



# Projections in enlargements of filtrations under Jacod's hypothesis and examples

Pavel V Gapeev, Monique Jeanblanc, Dongli Wu

## ► To cite this version:

Pavel V Gapeev, Monique Jeanblanc, Dongli Wu. Projections in enlargements of filtrations under Jacod's hypothesis and examples. 2019. hal-02120343

**HAL Id: hal-02120343**

**<https://hal.science/hal-02120343>**

Preprint submitted on 5 May 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Projections in enlargements of filtrations under Jacod's hypothesis and examples\*

Pavel V. Gapeev<sup>†</sup>      Monique Jeanblanc<sup>‡</sup>      Dongli Wu

In this paper, we consider two kinds of enlargements of a Brownian filtration  $\mathbb{F}$ : the initial enlargement with a random time  $\tau$ , denoted by  $\mathbb{F}^{(\tau)}$ , and the progressive enlargement with  $\tau$ , denoted by  $\mathbb{G}$ . We assume Jacod's equivalence hypothesis, that is, the existence of a positive  $\mathbb{F}$ -conditional density for  $\tau$ . Then, starting with the predictable representation of an  $\mathbb{F}^{(\tau)}$ -martingale  $Y(\tau)$  in terms of a standard  $\mathbb{F}^{(\tau)}$ -Brownian motion, we consider its projection on  $\mathbb{G}$ , denoted by  $Y^{\mathbb{G}}$ , and on  $\mathbb{F}$ , denoted by  $y$ . We show how to obtain the coefficients which appear in the predictable representation property for  $Y^{\mathbb{G}}$  (and  $y$ ) in terms of  $Y(\tau)$  and its predictable representation. In the last part, we give examples of conditional densities.

## 1 Introduction

In this paper, we consider two kinds of enlargement of a Brownian filtration  $\mathbb{F}$  generated by a Brownian motion  $W$ : the initial enlargement with a random time (a positive random variable)  $\tau$ , denoted by  $\mathbb{F}^{(\tau)}$ , and the progressive enlargement with  $\tau$ , denoted by  $\mathbb{G}$ . We assume Jacod's

---

\*This research benefited from the support of the 'Chaire Marchés en Mutation', French Banking Federation and ILB, Labex ANR 11-LABX-0019.

<sup>†</sup>London School of Economics, Department of Mathematics, Houghton Street, London WC2A 2AE, United Kingdom; e-mail: p.v.gapeev@lse.ac.uk

<sup>‡</sup>Laboratoire de Mathématiques et Modélisation d'Évry (LaMME), UMR CNRS 8071; Univ Evry-Université Paris Saclay, 23 Boulevard de France, 91037 Évry cedex France; e-mail: monique.jeanblanc@univ-evry.fr

*Mathematics Subject Classification 2000:* Primary 91B70, 60J60, 62M20. Secondary 91B28, 60J65, 60J25.

*Key words and phrases:* Brownian motion, conditional probability density.

*Date:* May 5, 2019

equivalence hypothesis, that is, the existence of a positive conditional density for  $\tau$  (see (2.1) below in Section 2 for a precise definition). Processes considered in the filtration  $\mathbb{F}^{(\tau)}$  will be denoted as  $Y(\tau)$  since, under Jacod's condition, any  $\mathcal{F}_t^{(\tau)}$ -measurable random variable is of the form  $Y_t(\omega, \tau(\omega))$  for some  $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$  measurable function  $(\omega, u) \rightarrow Y_t(\omega, u)$  (see [1; Proposition 4.22]), processes considered in  $\mathbb{G}$  will be indicated with the superscript  $\mathbb{G}$ , as  $Y^{\mathbb{G}}$ . Processes denoted without these symbols are  $\mathbb{F}$ -adapted as  $Y$  or  $y(u)$ .

In Section 2, we recall basic facts on enlargements of filtrations. In Section 3, we start with an  $\mathbb{F}^{(\tau)}$ -martingale  $Y(\tau)$ , which admits a decomposition

$$Y_t(\tau) = Y_0(\tau) + \int_0^t y_s(\tau) dW_s(\tau) \quad (1.1)$$

for all  $t \geq 0$ , where  $W(\tau)$  is an  $\mathbb{F}^{(\tau)}$ -Brownian motion and, for any  $u \geq 0$  the process  $y(u)$  is  $\mathbb{F}$ -predictable and for any  $t$  the map  $u \rightarrow y_t(u)$  is Borelian, and we study its  $\mathbb{G}$ -optional projection  $Y^{\mathbb{G}}$  and its  $\mathbb{F}$ -optional projection  $y$ . From the predictable representation properties [5], we obtain the existence of two  $\mathbb{G}$ -predictable processes  $\beta^{\mathbb{G}}$  and  $\gamma^{\mathbb{G}}$  such that

$$Y_t^{\mathbb{G}} = \mathbb{E}[Y_t(\tau) | \mathcal{G}_t] = Y_0^{\mathbb{G}} + \int_0^t \beta_s^{\mathbb{G}} dW_s^{\mathbb{G}} + \int_0^t \gamma_s^{\mathbb{G}} dM_s^{\mathbb{G}} \quad (1.2)$$

for all  $t \geq 0$ , where  $M^{\mathbb{G}}$  is the compensated  $\mathbb{G}$ -martingale of the default process  $\mathbb{1}_{\{\tau \leq t\}}$ , and the existence of an  $\mathbb{F}$ -predictable process  $\sigma$  such that

$$Y_t = \mathbb{E}[Y_t(\tau) | \mathcal{F}_t] = y_0 + \int_0^t \sigma_s dW_s.$$

We show how to compute  $\beta^{\mathbb{G}}, \gamma^{\mathbb{G}}$  and  $\sigma$  in terms of  $Y(\tau)$  and  $y(\tau)$ . We apply these computations to compare equivalent martingale measures for a financial market with price process following an  $\mathbb{F}$ -geometric Brownian motion considered in the two enlarged filtrations in Section 4. In the last section, we give a family of example of conditional densities.

In the whole paper, "positive" means "strictly positive".

## 2 Enlargement of filtrations

We recall that, if  $\mathbb{H} \subset \mathbb{K}$ , the  $\mathbb{H}$ -optional projection of a  $\mathbb{K}$ -martingale  $\mu$  is the  $\mathbb{H}$ -optional process  $\nu$  such that  $\mathbb{E}[\mu_{\vartheta} \mathbb{1}_{\{\vartheta < \infty\}} | \mathcal{H}_{\vartheta}] = \nu_{\vartheta} \mathbb{1}_{\{\vartheta < \infty\}}$  for any  $\mathbb{H}$ -stopping time  $\vartheta$ . This optional projection satisfies  $\mathbb{E}(\mu_t | \mathcal{H}_t) = \nu_t$ . By abuse of language, we shall call  $\mathbb{E}(\mu_t | \mathcal{H}_t)$  the  $\mathbb{H}$ -optional projection of  $\mu$ .

We consider a probability space  $(\Omega, \mathcal{G}, \mathbb{P})$  endowed with a standard Brownian motion  $W = (W_t)_{t \geq 0}$  and a positive random variable  $\tau$  with the positive density  $g$  on  $\mathbb{R}^+$  under  $\mathbb{P}$ . We denote by  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  the natural right-continuous and completed filtration of  $W$  and  $\mathbb{G}$  is the progressive enlargement of  $\mathbb{F}$  with  $\tau$ . We assume Jacod's equivalence hypothesis (see [1; Chapter IV; Section 5]), that is, the existence of a positive function  $(\omega, t, u) \rightarrow p_t(\omega, u)$  such that

- (i): the map  $u \mapsto p_t(u; \omega)$  is Borelian,  $(dt \times d\mathbb{P})$  a.s.
- (ii): for each  $u \geq 0$ , the process  $p(u) = (p_t(u))_{t \geq 0}$  is a continuous  $\mathbb{F}$ -martingale,
- (iii): for each  $t \geq 0$ , for every bounded Borel function  $f$ , we have

$$\mathbb{E}[f(\tau) \mid \mathcal{F}_t] = \int_0^\infty f(u) p_t(u) g(u) du. \quad (2.1)$$

Note that  $\int_0^\infty p_t(u) g(u) du = 1$ , for all  $t \geq 0$ , and  $p_0(u) = 1$ , for each  $u \geq 0$ . Moreover, it follows from the predictable representation theorem in a Brownian filtration that, for each  $u \geq 0$ , there exists an  $\mathbb{F}$ -predictable process  $\varphi(u) = (\varphi_t(u))_{t \geq 0}$  such that the positive martingale  $p(u)$  admits the representation

$$p_t(u) = p_0(u) + \int_0^t p_s(u) \varphi_s(u) dW_s \quad (2.2)$$

so that it takes the form of the Doléans-Dade stochastic exponential

$$p_t(u) = p_0(u) \exp \left( \int_0^t \varphi_s(u) dW_s - \frac{1}{2} \int_0^t \varphi_s^2(u) ds \right) \quad (2.3)$$

for all  $t \geq 0$ . It can be shown that  $u \mapsto \varphi_t(u)$  is a Borel function on  $[0, \infty)$ , for all  $t \geq 0$ .

