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Fractal graphs

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Abstract

The lexicographic sum of graphs is defined as follows. Let G be a graph. With each $v \in V(G)$ associate a graph H_v . The lexicographic sum of the graphs H_v over G is obtained from G by substituting each $v \in V(G)$ by H_v . Given distinct $v, w \in V(G)$, we have all the possible edges in the lexicographic sum between $V(H_v)$ and $V(H_w)$ if $vw \in E(G)$, and none otherwise. When all the graphs H_v are isomorphic to some graph H , the lexicographic sum of the graphs H_v over G is called the lexicographic product of H by G , and denoted by $G \wr H$.

We say that a graph G is fractal if there exists a graph Γ , with at least two vertices, such that $G \simeq \Gamma \wr G$. There is a simple way to construct fractal graphs. Let Γ be a graph with at least two vertices. The graph Γ^ω is defined on the set $V(\Gamma)^\omega$ of functions from ω to $V(\Gamma)$ as follows. Given distinct $f, g \in V(\Gamma)^\omega$, fg is an edge of Γ^ω if $f(m)g(m)$ is an edge of Γ , where m is the smallest integer such that $f(m) \neq g(m)$. The graph Γ^ω is fractal because $\Gamma \wr \Gamma^\omega \simeq \Gamma^{1+\omega} \simeq \Gamma^\omega$.

We prove that a fractal graph is isomorphic to a lexicographic sum over an induced subgraph of Γ^ω , which is itself fractal.

Mathematics Subject Classifications (2010): 05C63, 05C76, 05C75

Key words: lexicographic sum, lexicographic product, fractal graph

1 Introduction

We introduce a notion of a fractal graph. It comes from the idempotency under lexicographic product, which was studied by Sabidussi [8]. Precisely, let us consider graphs G and H . The lexicographic product $G \wr H$ of H by G is obtained from G by replacing each vertex of G by a copy of H . We retain the edges inside the copies, and we add all the possible edges between two copies

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just when the corresponding vertices of G form an edge of G . A graph G is said to be idempotent under the lexicographic graph if G is isomorphic to the lexicographic product of G by itself. Hence, a graph idempotent (under the lexicographic graph) is a fixed point of the mapping, which associates $G \wr G$ with each graph G . (To avoid trivialities, we require that an idempotent graph has at least two vertices and hence is infinite.) Sabidussi [8] built idempotent graphs from linear orders, that are idempotent under the usual sum of linear orders, like the usual linear order on the rationals or on the reals. We do not know if there are other examples of idempotent graphs. The study of graphs built by Sabidussi reveals a self replication in their modular structures described in [2].

Initially, Sabidussi [8] considered the relationship between the wreath product of automorphism groups of graphs and the automorphism group of the lexicographic product of graphs. Obviously, the wreath product of automorphism groups of graphs is a subgroup of the automorphism group of the lexicographic product of graphs. In the special case of an idempotent graph G , Sabidussi asked if the automorphism group of $G \wr G$ is larger than the wreath product of the automorphism group of G by itself. Ille [3] answers the question positively.

To analyze the structure of an idempotent graph, we need the notion of a quotient. Let G be a graph. A subset M of $V(G)$ is a module of G if each vertex outside M is adjacent to all the elements of M or to none of them. A modular partition of G is a partition of $V(G)$, all the blocks of which are modules of G . The links between two disjoint modules are the same, all are edges or none are. This property justifies the following notion of a quotient. With each modular partition P of G , we associate the quotient G/P of G by P , defined on P , which is obtained from G by reducing each block of P to a single vertex, a vertex of the quotient.

Let G be an idempotent graph. Since G is idempotent, there exists a modular partition P of G such that the corresponding quotient G/P is isomorphic to G , and for each $X \in P$, $G[X]$ is isomorphic to G . Let $X \in P$. Since $G[X]$ is isomorphic to G , $G[X]$ is decomposable into $G \wr G$. In other words, G is isomorphic to $G \wr (G \wr G)$. By continuing this decomposition, we obtain a tree, from which the fractal structure of G appears. In fact, to obtain such a fractal structure through this decomposition process, we do not need to require that the quotient G/P is isomorphic to G . Indeed, it is sufficient to require that there exists a modular partition P of G , with at least two blocks, such that for each $X \in P$, $G[X]$ is isomorphic to G . Equivalently, we say that a graph G is fractal if there exists a graph Γ , with at least two vertices, such that G is isomorphic to $\Gamma \wr G$. Hence, a fractal graph is a fixed point of the mapping, which associates $\Gamma \wr G$ with each graph G , for some graph Γ with at least two vertices.

It is easy to construct fractal graphs. Consider any graph Γ with at least two vertices. The graph Γ^ω is defined on the set $V(\Gamma)^\omega$ of functions from ω to $V(\Gamma)$ in the following manner. Given distinct $f, g \in V(\Gamma)^\omega$, fg is an edge of Γ^ω if $f(m)g(m)$ is an edge of Γ , where m is the smallest integer such that $f(m) \neq g(m)$. The graph Γ^ω is fractal because $\Gamma \wr (\Gamma^\omega) \simeq \Gamma^{1+\omega} \simeq \Gamma^\omega$.

Finally, we characterize fractal graphs by using the lexicographic sum, of

which the lexicographic product is a special case. Let G be a graph. With each $v \in V(G)$ associate a graph H_v . The lexicographic sum of the graphs H_v over G is obtained from G by substituting each $v \in V(G)$ by H_v . Given distinct $v, w \in V(G)$, we have all the possible edges in the lexicographic sum between $V(H_v)$ and $V(H_w)$ if $vw \in E(G)$, and none otherwise. Let G be a fractal graph. Let Γ be a graph such that $G \simeq \Gamma \wr G$. We prove that G is isomorphic to a lexicographic sum over an induced subgraph of Γ^ω , which is itself fractal.

At present, we formalize our presentation. We use the following notation. Let G be a graph. For distinct $v, w \in V(G)$, set

$$[v, w]_G = \begin{cases} 1 & \text{if } vw \in E(G) \\ \text{or} \\ 0 & \text{if } vw \notin E(G). \end{cases}$$

Let G be a graph. A subset M of $V(G)$ is a *module* [1, 6, 9] of G if for any $x, y \in M$ and $v \in V(G) \setminus M$, we have $[x, v]_G = [y, v]_G$. (Gallai [1] used *closed set* instead of module.) A partition P of $V(G)$ is a *modular partition* of G if each block of P is a module of G . Given disjoint modules M and M' of G , we have $[x, x']_G = [y, y']_G$ for $x, y \in M$ and $x', y' \in M'$. This property justifies the following definition of a quotient. Let G be a graph. We associate with each modular partition P of G the *quotient* G/P of G by P defined on $V(G/P) = P$ as follows. For distinct $M, N \in P$,

$$[M, N]_{G/P} = [x, y]_G,$$

where $x \in M$ and $y \in N$.

