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A constructive version of Warfield's Theorem

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Abstract: Within the algebraic analysis approach to linear system theory, a multidimensional linear system can be studied by means of its associated finitely presented left module. Deep connections exist between module isomorphisms and equivalent matrices. In the present paper, we introduce a constructive proof of a result due to Warfield which controls the size of equivalent matrices involved in the study of isomorphic modules. We illustrate our constructive proof with an example coming from differential equations with constant coefficients.

Keywords: Linear functional systems, module isomorphisms, matrix equivalence

1. INTRODUCTION

A linear multidimensional system, such as a linear system of differential equations or partial derivative equations, maybe described by a matrix of functional operators. Indeed, a system of q equations with p unknown functions η_1, \dots, η_p over a ring of operators D is written

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p \mid R\eta = 0\}, \quad (1)$$

where $R \in D^{q \times p}$ and \mathcal{F} is the functional space where we are looking for the solutions. The system (1) can be studied by mean of algebraic tools, using the finitely presented left module $M := D^{1 \times p} / (D^{1 \times q} R)$ with p generators submitted to the q relations specified by the lines of R . Indeed, from the properties of free and quotient modules, the abelian group $\ker_{\mathcal{F}}(R.)$ is isomorphic to $\text{hom}_D(M, \mathcal{F})$, that is the abelian group of left D -linear maps from M to \mathcal{F} , see Malgrange (1963). Hence, systemic properties of (1) can be studied by mean of modules properties, which can be computed using effective homological algebra and Groebner bases theory, see Chyzak et al. (2005).

The abelian group $\text{hom}_D(M, \mathcal{F})$ only depend on M and \mathcal{F} , in particular it does not depend on the matrix R . That means that two matrices R and R' defining isomorphic modules $M \simeq M'$ have the same algebraic properties. A particular example for the isomorphism $M \simeq M'$ is the case where R and R' are *equivalent*, that is $R = YR'X^{-1}$ for invertible matrices X and Y . A result due to Fitting (1936) asserts a weak converse implication: if R and R' define isomorphic modules, then they can be enlarged by 0 and identity blocs leading to equivalent matrices L and L' . An effective version of this result was obtained in Cluzeau and Quadrat (2011).

The purpose of the present paper is to introduce an effective version of a result of Warfield (1978), which asserts that the number of 0 and the size of identity blocs in L and L' maybe reduced, while keeping equivalent matrices. This result is based on an algebraic invariant of D , called the *stable rank*, see McConnell and Robson (2001)

The paper is organised as follows. In Section 2, we recall the notions of isomorphic left D -modules and equivalent matrices, as well as the effective version of Fitting's result and the statement of Warfield's result. In Section 3, we present the constructive version of Warfield's result for removing 0 and identity blocs. In Section 4, we illustrate this constructive version with a differential equation with constant coefficients. Section 5 contains proofs of formulas used in Section 3.

2. ISOMORPHISMS AND EQUIVALENT MATRICES

In this section, we recall the characterization of morphisms between finitely presented left D -modules, as well as results of Fitting and Warfield which rely isomorphic left D -modules to matrix conjugation.

2.1 Effective version of Fitting's Theorem

Consider two left D -modules M and M' with finite presentations:

$$D^{1 \times q} \xrightarrow{\cdot R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0,$$

$$D^{1 \times q'} \xrightarrow{\cdot R'} D^{1 \times p'} \xrightarrow{\pi'} M' \longrightarrow 0,$$

namely, *exact sequences* (see Rotman (2009)), where $R \in D^{q \times p}$, $(\cdot R)(\mu) = \mu R$, for every $\mu \in D^{1 \times q}$ and π is the natural projection on $M = D^{1 \times p} / (D^{1 \times q} R)$ (similarly for R' and π').

From Rotman (2009), there exists $f \in \text{hom}_D(M, M')$ if and only if there exist matrices $P \in D^{p \times p'}$ and $Q \in D^{q \times q'}$ such that $RP = QR'$ and

$$\forall \lambda \in D^{1 \times p}, f(\pi(\lambda)) = \pi'(\lambda \pi).$$

Hence, the following diagram is exact and commutative:

$$\begin{array}{ccccccc}
D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\cdot Q \downarrow & & \cdot P \downarrow & & f \downarrow & & \\
D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0
\end{array}$$

We let $n := q + p' + p + q'$ and $m := p + p'$. The two $n \times m$ matrices

$$L := \begin{pmatrix} R & 0 \\ 0 & \text{id}_{p'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad L' := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \text{id}_p & 0 \\ 0 & R' \end{pmatrix}, \quad (2)$$

induce finite presentations: $M \simeq D^{1 \times m} / (D^{1 \times n} L)$ and $M' \simeq D^{1 \times m} / (D^{1 \times n} L')$. In Cluzeau and Quadrat (2011), an effective version of a result due to Fitting (1936) is given. If f is an isomorphism, then L and L' are equivalent: there exist 6 matrices $R_2 \in D^{r \times q}$, $R'_2 \in D^{r' \times q'}$, $Z_2 \in D^{p \times q}$, $Z_2 \in D^{q \times r}$, $Z'_2 \in D^{q' \times r'}$, $Z \in D^{p \times q}$ and $Z' \in D^{p' \times q'}$ and two invertible matrices of size m and n

$$\begin{aligned}
X_F &:= \begin{pmatrix} \text{id}_p & P \\ -P' & \text{id}_{p'} - P'P \end{pmatrix} \\
Y_F &:= \begin{pmatrix} \text{id}_q & 0 & R & Q \\ 0 & \text{id}_{p'} & -P' & Z' \\ -Z & P & 0 & PZ' - ZQ \\ -Q' & -R' & 0 & Z'_2 R'_2 \end{pmatrix}, \quad (3)
\end{aligned}$$

