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STABILITY AND HOLOMORPHIC CONNECTIONS ON VECTOR BUNDLES OVER LVMB MANIFOLDS

INDRANIL BISWAS, SORIN DUMITRESCU, AND LAURENT MEERSSEMAN

ABSTRACT. We characterize all LVMB manifolds X such that the holomorphic tangent bundle TX is spanned at the generic point by a family of global holomorphic vector fields, each of them having non-empty zero locus. We deduce that holomorphic connections on semi-stable holomorphic vector bundles over LVMB manifolds with this previous property are always flat.

Résumé. **Fibrés vectoriels holomorphes sur les variétés LVMB.** Nous caractérisons les variétés LVMB qui ont la propriété de positivité \mathcal{P} suivante : le fibré tangent holomorphe est engendré au point générique par une famille de champs de vecteurs holomorphes (globalement définis) $\{v_i\}$, tel que chaque v_i s'annule en au moins un point de X . Nous en déduisons que, sur les variétés LVMB avec la propriété \mathcal{P} , les connexions holomorphes sur les fibrés vectoriels holomorphes semi-stables sont nécessairement plates.

Version française abrégée

Les variétés LVMB [10, 11, 3, 4] (voir aussi [14]) forment une classe de variétés complexes compactes (en général, non-Kähler) qui généralise les variétés toriques et aussi les variétés de Hopf diagonales. Les variétés LVMB sont des compactifications équivariantes lisses de groupes de Lie complexes abéliens.

Cet article étudie la géométrie complexe des fibrés holomorphes au-dessus d'une variété LVMB.

Rappelons qu'une variété LVMB X est biholomorphe à un quotient d'un ouvert V de l'espace projectif complexe \mathbb{P}^{n-1} par une action holomorphe, libre et propre du groupe de Lie complexe \mathbb{C}^m , engendré par m (avec $n > 2m$) champs de vecteurs linéaires diagonaux qui commutent.

L'ouvert V est le complémentaire dans \mathbb{P}^{n-1} d'une union E de k hyperplanes de coordonnées. Par conséquent, l'ouvert V contient le tore $\mathbb{T} := (\mathbb{C}^*)^{n-1}$ et est invariant par l'action naturelle de \mathbb{T} sur \mathbb{P}^{n-1} . Le quotient de \mathbb{T} par l'action de \mathbb{C}^m s'identifie à un groupe de Lie complexe abélien G . Cela implique que X est une compactification équivariante lisse (de dimension $\nu = n - m - 1$) de G [10, 11, 3, 4].

Le premier résultat principal de l'article (Théorème 2.4) est une caractérisation de variétés LVMB X qui ont la propriété de positivité \mathcal{P} suivante : le fibré tangent holomorphe TX est engendré au point générique par une famille de champs de vecteurs holomorphes (globalement définis) (v_1, \dots, v_{n-m-1}) , tel que chaque v_i s'annule en au moins un point de X . Le théorème 2.4 affirme qu'une variété LVMB a la propriété \mathcal{P} si et seulement si $k \leq m + 1$.

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Le cas $k = m + 1$ est particulièrement intéressant car les variétés LVMB correspondantes possèdent des structures affines complexes (notez que la preuve du Lemme 12.1 faite dans [4] pour les variétés LVM se généralise sans difficulté au cadre LVMB). Les variétés LVMB les plus simples avec $k = m + 1$ sont les variétés de Hopf diagonales, mais il y a également des exemples dont la topologie peut-être compliquée (variétés difféomorphes à un produit entre $(\mathbb{S}^1)^m$ et une somme connexe de produits de sphères, ainsi que des variétés avec torsion arbitraire dans le sens du Théorème 14.1 et du Corollaire 14.3 dans [4]).

Dans la deuxième partie de l'article nous étudions les fibrés holomorphes E munis de connexions holomorphes au-dessus des variétés LVMB qui ont la propriété \mathcal{P} . Plus précisément, nous faisons des liens avec la notion de stabilité au sens des pentes. Mentionnons que les notions de degré, pente et (semi)-stabilité sont considérées ici par rapport à une métrique de Gauduchon. En particulier, le degré n'est pas, en général, un invariant topologique (comme c'est le cas dans le contexte Kähler).