Now, we give a definition of fractal in terms of quotient, which is shown in Proposition 2 to be equivalent to the definition in terms of lexicographic product. A graph G is *fractal* if there exists a modular partition P of G satisfying

$$\begin{cases} |P| \geq 2, \\ \text{and} \\ \text{for each } M \in P, G[M] \simeq G. \end{cases} \quad (1)$$

The following weakening of the notion of isomorphism is useful (see Remark 1). Let G and H be graphs. A function $f : V(G) \rightarrow V(H)$ is a (strong) *egamorphism* [5] from G to H if for $v, w \in V(G)$ such that $f(v) \neq f(w)$, we have

$$[v, w]_G = [f(v), f(w)]_H.$$

Remark 1. Given graphs G and H , consider an egamorphism f from G to H . Set $W = f(V(G))$, and

$$\Pi(f) = \{f^{-1}(w) : w \in W\}.$$

The set $\Pi(f)$ is a modular partition of G . Moreover, the function

$$\begin{aligned} f/\Pi(f) : \quad G/\Pi(f) &\longrightarrow H[W] \\ f^{-1}(w) \ (w \in W) &\longmapsto w, \end{aligned}$$

is an isomorphism from $G/\Pi(f)$ onto $H[W]$.

The inverse operation of the quotient is the lexicographic sum defined in the following manner. Consider a graph G , and associate with each $v \in V(G)$ a graph H_v . Suppose that the vertex sets $V(H_v)$ are pairwise disjoint. We consider the function

$$p: \bigcup_{v \in V(G)} V(H_v) \longrightarrow V(G),$$

which maps each $x \in \bigcup_{v \in V(G)} V(H_v)$ to the unique vertex $p(x)$ of G such that $x \in V(H_{p(x)})$. The *lexicographic sum*

$$\sum_G H_v$$

of the graphs H_v over G (or the G -join of the family $\{H_v ; v \in V(G)\}$ [7]) is defined on $\bigcup_{v \in V(G)} V(H_v)$ as follows. Given distinct $x, y \in \bigcup_{v \in V(G)} V(H_v)$,

$$[x, y]_{(\sum_G H_v)} = \begin{cases} [p(x), p(y)]_G & \text{if } p(x) \neq p(y) \\ \text{or} \\ [x, y]_{H_{p(x)}} & \text{if } p(x) = p(y). \end{cases}$$

If there exists a graph H such that H_w is isomorphic to H for every $w \in V(G)$, then the lexicographic sum $\sum_G H_v$ of the graphs H_v over G is called the *lexicographic product* of H by G , and is denoted by $G \wr H$. Precisely, we can define $G \wr H$ on $V(G) \times V(H)$ as follows. Given distinct $(v, w), (v', w') \in V(G) \times V(H)$,

$$[(v, w), (v', w')]_{(G \wr H)} = \begin{cases} [v, v']_G & \text{if } v \neq v' \\ \text{or} \\ [w, w']_H & \text{if } v = v'. \end{cases}$$

We obtain the following characterization of a fractal graph in terms of lexicographic product.

Proposition 2. *For a graph G , G is fractal if and only if there exists a graph Γ satisfying*

$$\begin{cases} v(\Gamma) \geq 2, \\ \text{and} \\ G \simeq \Gamma \wr G. \end{cases} \quad (2)$$

Let G be a fractal graph. A graph Γ satisfying (2) is called a *fractal factor* of G . Now, consider a fractal factor Γ of G , and an isomorphism ψ from G onto $\Gamma \wr G$. We say that a subset W of $V(G)$ is ψ -invariant if

$$\psi(W) = V(\Gamma) \times W.$$

Remark 3. Let G be a fractal graph. Consider a fractal factor Γ of G , and an isomorphism ψ from G onto $\Gamma \wr G$. Given $W \subseteq V(G)$, W is ψ -invariant if and only if $\psi|_W$ is an isomorphism from $G[W]$ onto $\Gamma \wr G[W]$. Therefore, if a subset W of $V(G)$ is ψ -invariant, then $G[W]$ is fractal, and Γ is a fractal factor of $G[W]$.

1.1 A simple construction of fractal graphs

Consider any graph Γ such that $v(\Gamma) \geq 2$. (Observe that Γ can be infinite.) Consider the set $V(\Gamma)^\omega = \{f : \omega \longrightarrow V(\Gamma)\}$. Given distinct $f, g \in V(\Gamma)^\omega$, set

$$\mu(f, g) = \min(\{n \in \omega : f(n) \neq g(n)\}).$$

The graph Γ^ω is defined on $V(\Gamma)^\omega$ as follows. Given distinct $f, g \in V(\Gamma)^\omega$,

$$[f, g]_{\Gamma^\omega} = [f(\mu(f, g)), g(\mu(f, g))]_\Gamma.$$

The graph Γ^ω is fractal because

$$\Gamma \wr \Gamma^\omega \simeq \Gamma^{1+\omega} \simeq \Gamma^\omega.$$

Precisely, consider the shift function

$$\begin{aligned} \sigma : \quad \omega &\longrightarrow \omega \\ n &\longmapsto n + 1. \end{aligned}$$

The bijection

$$\begin{aligned} \varphi : \quad V(\Gamma)^\omega &\longrightarrow V(\Gamma) \times V(\Gamma)^\omega \\ f &\longmapsto (f(0), f \circ \sigma) \end{aligned} \tag{3}$$

is an isomorphism from Γ^ω onto $\Gamma \wr \Gamma^\omega$. By Proposition 2, Γ^ω is fractal. Furthermore, Γ is a fractal factor of Γ^ω .

1.2 Results

To begin, we provide a sufficient condition for a lexicographic sum over a fractal graph to be fractal as well.

Proposition 4. *Let G be a fractal graph. Consider a fractal factor Γ of G , and an isomorphism ψ from G onto $\Gamma \wr G$. With each $v \in V(G)$ associate a graph H_v . Suppose that the vertex sets $V(H_v)$ are pairwise disjoint.*

If for each $v \in V(G)$, $\{w \in V(G) : H_w \simeq H_v\}$ is ψ -invariant, then the lexicographic sum $\sum_G H_v$ is fractal, and Γ is a fractal factor of $\sum_G H_v$.

