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Abstract: Within the algebraic analysis approach to linear system theory, a multidimensional linear system can be studied by means of its associated finitely presented left module. Deep connections exist between module isomorphisms and equivalent matrices. In the present paper, we introduce a constructive proof of a result due to Warfield which controls the size of equivalent matrices involved in the study of isomorphic modules. We illustrate our constructive proof with an example coming from differential equations with constant coefficients.

Keywords: Linear functional systems, module isomorphisms, matrix equivalence

1. INTRODUCTION

A linear multidimensional system, such as a linear system of differential equations or partial derivative equations, maybe described by a matrix of functional operators. Indeed, a system of q equations with p unknown functions $\eta_1, \cdots, \eta_p$ over a ring of operators D is written

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p \mid R\eta = 0\}, \qquad (1)$$

where $R \in D^{q\times p}$ and $\mathcal{F}$ is the functional space where we are looking for the solutions. The system (1) can be studied by mean of algebraic tools, using the finitely presented left module $M := D^{1\times p}/(D^{1\times q}R)$ with p generators submitted to the q relations specified by the lines of R. Indeed, from the properties of free and quotient modules, the abelian group $\ker_{\mathcal{F}}(R.)$ is isomorphic to $\hom_D(M, \mathcal{F})$, that is the abelian group of left D-linear maps from M to $\mathcal{F}$, see Malgrange (1963). Hence, systemic properties of (1) can be studied by mean of modules properties, which can be computed using effective homological algebra and Groebner bases theory, see Chyzak et al. (2005).

The abelian group $\hom_D(M, \mathcal{F})$ only depend on M and $\mathcal{F}$, in particular it does not depend on the matrix R. That means that two matrices R and R' defining isomorphic modules $M \simeq M'$ have the same algebraic properties. A particular example for the isomorphism $M \simeq M'$ is the case where R and R' are *equivalent*, that is $R = YR'X^{-1}$ for invertible matrices X and Y. A result due to Fitting (1936) asserts a weak converse implication: if R and R' define isomorphic modules, then they can be enlarged by 0 and identity blocs leading to equivalent matrices L and L'. An effective version of this result was obtained in Cluzeau and Quadrat (2011).

The purpose of the present paper is to introduce an effective version of a result of Warfield (1978), which asserts that the number of 0 and the size of identity blocs in L and L' maybe reduced, while keeping equivalent matrices. This result is based on an algebraic invariant of D, called the *stable rank*, see McConnell and Robson (2001). For a general presentation of the problem, we refer to Nicolas (2014).

The paper is organised as follows. In Section 2, we recall the notions of isomorphic left D-modules and equivalent matrices, as well as the effective version of Fitting's result and the statement of Warfield's result. In Section 3, we present the constructive version of Warfield's result for removing 0 and identity blocs. In Section 4, we illustrate this constructive version with a differential equation with constant coefficients. Section 5 contains proofs of formulas used in Section 3.

2. ISOMORPHISMS AND EQUIVALENT MATRICES

In this section, we recall the characterization of morphisms between finitely presented left D-modules, as well as results of Fitting and Warfield which rely isomorphic left D-modules to matrix conjugation.

2.1 *Effective version of Fitting's Theorem*

Consider two left D-modules M and M' with finite presentations:

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0,$$

$$D^{1\times q'} \xrightarrow{.R'} D^{1\times p'} \xrightarrow{\pi'} M' \longrightarrow 0,$$

namely, *exact sequences* (see Rotman (2009)), where $R \in D^{q\times p}$, $(.R)(\mu) = \mu R$, for every $\mu \in D^{1\times q}$ and π is the natural projection on $M = D^{1\times p}/(D^{1\times q}R)$ (similarly for R' and π').

From Rotman (2009), there exists $f \in \hom_D(M, M')$ if and only if there exist matrices $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ such that $RP = QR'$ and

$$\forall \lambda \in D^{1\times p},\ f(\pi(\lambda)) = \pi'(\lambda\pi).$$

Hence, the following diagram is exact and commutative:

$$\begin{tikzcd} D^{1\times q} \arrow[r, ".R"] \arrow[d, ".Q"'] & D^{1\times p} \arrow[r, "\pi"] \arrow[d, ".P"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\ D^{1\times q'} \arrow[r, ".R'"] & D^{1\times p'} \arrow[r, "\pi'"] & M' \arrow[r] & 0 \end{tikzcd}$$

We let $n := q+p'+p+q'$ and $m := p+p'$. The two $n\times m$ matrices

$$L := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix},\ L' := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \mathrm{id}_p & 0 \\ 0 & R' \end{pmatrix} \tag{2}$$

induce finite presentations: $M \simeq D^{1\times m}/(D^{1\times n}L)$ and $M' \simeq D^{1\times m}/(D^{1\times n}L')$. In Cluzeau and Quadrat (2011), an effective version of a result due to Fitting (1936) is given. If f is an isomorphism, then L and L' are equivalent: there exist 6 matrices $R_2 \in D^{r\times q}$, $R_2' \in D^{r'\times q'}$, $Z_2 \in D^{p\times q}$, $Z_2 \in D^{q\times r}$, $Z_2' \in D^{q'\times r'}$, $Z \in D^{p\times q}$ and $Z' \in D^{p'\times q'}$ and two pairs of invertible matrices (X, X') and (Y, Y') of size m and n

$$X := \begin{pmatrix} \mathrm{id}_p & P \\ -P' & \mathrm{id}_{p'} - P'P \end{pmatrix},\ Y := \begin{pmatrix} \mathrm{id}_q & 0 & R & Q \\ 0 & \mathrm{id}_{p'} & -P' & Z' \\ -Z & P & 0 & PZ'-ZQ \\ -Q' & -R' & 0 & Z_2'R_2' \end{pmatrix}, \tag{3}$$

$$X' := \begin{pmatrix} \mathrm{id}_p - PP' & -P \\ P' & \mathrm{id}_{p'} \end{pmatrix},\ Y' := \begin{pmatrix} Z_2R_2 & 0 & -R & -Q \\ P'Z - Z'Q' & 0 & P' & -Z' \\ Z & -P & \mathrm{id}_p & 0 \\ Q' & R' & 0 & \mathrm{id}_{q'} \end{pmatrix},$$

such that $L' = Y'LX$. In other words, the following diagram is exact and commutative

$$\begin{tikzcd} D^{1\times n} \arrow[r, ".L"] \arrow[d, shift right, ".Y"'] & D^{1\times m} \arrow[r, "\pi\oplus 0"] \arrow[d, shift right, ".X"'] & M \arrow[r] \arrow[d, shift right, "f"'] & 0 \\ D^{1\times n} \arrow[r, ".L'"] \arrow[u, shift right, ".Y'"'] & D^{1\times m} \arrow[r, "0\oplus\pi'"] \arrow[u, shift right, ".X"'] & M' \arrow[r] \arrow[u, shift right, "f'"'] & 0 \end{tikzcd}$$

2.2 Warfield's Theorem

A result due to Warfield (1978) asserts that that the size of 0 an id blocs in (2) can be reduced, whereas the new matrices are still equivalent. This result is based on the notion of *stable rank*. The definition of the latter requires to introduce various notions that we present now.