Nous montrons que chaque sous-faisceau E_i dans la filtration de Harder-Narasimhan de E

$$0 = E_0 \subset E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_b \subsetneq E_{b+1} = E$$

est invariant par la connexion holomorphe et est, par conséquent, localement libre. Nous utilisons ensuite la propriété de positivité \mathcal{P} de TX pour en déduire que la connexion holomorphe induite sur chaque quotient semi-simple E_i/E_{i-1} est nécessairement plate. Par conséquent, nous obtenons le Corollaire 3.5 qui affirme que toute connexion holomorphe sur un fibré semi-stable E au-dessus d'une variété LVMB avec la propriété \mathcal{P} est nécessairement plate.

1. INTRODUCTION

An important class of compact complex (in general, non-Kähler) manifolds generalizing toric varieties and diagonal Hopf manifolds are the so-called LVMB manifolds [10, 11, 3, 4] (see also [14]). They are smooth equivariant compactifications of complex abelian Lie groups.

This article deals with the geometry of holomorphic vector bundles over LVMB manifolds.

Our first result (Theorem 2.4) is a geometric characterization of LVMB manifolds X admitting a family of global holomorphic vector fields spanning the holomorphic tangent bundle TX at the generic point and such that each vector field in the family has a non-empty zero locus.

We use this positivity result and stability arguments to show that holomorphic connections in semi-stable holomorphic vector bundles over those LVMB manifolds are always flat (see Corollary 3.5).

2. GEOMETRY OF THE TANGENT BUNDLE OF LVMB MANIFOLDS

We refer to [11] and [3] for more details on the properties of LVMB manifolds used below. It should be clarified that although the article [11] deals with LVM manifolds, most results there hold verbatim, and with the same proof, for the more general LVMB case.

A LVMB manifold X is a quotient of some open and dense subset V of \mathbb{P}^{n-1} by a free and proper holomorphic \mathbb{C}^m -action generated by m commuting linear diagonal vector fields ξ_1, \dots, ξ_m (with $n > 2m$). More precisely, the action is given by

$$T \cdot [z] := [z_i e^{(\Lambda_i, T)}]_{i=1}^n \quad (2.1)$$

where the m rows of the matrix $(\Lambda_1, \dots, \Lambda_n)$ are the coefficients of the m vector fields ξ_1, \dots, ξ_m as linear combinations of $z_1 \frac{\partial}{\partial z_1}, \dots, z_n \frac{\partial}{\partial z_n}$.

The subset V is the complement in \mathbb{P}^{n-1} of a union E of coordinate subspaces. It follows that it contains a torus $\mathbb{T} := (\mathbb{C}^*)^{n-1}$ and is preserved under the natural action of \mathbb{T} onto \mathbb{P}^{n-1} . The quotient of \mathbb{T} by the \mathbb{C}^m -action is a complex abelian Lie group, say G , and X is a smooth equivariant compactification of G .

The following alternative description of G can be found in [10, p.27] (and which itself is a straightforward generalization of Theorem 1 in [11]).

Proposition 2.1. *Assume that*

$$\text{rank}_{\mathbb{C}} \begin{pmatrix} \Lambda_1 & \cdots & \Lambda_{m+1} \\ 1 & \cdots & 1 \end{pmatrix} = m + 1. \quad (2.2)$$

Then G_{Λ} is isomorphic to the quotient of \mathbb{C}^{n-m-1} by the \mathbb{Z}^{n-1} abelian subgroup generated by $(Id, B_{\Lambda} A_{\Lambda}^{-1})$ where

$$A_{\Lambda} = {}^t(\Lambda_2 - \Lambda_1, \dots, \Lambda_{m+1} - \Lambda_1)$$

and

$$B_{\Lambda} = {}^t(\Lambda_{m+2} - \Lambda_1, \dots, \Lambda_{n-1} - \Lambda_1).$$

Remark 2.2. It is easy to prove that

$$\text{rank}_{\mathbb{C}} \begin{pmatrix} \Lambda_1 & \cdots & \Lambda_n \\ 1 & \cdots & 1 \end{pmatrix} = m + 1.$$

(cf. [12, Lemma 1.1] in the LVM case). Hence, up to a permutation, condition (2.2) is always fulfilled.