We introduce the supermartingale  $G$  defined as  $G_t := \mathbb{P}(\tau > t \mid \mathcal{F}_t)$  which can be written in terms of the conditional density as

$$G_t = \int_t^\infty p_t(u) g(u) du, \quad t \geq 0. \quad (2.4)$$

Note that  $G$  is a positive continuous  $\mathbb{F}$ -supermartingale called the Azéma supermartingale of  $\tau$  which admits a Doob-Meyer decomposition  $G_t = N_t - A_t$ , where  $N = (N_t)_{t \geq 0}$  is an  $\mathbb{F}$ -martingale under  $\mathbb{P}$  and  $A = (A_t)_{t \geq 0}$  is a nondecreasing  $\mathbb{F}$ -predictable process. Then, by means of Jacod's equivalence hypothesis, we have

$$A_t = \int_0^t p_u(u) g(u) du \quad \text{and} \quad N_t = 1 - \int_0^t (p_t(u) - p_u(u)) g(u) du \quad (2.5)$$

for all  $t \geq 0$  (see [7; Subsection 4.2.1]). Moreover, by means of Itô-Wentzell's lemma (see, e.g. Kunita [19], Wentzell [22], or [14; Theorem 1.5.3.2]), we have

$$dN_t = - \int_0^t d_t p_t(u) g(u) du = - \left( \int_0^t \varphi_t(u) p_t(u) g(u) du \right) dW_t \quad (2.6)$$

where  $d_t$  means that one makes use of the stochastic differential, and the associated predictable covariation processes have the form

$$d\langle W, N \rangle_t = - \left( \int_0^t \varphi_t(u) p_t(u) g(u) du \right) dt \quad \text{and} \quad d\langle W, p(u) \rangle_t = p_t(u) \varphi_t(u) dt \quad (2.7)$$

for all  $t, u \geq 0$ .

It follows from Jacod's theorem for initial enlargements of filtrations ([13; Corollaire 1.11] or [1; Chapter IV, Proposition 4.40]) that the process  $W(\tau) = (W_t(\tau))_{t \geq 0}$  defined by

$$W_t(\tau) := W_t - \int_0^t \frac{d\langle p(u), W \rangle_s|_{u=\tau}}{p_s(\tau)} = W_t - \int_0^t \varphi_s(\tau) ds \quad (2.8)$$

for all  $t \geq 0$ , is a  $\mathbb{P}$ -Brownian motion with respect to the initially enlarged filtration  $\mathbb{F}^{(\tau)}$  defined by  $\mathbb{F}^{(\tau)} = (\mathcal{F}_t \vee \sigma(\tau))_{t \geq 0}$ . Note that, according to the results of Fontana [9], the filtration  $\mathbb{F}^{(\tau)}$  is right-continuous.

It follows from the results of Grorud and Pontier [12] and also Amendinger [2], that the process  $(1/p_t(\tau))_{t \geq 0}$  forms an  $\mathbb{F}^{(\tau)}$ -martingale under  $\mathbb{P}$ , and thus, since  $p_0(\tau) = 1$  holds, we have  $\mathbb{E}[1/p_t(\tau)] = 1$ , for all  $t \geq 0$ . Therefore, we can define a probability measure  $\mathbb{P}^*$  on  $\mathcal{F}_t^{(\tau)}$  by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} \Big|_{\mathcal{F}_t^{(\tau)}} = \frac{1}{p_t(\tau)} = \exp \left( - \int_0^t \varphi_s(\tau) dW_s(\tau) - \frac{1}{2} \int_0^t \varphi_s^2(\tau) ds \right) \quad (2.9)$$

for all  $t \geq 0$ . It is proved in [12] (see also [1; Chapter IV, Proposition 4.37]) that the probability measure  $\mathbb{P}^*$  defined in (2.9) coincides with  $\mathbb{P}$  on  $\mathbb{F}$  and on  $\sigma(\tau)$ , and the random time  $\tau$  is independent of  $\mathbb{F}$  under  $\mathbb{P}^*$ . This fact particularly implies that  $\mathbb{P}^*(\tau > u | \mathcal{F}_t) = \mathbb{P}^*(\tau > u) = \mathbb{P}(\tau > u)$  holds, for all  $t, u \geq 0$ , as well as that the process  $W$  is a  $(\mathbb{P}, \mathbb{F})$  standard Brownian motion with respect to  $\mathbb{F}$  under  $\mathbb{P}^*$ . Note that, by Girsanov's theorem, following the arguments of Callegaro et al. [5], we can recover that the process  $W(\tau)$  from (2.8) is a  $(\mathbb{P}, \mathbb{F}^{(\tau)})$ -Brownian motion.

Let us now introduce the progressively enlarged filtration  $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$  which is the smallest right-continuous filtration containing  $\mathbb{F}$  and making  $\tau$  a stopping time.

Note that, according to the hypothesis that the positive random variable  $\tau$  has a positive density with support  $\mathbb{R}^+$ , the  $\sigma$ -algebra  $\mathcal{G}_0$  is trivial. It is known (see Jeulin [17; Chapter III] or Callegaro et al. [5; Chapter I, Section 2]) that any  $\mathbb{G}$ -predictable process  $Z^\mathbb{G}$  can be written as

$$Z_t^\mathbb{G} = Z_t^0 \mathbb{1}_{\{t \leq \tau\}} + Z_t^1(\tau) \mathbb{1}_{\{\tau < t\}} \quad (2.10)$$

for all  $t \geq 0$ , where the process  $Z^0$  is  $\mathbb{F}$ -predictable and, for any  $u \geq 0$ , the process  $Z^1(u)$  is  $\mathbb{F}$ -predictable. In that case, using the key lemma for the computation of  $Z^0$  (see, e.g. [1; Chapter II, Lemma 2.9]), we get

$$Z_t^1(u) = Z_t(u), \text{ for } t \geq u, \quad \text{and} \quad Z_t^0 = \frac{1}{G_t} \int_t^\infty Z_t(u) p_t(u) g(u) du. \quad (2.11)$$

We shall say that  $Z^0$  is the  $\mathbb{F}$ -predictable reduction of  $Z^\mathbb{G}$ .

The decomposition (2.10), proved for any progressive enlargement in Jeulin does not extend in general to optional processes (see Barlow [3] for a counter example). However, under Jacod's equivalence hypothesis, any  $\mathbb{G}$ -optional process  $U^\mathbb{G}$  can be written as

$$U_t^\mathbb{G} = U_t^0 \mathbb{1}_{\{t < \tau\}} + U_t^1(\tau) \mathbb{1}_{\{\tau \leq t\}} \quad (2.12)$$

for all  $t \geq 0$ , where  $U^0$  is  $\mathbb{F}$ -optional and, for each  $u \geq 0$ ,  $U^1(u)$  is  $\mathbb{F}$ -optional (see Song [21]). In particular, we shall use that result for the  $\mathbb{G}$ -optional projection of an integrable  $\mathbb{F}^{(\tau)}$ -optional process  $U(\tau)$ , i.e., for

$$U_t^\mathbb{G} = \mathbb{E}[U_t(\tau) \mid \mathcal{G}_t] \quad (2.13)$$

In that case, similarly to (2.11), we have

$$U_t^1(u) = U_t(u), \text{ for } t \geq u, \quad \text{and} \quad U_t^0 = \frac{1}{G_t} \int_t^\infty U_t(u) p_t(u) g(u) du \quad (2.14)$$

for all  $t \geq 0$ . The process  $U^0$  is called the optional reduction of  $U$ . Note that, working in a Brownian filtration under Jacod's equivalence hypothesis,  $U^0$  and  $U^1(u)$  being continuous are also  $\mathbb{F}$ -predictable.

Then, it follows from the results of Jeanblanc and Le Cam [16] that the process  $W^\mathbb{G} = (W_t^\mathbb{G})_{t \geq 0}$  defined by

$$\begin{aligned} W_t^\mathbb{G} &:= W_t - \int_0^{t \wedge \tau} \frac{d\langle W, N \rangle_s}{G_s} - \int_{t \wedge \tau}^t \frac{d\langle W, p(u) \rangle_s|_{u=\tau}}{p_s(\tau)} \\ &= W_t + \int_0^{t \wedge \tau} \frac{1}{G_s} \left( \int_0^s \varphi_s(u) p_s(u) g(u) du \right) ds - \int_{t \wedge \tau}^t \varphi_s(\tau) ds \\ &= W_t + \int_0^t \alpha_s^\mathbb{G} ds \end{aligned} \quad (2.15)$$

$$= W_t + \int_0^t \alpha_s^\mathbb{G} ds \quad (2.16)$$

is a  $(\mathbb{P}, \mathbb{G})$ -standard Brownian motion, where  $\alpha^\mathbb{G}$  is the  $\mathbb{G}$ -predictable process with the decomposition

$$\alpha_t^\mathbb{G} = \mathbb{1}_{\{t \leq \tau\}} \frac{1}{G_t} \int_0^t \varphi_t(u) p_t(u) g(u) du \mathbb{1}_{\{\tau < t\}} \varphi_t(u) := \mathbb{1}_{\{t \leq \tau\}} \alpha_t^0 + \mathbb{1}_{\{\tau < t\}} \alpha_t^1(\tau) \quad (2.17)$$

for all  $t \geq 0$ . Observe from (2.8) and (2.15) that

$$W_t(\tau) = W_t^\mathbb{G} - \int_0^{t \wedge \tau} (\alpha_s^0 + \varphi_s(\tau)) ds \quad (2.18)$$

for all  $t \geq 0$ .