A column vector $u := (u_1 \cdots u_k)^T \in D^{k\times 1}$ is called *unimodular* if there exists a line $v \in D^{1\times k}$ such that $vu = 1$. Moreover, u is said to be *stable* if there exist $d_1, \cdots, d_{k-1} \in D$ such that $(u_1 + d_1u_k \cdots u_{k-1} + d_{k-1}u_k)$ is unimodular. An integer r is said to be in the stable rank of D if whenever $k > r$, every column $u \in D^{k\times 1}$ is stable. The stable rank $\mathrm{sr}(D)$ of D is the smallest integer in the stable rank of D.

Assume that the two matrices (2) are equivalent, then Warfield's Theorem asserts that if there exist two integers r and s such that

$$\begin{cases} s \le \min(p+q', q+p') \\ \mathrm{sr}(D) \le \max(p+q'-s, q+p'-s) \\ r \le \min(p, p') \\ \mathrm{sr}(D) \le \max(p-r, p'-r) \end{cases}$$

then the following $(n-r-s)\times(m-r)$ matrices are equivalent

$$\overline{L} := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'-r} \\ 0 & 0 \end{pmatrix},\ \overline{L'} := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{p-r} & 0 \\ 0 & R' \end{pmatrix},$$

and induce finite presentations of M and M', respectively.

In the next section, we introduce a procedure which computes two pairs of invertible matrices $(\overline{X}, \overline{X}')$ and $(\overline{Y}, \overline{Y}')$ such that $\overline{L'} = \overline{Y}'\overline{L}\overline{X}$.

3. EFFECTIVE WARFIELD'S THEOREM

Throughout this Section, we fix some notations. Let M and M' be two left D-modules, isomorphic with $f : M \xrightarrow{\sim} M'$, with inverse f', finitely presented by matrices $R \in D^{q\times p}$ and $R' \in D^{q'\times p'}$, respectively, and let L, L', X, Y, X' and Y' be the matrices defined in (2) and (3).

Given a nonzero integer k, we let $\overline{k} := k-1$. For $1 \le i \le k$, the i-th vector of the canonical basis of $D^{1\times k}$ is written e_i^k. Moreover, the i-th component of $v \in D^{1\times k}$ or $v \in D^{k\times 1}$ is written v_i. Finally, for a matrix $A \in D^{k\times k'}$, the coefficient at position (i,j), the i-th row and the j-th column are written A_{ij}, $A_{i.}$ and $A_{.j}$, respectively.

3.1 Reduction of the zero bloc

In this section, we present the procedure for removing s lines of 0 in L and L', where s is such that

$$\begin{cases} s \le \min(p+q', q+p'), \\ \mathrm{sr}(D) \le \max(p+q'-s, q+p'-s). \end{cases} \tag{4}$$

Without lost of generalities, we suppose $q+p' \le p+q'$.

We let $n := (q+p'+p+q') - \overline{s}$, so that $\overline{n} = (q+p'+p+q') - s$, and we define $L_s, L_s' \in D^{\overline{n}\times m}$ as follows:

$$L_s := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'} \\ 0 & 0 \end{pmatrix},\ L_s' := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_p & 0 \\ 0 & R' \end{pmatrix},$$

that is, L_s and L_s' have respectively $p+q'-s$ and $q+p'-s$ lines of zeros.

Our objective is to construct $\overline{n}$-square matrices Y_s and Y_s', inverse to each other and such that the following diagram is exat and commutative:

$$\begin{tikzcd} D^{1\times \overline{n}} \arrow[r, ".L_s"] \arrow[d, shift right, ".Y_s"'] & D^{1\times m} \arrow[r, "\pi\oplus 0"] \arrow[d, shift right, ".X"'] & M \arrow[r] \arrow[d, shift right, "f"'] & 0 \\ D^{1\times \overline{n}} \arrow[r, ".L_s'"] \arrow[u, shift right, ".Y_s'"'] & D^{1\times m} \arrow[r, "0\oplus\pi'"] \arrow[u, shift right, ".X'"'] & M' \arrow[r] \arrow[u, shift right, "f'"'] & 0 \end{tikzcd} \tag{5}$$

For that, we let $Y_0 := Y$, $Y_0' := Y'$ and we assume by induction that $Y_{\overline{s}}$ and $Y_{\overline{s}}'$ have been constructed and are such that (5) is exact and commutative. We decompose $Y_{\overline{s}}$ and $Y_{\overline{s}}'$ as follows:

$$Y_{\overline{s}} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix},\ Y_{\overline{s}}' = \begin{pmatrix} Y_1' & Y_2' & Y_3' \end{pmatrix},$$

where Y_1, Y_2 and Y_3 (respectively, Y_1', Y_2' and Y_3') have $q+p'$, $p+q'-s$ and 1 lines (respectively, columns), respectively. Finally, we let

$$k := q + p' - \overline{s}.$$

Proposition 1. There exist $c \in D$ and $\mathbf{d}, \mathbf{u} \in D^{1\times(p+q'-s)}$ such that

$$\begin{pmatrix} c(Y_1')_{k.} & \mathbf{d} \end{pmatrix} \begin{pmatrix} (Y_1)_{.k} \\ (Y_2)_{.k} + \mathbf{u}^T (Y_3)_k \end{pmatrix} = 1. \tag{6}$$

Proof. Getting the coefficient at position (k,k) in the relation $Y_{\overline{s}}' Y_{\overline{s}} = \mathrm{id}_n$, we get the following relation: $(Y_1')_{k.}(Y_1)_{.k} + (Y_2')_{k.}(Y_2)_{.k} + (Y_3')_k (Y_3)_k = 1$. Hence, the left D-module $N := D/D\Big((Y_1')_{k.}(Y_1)_{.k}\Big)$ is generated by $[(Y_2)_{ik}]_N$, $1 \le i \le p+q'-s$, and $[(Y_3)_k]$. From (4), we have $\mathrm{sr}(D) \le p+q'-\overline{s}$, and from (McConnell and Robson, 2001, Lemma 11.4.6), $\mathrm{sr}(D)$ is in the stable range of N. Hence, there exists $\mathbf{u} := (u_1, \cdots, u_{p+q'-s})$ such that N is generated by $[(Y_2)_{ik} + u_i (Y_3)_k]$, $1 \le i \le p+q'-s$. Thus, there exist $c \in D$ and $\mathbf{d} := (d_1, \cdots, d_{p+q'-s})$ such that (6) holds.

