

$$I(\varphi) = \sup_{\xi\in\Xi}(\langle\xi,\varphi\rangle - \lambda_\xi).$$

Relation (2.19) thus translates into the FR

$$I(-\varphi) - I(\varphi) = -\langle\vartheta^{-1},\varphi\rangle, \tag{2.20}$$

where we recognize, in the right-hand side, the entropy production rate corresponding to the heat flux φ.

There are several issues with the above formal derivation. In particular, we can't expect the fluctuation relation (2.20) (resp. (2.19)) to hold for all values of $\varphi \in \Xi$ (resp. for all values of $\xi \in \Xi$). Indeed, by the contraction principle, the validity of (2.20) for all $\varphi \in \Xi$ would entail the validity of (2.11) for all $s \in \mathbb{R}$, in contradiction with the above mentioned result of [JPS]. The main contribution of the present work is to provide a rigorous proof of a large deviation principle for heat fluxes, an explicit formula for the rate function I and a description of the domain of validity of the universal relation (2.20).

3 Main results

3.1 The limiting cumulant generating function

In this paragraph, we first formulate the generalized detailed balance relation (see [EPR, Section 4.3] and [BL, Section 2.2]) which plays a central role in our analysis. Then we state our main result on the large time asymptotics of the cumulant generating function (2.13).

Proposition 3.1. *Given $\xi \in \Xi$, we shall write $\widetilde{\xi} \rhd \xi$ whenever $\widetilde{\xi} \in L(\Gamma)$ is self-adjoint and satisfies*

$$\widetilde{\xi}Q = Q\xi, \quad \theta\widetilde{\xi}\theta = \widetilde{\xi}. \tag{3.1}$$

To such a $\widetilde{\xi}$, we associate the quadratic forms

$$\mathcal{Q}_{\widetilde{\xi}}(x) = \frac{1}{2}x\cdot\widetilde{\xi}x,$$

[4] Here, we used the fact that the steady state covariance M satisfies $M^{-1} = \theta X_{\vartheta^{-1}}\theta$.

and

$$\sigma_{\widetilde{\xi}}(x) = \frac{1}{2} x \cdot \Sigma_{\widetilde{\xi}} x, \qquad \Sigma_{\widetilde{\xi}} = [\Omega, \widetilde{\xi}],$$

and the measure $\mu_{\widetilde{\xi}}$ *on* Γ *defined by*

$$\frac{\mathrm{d}\mu_{\widetilde{\xi}}}{\mathrm{d}x}(x) = \mathrm{e}^{-\mathscr{Q}_{\widetilde{\xi}}(x)}.$$

Then, the following assertions hold:

(1) $\mu_{\widetilde{\xi}}$ *and* $\sigma_{\widetilde{\xi}}$ *satisfy*

$$\mu_{\widetilde{\xi}} \circ \theta = \mu_{\widetilde{\xi}}, \qquad \sigma_{\widetilde{\xi}} \circ \theta = -\sigma_{\widetilde{\xi}}.$$

(2) Denote by $L^{\widetilde{\xi}}$ *the formal adjoint of the Markov generator* (2.4) *w.r.t. the inner product of the Hilbert space* $L^2(\Gamma, \mu_{\widetilde{\xi}})$*. Then the generalized detailed balance relation*

$$\Theta L^{\widetilde{\xi}} \Theta = L + \sigma_{\widetilde{\xi}}$$

holds.

(3) There exists $\widetilde{\xi} \in L(\Gamma)$ *satisfying* (3.1) *and such that* $\Sigma_{\widetilde{\xi}} = 0$ *iff*

$$\mathrm{e}^{-\mathscr{Q}_{\widetilde{\xi}}(x)} L_\eta \mathrm{e}^{\mathscr{Q}_{\widetilde{\xi}}(x)} = L_{\eta+\xi}$$

holds for all $\eta \in \Xi$*. Moreover, under Condition* **(C)***, such a* $\widetilde{\xi}$*, if it exists, is unique and satisfies*

$$\mathrm{sp}(\widetilde{\xi}) = \mathrm{sp}(\xi).$$

(4) The functional (2.14) *can be written as*

$$\langle \xi, \Phi(t) \rangle = \mathscr{Q}_{\widetilde{\xi}}(x(t)) - \mathscr{Q}_{\widetilde{\xi}}(x(0)) + \int_0^t \sigma_{\widetilde{\xi}}(x(s)) \mathrm{d}s.$$

Given the structure of the map Q and the diagonal nature of ξ, the existence of $\widetilde{\xi} \in L(\Gamma)$ satisfying (3.1) is obvious. Apart from Part (3), the proof of the previous proposition is identical to the elementary proof of Proposition 3.5 in [JPS] and we omit it. The first statement in (3) follows from an explicit calculation. One easily checks that $\Sigma_{\widetilde{\xi}} = 0$ is equivalent to $[A, \widetilde{\xi}] = 0$, which implies that $\widetilde{\xi} A^n Q = A^n Q \xi$ for any $n \geq 0$. Thus, Condition **(C)** immediately yields the uniqueness of $\widetilde{\xi}$. Similarly, one deduces from the relation $(\widetilde{\xi} - z)^{-1} A^n Q = Q A^n (\xi - z)^{-1}$, obviously valid for $z \in \mathbb{C} \setminus \mathbb{R}$, that $\widetilde{\xi}$ and ξ have the same spectrum.

Proposition 3.2. *Assume that Condition* **(C)** *holds.*

(1) For $\xi \in \Xi$ *and* $\omega \in \mathbb{R}$*, the operator*

$$E_\xi(\omega) = Q^* (A^* - \mathrm{i}\omega)^{-1} \Sigma_{\widetilde{\xi}} (A + i\omega)^{-1} Q \tag{3.2}$$

is self-adjoint on the complexification of Ξ *and does not depend on the choice of* $\widetilde{\xi} \rhd \xi$*. Moreover, the map* $\mathbb{R} \times \Xi \ni (\omega, \xi) \mapsto E_\xi(\omega)$ *is continuous.*

(2) The set

$$\mathscr{D} = \bigcap_{\omega \in \mathbb{R}} \{\xi \in \Xi \mid I - E_\xi(\omega) > 0\}$$

is open, convex, centrally symmetric around the point $(2\vartheta)^{-1}$ *and contains*

$$\mathscr{D}_0 = \{\xi \in \Xi \mid 0 < \xi < \vartheta^{-1}\}.$$

Its lineality space[5] *is given by*

$$\mathscr{L} = \bigcap_{\omega\in\mathbb{R}} \{\xi \in \Xi \,|\, E_\xi(\omega) = 0\} = \{\xi \in \Xi \,|\, \Sigma_{\widetilde{\xi}} = 0 \text{ for some } \widetilde{\xi} \rhd \xi\}, \tag{3.3}$$

and in particular $\mathbf{1} = (1, 1, \ldots, 1) \in \mathscr{L}$.

(3) *The function*

$$g(\xi) = -\int_{-\infty}^{+\infty} \log\det(I - E_\xi(\omega)) \frac{\mathrm{d}\omega}{4\pi},$$

is convex and real analytic on $\mathscr{D}$. *It is centrally symmetric w.r.t. the point* $(2\vartheta)^{-1}$ *and translation invariant in the direction* $\mathscr{L}$, *i.e.,*

$$g(\vartheta^{-1} - \xi) = g(\xi) = g(\xi + \eta) \tag{3.4}$$

for all $\xi \in \mathscr{D}$ *and* $\eta \in \mathscr{L}$. *In particular,* $g(0) = g(\vartheta^{-1}) = 0$.

(4) *One has*

$$\mathscr{L} = \nabla g(\mathscr{D})^\perp,$$

and the following alternative holds: Either $\mathscr{D} = \mathscr{L} = \Xi$ *and* g *vanishes identically, or* $\mathscr{L}^\perp \neq \{0\}$ *and* g *is strictly convex on the section* $\mathscr{S} = \mathscr{D} \cap \mathscr{L}^\perp$, *the closure of* $\mathscr{S}$ *being a compact convex subset of* $\mathscr{L}^\perp$.

(5) *For* $\xi \in \Xi$, *define*

$$K_\xi = \begin{bmatrix} -A_\xi & QQ^* \\ C_\xi & A_\xi^* \end{bmatrix}, \tag{3.5}$$

where A_ξ *and* C_ξ *are given by* (2.15). *Then* $\mathscr{D}$ *is the connected component of the point* $\xi = 0$ *in the set*

$$\{\xi \in \Xi \,|\, \mathrm{sp}(K_\xi) \cap \mathrm{i}\mathbb{R} = \emptyset\}.$$

Moreover, for any $(\omega, \xi) \in \mathbb{R} \times \mathscr{D}$ *one has*

$$\det(K_\xi - \mathrm{i}\omega) = |\det(A + \mathrm{i}\omega)|^2 \det(I - E_\xi(\omega)).$$

(6) *The function* g *has a bounded continuous extension to the closed set* $\overline{\mathscr{D}}$ *which is given by*

$$g(\xi) = \frac{1}{4}\mathrm{tr}(Q\vartheta^{-1}Q^*) - \frac{1}{4}\sum_{\lambda\in\mathrm{sp}(K_\xi)} |\mathrm{Re}\,\lambda| m_\lambda, \tag{3.6}$$

where m_λ *denotes the algebraic multiplicity of* $\lambda \in \mathrm{sp}(K_\xi)$.

(7) *For any finite* $\xi_0 \in \partial\mathscr{D}$ *one has*

$$\lim_{\mathscr{D}\ni\xi\to\xi_0} |\nabla g(\xi)| = \infty.$$

Thus, setting $g(\xi) = +\infty$ *for* $\xi \in \Xi \setminus \overline{\mathscr{D}}$ *yields an essentially smooth, essentially strictly convex, closed, proper convex function* $g : \Xi \to]-\infty, +\infty]$.

(8) *For all* $\xi \in \overline{\mathscr{D}}$ *the Riccati equation* (2.16) *has a maximal self-adjoint solution* X_ξ. *The map* $\xi \mapsto X_\xi$ *is continuous and concave on* $\overline{\mathscr{D}}$, *and*

$$g(\xi) = -\frac{1}{2}\mathrm{tr}(Q^*(X_\xi - \widetilde{\xi})Q). \tag{3.7}$$

Moreover, setting $D_\xi = A_\xi - BX_\xi$, *the pair* (D_ξ, Q) *is controllable and* $\mathrm{sp}(D_\xi) = \mathrm{sp}(K_\xi) \cap \overline{\mathbb{C}_-}$.

[5] The lineality space of a convex set $\mathscr{C} \subset \mathbb{R}^n$ is the set of vectors $y \in \mathbb{R}^n$ such that $x + \lambda y \in \mathscr{C}$ for all $x \in \mathscr{C}$ and all $\lambda \in \mathbb{R}$, see [Ro]

The lineality subspace $\mathscr{L}$ is related to conservation laws of the harmonic network. Indeed, under its Hamiltonian dynamics, the network evolves according to $x_t = \mathrm{e}^{t\Omega}x_0$ and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{Q}_{\widetilde{\xi}}(x_t) = \sigma_{\widetilde{\xi}}(x_t) = 0$$

for $\xi \in \mathscr{L}$. It follows that, for any $\xi \in \mathscr{L}$, the quadratic form $\mathscr{Q}_{\widetilde{\xi}}$ is a first integral of the Hamiltonian flow. In particular the direction $\mathbf{1} \in \mathscr{L}$ and the invariance $g(\xi + \lambda\mathbf{1}) = g(\xi)$ is related to the conservation of the total energy of the network, $h = \mathscr{Q}_I$. This symmetry of the cumulant generating function of currents was already described, in the quantum setting, in [AGMT], see also [BPP] for a detailed discussion.

It follows from [JPS, Theorem 3.13] that the entropy production rate of the network (2.10) is related to the function g by

$$\mathrm{ep} = -\langle \vartheta^{-1}, \nabla g(0)\rangle.$$

Thus, $\mathrm{ep} = 0$ whenever the alternative $\mathscr{L} = \Xi$ in Part (4) of Proposition 3.2 holds. In the following we shall avoid trivialities assuming, without further notice, that $\mathrm{ep} > 0$ and hence $\mathscr{L}^\perp \neq \{0\}$ and

$$\mathscr{D} = \mathscr{S} \oplus \mathscr{L}$$

with $\mathscr{S} = \mathscr{D} \cap \mathscr{L}^\perp$. By Proposition 3.2 (8) the functions

$$\Lambda_-(\xi) = -\min \mathrm{sp}(X_\xi + \theta X_{\vartheta^{-1}}\theta), \qquad \Lambda_+(\xi) = \min \mathrm{sp}(X_{\vartheta^{-1}-\xi}),$$

are continuous and respectively convex/concave on $\overline{\mathscr{D}}$. The function g of the preceding proposition is related to the limiting cumulant generating function (2.12) by the following

Proposition 3.3. *Under Assumption* **(C)** *one has*

$$e(\xi) = \lim_{t\to\infty}\frac{1}{t}g_t(\xi) = \begin{cases} g(\xi) & \text{for } \xi \in \mathscr{D}_\infty \\ +\infty & \text{for } \xi \in \Xi \setminus \overline{\mathscr{D}_\infty}, \end{cases} \tag{3.8}$$

where (compare this with the right-hand side of (2.18)*)*

$$\mathscr{D}_\infty = \{\xi \in \mathscr{D} \mid \Lambda_-(\xi) < 0 < \Lambda_+(\xi)\}$$

is a bounded, open, convex subset of $\mathscr{D}$ such that

$$\overline{\mathscr{D}_0} \setminus \{\vartheta^{-1}\} \subset \mathscr{D}_\infty.$$

In particular, $\mathscr{D}_\infty$ contains a neighborhood of 0.