Let \mathfrak{G} be the Lie algebra of G . Given $a \in \mathfrak{G}$, let

$$v(x) := \frac{d}{dt} (x \cdot \exp(ta))|_{t=0}, \quad x \in X$$

be the corresponding fundamental vector field.

Definition 2.3. We say that X has the non-zero vanishing property if we may find a basis (a_1, \dots, a_{n-m-1}) of \mathfrak{G} such that the corresponding fundamental vector fields (v_1, \dots, v_{n-m-1}) all have a non-empty vanishing locus.

Let k denote the number of coordinate hyperplanes in E . We have

Theorem 2.4. *A LVMB manifold X has the non-zero vanishing property if and only if the inequality $k \leq m + 1$ is fulfilled.*

Proof. Observe that the natural map $V \rightarrow X$ induces a Lie group morphism $\mathbb{T} \rightarrow G$. Hence, using homogeneous coordinates $[z_1; \dots; z_n]$ in \mathbb{P}^{n-1} , the image of the vector fields $z_1 \frac{\partial}{\partial z_1}, \dots, z_n \frac{\partial}{\partial z_n}$ form a generating family of fundamental vector fields of X .

Observe also that such a vector field $z_i \frac{\partial}{\partial z_i}$ vanishes on some point of X if and only if (the projectivization of) $\{z_i = 0\}$ is not included in E .

Assume firstly that $k \leq m + 1$. For simplicity, assume that the coordinate hyperplanes belonging to E are $\{z_{n-k-1} = 0\}, \dots, \{z_n = 0\}$. Observe that the \mathbb{C}^m -action (2.1) preserves the zero and the non-zero coordinates. Given $[z] \in V$, since $k \leq m + 1$, we may thus find some $T \in \mathbb{C}^m$ such that the image w of z under the action of T has the form

$$[w_1, \dots, w_{n-k-2}, 1, \dots, 1].$$

Such a T is not unique but unique up to some additive subgroup H of \mathbb{C}^m of dimension $m - k - 1$, cf. the statement of Proposition 2.1 whose proof follows the same line of arguments. In other words, setting

$$V_0 := \{[w_1, \dots, w_{n-k-2}, 1, \dots, 1] \in V\}$$

then X is isomorphic to V_0 quotiented by the action of H . As a consequence, the smaller family $z_1 \frac{\partial}{\partial z_1}, \dots, z_{n-k-2} \frac{\partial}{\partial z_{n-k-2}}$ still descends as a generating family of fundamental vector fields of X . Hence we may find $n - m - 1$ vector fields in this family which form a basis of fundamental vector fields. Now they all vanish on X by the above observation.

To prove the converse, assume that $k > m + 1$. Then in the generating family of fundamental vector fields, we have strictly less than $n - m - 1$ vector fields with non-empty vanishing locus. Moreover, a linear combination of vector fields of the family, say

$$\sum_{i=1}^n \alpha_i z_i \frac{\partial}{\partial z_i}$$

has vanishing locus

$$V \cap \mathbb{P}(\{z_i = 0 \mid \alpha_i \neq 0\})$$

Hence to have non-empty vanishing locus, a fundamental vector field must be a linear combination of the $n - k$ vector fields with non-empty vanishing locus of the generating family. Since $n - k < n - m - 1$, we cannot find a basis of such vector fields. \square

Remark 2.5. The case $k = m + 1$ in Theorem 2.4 is of special interest, because the corresponding LVMB manifolds admit an affine atlas, see Lemma 12.1 of [4] for a proof in the LVM case, which trivially generalizes to LVMB manifolds. Examples of LVMB fulfilling $k = m + 1$ include diagonal Hopf manifolds, but also manifolds diffeomorphic to the product of $(\mathbb{S}^1)^m$ with connected sums of sphere products and even manifolds with arbitrary torsion in the sense of Theorem 14.1 and Corollary 14.3 of [4].