with inverses

$$\begin{aligned}
X_F^- &:= \begin{pmatrix} \text{id}_p - PP' & -P \\ P' & \text{id}_{p'} \end{pmatrix} \\
Y_F^- &:= \begin{pmatrix} Z_2 R_2 & 0 & -R & -Q \\ P'Z - Z'Q' & 0 & P' & -Z' \\ Z & -P & \text{id}_p & 0 \\ Q' & R' & 0 & \text{id}_{q'} \end{pmatrix}, \quad (4)
\end{aligned}$$

such that

$$L' = Y_F^- L X_F. \quad (5)$$

In other words, the following diagram is exact commutative

$$\begin{array}{ccccccc}
D^{1 \times n} & \xrightarrow{\cdot L} & D^{1 \times m} & \xrightarrow{\pi \oplus 0} & M & \longrightarrow & 0 \\
\cdot Y_W \downarrow \uparrow \cdot Y_W^- & & \cdot X_W \downarrow \uparrow \cdot X_W^- & & f \downarrow \uparrow f^- & & \\
D^{1 \times n} & \xrightarrow{\cdot L'} & D^{1 \times m} & \xrightarrow{0 \oplus \pi'} & M' & \longrightarrow & 0
\end{array}$$

2.2 Warfield's Theorem

A result due to Warfield (1978) asserts that that the size of 0 an id blocs in (2) can be reduced, whereas the new matrices are still equivalent. This result is based on the notion of *stable rank*. The definition of the latter requires to introduce various notions that we present now.

A column vector $u := (u_1 \cdots u_k)^T \in D^{k \times 1}$ is called *unimodular* if there exists a line $v \in D^{1 \times k}$ such that

$vu = 1$. Moreover, u is said to be *stable* if there exist $d_1, \dots, d_{k-1} \in D$ such that $(u_1 + d_1 u_k \cdots u_{k-1} + d_{k-1} u_k)$ is unimodular. An integer r is said to be in the *stable rank* of D if whenever $k > r$, every column $u \in D^{k \times 1}$ is stable. The *stable rank* $\text{sr}(D)$ of D is the smallest integer in the *stable rank* of D .

Assume that the two matrices (2) are equivalent, then Warfield's Theorem asserts that if there exist two integers r and s such that

$$\begin{cases} s \leq \min(p + q', q + p'), \\ \text{sr}(D) \leq \max(p + q' - s, q + p' - s), \\ r \leq \min(p, p'), \\ \text{sr}(D) \leq \max(p - r, p' - r), \end{cases} \quad (6)$$

then the following $(n - r - s) \times (m - r)$ matrices are equivalent

$$\bar{L} := \begin{pmatrix} R & 0 \\ 0 & \text{id}_{p'-r} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \bar{L}' := \begin{pmatrix} 0 & 0 \\ \text{id}_{p-r} & 0 \\ 0 & R' \end{pmatrix}, \quad (7)$$

and induce finite presentations of M and M' , respectively.

In the next section, we introduce a procedure which computes invertible matrices X_W and Y_W such that

$$\bar{L}' = Y_W^- \bar{L} X_W.$$

3. EFFECTIVE WARFIELD'S THEOREM

Throughout this Section, we fix some notations. Let M and M' be two left D -modules, isomorphic with $f : M \xrightarrow{\sim} M'$, finitely presented by matrices $R \in D^{q \times p}$ and $R' \in D^{q' \times p'}$, respectively, and let L, L', X_F, Y_F, X_F^- and Y_F^- be the matrices defined in (2) (3) and (4). Let $n := q + p' + p + q'$ and $m := p + p'$. Given a nonzero integer k , we let $\bar{k} := k - 1$. For $1 \leq i \leq k$, the i -th vector of the canonical basis of $D^{1 \times k}$ is written e_i^k .

3.1 Reduction of the zero bloc

In this section, we present the procedure for removing one 0 in L and L' . Inductive applications of this procedure enables us to remove many 0. Without lost of generalities, we may suppose that $q + p' \leq p + q'$, and we assume that $\text{sr}(D) \leq p + q'$, so that the hypotheses of (6) are fulfilled for $s = 1$. Our purpose is to show that the following $\bar{n} \times m$ matrices are equivalent:

$$\tilde{L} := \begin{pmatrix} R & 0 \\ 0 & \text{id}_{p'} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{L}' := \begin{pmatrix} 0 & 0 \\ \text{id}_p & 0 \\ 0 & R' \end{pmatrix}.$$

Proposition 1. There exist $\mathbf{c}, \mathbf{u} \in D^{1 \times p}$ and $\mathbf{d}, \mathbf{v} \in D^{1 \times \bar{q}}$ such that

$$(0 \ \mathbf{c} \ \mathbf{d}) \begin{pmatrix} \text{id}_{q+p'} & 0 & 0 & 0 \\ 0 & \text{id}_p & 0 & \mathbf{u}^T \\ 0 & 0 & \text{id}_{\bar{q}} & \mathbf{v}^T \end{pmatrix} Y_F (e_{q+p'}^n)^T = 1. \quad (8)$$

Proof. Getting terms of the p' -th column and p' -th line in the relation $\text{id}_{p'} = P'P + Z'R'$, we get the following:

$$\sum_{k=1}^p P'_{p'k} P_{kp'} + \sum_{k=1}^{q'} Z'_{p'k} R'_{kp'} = 1.$$

From $\text{sr}(D) \leq \overline{p+q'}$, we deduce that there exist $c_1, \dots, c_p, d_1, \dots, d_{\overline{q'}}, u_1, \dots, u_p, v_1, \dots, v_{\overline{q'}} \in D$ such that

$$\sum_{k=1}^p c_k (P_{kp'} + u_k R'_{q'p'}) + \sum_{k=1}^{\overline{q'}} d_k (R'_{kp'} + v_k R'_{q'p'}) = 1. \quad (9)$$

Letting $\mathbf{c} := (c_1, \dots, c_p)$, $\mathbf{d} := (-d_1, \dots, -d_{\overline{q'}})$, $\mathbf{u} := (-u_1, \dots, -u_p)$ and $\mathbf{v} := (v_1, \dots, v_{\overline{q'}})$, the hand side of (9) is the left hand side of (8), which proves Proposition 1.