3.2 Fluctuations of conserved quantities

As mentioned above, each $\xi \in \mathscr{L}$ is associated to a first integral $\mathscr{Q}_{\widetilde{\xi}}$ of the harmonic network. In this section, we briefly focus on these conserved quantities. Since $\xi_\lambda = \xi + \lambda\mathbf{1} \in \mathscr{L}$ and $\widetilde{\xi}_\lambda = \widetilde{\xi} + \lambda I > 0$ for $\lambda \in \mathbb{R}$ large enough, there is no loss of generality in assuming that $\mathscr{Q}_{\widetilde{\xi}} \geq 0$. It follows from Proposition 3.1 (4) and [BJP, Proposition 2.2] that the law of

$$\langle \xi, \Phi(t)\rangle = \mathscr{Q}_{\widetilde{\xi}}(x(t)) - \mathscr{Q}_{\widetilde{\xi}}(x(0))$$

under $\mathbb{P}_\mu$ converges towards a *variance-gamma* distribution

$$\lim_{t\to\infty}\mathbb{P}_\mu[\langle \xi, \Phi(t)\rangle \in S] = \int_S f_{\mathrm{vg}}(q)\mathrm{d}q,$$

with density

$$f_{\mathrm{vg}}(q) = |q|^{(m-1)/2} \int_{S^{m-1}} K_{(m-1)/2}\left(\frac{|q|}{|Nk|}\right) \frac{\mathrm{d}\sigma(k)}{(2\pi|Nk|)^{(m+1)/2}},$$

where[6] $m = 2|\mathscr{I}|$, $N = \widetilde{\xi}^{1/2} M \widetilde{\xi}^{1/2}$, σ is the Lebesgue measure on the unit sphere S^{m-1} of Γ, and K denotes a modified Bessel function [W]. As mentioned in the Introduction, the variance-gamma distribution as full support on $\mathbb{R}$. Moreover, the above convergence is accompanied by a LDP: for any open set $O \subset \mathbb{R}$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_\mu[t^{-1}\langle \xi, \Phi(t)\rangle \in O] = -\inf_{q\in O} I(q)$$

with the rate function

$$I(q) = \frac{|q|}{\max \mathrm{sp}(N)}.$$

This applies, in particular, to the fluctuations of the total energy which was the primary concern in [BJP].

3.3 A local Fluctuation Theorem

While the results of the previous section quantify departures from the conservation laws, the main results of this paper deal with the component of the heat currents fluctuations which do not violate these conservation laws.

Recall that $\mathscr{S} = \mathscr{D} \cap \mathscr{L}^\perp$ is the (precompact, convex) base of the cylinder $\mathscr{D} \subset \Xi$. Denoting by $\Pi \in L(\Xi)$ the orthogonal projection on $\mathscr{L}^\perp$, setting $\mathscr{S}_\infty = \Pi \mathscr{D}_\infty$ and defining the function $I : \mathscr{L}^\perp \to [0, +\infty[$ by

$$I(\varphi) = \sup_{\xi \in \mathscr{S}_\infty} \big(\langle \xi, \varphi\rangle - g(\xi)\big), \tag{3.9}$$

a direct application of the Gärtner-Ellis theorem yields the following

Theorem 3.4. *Assume that Condition* **(C)** *holds. Then, under the law* $\mathbb{P}_\mu$*, the family* $\{\Pi\Phi(t)\}_{t\geq 0}$ *satisfies a local LDP with the good rate function* I *given by* (3.9)*, i.e., for any Borel set* $F \subset \mathscr{L}^\perp$*, one has*

$$-\inf_{\varphi\in \dot{F}\cap\mathscr{F}} I(\varphi) \leq \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}_\mu[t^{-1}\Pi\Phi(t) \in F] \leq \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_\mu[t^{-1}\Pi\Phi(t) \in F] \leq -\inf_{\varphi\in\bar{F}} I(\varphi), \tag{3.10}$$

where $\dot{F}$ *and* $\bar{F}$ *denote respectively the interior and the closure of* F *and*

$$\mathscr{F} = \nabla g(\mathscr{S}_\infty).$$

Moreover, for $\varphi \in \mathscr{F}_0 = \{\nabla g(\xi) \,|\, \xi \in \mathscr{S}_\infty \text{ and } \vartheta^{-1} - \xi \in \mathscr{S}_\infty\} \supset \nabla g(\mathscr{D}_0)$*, the fluctuation relation*

$$I(-\varphi) = I(\varphi) - \langle \vartheta^{-1}, \varphi\rangle,$$

holds.

3.4 A global Fluctuation Theorem

To improve on Theorem 3.4 and obtain a global LDP on $\mathscr{L}^\perp$, we impose a further condition on the network:

(R) $\min_{\xi\in\partial\mathscr{S}} (\Lambda_+(\xi) - \Lambda_-(\xi)) > 0.$

Since $\Lambda_+ - \Lambda_-$ is a concave function of ξ on $\mathscr{S}$, Condition **(R)** ensures that it is positive on $\mathscr{S}$.

[6] Recall that M, given in (2.8), is the covariance of the invariant measure μ.

Theorem 3.5. *Assume that Conditions* **(C)** *and* **(R)** *hold. Then, under the law* $\mathbb{P}_\mu$*, the family* $\{\Pi W(t)\}_{t\geq 0}$ *satisfies a global LDP with the good rate function* I *given by* (3.9)*, i.e., for any Borel set* $F \subset \mathscr{L}^\perp$*, one has*

$$-\inf_{\varphi\in\dot{F}} I(\varphi) \leq \liminf_{t\to\infty} \frac{1}{t}\log\mathbb{P}_\mu[t^{-1}\Pi W(t)\in F] \leq \limsup_{t\to\infty} \frac{1}{t}\log\mathbb{P}_\mu[t^{-1}\Pi W(t)\in F] \leq -\inf_{\varphi\in\bar{F}} I(\varphi), \tag{3.11}$$

where $\dot{F}$ *and* $\bar{F}$ *denote respectively the interior and the closure of* F*. Moreover, the fluctuation relation*

$$I(-\varphi) = I(\varphi) - \langle \vartheta^{-1}, \varphi\rangle, \tag{3.12}$$

holds for all $\varphi \in \mathscr{L}^\perp$*.*

Remark. We stress that Theorem 3.5 only gives sufficient conditions for the global validity of the LDP. We conjecture that Condition **(C)** alone is sufficient for (3.11) to hold for all Borel sets $F \subset \mathscr{L}^\perp$ (with the rate function I given by (3.9)). However, we were not able to prove this claim, and in particular we are not aware of any general result in the theory of large deviations which would imply it. We leave this conjecture as an interesting open problem.

Discussion. 1. The quantity $-\langle\vartheta^{-1},\varphi\rangle$ is the entropy production rate associated with a given heat current $\varphi \in \mathscr{L}^\perp$ (compare with (2.9)). Thus, the FR (3.12) implies that current fluctuations φ with negative entropy production rate are exponentially suppressed, as $t \to \infty$, relative to the opposite fluctuation $-\varphi$ (which has positive entropy production rate).

2. We note that, according to the results of [JPS], the entropy production (2.9) *never* satisfies the FR (2.11) globally (i.e., for all $s \in \mathbb{R}$). This is in sharp contrast with the component of the heat flux along $\mathscr{L}^\perp$. Indeed, we will see in the next section that a global FR is possible in this case.

3. Suppose that, in view of the above conjecture, Condition **(R)** being violated, the LDP (3.11) holds for all Borel sets $F \subset \mathscr{L}^\perp$. It follows that the FR (3.12) does not hold for all $\varphi \in \mathscr{L}^\perp$ but is replaced by an extended – i.e., non-universal – FR on the unbounded set

$$\{\varphi \in \mathscr{L}^\perp \,|\, \text{either } \varphi \notin \nabla g(\mathscr{S}_\infty) \text{ or } -\varphi \notin \nabla g(\mathscr{S}_\infty)\},$$

as illustrated in Figure 3, below. The graph of the rate function in the region $\mathscr{L}^\perp \setminus \nabla g(\mathscr{S}_\infty)$ has the peculiar form of a ruled surface. More precisely, for $\xi \in \partial\mathscr{S}_\infty$ let $\varphi_0 = \nabla g(\xi)$ and denote by η the exterior normal to $\mathscr{S}_\infty$ at ξ. Then, one has $I(\varphi_0 + \lambda\eta) = I(\varphi_0) + \lambda\eta\cdot\xi$ for all $\lambda > 0$. This behavior generalizes to the multivariate case the affine character of the rate function of scalar observables (heat, entropy production,...) found in [CZ, BJTM, Fa, RH, Vi, JPS]. In these circumstances, the rate function is convex on $\mathscr{L}^\perp$ but fails to be strictly convex on the complement of $\nabla g(\mathscr{S}_\infty)$. As in the scalar cases mentioned above, the somewhat striking consequence of these *extended FRs* is an increase of the probability ratio for current fluctuations with negative to positive entropy production rate.

4. From a mathematical perspective, deviations from the FR (3.12) are due to the lack of essential smoothness of the limiting cumulant generating function (3.8). In the setting of the present work, this can be traced back to the divergence of a pre-exponential factor in the asymptotic expansion of the cumulant generating function (the term d_t in (5.13) below). The analysis of [EN, RH] suggests that this phenomenon may remain relevant beyond the harmonic/Gaussian setting, however we are not aware of any rigorous result in this direction.

5. From the physical point of view, the circumstances leading to the failure of the global FR (3.12) are still not well understood. However, the examples below tend to indicate that the strength of the thermal drive (and hence large currents) is one determinant factor.

4 Examples

Observe that Eq. (3.2) implies that the matrix $E_\xi(\omega)$ and thence the function $g(\xi)$ are invariant under the simultaneous rescaling

$$\vartheta_i \mapsto \lambda\vartheta_i, \qquad \xi_i \mapsto \lambda^{-1}\xi_i$$

with $\lambda > 0$. Furthermore, one easily checks that, under the same rescaling, the maximal solution to the Riccati equation (2.16) obeys $X_\xi \mapsto \lambda X_\xi$, so that $\Lambda_\pm(\xi) \mapsto \lambda\Lambda_\pm(\xi)$. Consequently, without losing in generality, we shall fix the average temperature according to

$$\frac{1}{|\partial\mathscr{I}|}\sum_{i\in\partial\mathscr{I}} \vartheta_i^{-1} = 1$$

in all our examples, denoting temperature ratios by $[\vartheta_1 : \vartheta_2 : \cdots]$. For systems out of thermal equilibrium, we shall also use a special system of cartesian coordinates on the space $\mathscr{L}^\perp$: we set its origin at the orthogonal projection of the symmetry center $\big((2\vartheta_i)^{-1}\big)_{i\in\partial\mathscr{I}}$ on $\mathscr{L}^\perp$, and chose the first basis vector along the same direction. Finally, we note that in all the examples below, it is straightforward to check that Condition **(C)** is satisfied and that $\mathscr{L} = \mathbb{R}\mathbf{1}$. We shall therefore concentrate our discussions on the validity of Condition **(R)**.

4.1 A lozenge network

As a first example of numerical exploitation of our scheme, we investigate some properties of the Lozenge network of Figure 1. With $|\mathscr{I}| = 4$ and $|\partial\mathscr{I}| = 3$, the parameters of the model are given by

$$\kappa^2 = \begin{bmatrix} 1 & 0 & \varepsilon & \varepsilon \\ 0 & 1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 & 0 \\ \varepsilon & \varepsilon & 0 & 1 \end{bmatrix}, \qquad \varepsilon = \frac{1}{2\sqrt{2}}, \qquad \gamma_1 = \gamma_2 = \gamma_3 = 1.$$

Consider first the case of thermal equilibrium: $[1 : 1 : 1]$. The mean heat fluxes vanish, $\bar{\varphi} = 0$. From the right pane of figure 1, which shows the functions $\mathscr{S} \ni \xi \to \Lambda_\pm(\xi)$, one infers that Condition **(R)** is verified so that, by Theorem 3.5, the global LDP (3.11) holds with a rate function I satisfying the FR (2.20) on $\mathscr{L}^\perp$.

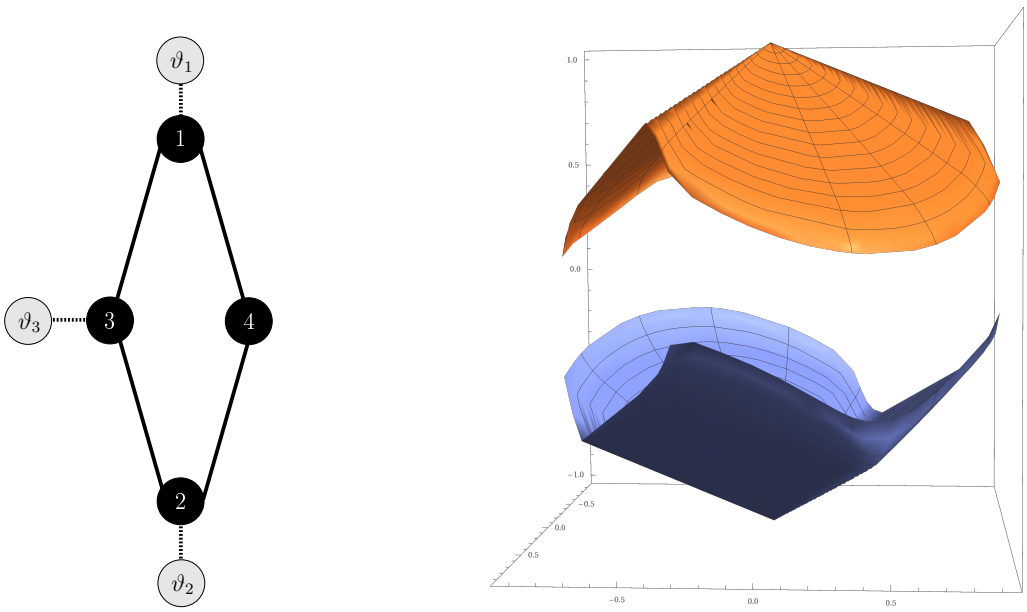

Figure 1: The lozenge network (left) and a plot (right) of the functions $\mathscr{S} \ni \xi \mapsto \Lambda_\pm(\xi)$ in thermal equilibrium (for the purpose of this representation, the set $\mathscr{S}$ has been mapped to the open unit disk).

By continuity, Condition **(R)** persists sufficiently near thermal equilibrium so that the same conclusions hold there. Figure 2 shows the "spectral gap" $\Lambda_+ - \Lambda_-$ on the boundary of the set $\mathscr{S}$ for different temperature ratios. It appears that this gap eventually closes (i.e., takes non-positive values)

when the temperature differences become large. We conclude that the FR (2.20) breaks down in this regime. This is illustrated on Figure 3 where the rate function I and the anomalous fluctuation function $\Delta(\varphi) = I(\varphi) - I(-\varphi) - \vartheta^{-1} \cdot \varphi$ are plotted for the temperature ratios $[1:2:64]$.