We define the  $\mathbb{G}$ -martingale

$$M_t^\mathbb{G} := \mathbb{1}_{\{\tau \leq t\}} - \int_0^t \lambda_s^\mathbb{G} ds = \mathbb{1}_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \lambda_s^0 ds \quad (2.19)$$

for all  $t \geq 0$ , where  $\lambda_s^\mathbb{G} = \mathbb{1}_{\{s \leq \tau\}} \lambda_s^0$  and  $\lambda_s^0 = p_s(s)g(s)/G_s$  (see, e.g. [5; Subsection 1.2] or [1; Chapter II]).

We recall that the process  $W(\tau)$  enjoys the  $\mathbb{F}^\tau$ -predictable representation property and that the pair  $(W^\mathbb{G}, M^\mathbb{G})$  enjoys the  $\mathbb{G}$ -predictable representation property (see, e.g. [5; Proposition 4.3]):

Any  $\mathbb{F}^{(\tau)}$ -martingale  $Y(\tau)$  has the form

$$Y_t(\tau) = Y_0(\tau) + \int_0^t y_s(\tau) dW_s(\tau) \quad (2.20)$$

for all  $t \geq 0$ , where  $Y_0$  is a (deterministic) function and  $y(u)$ ,  $u \geq 0$ , is a family of  $\mathbb{F}$ -predictable processes. In particular, any  $(\mathbb{P}, \mathbb{F}^{(\tau)})$ -martingale is continuous, and if  $Y(\tau)$  is square integrable on  $[0, T]$ , then  $\mathbb{E}[\int_0^T (y_s(\tau))^2 ds] < \infty$ .

Any  $\mathbb{G}$ -martingale admits the representation

$$Y_t^\mathbb{G} = Y_0^\mathbb{G} + \int_0^t \beta_s^\mathbb{G} dW_s^\mathbb{G} + \int_0^t \gamma_s^\mathbb{G} dM_s^\mathbb{G} \quad (2.21)$$

where  $\beta^\mathbb{G}$  and  $\gamma^\mathbb{G}$  are  $\mathbb{G}$ -predictable processes.

### 3 Martingales and projections

#### 3.1 Projections of $\mathbb{F}^{(\tau)}$ martingales on $\mathbb{G}$

**Proposition 3.1** *Let  $Y(\tau)$  be an  $\mathbb{F}^{(\tau)}$ -martingale with representation (2.20). Its  $\mathbb{G}$ -optional projection  $Y^\mathbb{G}$  admits the following representation*

$$Y_t^\mathbb{G} = \mathbb{E}[Y_t(\tau) | \mathcal{G}_t] = Y_0^\mathbb{G} + \int_0^t \beta_s^\mathbb{G} dW_s^\mathbb{G} + \int_0^t \gamma_s^\mathbb{G} dM_s^\mathbb{G} \quad (3.1)$$

where  $Y_0^{\mathbb{G}} = \mathbb{E}[Y_0(\tau)]$ , and the  $\mathbb{G}$ -predictable processes  $\beta^{\mathbb{G}}$  and  $\gamma^{\mathbb{G}}$  admit, from (2.10), the decomposition

$$\beta_t^{\mathbb{G}} = \beta_t^0 \mathbb{1}_{\{t \leq \tau\}} + \beta_t^1(\tau) \mathbb{1}_{\{\tau < t\}} \quad (3.2)$$

$$\gamma_t^{\mathbb{G}} = \gamma_t^0 \mathbb{1}_{\{t \leq \tau\}} + \gamma_t^1(\tau) \mathbb{1}_{\{\tau < t\}}. \quad (3.3)$$

Then,

$$\beta_t^0 = \frac{1}{G_t} \int_t^\infty (y_t(u) + (\alpha_t^0 + \varphi_t(u)) Y_t(u)) p_t(u) g(u) du, \quad \beta_t^1(u) = y_t(u), \quad \text{for } t \geq u \quad (3.4)$$

$$\gamma_t^0 = Y_t(t) - Y_t^0, \quad (3.5)$$

for all  $t \geq 0$ , and  $Y^0$  is given by (2.11). Note that for any choice of  $\gamma^1$ , the property  $\int_0^t \gamma_s^{\mathbb{G}} dM_s^{\mathbb{G}} = \int_0^t \gamma_s^0 dM_s^{\mathbb{G}}$  holds.

Proof: In a first step, we assume that  $Y(\tau)$  (hence  $Y^{\mathbb{G}}$ ), are square integrable on  $[0, T]$ . We start to determine  $\beta^{\mathbb{G}}$  (which is square integrable). For this purpose, we observe that for any bounded  $\mathbb{G}$ -predictable process  $n^{\mathbb{G}}$ , using the tower property and the fact that  $W^{\mathbb{G}}$  is a  $\mathbb{G}$ -martingale orthogonal to  $M^{\mathbb{G}}$ , we have

$$\begin{aligned} \mathbb{E} \left[ Y_t(\tau) \int_0^t n_s^{\mathbb{G}} dW_s^{\mathbb{G}} \right] &= \mathbb{E} \left[ Y_t^{\mathbb{G}} \int_0^t n_s^{\mathbb{G}} dW_s^{\mathbb{G}} \right] \\ &= \mathbb{E} \left[ \int_0^t \beta_s^{\mathbb{G}} dW_s^{\mathbb{G}} \int_0^t n_s^{\mathbb{G}} dW_s^{\mathbb{G}} \right] = \mathbb{E} \left[ \int_0^t \beta_s^{\mathbb{G}} n_s^{\mathbb{G}} ds \right] \end{aligned} \quad (3.6)$$

for all  $t \geq 0$ . Recall that, from (2.18), we have

$$W_t^{\mathbb{G}} = W_t(\tau) + \int_0^{t \wedge \tau} (\alpha_s^0 + \varphi_s(\tau)) ds \quad (3.7)$$

for all  $t \geq 0$ . We introduce the continuous  $\mathbb{G}$ -martingale

$$V_t^{\mathbb{G}} = \int_0^t n_s^{\mathbb{G}} dW_s^{\mathbb{G}} = \int_0^t n_s^{\mathbb{G}} dW_s(\tau) + \int_0^t (\alpha_s^0 + \varphi_s(\tau)) \mathbb{1}_{\{s \leq \tau\}} n_s^{\mathbb{G}} ds \quad (3.8)$$

for all  $t \geq 0$ . By means of the integration by parts, we have

$$\mathbb{E} [Y_t(\tau) V_t^{\mathbb{G}}] = \mathbb{E} \left[ + \int_0^t Y_s(\tau) dV_s^{\mathbb{G}} + \langle Y(\tau), V^{\mathbb{G}} \rangle_t^{\mathbb{F}(\tau)} \int_0^t V_s^{\mathbb{G}} dY_s(\tau) \right] \quad (3.9)$$

for all  $t \geq 0$ .

---

A martingale  $X$  is square integrable if  $\sup_{t \leq T} \mathbb{E}(X_t^2) < \infty$ .



We recall that a local martingale  $M$  satisfying  $\mathbb{E}[(\langle M \rangle_T)^{1/2}] < \infty$  is a martingale. Let us prove that the  $\mathbb{F}^{(\tau)}$ -local martingale  $M_t = \int_0^t V_s^\mathbb{G} dY_s(\tau)$  is a martingale. One has

$$\begin{aligned} \mathbb{E}[(\langle M \rangle_T)^{1/2}] &= \mathbb{E} \left[ \left( \int_0^T (V_s^\mathbb{G})^2 (y_s(\tau))^2 ds \right)^{1/2} \right] \leq \mathbb{E} \left[ \sup_{s \leq T} |V_s^\mathbb{G}| \left( \int_0^T (y_s(\tau))^2 ds \right)^{1/2} \right] \\ &\leq \mathbb{E}[\sup_{s \leq T} |V_s^\mathbb{G}|^2] + \mathbb{E} \left[ \int_0^T y_s(\tau)^2 ds \right]. \end{aligned}$$

From Burkholder's inequality {footnote Burkholder's inequality states that for any  $p \geq 1$ ,  $\mathbb{E}[\sup_{t \leq T} |M_t|^p] \leq C_p \mathbb{E}[\langle M \rangle_T^{p/2}]$  where  $C_p$  is a constant depending only on  $p$ . the quantity  $\mathbb{E}[\sup_{s \leq T} |V_s^\mathbb{G}|^2]$  is smaller than  $C_2 \mathbb{E}[\int_0^T (n_s^\mathbb{G})^2 ds]$  which is bounded. Furthermore,  $\mathbb{E}[\int_0^T y_s(\tau)^2 ds]$  is bounded. Hence,  $\int_0^\cdot V_s^\mathbb{G} dY_s(\tau)$  is a martingale and its expectation is null. The boundeness of  $n^\mathbb{G}$  and the square integrability of  $Y(\tau)$  imply that  $\int_0^t Y_s(\tau) n_s^\mathbb{G} dW_s(\tau)$  is a martingale, so that

$$\mathbb{E} \left[ \int_0^t V_s^\mathbb{G} dY_s(\tau) + \int_0^t Y_s(\tau) dV_s^\mathbb{G} \right] = \mathbb{E} \left[ \int_0^t Y_s(\tau) (\alpha_s^0 + \varphi_s(\tau)) \mathbb{1}_{\{s \leq \tau\}} n_s^\mathbb{G} ds \right]. \quad (3.10)$$