With the notations of Proposition 1, we introduce the lines $\tilde{\ell} \in D^{1\times\overline{n}}$ and $\ell \in D^{1\times n}$ defined as follows:

$$\tilde{\ell} := \begin{pmatrix} c(Y_1')_{k.} & \mathbf{d} \end{pmatrix}, \ \ell := \begin{pmatrix} c(Y_1')_{k.} & \mathbf{d} & 0 \end{pmatrix},$$

as well as the matrices $U, U' \in D^{n\times n}$ and $F \in D^{\overline{n}\times n}$ defined as follows:

$$U := \begin{pmatrix} \mathrm{id}_{q+p'} & 0 & 0 \\ 0 & \mathrm{id}_{p+q'-s} & \mathbf{u}^T \\ 0 & 0 & 1 \end{pmatrix}, \ U' := \begin{pmatrix} \mathrm{id}_{q+p'} & 0 & 0 \\ 0 & \mathrm{id}_{p+q'-s} & -\mathbf{u}^T \\ 0 & 0 & 1 \end{pmatrix},$$

$$F := \begin{pmatrix} Y_1 \\ Y_2 + \mathbf{u}^T Y_3 \end{pmatrix}.$$

We point out that U and U' are inverses to each other, and that $F = (\mathrm{id}_{\overline{n}}\ 0)\, U Y_{\overline{s}}$. Moreover, from $\ell = \tilde{\ell}(\mathrm{id}_{\overline{n}}\ 0)$ and (6), we get

$$1 = \tilde{\ell} F (e_k^n)^T = \ell U Y_{\overline{s}} (e_k^n)^T. \tag{7}$$

Finally, we consider $\psi,\ \psi' \in D^{n\times\overline{n}}$, $\iota,\ \iota' \in D^{\overline{n}\times n}$ defined as follows

$$\psi := \begin{pmatrix} \mathrm{id}_{\overline{n}} - F(e_k^n)^T \tilde{\ell} \\ \tilde{\ell} \end{pmatrix}, \qquad \iota := \begin{pmatrix} \mathrm{id}_{\overline{n}} - F(e_k^n)^T \tilde{\ell} & F(e_k^n)^T \end{pmatrix},$$

$$\psi' := \begin{pmatrix} \mathrm{id}_n - (e_k^n)^T \ell U Y_{\overline{s}} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{k-1} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{p+q'} \end{pmatrix}, \qquad \iota' := \begin{pmatrix} \mathrm{id}_{\overline{k}} & 0 & 0 \\ 0 & 0 & \mathrm{id}_{p+q'} \end{pmatrix}.$$

Proposition 2. We have the following relations:

(1) $\iota\psi = \mathrm{id}_{\overline{n}}$,
(2) $\iota'\psi' = \mathrm{id}_{\overline{n}}$,
(3) $\ker(.\psi) = D\ell$,
(4) $\ker(.\psi') = D\ell U Y_{\overline{s}}$.

Proof. By computing matrix products, $\iota\psi$ is equal to $(\mathrm{id}_{\overline{n}} - F(e_k^n)^T\tilde{\ell})^2 + F(e_k^n)^T\tilde{\ell}$. Moreover, from (7), $F(e_k^n)^T\tilde{\ell}$ is a projector, so that $\mathrm{id}_{\overline{n}} - F(e_k^n)^T\tilde{\ell}$ is also a projector, whence Point 1.

We have $\iota'(e_k^n)^T = 0$, from which we deduce Point 2 by computing the matrix product.

Let us show Point 3. Considering the isomorphism $D^{1\times n} \simeq D^{1\times\overline{n}} \oplus D$, we have $\psi = \psi_1\psi_2$, where $\psi_1 \in D^{n\times n}$ and $\psi_2 \in D^{n\times\overline{n}}$ are defined as follows:

$$\psi_1 := \begin{pmatrix} \mathrm{id}_{\overline{n}} - F(e_k^n)^T\tilde{\ell} & 0 \\ 0 & 1 \end{pmatrix}, \ \psi_2 := \begin{pmatrix} \mathrm{id}_{\overline{n}} \\ \tilde{\ell} \end{pmatrix}.$$

From (7), $\mathrm{im}(.\psi_1)$ is included in $\ker(.F(e_k^n)^T) \oplus D$ and the restriction of $.\psi_2$ to the latter is injective: for $(u,x) \in \big(\ker(.F(e_k^n)^T) \oplus D\big) \cap \ker(.\psi_2)$, we have $u + x\tilde{\ell} = 0$, so that $x\tilde{\ell}F(e_k^n)^T = 0$, which, by (7), gives $x = 0$ and $u = 0$. Hence, we have $\ker(.\psi) = \ker(.\psi_1)$, that is $\ker\Big(\mathrm{id}_{\overline{n}} - .F(e_k^n)^T\tilde{\ell}\Big) \oplus 0$. We conclude by showing $\ker\Big(\mathrm{id}_{\overline{n}} - .F(e_k^n)^T\tilde{\ell}\Big) = D\tilde{\ell}$: the right to left inclusion is due to (7), and the other is due to $x = \big(xF(e_k^n)^T\big)\,\tilde{\ell}$, for every $x \in \ker\Big(\mathrm{id}_{\overline{n}} - .F(e_k^n)^T\tilde{\ell}\Big)$.

Let us show Point 4. From (7), we have $D\ell U Y_{\overline{s}} \subseteq \ker(.\psi')$. The converse inclusion is due to the relation $x = x_k \ell U Y_{\overline{s}}$, for every $x \in \ker(.\psi')$. Indeed, the first $k-1$ and the last $p+q'$ columns of x and $x_k\ell U Y_{\overline{s}}$ are equal since $x \in \ker(.\psi')$ implies

$$x \begin{pmatrix} \mathrm{id}_{k-1} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{p+q'} \end{pmatrix} = x_k \ell U Y_{\overline{s}} \begin{pmatrix} \mathrm{id}_{k-1} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{p+q'} \end{pmatrix}.$$

Moreover, the k-th column of $x_k\ell U Y_{\overline{s}}$ is $x_k \ell U Y_{\overline{s}} (e_k^n)^T$, that is x_k from (7).

Theorem 3. With the previous notations, we let

$$Y_s := \iota U Y_{\overline{s}} \psi', \quad Y_s' := \iota' Y_{\overline{s}}' U' \psi.$$

The following diagram is exact and commutative:

$$\begin{tikzcd} D^{1\times\overline{n}} \arrow[r, ".L_s"] \arrow[d, shift right, ".Y_s"'] & D^{1\times m} \arrow[r, "\pi\oplus 0"] \arrow[d, shift right, ".X"'] & M \arrow[r] \arrow[d, shift right, "f"'] & 0 \\ D^{1\times\overline{n}} \arrow[r, ".L_s'"] \arrow[u, shift right, ".Y_s'"'] & D^{1\times m} \arrow[r, "0\oplus\pi'"] \arrow[u, shift right, ".X'"'] & M' \arrow[r] \arrow[u, shift right, "f'"'] & 0 \end{tikzcd}$$

In particular, we have

$$L_s' = Y_s' L_s X.$$

Proof. We only have to show that the diagram is commutative.