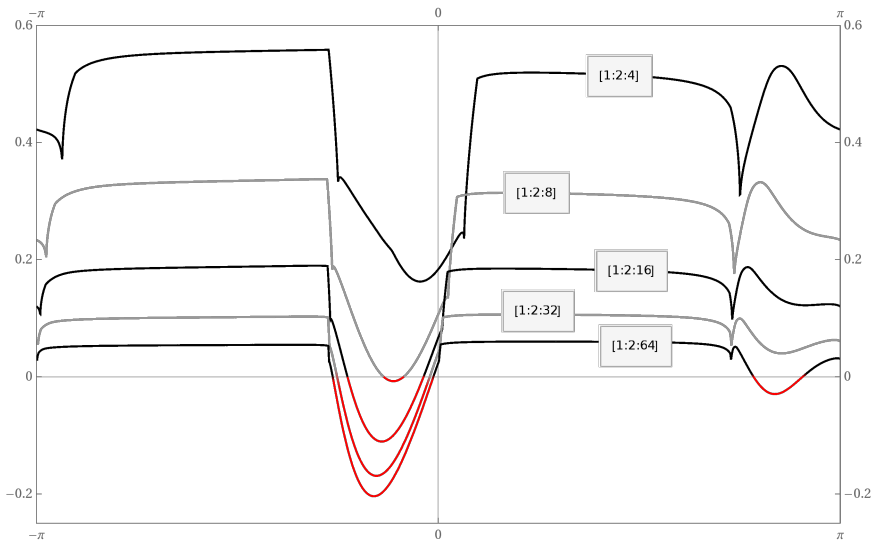

Figure 2: Plot of the spectral gap $\partial\mathscr{S} \ni \xi \mapsto \Lambda_+(\xi) - \Lambda_-(\xi)$ of the lozenge network for different temperature ratios (here, the set $\partial\mathscr{S}$ has been mapped to a circle and the polar angle 0 corresponds to the direction ϑ^{-1}).

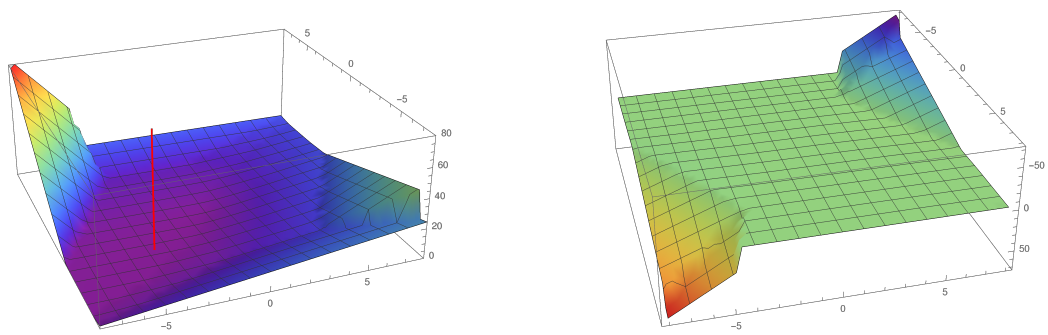

Figure 3: The rate function I (left, the vertical line denotes the position of the average current $\bar{\varphi}$) and the anomalous fluctuation function Δ (right) for the lozenge network (see the main text for details).

4.2 A triangular network

Our second example is the triangular network already considered in [JPS] and illustrated on the left pane of Figure 4. Here we have $|\mathscr{I}| = 6$, $|\partial\mathscr{I}| = 3$ and the parameters are

$$\kappa^2 = \begin{bmatrix} 1/2 & a & 0 & 0 & 0 & a \\ a & 1/2 & a & b & 0 & b \\ 0 & a & 1/2 & a & 0 & 0 \\ 0 & b & a & 1/2 & a & b \\ 0 & 0 & 0 & a & 1/2 & a \\ a & b & 0 & b & a & 1/2 \end{bmatrix}, \qquad a = \frac{1}{2\sqrt{2}}, \qquad b = \frac{1}{4}, \qquad \gamma_1 = \gamma_3 = \gamma_5 = 1.$$

In thermal equilibrium one finds that $\mathscr{S}$ is the disk of radius $\sqrt{3}/2$ centered at 0 on $\mathscr{L}^\perp$. The spectral gap $\Lambda_+ - \Lambda_-$ is open, as seen on the right pane of Figure 4. Hence, here again, Theorem 3.5 applies: the

global LDP (3.11) and the FR (2.20) hold on $\mathscr{L}^{\perp}$ near equilibrium. The spectral gap on the boundary

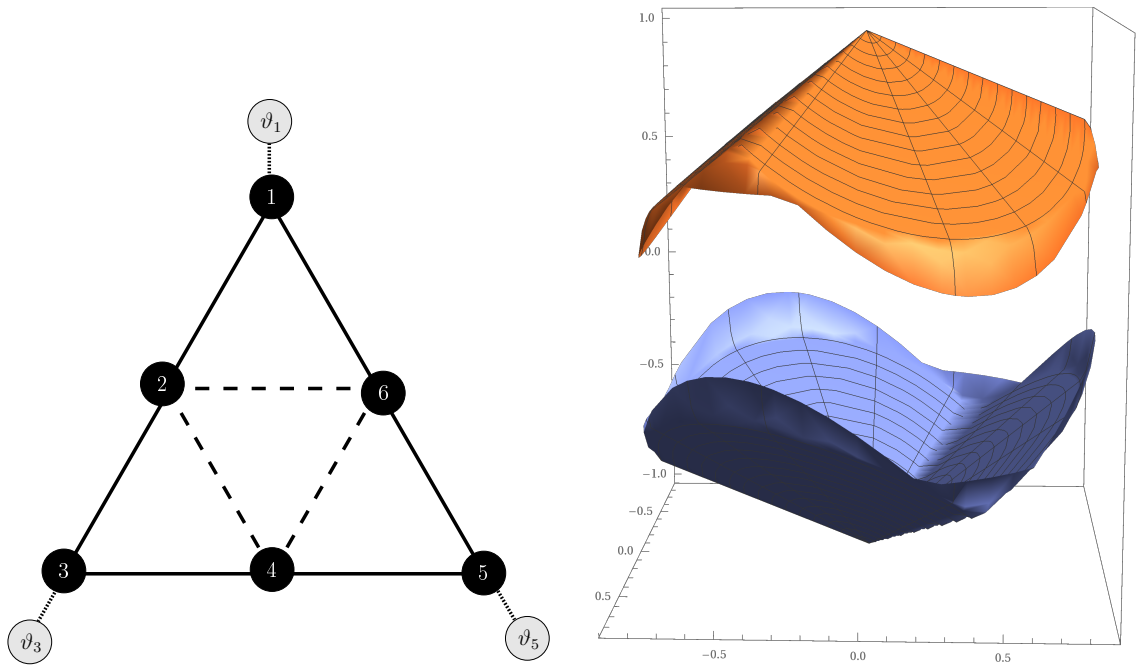

Figure 4: A triangular network (left) and a plot (right) of the functions $\mathscr{S} \ni \xi \mapsto \Lambda_+ \pm (\xi)$ in thermal equilibrium.

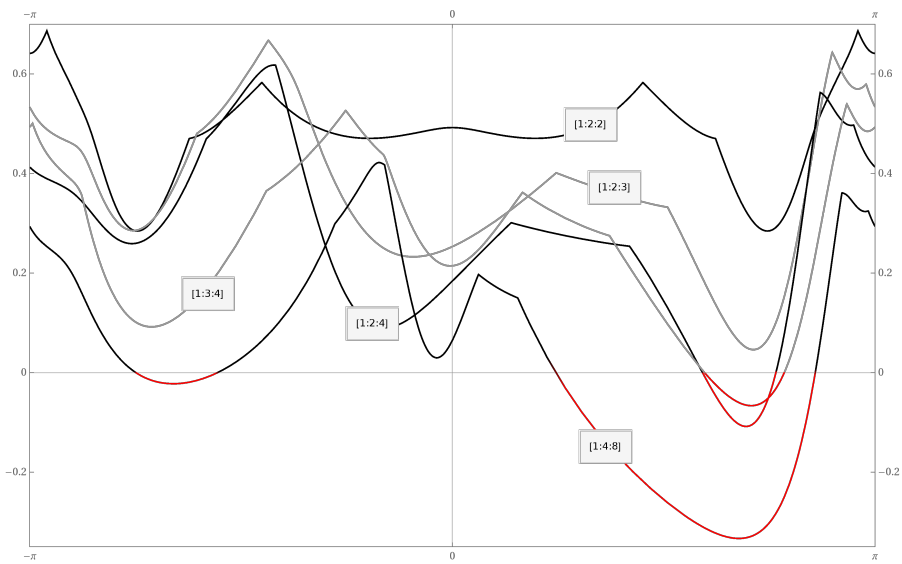

Figure 5: The spectral gap $\Lambda_+ - \Lambda_-$ of the triangular network on the boundary $\partial\mathscr{S}$ for different temperature ratios.

$\partial\mathscr{S}$ is plotted in Figure 5 for various temperature ratios $[\vartheta_1 : \vartheta_2 : \vartheta_3]$. One observes a similar behavior as in our first example.

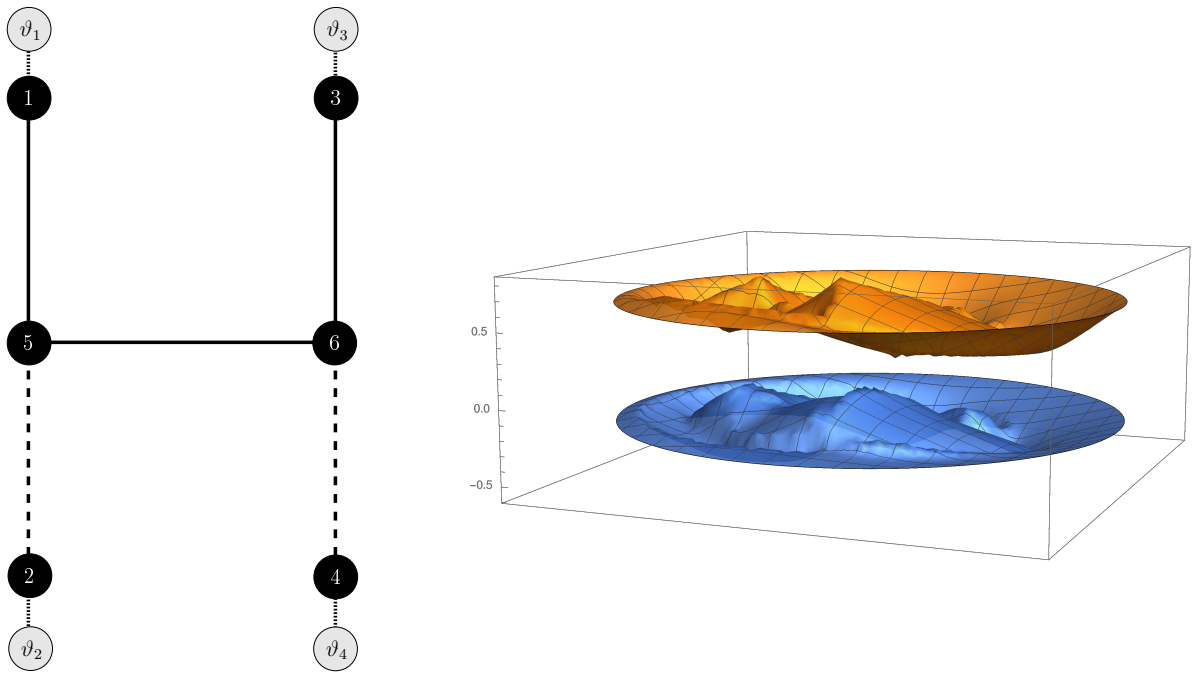

Figure 6: The heat pump network (left) and a plot of the functions $\partial\mathscr{S} \ni \xi \mapsto \Lambda_\pm(\xi)$ for the temperature ratios $[10:3.6:7:6.8]$.

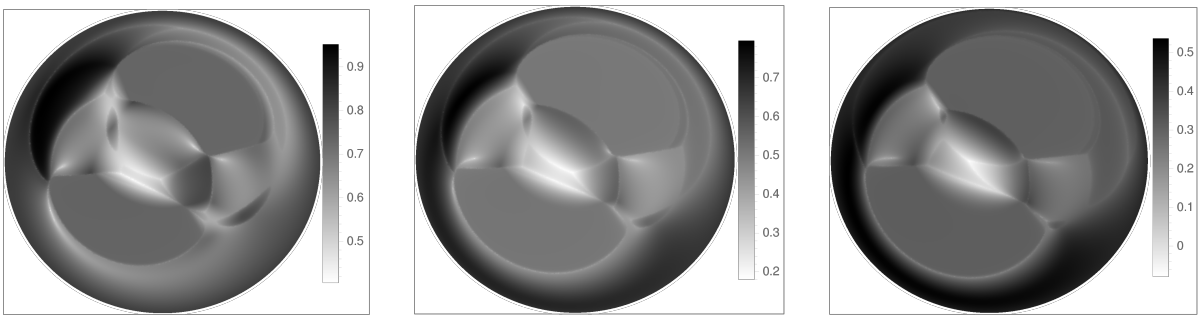

Figure 7: Density plots of the spectral gap $\Lambda_+ - \Lambda_-$ as a function of $\xi \in \partial\mathscr{S}$ for the temperature ratios $[10:3.6:7:6.8]$, $[20:3.6:7:6.8]$ and $[40:3.6:7:6.8]$.

4.3 A heat pump network

Our last example is the heat pump network of [EZ], see Figure 6. With $|\mathscr{I}| = 6$ and $|\partial\mathscr{I}| = 4$, the parameters:

$$\kappa^2 = \begin{bmatrix} 1-a & 0 & 0 & 0 & a & 0 \\ 0 & 1-b & 0 & 0 & b & 0 \\ 0 & 0 & 1-a & 0 & 0 & a \\ 0 & 0 & 0 & 1-b & 0 & b \\ a & b & 0 & 0 & 1-2a-b & b \\ 0 & 0 & a & b & a & 1-2a-b \end{bmatrix} \qquad \begin{array}{c} a=-40, \quad b=-20, \\ \gamma_1=\gamma_2=\gamma_3=\gamma_4=1, \\ \vartheta_1=10, \quad \vartheta_2=3.6, \quad \vartheta_3=7, \quad \vartheta_4=6.8, \end{array}$$

were chosen in [EZ] in such a way that the mean steady heat current between the vertices 5 and 6 vanishes while the heat flows from the hot reservoir to the cold one on the left side, and from the cold to the hot one on the right side. Thus, the right side of the device acts as a heat pump. On the right

pane of Figure 6 we plot the two functions $\Lambda_\pm$ on the boundary $\partial\mathscr{S}$.[7] Condition **(R)** is satisfied, so that the global large deviation principle (3.11) and the FR (2.20) hold on $\mathscr{L}^\perp$ near this non-equilibrium heat pump regime.

