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# Comments on "Series for Collision Probability in Short-Encounter Model"

Denis Arzelier,\*

*LAAS-CNRS, Université de Toulouse, CNRS, Toulouse, France.*

Florent Bréhard†

*LIP & LAAS-CNRS, ENS de Lyon & Université de Toulouse, CNRS, Lyon & Toulouse, France.*

Mioara Joldes‡

*LAAS-CNRS, Université de Toulouse, CNRS, Toulouse, France.*

## I. Introduction

An analytic solution based on a series expansion of the collision probability of two spheres under the assumption of short-term encounters has recently been proposed in [1]. Relying on the analyticity of the Gaussian distribution of the relative positions and its Taylor expansion, this approach yields a series expansion for the classical 2D formulation of the collision probability, where the terms are obtained as a product of Hermite polynomials. Unfortunately, this work cannot be considered as a new contribution to the field since the method itself is nothing but a crude variety of the method published shortly before in [2] (see also the early conference version [3]) and referenced in [1]. Indeed, this note shows that the series expansion formulation of the probability collision given in [1] is exactly the one exposed in [2] without a preconditionner and therefore suffers from numerical issues clearly identified in this article and due to the so-called cancellation phenomenon. Classic and real-case numerical examples further demonstrate that the results of [1] are of limited interest.

## II. Theoretical analysis

The short-term encounter model (whose relevant assumptions are recalled in detail in [4] or [2]) for the computation of the probability of collision between two objects considered as spheres mainly consists in combining objects' uncertainty and projecting it onto the encounter plane defined to be perpendicular to the relative velocity when assuming linear relative motion. Let  $(x, y, z)$  denote the mean coordinates of the relative position of the secondary object with respect to the primary in the encounter frame (see [2] for its definition), the relative position uncertainty in the encounter plane is described by the following bivariate Gaussian density function:

$$G(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp \left[ -\frac{1}{2} \left( \frac{(x - x_m)^2}{\sigma_x^2} + \frac{(y - y_m)^2}{\sigma_y^2} \right) \right], \quad (1)$$

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\*Senior researcher, ROC Team, LAAS-CNRS, 07 avenue Colonel Roche, 31400 Toulouse, France, arzelier@laas.fr

†Ph.D., AriC & MAC teams, ENS de Lyon, 15 Parvis René Descartes, BP 7000, 69342 Lyon & LAAS-CNRS, 07 avenue Colonel Roche, 31400 Toulouse, France, florent.brehard@ens-lyon.fr

‡Researcher, MAC Team, LAAS-CNRS, 07 avenue Colonel Roche, 31400 Toulouse, France, joldes@laas.fr

where it is assumed without loss of generality that the standard deviations of the relative coordinates in the encounter plane,  $\sigma_x, \sigma_y \in \mathbb{R}_*^+$  are such that  $\sigma_x \geq \sigma_y$  and  $(x_m, y_m)$  are the coordinates of the position of the secondary object relative to the primary object in the covariance frame.

Under the assumptions of the short-term encounter model, the probability of collision involving two spherical objects is given as a two-dimensional integral on a disk  $\bar{B}(0, 0, R)$ , parameterized by the radius  $R$  of the combined spherical object in the encounter plane:

$$g(R^2) = \frac{1}{2\pi\sigma_x\sigma_y} \int_{\bar{B}(0,0,R)} \exp \left[ -\frac{1}{2} \left( \frac{(x-x_m)^2}{\sigma_x^2} + \frac{(y-y_m)^2}{\sigma_y^2} \right) \right] dx dy = \int_{\bar{B}(0,0,R)} G(x, y) dx dy. \quad (2)$$

Several approaches have been developed for the accurate and fast computation of this two-dimensional integral. Recently, a fully analytic method based on the power series expansion of (2) and involving Hermite polynomials has been proposed in [1]:

$$g(R^2) = \sum_{k=0}^{+\infty} \frac{4\pi}{2^{2(k+1)}(k+1)(k!)^2} \sum_{j=0}^k \binom{k}{j} \frac{H_{2(k-j)}(x_m/\sigma_x) H_{2j}(y_m/\sigma_y)}{\sigma_x^{2(k-j)} \sigma_y^{2j}} G(0, 0) R^{2(k+1)}, \quad (3)$$

where  $H_j$  is the  $j$ th probabilists' Hermite polynomials. The Authors of [1] claim that the computational algorithm based on this expansion is original and the fastest to compute the probability of collision and that only two terms of the series expansion are necessary for all practical cases. In this note it is firstly shown below that the proposed series expansion may be regarded as the one proposed in [2], but without a preconditionner which is necessary for an efficient and accurate numerical evaluation of the series. Secondly, it is shown based on [2], that the terms of the unpreconditioned series can be more efficiently evaluated than the method proposed in [1]. However, as already discussed in [2], these terms are alternating in sign and the series cannot be accurately evaluated without the preconditionner, as finally shown on practical cases.