First, we show that Y_s and Y_s' are inverse to each other. From Proposition 2, the lines of the following diagram are exact

$$\begin{tikzcd} D \arrow[r, ".\ell"] \arrow[d, "\mathrm{id}_D"'] & D^{1\times n} \arrow[r, "\psi"] \arrow[d, shift right, ".UY_{\overline{s}}"'] & D^{1\times\overline{n}} \arrow[r] \arrow[d, shift right, ".Y_s"'] & 0 \\ D \arrow[r, ".\ell U Y_{\overline{s}}"'] & D^{1\times n} \arrow[r, "\psi'"'] \arrow[u, shift right, ".Y_{\overline{s}}' U'"'] & D^{1\times\overline{n}} \arrow[r] \arrow[u, shift right, ".Y_s'"'] & 0 \end{tikzcd} \tag{8}$$

Moreover, it is also commutative. Indeed, ψY_s is equal to $\psi\iota U Y_{\overline{s}} \psi'$ and from 1 of Proposition 2, we have $\mathrm{im}(.\psi\iota - \mathrm{id}_n) \subseteq \ker(.\psi)$. By commutativity of the left rectangle and by exactness of the lines of (8), we have $(\psi\iota - \mathrm{id}_n)UY_{\overline{s}}\psi' = 0$, so that $\psi Y_s = UY_{\overline{s}}\psi'$. In the same manner, we show that $Y'_{\overline{s}}U'\psi = \psi' Y'_s$. By commutativity and exactness of (8) and from the equations $UY_{\overline{s}}Y'_{\overline{s}}U' = Y'_{\overline{s}}U'UY_{\overline{s}} = \mathrm{id}_n$, we get $Y_sY'_s = Y'_sY_s = \mathrm{id}_{\overline{n}}$.

In Section 5.1 we prove the following relations:

$$\begin{array}{llll} \iota L_{\overline{s}} = L_s, & UL_{\overline{s}} = L_{\overline{s}}, & Y_{\overline{s}}L'_{\overline{s}} = L_{\overline{s}}X, & \psi' L'_s = L'_{\overline{s}}, \\ \iota' L'_{\overline{s}} = L'_s, & Y'_{\overline{s}}L_{\overline{s}} = L'_{\overline{s}}X', & U'L_{\overline{s}} = L_{\overline{s}}, & \psi L_s = L_{\overline{s}}. \end{array} \tag{9}$$

Hence, $Y_sL'_s = L_sX$ and $Y'_sL_s = L'_sX'$ follow from commutativity of

$$\begin{tikzcd}
D^{1\times\overline{n}} \arrow[r, shift left, "\iota"] \arrow[d, "L_s"'] & D^{1\times n} \arrow[l, shift left, "\psi"] \arrow[r, shift left, "U"] \arrow[d, "L_{\overline{s}}"] & D^{1\times n} \arrow[l, shift left, "U'"] \arrow[r, shift left, "Y_{\overline{s}}"] \arrow[d, "L_{\overline{s}}"] & D^{1\times n} \arrow[l, shift left, "Y'_{\overline{s}}"] \arrow[r, shift left, "\psi'"] \arrow[d, "L'_{\overline{s}}"] & D^{1\times\overline{n}} \arrow[l, shift left, "\iota'"] \arrow[d, "L'_s"] \\
D^{1\times m} \arrow[r, "\mathrm{id}_m"] & D^{1\times m} \arrow[r, "\mathrm{id}_m"] & D^{1\times m} \arrow[r, shift left, "X"] & D^{1\times m} \arrow[l, shift left, "X'"] \arrow[r, "\mathrm{id}_m"] & D^{1\times m}
\end{tikzcd}$$

3.2 Reduction of the identity bloc

In this section, we assume that s zero lines have already been removed from L and L', where s is as in (4). For simplicity, we write K, K', Z and Z' instead of L_s, L'_s, Y_s and Y'_s, respectively. Hence, we have

$$K = \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'} \\ 0 & 0 \end{pmatrix}, \; K' = \begin{pmatrix} 0 & 0 \\ \mathrm{id}_p & 0 \\ 0 & R' \end{pmatrix},$$

where K and K' have respectively $p+q'-s$ and $q+p'-s$ lines of zeros, and Z, Z' are $(q+p'+p+q'-s)$-square matrices such as in (5).

We present the procedure for removing r lines of identity blocs of K and K', where r is such that

$$\begin{cases} r \le \min(p, p') \\ \mathrm{sr}(D) \le \max(p-r, p'-r). \end{cases} \tag{10}$$

Without lost of generality, we assume that $p \le p'$. We let $n_1 := (q+p'+p+q'-s) - \overline{r}$ and $n_2 := p+p'-\overline{r}$, so that $\overline{n_1} = (q+p'+p+q'-s) - r$, $\overline{n}_2 = p+p'-r$, and we define $K_r, K'_r \in D^{\overline{n_1}\times\overline{n}_2}$ as follows:

$$K_r := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'-r} \\ 0 & 0 \end{pmatrix}, \; K'_r := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{p-r} & 0 \\ 0 & R' \end{pmatrix}.$$

Our objective is to construct two pairs of square matrices (Z_r, Z'_r), (X_r, X'_r), of size $\overline{n_1}$ and $\overline{n}_2$, respectively, invertible to each other and such that the following diagram is exact and commutative:

$$\begin{tikzcd}
D^{1\times\overline{n_1}} \arrow[r, ".K_r"] \arrow[d, shift right, ".Z_r"'] & D^{1\times\overline{n}_2} \arrow[r, "\pi\oplus 0"] \arrow[d, shift right, ".X_r"'] & M \arrow[r] \arrow[d, shift right, "f"'] & 0 \\
D^{1\times\overline{n_1}} \arrow[r, ".K'_r"] \arrow[u, shift right, ".Z'_r"'] & D^{1\times\overline{n}_2} \arrow[r, "0\oplus\pi'"] \arrow[u, shift right, ".X'_r"'] & M' \arrow[r] \arrow[u, shift right, "f'"'] & 0
\end{tikzcd} \tag{11}$$

We let $Z_0 := Z$, $Z'_0 := Z'$, $X_0 = X$, $X'_0 := X'$ and we assume by induction that $Z_{\overline{r}}$, $Z'_{\overline{r}}$, $X_{\overline{r}}$ and $X'_{\overline{r}}$ have been constructed and are such that (11) is exact and commutative. We decompose these matrices as follows:

$$X_{\overline{r}} = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \\ X_5 & X_6 \end{pmatrix}, \; Z_{\overline{r}} = \begin{pmatrix} Z_1 & Z_2 & Z_3 \\ Z_4 & Z_5 & Z_6 \\ Z_7 & Z_8 & Z_9 \\ Z_{10} & Z_{11} & Z_{12} \end{pmatrix},$$

$$X'_{\overline{r}} = \begin{pmatrix} X'_1 & X'_2 & X'_3 \\ X'_4 & X'_5 & X'_6 \end{pmatrix}, \; Z'_{\overline{r}} = \begin{pmatrix} Z'_1 & Z'_2 & Z'_3 & Z'_4 \\ Z'_5 & Z'_6 & Z'_7 & Z'_8 \\ Z'_9 & Z'_{10} & Z'_{11} & Z'_{12} \end{pmatrix},$$

where the column and line separations are the following:

- the column (respectively, line) blocs of $X_{\overline{r}}$ from left to right (respectively, top to bottom) have $p-\overline{r}$ and p' columns (respectively, p, $p'-r$ and 1 lines),
- the column (respectively, line) blocs of $X'_{\overline{r}}$ from left to right (respectively, top to bottom) have p, $p'-r$ and 1 columns (respectively, $p-\overline{r}$ and p' lines),
- the column (respectively, line) blocs of $Z_{\overline{r}}$ from left to right (respectively, top to bottom) have $q+p'-s$, $p-\overline{r}$ and q' colums (respectively, q, $p'-r$, 1 and $p+q'-s$ lines),
- the column (respectively, line) blocs of $Z'_{\overline{r}}$ from left to right (respectively, top to bottom) have q, $p'-r$, 1 and $p+q'-s$ columns (respectively, $q+p'-s$, $p-\overline{r}$ and q' lines).