Recall that  $\langle Y(\tau), V_t^\mathbb{G} \rangle_t^{\mathbb{F}^{(\tau)}} = \int_0^t y_s(\tau) n_s^\mathbb{G} ds$ , for all  $t \geq 0$ . Then, we have

$$\mathbb{E}[Y_t(\tau) V_t^\mathbb{G}] = \mathbb{E} \left[ \int_0^t \left( Y_s(\tau) (\alpha_s^0 + \varphi_s(\tau)) \mathbb{1}_{\{s \leq \tau\}} + y_s(\tau) \right) n_s^\mathbb{G} ds \right] \quad (3.11)$$

hence, using the tower property

$$\mathbb{E} \left[ \int_0^t \beta_s^\mathbb{G} n_s^\mathbb{G} ds \right] = \mathbb{E} \left[ \int_0^t \mathbb{E}[y_s(\tau) + Y_s(\tau) (\alpha_s^0 + \varphi_s(\tau)) \mathbb{1}_{\{s \leq \tau\}} | \mathcal{G}_s] n_s^\mathbb{G} ds \right] \quad (3.12)$$

for all  $t \geq 0$  and for any bounded  $n^\mathbb{G}$ . It follows that

$$\beta_t^\mathbb{G} = \mathbb{E}[y_t(\tau) + Y_t(\tau) (\alpha_t^0 + \varphi_t(\tau)) \mathbb{1}_{\{t \leq \tau\}} | \mathcal{G}_t], \quad (3.13)$$

and the  $\mathbb{F}$ -predictable reduction of  $\beta^\mathbb{G}$  is

$$\beta_t^0 = \frac{1}{G_t} \int_t^\infty \left( y_t(u) + (\alpha_t^0 + \varphi_t(u)) Y_t(u) \right) p_t(u) g(u) du \quad (3.14)$$

and on  $\{t > \tau\}$ , one has, as expected, that  $\beta_t^\mathbb{G} = \mathbb{E}[y_t(\tau) | \mathcal{G}_t] = y_t(\tau)$  so that  $\beta_t^1(u) = y_t(u)$ .

In the second step, we determine  $\gamma^\mathbb{G}$ . For this purpose, on the one hand, for any bounded  $\mathbb{G}$ -predictable process  $n^\mathbb{G}$ , using the fact that  $M^\mathbb{G}$  is a  $\mathbb{G}$ -martingale orthogonal to  $W^\mathbb{G}$  and

that  $M^\mathbb{G}$  is flat after  $\tau$  ( $M_t^\mathbb{G} = M_{t \wedge \tau}$ ), and that the equality  $d\langle M^\mathbb{G} \rangle_t = \lambda_t^0 \mathbb{1}_{\{t \leq \tau\}} dt$  holds, we have

$$\begin{aligned} \mathbb{E} \left[ Y_t(\tau) \int_0^t n_s^\mathbb{G} dM_s^\mathbb{G} \right] &= \mathbb{E} \left[ Y_t(\tau) \int_0^t n_s^0 dM_s^\mathbb{G} \right] = \mathbb{E} \left[ Y_t^\mathbb{G} \int_0^t n_s^0 dM_s^\mathbb{G} \right] \\ &= \mathbb{E} \left[ \int_0^t \gamma_s^0 dM_s^\mathbb{G} \int_0^t n_s^0 dM_s^\mathbb{G} \right] = \mathbb{E} \left[ \int_0^t \gamma_s^0 \lambda_s^0 \mathbb{1}_{\{s < \tau\}} n_s^0 ds \right] = \mathbb{E} \left[ \int_0^t \gamma_s^0 \lambda_s^0 G_s n_s^0 ds \right] \end{aligned} \quad (3.15)$$

for all  $t \geq 0$ , where  $n^0$  is the  $\mathbb{F}$ -predictable reduction of  $n^\mathbb{G}$  and where the last equality comes from the tower property.

Recall that  $M^\mathbb{G}$  is a predictable bounded variation process in the initially enlarged filtration. Let  $U_t^\mathbb{G} = \int_0^t n_s^0 dM_s^\mathbb{G}$ . By integration by parts, using the fact that  $Y(\tau)$  is continuous, its bracket (in  $\mathbb{F}^\mathbb{G}$  with the  $\mathbb{F}^{(\tau)}$ -predictable bounded variation process  $U^\mathbb{G}$  is null

$$\mathbb{E}[Y_t(\tau)U_t^\mathbb{G}] = \mathbb{E} \left[ \int_0^t Y_s(\tau) n_s^0 dM_s^\mathbb{G} + \int_0^t U_s^\mathbb{G} dY_s(\tau) \right].$$

The same methodology than in the first part for  $\int V^\mathbb{G} dY_s(\tau)$  proves that the process  $\int_0^\cdot U_s^\mathbb{G} dY_s(\tau)$  is a martingale. Furthermore, setting  $H_t = \mathbb{1}_{\{t \leq \tau\}}$ , we have

$$\begin{aligned} \mathbb{E} \left[ \int_0^t Y_s(\tau) n_s^0 dM_s^\mathbb{G} \right] &= \mathbb{E} \left[ \int_0^t Y_s(\tau) n_s^0 dH_s - \int_0^t Y_s(\tau) n_s^0 \lambda_s^0 \mathbb{1}_{\{s < \tau\}} ds \right] \\ &= \mathbb{E} \left[ \mathbb{1}_{\{\tau \leq t\}} Y_\tau(\tau) n_\tau^0 - \int_0^t \mathbb{E}[Y_s(\tau) \mathcal{G}_s] n_s^0 \lambda_s^0 \mathbb{1}_{\{s < \tau\}} ds \right] \\ &= \mathbb{E} \left[ \int_0^t Y_s() n_u^0 p_t(s) g(s) ds - \int_0^t Y_s^0 n_s^0 \lambda_s^0 \mathbb{1}_{\{s < \tau\}} ds \right] \\ &= \mathbb{E} \left[ \int_0^t [Y_s(s) p_s(s) - Y_u^0 \lambda_u^0 G_u] n_u^0 du \right] \end{aligned}$$

where in the last equality, we used the fact that  $p(u)$  is an  $\mathbb{F}$ -martingale. To conclude, we have

$$\mathbb{E} \left[ \int_0^t \gamma_s^0 \lambda_s^0 G_s n_s^0 ds \right] = \mathbb{E} \left[ \int_0^t (Y_s(s) p_s(s) g(s) - \lambda_s^0 G_s Y_s^0) n_s^0 ds \right] \quad (3.16)$$

and using the fact that  $\lambda_s^0 G_s = p_s(s) g(s)$ , one has, for any  $\mathbb{F}$  adapted bounded process  $n^0$

$$\mathbb{E} \left[ \int_0^t (\gamma_s^0 - Y_s(s) + Y_s^0) \lambda_s^0 G_s n_s^0 ds \right] = 0 \quad (3.17)$$

for all  $t \geq 0$ , so that the expression in (3.5) holds. The result obtained for square integrable martingales  $Y(\tau)$  extends to all martingales.  $\square$

### 3.2 Projection of $\mathbb{F}^{(\tau)}$ -martingales on $\mathbb{F}$

**Proposition 3.2** *Let  $Y(\tau)$  be an  $\mathbb{F}^{(\tau)}$ -martingale of the form  $dY_t(\tau) = y_t(\tau)dW_t(\tau)$ . Its  $\mathbb{F}$ -optional projection is  $Y_t = \mathbb{E}[Y_t(\tau) | \mathcal{F}_t] = Y_0 + \int_0^t \sigma_s dW_s$ , where the  $\mathbb{F}$ -predictable process  $\sigma$  is given by*

$$\sigma_t = \int_0^\infty (y_t(u) + Y_t(u) \varphi_t(u)) p_t(u) g(u) du \quad (3.18)$$

for all  $t \geq 0$ .