As shown in Figure 7, here again the spectral gap closes as the temperatures differences increase.

5 Proofs

5.1 Proof of Proposition 3.2

In order to prove the proposition, we shall need the following two Lemmas.

Lemma 5.1. *Let A and B be linear operators on a finite dimensional Hilbert space and $\mathbb{C} = V_- \cup \ell \cup V_+$ a partition of the plane into two open half-planes $V_\pm$ and a separating line ℓ. Assume that $(\mathrm{sp}(A) \cup \mathrm{sp}(B)) \cap \ell = \emptyset$ and that A and B have the same number of repeated eigenvalues in V_+. Denote by $A_\pm$ and $B_\pm$ the parts of A and B corresponding to their spectra in $V_\pm$. Let K be a compact set such that $(\mathrm{sp}(A) \cup \mathrm{sp}(B)) \cap V_+ \subset K \subset V_+$. Then the following holds:*

(1) The function $f(z) = \log\det\big((z-A)^{-1}(z-B)\big)$ is analytic in $V_+ \setminus K$.

(2) For any Jordan curve γ in $V_+ \setminus K$ "enclosing" K

$$\oint_\gamma f(z)\frac{\mathrm{d}z}{2\pi\mathrm{i}} = -\oint_\gamma z f'(z)\frac{dz}{2\pi\mathrm{i}} = \mathrm{tr}(A_+ - B_+). \tag{5.1}$$

Proof. (1) By assumption we can enumerate the repeated eigenvalues of A and B in such a way that

$$\mathrm{sp}(A) = \{\lambda_j^+ \mid j \in J\} \cup \{\lambda_i^- \mid i \in I\}, \qquad \mathrm{sp}(B) = \{\mu_j^+ \mid j \in J\} \cup \{\mu_i^- \mid i \in I\}$$

with

$$\lambda_j^+, \mu_j^+ \in K \subset V_+, \qquad \lambda_j^-, \mu_j^- \in V_-.$$

In terms of these eigenvalues, we have

$$\det\big((z-A)^{-1}(z-B)\big) = \left(\prod_{j\in J}\frac{z-\mu_j^+}{z-\lambda_j^+}\right)\left(\prod_{i\in I}\frac{z-\mu_i^-}{z-\lambda_i^-}\right).$$

The Möbius transformation $z \mapsto \frac{z-b}{z-a}$ maps the interior of the complement of any open neighborhood of the line segment joining a to b to a simply connected open subset of $\mathbb{C}\setminus\{0\}$. It follows that the function $\log\frac{z-b}{z-a}$ is analytic on the complement of any neighborhood of the segment joining a to b. Thus, the functions $z \mapsto \log\frac{z-\mu_j^+}{z-\lambda_j^+}$ and $z \mapsto \log\frac{z-\mu_i^-}{z-\lambda_i^-}$ are analytic in $V_+ \setminus K$ and so is $f(z)$.

(2) The first identity in (5.1) now follows from integration by parts. Finally, noticing that

$$f'(z) = \sum_{j\in J}\left(\frac{1}{z-\mu_j^+} - \frac{1}{z-\lambda_j^+}\right) + \sum_{i\in I}\left(\frac{1}{z-\mu_i^-} - \frac{1}{z-\lambda_i^-}\right) = \mathrm{tr}(z-B)^{-1} - \mathrm{tr}(z-A)^{-1},$$

the second identity follows from the Riesz formula

$$-\oint_\gamma z f'(z)\frac{\mathrm{d}z}{2\pi\mathrm{i}} = -\mathrm{tr}\left(\oint_\gamma z(z-B)^{-1}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\right) + \mathrm{tr}\left(\oint_\gamma z(z-A)^{-1}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\right) = \mathrm{tr}(A_+) - \mathrm{tr}(B_+).$$

□

[7] In Figures 6 and 7 the set $\partial\mathscr{S}$ is mapped to the closed unit disk by first mapping $\partial\mathscr{S}$ to the unit sphere and then mapping the point with spherical coordinates $(\varphi,\theta) \in [0,2\pi]\times[0,\pi]$ on this sphere to the point $\frac{\theta}{\pi}(\cos\varphi, \sin\varphi)$ of the plane.

Lemma 5.2. *Let A and B be as in the previous Lemma, where ℓ is the imaginary axis and V_+ the right half-plane. Then*

$$\int_{-\infty}^{+\infty} \log\det\big((\mathrm{i}\omega - A)^{-1}(\mathrm{i}\omega - B)\big)\frac{\mathrm{d}\omega}{4\pi} = \frac{1}{4}\left(\mathrm{tr}(B_+ - B_-) - \mathrm{tr}(A_+ - A_-)\right).$$

In particular, if the spectra of A and B are symmetric w.r.t. ℓ, then $\mathrm{tr}(A_-) = -\overline{\mathrm{tr}(A_+)}$ and similarly for B, so

$$\begin{aligned}\int_{-\infty}^{+\infty} \log\det\big((\mathrm{i}\omega - A)^{-1}(\mathrm{i}\omega - B)\big)\frac{\mathrm{d}\omega}{4\pi} &= \frac{1}{2}\mathrm{Re}(\mathrm{tr}(B_+ - A_+)) \\ &= \frac{1}{4}\sum_{\lambda\in\mathrm{sp}(B)} |\mathrm{Re}(\lambda)| m_\lambda - \frac{1}{4}\sum_{\lambda\in\mathrm{sp}(A)} |\mathrm{Re}(\lambda)| m_\lambda,\end{aligned}$$

where m_λ denotes the algebraic multiplicity of the eigenvalue λ.

Proof. Denote by γ_R the positively oriented boundary of the intersection of the disk of radius R centered at 0 with the right half-plane. Applying the previous Lemma and observing that f is analytic in a neighborhood of ℓ we get, for R large enough,

$$\mathrm{tr}(A_+) - \mathrm{tr}(B_+) = \oint_{\gamma_R} f(z)\frac{\mathrm{d}z}{2\pi\mathrm{i}} = -\int_{-R}^{R} f(\mathrm{i}\omega)\frac{\mathrm{d}\omega}{2\pi} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(R\mathrm{e}^{\mathrm{i}\varphi})R\mathrm{e}^{\mathrm{i}\varphi}\frac{\mathrm{d}\varphi}{2\pi}.$$

To evaluate the second integral on the right-hand side we note that, as $R\to\infty$,

$$\frac{R\mathrm{e}^{\mathrm{i}\varphi} - b}{R\mathrm{e}^{\mathrm{i}\varphi} - a} = (1 - bR^{-1}\mathrm{e}^{-\mathrm{i}\varphi})(1 + aR^{-1}\mathrm{e}^{-\mathrm{i}\varphi} + O(R^{-2})) = 1 + (a-b)R^{-1}\mathrm{e}^{-\mathrm{i}\varphi} + O(R^{-2})$$

so

$$\log\frac{R\mathrm{e}^{\mathrm{i}\varphi} - a}{R\mathrm{e}^{\mathrm{i}\varphi} - b} = (a-b)R^{-1}\mathrm{e}^{-\mathrm{i}\varphi} + O(R^{-2}),$$

and

$$f(R\mathrm{e}^{\mathrm{i}\varphi}) = R^{-1}\mathrm{e}^{-i\varphi}\,\mathrm{tr}(A-B) + O(R^{-2}).$$

It follows that

$$\lim_{R\to\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(R\mathrm{e}^{\mathrm{i}\varphi})R\mathrm{e}^{\mathrm{i}\varphi}\frac{\mathrm{d}\varphi}{2\pi} = \frac{1}{2}\mathrm{tr}(A-B),$$

and we conclude that

$$\begin{aligned}\int_{-\infty}^{+\infty} \log\det\big((\mathrm{i}\omega - A)^{-1}(\mathrm{i}\omega - B)\big)\frac{\mathrm{d}\omega}{4\pi} &= \frac{1}{4}\mathrm{tr}(A-B) - \frac{1}{2}(\mathrm{tr}(A_+) - \mathrm{tr}(B_+)) \\ &= -\frac{1}{4}\mathrm{tr}(A_+) + \frac{1}{4}\mathrm{tr}(A_-) + \frac{1}{4}\mathrm{tr}(B_+) - \frac{1}{4}\mathrm{tr}(B_-).\end{aligned}$$

The last two statements follow from elementary calculations. □

We now turn to the proof of Proposition 3.2

(1) Set $R(\omega) = \vartheta^{-1}Q^*(A + \mathrm{i}\omega)^{-1}Q$. Assumption **(C)** implies that A is stable (see (2.5)) so that the map $\mathbb{R}\ni\omega\mapsto R(\omega)\in L(\Xi)$ is continuous. Using the identities

$$\Omega = A + \frac{1}{2}Q\vartheta^{-1}Q^* = -A^* - \frac{1}{2}Q\vartheta^{-1}Q^*$$

a simple calculation yields that for any $\widetilde{\xi}\rhd\xi\in\Xi$ one has

$$E_\xi(\omega) = Q^*(A^* - \mathrm{i}\omega)^{-1}[\Omega, \widetilde{\xi}](A + \mathrm{i}\omega)^{-1}Q = -\zeta R(\omega) - R(\omega)^*\zeta - R(\omega)^*\zeta R(\omega), \tag{5.2}$$

with $\zeta = \vartheta^{1/2}\xi\vartheta^{1/2}$. All the stated properties immediately follow.

(2) It will be convenient to introduce $U(\omega) = I + R(\omega)$ and to rescale ξ by setting $\zeta = (\xi_i \vartheta_i)_{i\in\partial\mathscr{I}}$. With this change of variable

$$I - E_\xi(\omega) = I - F_\zeta(\omega) = I - \zeta + U(\omega)^*\zeta U(\omega),$$

and

$$\mathscr{D} = \bigcap_{\omega\in\mathbb{R}} \{\zeta \in \Xi \,|\, I - F_\zeta(\omega) > 0\}, \qquad \mathscr{D}_0 = \{\zeta \in \Xi \,|\, 0 < \zeta_i < 1, i \in \partial\mathscr{I}\}.$$

It immediately follows that $\mathscr{D}_0 \subset \mathscr{D}$. Moreover, since $I - F_\zeta(\omega)$ is an affine function of ζ, $\mathscr{D}$ is convex.

Using (2.5), elementary calculations (see [JPS, Section 5.5]) show that, for any $\omega \in \mathbb{R}$, $U(\omega)^{-1} = U(-\omega)$ and $|\det(U(\omega))| = 1$. The first relation allows us to derive

$$I - F_\zeta(\omega) = U(\omega)^*(U(-\omega)^*(I-\zeta)U(-\omega) - (I-\zeta) + I)U(\omega) = U(\omega)^*(I - F_{I-\zeta}(-\omega))U(\omega), \tag{5.3}$$

and the second one yields that $\zeta \in \mathscr{D} \Longleftrightarrow I - \zeta \in \mathscr{D}$, which shows that $\mathscr{D}$ is centrally symmetric around the point $I/2$. To show that it is open, we now argue that its complement $\mathscr{D}^{\mathrm{c}}$ is closed. Indeed, $\xi \in \mathscr{D}^{\mathrm{c}}$ iff there exists $\omega \in \mathbb{R}$ such that $E_\xi(\omega)$ has an eigenvalue $\lambda \geq 1$. Thus, if ξ_n is a sequence in $\mathscr{D}^{\mathrm{c}}$ which converges to ξ, there exist $\omega_n \in \mathbb{R}$, $\lambda_n \geq 1$ and unit vectors u_n such that $E_{\xi_n}(\omega_n)u_n = \lambda_n u_n$. Given the fact that

$$1 \leq \lambda_n \leq \|E_{\xi_n}(\omega_n)\| \leq C(1+|\omega_n|)^{-2} \leq C < \infty$$

one concludes that $|\omega_n| \leq C^{1/2} - 1$ and $\lambda_n \in [1, C]$. Thus the sequence $(\xi_n, \omega_n, \lambda_n, u_n)$ has a convergent subsequence with limit $(\xi, \omega, \lambda, u)$ satisfying $E_\xi(\omega)u = \lambda u$, $\lambda \geq 1$ and $u \neq 0$. Hence, $\xi \in \mathscr{D}^{\mathrm{c}}$ and consequently $\mathscr{D}^{\mathrm{c}}$ is closed.

If the map $\omega \mapsto E_\eta(\omega)$ vanishes identically on $\mathbb{R}$, then it follows from the identity $I - E_{\xi+\lambda\eta}(\omega) = I - E_\xi(\omega)$ that $\mathscr{D} + \lambda\eta \subset \mathscr{D}$ for all $\lambda \in \mathbb{R}$. Reciprocally, if the later condition holds, then for all $\xi \in \mathscr{D}$ and all $\lambda > 0$, one has

$$-\frac{1}{\lambda}(I - E_\xi(\omega)) < E_\eta(\omega) < \frac{1}{\lambda}(I - E_\xi(\omega)).$$

Letting $\lambda \to \infty$ we deduce that $E_\eta(\omega)$ vanishes identically. Note that the later condition is satisfied for any η such that there exists $\widetilde{\eta} \rhd \eta$ with $\Sigma_{\widetilde{\eta}} = 0$. This is in particular the case for $\eta = \mathbf{1}$ and $\widetilde{\eta} = I_\Gamma$. Reciprocally, since it follows from Condition **(C)** that the set $\{(A + \mathrm{i}\omega)^{-1}Qu \,|\, \omega \in \mathbb{R}, u \in \mathbb{C}^{\partial\mathscr{I}}\}$ is total in $\mathbb{C}^{\mathscr{I}}$, one concludes that $E_\xi(\omega) = 0$ for all $\omega \in \mathbb{R}$ implies $\Sigma_{\widetilde{\xi}} = 0$ for any $\widetilde{\xi} \rhd \xi$.