Finally, we let

$$k_1 := q+p'-s+p-\overline{r}, \; k_2 := p-\overline{r}.$$

Lemma 4. We have

$$X_3 = Z_5, \; X_5 = Z_8 \tag{12}$$

and

$$(Z'_5)_{k_2.}(Z_2)_{.k_2} + (Z'_8)_{k_2.}(Z_{11})_{.k_2} = (X'_1)_{k_2.}(X_1)_{.k_2}. \tag{13}$$

Proof. From $K_{\overline{r}}X_{\overline{r}} = Z_{\overline{r}}K'_{\overline{r}}$ and $K'_{\overline{r}}X'_{\overline{r}} = Z'_{\overline{r}}K_{\overline{r}}$, we get respectively

$$\begin{pmatrix} RX_1 & RX_2 \\ X_3 & X_4 \\ X_5 & X_6 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} Z_2 & Z_3R' \\ Z_5 & Z_6R' \\ Z_8 & Z_9R' \\ Z_{11} & Z_{12}R' \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ X'_1 & X'_2 & X'_3 \\ R'X'_4 & R'X'_5 & R'X'_6 \end{pmatrix} = \begin{pmatrix} Z'_1R & Z'_2 & Z'_3 \\ Z'_5R & Z'_6 & Z'_7 \\ Z'_9R & Z'_{10} & Z'_{11} \end{pmatrix}. \tag{14}$$

Hence, (12) holds. Moreover, from $Z'_{\overline{r}}Z_{\overline{r}} = \mathrm{id}_{n_1}$ and $X'_{\overline{r}}X_{\overline{r}} = \mathrm{id}_{n_2}$, the coefficients at positions (k_1, k_1) and (k_2, k_2) of $Z'_{\overline{r}}Z_{\overline{r}}$ and $X'_{\overline{r}}X_{\overline{r}}$, respectively, are equal to 1. Hence, we get

$$\begin{aligned} &(Z'_5)_{k_2.}(Z_2)_{.k_2} + (Z'_6)_{k_2.}(Z_5)_{.k_2} + (Z'_7)_{k_2.}(Z_8)_{.k_2} + (Z'_8)_{k_2.}(Z_{11})_{.k_2} \\ =&(X'_1)_{k_2.}(X_1)_{.k_2} + (X'_2)_{k_2.}(X_3)_{.k_2} + (X'_3)_{k_2.}(X_5)_{.k_2}. \end{aligned}$$

From (14), we also have $X'_2 = Z'_6$ and $X'_3 = Z'_7$, so that (13) holds.

Proposition 5. There exist $c \in D$ and $\mathbf{d}, \mathbf{u} \in D^{1\times(p'-r)}$ such that

$$\begin{pmatrix} c(Z'_5)_{k_2.} & \mathbf{d} & c(Z'_8)_{k_2.} \end{pmatrix} \begin{pmatrix} (Z_2)_{.k_2} \\ (Z_5)_{.k_2} + \mathbf{u}^T(Z_8)_{k_2} \\ (Z_{11})_{.k_2} \end{pmatrix} \tag{15}$$

and

$$\Big(c(X_1')_{k_2.}\ \mathbf{d}\Big)\begin{pmatrix}(X_1)_{.k_2}\\(X_3)_{.k_2}+\mathbf{u}^T(X_5)_{k_2}\end{pmatrix}=1. \tag{16}$$

Proof. By computing the matrix products and from Lemma 4, the left-hand sides of (15) and (16) are equal. Moreover, we show (15) as we did for Proposition 1 by taking the coefficient at position (k_1,k_1) of $Z'_{\overline{r}}Z_{\overline{r}}$, and considering the left D-module

$$N := D/D\Big((Z_5')_{k_2.}(Z_2)_{.k_2}+(Z_8')_{k_2.}(Z_{11})_{.k_2}\Big).$$

With the notations of Proposition 5, we introduce the lines $\tilde{\ell_1}\in D^{1\times\overline{n_1}}$, $\ell_1\in D^{1\times n_1}$, $\tilde{\ell_2}\in D^{1\times\overline{n}_2}$ and $\ell_2\in D^{1\times n_2}$ defined as follows:

$$\tilde{\ell_1} := \Big(c(Z_5')_{k_2.}\ \mathbf{d}\ c(Z_8')_{k_2.}\Big),\qquad \ell_1 := \Big(c(Z_5')_{k_2.}\ \mathbf{d}\ 0\ c(Z_8')_{k_2.}\Big),$$

$$\tilde{\ell_2} := \Big(c(X_1')_{k_2.}\ \mathbf{d}\Big),\qquad \ell_2 := \Big(c(X_1')_{k_2.}\ \mathbf{d}\ 0\Big),$$

as well as the matrices $U_1,U_1'\in D^{n_1\times n_1}$, $F_1\in D^{\overline{n_1}\times n_1}$, $U_2,U_2'\in D^{n_2\times n_2}$ and $F_2\in D^{\overline{n}_2\times n_2}$:

$$U_1 := \begin{pmatrix}\mathrm{id}_q & 0 & 0 & 0\\ 0 & \mathrm{id}_{p'-r} & \mathbf{u}^T & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \mathrm{id}_{p+q'-s}\end{pmatrix},\quad U_2 := \begin{pmatrix}\mathrm{id}_p & 0 & 0\\ 0 & \mathrm{id}_{p'-r} & \mathbf{u}^T\\ 0 & 0 & 1\end{pmatrix},$$

$$U_1' : \begin{pmatrix}\mathrm{id}_q & 0 & 0 & 0\\ 0 & \mathrm{id}_{p'-r} & -\mathbf{u}^T & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \mathrm{id}_{p+q'-s}\end{pmatrix},\quad U_2' := \begin{pmatrix}\mathrm{id}_p & 0 & 0\\ 0 & \mathrm{id}_{p'-r} & -\mathbf{u}^T\\ 0 & 0 & 1\end{pmatrix},$$

$$F_1 := \begin{pmatrix}Z_1 & Z_2 & Z_3\\ Z_4+\mathbf{u}^TZ_7 & Z_5+\mathbf{u}^TZ_8 & Z_6+\mathbf{u}^TZ_9\\ Z_{10} & Z_{11} & Z_{12}\end{pmatrix},$$

$$F_2 := \begin{pmatrix}X_1 & X_2\\ X_3+\mathbf{u}^TX_5 & X_4+\mathbf{u}^TX_6\end{pmatrix}.$$

We point out that U_i and U_i', $i\in\{1,2\}$, are inverses to each other and

$$F_1=\begin{pmatrix}\mathrm{id}_q & 0 & 0 & 0\\ 0 & \mathrm{id}_{p'-r} & 0 & 0\\ 0 & 0 & 0 & \mathrm{id}_{p+q'-s}\end{pmatrix}U_1Z_{\overline{r}},\ F_2=\begin{pmatrix}\mathrm{id}_p & 0 & 0\\ 0 & \mathrm{id}_{p'-r} & 0\end{pmatrix}U_2X_{\overline{r}}.$$