Proof: It follows from the predictable representation property in the filtration  $\mathbb{F}$  that there exists an  $\mathbb{F}$ -predictable process  $\sigma$  such that

$$\mathbb{E}[Y_t(\tau) | \mathcal{F}_t] = Y_0 + \int_0^t \sigma_s dW_s \quad (3.19)$$

holds for all  $t \geq 0$ . On the one hand, for any bounded  $\mathbb{F}$ -adapted process  $n$ , we have

$$\mathbb{E}\left[Y_t(\tau) \int_0^t n_s dW_s\right] = \mathbb{E}\left[Y_t \int_0^t n_s dW_s\right] = \mathbb{E}\left[\int_0^t \sigma_s n_s ds\right] \quad (3.20)$$

for all  $t \geq 0$ . On the other hand, using (2.8) and an integration by parts on the left-hand side lead to, assuming that  $Y(\tau)$  is square integrable

$$\begin{aligned} \mathbb{E}\left[Y_t(\tau) \int_0^t n_s dW_s\right] &= \mathbb{E}\left[Y_t(\tau) \left(\int_0^t n_s dW_s(\tau) + \int_0^t n_s \varphi_s(\tau) ds\right)\right] \\ &= \mathbb{E}\left[\int_0^t (y_s(\tau) + Y_s(\tau) \varphi_s(\tau)) n_s ds\right] = \mathbb{E}\left[\int_0^t \mathbb{E}[y_s(\tau) + Y_s(\tau) \varphi_s(\tau) | \mathcal{F}_s] n_s ds\right] \end{aligned} \quad (3.21)$$

for all  $t \geq 0$ . Hence, we have

$$\sigma_t = \mathbb{E}[y_t(\tau) + Y_t(\tau) \varphi_t(\tau) | \mathcal{F}_t] = \int_0^\infty (y_t(u) + Y_t(u) \varphi_t(u)) p_t(u) g(u) du \quad (3.22)$$

for all  $t \geq 0$ . □

### 3.3 Projection of $\mathbb{G}$ -martingales on $\mathbb{F}$

**Proposition 3.3** *Consider the  $\mathbb{G}$ -martingale  $Y^\mathbb{G} = (Y_t^\mathbb{G})_{t \geq 0}$  defined by*

$$Y_t^\mathbb{G} = Y_0^\mathbb{G} + \int_0^t \beta_s^\mathbb{G} dW_s^\mathbb{G} + \int_0^t \gamma_s^\mathbb{G} dM_s^\mathbb{G} = Y_t^0 \mathbb{1}_{\{t < \tau\}} + Y_t^1(\tau) \mathbb{1}_{\{\tau \leq t\}}. \quad (3.23)$$

Then, its  $\mathbb{F}$ -optional projection is  $Y_t = \mathbb{E}[Y_t^{\mathbb{G}} | \mathcal{F}_t] = Y_0 + \int_0^t \eta_s dW_s$ , where the  $\mathbb{F}$ -predictable process  $\eta$  satisfies

$$\eta_t = \mathbb{E}[\beta_t^{\mathbb{G}} - Y_t^{\mathbb{G}} \alpha_t^{\mathbb{G}} | \mathcal{F}_t] = (\beta_t^0 - Y_t^0 \alpha_t^0) G_t + \int_0^t (\beta_t^1(u) - Y_t^1(u) \alpha_t^1(u)) p_t(u) g(u) du \quad (3.24)$$

for all  $t \geq 0$ .

Proof: We proceed as before and consider, for a bounded  $\mathbb{F}$ -adapted process  $n$ , the quantity  $\mathbb{E} \left[ Y_t^{\mathbb{G}} \int_0^t n_s dW_s \right]$ , for all  $t \geq 0$ . On the one hand, using the same procedure as in the analysis of (3.12), we get

$$\mathbb{E} \left[ Y_t^{\mathbb{G}} \int_0^t n_s dW_s \right] = \mathbb{E} \left[ \int_0^t \eta_s n_s ds \right] \quad (3.25)$$

for all  $t \geq 0$ . On the other hand, setting  $U_t = \int_0^t n_s dW_s$ , by means of integration by parts and (2.15), we obtain, using the same methodology as before

$$\begin{aligned} \mathbb{E} \left[ Y_t^{\mathbb{G}} \int_0^t n_s dW_s \right] &= \mathbb{E} [Y_t^{\mathbb{G}} U_t] = \mathbb{E} \left[ \int_0^t U_s dY_s^{\mathbb{G}} + \int_0^t Y_s^{\mathbb{G}} n_s dW_s + \int_0^t \beta_s^{\mathbb{G}} n_s ds \right] \\ &= \mathbb{E} \left[ \int_0^t (\beta_s^{\mathbb{G}} - Y_s^{\mathbb{G}} \alpha_s^{\mathbb{G}}) n_s ds \right] = \mathbb{E} \left[ \int_0^t \mathbb{E} [\beta_s^{\mathbb{G}} - Y_s^{\mathbb{G}} \alpha_s^{\mathbb{G}} | \mathcal{F}_s] n_s ds \right] \end{aligned} \quad (3.26)$$

for all  $t \geq 0$ . □

**Remark 3.4** Since  $\mathbb{E}[Y_t(\tau) | \mathcal{F}_t] = \mathbb{E}[\mathbb{E}[Y_t(\tau) | \mathcal{G}_t] | \mathcal{F}_t]$  we have  $\sigma = \eta$  where these quantities are defined in (3.18) and (3.24). Indeed, from (3.13),

$$\begin{aligned} \mathbb{E}[\beta_t^{\mathbb{G}} - \alpha_t^{\mathbb{G}} Y_t^{\mathbb{G}} | \mathcal{F}_t] &= \mathbb{E}[y_t(\tau) + Y_t(\tau)(\alpha_t^0 + \varphi_t(\tau)) \mathbb{1}_{\{t \leq \tau\}} - \alpha_t^0 Y_t^0 \mathbb{1}_{\{t \leq \tau\}} - \alpha^1(\tau) Y_t^1(\tau) \mathbb{1}_{\{\tau < t\}} | \mathcal{F}_t] \\ &= \mathbb{E}[y_t(\tau) + Y_t(\tau) \varphi_t(\tau) | \mathcal{F}_t]. \end{aligned} \quad (3.27)$$

## 3.4 Change of probability measures

### 3.4.1 $\mathbb{F}^{(\tau)}$ versus $\mathbb{G}$

Note that, from PRP, and positive  $\mathbb{F}^{(\tau)}$ -martingale has the form

$$L_t(\tau) = L_0(\tau) \mathcal{E}(\zeta(\tau) \cdot W(\tau))_t = L_0(\tau) \exp \left( \int_0^t \zeta_s(\tau) dW_s(\tau) - \frac{1}{2} \int_0^t (\zeta_s(\tau))^2 ds \right), \quad t \geq 0 \quad (3.28)$$

where  $L_0$  is a positive function and  $\zeta(\tau)$  is  $\mathbb{F}^{(\tau)}$ -predictable, or in a closed form

$$L_t(\tau) = L_0(\tau) \exp \left( \int_0^t \zeta_s(\tau) dW_s(\tau) - \frac{1}{2} \int_0^t (\zeta_s(\tau))^2 ds \right)$$

and any positive  $\mathbb{G}$ -martingale has the form

$$C\mathcal{E}(\mu^{\mathbb{G}} \cdot W^{\mathbb{G}})_t \mathcal{E}(\psi^{\mathbb{G}} \cdot M^{\mathbb{G}})_t, \quad (3.29)$$

where  $C$  is a positive constant and  $\mu^{\mathbb{G}}$  and  $\psi^{\mathbb{G}}$  are  $\mathbb{G}$  predictable and  $\psi^{\mathbb{G}} > -1$ . We recall that

$$\mathcal{E}(\psi^{\mathbb{G}} \cdot M^{\mathbb{G}})_t = \exp \left( \int_0^{t \wedge \tau} \psi_s^{\mathbb{G}} \lambda_s^0 ds \right) (1 + \psi_{\tau}^{\mathbb{G}})^{H_t}.$$

**Proposition 3.5** *Let  $L(\tau) = (L_t(\tau))_{t \geq 0}$  be of the form (3.28). Then, its  $\mathbb{G}$ -optional projection  $L^{\mathbb{G}}$  satisfies*

$$L_t^{\mathbb{G}} = \mathbb{E}[L_t(\tau) | \mathcal{G}_t] = \mathbb{E}[L_0(\tau)] + \int_0^t L_{s-}^{\mathbb{G}} \mu_s^{\mathbb{G}} dW_s^{\mathbb{G}} + \int_0^t L_{s-}^{\mathbb{G}} \psi_s^{\mathbb{G}} dM_s^{\mathbb{G}} \quad (3.30)$$

where the  $\mathbb{G}$ -predictable processes  $\mu^{\mathbb{G}}$  and  $\psi^{\mathbb{G}}$  are given by

$$\mu_t^{\mathbb{G}} = \mathbb{1}_{\{t \leq \tau\}} \frac{1}{L_t^0 G_t} \int_t^{\infty} L_t(u) (\zeta_t(u) + \varphi_t(u) + \alpha_t^0) p_t(u) g(u) du + \mathbb{1}_{\{\tau < t\}} \zeta_t(\tau) \quad (3.31)$$

$$\psi_t^{\mathbb{G}} = \frac{L_t(t)}{L_t^0} - 1 \quad (3.32)$$

where  $L^0$  is the optional reduction of  $L^{\mathbb{G}}$ .

Proof: This is an application of Proposition 3.2.  $\square$

Note that  $\psi_t^{\mathbb{G}} > -1$ , as it must be. If  $\mathbb{E}[L_0(\tau)] = 1$ , and  $L(\tau)$  is a positive  $(\mathbb{P}, \mathbb{F}^{(\tau)})$ -martingale with initial value  $\ell(\tau)$ , one can associate to it the change of probability measure defined by  $d\tilde{\mathbb{P}} = L_t(\tau) d\mathbb{P}$  on  $\mathcal{F}_t^{(\tau)}$ , for any  $t \geq 0$ . The choice  $L_0 \equiv 1$  is equivalent to  $\tilde{\mathbb{P}}(\tau > u) = \mathbb{P}(\tau > u)$ , for each  $u \geq 0$ . Indeed, since  $\tau$  is  $\mathcal{F}_0 \vee \sigma(\tau)$ -measurable, we have, using tower property  $\tilde{\mathbb{P}}(\tau > u) = \mathbb{E}[\ell(\tau) \mathbb{1}_{\{\tau > u\}}]$  and the equality  $\mathbb{E}[L_0(\tau) \mathbb{1}_{\{\tau > u\}}] = \mathbb{E}[\mathbb{1}_{\{\tau > u\}}]$  holds, for each  $u \geq 0$ , which implies that  $L_0 \equiv 1$ .