With the notations of Proposition 1, we introduce the lines $\tilde{\ell} \in D^{1 \times \overline{n}}$ and $\ell \in D^{1 \times n}$ defined as follows:

$$\tilde{\ell} := (0 \ \mathbf{c} \ \mathbf{d}) \quad \text{and} \quad \ell := (0 \ \mathbf{c} \ \mathbf{d} \ 0),$$

as well as the matrices $U \in D^{n \times n}$, $F \in D^{\overline{n} \times n}$ defined as follows:

$$U := \begin{pmatrix} \text{id}_{q+p'} & 0 & 0 & 0 \\ 0 & \text{id}_p & 0 & \mathbf{u}^T \\ 0 & 0 & \text{id}_{\overline{q'}} & \mathbf{v}^T \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad F := (\text{id}_{\overline{n}} \ 0) U Y_F.$$

From Relation (8) and

$$F = \begin{pmatrix} \text{id}_{q+p'} & 0 & 0 & 0 \\ 0 & \text{id}_p & 0 & \mathbf{u}^T \\ 0 & 0 & \text{id}_{\overline{q'}} & \mathbf{v}^T \end{pmatrix} Y_F,$$

we get:

$$1 = \tilde{\ell} F (e_{q+p'}^n)^T = \ell U Y_F (e_{q+p'}^n)^T. \quad (10)$$

We consider the matrices $\text{pr}, \text{pr}' \in D^{n \times \overline{n}}$

$$\text{pr} := \begin{pmatrix} \text{id}_{\overline{n}} - F(e_{q+p'}^n)^T \tilde{\ell} \\ \tilde{\ell} \end{pmatrix},$$

$$\text{pr}' := (\text{id}_n - (e_{q+p'}^n)^T \ell U Y_F) \begin{pmatrix} \text{id}_{q+p'} & 0 \\ 0 & 0 \\ 0 & \text{id}_{p+q'} \end{pmatrix},$$

and $\iota, \iota' \in D^{\overline{n} \times n}$

$$\iota := (\text{id}_{\overline{n}} - F(e_{q+p'}^n)^T \tilde{\ell} \ F(e_{q+p'}^n)^T), \quad \iota' := \begin{pmatrix} \text{id}_{q+p'} & 0 & 0 \\ 0 & 0 & \text{id}_{p+q'} \end{pmatrix}.$$

Proposition 2. We have the following relations:

- (1) $\iota \text{pr} = \text{id}_{\overline{n}}$,
- (2) $\iota' \text{pr}' = \text{id}_{\overline{n}}$,
- (3) $\ker(\text{pr}) = D\ell$,
- (4) $\ker(\text{pr}') = D\ell U Y_F$.

Proof.

- (1) From (10), we have $(F(e_{q+p'}^n)^T \tilde{\ell})^2 = F(e_{q+p'}^n)^T \tilde{\ell}$, from which we deduce $\iota \text{pr} = \text{id}_{\overline{n}}$ by computing the matrix product.

- (2) We have $\iota'(e_{q+p'}^n)^T = 0$, from which we deduce $\iota' \text{pr}' = \text{id}_{\overline{n}}$ by computing the matrix product.
- (3) Considering the isomorphism $D^{1 \times n} \simeq D^{1 \times \overline{n}} \oplus D$, we have $\text{pr} = \text{pr}_1 \text{pr}_2$, where $\text{pr}_1 \in D^{n \times n}$ and $\text{pr}_2 \in D^{n \times \overline{n}}$ are defined as follows:

$$\text{pr}_1 := \begin{pmatrix} \text{id}_{\overline{n}} - F(e_{q+p'}^n)^T \tilde{\ell} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \text{pr}_2 := \begin{pmatrix} \text{id}_{\overline{n}} \\ \tilde{\ell} \end{pmatrix}.$$

From (10), $\text{im}(\cdot \text{pr}_1)$ is included in $\ker(\cdot F(e_{q+p'}^n)^T) \oplus D$

and the restriction of $\cdot \text{pr}_2$ to the latter is injective: for $(u, x) \in (\ker(\cdot F(e_{q+p'}^n)^T) \oplus D) \cap \ker(\cdot \text{pr}_2)$, we have $u + x\tilde{\ell} = 0$, which gives $x = 0$ and $u = 0$ by (10). Hence, we have $\ker(\cdot \text{pr}) = \ker(\cdot \text{pr}_1)$, that is $\ker(\text{id}_{\overline{n}} - \cdot F(e_{q+p'}^n)^T \tilde{\ell}) \oplus 0$. We conclude by showing $\ker(\text{id}_{\overline{n}} - \cdot F(e_{q+p'}^n)^T \tilde{\ell}) = D\tilde{\ell}$: the right to left inclusion is due to (10), and the other one is due to the relation $x = (x F(e_{q+p'}^n)^T) \tilde{\ell}$, for every x in $\ker(\text{id}_{\overline{n}} - \cdot F(e_{q+p'}^n)^T \tilde{\ell})$.