(3) Consider first the map $\mathscr{D} \ni \eta \mapsto -\log\det(I - F_\eta(\omega))$ for fixed $\omega \in \mathbb{R}$. The previous discussion clearly implies that it is real analytic. Its convexity follows from an elementary calculation which yields that

$$-\sum_{i,j\in\partial\mathscr{I}} \bar{z}_i \frac{\partial^2 \log\det(I - F_\eta(\omega))}{\partial\eta_i\partial\eta_j} z_j = \mathrm{tr}\big((I - F_\eta(\omega))^{-1/2} F_z(\omega)^*(I - F_\eta(\omega))^{-1} F_z(\omega)(I - F_\eta(\omega))^{-1/2}\big) \geq 0$$

for $\eta \in \mathscr{D}$ and $z \in \mathbb{C}^{\partial\mathscr{I}}$. From (5.2) one further deduces that $F_\eta(\omega) = O(\omega^{-2})$ as $|\omega| \to \infty$, locally uniformly in $\eta \in \mathscr{D}$. It follows that

$$f(\eta) = -\int_{-\infty}^{\infty} \log\det(I - F_\eta(\omega)) \frac{\mathrm{d}\omega}{4\pi} = g(\xi)$$

is convex and real analytic on $\mathscr{D}$. The identity (5.3) leads to

$$\det(I - F_\eta(\omega)) = \det(I - F_{\mathbf{1}-\eta}(-\omega)),$$

and in particular $\det(I - F_{\mathbf{1}}(\omega)) = 1$. This proves the first equality in (3.4). The second one follows from (3.3) and the linearity of the map $\eta \mapsto F_\eta$.

(4) The second equality in (3.4) implies that $\eta \cdot \nabla g(\xi) = 0$ for all $\xi \in \mathscr{D}$ and all $\eta \in \mathscr{L}$. To establish the reciprocal property, note that since $\mathscr{D}$ is open, for any $\xi \in \mathscr{D}$ and any $\eta \in \nabla g(\mathscr{D})^\perp$ there exists $\epsilon > 0$ such that $\xi + \alpha\eta \in \mathscr{D}$ for $|\alpha| < \epsilon$. It follows that the function $\alpha \mapsto g(\xi + \alpha\eta)$ is constant in a real neighborhood

of 0 and hence extends by analyticity to the constant function on the line $\xi + \mathbb{R}\eta$. Since, by Part (7), g is singular on $\partial\mathscr{D}$, it follows that $\xi + \mathbb{R}\eta \subset \mathscr{D}$, i.e., $\eta \in \mathscr{L}$.

Consequently, g vanishes identically whenever $\mathscr{L} = \Xi$. In the opposite case, the calculation in Part (3) gives that the Hessian of g satisfies

$$\eta \cdot g''(\xi)\eta = \int_{-\infty}^{\infty} \mathrm{tr}\Big(\big[(I - E_\xi(\omega))^{-1/2} E_\eta(\omega)(I - E_\xi(\omega))^{-1/2}\big]^2\Big)\,\frac{\mathrm{d}\omega}{4\pi} > 0$$

for non-zero $\eta \notin \mathscr{L}$. It follows that the restriction of $g''(\xi)$ to $\mathscr{L}^\perp$ is positive definite which implies that the restriction of g to $\mathscr{S}$ is strictly convex. To show that the closure of $\mathscr{S}$ is compact, let us assume that $\mathscr{S}$ is unbounded. Since $\mathscr{S}$ is convex and centrally symmetric w.r.t. the orthogonal projection ξ_0 of $(2\vartheta)^{-1}$ onto $\mathscr{L}^\perp$, it follows that for some non-vanishing $\xi \in \mathscr{L}^\perp$ one has $\xi_0 + \lambda\xi \in \mathscr{S}$ for all $\lambda \in \mathbb{R}$, i.e.,

$$-\frac{1}{|\lambda|}(I - E_{\xi_0}(\omega)) \le E_\xi(\omega) \le \frac{1}{|\lambda|}(I - E_{\xi_0}(\omega))$$

for all $\omega \in \mathbb{R}$. Letting $|\lambda| \to \infty$ yields that $\xi \in \mathscr{L}$ which contradicts the fact that $0 \neq \xi \in \mathscr{L}^\perp$.

(5) We start with some simple consequences of Condition **(C)**. For a short introduction to the necessary elementary material, we refer the reader to [LR, Section 4]. Since $A_\xi = A + Q\xi Q^*$, the pair (A_ξ, Q) is controllable for all ξ. The relation $A_\xi^* = -A_{\vartheta^{-1}-\xi}$ shows that the same is true for the pair (A_ξ^*, Q). Thus, one has

$$\bigcap_{n\ge 0} \mathrm{Ker}(Q^* A_\xi^n) = \bigcap_{n\ge 0} \mathrm{Ker}(Q^* A_\xi^{*n}) = \{0\}$$

for all ξ. This implies that if $Q^* u = 0$ and $(A_\xi - z)u = 0$ or $(A_\xi^* - z)u = 0$, then $u = 0$, i.e., no eigenvector of A_ξ or A_ξ^* can live in $\mathrm{Ker}\, Q^*$. Assume now $z \in \mathrm{sp}(A_\xi)$ and let $u \neq 0$ be a corresponding eigenvector. Since

$$A_\xi + A_\xi^* = 2Q(\xi - (2\vartheta)^{-1})Q^*,$$

taking the real part of $\langle u, (A_\xi - z)u\rangle = 0$ we infer

$$\langle Q^* u, (\xi - (2\vartheta)^{-1})Q^* u\rangle = \mathrm{Re}(z)|u|^2.$$

Thus, controllability of (A_ξ, Q) implies that for $\pm(\xi - (2\vartheta)^{-1}) > 0$ one has $\mathrm{sp}(A_\xi) \subset \mathbb{C}_\pm$ and in particular $\mathrm{sp}(A_\xi) \cap \mathrm{i}\mathbb{R} = \emptyset$. Hence, for $\xi > (2\vartheta)^{-1}$ and $\omega \in \mathbb{R}$, Schur's complement formula yields

$$\det(K_\xi - \mathrm{i}\omega) = |\det(A_\xi + \mathrm{i}\omega)|^2 \det\big(I + r_\xi(\omega)^* \xi(\vartheta^{-1} - \xi) r_\xi(\omega)\big) \tag{5.4}$$

where we have set

$$r_\xi(\omega) = Q^*(A_\xi + \mathrm{i}\omega)^{-1} Q.$$

One easily checks that $r_\xi(\omega) = r_0(\omega)(I + \xi r_0(\omega))^{-1}$ from which a simple calculation gives

$$I + r_\xi(\omega)^* \xi(\vartheta^{-1} - \xi) r_\xi(\omega) = (I + r_0(\omega)^* \xi)^{-1}(I - E_\xi(\omega))(I + \xi r_0(\omega))^{-1}.$$

Inserting the last identity into the right-hand side of (5.4) and using the fact that

$$\det(A_\xi + \mathrm{i}\omega) = \det(A + \mathrm{i}\omega)\det(I + \xi r_0(\omega)),$$

we obtain

$$\det(K_\xi - \mathrm{i}\omega) = |\det(A + \mathrm{i}\omega)|^2 \det(I - E_\xi(\omega)). \tag{5.5}$$

Both sides of this identity being polynomials in ξ, it extends to all $\xi \in \Xi$. It follows that

$$\bigcap_{\omega\in\mathbb{R}} \{\xi \in \Xi \,|\, 1 \notin \mathrm{sp}(E_\xi(\omega))\} = \{\xi \in \Xi \,|\, \mathrm{sp}(K_\xi) \cap \mathrm{i}\mathbb{R} = \emptyset\}.$$

By continuity of the function $\xi \mapsto \min_{\omega\in\mathbb{R}} \det(I - E_\xi(\omega))$, $\mathscr{D}$ is the connected component of the point $\xi = 0$ in the left-hand side of this identity.

(6) For $\xi \in \Xi$, K_ξ is $\mathbb{R}$-linear on the real vector space $\Gamma \oplus \Gamma$. Thus, its spectrum is symmetric w.r.t. the real axis. Observing that $JK_\xi + K_\xi^* J = 0$, where J is the unitary operator

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

we conclude that the spectrum of K_ξ is also symmetric w.r.t. the imaginary axis. Assume now that $\xi \in \mathscr{D}$. Since the eigenvalues of K_ξ are continuous functions of ξ, K_ξ and K_0 have the same number of repeated eigenvalues in the left/right half-plane. From (5.5) we deduce

$$g(\xi) = \int_{-\infty}^{+\infty} \log\det\big((\mathrm{i}\omega - K_\xi)^{-1}(\mathrm{i}\omega - K_0)\big)\,\frac{\mathrm{d}\omega}{4\pi},$$

and Lemma 5.2 allows us to conclude that

$$g(\xi) = \frac{1}{4}\sum_{\lambda\in\mathrm{sp}(K_0)} |\mathrm{Re}(\lambda)| m_\lambda - \frac{1}{4}\sum_{\lambda\in\mathrm{sp}(K_\xi)} |\mathrm{Re}(\lambda)| m_\lambda.$$

Since $\mathrm{sp}(K_0) = \overline{\mathrm{sp}(A)} \cup \mathrm{sp}(-A)$ and A is stable, we have

$$\sum_{\lambda\in\mathrm{sp}(K_0)} |\mathrm{Re}(\lambda)| m_\lambda = -2\sum_{\lambda\in\mathrm{sp}(A)} \mathrm{Re}(\lambda) m_\lambda = -2\,\mathrm{Re}\,\mathrm{tr}(A) = \mathrm{tr}(Q\vartheta^{-1}Q^*),$$

and (3.6) follows for $\xi \in \mathscr{D}$. Since the eigenvalues of K_ξ are continuous functions of $\xi \in \Xi$, this relation extends to $\xi \in \overline{\mathscr{D}}$. The boundedness of this extension follows from the translation invariance along $\mathscr{L}$ and the precompactness of $\mathscr{S}$.

(7) The idea of the proof is that $\xi_0 \in \partial\mathscr{D}$ iff at least one eigenvalue of $E_{\xi_0}(\omega)$ reaches its global maximum 1 at some $\omega_0 \in \mathbb{R}$. Since $E_{\xi_0}(\omega)$ is a real analytic function of ω, the function $\mathrm{tr}\big((1 - E_{\xi_0}(\omega))^{-1}\big)$ has a pôle at $\omega = \omega_0$, and since this function is non-negative the order of this pôle must be even. Consequently,

$$\int_{\omega_0-\epsilon}^{\omega_0+\epsilon} \mathrm{tr}\big((1 - E_{\xi_0}(\omega))^{-1}\big)\,\mathrm{d}\omega = +\infty.$$

For $\xi \in \mathscr{D}$, a simple calculation and Cauchy–Schwarz inequality yield

$$|\nabla g(\xi)| \geq \frac{\xi}{|\xi|}\cdot\nabla g(\xi) = \frac{1}{|\xi|}\int_{-\infty}^{+\infty} \mathrm{tr}\big((I - E_\xi(\omega))^{-1}E_\xi(\omega)\big)\,\frac{\mathrm{d}\omega}{4\pi}.$$

Since $0 \notin \partial\mathscr{D}$, it suffices to show that the integral on the right-hand side diverges to $+\infty$ as $\xi \to \xi_0 \in \partial\mathscr{D}$. Let us fix $\xi_0 \in \partial\mathscr{D}$ and set $V_\delta = \{\xi \in \mathscr{D} \,|\, |\xi - \xi_0| < \delta\}$. Elementary considerations show that for sufficiently small $\delta > 0$ and sufficiently large $M > 0$ there exists a constant C such that

$$\int_{-\infty}^{+\infty} \mathrm{tr}\big((I - E_\xi(\omega))^{-1}E_\xi(\omega)\big)\,\frac{\mathrm{d}\omega}{4\pi} \geq C\left(-1 + \int_{-M}^{M} \mathrm{tr}\big((I - E_\xi(\omega))^{-1}\big)\,\mathrm{d}\omega\right)$$

for any $\xi \in V_\delta$. Making δ smaller and M larger if necessary, we can assume that $J_\xi(\omega) = (I - E_{\xi-\xi_0}(\omega))^{-1/2}$ satisfies $1/\sqrt{2} \leq J_\xi(\omega) \leq \sqrt{2}$ for $(\omega, \xi) \in [-M, M] \times V_\delta$. Writing

$$(I - E_\xi(\omega))^{-1} = J_\xi(\omega)(I - J_\xi(\omega)E_{\xi_0}(\omega)J_\xi(\omega))^{-1}J_\xi(\omega)$$

and observing that this implies, in particular, that $I - J_\xi(\omega)E_{\xi_0}(\omega)J_\xi(\omega) > 0$, we derive

$$\mathrm{tr}\big((I - E_\xi(\omega))^{-1}\big) \geq \frac{1}{2}\,\mathrm{tr}\big((I - J_\xi(\omega)E_{\xi_0}(\omega)J_\xi(\omega))^{-1}\big) > 0.$$

By Fatou's lemma

$$\liminf_{\mathscr{D}\ni\xi\to\xi_0} \int_{-M}^{M} \mathrm{tr}\big((I - E_\xi(\omega))^{-1}\big)\,\mathrm{d}\omega \geq \frac{1}{2}\int_{-M}^{M} \mathrm{tr}\big((I - E_{\xi_0}(\omega))^{-1}\big)\,\mathrm{d}\omega$$

and by the above argument, the last integral is $+\infty$.

(8) The existence and uniqueness of the maximal solutions of Eq. (2.16) as well as the stated properties of D_ξ follow from [LR, Theorems 7.3.7 and 7.5.1], Part (5), and the relation $D_\xi = A + Q(\xi Q^* - Q^* X_\xi)$. It further follows from (3.6) and the symmetries of $\mathrm{sp}(K_\xi)$ discussed in the proof of Part (5) that

$$g(\xi) = \frac{1}{4}\mathrm{tr}(Q\vartheta^{-1}Q^*) + \frac{1}{2}\mathrm{tr}(D_\xi) = \frac{1}{2}\mathrm{tr}(D_\xi - D_0) = -\frac{1}{2}\mathrm{tr}(Q^*(X_\xi - \widetilde{\xi})Q).$$

The proof of Proposition 3.2 is complete.