The next result is an immediate consequence of Proposition 4.

Corollary 5. *Let G be a fractal graph. For every graph H , $G \wr H$ is fractal, and the fractal factors of G are fractal factors of $G \wr H$.*

The opposite direction in Corollary 5 is not true. Nevertheless, we obtain the next result (see Proposition 6). We need the following notion of primality. Let G be a graph. The empty set, the vertex set of G , and the singletons are modules of G , called *trivial*. A graph G is *prime* if $v(G) \geq 3$, and all the modules of G are trivial. (Observe that there is no prime graph on 3 vertices.)

Proposition 6. *Given graphs G and H , if $G \wr H$ is fractal and H is prime, then G is fractal, and the fractal factors of $G \wr H$ are fractal factors of G .*

The main result follows.

Theorem 7. *Let G be a fractal graph. Consider a fractal factor Γ of G , and an isomorphism ψ from G onto $\Gamma \wr G$.*

There exists a unique function $\theta : V(G) \longrightarrow V(\Gamma)^\omega$ such that the following diagram commutes.

$$\begin{array}{ccc} G & \xrightarrow{\psi} & \Gamma \wr G \\ \theta \downarrow & & \downarrow \text{Id}_{V(\Gamma)} \times \theta \\ \Gamma^\omega & \xrightarrow[\varphi]{} & \Gamma \wr \Gamma^\omega \end{array} \quad (\mathcal{D}_1)$$

Moreover, the following assertions hold, where $W = \theta(V(G))$ and

$$\Pi(\theta) = \{\theta^{-1}(f) : f \in W\}.$$

(A1) *The function θ is an egamorphism from G to Γ^ω .*

(A2) *The function θ induces an isomorphism $\theta/\Pi(\theta)$ from $G/\Pi(\theta)$ onto $\Gamma^\omega[W]$.*

(A3) *We have*

$$G = \sum_{\Gamma^\omega[W]} G[\theta^{-1}(f)].$$

(A4) *The subset W of Γ^ω is φ -invariant.*

(A5) *For each $f \in W$, $\{g \in W : G[\theta^{-1}(g)] \simeq G[\theta^{-1}(f)]\}$ is φ -invariant.*

(A6) *The function*

$$\Psi : \begin{array}{ccc} \Pi(\theta) & \longrightarrow & V(\Gamma) \times \Pi(\theta) \\ \theta^{-1}(f) \ (f \in W) & \longmapsto & (f(0), \theta^{-1}(f \circ \sigma)) \end{array}$$

is an isomorphism from $G/\Pi(\theta)$ onto $\Gamma \wr (G/\Pi(\theta))$. Moreover, the following diagram commutes.

$$\begin{array}{ccc} G/\Pi(\theta) & \xrightarrow{\Psi} & \Gamma \wr (G/\Pi(\theta)) \\ \theta/\Pi(\theta) \downarrow & & \downarrow \text{Id}_{V(\Gamma)} \times (\theta/\Pi(\theta)) \\ \Gamma^\omega[W] & \xrightarrow[\varphi|_W]{} & \Gamma \wr (\Gamma^\omega[W]) \end{array} \quad (\mathcal{D}_2)$$

Finally, all the functions occurring in Diagram (\mathcal{D}_2) are isomorphisms.

2 Proofs

We begin with the easy proof of Proposition 2.

Proof of Proposition 2. To begin, suppose that G is fractal. There exists a modular partition P of G , which satisfies (1). Since $G[M] \simeq G$ for each $M \in P$, we get

$$G \simeq (G/P) \wr G.$$

Thus G/P satisfies (2).

Conversely, suppose that there exists a graph Γ satisfying (2). Set

$$P = \{\{x\} \times V(G) : x \in V(\Gamma)\}.$$

Clearly, P is a modular partition of $\Gamma \wr G$. Since Γ satisfies (2), there exists an isomorphism ψ from G onto $\Gamma \wr G$. Therefore,

$$\psi^{-1}(P) = \{\psi^{-1}(M) : M \in P\}$$

is a modular partition of G . Since $v(\Gamma) \geq 2$ by (2), we have $|P| \geq 2$, so $|\psi^{-1}(P)| \geq 2$. Furthermore, consider $N \in \psi^{-1}(P)$. There exists $x \in V(\Gamma)$ such that $N = \psi^{-1}(\{x\} \times V(G))$. Therefore, we have

$$\begin{aligned} G[N] &= G[\psi^{-1}(\{x\} \times V(G))] \\ &\simeq (\Gamma \wr G)[\{x\} \times V(G)] \\ &\simeq G \end{aligned} \quad (\text{by definition of } \Gamma \wr G).$$

It follows that $\psi^{-1}(P)$ satisfies (1). Hence G is fractal. \square

2.1 Proof of Proposition 4 and Corollary 5

We use the following notation in the proof of Proposition 4.

Notation 8. Let G and H be graphs. The function from $V(G) \times V(H)$ to $V(G)$, which maps $(v, w) \in V(G) \times V(H)$ to v , is denoted by q_G . Note that q_G is an egamorphism from $G \wr H$ to G . Similarly, the function from $V(G) \times V(H)$ to $V(H)$, which maps $(v, w) \in V(G) \times V(H)$ to w , is denoted by q_H .

Proof of Proposition 4. Consider the function $p : \bigcup_{v \in V(G)} V(H_v) \longrightarrow V(G)$, which maps each $x \in \bigcup_{v \in V(G)} V(H_v)$ to the unique vertex $p(x)$ of G such that $x \in V(H_{p(x)})$.

Let $v \in V(G)$. Since $\{w \in V(G) : H_w \simeq H_v\}$ is ψ -invariant, we have $(q_G \circ \psi)(v) \in \{w \in V(G) : H_w \simeq H_v\}$. Thus

$$H_{(q_G \circ \psi)(v)} \simeq H_v.$$

By the axiom of choice, there exists a function ν , which associates with each $v \in V(G)$ an isomorphism $\nu(v)$ from H_v onto $H_{(q_G \circ \psi)(v)}$.