- (4) From (10), we have $D\ell U Y_F \subseteq \ker(\cdot \text{pr}')$. The converse inclusion is due to the relation $x = x_{q+p'} \ell U Y_F$, for every $x \in \ker(\cdot \text{pr}')$. Indeed, the first $q+p'$ and the last $p+q'$ columns of x and $x_{q+p'} \ell U Y_F$ are equal since $x \in \ker(\cdot \text{pr}')$ implies

$$x \begin{pmatrix} \text{id}_{q+p'} & 0 \\ 0 & 0 \\ 0 & \text{id}_{p+q'} \end{pmatrix} = x_{q+p'} \ell U Y_F \begin{pmatrix} \text{id}_{q+p'} & 0 \\ 0 & 0 \\ 0 & \text{id}_{p+q'} \end{pmatrix}.$$

Moreover, the $q+p'$ -th columns of $x_{q+p'} \ell U Y_F$ is computed by right multiplication by $(e_{q+p'}^n)^T$ and is equal to $x_{q+p'}$ from (10).

Theorem 3. With the previous notations, we let

$$X_W := X_F, \quad Y_W := \iota U Y_F \text{pr}',$$

$$X_W^- := X_F, \quad Y_W^- := \iota' Y_F^- U^- \text{pr}.$$

The following diagram is exact and commutative:

$$\begin{array}{ccccccc} D^{1 \times \overline{n}} & \xrightarrow{\cdot \tilde{L}} & D^{1 \times m} & \xrightarrow{\pi \oplus 0} & M & \longrightarrow & 0 \\ \cdot Y_W \downarrow & \uparrow \cdot Y_W^- & \cdot X_W \downarrow & \uparrow \cdot X_W^- & f \downarrow & \uparrow f^- & \\ D^{1 \times \overline{n}} & \xrightarrow{\cdot \tilde{L}'} & D^{1 \times m} & \xrightarrow{0 \oplus \pi'} & M' & \longrightarrow & 0 \end{array}$$

In particular, we have

$$\tilde{L}' = Y_W^- \tilde{L} X_W.$$

Proof. We only have to show that the diagram is commutative.

First, we show that Y_W and Y_W^- are inverse to each other. From Proposition 2, the lines of the following diagram are exact

$$\begin{array}{ccccccc}
D & \xrightarrow{\ell} & D^{1 \times n} & \xrightarrow{\text{pr}} & D^{1 \times \bar{n}} & \longrightarrow & 0 \\
\text{id}_D \downarrow & & \downarrow \cdot UY_F & \downarrow \cdot Y_F^- U^- & \downarrow \cdot Y_W & \downarrow \cdot Y_W^- & \\
D & \xrightarrow{\ell UY_F} & D^{1 \times n} & \xrightarrow{\text{pr}'} & D^{1 \times \bar{n}} & \longrightarrow & 0
\end{array} \quad (11)$$

Moreover, it is also commutative. Indeed, $\text{pr}Y_W$ is equal to $\text{pr}UY_F\text{pr}'$ and from 1 of Proposition 2, we have $\text{im}(\text{pr}\ell - \text{id}_n) \subseteq \ker(\cdot\text{pr})$. By commutativity of the left rectangle and by exactness of the lines of (11), we have $(\text{pr}\ell - \text{id}_n)UY_F\text{pr}' = 0$, so that $\text{pr}Y_W = UY_F\text{pr}'$. In the same manner, we show that $Y_F^-U^-\text{pr} = \text{pr}'Y_W^-$. By commutativity and exactness of (11) and from the equations $UY_FY_F^-U^- = Y_F^-U^-UY_F = \text{id}_n$, we get $Y_WY_W^- = Y_W^-Y_W = \text{id}_{\bar{n}}$.

Moreover, $Y_W\tilde{L}' = \tilde{L}'X_F$ and $Y_W^-\tilde{L} = \tilde{L}'X_F^-$ follow from the following commutative diagram:

$$\begin{array}{ccccccccc}
D^{1 \times \bar{n}} & \xleftarrow{\ell} & D^{1 \times n} & \xleftarrow{U} & D^{1 \times n} & \xleftarrow{Y_F} & D^{1 \times n} & \xleftarrow{\text{pr}'} & D^{1 \times \bar{n}} \\
\tilde{L} \downarrow & & \downarrow L & & \downarrow L & & \downarrow L' & & \downarrow L' \\
D^{1 \times m} & \xrightarrow{\text{id}_m} & D^{1 \times m} & \xrightarrow{\text{id}_m} & D^{1 \times m} & \xrightarrow{X_F} & D^{1 \times m} & \xrightarrow{\text{id}_m} & D^{1 \times m} \\
& & & & & \xleftarrow{X_F^-} & & &
\end{array}$$

Indeed by computing the matrix products and from (5), we have the following relations, proven in Section 5.1:

$$\begin{aligned}
\ell L &= \tilde{L}, & UL &= L, & Y_F L' &= L X_F, & \text{pr}' \tilde{L}' &= L', \\
\ell' L' &= \tilde{L}', & Y_F^- L &= L' X_F^-, & U^- L &= L, & \text{pr} \tilde{L} &= L.
\end{aligned} \quad (12)$$

3.2 Reduction of the identity bloc

In this section, we present the procedure for removing on 1 in L and L' , many 1 are removing by induction. Without lost of generality, we assume that $p \leq p'$ and $\text{sr}(D) \leq \bar{p}'$, hence (6) is fulfilled for $r = 1$. Our purpose is to show that the following $\bar{n} \times \bar{m}$ matrices are equivalent:

$$\tilde{L} := \begin{pmatrix} R & 0 \\ 0 & \text{id}_{\bar{p}'} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{L}' := \begin{pmatrix} 0 & 0 \\ \text{id}_{\bar{p}'} & 0 \\ 0 & R' \end{pmatrix}.$$