5.2 Proof of Proposition 3.3

5.2.1 Some properties of the algebraic Riccati equations (2.16)

In order to prove Proposition 3.3 we shall need some properties of the algebraic Riccati equation

$$\mathscr{R}_\xi(X) \equiv XBX - XA_\xi - A_\xi^* X - C_\xi = 0. \tag{5.6}$$

This is the purpose of the following proposition which provides a generalization of [JPS, Proposition 5.5]. In the sequel, whenever we mention a solution of (5.6), we always mean a *self-adjoint* $X \in L(\Gamma)$ such that $\mathscr{R}_\xi(X) = 0$. We say that such a solution X is maximal (resp. minimal) if any other solution $X' \in L(\Gamma)$ satisfies $X' \le X$ (resp. $X' \ge X$).

Proposition 5.3. *Assume that Condition* **(C)** *holds.*

(1) For $\xi \in \mathscr{D}$ the Riccati equation (5.6) has a unique maximal solution X_ξ and a unique minimal solution $-\theta X_{\vartheta^{-1}-\xi}\theta$. Moreover, the matrix

$$D_\xi = A_\xi - BX_\xi$$

is stable and satisfies

$$Y_\xi = X_\xi + \theta X_{\vartheta^{-1}-\xi}\theta > 0.$$

(2) If $\xi_0 \in \partial\mathscr{D}$ is finite, then the non-tangential limit

$$X_{\xi_0} = \lim_{\mathscr{D}\ni\xi\to\xi_0} X_\xi$$

exists and is the maximal solution of the corresponding limiting Riccati equation $\mathscr{R}_{\xi_0}(X) = 0$.

(3) The function $\mathscr{D} \ni \xi \mapsto X_\xi \in L(\Gamma)$ is real analytic and concave. Moreover, $X_\xi < 0$ for $\xi < 0$, $X_\xi > 0$ for ξ in the convex hull of the set $\mathscr{D}_0 \cup \{\xi \in \mathscr{D} \mid \xi > \vartheta^{-1}\}$, $X_0 = 0$ and $X_{\vartheta^{-1}} = \theta M^{-1}\theta$.

(4) For any $\xi \in \overline{\mathscr{D}}$ and $\eta \in \mathscr{L}$ one has

$$X_{\xi+\eta} = X_\xi + \widetilde{\eta}.$$

(5) For $t > 0$, set

$$M_{\xi,t} = \int_0^t \mathrm{e}^{sD_\xi} B \mathrm{e}^{sD_\xi^*}\,\mathrm{d}s > 0.$$

Then, for all $\xi \in \overline{\mathscr{D}}$ one has

$$\lim_{t\to\infty} M_{\xi,t}^{-1} = \inf_{t>0} M_{\xi,t}^{-1} = Y_\xi \ge 0,$$

and $\ker(Y_\xi)$ is the spectral subspace of D_ξ corresponding to its imaginary eigenvalues.

(6) Set $\Delta_{\xi,t} = M_{\xi,t}^{-1} - Y_\xi$. *For all* $\xi \in \overline{\mathscr{D}}$, *one has*

$$\mathrm{e}^{tD_\xi^*} M_{\xi,t}^{-1} \mathrm{e}^{tD_\xi} = \theta \Delta_{\vartheta^{-1}-\xi,t}\theta, \tag{5.7}$$

and

$$\lim_{t\to\infty} \frac{1}{t} \log\det(\Delta_{\xi,t}) = 4g(\xi) - \mathrm{tr}(Q\vartheta^{-1}Q^*).$$

In particular, for $\xi \in \mathscr{D}$, $\Delta_{\xi,t} \to 0$ *exponentially fast as* $t \to \infty$.

(7) Let $\widetilde{D}_\xi = \theta D_{\vartheta^{-1}-\xi}\theta$. *Then*

$$Y_\xi \mathrm{e}^{t\widetilde{D}_\xi^*} = \mathrm{e}^{tD_\xi^*} Y_\xi$$

for all $\xi \in \overline{\mathscr{D}}$ *and* $t \in \mathbb{R}$.

(8) For $\xi \in \mathscr{D}$ *and* $\eta \in \Xi$

$$\eta \cdot \nabla g(\xi) = \frac{1}{2} \mathrm{tr}\left(\Sigma_{\widetilde{\eta}} Y_\xi^{-1}\right)$$

Proof. We refer to [LR] for a detailed introduction to algebraic Riccati equations (see also the Appendix in [JPS] for a summary of the necessary basic facts). Our proof is similar to that of [JPS, Proposition 5.5]. The Hamiltonian matrix K_ξ associated to the Riccati equation (5.6) is given by Eq. (3.5). Let $\mathscr{H}$ be the complex Hilbert space $\mathbb{C}\Xi \oplus \mathbb{C}\Xi$ on which K_ξ acts. The operator

$$\Theta = \begin{bmatrix} 0 & \theta \\ \theta & 0 \end{bmatrix}$$

acts unitarily on $\mathscr{H}$. We have already observed in the proof of Proposition 3.2 (6) that for $\xi \in \Xi$, the spectrum of K_ξ is symmetric w.r.t. the real axis and the imaginary axis. The time-reversal covariance relations

$$\theta A_\xi \theta = A_\xi^* = -A_{\vartheta^{-1}-\xi}, \quad \theta B \theta = B^* = B, \quad \theta C_\xi \theta = C_\xi^* = C_\xi = C_{\vartheta^{-1}-\xi}, \tag{5.8}$$

which follow easily from the definitions of the operators A_ξ, B, C_ξ, further yield $\Theta K_\xi - K_{\vartheta^{-1}-\xi}\Theta = 0$ which implies

$$\mathrm{sp}(K_\xi) = \mathrm{sp}(K_{\vartheta^{-1}-\xi}). \tag{5.9}$$

Let $\mathscr{H}_-(K_\xi)$ be the spectral subspace of K_ξ for the part of its spectrum in the open left half-plane $\mathbb{C}_-$.

(1) By Proposition 3.2 (5), $\mathrm{sp}(K_\xi) \cap \mathrm{i}\mathbb{R} = \emptyset$ for $\xi \in \mathscr{D}$ and the existence and uniqueness of the maximal and minimal solutions of the Riccati equation (5.6) follow from [LR, Theorems 7.3.7 and 7.5.1]. The relation between minimal and maximal solutions is a consequence of the Relations (5.8) which imply that

$$\mathscr{R}_\xi(\theta X \theta) = \theta \mathscr{R}_{\vartheta^{-1}-\xi}(-X)\theta.$$

By [LR, Theorems 7.5.1], the maximal solution X_ξ is related to the spectral subspace $\mathscr{H}_-(K_\xi)$ by

$$\mathscr{H}_-(K_\xi) = \mathrm{Ran}\begin{bmatrix} I \\ X_\xi \end{bmatrix},$$

moreover, $\mathrm{sp}(D_\xi) = \mathrm{sp}(K_\xi) \cap \mathbb{C}_-$.

$Y_\xi = X_\xi + \theta X_{\vartheta^{-1}-\xi}\theta$ is called the gap of Eq. (5.6). As the difference between its maximal and minimal solutions, it is obviously non-negative. It further has the remarkable property that for any solution X, $\ker(Y_\xi)$ is the spectral subspace of $A_\xi - BX$ for the part of its spectrum in $\mathrm{i}\mathbb{R}$ [LR, Theorem 7.5.3]. Since $\mathrm{sp}(D_\xi) \subset \mathbb{C}_-$, we must have $Y_\xi > 0$.

(2) Let $\xi_0 \in \partial\mathscr{D}$ be finite and $\eta \neq 0$ be non-tangential to $\partial\mathscr{D}$ at ξ_0. Set $\xi_t = \xi_0 - t\eta$. W.l.o.g. we may assume that $\xi_1 \in \mathscr{D}$. The function

$$]0,1] \ni t \mapsto Z_t = X_{\xi_t} + tX'_{\xi_1}[\eta]$$

is concave and its first derivative vanishes at $t = 1$. Hence, it is monotone non-decreasing. We claim that the set $\{X_\xi \,|\, \xi \in \mathscr{D}, |\xi| < r\}$ is bounded in $L(\Gamma)$ for any finite r. It thus follows that

$$X = \lim_{t\downarrow 0} X_{\xi_t} = \lim_{t\downarrow 0} Z_t = \inf_{t\in]0,1]} Z_t$$

exists. By continuity, one has $\mathscr{R}_{\xi_0}(X) = 0$ and $\mathrm{sp}(A_{\xi_0} - BX) \subset \overline{\mathbb{C}}_-$, and it follows from [LR, Theorem 7.5.1] that X is the maximal solution of the limiting Riccati equation. In particular, the non-tangential limit exists (i.e., does not depend on the direction).

To prove our claim, we first derive a bound on $\hat{X}_\xi = Q^* X_\xi Q$. Using $(Q^* X_\xi Q)^2 \le \|Q\|^2 Q^* X_\xi^2 Q$, one easily deduces from (5.6) and Cauchy-Schwarz inequality

$$\mathrm{tr}(\hat{X}_\xi^2) \le \|Q\|^2 \,\mathrm{tr}(Q^* X_\xi^2 Q) = \|Q\|^2 \left(\mathrm{tr}(C_\xi) + 2\,\mathrm{tr}(\hat{X}_\xi(\xi - (2\vartheta)^{-1}))\right) \le b_\xi + a_\xi \,\mathrm{tr}(\hat{X}_\xi^2)^{1/2},$$

where a_ξ and b_ξ are locally bounded functions of ξ. Solving the resulting quadratic inequality yields that $\mathrm{tr}(\hat{X}_\xi^2)$, and hence $\mathrm{tr}(Q^* X_\xi^2 Q)$ are locally bounded as functions of ξ. Rewriting (5.6) as the Lyapunov equation

$$X_\xi A + A^* X_\xi = F_\xi \equiv X_\xi B X_\xi - X_\xi Q\xi Q^* - Q\xi Q^* X_\xi - C_\xi,$$

and using the fact that A is stable, we get

$$X_\xi = -\int_0^\infty \mathrm{e}^{tA^*} F_\xi \mathrm{e}^{tA} \mathrm{d}t.$$

It follows that for any $T \in L(\Gamma)$

$$|\,\mathrm{tr}(T X_\xi)| \le \int_0^\infty \left|\mathrm{tr}\left(\mathrm{e}^{tA} T \mathrm{e}^{tA^*} F_\xi\right)\right| \mathrm{d}t,$$

from which one easily concludes that $\|X_\xi\|$ is locally bounded.

(3) The spectral projection of K_ξ for the part of its spectrum in $\mathbb{C}_+$ can be written as

$$P_\xi = \begin{bmatrix} I \\ X_\xi \end{bmatrix} Y_\xi^{-1} \begin{bmatrix} \theta X_{\vartheta^{-1}-\xi}\theta & I \end{bmatrix} = \begin{bmatrix} I - Y_\xi^{-1} X_\xi & Y_\xi^{-1} \\ X_\xi(I - Y_\xi^{-1} X_\xi) & X_\xi Y_\xi^{-1} \end{bmatrix}.$$

Since $\mathscr{D} \ni \xi \mapsto P_\xi$ is real analytic by regular perturbation theory, Y_ξ^{-1} and $X_\xi Y_\xi^{-1}$ are real analytic function of $\xi \in \mathscr{D}$. The same holds for Y_ξ and $X_\xi = X_\xi Y_\xi^{-1} Y_\xi$.

Invoking the implicit function theorem and using the stability of D_ξ, one easily computes derivatives of the map $\mathscr{D} \ni \xi \mapsto X_\xi$. The first derivative is the linear map

$$\Xi \ni \eta \mapsto X'_\xi[\eta] = \widetilde{\eta} - \int_0^\infty \mathrm{e}^{tD_\xi^*} \Sigma_{\widetilde{\eta}} \mathrm{e}^{tD_\xi} \mathrm{d}t, \tag{5.10}$$

where, as usual, we identify $\eta \in \Xi$ with the corresponding diagonal matrix in $L(\Xi)$ and $\widetilde{\eta} \rhd \eta$. The second derivative is the quadratic form

$$\Xi \ni \eta \mapsto X''_\xi[\eta] = -2\int_0^\infty \mathrm{e}^{tD_\xi^*} (X'_\xi[\eta] - \widetilde{\eta}) B (X'_\xi[\eta] - \widetilde{\eta}) \mathrm{e}^{tD_\xi} \mathrm{d}t,$$

and concavity follows from the obvious fact that $X''_\xi[\eta] \le 0$.

To prove the inequalities let us rewrite the Riccati equation (5.6) as a Lyapunov equation

$$X_\xi A_\xi + A_\xi^* X_\xi = X_\xi B X_\xi - C_\xi,$$

and recall that, as established in the proof of Proposition 3.2 (5), $\mp A_\xi$ is stable for $\pm(\xi - (2\vartheta)^{-1}) > 0$. Thus, we have

$$\mp X_\xi = -\int_0^\infty \mathrm{e}^{\mp tA_\xi^*} (X_\xi B X_\xi - C_\xi) \mathrm{e}^{\mp tA_\xi} \mathrm{d}t \le \int_0^\infty \mathrm{e}^{\mp tA_\xi^*} C_\xi \mathrm{e}^{\mp tA_\xi} \mathrm{d}t,$$

and since $C_\xi \le 0$ for $|\xi - (2\vartheta)^{-1}| \ge (2\vartheta)^{-1}$, we conclude that $X_\xi \le 0$ for $\xi < 0$ and $X_\xi \ge 0$ for $\xi > \vartheta^{-1}$. The controllability of $(\mp A_\xi, Q)$ yields that these inequalities for X_ξ are strict. Writing (5.6) as

$$X_\xi D_\xi + D_\xi^* X_\xi = -X_\xi B X_\xi - C_\xi,$$

the stability of D_ξ gives

$$X_\xi = \int_0^\infty \mathrm{e}^{tD_\xi^*}(X_\xi B X_\xi + C_\xi)\mathrm{e}^{tD_\xi}\mathrm{d}t \ge \int_0^\infty \mathrm{e}^{tD_\xi^*} C_\xi \mathrm{e}^{tD_\xi}\mathrm{d}t,$$

and since $C_\xi \ge 0$ for $\xi \in \mathscr{D}_0$, we can conclude that $X_\xi \ge 0$ for such ξ. The controllability of (D_ξ, Q) again yields the strict inequality. The concavity of the map $\xi \mapsto X_\xi$ implies that the subset of all $\xi \in \mathscr{D}$ such that $X_\xi > 0$ is convex, so that it contains the convex hull of $\mathscr{D}_0 \cup \{\xi \in \mathscr{D} \,|\, \xi > \vartheta^{-1}\}$.