Consider the function

$$\begin{aligned} \tilde{\psi}: \bigcup_{v \in V(G)} V(H_v) &\longrightarrow V(\Gamma) \times \bigcup_{v \in V(G)} V(H_v) \\ x &\longmapsto ((q_\Gamma \circ \psi \circ p)(x), \nu(p(x))(x)). \end{aligned}$$

We show that $\tilde{\psi}$ is an isomorphism from $\sum_G H_v$ onto $\Gamma \wr (\sum_G H_v)$. For distinct $x, y \in \bigcup_{v \in V(G)} V(H_v)$, we have to verify that

$$\begin{aligned} [x, y]_{(\sum_G H_v)} = \\ [((q_\Gamma \circ \psi \circ p)(x), \nu(p(x))(x)), ((q_\Gamma \circ \psi \circ p)(y), \nu(p(y))(y))]_{\Gamma \wr (\sum_G H_v)}. \end{aligned} \quad (4)$$

We distinguish the following cases.

1. Suppose that $p(x) \neq p(y)$. We obtain

$$[x, y]_{\sum_G H_v} = [p(x), p(y)]_G = [(\psi \circ p)(x), (\psi \circ p)(y)]_{(\Gamma \wr G)}. \quad (5)$$

(a) Suppose that $(q_\Gamma \circ \psi \circ p)(x) \neq (q_\Gamma \circ \psi \circ p)(y)$. We obtain

$$\begin{aligned} & [((q_\Gamma \circ \psi \circ p)(x), \nu(p(x))(x)), \\ & \quad ((q_\Gamma \circ \psi \circ p)(y), \nu(p(y))(y))]_{\Gamma \wr (\sum_G H_v)} \\ &= [(q_\Gamma \circ \psi \circ p)(x), (q_\Gamma \circ \psi \circ p)(y)]_\Gamma \\ &= [(\psi \circ p)(x), (\psi \circ p)(y)]_{(\Gamma \wr G)}. \end{aligned}$$

It follows from (5) that (4) holds.

(b) Suppose that $(q_\Gamma \circ \psi \circ p)(x) = (q_\Gamma \circ \psi \circ p)(y)$. We obtain

$$\begin{aligned} & [((q_\Gamma \circ \psi \circ p)(x), \nu(p(x))(x)), \\ & \quad ((q_\Gamma \circ \psi \circ p)(y), \nu(p(y))(y))]_{\Gamma \wr (\sum_G H_v)} \\ &= [\nu(p(x))(x), \nu(p(y))(y)]_{(\sum_G H_v)} \end{aligned}$$

We have

$$\begin{cases} \nu(p(x))(x) \in V(H_{(q_G \circ \psi \circ p)(x)}) \\ \text{and} \\ \nu(p(y))(y) \in V(H_{(q_G \circ \psi \circ p)(y)}) \end{cases}$$

Since $p(x) \neq p(y)$, $(\psi \circ p)(x) \neq (\psi \circ p)(y)$. Since $(q_\Gamma \circ \psi \circ p)(x) = (q_\Gamma \circ \psi \circ p)(y)$, we obtain $(q_G \circ \psi \circ p)(x) \neq (q_G \circ \psi \circ p)(y)$. It follows that

$$\begin{aligned} & [\nu(p(x))(x), \nu(p(y))(y)]_{(\sum_G H_v)} \\ &= [(q_G \circ \psi \circ p)(x), (q_G \circ \psi \circ p)(y)]_G \\ &= [(\psi \circ p)(x), (\psi \circ p)(y)]_{\Gamma \wr G}. \end{aligned}$$

It follows from (5) that (4) holds.

2. Suppose that $p(x) = p(y)$. We obtain

$$\begin{aligned}
& [((q_\Gamma \circ \psi \circ p)(x), \nu(p(x))(x)), \\
& \quad ((q_\Gamma \circ \psi \circ p)(y), \nu(p(y))(y)))]_{\Gamma \wr (\Sigma_G H_v)} \\
&= [\nu(p(x))(x), \nu(p(y))(y)]_{(\Sigma_G H_v)} \\
&= [\nu(p(x))(x), \nu(p(x))(y)]_{H_{(q_G \circ \psi \circ p)(x)}} \\
&= [x, y]_{H_{p(x)}} \\
&= [x, y]_{(\Sigma_G H_v)}. \quad \square
\end{aligned}$$

As indicated in Subsection 1.2, Corollary 5 is an immediate consequence of Proposition 4. Also, a direct proof of Corollary 5 is immediate by using the associativity of the lexicographic product.

Proof of Corollary 5. Let Γ be a fractal factor of G . We have $G \simeq \Gamma \wr G$. For any graph H , we obtain

$$\Gamma \wr (G \wr H) \simeq (\Gamma \wr G) \wr H \simeq G \wr H. \quad \square$$

As seen in Proposition 4, the ψ -invariant subsets of $V(G)$ play an important role. We complete the subsection with their examination.

Notation 9. Let G be a fractal graph. Consider a fractal factor Γ of G . Let ψ be an isomorphism from G onto $\Gamma \wr G$. The set of the ψ -invariant subsets of $V(G)$ is denoted by \mathcal{I}_ψ .

Remark 10. Let G be a fractal graph. Consider a fractal factor Γ of G . Let ψ be an isomorphism from G onto $\Gamma \wr G$. Clearly, \mathcal{I}_ψ is closed under complementation and intersection. Hence \mathcal{I}_ψ is a boolean algebra. For each $v \in V(G)$, set

$$\mathcal{I}_\psi(v) = \{W \in \mathcal{I}_\psi : v \in W\}.$$

Clearly $\mathcal{I}_\psi(v) \neq \emptyset$ because $V(G) \in \mathcal{I}_\psi(v)$. It follows that

$$\bigcap_{W \in \mathcal{I}_\psi(v)} W$$

is ψ -invariant. Set

$$\langle v \rangle_\psi = \bigcap_{W \in \mathcal{I}_\psi(v)} W.$$

Clearly, $\langle v \rangle_\psi$ is the smallest ψ -invariant subset of $V(G)$ containing v . Therefore, the atoms of \mathcal{I}_ψ are $\langle v \rangle_\psi$ for $v \in V(G)$. Given $W \subseteq V(G)$, it follows that W is ψ -invariant if and only if for each $v \in W$, $\langle v \rangle_\psi \subseteq W$.

Lastly, let Γ be a graph such that $v(\Gamma) \geq 2$. Consider the isomorphism φ defined in (3). We characterize the atoms $\langle f \rangle_\varphi$, where $f \in V(\Gamma)^\omega$, of the boolean algebra \mathcal{I}_φ (see Lemma 11 below).