Proposition 4. We let $k := q + p' + p$ and $l := p + q'$. There exist $\mathbf{c} \in D$ and lines $\mathbf{d}, \mathbf{u} \in D^{1 \times \bar{p}'}$ such that

$$\left(\mathbf{c} e_k^n Y_F^- \begin{pmatrix} \text{id}_q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \text{id}_l \end{pmatrix} + (0 \ \mathbf{d} \ 0) \right) \begin{pmatrix} \text{id}_q & 0 & 0 & 0 \\ 0 & \text{id}_{\bar{p}'} & \mathbf{u}^T & 0 \\ 0 & 0 & 0 & \text{id}_l \end{pmatrix} Y_F (e_k^n)^T = 1, \quad (13)$$

and

$$\left(\mathbf{c} e_p^m X_F^- \begin{pmatrix} \text{id}_p & 0 \\ 0 & 0 \end{pmatrix} + (0 \ \mathbf{d}) \right) \begin{pmatrix} \text{id}_p & 0 & 0 \\ 0 & \text{id}_{\bar{p}'} & \mathbf{u}^T \end{pmatrix} X_F (e_p^m)^T = 1. \quad (14)$$

Proof. We have

$$(1 - PP'_{pp}) + \sum_{i=1}^{p'} P_{pi} P'_{ip} = 1.$$

By projecting this equality on the finitely presented left module $N := D/D(1 - PP'_{pp})$, the latter is spanned by $[P'_{1p}]_N, \dots, [P'_{p'p}]_N$. From McConnell and Robson (2001),

$\text{sr}(D)$ is in the stable range of N , so that there exists $\mathbf{u} := (u_1, \dots, u_{p'}) \in D^{1 \times \bar{p}'}$ such that N is spanned by $[P'_{1p} + u_1 P'_{p'p}]_N$. Hence, $\mathbf{c} \in D$ and $d_1, \dots, d_{\bar{p}'}$ exist such that

$$\mathbf{c} (1 - PP'_{pp}) + \sum_{k=1}^{\bar{p}'} d_k (P'_{kp} + u_k P'_{p'p}) = 1. \quad (15)$$

Letting $\mathbf{d} := (-d_1, \dots, -d_{\bar{p}'}) \in D^{1 \times \bar{p}'}$ and from the relation $\text{id}_p = ZR + PP'$, the left hand sides of (13) and (14) are both equal to the left hand side of (15), which proves Proposition 4.

With the notations of Proposition 4, we introduce the lines $\tilde{\ell}_r \in D^{1 \times \bar{n}}, \ell_r \in D^{1 \times n}, \tilde{\ell}_g \in D^{1 \times \bar{m}}$ and $\ell_g \in D^{1 \times m}$ defined as follows:

$$\begin{aligned}
\tilde{\ell}_r &:= \mathbf{c} e_k^n Y_F^- \begin{pmatrix} \text{id}_q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \text{id}_l \end{pmatrix} + (0 \ \mathbf{d} \ 0), & \ell_r &:= \tilde{\ell}_r \begin{pmatrix} \text{id}_{q+p'} & 0 & 0 \\ 0 & 0 & \text{id}_l \end{pmatrix}, \\
\tilde{\ell}_g &:= \mathbf{c} e_p^m X_F^- \begin{pmatrix} \text{id}_p & 0 \\ 0 & 0 \end{pmatrix} + (0 \ \mathbf{d}), & \ell_g &:= (\tilde{\ell}_g \ 0).
\end{aligned}$$

as well as the matrices $U_r \in D^{n \times n}, U_g \in D^{m \times m}, F_r \in D^{\bar{n} \times n}$ and $F_g \in D^{\bar{m} \times m}$ defined as follows:

$$\begin{aligned}
U_r &:= \begin{pmatrix} \text{id}_q & 0 & 0 & 0 \\ 0 & \text{id}_{\bar{p}'} & \mathbf{u}^T & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \text{id}_l \end{pmatrix}, & F_r &:= \begin{pmatrix} \text{id}_{q+p'} & 0 & 0 \\ 0 & 0 & \text{id}_l \end{pmatrix} U_r Y_F, \\
U_g &:= \begin{pmatrix} \text{id}_p & 0 & 0 \\ 0 & \text{id}_{\bar{p}'} & \mathbf{u}^T \\ 0 & 0 & 1 \end{pmatrix}, & F_g &:= (\text{id}_{\bar{m}} \ 0) U_g X_F.
\end{aligned}$$

Explicitly, we have:

$$\begin{aligned}
\tilde{\ell}_r &= (\mathbf{c} Z_p \ \mathbf{d} \ \mathbf{c} e_p^l) & \ell_r &= (\mathbf{c} Z_p \ \mathbf{d} \ 0 \ \mathbf{c} e_p^l), \\
\tilde{\ell}_g &= (\mathbf{c} (\text{id}_p - PP')_p \ \mathbf{d}) & \ell_g &= (\mathbf{c} (\text{id}_p - PP')_p \ \mathbf{d} \ 0),
\end{aligned}$$

where the index p denotes the p -th line of the considered matrix.

As we did for Relation (10), we obtain the following relations:

$$1 = \tilde{\ell}_r F_r (e_k^n)^T = \ell_r U_r Y_F (e_k^n)^T, \quad (16)$$

and

$$1 = \tilde{\ell}_g F_g (e_p^m)^T = \ell_g U_g X_F (e_p^m)^T. \quad (17)$$

Let us consider $\text{pr}_r, \text{pr}'_r \in D^{n \times \bar{n}}, \text{pr}_g, \text{pr}'_g \in D^{m \times \bar{m}}, \iota_r, \iota'_r \in D^{\bar{n} \times n}$ and $\iota_g, \iota'_g \in D^{\bar{m} \times m}$ defined as follows:

$$\text{pr}_r := \begin{pmatrix} \text{id}_{q+p'} & 0 \\ 0 & 0 \\ 0 & \text{id}_l \end{pmatrix} \left(\text{id}_{\bar{n}} - F_r (e_k^n)^T \tilde{\ell}_r \right) + (e_k^n)^T \tilde{\ell}_r,$$