In particular, in the case  $L_t(\tau) = p_0(\tau)/p_t(\tau)$ , which satisfies  $dL_t(\tau) = -L_t(\tau)\varphi_t(\tau)dW_t(\tau)$ , one obtains (recall that  $P^*$  is defined in (2.9))  $d\mathbb{P}^*|_{\mathcal{G}_t} = L_t^{\mathbb{G},*} d\mathbb{P}|_{\mathcal{G}_t}$  with

$$L_t^{\mathbb{G},*} = \mathbb{E}\left(\frac{p_0(\tau)}{p_t(\tau)} | \mathcal{G}_t\right) = \mathcal{E}(\mu^{\mathbb{G},*} \cdot W^{\mathbb{G}})_t \mathcal{E}(\psi^{\mathbb{G},*} \cdot M^{\mathbb{G}})_t \quad (3.33)$$

with

$$\mu_t^{\mathbb{G},*} = \mathbb{1}_{\{t \leq \tau\}} \alpha_t^0 - \mathbb{1}_{\{\tau < t\}} \varphi_t(\tau) \quad \text{and} \quad \psi_t^{\mathbb{G},*} = \mathbb{1}_{\{t \leq \tau\}} \left( \frac{G_t}{p_t(t)(1 - F(t))} - 1 \right) \quad (3.34)$$

where  $F(t) = \int_0^t g(u)du$ .

From the definition of  $L^{\mathbb{G},*}$  one has

$$L_t^{\mathbb{G},*} = \mathbb{1}_{\{t \leq \tau\}} \frac{1 - F(t)}{G_t} + \mathbb{1}_{\{\tau < t\}} L_t(\tau) \quad (3.35)$$

for all  $t \geq 0$ .

### 3.4.2 $\mathbb{G}$ versus $\mathbb{F}$

From PRP, any positive  $(\mathbb{P}, \mathbb{G})$ -martingale  $L^{\mathbb{G}} = (L_t^{\mathbb{G}})_{t \geq 0}$  can be written as

$$L_t^{\mathbb{G}} = L_0^{\mathbb{G}} + \int_0^t L_s^{\mathbb{G}} \mu_s^{\mathbb{G}} dW_s^{\mathbb{G}} + \int_0^t L_{s-}^{\mathbb{G}} \psi_s^{\mathbb{G}} dM_s^{\mathbb{G}} = L_t^0 \mathbb{1}_{\{t < \tau\}} + L_t^1(\tau) \mathbb{1}_{\{\tau \leq t\}}. \quad (3.36)$$

Let  $L = (L_t)_{t \geq 0}$  be its  $\mathbb{F}$ -optional projection  $L_t = \mathbb{E}[L_t^{\mathbb{G}} | \mathcal{F}_t]$ , for all  $t \geq 0$ . Then, we have  $L_t = L_0 + \int_0^t L_s \eta_s dW_s$ , where  $L_0 = \mathbb{E}[L_0(\tau)]$  and  $\eta$  is the  $\mathbb{F}$ -predictable process

$$\eta_t = \frac{L_t^0}{L_t} (\mu_t^0 - \alpha_t^0) G_t + \int_0^t L_t^1(u) (\mu_t^1(u) - \alpha_t^1(u)) p_t(u) g(u) du \quad (3.37)$$

for all  $t \geq 0$ .

### 3.4.3 $\mathbb{F}^{(\tau)}$ versus $\mathbb{F}$

Let  $L(\tau)$  be a positive  $\mathbb{F}^{(\tau)}$ -martingale of the form (3.28), then its  $\mathbb{F}$ -optional projection is  $L_t = \mathbb{E}[L_t(\tau) | \mathcal{F}_t] = L_0 + \int_0^t L_s \sigma_s dW_s$  where  $L_0 = \mathbb{E}[L_0(\tau)]$  and  $\sigma$  is the  $\mathbb{F}$ -predictable process

$$\sigma_t = \frac{1}{\ell_t} \int_0^\infty L_t(u) (\zeta_t(u) + \varphi_t(u)) p_t(u) g(u) du \quad (3.38)$$

for all  $t \geq 0$ .

## 3.5 Stability of the Brownian property

As we mentioned before, due to the fact that the processes  $W^{\mathbb{G}}$  and  $M^{\mathbb{G}}$  enjoy the predictable representation property in  $\mathbb{G}$ , any locally equivalent probability measure  $\mathbb{Q}$  which is equivalent to  $\mathbb{P}$  on  $\mathcal{G}_t$  is given by

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{G}_t} = \mathcal{E}(\mu^{\mathbb{G}} \cdot W^{\mathbb{G}})_t \mathcal{E}(\psi^{\mathbb{G}} \cdot M^{\mathbb{G}})_t \quad (3.39)$$

for all  $t \geq 0$ , where  $\mu^{\mathbb{G}}$  and  $\psi^{\mathbb{G}}$  are  $\mathbb{G}$  predictable and  $\psi^{\mathbb{G}} > -1$ . Note that the change of probability measure does not affect the intensity if and only if  $\psi^{\mathbb{G}} = 0$ .

For a probability measure  $\mathbb{Q}$  which is locally equivalent to  $\mathbb{P}$  on  $\mathbb{G}$ , let us denote by  $\mathbb{Q}^{\mathbb{F}}$  its restriction to  $\mathbb{F}$ , which is locally equivalent to  $\mathbb{P}$  on  $\mathbb{F}$ , with its density being of the form  $d\mathbb{Q}^{\mathbb{F}}/d\mathbb{P} = \mathcal{E}(\eta \cdot W)_t$ , for all  $t \geq 0$ . Under  $\mathbb{Q}^{\mathbb{F}}$ , the process  $W^{\mathbb{Q},\mathbb{F}}$  defined by  $W_t^{\mathbb{Q},\mathbb{F}} = W_t - \int_0^t \eta_s ds$  is a  $(\mathbb{Q}^{\mathbb{F}}, \mathbb{F})$ -standard Brownian motion. Let us provide a set of probability measures  $\mathbb{Q}$  which are equivalent to  $\mathbb{P}$  on  $\mathbb{G}$  such that the  $(\mathbb{Q}^{\mathbb{F}}, \mathbb{F})$ -standard Brownian motion  $W^{\mathbb{Q},\mathbb{F}}$  is a  $(\mathbb{Q}, \mathbb{G})$ -standard Brownian motion.

**Proposition 3.6** *Let  $L^{\mathbb{G}}$  be a positive  $(\mathbb{P}, \mathbb{G})$ -martingale of the form  $\mathcal{E}(\mu^{\mathbb{G}} \cdot W^{\mathbb{G}})$ , and define the probability measure  $\mathbb{Q}$  by*

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{G}_t} = L_t^{\mathbb{G}} \quad (3.40)$$

*for all  $t \geq 0$ . The process  $W^{\mathbb{Q},\mathbb{F}}$  is a  $(\mathbb{Q}, \mathbb{G})$ -standard Brownian motion if and only if  $\mu^{\mathbb{G}} - \alpha^{\mathbb{G}}$  is  $\mathbb{F}$ -adapted. Then, we have  $\eta_t = \mu_t^{\mathbb{G}} - \alpha_t^{\mathbb{G}}$ , for all  $t \geq 0$ .*

Proof: On the one hand, we have seen in (2.15) that  $W_t^{\mathbb{G}} = W_t + \int_0^t \alpha_s^{\mathbb{G}} ds$ , for all  $t \geq 0$ . Hence, we have  $W_t^{\mathbb{G}} = W_t^{\mathbb{Q},\mathbb{F}} + \int_0^t \sigma_s ds + \int_0^t \alpha_s^{\mathbb{G}} ds$ , for all  $t \geq 0$ . On the other hand, we see that  $W_t^{\mathbb{G}} - \int_0^t \mu_s^{\mathbb{G}} ds$  is a  $(\mathbb{Q}, \mathbb{G})$ -standard Brownian motion. It follows that  $W_t^{\mathbb{Q},\mathbb{F}} + \int_0^t \sigma_s ds + \int_0^t \alpha_s^{\mathbb{G}} ds - \int_0^t \mu_s^{\mathbb{G}} ds$  is a  $\mathbb{G}$ -standard Brownian motion. Therefore,  $W^{\mathbb{Q},\mathbb{F}}$  is a  $(\mathbb{Q}, \mathbb{G})$ -standard Brownian motion if and only if  $\sigma_t + \alpha_t^{\mathbb{G}} - \mu_t^{\mathbb{G}} = 0$ , for all  $t \geq 0$ . Note that indeed, if  $\alpha_t^{\mathbb{G}} - \mu_t^{\mathbb{G}}$  is  $\mathbb{F}$  adapted, then, from (3.24)  $L_t \sigma_t = \mathbb{E}(L_t^{\mathbb{G}}(\alpha_t^{\mathbb{G}} - \mu_t^{\mathbb{G}}) | \mathcal{F}_t) = L_t(\alpha_t^{\mathbb{G}} - \mu_t^{\mathbb{G}})$  where  $L_t = \mathbb{E}(L_t^{\mathbb{G}} | \mathcal{F}_t)$ .  $\square$