Let $f \in V(\Gamma)^\omega$. Given $n \geq 1$, consider $u \in V(\Gamma)^n$, that is,

$$u : \{0, \dots, n-1\} \longrightarrow V(\Gamma).$$

We define the concatenation $u \bullet f : \omega \longrightarrow V(\Gamma)$ as follows. For $p \geq 0$,

$$(u \bullet f)(p) = \begin{cases} u(p) & \text{if } p \leq n-1 \\ \text{or} \\ f(p-n) & \text{if } p \geq n. \end{cases}$$

The set $V(\Gamma)^0$ contains a unique function denoted by $\mathbb{1}$. For every $f \in V(\Gamma)^\omega$, we have $\mathbb{1} \bullet f = f$. Furthermore, for every $f \in V(\Gamma)^\omega$, $f \circ \sigma^0 = f$.

Since the proof of the next lemma is a straightforward verification, we leave the details to the reader.

Lemma 11. *For every $f \in V(\Gamma)^\omega$,*

$$\langle f \rangle_\varphi = \bigcup_{m, n \geq 0} \{u \bullet (f \circ \sigma^m) : u \in V(\Gamma)^n\}. \quad (6)$$

Using a φ -invariant subset of $V(\Gamma)^\omega$, we define a more complicated fractal graph than Γ^ω , which is decomposed into a lexicographical sum over an induced subgraph of Γ^ω , which is itself fractal. Consider a nonempty set \mathcal{C} of nonempty φ -invariant subsets of $V(\Gamma)^\omega$ that are pairwise disjoint. Set

$$W = \bigcup_{C \in \mathcal{C}} C.$$

As already noticed, W is a φ -invariant subset of $V(\Gamma)^\omega$. By Remark 3, $\Gamma^\omega[W]$ is fractal. With each $C \in \mathcal{C}$, we associate a graph H_C . For each $f \in C$, set

$$H_f = \alpha_f \wr H_C,$$

where α_f is the graph defined on $\{f\}$. (In this manner, the vertex sets $V(H_f)$, where $f \in W$, are pairwise disjoint.) It follows from Proposition 4 that the lexicographic sum

$$G = \sum_{\Gamma^\omega[W]} H_f$$

is fractal. The purpose of Theorem 7 is to establish that any fractal graph is isomorphic to such a lexicographic sum.

2.2 Proof of Proposition 6

The following strengthening of the notion of a module is helpful to establish Proposition 6. Given a graph G , a subset M of $V(G)$ is a *strong module* [1, 6] of G if M is a module of G satisfying: for every module N of G , if $M \cap N \neq \emptyset$, then $M \subseteq N$ or $N \subseteq M$. (Gallai [1] used *strongly closed set* instead of strong module.)

Remark 12. Let G be a graph. Consider a module M of G . Suppose that M is not a strong module of G . There exists a module N of G such that $M \cap N \neq \emptyset$, $M \setminus N \neq \emptyset$, and $N \setminus M \neq \emptyset$. It follows that $\{M \cap N, M \setminus N\}$ is a modular partition of $G[M]$. Thus, $G[M]$ (or its complement) is disconnected. Consequently, given a module M of G , if $G[M]$ is connected and coconnected, then M is a strong module. In particular, given a module M of G , if $G[M]$ is prime, then M is a strong module.

We complete the remark with the following property. Let G be a graph. Given modules M and N of G , if $G[M]$ and $G[N]$ are prime, then $M \cap N = \emptyset$ or $M = N$.

Proposition 6 is a consequence of the next result.

Lemma 13. *Given graphs G, G' and H , if $G \wr H \simeq G' \wr H$ and H is prime, then $G \simeq G'$.*

Proof. Let ρ be an isomorphism from $G \wr H$ onto $G' \wr H$. With each $x \in V(G)$ associate the subset

$$W'(x) = \{x' \in V(G') : \rho(\{x\} \times V(H)) \cap (\{x'\} \times V(H)) \neq \emptyset\}$$

of $V(G')$. Let $x \in V(G)$. Since H is prime, $(G \wr H)[\{x\} \times V(H)]$ is prime, and hence $(G' \wr H)[\rho(\{x\} \times V(H))]$ is as well. Moreover, since $\{x\} \times V(H)$ is a module of $G \wr H$, $\rho(\{x\} \times V(H))$ is a module of $G' \wr H$. Similarly, for $x' \in V(G')$, $(G' \wr H)[\{x'\} \times V(H)]$ is prime, and $\{x'\} \times V(H)$ is a module of $G' \wr H$.

Now, consider $x \in V(G)$. Let $x' \in W'(x)$. We have $\rho(\{x\} \times V(H)) \cap (\{x'\} \times V(H)) \neq \emptyset$. Since $\rho(\{x\} \times V(H))$ and $\{x'\} \times V(H)$ are modules of $G' \wr H$ such that $(G' \wr H)[\rho(\{x\} \times V(H))]$ and $(G' \wr H)[\{x'\} \times V(H)]$ are prime, it follows from Remark 12 that $\rho(\{x\} \times V(H)) = \{x'\} \times V(H)$. Hence $W'(x) = \{x'\}$. Consequently

$$|W'(x)| = 1$$

for every $x \in V(G)$.

It is easy to verify that the function from $V(G)$ to $V(G')$ that maps each $x \in V(G)$ to the unique element of $W'(x)$ is an isomorphism from G onto G' . \square

Proof of Proposition 6. Since $G \wr H$ is fractal, it follows from Proposition 2 that there exists a graph Γ , with $v(\Gamma) \geq 2$, such that $G \wr H \simeq \Gamma \wr (G \wr H)$. Since $\Gamma \wr (G \wr H) \simeq (\Gamma \wr G) \wr H$, we obtain $G \wr H \simeq (\Gamma \wr G) \wr H$. Since H is prime, it follows from Lemma 13 that $G \simeq \Gamma \wr G$. By Proposition 2, G is fractal. \square

2.3 Proof of Theorem 7

Let G be a fractal graph. Consider a fractal factor Γ of G . There exists an isomorphism ψ from G onto $\Gamma \wr G$.