$$\text{pr}'_r := \left(\text{id}_n - (e_k^n)^T \ell_r U_r Y_F \right) \begin{pmatrix} \text{id}_k & 0 \\ 0 & 0 \\ 0 & \text{id}_{q'} \end{pmatrix},$$

$$\iota_r := \left(\text{id}_{\bar{n}} - F_r (e_k^n)^T \tilde{\ell}_r \right) \begin{pmatrix} \text{id}_{q+p'} & 0 & 0 \\ 0 & 0 & \text{id}_l \end{pmatrix} + F_r (e_k^n)^T e_{q+p'}^n,$$

$$\iota'_r := \begin{pmatrix} \text{id}_k & 0 & 0 \\ 0 & 0 & \text{id}_{q'} \end{pmatrix},$$

$$\text{pr}_g := \begin{pmatrix} \text{id}_{\bar{m}} - F_g (e_p^m)^T \tilde{\ell}_g \\ \tilde{\ell}_g \end{pmatrix},$$

$$\text{pr}'_g := \left(\text{id}_m - (e_p^m)^T \ell_g U_g X_F \right) \begin{pmatrix} \text{id}_{\bar{p}} & 0 \\ 0 & 0 \\ 0 & \text{id}_{p'} \end{pmatrix},$$

$$\iota_g := \left(\text{id}_{\bar{m}} - F_g (e_p^m)^T \tilde{\ell}_g \ F_g (e_p^m)^T \right),$$

$$\iota'_g := \begin{pmatrix} \text{id}_{\bar{p}} & 0 & 0 \\ 0 & 0 & \text{id}_{p'} \end{pmatrix}.$$

By adapting the arguments of the proof of Proposition 2, we get the following.

Proposition 5. We have the following relations

- | | |
|--|---|
| (1) $\iota_r \text{pr}_r = \text{id}_{\bar{n}}$, | (5) $\ker(\cdot, \text{pr}_r) = D\ell_r$, |
| (2) $\iota_r \text{pr}'_r = \text{id}_{\bar{n}}$, | (6) $\ker(\cdot, \text{pr}'_r) = D\ell_r U_r Y_F$, |
| (3) $\iota_g \text{pr}_g = \text{id}_{\bar{m}}$, | (7) $\ker(\cdot, \text{pr}_g) = D\ell_g$, |
| (4) $\iota_g \text{pr}'_g = \text{id}_{\bar{m}}$, | (8) $\ker(\cdot, \text{pr}'_g) = D\ell_g U_g X_F$. |

Theorem 6. With the previous notations, we let

$$Y_W := \iota_r U_r Y_F \text{pr}'_r, \quad X_W := \iota_g U_g X_F \text{pr}'_g,$$

$$Y_W^- := \iota'_r Y_F^- U_r^- \text{pr}_r, \quad X_W^- := \iota'_g U_g X_F^- \text{pr}_g.$$

The following diagram is exact and commutative:

$$\begin{array}{ccccccc} D^{1 \times \bar{n}} & \xrightarrow{\tilde{L}} & D^{1 \times \bar{m}} & \xrightarrow{\pi \oplus 0} & M & \longrightarrow & 0 \\ \cdot Y_W \downarrow \uparrow \cdot Y_W^- & & \cdot X_W \downarrow \uparrow \cdot X_W^- & & f \downarrow \uparrow f^- & & \\ D^{1 \times \bar{n}} & \xrightarrow{\tilde{L}'} & D^{1 \times \bar{m}} & \xrightarrow{0 \oplus \pi'} & M' & \longrightarrow & 0 \end{array}$$

In particular, we have

$$\tilde{L}' = Y_W^- \tilde{L} X_W.$$

Proof. We only have to show that the diagram is commutative.

We show that Y_W and Y_W^- (respectively, X_W and X_W^-) are inverse to each other in the same manner that we did in the proof of Theorem 3.

Moreover, $Y_W \tilde{L}' = \tilde{L} X_W$ and $Y_W^- \tilde{L} = \tilde{L}' X_W^-$ follow from the following commutative diagram:

$$\begin{array}{ccccccc} D^{1 \times \bar{n}} & \xleftarrow{\text{pr}_r} & D^{1 \times n} & \xleftarrow{U_r} & D^{1 \times n} & \xleftarrow{Y_F} & D^{1 \times n} & \xleftarrow{\text{pr}'_r} & D^{1 \times \bar{n}} \\ \tilde{L} \downarrow & & \downarrow L & & \downarrow L & & \downarrow L' & & \downarrow \tilde{L}' \\ D^{1 \times \bar{m}} & \xleftarrow{\text{pr}_g} & D^{1 \times m} & \xleftarrow{U_g} & D^{1 \times m} & \xleftarrow{X_F} & D^{1 \times m} & \xleftarrow{\text{pr}'_g} & D^{1 \times \bar{m}} \\ & & \xleftarrow{U_g^-} & & \xleftarrow{X_F^-} & & \xleftarrow{\text{pr}'_g} & & \end{array}$$

Indeed by computing the matrix products and from (5), we have the following relations, proven in Section 5.2:

$$\begin{aligned} \iota_r L &= \tilde{L} \iota_g, & U_r L &= L U_g, & Y_F L' &= L X_F, & \text{pr}_r \tilde{L} &= L \text{pr}_g, \\ \iota'_r L' &= \tilde{L}' \iota'_g, & Y_F^- L &= L' X_F^-, & U_r^- L &= L U_g^-, & \text{pr}'_r \tilde{L}' &= L' \text{pr}'_g. \end{aligned} \quad (18)$$

4. EXAMPLE

In this Section, we illustrate Theorem 3.