Adapting the arguments for proving (7), we show

$$\begin{aligned}1&=\tilde{\ell}_1F_1(e^n_{k_1})^T=\ell_1U_1Z_{\overline{r}}(e^n_{k_1})^T\\ 1&=\tilde{\ell}_2F_2(e^n_{k_2})^T=\ell_2U_2X_{\overline{r}}(e^n_{k_2})^T.\end{aligned}\tag{17}$$

We decompose $\mathrm{id}_{\overline{n_1}}-F_1\left(e^n_{k_1}\right)^T\tilde{\ell}_1\in D^{\overline{n_1}\times\overline{n_1}}$ as follows

$$\mathrm{id}_{\overline{n_1}}-F_1\left(e^n_{k_1}\right)^T\tilde{\ell}_1=\begin{pmatrix}\varphi_1 & \varphi_2\end{pmatrix}=\begin{pmatrix}\varphi_3\\ \varphi_4\end{pmatrix},$$

where φ_1 and φ_2 (respectively, φ_3 and φ_4) have $q+p'-r$ and $p+q'-s$ columns (respectively, lines). Finally, we consider $\psi_1,\ \psi_1'\in D^{n_1\times\overline{n_1}}$, $\iota_1,\ \iota_1'\in D^{\overline{n_1}\times n_1}$, $\psi_2,\ \psi_2'\in D^{n_2\times\overline{n}_2}$, and $\iota_2,\ \iota_2'\in D^{\overline{n}_2\times n_2}$ defined as follows:

$$\psi_1 := \begin{pmatrix}\varphi_3\\ \tilde{\ell}_1\\ \varphi_4\end{pmatrix},\qquad \psi_1' := \Big(\mathrm{id}_{n_1}-(e^{n_1}_{k_1})^T\ell_1U_1Z_{\overline{r}}\Big)\begin{pmatrix}\mathrm{id}_{\overline{k_1}} & 0\\ 0 & 0\\ 0 & \mathrm{id}_{q'}\end{pmatrix},$$

$$\psi_2 := \begin{pmatrix}\mathrm{id}_{\overline{n}_2}-F_2(e^{n_2}_{k_2})^T\tilde{\ell}_2\\ \tilde{\ell}_2\end{pmatrix},\qquad \psi_2' := \Big(\mathrm{id}_{n_2}-(e^{n_2}_{k_2})^T\ell_2U_2X_{\overline{r}}\Big)\begin{pmatrix}\mathrm{id}_{\overline{k_2}} & 0\\ 0 & 0\\ 0 & \mathrm{id}_{p'}\end{pmatrix},$$

$$\iota_1 := \Big(\varphi_1\ \ F_1(e^{n_1}_{k_1})^T\ \ \varphi_2\Big),\qquad \iota_2 := \Big(\mathrm{id}_{\overline{n}_2}-F_2(e^{n_2}_{k_2})^T\tilde{\ell}_2\ \ F_2(e^{n_2}_{k_2})^T\Big),$$

$$\iota_1' := \begin{pmatrix}\mathrm{id}_{\overline{k_1}} & 0 & 0\\ 0 & 0 & \mathrm{id}_{q'}\end{pmatrix},\qquad \iota_2' := \begin{pmatrix}\mathrm{id}_{\overline{k_2}} & 0 & 0\\ 0 & 0 & \mathrm{id}_{p'}\end{pmatrix}.$$

By adapting the arguments of the proof of Proposition 2, we get:

Proposition 6. We have the following relations

(1) $\iota_1\psi_1=\mathrm{id}_{\overline{n_1}}$, $\iota_2\psi_2=\mathrm{id}_{\overline{n}_2}$,
(2) $\iota_1'\psi_1'=\mathrm{id}_{\overline{n_1}}$, $\iota_2'\psi_2'=\mathrm{id}_{\overline{n}_2}$,
(3) $\ker(.\psi_1)=D\ell_1$, $\ker(.\psi_2)=D\ell_2$,
(4) $\ker(.\psi_1')=D\ell_1U_1Z_{\overline{r}}$, $\ker(.\psi_2')=D\ell_2U_2X_{\overline{r}}$.

Theorem 7. With the previous notations, we let

$$Z_r := \iota_1U_1Z_{\overline{r}}\psi_1',\quad X_r := \iota_2U_2X_{\overline{r}}\psi_2',$$

$$Z_r' := \iota_1'Z'_{\overline{r}}U_1'\psi_1,\quad X_r' := \iota_2'X'_{\overline{r}}U_2'\psi_2.$$

The following diagram is exact and commutative:

$$\begin{tikzcd}
D^{1\times\overline{n_1}} \arrow[r, ".K_r"] \arrow[d, shift right, ".Z_r"'] & D^{1\times\overline{n}_2} \arrow[r, "\pi\oplus 0"] \arrow[d, shift right, ".X_r"'] & M \arrow[r] \arrow[d, shift right, "f"'] & 0\\
D^{1\times\overline{n_1}} \arrow[r, ".K_r'"] \arrow[u, shift right, ".Z_r'"'] & D^{1\times\overline{n}_2} \arrow[r, "0\oplus\pi'"] \arrow[u, shift right, ".X_r'"'] & M' \arrow[r] \arrow[u, shift right, "f^-"'] & 0
\end{tikzcd}$$

In particular, we have

$$K_r'=Z_r'K_rX_r.$$

Proof. We only have to show that the diagram is commutative.

We show that Z_r' and Z_r' (respectively, X_r and X_r') are inverse to each over in the same manner that we did in the proof of Theorem 3.

In Section 5.1 we prove the following relations:

$$\begin{aligned}&\iota_1K_{\overline{r}}=K_r\iota_2, && \iota_1'K'_{\overline{r}}=K_r'\iota_2',\\ &U_1K_{\overline{r}}=K_{\overline{r}}U_2, && Z'_{\overline{r}}K_{\overline{r}}=K'_{\overline{r}}X'_{\overline{r}},\\ &Z_{\overline{r}}K'_{\overline{r}}=K_{\overline{r}}X_{\overline{r}}, && U_1'K_{\overline{r}}=K_{\overline{r}}U_2',\\ &\psi_1'K_r'=K'_{\overline{r}}\psi_2', && \psi_1K_r=K_{\overline{r}}\psi_2.\end{aligned}\tag{18}$$

Hence, we deduce $Z_rK_r'=K_rX_r$ and $Z_r'K_r=K_r'X_r'$ from commutativity of:

$$\begin{tikzcd}
D^{1\times\overline{n_1}} \arrow[r, shift left, "\iota_1"] \arrow[d, "K_r"'] & D^{1\times n_1} \arrow[l, shift left, "\psi_1"] \arrow[r, shift left, "U_1"] \arrow[d, "K_{\overline{r}}"] & D^{1\times n_1} \arrow[l, shift left, "U_1'"] \arrow[r, shift left, "Z_{\overline{r}}"] \arrow[d, "K_{\overline{r}}"] & D^{1\times n_1} \arrow[l, shift left, "Z'_{\overline{r}}"] \arrow[r, shift left, "\psi_1'"] \arrow[d, "K'_{\overline{r}}"] & D^{1\times\overline{n}} \arrow[l, shift left, "\iota_1'"] \arrow[d, "K_r'"]\\
D^{1\times\overline{n}_2} \arrow[r, shift left, "\iota_2"] & D^{1\times n_2} \arrow[l, shift left, "\psi_2"] \arrow[r, shift left, "U_2"] & D^{1\times n_2} \arrow[l, shift left, "U_2'"] \arrow[r, shift left, "X_{\overline{r}}"] & D^{1\times n_2} \arrow[l, shift left, "X'_{\overline{r}}"] \arrow[r, shift left, "\psi_2'"] & D^{1\times\overline{n}_2} \arrow[l, shift left, "\iota_2'"]
\end{tikzcd}$$

4. EXAMPLE

In this Section, we illustrate Theorem 3.