Along the lines of the derivations presented in [2], the series expansion (3) may be obtained as follows:

- Compute the closed-form of the Laplace transform  $h(\lambda) = \mathcal{L}_g(\lambda)$ , of the function  $g$  of the variable  $R^2$ , for all  $\lambda$ ,  $\text{Re}(\lambda) \geq \kappa$  for any  $\kappa \in \mathbb{R}^{+*}$ ;
- Derive the Linear Ordinary Differential Equation (LODE) with polynomial coefficients satisfied by  $h$  and the associated initial condition ( $h$  is a so-called differentially finite function);
- After analyzing the singularities of  $h(\lambda)$ , consider its Laurent series expansion  $h(\lambda) = \sum_{k=0}^{\infty} \alpha_k \lambda^{-(k+2)}$ ;
- From the previous LODE, deduce the recurrence satisfied by the coefficients  $\alpha_k$ ;
- The term by term inverse Laplace transform of the expansion of  $h(\lambda)$  provides the power series expansion (4) of  $g$ ;

$$g(R^2) = \sum_{k=0}^{+\infty} r_k R^{2(k+1)}. \quad (4)$$

The advantage of the above derivation is that the recurrence for the coefficients  $r_k$  of the power series expansion may be

readily obtained as

$$\begin{aligned}
(k+5)(k+4)^2(k+3)(k+2)r_{k+4} &= [n_3 - kd_4]r_k + [n_2 - (k+1)d_3](k+2)r_{k+1} \\
&+ [n_1 - (k+2)d_2](k+3)(k+2)r_{k+2} \\
&+ [n_0 - (k+3)d_1](k+4)(k+3)(k+2)r_{k+3}.
\end{aligned} \tag{5}$$

with the initial conditions

$$\begin{aligned}
r_0 &= \pi G(0, 0), \\
r_1 &= \frac{\pi G(0, 0)}{2!} n_0, \\
r_2 &= \frac{\pi G(0, 0)}{2 \cdot 3!} [n_1 + n_0(n_0 - d_1)], \\
r_3 &= \frac{\pi G(0, 0)}{2 \cdot 3 \cdot 4!} [2n_2 - 2d_1n_1 + n_0(d_1(2d_1 - 3n_0) + n_0^2 - 2d_2 + 3n_1)],
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
n_3 &= -q^4(1-\phi)^2, & n_2 &= \omega_y q^2(1-\phi)^2 + \omega_x q^2 - 3q^3(1-\phi) \left(1 - \frac{\phi}{2}\right), \\
n_1 &= 2q(\omega_y(1-\phi) + \omega_x) - q^2 \left(3 - 3\phi + \frac{\phi^2}{2}\right), & n_0 &= \omega_x + \omega_y - \frac{q}{2}(2-\phi), \\
d_4 &= q^4(1-\phi)^2, & d_3 &= 2q^3(1-\phi)(2-\phi), \\
d_2 &= q^2(\phi^2 - 6\phi + 6), & d_1 &= 2q(2-\phi),
\end{aligned} \tag{7}$$

and

$$q = \frac{1}{2\sigma_y^2}, \quad \phi = 1 - \frac{\sigma_y^2}{\sigma_x^2}, \quad \omega_x = \frac{x_m^2}{4\sigma_x^4}, \quad \omega_y = \frac{y_m^2}{4\sigma_y^4}. \tag{8}$$

In the reference [2] (see also [5, Propositions 1.1.10 and 1.6.3]), it is shown that the function  $g$  is an entire function (or a real analytic function on  $\mathbb{R}$ ) and therefore its power series expansion at  $R^2$  is unique and we have that

$$r_k = \frac{4\pi}{2^{2(k+1)}(k+1)(k!)^2} \sum_{j=0}^k \binom{k}{j} \frac{H_{2(k-j)}(x_m/\sigma_x) H_{2j}(y_m/\sigma_y)}{\sigma_x^{2(k-j)} \sigma_y^{2j}} G(0, 0). \tag{9}$$

**Remark 1** *The equality between the power series expansion (3) and (4) may also be checked after simple but lengthy computations by showing that the coefficients of (3) satisfy the recurrence (5) and its initial conditions (6).*

The linear recurrence (5) obtained for the computation of the coefficients of the power series of (2) is interesting in itself since it simply leads to a computational algorithm linear in the number of terms of the truncated sum, while the

formulation (3) induces a quadratic algorithm. In addition, it allows to readily check that not all the coefficients  $r_k$  of the series (3) are positive, preventing to get a lower bound for the exact probability of collision by computing any arbitrary partial sum of order  $N$ . This, as a matter of fact, implies that the formulation (3) may suffer from numerical instability due to the fact that the computation in finite precision arithmetic of such a partial sum is prone to cancellation [6]. This means that for consecutive terms that are close in magnitude, but of different signs, their sum in finite precision arithmetic contains very few correct significant digits. This makes the power series evaluation impractical for large values of  $R^2$ . Instead of directly expanding (2), it has been therefore proposed in the reference [2] to expand the function  $\psi \cdot g$  where  $\psi : R^2 \mapsto \exp[pR^2]$  is a so-called preconditionner, in order to remedy these observed numerical shortcomings. Some insights about the right choice for the parameter  $p$  are given in [7]. The function (2) is finally obtained as a simple product between an exponential term and a convergent power series with positive coefficients. As a very interesting by-product, it is worth mentioning that analytic bounds on the truncation error may also be derived (only for the preconditionned series, whose terms are positive).