4.1 Two presentations of one ODE

We consider a linear differential equation with constant coefficients:

$$y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 \dot{y}(t) + a_0 y(t) = 0, \quad (19)$$

where $a_i \in \mathbb{R}$. Letting

$$x_1 := y, \quad x_2 := \dot{x}_1 = \dot{y}, \quad \cdots, \quad x_n := \dot{x}_{n-1} = y^{(n-1)},$$

(19) rewrites as follows:

$$\begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_1 & -a_2 & \cdots & \cdots & -a_{n-1} \end{pmatrix} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}. \quad (20)$$

Hence, the system (19)–(20) is described by two matrices over the ring $D := \mathbb{R}[\partial]$ of differential polynomials with constant coefficients: $Ry = 0$ and $R'x = 0$, $R \in D^{1 \times 1} = D$ and $R' \in D^{n \times n}$ defined as follows

$$R := \partial^n + a_{n-1} \partial^{n-1} + \cdots + a_1 \partial + a_0$$

$$R' := \begin{pmatrix} \partial & -1 & 0 & \cdots & \cdots & 0 \\ 0 & \partial & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \partial & -1 \\ a_0 & a_1 & \cdots & \cdots & a_{n-1} & \partial + a_n \end{pmatrix}$$

Hence, letting $p = q = 1$ and $p' = q' = n$, we associate with this system the two modules $M := D/(DR)$ and $M' := D^{1 \times n}/(D^{1 \times n} R')$, isomorphic as follows

$$\begin{array}{ccccccc} D^{1 \times p} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \cdot Q \downarrow \uparrow \cdot Q' & & \cdot P \downarrow \uparrow \cdot P' & & f \downarrow \uparrow f^- & & \\ D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0 \end{array}$$

where $P, Q \in D^{1 \times n}$ and $P', Q' \in D^{n \times 1}$ are:

$$P := (1 \ 0 \ \dots \ 0 \ 0)$$

$$Q := \left(\partial^{n-1} + \sum_{i=1}^{n-1} a_i \partial^{(i-1)} \ \partial^{n-2} + \sum_{i=2}^{n-2} a_i \partial^{(i-1)} \ \dots \ \partial + a_{n-1} \ 1 \right)$$

$$P' := \begin{pmatrix} 1 \\ \partial \\ \vdots \\ \partial^{n-1} \end{pmatrix} \quad Q' := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Let L and L' be the matrices of (2).

4.2 Reduction of L and L'

For simplicity, we assume that $n = 3$. We have

$$Z = (0) \quad \text{and} \quad Z' = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -\partial & -1 & 0 \end{pmatrix}$$

and the matrices Z_2, R_2, Z'_2 and R'_2 are the zero matrices. The expressions of $X_F, X_F^- \in D^{4 \times 4}$ and $Y_F, Y_F^- \in D^{8 \times 8}$ come from (3) and (4).

We have $\text{sr}(D) = 2$, see McConnell and Robson (2001). From (6), we may remove two lines of 0 in L and L' . We give the details for removing the first zero lines of L and L' . With the notations of Proposition 1, we may choose

$$c = u = 0 \in D, \quad v = (0 \ 0) \in D^{1 \times 2} \quad \text{and} \quad d = (0 \ 1) \in D^{1 \times 2}.$$

We get $U = \text{id}_8$ and

$$\text{pr} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \iota = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\pi' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \iota' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

From this, we get the matrices Y_W and Y_W^- in (21). We remove the second zero lines with the matrices Y_W and Y_W^- of (22), where we use the following notations:

$$P(a, \partial) := -(\partial^3 + a_2 \partial^2 + (a_1 - 1) \partial + a_0)$$

$$Q(a, \partial) := \partial^2 + (a_2 - 1) \partial + a_1 - 1$$

$$R(a, \partial) := \partial^5 + (2a_2 - 1) \partial^4 + (a_2^2 + 2a_1 - a_2 - 1) \partial^3 \\ + ((2a_2 - 1)a_1 + a_0 - a_2 + 1) \partial^2 \\ + (a_1(a_1 - 1) + a_0(a_2 - 1)) \partial + a_0(a_1 - 1)$$

5. PROOFS OF FORMULAS (12) AND (18)

5.1 Proof of Formulas (12)

We have to show the following relations

$$\iota L = \tilde{L}, \quad (23) \quad Y_F L' = L X_F, \quad (25)$$

$$U L = L, \quad (24) \quad \text{pr}' \tilde{L}' = L', \quad (26)$$

$$\iota' L' = \tilde{L}', \quad (27) \quad U^- L = L, \quad (29)$$

$$Y_F^- L = L' X_F^-, \quad (28) \quad \text{pr} \tilde{L} = L. \quad (30)$$

The two relations (25) and (28) come from Cluzeau and Quadrat (2011), (24) and (27) are proven by direct computations, (23) and (29) are proven by direct computations using respectively the following two relations:

$$\ell L = 0 \quad \text{and} \quad U^- = \begin{pmatrix} \text{id}_{q+p'} & 0 & 0 & 0 \\ 0 & \text{id}_p & 0 & -\mathbf{u}^T \\ 0 & 0 & \text{id}_{q'} & -\mathbf{v}^T \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In order to prove (26), we first show $\text{pr}' \iota' L' = L'$: from 2 and 4 of Proposition 2, $\text{im}(\text{pr}' \iota' - \text{id}_n)$ is included $D \ell U Y_F$ and using (24), (25) and $\ell L = 0$, $\ell U Y_F L'$ and $\ell L U X_F$ are both equal to 0, which proves the desired relation. Moreover, from (27), we get $\text{pr}' \iota' L' = \text{pr}' \tilde{L}'$ which, with $\text{pr}' \iota' L' = L'$, gives (26). We show (30) in the same manner using (23).