3. STABILITY AND HOLOMORPHIC CONNECTIONS ON LVMB MANIFOLDS

As in Theorem 2.4, let X be a LVMB manifold, of complex dimension $\nu := n - m - 1$, such that $k \leq m + 1$. A Gauduchon metric on X is a Hermitian structure h such that the corresponding $(1, 1)$ -form ω_h satisfies the equation $\partial \bar{\partial} \omega_h^{\nu-1} = 0$; Gauduchon metrics exist [6]. Fix a Gauduchon form ω_h on X . The degree of a torsionfree coherent analytic sheaf F on X is defined to be

$$\text{degree}(F) := \frac{\sqrt{-1}}{2\pi} \int_X K(\det F) \wedge \omega_h^{\nu-1} \in \mathbb{R},$$

where $\det F$ is the determinant line bundle for F [8, Ch. V, § 6] (or Definition 1.34 in [5]) and K is the curvature for a hermitian connection on $\det F$ compatible with the Dolbeault operator $\bar{\partial}_{\det F}$. This degree is independent of the Hermitian metric on $\det F$, because any two such curvature forms differ by a $\partial\bar{\partial}$ -exact 2-form on X :

$$\int_X (\partial\bar{\partial}u) \wedge \omega_h^{\nu-1} = - \int_X u \wedge \partial\bar{\partial}\omega_h^{\nu-1} = 0.$$

A torsionfree coherent analytic sheaf F on X is called *semistable* if

$$\frac{\text{degree}(V)}{\text{rank}(V)} \leq \frac{\text{degree}(F)}{\text{rank}(F)}$$

for all nonzero coherent analytic subsheaf $V \subset F$ (see [9, p. 44, Definition 1.4.3], [8, Ch. V, § 7]).

Let

$$T_0 \subsetneq T_1 \subsetneq \cdots \subsetneq T_\ell \subsetneq T_{\ell+1} = TX \tag{3.1}$$

be the Harder–Narasimhan filtration of the holomorphic tangent bundle TX . We note that TX is semistable if and only if $\ell = 0$.

Lemma 3.1. *The subsheaf T_ℓ in (3.1) satisfies the following condition:*

$$\text{degree}(TX/T_\ell) > 0.$$

Proof. Fix a basis (a_1, \dots, a_ν) of \mathfrak{G} such that the corresponding fundamental vector fields (v_1, \dots, v_ν) have the property that each of them has a non-empty vanishing locus. Theorem 2.4 ensures that such a basis exists. Let r be the rank of TX/T_ℓ . Fix r elements from (v_1, \dots, v_ν) satisfying the following condition: their projections to TX/T_ℓ together generate TX/T_ℓ over a Zariski open subset. Let (w_1, \dots, w_r) denote this subset of (v_1, \dots, v_ν) . Therefore, $w_1 \wedge \cdots \wedge w_r$ is a holomorphic section of $\bigwedge^r TX$ which is not identically zero.

Consider the holomorphic line bundle $\det(TX/T_\ell)$ over X . The quotient map $TX \rightarrow TX/T_\ell$ produces a natural projection $q : \bigwedge^r TX \rightarrow \det(TX/T_\ell)$. Let

$$\sigma := q(w_1 \wedge \cdots \wedge w_r) \in H^0(X, \det(TX/T_\ell))$$

be the nonzero holomorphic section given by $w_1 \wedge \cdots \wedge w_r$. If any of (w_1, \dots, w_r) vanishes at a point $x \in X$, then clearly we have $\sigma(x) = 0$. In particular, the divisor $\text{div}(\sigma)$ is nonzero. Consequently, we have (see [5, Proposition 5.23])

$$\text{degree}(TX/T_\ell) = \text{Volume}_h(\text{div}(\sigma)) > 0.$$

This completes the proof. □

Let E be a holomorphic vector bundle on X equipped with a holomorphic connection D (see [1] for holomorphic connections). Let

$$0 = E_0 \subset E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_b \subsetneq E_{b+1} = E \tag{3.2}$$

be the Harder–Narasimhan filtration of E , so E is semistable if and only if $b = 0$.