From

$$X_0 = \lim_{0>\xi\to 0} X_\xi \le 0, \qquad X_0 = \lim_{\mathscr{D}_0 \ni \xi \to 0} X_\xi \ge 0,$$

we deduce $X_0 = 0$. To prove the last assertion, starting from Eq. (5.6) and invoking Relations (5.8) one shows that $\widehat{M} = \theta X_{\vartheta^{-1}}^{-1}\theta$ satisfies the Lyapunov equation $A\widehat{M} + \widehat{M}A^* + B = 0$. Since A is stable, this equation has a unique solution given by (2.8), hence $\widehat{M} = M$.

(4) A simple calculation yields

$$\mathscr{R}_{\xi+\eta}(X + \widetilde{\eta}) = \mathscr{R}_\xi(X) + \Sigma_{\widetilde{\eta}}$$

for any $X \in L(\Gamma)$ and $\xi, \eta \in \Xi$. Thus, $\mathscr{R}_{\xi+\eta}(X_\xi + \widetilde{\eta}) = \Sigma_{\widetilde{\eta}}$ and since $A_{\xi+\eta} - B(X_\xi + \widetilde{\eta}) = D_\xi$, we conclude that whenever $\Sigma_{\widetilde{\eta}} = 0$ one has $X_{\xi+\eta} = X_\xi + \widetilde{\eta}$.

(5)–(7) The proof follows line by line the one of the corresponding Parts of [JPS, Proposition 5.5].

(8) Upon differentiating Eq. (3.7) one gets

$$\eta \cdot \nabla g(\xi) = -\frac{1}{2}\operatorname{tr}\Big(Q^*(X_\xi'[\eta] - \widetilde{\eta})Q\Big).$$

Further, using (5.10) leads to

$$\eta \cdot \nabla g(\xi) = \frac{1}{2}\int_0^\infty \operatorname{tr}\Big(\Sigma_{\widetilde{\eta}}\mathrm{e}^{tD_\xi} B \mathrm{e}^{tD_\xi^*}\Big)\mathrm{d}t,$$

and the result now follows from Part (5). □

5.2.2 Proof of Proposition 3.3

By Proposition 5.3, for $\xi \in \overline{\mathscr{D}}$, we have $A = D_\xi + QQ^*(X_\xi - \widetilde{\xi})$ with $\widetilde{\xi} \rhd \xi$, and we can rewrite the equation of the motion (2.2) as

$$\mathrm{d}x(t) = D_\xi x(t)\mathrm{d}t + Q\mathrm{d}w_\xi(t), \tag{5.11}$$

where

$$w_\xi(t) = w(t) - \int_0^t Q^*(\widetilde{\xi} - X_\xi)x(s)\mathrm{d}s.$$

Let $Z_\xi(t)$ be the stochastic exponential of the local martingale

$$\eta_\xi(t) = \int_0^t Q^*(\widetilde{\xi} - X_\xi)x(s) \cdot \mathrm{d}w(s).$$

Combining Eq. (5.6) with the relations $\widetilde{\xi}QQ^* = QQ^*\widetilde{\xi} = Q\xi Q^*$ and $\widetilde{\xi}QQ^*\widetilde{\xi} = Q\xi^2 Q^*$, we derive

$$\frac{1}{2}|Q^*(\widetilde{\xi} - X_\xi)x|^2 = -\sigma_{\widetilde{\xi}}(x) - (\widetilde{\xi} - X_\xi)x \cdot Ax,$$

and we can write the quadratic variation of η_ξ as

$$\frac{1}{2}[\eta_\xi](t) = -\int_0^t \sigma_{\widetilde{\xi}}(x(s))\mathrm{d}s - \int_0^t (\widetilde{\xi} - X_\xi)x(s)\cdot Ax(s)\mathrm{d}s.$$

Itô calculus and Proposition 3.1 (4) give

$$\log Z_\xi(t) = \eta_\xi(t) - \frac{1}{2}[\eta_\xi](t) = \xi\cdot W(t) - \chi_\xi(x(t)) + \chi_\xi(x(0)) - t\lambda_\xi,$$

with

$$\chi_\xi(x) = \frac{1}{2}x\cdot X_\xi x.$$

and, taking (3.7) into account,

$$\lambda_\xi = -\frac{1}{2}\operatorname{tr}(Q^*(X_\xi - \widetilde{\xi})Q) = g(\xi).$$

The proof of the following Lemma is identical to the one of [JPS, Lemma 5.7], and we omit it.

Lemma 5.4. *For $\xi \in \overline{\mathscr{D}}$, the process*

$$Z_\xi(t) = \mathrm{e}^{-tg(\xi)+\langle\xi,\Phi(t)\rangle-\chi_\xi(x(t))+\chi_\xi(x(0))} \tag{5.12}$$

is a $\mathbb{P}_x$-martingale for all $x \in \Gamma$.

Applying Girsanov theorem, we conclude that $\{w_\xi(t)\}_{t\in[0,\tau]}$ is a standard Wiener process under the law $\mathbb{Q}^\tau_{\xi,\mu}[\cdot] = \mathbb{E}_\mu[Z_\xi(\tau)\cdot]$. It follows that the finite-time cumulant generating function can be written as

$$g_t(\xi) = tg(\xi) + \log\mathbb{E}_\mu\left[Z_\xi(t)\mathrm{e}^{\chi_\xi(x(t))-\chi_\xi(x(0))}\right] = tg(\xi) + \log\mathbb{Q}^t_{\xi,\mu}\left[\mathrm{e}^{\chi_\xi(x(t))-\chi_\xi(x(0))}\right]$$

for $\xi \in \overline{\mathscr{D}}$, i.e.,

$$g_t(\xi) = tg(\xi) + \log d_t(\xi), \qquad d_t(\xi) = \langle\eta^1_\xi, Q^t_\xi\eta^2_\xi\rangle, \tag{5.13}$$

where

$$\eta^1_\xi(x) = \det(2\pi M)^{-\frac{1}{2}}\mathrm{e}^{-\chi_\xi(x)-\frac{1}{2}|M^{-\frac{1}{2}}x|^2}, \qquad \eta^2_\xi(x) = \mathrm{e}^{\chi_\xi(x)},$$

and Q^t_ξ is the Markov semigroup associated with the SDE (5.11). From the explicit solution

$$x(t) = \mathrm{e}^{tD_\xi}x(0) + \int_0^t \mathrm{e}^{(t-s)D_\xi}Q\mathrm{d}w_\xi(s)$$

to this SDE we easily obtain the representation

$$(Q^t_\xi f)(x) = \det(2\pi M_{\xi,t})^{-\frac{1}{2}}\int \mathrm{e}^{-\frac{1}{2}|M^{-\frac{1}{2}}_{\xi,t}(y-\mathrm{e}^{tD_\xi}x)|^2}f(y)\mathrm{d}y.$$

Setting

$$N_{\xi,t} = \begin{bmatrix} X_\xi + \theta X_{\theta^{-1}}\theta + \mathrm{e}^{tD^*_\xi}M^{-1}_{\xi,t}\mathrm{e}^{tD_\xi} & -\mathrm{e}^{tD^*_\xi}M^{-1}_{\xi,t} \\ -M^{-1}_{\xi,t}\mathrm{e}^{tD_\xi} & M^{-1}_{\xi,t} - X_\xi \end{bmatrix},$$

and $\mathscr{D}_t = \{\xi \in \mathscr{D} \mid N_{\xi,t} > 0\}$, an elementary calculation then leads to

$$\begin{aligned} d_t(\xi) &= \det(2\pi M_{\xi,t})^{-\frac{1}{2}}\det(2\pi M)^{-\frac{1}{2}}\int \mathrm{e}^{-\frac{1}{2}z\cdot N_{\xi,t}z}\mathrm{d}z \\ &= \begin{cases} \det(M^{-1}_{\xi,t})^{\frac{1}{2}}\det(M^{-1})^{\frac{1}{2}}\ \det(N_{\xi,t})^{-\frac{1}{2}} & \text{for } \xi \in \mathscr{D}_t; \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Schur's complement formula and Proposition 5.3 (5–6) lead to the factorization

$$\det(N_{\xi,t}) = d^-_t(\xi)d^+_t(\xi)$$

where

$$d_t^-(\xi) = \det(X_\xi + \theta X_{\vartheta^{-1}}\theta - \widetilde{\Delta}_{\xi,t}), \qquad d_t^+(\xi) = \det(X_{\vartheta^{-1}-\xi} + \theta\Delta_{\xi,t}\theta),$$

$\Delta_{\xi,t}$, as defined in Proposition 5.3 (6), and $\widetilde{\Delta}_{\xi,t} = \mathrm{e}^{tD_\xi^*}(X_\xi + X_\xi(\theta X_{\vartheta^{-1}-\xi}\theta + \Delta_{\xi,t})^{-1}X_\xi)\mathrm{e}^{tD\xi}$ are strictly positive for $t > 0$ and vanish exponentially as $t \to \infty$. Thus, for $\xi \in \mathscr{D}$,

$$d^-(\xi) = \lim_{t\to\infty} d_t^-(\xi) = \det(X_\xi + \theta X_{\vartheta^{-1}}\theta), \qquad d^+(\xi) = \lim_{t\to\infty} d_t^+(\xi) = \det(X_{\vartheta^{-1}-\xi}),$$

and setting

$$\mathscr{D}_t^- = \{\xi \in \mathscr{D} \mid X_\xi + \theta X_{\vartheta^{-1}}\theta > \widetilde{\Delta}_{\xi,t}\}, \qquad \mathscr{D}^- = \{\xi \in \mathscr{D} \mid X_\xi + \theta X_{\vartheta^{-1}}\theta > 0\},$$
$$\mathscr{D}_t^+ = \{\xi \in \mathscr{D} \mid \theta X_{\vartheta^{-1}-\xi}\theta > -\Delta_{\xi,t}\}, \qquad \mathscr{D}^+ = \{\xi \in \mathscr{D} \mid X_{\vartheta^{-1}-\xi} > 0\},$$

one has

$$\mathscr{D}_\infty = \mathscr{D}^- \cap \mathscr{D}^+ \subset \bigcup_{t>0}\bigcap_{s\geq t}(\mathscr{D}_s^- \cap \mathscr{D}_s^+).$$

It follows that, for all $\xi \in \mathscr{D}_\infty$, the limit

$$\lim_{t\to\infty} d_t(\xi) = \frac{\det(Y_\xi)^{1/2}\det(X_{\vartheta^{-1}})^{1/2}}{d^-(\xi)^{1/2}d^+(\xi)^{1/2}}$$

is finite and positive, which yields the first part of (3.8). To deal with the second part, we note that whenever $\xi \in \Xi \setminus \overline{\mathscr{D}_\infty}$, then either $\xi \in \Xi \setminus \overline{\mathscr{D}}$ or $\xi \in \overline{\mathscr{D}} \setminus \overline{\mathscr{D}_\infty}$. In the latter case, either $X_\xi + \theta X_{\vartheta^{-1}}\theta$ or $X_{\vartheta^{-1}-\xi}$ has a negative eigenvalue and the matrix $N_{\xi,t}$ loses its positiveness as $t \to \infty$. It follows that $d_t(\xi) = +\infty$ and hence $g_t(\xi) = +\infty$ for large enough t. For $\xi \in \Xi \setminus \overline{\mathscr{D}}$, applying [JPS, Lemma 5.8] to the functions $f_t(\alpha) = g_t(\alpha\xi)$ yields the desired result.

Finally, we note that the continuity and concavity of the map $\mathscr{D} \ni \xi \mapsto X_\xi$ imply that $\mathscr{D}_\infty$ is an open convex subset of $\mathscr{D}$. The positiveness of N_ξ for $\xi \in \overline{\mathscr{D}_0} \setminus \{\vartheta^{-1}\}$ is a consequence of its continuity and proves the last statement, concluding the proof of Proposition 3.3.

5.3 Proof of Theorem 3.4

Since $X_0 = 0$ and $X_{\vartheta^{-1}} = \theta M^{-1}\theta > 0$, it follows from the continuity of the map $\overline{\mathscr{D}} \ni \xi \mapsto X_\xi$ that the open set $\mathscr{D}_\infty$ contains 0. By Proposition 3.3, $\mathscr{D}_\infty$ is the interior of the essential domain of the limiting cumulant generating function (3.8). The Gärtner-Ellis theorem thus implies that the LDP upper bound in (3.10) hold for any Borel $F \subset \mathscr{L}^\perp$, with the rate function

$$I(\varphi) = \sup_{\xi\in\mathscr{D}_\infty}\big(\xi\cdot\varphi - g(\xi)\big) = \sup_{\xi\in\mathscr{S}_\infty}\big(\xi\cdot\varphi - g(\xi)\big).$$

Moreover, the corresponding lower bound holds for any subset F of the set $\mathscr{F}$ of exposed points of this function. Let us set $\mathscr{E} = \nabla g(\mathscr{S}_\infty)$. We have to show that $\mathscr{F} = \mathscr{E}$.

By Proposition 3.2 (3+4+6) , for all $\varphi \in \mathscr{L}^\perp$ one has

$$0 \leq J(\varphi) = \sup_{\xi\in\mathscr{S}}\big(\xi\cdot\varphi - g(\xi)\big) < \infty.$$

It follows from Proposition 3.2 (4+7) and [Ro, Theorem 26.5] that, as a function on $\mathscr{L}^\perp$, J is the Legendre conjugate of the restriction of g to $\mathscr{S}$. In particular, it is strictly convex and differentiable on $\mathscr{L}^\perp$. Moreover, $\nabla g : \mathscr{S} \to \mathscr{L}^\perp$ is a homeomorphism whose inverse is $\nabla J : \mathscr{L}^\perp \to \mathscr{S}$. Since $\mathscr{D}_\infty$ is open and convex, so is $\mathscr{S}_\infty$, and its image $\nabla g(\mathscr{S}_\infty) = \nabla g(\mathscr{D}_\infty) = \mathscr{E}$ is open and connected. We note that

$$I(\varphi) = \sup_{\xi\in\mathscr{S}_\infty}\big(\xi\cdot\varphi - g(\xi)\big) \leq J(\varphi)$$

for $\varphi \in \mathscr{L}^\perp$. For $\varphi = \nabla g(\xi) \in \mathscr{E}$ one has

$$J(\varphi) = \xi\cdot\nabla g(\xi) - g(\xi) = I(\varphi),$$

i.e., I and J coincide on $\mathscr{E}$. In particular, I is strictly convex on any convex subset of $\mathscr{E}$. Suppose that $\varphi \in \mathscr{E}$ is not an exposed point of I. Since $\varphi = \nabla g(\xi)$ with $\xi \in \mathscr{S}_\infty$, there exists $\psi \in \mathscr{L}^\perp$ such that $\psi \neq \varphi$ and $I(\psi) = I(\varphi) + \xi \cdot (\psi - \varphi)$. Invoking convexity, one shows that $I(\psi_\lambda) = I(\varphi) + \xi \cdot (\psi_\lambda - \varphi)$ with $\psi_\lambda = \lambda\psi + (1-\lambda)\varphi$ and $\lambda \in [0,1]$, which contradicts the strict convexity of I in a convex neighborhood of φ.