**Examples 3.7** (i) If the process  $\mu^{\mathbb{G}}$  is such that

$$\mu_t^0 = 0 \quad (3.41)$$

then  $\mathbb{Q}$  is equal to  $\mathbb{P}$  on  $\mathcal{G}_\tau$  and the immersion holds for  $\mathbb{F}$  and  $\mathbb{G}$  under  $\mathbb{Q}$ . The choice  $\mu_t^0 = 0$  leads to  $\eta = \alpha^0$ , hence  $\mu_t^1(u) = \alpha_t^1(u) - \alpha_t^0 = -\varphi_t(u) - \alpha_t^0$ , for all  $t \geq u$ . Therefore, we have

$$\begin{aligned} & \mathcal{E}(\mu^1 \cdot W^{\mathbb{G}})_t \\ &= \mathbb{1}_{\{t < \tau\}} + \mathbb{1}_{\{\tau \leq t\}} \exp \left( \int_\tau^t \alpha_s^1(\tau) dW_s^{\mathbb{G}} - \frac{1}{2} \int_\tau^t (\alpha_s^1)^2 ds \right) \exp \left( - \int_\tau^t \alpha_s^0 dW_s^{\mathbb{G}} - \frac{1}{2} \int_\tau^t (\alpha_s^0)^2 ds \right) \\ &= \mathbb{1}_{\{t < \tau\}} + \mathbb{1}_{\{\tau \leq t\}} \frac{p_\tau(\tau)}{p_t(\tau)} \frac{Z_t}{Z_\tau} \end{aligned} \quad (3.42)$$

where we have used the fact that, on  $\{\tau < t\}$ , one has  $p_\tau(\tau)/p_t(\tau) = \exp\left(\int_\tau^t \alpha_s^1 dW_s + \frac{1}{2} \int_\tau^t (\alpha_s^1)^2 ds\right)$  and  $Z_t = \mathcal{E}(-\alpha^0 \cdot W)_t = G_t e^{\Lambda_t}$ , for all  $t \geq 0$ . More precisely, we have

$$L_t^{\mathbb{G}} = \mathbb{1}_{\{t < \tau\}} + \mathbb{1}_{\{\tau \leq t\}} \frac{p_\tau(\tau) e^{\Lambda_t} G_t}{p_t(\tau) e^{\Lambda_\tau} G_\tau} \quad (3.43)$$

and

$$\eta_t = \frac{G_t}{1 - F(t)} \quad (3.44)$$

This example was presented in [1; Chapter V].

(ii) the case  $\mu^0 = \alpha^0$  leads to  $\eta = 0$  and  $\mu^1 = \alpha^1$ , that is, this is the case in which  $\mathbb{Q} = \mathbb{P}^*$ .

## 4 Equivalent Martingale Measures

consider the case where  $\nu$  and  $\sigma$  are processes Let us consider a model of a financial market in which the stock price process  $S = (S_t)_{t \geq 0}$  started at  $S_0 = 1$  satisfies the stochastic differential equations (written in various filtrations)

$$dS_t = S_t (\nu dt + \sigma dW_t) = S_t (\nu_t^{\mathbb{G}} dt + \sigma dW_t^{\mathbb{G}}) = S_t ((\nu + \sigma \varphi_t(\tau)) dt + \sigma dW_t(\tau)) \quad (4.1)$$

for some  $\sigma > 0$  fixed, where, from (2.15), we have

$$\nu_t^{\mathbb{G}} = \nu + \sigma \alpha_t^{\mathbb{G}} = \nu + \sigma \alpha_t^0 \mathbb{1}_{\{t \leq \tau\}} - \varphi_t(\tau) \mathbb{1}_{\{\tau < t\}}. \quad (4.2)$$

We assume that a riskless asset is traded with a null interest rate. It is straightforward to show that, for positive function  $\ell$  satisfying  $\mathbb{E}[\ell(\tau)] = 1$ , the positive martingale

$$Y_t(\tau) = \ell(\tau) \exp \left( \int_0^t y_s(\tau) dW_s(\tau) - \frac{1}{2} \int_0^t y_s^2(\tau) ds \right) \quad (4.3)$$

is a Radon-Nikodým density of an equivalent martingale measure on  $\mathbb{F}^{(\tau)}$  (i.e. such that  $SY(\tau)$  is an  $\mathbb{F}^{(\tau)}$ -martingale) if and only if

$$y_t(\tau) = -\varphi_t(\tau) - \frac{\nu}{\sigma} \quad (4.4)$$

for all  $t \geq 0$ . We denote by  $L(\tau) = (L_t(\tau))_{t \geq 0}$  such a positive martingale defined by

$$L_t(\tau) = L_0(\tau) \exp \left( - \int_0^t \left( \varphi_s(\tau) + \frac{\nu}{\sigma} \right) dW_s(\tau) - \frac{1}{2} \int_0^t \left( \varphi_s(\tau) + \frac{\nu}{\sigma} \right)^2 ds \right) \quad (4.5)$$



for all  $t \geq 0$ . Then, we define a new probability measure  $\tilde{\mathbb{P}}$  which is locally equivalent to  $\mathbb{P}$  on  $\mathbb{F}^{(\tau)}$  by

$$\left. \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{F}_t^{(\tau)}} = L_t(\tau) \quad (4.6)$$

for all  $t \geq 0$ . Actually, there exists infinitely many such probabilities, which differ from each other by the choice of the initial value  $L_0$ , that is, by the choice of the law of  $\tau$  (under  $\tilde{\mathbb{P}}$ ):  $\tilde{\mathbb{P}}(\tau > u) = \mathbb{E} [L_0(\tau) \mathbb{1}_{\{\tau > u\}}] = \int_u^\infty L_0(v) g(v) dv$ , for all  $u \geq 0$ . The case  $L_0 \equiv 1$  corresponds to the situation in which  $\tilde{\mathbb{P}}(\tau > u) = \mathbb{P}(\tau > u)$ , for any  $u \geq 0$ . Note that, by virtue of Girsanov's theorem, the process  $\tilde{W}(\tau)$  defined as

$$\tilde{W}_t(\tau) = W_t(\tau) + \int_0^t \left( \varphi_s(\tau) + \frac{\nu}{\sigma} \right) ds \quad (4.7)$$

is a  $(\tilde{\mathbb{P}}, \mathbb{F}^{(\tau)})$ -standard Brownian motion.

Working in the filtration  $\mathbb{G}$ , it easily seen that the set of  $\mathbb{G}$ -equivalent martingale measures (i.e. the set of positive  $\mathbb{G}$ -martingales  $Y^\mathbb{G}$  such that  $SY^\mathbb{G}$  is a  $\mathbb{G}$ -martingale) is

$$dY_t^\mathbb{G} = Y_{t-}^\mathbb{G} \left( -\frac{\nu_t^\mathbb{G}}{\sigma} dW_t^\mathbb{G} + \gamma_t^\mathbb{G} dM_t^\mathbb{G} \right) \quad (4.8)$$

so that

$$Y_t^\mathbb{G} = \mathcal{E}(-(\nu^\mathbb{G}/\sigma) \cdot W^\mathbb{G})_t \mathcal{E}(\gamma^\mathbb{G} \cdot M^\mathbb{G})_t \quad (4.9)$$

where  $\gamma$  is any  $\mathbb{G}$ -predictable process, with  $\gamma > -1$ .

Let us now consider  $L^\mathbb{G}$ , the  $\mathbb{G}$ -optional projection of  $L_t(\tau)$ . The process  $L^\mathbb{G}$  defines an equivalent martingale measure for  $S$  and can be written as in Section 3.4

$$L_t^\mathbb{G} = 1 + \int_0^t L_{s-}^\mathbb{G} \mu_s^\mathbb{G} dW_s^\mathbb{G} + \int_0^t L_{s-}^\mathbb{G} \psi_s^\mathbb{G} dM_s^\mathbb{G} = \mathcal{E}(\mu^\mathbb{G} \cdot W^\mathbb{G})_t \mathcal{E}(\psi^\mathbb{G} \cdot M^\mathbb{G})_t \quad (4.10)$$

where the processes  $\mu^\mathbb{G}$  and  $\psi^\mathbb{G}$  are given in (3.31). Using the fact that  $L_t^0 = \frac{1}{G_t} \int_0^t L_t(u) p_t(u) g(u) du$ , one has

$$\mu_t^\mathbb{G} = \mathbb{1}_{\{t \leq \tau\}} \frac{1}{G_t L_t^0} \left( \int_0^t \left( -\frac{\nu}{\sigma} - \alpha_t^0 \right) L_t(u) p_t(u) g(u) du + \mathbb{1}_{\{\tau < t\}} \varphi_t(\tau) \right) \quad (4.11)$$

$$\begin{aligned} &= -\mathbb{1}_{\{t \leq \tau\}} \left( \frac{\nu_t^\mathbb{G}}{\sigma} + \alpha_t^0 \right) + \mathbb{1}_{\{\tau < t\}} \varphi_t(\tau) = -\mathbb{1}_{\{t \leq \tau\}} \frac{\nu_t^\mathbb{G}}{\sigma} + \mathbb{1}_{\{\tau < t\}} \varphi_t(\tau) \\ \psi_t^\mathbb{G} &= \mathbb{1}_{\{t \leq \tau\}} \left( \frac{L_t(t)}{L_t^0} - 1 \right). \end{aligned} \quad (4.12)$$

We obtain here an equivalent martingale measures.