With each $v \in V(G)$, we associate a sequence $(f_v(n), x_v(n))_{n \geq 0}$ of elements of $V(\Gamma) \times V(G)$ defined by recursion as follows. First, set

$$(f_v(0), x_v(0)) = \psi(v).$$

Second, given $(f_v(n), x_v(n))$, where $n \geq 0$, set

$$(f_v(n+1), x_v(n+1)) = \psi(x_v(n)).$$

Consider the function

$$\begin{array}{ccc} \theta : V(G) & \longrightarrow & V(\Gamma)^\omega \\ v & \longmapsto & f_v. \end{array}$$

Claim 14. *Given $\gamma \in V(\Gamma)$ and $v, w \in V(G)$, if $\psi(v) = (\gamma, w)$, then $\gamma = \theta(v)(0)$, and $\theta(w) = \theta(v) \circ \sigma$.*

Proof. We have $\psi(v) = (f_v(0), x_v(0))$, that is, $\psi(v) = (\theta(v)(0), x_v(0))$. Hence $\gamma = \theta(v)(0)$ and $w = x_v(0)$. It is easy to verify by induction on $n \geq 0$ that

$$(f_v(n+1), x_v(n+1)) = (f_w(n), x_w(n)).$$

It follows that $f_v(n+1) = f_w(n)$ for every $n \geq 0$. Therefore $f_v \circ \sigma = f_w$, that is, $\theta(v) \circ \sigma = \theta(w)$. \square

The next claim follows from Claim 14.

Claim 15. *For every $v \in V(G)$, we have*

$$(\varphi \circ \theta)(v) = ((\text{Id}_{V(\Gamma)} \times \theta) \circ \psi)(v).$$

Proof. Let $v \in V(G)$. We have

$$\varphi(\theta(v)) = (\theta(v)(0), \theta(v) \circ \sigma).$$

Furthermore, there exist $\gamma \in V(\Gamma)$ and $w \in V(G)$ such that $\psi(v) = (\gamma, w)$. We get $(\text{Id}_{V(\Gamma)} \times \theta)(\psi(v)) = (\text{Id}_{V(\Gamma)} \times \theta)(\gamma, w)$. By Claim 14, $\gamma = \theta(v)(0)$. Thus

$$\begin{aligned} (\text{Id}_{V(\Gamma)} \times \theta)(\psi(v)) &= (\text{Id}_{V(\Gamma)} \times \theta)(\theta(v)(0), w) \\ &= (\theta(v)(0), \theta(w)). \end{aligned}$$

By Claim 14, $\theta(w) = \theta(v) \circ \sigma$. It follows that

$$(\text{Id}_{V(\Gamma)} \times \theta)(\psi(v)) = (\theta(v)(0), \theta(v) \circ \sigma). \quad \square$$

By Claim 15, Diagram (\mathcal{D}_1) commutes. Now, we prove that θ is unique.

Claim 16. *Let $\theta' : V(G) \longrightarrow V(\Gamma)^\omega$. If $\varphi \circ \theta' = (\text{Id}_{V(\Gamma)} \times \theta') \circ \psi$, then $\theta' = \theta$.*

Proof. Consider $\theta' : V(G) \longrightarrow V(\Gamma)^\omega$ such that $\varphi \circ \theta' = (\text{Id}_{V(\Gamma)} \times \theta') \circ \psi$. Let $v \in V(G)$. We have

$$(\varphi \circ \theta')(v) = (\theta'(v)(0), \theta'(v) \circ \sigma)$$

and

$$((\text{Id}_{V(\Gamma)} \times \theta') \circ \psi)(v) = (\text{Id}_{V(\Gamma)} \times \theta')(f_v(0), x_v(0)) = (f_v(0), \theta'(x_v(0))).$$

Thus, for each $v \in V(G)$, we have

$$\begin{cases} \theta'(v)(0) = f_v(0) \\ \text{and} \\ \theta'(v) \circ \sigma = \theta'(x_v(0)). \end{cases} \quad (7)$$

We prove by induction on $n \geq 0$ that for each $v \in V(G)$,

$$\theta'(v)(n) = f_v(n). \quad (8)$$

It follows from (7) that (8) holds for $n = 0$. Now, suppose that (8) holds for $n \geq 0$. Consider $v \in V(G)$. Set

$$w = x_v(0).$$

We have

$$\begin{aligned} \theta'(v)(n+1) &= (\theta'(v) \circ \sigma)(n) \\ &= \theta'(w)(n) && \text{by (7)} \\ &= f_w(n) && \text{because (8) holds for } n. \end{aligned}$$

As observed in the proof of Claim 14, we have

$$(f_v(m+1), x_v(m+1)) = (f_w(m), x_w(m))$$

for every $m \geq 0$. It follows that

$$\theta'(v)(n+1) = f_v(n+1).$$

Therefore, (8) holds for every $n \geq 0$. It follows that for each $v \in V(G)$, $\theta'(v) = f_v$, so $\theta'(v) = \theta(v)$. \square

We prove that Assertion (A1) of Theorem 7 holds.

Claim 17. *For $v, w \in V(G)$, if $\theta(v) \neq \theta(w)$, then $[v, w]_G = [\theta(v), \theta(w)]_{\Gamma^\omega}$.*

Proof. Let $v, w \in V(G)$ such that $\theta(v) \neq \theta(w)$. Set

$$m = \mu(\theta(v), \theta(w)).$$

To begin, suppose that $m = 0$. We have

$$\begin{aligned} [v, w]_G &= [\psi(v), \psi(w)]_{\Gamma \wr G} \\ &= [(\theta(v)(0), x_v(0)), (\theta(w)(0), x_w(0))]_{\Gamma \wr G} \\ &= [(\theta(v)(0), \theta(w)(0))]_{\Gamma} \\ &= [\theta(v), \theta(w)]_{\Gamma^\omega}. \end{aligned}$$

Now, suppose that $m \geq 1$. We prove by induction on $i \in \{0, \dots, m-1\}$ that

$$[v, w]_G = [x_v(i), x_w(i)]_G.$$

For $i = 0$, we have

$$\begin{aligned} [v, w]_G &= [\psi(v), \psi(w)]_{\Gamma \wr G} \\ &= [(\theta(v)(0), x_v(0)), (\theta(w)(0), x_w(0))]_{\Gamma \wr G} \\ &= [x_v(0), x_w(0)]_G. \end{aligned}$$

Suppose that $i \leq m - 2$. By induction hypothesis, we have

$$[v, w]_G = [x_v(i), x_w(i)]_G.$$

Thus, we obtain

$$\begin{aligned} [v, w]_G &= [x_v(i), x_w(i)]_G \\ &= [\psi(x_v(i)), \psi(x_w(i))]_{\Gamma \wr G} \\ &= [(\theta(v)(i+1), x_v(i+1)), (\theta(w)(i+1), x_w(i+1))]_{\Gamma \wr G} \\ &= [x_v(i+1), x_w(i+1)]_G. \end{aligned}$$