4.1 Two presentations of one ODE

We consider a linear differential equation with constant coefficients:

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1\dot{y}(t) + a_0 y(t) = 0, \quad (19)$$

where $a_i \in \mathbb{R}$. Letting

$$x_1 := y,\ x_2 := \dot{x}_1 = \dot{y},\ \cdots,\ x_n := \dot{x}_{n-1} = y^{(n-1)},$$

(19) rewrites as follows:

$$\begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_1 & -a_2 & \cdots & \cdots & -a_{n-1} \end{pmatrix} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}. \quad (20)$$

Hence, the system (19)–(20) is described by two matrices over the ring $D := \mathbb{R}[\partial]$ of differential polynomials with constant coefficients: $Ry = 0$ and $R'x = 0$, $R \in D^{1\times 1} = D$ and $R' \in D^{n\times n}$ defined as follows

$$R := \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0,$$

$$R' := \begin{pmatrix} \partial & -1 & 0 & \cdots & \cdots & 0 \\ 0 & \partial & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \partial & -1 \\ a_0 & a_1 & \cdots & \cdots & a_{n-1} & \partial + a_n \end{pmatrix}.$$

Hence, letting $p = q = 1$ and $p' = q' = n$, we associate with this system the two modules $M := D/(DR)$ and $M' := D^{1\times n}/(D^{1\times n}R')$, isomorphic as follows

$$\begin{tikzcd} D^{1\times p} \arrow[r, ".R"] \arrow[d, shift right, ".Q"'] & D^{1\times p} \arrow[r, "\pi"] \arrow[d, shift right, ".P"'] & M \arrow[r] \arrow[d, shift right, "f"'] & 0 \\ D^{1\times q'} \arrow[r, ".R'"] \arrow[u, shift right, ".Q'"'] & D^{1\times p'} \arrow[r, "\pi'"] \arrow[u, shift right, ".P'"'] & M' \arrow[r] \arrow[u, shift right, "f^-"'] & 0 \end{tikzcd}$$

where $P, Q \in D^{1\times n}$ and $P', Q' \in D^{n\times 1}$ are:

$$P := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$Q := \left(\partial^{n-1} + \sum_{i=1}^{n-1} a_i \partial^{(i-1)} \ \ \partial^{n-2} + \sum_{i=2}^{n-2} a_i\partial^{(i-1)} \ \cdots \ \partial + a_{n-1} \ \ 1 \right),$$

$$P' := \begin{pmatrix} 1 \\ \partial \\ \vdots \\ \partial^{n-1} \end{pmatrix}, \qquad Q' := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

4.2 Reduction of L and L'

Let L and L' be the matrices as in (2). For simplicity, we assume that $n = 3$. We have

$$Z = \begin{pmatrix} 0 \end{pmatrix},\ Z' = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -\partial & -1 & 0 \end{pmatrix}$$

and the matrices Z_2, R_2, Z_2' and R_2' are the zero matrices. The expressions of $X, X' \in D^{4\times 4}$ and $Y, Y' \in D^{8\times 8}$ come from (3).

We have $\mathrm{sr}(D) = 2$, see McConnell and Robson (2001). From (10), we may remove two lines of 0 in L and L'. We give the details for removing the first zero lines of L and L'. With the notations of Proposition 1, we may choose

$$c = 0 \in D, \quad u = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \in D^{1\times 3} \quad \text{and} \quad d = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \in D^{1\times 3}.$$

We get $U = \mathrm{id}_8$ and

$$\psi = \begin{pmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&-1 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1 \end{pmatrix}, \quad \iota = \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&1 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&1 \end{pmatrix},$$

$$\pi' = \begin{pmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&\partial&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{pmatrix}, \quad \iota' = \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix}.$$

From this, we get the matrices Y_1 and Y_1' in (21). We remove the second zero lines with the matrices Y_2 and Y_2' of (22), where we use the following notations:

$$P(a,\partial) := -(\partial^3 + a_2\partial^2 + (a_1 - 1)\partial + a_0),$$

$$Q(a,\partial) := \partial^2 + (a_2 - 1)\partial + a_1 - 1,$$

$$\begin{aligned} R(a,\partial) := {} & \partial^5 + (2a_2 - 1)\partial^4 + (a_2^2 + 2a_1 - a_2 - 1)\partial^3 \\ & + ((2a_2 - 1)a_1 + a_0 - a_2 + 1)\partial^2 \\ & + (a_1(a_1 - 1) + a_0(a_2 - 1))\partial + a_0(a_1 - 1). \end{aligned}$$

5. PROOFS OF FORMULAS (9) AND (18)

5.1 Proof of Formulas (9)

We have to show the following relations

$$\iota L_{\overline{s}} = L_s, \quad (23) \qquad Y_{\overline{s}} L_{\overline{s}}' = L_{\overline{s}} X, \quad (25)$$

$$U L_{\overline{s}} = L_{\overline{s}}, \quad (24) \qquad \psi' L_s' = L_{\overline{s}}', \quad (26)$$

$$\iota' L_{\overline{s}}' = L_s', \quad (27) \qquad U' L_{\overline{s}} = L_{\overline{s}}, \quad (29)$$

$$Y_{\overline{s}}' L_{\overline{s}} = L_{\overline{s}}' X', \quad (28) \qquad \psi L_s = L_{\overline{s}}. \quad (30)$$

The two relations (25) and (28) are assumed by induction hypothesis, (24), (27) and (29) are proven by direct computations. For proving (23), we first check that by definitions

$$Y_1 = \begin{pmatrix} 1 & 0 & 0 & \partial^3 + a_2\partial^2 + a_1\partial + a_0 & \partial^2 + a_2\partial + a_1 & \partial + a_2 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\partial & -1 & 0 & 0 \\ -1 & -a_0 & (-\partial^2 + (1-a_2)\partial - a_1) & -\partial^2 & -\partial & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\partial & 1 & 0 & 0 & 0 & 0 \\ -1 & -a_0 & -(\partial^2 + a_2\partial + a_1) & 0 & 0 & 0 & 0 \end{pmatrix},$$