Finally, we disagree with the claim of the Authors about formulation (3) that "its first two terms alone are sufficient for the computation of the probability of collision". Well-known numerical examples borrowed from the literature [8] and practical cases extracted from the database of CNES (French space agency) illustrate this particular point in the next section.

### III. Numerical analysis

In this section, it is shown on four examples that formulation (3) may fail to compute accurately the probability of collision when the parameters of the collision meet a specific configuration ( $R^2/2\sigma_y^2 \gg 1$ ). The chosen numerical examples are the test cases 3 and 5 from the reference [8] and CNES database cases 1,2 which are related to Smos (ocean and surface soil monitoring, Sun-synchronous orbit) and Jason (altimetry satellite, circular orbit) respectively.

Case label	$\sigma_x$	$\sigma_y$	$R$	$x_m$	$y_m$	$\sigma_x/\sigma_y$	$R/\sigma_y$
Alfano case 3	114.25852	1.41018	15	0.15916	-3.88721	81.02407	10.63694
Alfano case 5	177.81090	0.03733	10	2.12301	-1.22179	4763.21725	267.88107
Database case 1	218.27304	3.58024	20	164.4	30.19	56.69083	5.19449
Database case 2	129.79788	3.50240	20	25.61622	-0.15315	37.05967	5.71037

**Table 1** Inputs for test cases from [8] and CNES database.

The complete numerical results are given in Table 2, where the first column presents the probability of collision computed with a Monte Carlo approach applied to the 2D integral (2). To this end, a random sampling of  $N_{m.c.}$  relative position vectors in the encounter is generated according to the projected covariance matrix. For each sample  $i$ , one defines  $\delta_i = 1$  if  $(x, y) \in \bar{B}(0, R)$  and  $\delta_i = 0$  otherwise. The collision probability is then computed as  $\sum_{i=1}^{N_{m.c.}} \delta_i / N_{m.c.}$ . The sampling is chosen according to a confidence level of 95% and a relative accuracy of 0.1. Columns 2 and 3 of Table

2 show the results obtained by the recurrence proposed by [2] (with preconditionning) and the number of terms needed to have 5 correct significant digits. The results when applying (4) with two terms and finally the results of the recurrence (5) with the same number of terms as the one with the preconditionner are given in columns 4, 5 respectively. The last column gives the minimum number of terms needed to have 5 correct significant digits with the non preconditionned recurrence (5).

Case label	Monte Carlo	Rec. from [2]		Rec. from [1] ( $N = 2$ )	Rec. from [1]	
Alfano case 3	1.005E-01	1.004E-01	54	1.47436	1.355E+19	-
Alfano case 5	4.450E-02	4.451E-02	18530	1.76821E-225	-	-
Database case 1	6.578E-05	6.768E-05	30	1.63823E-12	6.7686E-05	30
Database case 2	1.181E-01	1.1823E-01	23	-1.32378	3.6063E+02	46

**Table 2 Results for test cases from [8] and CNES database obtained by algorithms from [2] and from [1].**

The first two examples of Alfano were originally chosen to compare the efficiency of several methods of the literature and are quite challenging. Indeed, the number of terms to be considered in the series expansion from [2] is important, in particular for case 5. Meaningless results are obtained in both Alfano's cases when using only the first two terms of the series expansion proposed in [1], contradicting the claim in the conclusion of the mentioned paper. Interestingly enough, the situation gets worse if an increasing number of terms is used in the series expansion to compute the probability as shown in the last column of Table 2. The same trend is noted for the two other numerical examples for the results obtained with the first two terms of the series expansion given in [1], providing evidence contrary to what is stated in [1] on real-case scenarios. Finally, if the same number of terms is required for the two recurrences in the first database case, the second shows clearly the defect of the non preconditionned series with twice the required number of terms to reach the same precision.

#### IV. Conclusion

The series expansion of the collision probability under the assumptions of short-term space encounters, presented in [1] is not genuinely new since an improved version has been already published in [3] and [2]. Moreover, the refinement used in these references appears to be mandatory to get a safe computation of the probability of collision for hard cases as illustrated on several examples. Finally, the claim that only two terms of the series expansion are sufficient from a practical point of view is contradicted by numerical real cases.

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