Proposition 3.2. *For every $1 \leq i \leq b - 1$, the subsheaf $E_i \subset E$ in (3.2) is preserved by the holomorphic connection D .*

Proof. Take any $1 \leq i \leq b - 1$. Let

$$S_i : TX \otimes E_i \longrightarrow E/E_i$$

be the homomorphism given by the second fundamental form of E_i for the connection D . We recall that this second fundamental form is given by the composition

$$E_i \xrightarrow{D} E \otimes \Omega_X^1 \xrightarrow{q_i \otimes \text{Id}} (E/E_i) \otimes \Omega_X^1,$$

where $q_i : E \longrightarrow E/E_i$ is the quotient map. Tensoring both sides of it with TX and taking trace, the above homomorphism S_i is obtained from the second fundamental form.

For any holomorphic vector field v on X , consider the composition

$$E_i \xrightarrow{v \otimes \text{Id}} TX \otimes E_i \xrightarrow{S_i} E/E_i. \quad (3.3)$$

From the properties of the Harder–Narasimhan filtration we know that there is no nonzero homomorphism from E_i to E/E_i . Hence the composition homomorphism in (3.3) vanishes identically. Since global holomorphic vector fields generate TX over a Zariski open subset, the proposition follows. \square

Corollary 3.3. *Each subsheaf E_i in (3.2) is a holomorphic subbundle of E .*

Proof. By Lemma 4.5 in [2], any coherent analytic subsheaf E_i invariant by the holomorphic connection D is locally free. \square

For each $1 \leq i \leq b$, let D_i be the holomorphic connection, induced by D , on the quotient bundle E_i/E_{i-1} in (3.2).

Proposition 3.4. *The holomorphic connection D_i on E_i/E_{i-1} is flat for every i .*

Proof. A holomorphic vector bundle V on X is semistable if and only if V admits an approximate Hermitian–Einstein structure [13, p. 629, Theorem 1.1] (set the Higgs field ϕ in [13, Theorem 1.1] to be zero). If V_1 and V_2 are semistable holomorphic vector bundles on X , then approximate Hermitian–Einstein structures on V_1 and V_2 together produce an approximate Hermitian–Einstein structure on $V_1 \otimes V_2$. Hence $V_1 \otimes V_2$ is also semistable.

Since E_i/E_{i-1} is semistable, from the above observation we conclude that $\text{End}(E_i/E_{i-1})$ is also semistable. Note that we have

$$\text{degree}(\text{End}(E_i/E_{i-1})) = 0,$$

because $\det \text{End}(E_i/E_{i-1}) = \mathcal{O}_X$.

Let

$$\mathcal{K}(D_i) \in H^0(X, \Omega_X^2 \otimes \text{End}(E_i/E_{i-1}))$$

be the curvature of the holomorphic connection D_i on E_i/E_{i-1} . For any holomorphic vector field v on X , the contraction $i_v \mathcal{K}(D_i)$ of this curvature form gives a holomorphic homomorphism

$$\widetilde{i_v \mathcal{K}(D_i)} : TX \longrightarrow \text{End}(E_i/E_{i-1}).$$

Now, since $\text{End}(E_i/E_{i-1})$ is semistable of degree zero, from Lemma 3.1 it follows immediately that there is no nonzero holomorphic homomorphism from TX to $\text{End}(E_i/E_{i-1})$. Hence the above homomorphism $\widetilde{i_v \mathcal{K}(D_i)}$ vanishes identically. Since global holomorphic vector fields generate TX over a Zariski open subset, we conclude that $\mathcal{K}(D_i) = 0$. \square

The following is an immediate consequence of Proposition 3.4.

Corollary 3.5. *Let E be a semistable holomorphic vector bundle on X , and let D be a holomorphic connection on E . Then D is flat.*

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