5.2 Proof of Formulas (18)

We have to show the following relations

$$\iota_r L = \tilde{L} \iota_g, \quad (31) \quad \iota'_r L' = \tilde{L}' \iota'_g, \quad (35)$$

$$U_r L = L U_g, \quad (32) \quad Y_F^- L = L' X_F^-, \quad (36)$$

$$Y_F L' = L X_F, \quad (33) \quad U_r^- L = L U_g^-, \quad (37)$$

$$\text{pr}'_r \tilde{L}' = L' \text{pr}'_g, \quad (34) \quad \text{pr}_r \tilde{L} = L \text{pr}_g. \quad (38)$$

The two relations (33) and (36) come from Cluzeau and Quadrat (2011), (32) and (35) are proven by direct computations, (37) is proven by a direct computation using the inverse formulas for U_g and U_r which are analog to the inverse of U given in Section 5.1.

Let us show (31). For that, we decompose ι_r and ι_g in 3 parts, as follows:

$$\iota_r^1 := \begin{pmatrix} \text{id}_{q+p'} & 0 & 0 \\ 0 & 0 & \text{id}_l \end{pmatrix}, \quad \iota_g^1 := (\text{id}_{\overline{m}} \ 0),$$

$$\iota_r^2 := F_r(e_k^n)^T \tilde{\ell}_r \begin{pmatrix} \text{id}_{q+p'} & 0 & 0 \\ 0 & 0 & \text{id}_l \end{pmatrix}, \quad \iota_g^2 := F_g(e_p^m)^T \ell_g,$$

$$\iota_r^3 := F_r(e_k^n)^T e_{q+p'}^n, \quad \iota_g^3 := F_g(e_p^m)^T e_{p+p'}^m,$$

$$Y_W = \begin{pmatrix} 1 & 0 & 0 & \partial^3 + a_2\partial^2 + a_1\partial + a_0 & \partial^2 + a_2\partial + a_1 & \partial + a_2 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\partial & -1 & 0 & 0 \\ -1 & -a_0 & (-\partial^2 + (1 - a_2)\partial - a_1) & -\partial^2 & -\partial & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\partial & 1 & 0 & 0 & 0 & 0 \\ -1 & -a_0 & -(\partial^2 + a_2\partial + a_1) & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (21)$$

$$Y_W^- = \begin{pmatrix} 0 & 0 & 0 & 0 & -(\partial^3 + a_2\partial^2 + a_1\partial + a_0) & -(\partial^2 + a_2\partial + a_1) & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \partial & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \partial & -1 & 0 & 0 & 1 \\ 1 & a_0 & a_1 & \partial + a_2 & 0 & 0 & -\partial - a_2 + 1 \end{pmatrix}.$$

$$Y_W = \begin{pmatrix} 1 & 0 & \partial^3 + a_2\partial^2 + a_1\partial + a_0 & (\partial^2 + a_2\partial + a_1) & \partial + a_2 & 1 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ -1 & P(a, \partial) & -\partial & -1 & 0 & 0 \\ Q(a, \partial) & R(a, \partial) & -\partial^2 & -\partial & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -(\partial^3 + a_2\partial^2 + a_1\partial + a_0) & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (22)$$

$$Y_W^- = \begin{pmatrix} 0 & 0 & 0 & 0 & -(\partial^3 + a_2\partial^2 + a_1\partial + a_0) & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & \partial & -1 & 0 & 0 & 1 \\ 0 & 0 & \partial & -1 & 0 & -(\partial^2 + a_2\partial + a_1 - 1) \\ 1 & a_0 & a_1 & \partial + a_2 & 0 & (\partial + a_2 - 1)(\partial^2 + a_2\partial + a_1 - 1) \end{pmatrix}.$$

so that we have $\iota_r = \iota_r^1 - \iota_r^2 + \iota_r^3$ and $\iota_g = \iota_g^1 - \iota_g^2 + \iota_g^3$. By computing the matrix products, we show that $\iota_r^1 L = \tilde{L}\iota_g^1$, that the first \bar{m} columns of $\iota_r^2 L$ and $\tilde{L}\iota_g^2$ are both equal to 0 and their m -th column are both equal to

$$\begin{pmatrix} R_{.p} \\ -(P'_{ip} + u_i P'_{ip'})_{1 \leq i \leq p'} \\ 0 \\ 0 \end{pmatrix}.$$

Finally, by computing the matrix products, we show that $\iota_r^3 L$ and $\tilde{L}\iota_g^3$ are respectively equal to

$$\begin{pmatrix} R_{.p} \\ (\text{id}_{p'} - P'_{ip'})_{1 \leq i \leq p'} \\ 0 \\ 0 \end{pmatrix} (\mathbf{c}(ZR)_p, \mathbf{d} \ 0) \text{ and}$$

$$\begin{pmatrix} R_{.p} \\ (\text{id}_{p'} - P'_{ip'})_{1 \leq i \leq p'} \\ 0 \\ 0 \end{pmatrix} (\mathbf{c}(\text{id}_p - PP')_p, \mathbf{d} \ 0)$$

so that they are equal from the relation $\text{id}_p = PP' + ZR$, which proves (31).

Let us show (34). By using Relation (35), and 1 of Proposition 5, we have $\text{pr}'_r \iota'_r L' \text{pr}'_g = \text{pr}'_r \tilde{L}'$. We proceed as in the proof of (26): we only show that $\text{im}(\text{pr}'_r \iota'_r - \text{id}_n) L'$ is included in $\ker(\text{pr}'_g)$. We have $\text{im}(\text{pr}'_r \iota'_r - \text{id}_n) \subseteq D\ell_r U_r Y_F$, $\ell_r U_r Y_F L' = \ell_r L U_g X_F$ and $\ell_r L = \ell_g$, so that $\ell_r U_r Y_F L'$ is equal to $\ell_g U_g X_F$, and it belongs to $\ker(\text{pr}'_g)$. With the same arguments, we show (34).

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