Whenever both $\pm\varphi \in \mathscr{E}$, we have $\varphi = \nabla g(\xi)$ and $-\varphi = \nabla g(\vartheta^{-1} - \xi)$ with $\xi \in \mathscr{S}_\infty$ and $\vartheta^{-1} - \xi \in \mathscr{S}_\infty$, and thus,

$$\begin{aligned} I(-\varphi) = I(\nabla g(\vartheta^{-1} - \xi)) &= (\vartheta^{-1} - \xi) \cdot \nabla g(\vartheta^{-1} - \xi) - g(\vartheta^{-1} - \xi) \\ &= -(\vartheta^{-1} - \xi) \cdot \nabla g(\xi) - g(\xi) \\ &= I(\nabla g(\xi)) - \vartheta^{-1} \cdot \nabla g(\xi) \\ &= I(\varphi) - \vartheta^{-1} \cdot \varphi. \end{aligned}$$

Finally, we note that $\mathscr{F}_0 = \{\nabla g(\xi) \,|\, \xi \in \mathscr{D}_\infty \text{ and } \vartheta^{-1} - \xi \in \mathscr{D}_\infty\}$. Proposition 5.3 (3) implies that $X_\xi > 0$ and $X_{\vartheta^{-1}-\xi} > 0$ for $\xi \in \mathscr{D}_0$, and hence that $\nabla g(\mathscr{D}_0) \subset \mathscr{F}_0$.

5.4 Proof of Theorem 3.5

By Theorem 3.4 it suffices to show that, under Condition **(R)**, one has $\nabla g(\mathscr{S}_\infty) = \mathscr{L}^\perp$. By Proposition 5.3 (4), for any $\xi \in \mathscr{S}$ and any $\eta \in \mathscr{L}$ one has

$$X_{\vartheta^{-1}-\xi-\eta} = X_{\vartheta^{-1}-\xi} - \widetilde{\eta}$$

so that

$$\mathscr{D}^+ = \{\xi \oplus \eta \,|\, \xi \in \mathscr{S}, \eta \in \mathscr{L}, \widetilde{\eta} < X_{\vartheta^{-1}-\xi}\}.$$

Similarly, from

$$X_{\xi+\eta} + \theta X_{\vartheta^{-1}} \theta = X_\xi + \theta X_{\vartheta^{-1}} \theta + \widetilde{\eta},$$

we deduce that

$$\mathscr{D}^- = \{\xi \oplus \eta \,|\, \xi \in \mathscr{S}, \eta \in \mathscr{L}, \widetilde{\eta} > -X_\xi - \theta X_{\vartheta^{-1}} \theta\}.$$

It follows that

$$\mathscr{D}_\infty = \{\xi \oplus \eta \,|\, \xi \in \mathscr{S}, \eta \in \mathscr{L}, -X_\xi - \theta X_{\vartheta^{-1}} \theta < \widetilde{\eta} < X_{\vartheta^{-1}-\xi}\},$$

and hence

$$\mathscr{S}_\infty = \{\xi \in \mathscr{S} \,|\, \text{there exists } \eta \in \mathscr{L} \text{ such that } -X_\xi - \theta X_{\vartheta^{-1}} \theta < \widetilde{\eta} < X_{\vartheta^{-1}-\xi}\}.$$

Thus, under Condition **(R)**, $\nabla g(\mathscr{S}_\infty) = \nabla g(\mathscr{S}) = \nabla g(\mathscr{D}) = \mathscr{L}^\perp$.

References

[ADJ] Atilgan, A.R., Durrell, S.R., Jernigan, R.L., Demirel, M.C., Keskin, O., and Bahar, I.: Anisotropy of fluctuation dynamics of proteins with an elastic network model. Biophys. J. **80**, 505–515 (2001).

[AGMT] Andrieux, D., Gaspard, P., Monnai, T., and Tasaki, S.: The fluctuation theorem for currents in open quantum systems. New J. Phys. **11**, 043014 (2009).

[BJTM] Baiesi, M., Jacobs, T., Maes, C., and Skantzos, N.S.: Fluctuation symmetries for work and heat. Phys. Rev. E **74**, 021111 (2006).

[BK] Bochkov, G.N., and Kuzovlev, Yu.E.: General theory of thermal fluctuations in nonlinear systems. Sov. Phys. JETP **45**, 125–130 (1977).

[BL] Bodineau, T., and Lefevere, R.: Large deviations of lattice Hamiltonian dynamics coupled to stochastic thermostats. J. Stat. Phys. **133**, 1–27 (2008).

[BGGZ] Bonetto, F., Gallavotti, G., Giuliani, A., and Zamponi, F.: Chaotic hypothesis, fluctuation theorem and singularities. J. Stat. Phys. **123**, 39–54 (2006).

[BJP] Benoist, T., Jakšić, V., and Pillet, C.-A.: Energy statistics in open harmonic networks. J. Stat. Phys. **168**, 1016–1030 (2017).

[BPP] Benoist, T., Panati, A., and Pautrat, Y.: Heat conservation and fluctuations for open quantum systems in the Two-Time Measurement picture. Preprint arXiv:1810.09999.

[CCZ] van Zon, R., Ciliberto, S., and Cohen, E.G.D.: Power and heat fluctuation theorems for electric circuits. Phys. Rev. Lett. **92**, 130601 (2004).

[CDF] Douarche, F., Joubaud, S., Garnier, N.B., Petrosyan, A. and Ciliberto, S.: Work fluctuation theorems for harmonic oscillators. Phys. Rev. Lett. **97**, 140603 (2006).

[CE] Cuneo, N., and Eckmann, J.-P.: Controlling general polynomial networks. Commun. Math. Phys. **328**, 1255–1274 (2014).

[CEHR] Cuneo, N., Eckmann, J.-P., Hairer, M., and Rey-Bellet, L.: Non-equilibrium steady states for networks of oscillators. Electronic J. Proba. **23**, 1–28 (2018).

[CG] Chetrite, R., and Gawędzki, K.: Fluctuation relations for diffusion processes. Commun. Math. Phys. **282**, 469–518 (2008).

[Cr] Crooks, G.E.: Nonequilibrium measurements of free energy differences for microscopically reversible Markovian systems. J. Stat. Phys. **90**, 1481–1487 (1998).

[CZ] Cohen, E.G.D., van Zon, R.: Extended heat-fluctuation theorems for a system with deterministic and stochastic forces. Phys. Rev. E **69**, 056121 (2004).

[dH] den Hollander, F.: *Large Deviations.* Fields Institute Monographs, AMS, Providence, Rhodes Island, (2000).

[DZ] Dembo, A., and Zeitouni, O.: *Large Deviations. Techniques and Applications.* Springer, Berlin, (1998).

[ECM] Evans, D.J., Cohen, E.G.D., and Morriss, G.P.: Probability of second law violation in shearing steady flows. Phys. Rev. Lett. **71**, 2401–2404 (1993).

[EN] Nickelsen, D., and Engel, A.: Asymptotics of work distributions: the pre-exponential factor. Eur. Phys. J. B **82**, 207–218 (2011).

[EPR] Eckmann, J.-P., Pillet, C.-A., and Rey-Bellet, L.: Non-equilibrium statistical mechanics of anharmonic chains coupled to two heat baths at different temperatures. Commun. Math. Phys. **201**, 657–697 (1999).

[ES] Evans, D.J., and Searles, D.J.: Equilibrium microstates which generate second law violating steady states. Phys Rev. E **50**, 1645–1648 (1994).

[EZ] Eckmann, J.-P., and Zabey, E.: Strange heat flux in (an)harmonic networks. J. Stat. Phys. **114**, 515–523 (2004).

[Fa] Farago, J.: Injected power fluctuations in Langevin equation. J. Stat. Phys. **107**, 781–803 (2002).

[Ga1] Gallavotti, G.: Extension of Onsager's reciprocity to large fields and the chaotic hypothesis. Phys. Rev. Lett. **77**, 4334–4337 (1996).

[Ga2] Gallavotti, G.: New methods in nonequilibrium gases and fluids. Open Systems and Information Dynamics **6**, 101–136 (1999).

[Ga3] Gallavotti, G.: *Statistical Mechanics. A short treatise.* Springer, Berlin, 2000.

[GC1] Gallavotti, G., and Cohen, E.G.D.: Dynamical ensembles in nonequilibrium statistical mechanics. Phys. Rev. Lett. **74**, 2694–2697 (1995).

[GC2] Gallavotti, G., and Cohen, E.G.D.: Dynamical ensembles in stationary states. J. Stat. Phys. **80**, 931–970 (1995).

[GMR] Granger, L., Mehlis, J., Roldán, É., Ciliberto, S., and Kantz, H.: Fluctuation theorem between non-equilibrium states in an RC circuit. New J. Phys. **17**, 065005 (2015)

[HE] Haliloglu, T.B., and Erman, B.: Gaussian dynamics of folded proteins. Phys. Rev. Lett. **79**, 3090–3093 (1997).

[Ja] Jarzynski, C.: A nonequilibrium equality for free energy differences. Phys. Rev. Lett. **78**, 2690–2693 (1997).

[JNPS1] Jakšić, V., Nersesyan, V., Pillet, C.-A., and Shirikyan, A.: Large deviations from a stationary measure for a class of dissipative PDEs with random kicks Commun. Pure Appl. Math. **68**, 2108–43 (2015).

[JNPS2] Jakšić, V., Nersesyan, V., Pillet, C.-A., and Shirikyan, A.: Large deviations and mixing for dissipative PDEs with unbounded random kicks. Nonlinearity **31**, 540–596 (2018).

[JNPS3] Jakšić, V., Nersesyan, V., Pillet, C.-A., and Shirikyan, A.: Large deviations and Gallavotti-Cohen principle for dissipative PDEs with rough noise. Commun. Math. Phys. **336**, 131–170 (2015).

[JNPS4] Jakšić, V., Nersesyan, V., Pillet, C.-A., and Shirikyan, A.: Large deviations and entropy production in viscous fluid flows. Preprint arXiv:1902.03278.

[JOPP] Jakšić, V., Ogata, Y., Pautrat, Y., and Pillet, C.-A.: Entropic fluctuations in quantum statistical mechanics–an introduction. In *Quantum Theory from Small to Large Scales.* J. Fröhlich, M. Salmhofer, V. Mastropietro, W. De Roeck and L.F. Cugliandolo editors. Oxford University Press, Oxford, 2012.

[JPC] Jop, P., Petrosyan, A., and Ciliberto, S.: Work and dissipation fluctuations near the stochastic resonance of a colloidal particle. EPL **81**, 50005 (2008).

[JPR] Jakšić, V., Pillet, C.-A., and Rey-Bellet, L.: Entropic fluctuations in statistical mechanics I. Classical dynamical systems. Nonlinearity **24**, 699–763 (2011).

[JPS] Jakšić, V., Pillet, C.-A., and Shirikyan, A.: Entropic fluctuations in thermally driven harmonic networks. J. Stat. Phys. **166**, 926–1015 (2017).

[KSD] Kundu, A., Sabhapandit, S., and Dhar, A.: Large deviations of heat flow in harmonic chains. J. Stat. Mech. P03007 (2011).

[LR] Lancaster, P., and Rodman, L.: *The Algebraic Riccati Equation.* Clarendon Press, Oxford, 1995.

[LS] Lebowitz, J.L., and Spohn, H.: A Gallavotti–Cohen-type symmetry in the large deviation functional for stochastic dynamics. J. Stat. Phys. **95**, 333–365 (1999).

[Ma] Maes, C.: The fluctuation theorem as a Gibbs property. J. Stat. Phys. **95**, 367–392 (1999).

[MN] Maes, C., Netočný, K., and Verschuere, M.: Heat conduction networks. J. Stat. Phys. **111**, 1219–1244 (2003).

[Ne] Nersesyan, V.: Large deviations for the Navier-Stokes equations driven by a white-in-time noise. To appear in Ann. H. Lebesgue, (2019).

[RH] Rákos, A., and Harris, R.J.: On the range of validity of the fluctuation theorem for stochastic Markovian dynamics. J. Stat. Mech. P05005 (2008).

[RM] Rondoni, L., and Mejía-Monasterio, C.: Fluctuations in non-equlibrium statistical mechanics: models, mathematical theory, physical mechanisms. Nonlinearity **20**, 1–37 (2007).

[Ro] Rockafellar, R.T.: *Convex Analysis.* Princeton University Press, Princeton, NJ, 1972.

[RT] Rey-Bellet, L., and Thomas, L.E.: Fluctuations of the entropy production in anharmonic chains. Ann. Henri Poincaré **3**, 483–502 (2002).

[Ru1] Ruelle, D.: *Statistical Mechanics: Rigorous Results (3rd edition).* Mathematical Physics Monograph Series. Benjamin, Reading, MA, 1977.

[Ru2] Ruelle, D.: Smooth dynamics and new theoretical ideas in nonequilibrium statistical mechanics J. Stat. Phys. **95**, 393–468 (1999).

[Se] Seifert, U.: Stochastic thermodynamics, fluctuation theorems and molecular machines. Rep. Prog. Phys. **75**, 126001 (2012).

[Vi] Visco, P.: Work fluctuations for a Brownian particle between two thermostats. J. Stat. Mech. P06006, (2006).

[W] Watson, G.N.: *A Treatise on the Theory of Bessel Functions,* 2nd edition. Cambridge University Press, Cambridge (1966).