Consequently, we have

$$[v, w]_G = [x_v(m-1), x_w(m-1)]_G.$$

Lastly, we obtain

$$\begin{aligned} [v, w]_G &= [x_v(m-1), x_w(m-1)]_G \\ &= [\psi(x_v(m-1)), \psi(x_w(m-1))]_{\Gamma \wr G} \\ &= [(\theta(v)(m), x_v(m)), (\theta(w)(m), x_w(m))]_{\Gamma \wr G} \\ &= [(\theta(v)(m), (\theta(w)(m))]_{\Gamma} \\ &= [\theta(v), \theta(w)]_{\Gamma^\omega}. \end{aligned} \quad \square$$

For convenience, set $W = \theta(V(G))$ and $\Pi(\theta) = \{\theta^{-1}(f) : f \in W\}$. By Remark 1, $\theta/\Pi(\theta)$ is an isomorphism from $G/\Pi(\theta)$ onto $\Gamma^\omega[W]$. Thus, Assertion (A2) of Theorem 7 holds.

Since $\Pi(\theta)$ is a modular partition of G , we have

$$G = \sum_{G/\Pi(\theta)} G[X].$$

Since $\theta/\Pi(\theta)$ is an isomorphism from $G/\Pi(\theta)$ onto $\Gamma^\omega[W]$, we obtain

$$G = \sum_{\Gamma^\omega[W]} G[\theta^{-1}(f)].$$

Therefore, Assertion (A3) of Theorem 7 holds.

We prove that Assertion (A4) of Theorem 7 holds.

Claim 18. *For each $f \in W$, we have $f \circ \sigma \in W$.*

Proof. Let $f \in W$. There exists $v \in V(G)$ such that $f = \theta(v)$. By Claim 14, $\psi(v) = (\theta(v)(0), w)$, where $w \in V(G)$ such that $\theta(w) = \theta(v) \circ \sigma$. It follows that $\theta(w) = f \circ \sigma$. Therefore $f \circ \sigma \in W$. \square

It follows from Claim 18 that

$$\varphi(W) \subseteq V(\Gamma) \times W. \quad (9)$$

Claim 19. For any $\gamma \in V(\Gamma)$ and $g \in W$, we have $\varphi^{-1}(\gamma, g) \in W$.

Proof. Consider $\gamma \in V(\Gamma)$ and $g \in W$. Since $g \in W$, there exists $w \in V(G)$ such that $\theta(w) = g$. Set $v = \psi^{-1}(\gamma, w)$. We have

$$((\text{Id}_{V(\Gamma)} \times \theta) \circ \psi)(v) = (\text{Id}_{V(\Gamma)} \times \theta)(\gamma, w) = (\gamma, g).$$

It follows from Claim 15 that

$$(\varphi \circ \theta)(v) = (\gamma, g).$$

Thus $\theta(v) = \varphi^{-1}(\gamma, g)$. Consequently $\varphi^{-1}(\gamma, g) \in W$. \square

It follows from Claim 19 that

$$\varphi^{-1}(V(\Gamma) \times W) \subseteq W. \quad (10)$$

Therefore, it follows from (9) and (10) that W is φ -invariant. Hence, Assertion (A4) of Theorem 7 holds. Since W is φ -invariant, the following diagram commutes.

$$\begin{array}{ccc} G & \xrightarrow{\varphi_G} & \Gamma \wr G \\ \theta \downarrow & & \downarrow \text{Id}_{V(\Gamma)} \times \theta \\ \Gamma^\omega[W] & \xrightarrow{\varphi_{\wr W}} & \Gamma \wr (\Gamma^\omega[W]) \end{array}$$

We prove that Assertion (A5) of Theorem 7 holds.

Claim 20. For each $f \in W$, we have $\psi(\theta^{-1}(f)) = \{f(0)\} \times \theta^{-1}(f \circ \sigma)$.

Proof. Let $f \in W$. Consider $v \in \theta^{-1}(f)$. We have $f = \theta(v)$. It follows from Claim 14 that $\psi(v) = (f(0), w)$, where $w \in \theta^{-1}(f \circ \sigma)$. Therefore

$$\psi(\theta^{-1}(f)) \subseteq \{f(0)\} \times \theta^{-1}(f \circ \sigma).$$

Conversely, consider $w \in \theta^{-1}(f \circ \sigma)$. We have $\theta(w) = f \circ \sigma$. It follows from Claim 14 that $\theta(\psi^{-1}(f(0), w))(0) = f(0)$ and

$$\theta(\psi^{-1}(f(0), w)) \circ \sigma = f \circ \sigma.$$

Consequently, for each $n \geq 1$, we have $\theta(\psi^{-1}(f(0), w))(n) = f(n)$. Since $\theta(\psi^{-1}(f(0), w))(0) = f(0)$, we obtain

$$\theta(\psi^{-1}(f(0), w)) = f.$$

Thus $\psi^{-1}(f(0), w) \in \theta^{-1}(f)$, so $(f(0), w) \in \psi(\theta^{-1}(f))$. It follows that

$$\{f(0)\} \times \theta^{-1}(f \circ \sigma) \subseteq \psi(\theta^{-1}(f)).$$

Hence $\psi(\theta^{-1}(f)) = \{f(0)\} \times \theta^{-1}(f \circ \sigma)$. \square

The next two claims follow from Claim 20.