(21)

$$Y_1' = \begin{pmatrix} 0 & 0 & 0 & 0 & -(\partial^3 + a_2\partial^2 + a_1\partial + a_0) & -(\partial^2 + a_2\partial + a_1) & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \partial & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \partial & -1 & 0 & 0 & 1 \\ 1 & a_0 & a_1 & \partial + a_2 & 0 & 0 & -\partial - a_2 + 1 \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} 1 & 0 & \partial^3 + a_2\partial^2 + a_1\partial + a_0 & (\partial^2 + a_2\partial + a_1) & \partial + a_2 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & P(a,\partial) & -\partial & -1 & 0 & 0 \\ Q(a,\partial) & R(a,\partial) & -\partial^2 & -\partial & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -(\partial^3 + a_2\partial^2 + a_1\partial + a_0) & 0 & 0 & 0 & 0 \end{pmatrix},$$

(22)

$$Y_2' = \begin{pmatrix} 0 & 0 & 0 & 0 & -(\partial^3 + a_2\partial^2 + a_1\partial + a_0) & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & \partial & -1 & 0 & 0 & 1 \\ 0 & 0 & \partial & -1 & 0 & -(\partial^2 + a_2\partial + a_1 - 1) \\ 1 & a_0 & a_1 & \partial + a_2 & 0 & (\partial + a_2 - 1)(\partial^2 + a_2\partial + a_1 - 1) \end{pmatrix}.$$

of ι, L_s and $L_{\overline{s}}$, we have $\iota L_{\overline{s}} = L_s - F(e_k^n)^T \tilde{\ell} L_s$. Moreover, we have $\tilde{\ell} L_s = 0$ from the following sequence of equalities:

$$\tilde{\ell} L_s = (Y_1')_{k.} \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_p \end{pmatrix} = (Y_{\overline{s}}' L_{\overline{s}})_{k.} = (L_{\overline{s}}' X')_{k.} = 0.$$

The first equality comes from definitions of $\tilde{\ell}$ and L_s, the second is due to $(Y_{\overline{s}}')_{k.} = ((Y_1')_{k.}\ (Y_2')_{k.}\ (Y_3')_{k.})$ and the definition of $L_{\overline{s}}$ and the last one to the induction hypothesis $Y_{\overline{s}}' L_{\overline{s}} = L_{\overline{s}}' X'$ and $(L_{\overline{s}}')_{k.} = 0$.

In order to prove (26), we first show $\psi' \iota' K_{\overline{s}}' = K_{\overline{s}}'$. From 2 and 4 of Proposition 2, $\mathrm{im}(\psi'\iota' - \mathrm{id}_n)$ is included $D\ell U Y_{\overline{s}}$. From (24) and (25), we have $\ell U Y_{\overline{s}} K_{\overline{s}}' = \ell K_{\overline{s}} X$, and from $\tilde{\ell} K_s = 0$, we have $\ell K_{\overline{s}} = 0$, so that $\ell U Y_{\overline{s}} K_{\overline{s}}' = 0$. Thus, $(\psi'\iota' - \mathrm{id}_n) K_{\overline{s}}' = 0$, which is the desired relation. Moreover, from (27), we get $\psi'\iota' K_{\overline{s}}' = \psi' K_s'$, which, with $\psi'\iota' K_{\overline{s}}' = K_{\overline{s}}'$, gives (26).

We show (30) in the same manner using (23).

5.2 Proof of Formulas (18)

We have to show the following relations

$$\iota_1 K_{\overline{r}} = K_r \iota_2, \qquad (31)$$

$$U_1 K_{\overline{r}} = K_{\overline{r}} U_2, \qquad (32)$$

$$Z_{\overline{r}} K_{\overline{r}}' = K_{\overline{r}} X_{\overline{r}}, \qquad (33)$$

$$\psi_1' K_r' = K_{\overline{r}}' \psi_2', \qquad (34)$$

$$\iota_1' K_{\overline{r}}' = K_r' \iota_2', \qquad (35)$$

$$Z_{\overline{r}}' K_{\overline{r}} = K_{\overline{r}}' X_{\overline{r}}', \qquad (36)$$

$$U_1' K_{\overline{r}} = K_{\overline{r}} U_2', \qquad (37)$$

$$\psi_1 K_r = K_{\overline{r}} \psi_2. \qquad (38)$$

The two relations (33) and (36) are assumed by induction hypothesis, (32), (35) and (37) are proven by direct computations.

Let us show (31). For that, we use that $\iota_1 K_{\overline{r}}$ is equal to $(\varphi_1\ 0\ \varphi_2) K_{\overline{r}} + (0\ F_1(e_{k_1}^{n_1})^T\ 0) K_{\overline{r}}$ and that, by computing matrix poducts, the first summand of this expression is equal to $\left(\mathrm{id}_{\overline{n_1}} - F_1(e_{k_1}^{n_1})^T \tilde{\ell}_1\right)(K_r\ 0)$, so that

$$\iota_1 K_{\overline{r}} = (K_r\ 0) - F_1(e_{k_1}^{n_1})^T \tilde{\ell}_1 (K_r\ 0) + (0\ F_1(e_{k_1}^{n_1})^T\ 0) K_{\overline{r}}.$$

By adapting the arguments, we also show

$$K_r \iota_2 = (K_r\ 0) - \left(\left(K_r F_2(e_{k_2}^{n_2})^T \tilde{\ell}_2\right)\ 0\right) + K_r (0\ F_2(e_{k_2}^{n_2})^T).$$

Now, we show that the last two summands of these expressions are equal by computing matrix porducts and using (14).

Let us show (34). For that, multiplyting (35) by ψ_1' on the left and ψ_2' on the right, and from Point 2 of Proposition 6, we get $\psi_1'\iota_1'K_{\overline{r}}'\psi_2' = \psi_1'K_r'$. It remains to show that $\psi_1'\iota_1'K_{\overline{r}}'\psi_2' = K_{\overline{r}}'\psi_2'$, for which it is sufficient to show that $\mathrm{im}(\psi_1'\iota_1' - \mathrm{id}_n)K_{\overline{r}}'$ is included in $\ker(.\psi_2')$. From Points 2 and 4 of Proposition 6, we have $\mathrm{im}(\psi_1'\iota_1' - \mathrm{id}_n) \subseteq \ker(.\psi_1') = D\ell_1U_1Z_{\overline{r}}$. From (33) and (32), we have $\ell_1U_1Z_{\overline{r}}K_{\overline{r}}' = \ell_1K_{\overline{r}}U_2X_{\overline{r}}$. By computing matrix products and using (14), we have $\ell_1K_{\overline{r}} = \ell_2$. Hence, $\ell_1U_1Z_{\overline{r}}K_{\overline{r}}'$ is equal to $\ell_2U_2X_{\overline{r}}$, and by computing matrix products and using (17), $\ell_2U_2X_{\overline{r}}$ is included in $\ker(.\psi_2')$. Hence, $D\ell_1U_1Z_{\overline{r}}$ is included in $\ker(.\psi_2')$, so that $\mathrm{im}(\psi_1'\iota_1' - \mathrm{id}_n)K_{\overline{r}}'$ is also included in $\ker(.\psi_2')$, which finishes the proof of (34).

With the same arguments, we show (38).

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