Claim 21. *For each $f \in W$, we have $f \circ \sigma \in W$ and*

$$G[\theta^{-1}(f)] \simeq G[\theta^{-1}(f \circ \sigma)].$$

Proof. Let $f \in W$. By Claim 18, $f \circ \sigma \in W$. Furthermore, it follows from Claim 20 that

$$\psi(\theta^{-1}(f)) = \{f(0)\} \times \theta^{-1}(f \circ \sigma).$$

Since ψ is an isomorphism from G onto $\Gamma \wr G$, we obtain

$$G[\theta^{-1}(f)] \simeq (\Gamma \wr G)[\{f(0)\} \times \theta^{-1}(f \circ \sigma)].$$

Since $(\Gamma \wr G)[\{f(0)\} \times \theta^{-1}(f \circ \sigma)] \simeq G[\theta^{-1}(f \circ \sigma)]$, we obtain

$$G[\theta^{-1}(f)] \simeq G[\theta^{-1}(f \circ \sigma)].$$
 \square

Claim 22. *For any $\gamma \in V(\Gamma)$ and $g \in W$, we have $\varphi^{-1}(\gamma, g) \in W$ and*

$$G[\theta^{-1}(g)] \simeq G[\theta^{-1}(\varphi^{-1}(\gamma, g))].$$

Proof. Consider $\gamma \in V(\Gamma)$ and $g \in W$. By Claim 19, $\varphi^{-1}(\gamma, g) \in W$. By Claim 21,

$$G[\theta^{-1}(\varphi^{-1}(\gamma, g))] \simeq G[\theta^{-1}(\varphi^{-1}(\gamma, g) \circ \sigma)].$$

Since $\varphi^{-1}(\gamma, g) \circ \sigma = g$, we obtain $G[\theta^{-1}(g)] \simeq G[\theta^{-1}(\varphi^{-1}(\gamma, g))]$. \square

Claim 23. *For each $f \in W$, we have*

$$\begin{aligned} \varphi(\{g \in W : G[\theta^{-1}(g)] \simeq G[\theta^{-1}(f)]\}) = \\ V(\Gamma) \times \{g \in W : G[\theta^{-1}(g)] \simeq G[\theta^{-1}(f)]\}. \end{aligned} \quad (11)$$

Proof. Let $f \in W$. To begin, consider $g \in W$ such that $G[\theta^{-1}(g)] \simeq G[\theta^{-1}(f)]$. Recall that $\varphi(g) = (g(0), g \circ \sigma)$. By Claim 21, $g \circ \sigma \in W$ and

$$G[\theta^{-1}(g)] \simeq G[\theta^{-1}(g \circ \sigma)].$$

Since $G[\theta^{-1}(g)] \simeq G[\theta^{-1}(f)]$, we obtain

$$G[\theta^{-1}(g \circ \sigma)] \simeq G[\theta^{-1}(f)].$$

Thus

$$\varphi(\{g \in W : G[\theta^{-1}(g)] \simeq G[\theta^{-1}(f)]\}) \subseteq V(\Gamma) \times \{g \in W : G[\theta^{-1}(g)] \simeq G[\theta^{-1}(f)]\}.$$

Conversely, consider $\gamma \in V(\Gamma)$ and $g \in W$ such that $G[\theta^{-1}(g)] \simeq G[\theta^{-1}(f)]$. By Claim 22, $\varphi^{-1}(\gamma, g) \in W$ and

$$G[\theta^{-1}(g)] \simeq G[\theta^{-1}(\varphi^{-1}(\gamma, g))].$$

Since $G[\theta^{-1}(g)] \simeq G[\theta^{-1}(f)]$, we obtain

$$G[\theta^{-1}(\varphi^{-1}(\gamma, g))] \simeq G[\theta^{-1}(f)].$$

Thus

$$\begin{aligned} \varphi^{-1}(V(\Gamma) \times \{g \in W : G[\theta^{-1}(g)] \simeq G[\theta^{-1}(f)]\}) \subseteq \\ \{g \in W : G[\theta^{-1}(g)] \simeq G[\theta^{-1}(f)]\}, \end{aligned}$$

so

$$V(\Gamma) \times \{g \in W : G[\theta^{-1}(g)] \simeq G[\theta^{-1}(f)]\} \subseteq \varphi(\{g \in W : G[\theta^{-1}(g)] \simeq G[\theta^{-1}(f)]\}).$$

Consequently (11) holds. \square

Remark 24. Consider $f \in W$. By Claim 23, $\{g \in W : G[\theta^{-1}(g)] \simeq G[\theta^{-1}(f)]\}$ is φ -invariant. It follows from Remark 10 that

$$\langle f \rangle_\varphi \subseteq \{g \in W : G[\theta^{-1}(g)] \simeq G[\theta^{-1}(f)]\}.$$

It follows from Lemma 11 that for $m, n \geq 0$ and $u \in V(\Gamma)^n$,

$$G[\theta^{-1}(u \bullet (f \circ \sigma^m))] \simeq G[\theta^{-1}(f)].$$

Finally, we prove that Assertion (A6) of Theorem 7 holds. Set

$$\psi(\Pi(\theta)) = \{\psi(\theta^{-1}(f)) : f \in W\}.$$

Since $\Pi(\theta)$ is a modular partition of G and ψ is an isomorphism from G onto $\Gamma \wr G$, $\psi(\Pi(\theta))$ is a modular partition of $\Gamma \wr G$. Moreover, the function

$$\begin{aligned} \psi/\Pi(\theta) : \quad \Pi(\theta) &\longrightarrow \psi(\Pi(\theta)) \\ \theta^{-1}(f) \ (f \in W) &\longmapsto \psi(\theta^{-1}(f)) \end{aligned}$$

is an isomorphism from $G/\Pi(\theta)$ onto $(\Gamma \wr G)/\psi(\Pi(\theta))$. Let $f \in W$. By Claim 20, $\psi(\theta^{-1}(f)) = \{f(0)\} \times \theta^{-1}(f \circ \sigma)$. It follows that

$$\begin{aligned} \psi(\Pi(\theta)) &= \{\psi(\theta^{-1}(f)) : f \in W\} \\ &= \{\{f(0)\} \times \theta^{-1}(f \circ \sigma) : f \in W\} \\ &= \{\{\gamma\} \times \theta^{-1}(g) : \gamma \in V(\Gamma), g \in W\}. \end{aligned}$$

Furthermore, the function

$$\begin{aligned} F : \quad \psi(\Pi(\theta)) &\longrightarrow V(\Gamma) \times \Pi(\theta) \\ \{\gamma\} \times \theta^{-1}(g) \ (\gamma \in V(\Gamma), g \in W) &\longmapsto (\gamma, \theta^{-1}(g)) \end{aligned}$$

is an isomorphism from $(\Gamma \wr G)/\psi(\Pi(\theta))$ onto $\Gamma \wr (G/\Pi(\theta))$. We obtain

$$\Psi = F \circ (\psi/\Pi(\theta)),$$

so Ψ is an isomorphism from $G/\Pi(\theta)$ onto $\Gamma \wr (G/\Pi(\theta))$. Clearly, Diagram (\mathcal{D}_2) commutes, and all the functions occurring in Diagram (\mathcal{D}_2) are isomorphisms.

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