However, the set of all equivalent martingale measures is larger, since it is the family  $\mathcal{E}(\mu^{\mathbb{G}} \cdot W^{\mathbb{G}}) \mathcal{E}(\gamma^{\mathbb{G}} \cdot M^{\mathbb{G}})$ , for any  $\mathbb{G}$ -predictable process  $\gamma^{\mathbb{G}} = (\gamma_t^{\mathbb{G}})_{t \geq 0}$  such that  $\gamma_t^{\mathbb{G}} > -1$ , for all  $t \geq 0$ . The compensated  $(\mathbb{P}, \mathbb{G})$ -martingale of  $H = (H_t)_{t \geq 0}$  with  $H_t = \mathbb{1}_{\{\tau \leq t\}}$ , for all  $t \geq 0$  is the (uniformly integrable)  $(\mathbb{P}, \mathbb{G})$ -martingale

$$M_t^{\mathbb{G}} = H_t - \int_0^{\tau \wedge t} \frac{p_s(s)g(s)}{G_s} ds = H_t - \int_0^{t \wedge \tau} \lambda_s^0 ds \quad (4.13)$$

(see, e.g., [5; Subsection 1.2] or [1]). We introduce  $L^{\mathbb{G}}$  as

$$L_t^{\mathbb{G}} = \mathcal{E}(\mu^{\mathbb{G}} \cdot W^{\mathbb{G}}) \mathcal{E}(\gamma^{\mathbb{G}} \cdot M^{\mathbb{G}})$$

and the measure  $\widehat{\mathbb{P}}$  on  $\mathbb{G}$  with

$$\left. \frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{G}_t} = L_t^{\mathbb{G}} \quad (4.14)$$

By virtue of Girsanov's theorem, we have that the process  $\widehat{W}$  defines as

$$\widehat{W}_t^{\mathbb{G}} = W_t^{\mathbb{G}} - \int_0^t \mu_s^{\mathbb{G}} ds \quad (4.15)$$

is a standard Brownian motion under  $\widehat{\mathbb{P}}$  with respect to  $\mathbb{G}$ . Moreover, the process

$$\widehat{M}_t^{\mathbb{G}} = M_t^{\mathbb{G}} - \int_0^{t \wedge \tau} \gamma_s^0 \lambda_s^0 ds = H_t - \int_0^{t \wedge \tau} (1 + \gamma_s^0) \lambda_s^0 ds \quad (4.16)$$

is a (uniformly integrable)  $(\widehat{\mathbb{P}}, \mathbb{G})$ -martingale. The above change of probability changes the Brownian motion and the intensity of the default. The specific choice  $\gamma = 0$  leads to a change of probability measure which does not affect the intensity.

**Remark 4.1** Assume for simplicity that  $S$  is a martingale under  $\mathbb{P}$ . By definition  $S$  is also a  $\mathbb{Q}^{\mathbb{F}}$  martingale, hence  $W = W^{\mathbb{G}}$ . Then, for any  $\zeta_T \in \mathcal{F}_T$  and any equivalent martingale measure  $\mathbb{Q}^{\mathbb{G}}$  on  $\mathbb{G}$ , one has  $\mathbb{E}_{\mathbb{Q}^{\mathbb{G}}}[\zeta_T | \mathcal{G}_t] = \mathbb{E}_{\mathbb{P}}[\zeta_T | \mathcal{F}_t]$ , for  $0 \leq t \leq T$ . The first reason is that  $\zeta_T$  is hedgeable in  $\mathbb{F}$ , and thus in  $\mathbb{G}$ . The second reason is that from  $\zeta_T = x + \int_0^T x_s dW_s + \int_0^T x_s dW_s^{\mathbb{G}}$  and using the orthogonality of  $W^{\mathbb{G}}$  and  $M^{\mathbb{G}}$ , setting  $\zeta_t = x + \int_0^t x_s dW_s^{\mathbb{G}}$ , the product  $\mathcal{E}(\gamma \cdot M^{\mathbb{G}})_t \zeta_t$  is a martingale, hence, by Bayes' formula

$$\mathbb{E}_{\mathbb{Q}^{\mathbb{G}}}[\zeta_T | \mathcal{G}_t] = \frac{1}{\mathcal{E}(\gamma \cdot M^{\mathbb{G}})_t} \mathbb{E}_{\mathbb{P}}[\mathcal{E}(\gamma \cdot M^{\mathbb{G}})_T \zeta_T | \mathcal{G}_t] = \zeta_t \quad (4.17)$$

for  $0 \leq t \leq T$ .

**Remark 4.2** We can extend easily the study to the case where the interest rate is  $\mathbb{G}$ -adapted with  $r_t(\tau) = r_t^0 \mathbb{1}_{\{t < \tau\}} + r_t^1(\tau) \mathbb{1}_{\{\tau \leq t\}}$ , where the processes  $r^0 = (r_t^0)_{t \geq 0}$  and  $r^1(u) = (r_t^1(u))_{t \geq 0}$  are  $\mathbb{F}$ -adapted, for each  $u \geq 0$  fixed, and we define the discounted process  $\tilde{S}(\tau) = (\tilde{S}_t(\tau))_{t \geq 0}$  by

$$\tilde{S}_t(\tau) = \exp \left( - \int_0^t r_s(\tau) ds \right) S_t \quad (4.18)$$

for all  $t \geq 0$ .

## 5 Examples of conditional density

In the literature, there are some explicit examples of random times satisfying Jacod's Hypothesis, among them the Gaussian ones given in Crépey et al. [4] and the one in [10]. Here, we shall work in the same framework as in [10], with the difference that we do not make use of a change of probability measure, and we are working under the given probability  $\mathbb{P}$ .

Let  $W = (W_t)_{t \geq 0}$  be a standard Brownian motion with its natural filtration  $(\mathcal{F}_t)_{t \geq 0}$ . Let  $F$  be the cumulative distribution function defined by  $F(t) = \int_0^t g(s) ds$ , with the given probability density function  $g$ . For some constants  $\mu$  and  $\sigma > 0$ , we define  $G = (G_t)_{t \geq 0}$  as the unique strong solution of the stochastic differential equation

$$dG_t = -G_t \frac{g(t)}{1 - F(t)} dt - \frac{\mu}{\sigma} G_t (1 - G_t) dW_t \quad (G_0 = 1). \quad (5.1)$$

for all  $t \geq 0$ . It is easily seen from the expressions in (5.1) that  $G$  is a supermartingale valued in  $[0, 1]$ . We also define the process  $X = (X_t)_{t \geq 0}$  by

$$X_t = \int_0^t \mu (1 - G_s) ds + \int_0^t \sigma dW_s \quad (5.2)$$

so that, we have  $\mathcal{F}_t^X \subseteq \mathcal{F}_t$  with  $\mathcal{F}_t^X = \sigma(X_s | 0 \leq s \leq t)$ , for all  $t \geq 0$ .

Let us now provide the multiplicative decomposition for the supermartingale  $G$ . For this purpose, we define the process  $Z = (Z_t)_{t \geq 0}$  by:

$$Z_t = \exp \left( \frac{\mu}{\sigma^2} X_t - \frac{\mu^2}{2\sigma^2} t \right) \quad (5.3)$$

and the process  $Y = (Y_t)_{t \geq 0}$  by

$$Y_t = \int_0^t \frac{g(s)}{Z_s} ds + \frac{1 - F(t)}{Z_t} \quad (5.4)$$

for all  $t \geq 0$ . Then, it is shown by means of standard arguments based on Itô's formula that the process  $Z$  solves the stochastic differential equation

$$dZ_t = Z_t \frac{\mu}{\sigma^2} dX_t = \left(\frac{\mu}{\sigma}\right)^2 (1 - G_t) Z_t dt + \frac{\mu}{\sigma} Z_t dW_t \quad (Z_0 = 1) \quad (5.5)$$

so that the process  $1/Z$  takes the expression

$$d\left(\frac{1}{Z_t}\right) = -\frac{1}{Z_t} \frac{\mu}{\sigma^2} dX_t + \frac{1}{Z_t} \left(\frac{\mu}{\sigma}\right)^2 dt \quad \left(\frac{1}{Z_0} = 1\right) \quad (5.6)$$

and thus, the process  $Y$  admits the representation

$$dY_t = \left(\frac{\mu}{\sigma}\right)^2 \frac{(1 - F(t))G_t}{Z_t} dt - \frac{\mu}{\sigma} \frac{(1 - F(t))}{Z_t} dW_t \quad (Y_0 = 1). \quad (5.7)$$

Observe from the expression in (5.1) that  $G_t = N_t - A_t$  with  $dN_t = -(\mu/\sigma)G_t(1 - G_t)dW_t$  and that  $p_t(t) = G_t/(1 - F(t))$ , so that the intensity of  $\tau$  is a deterministic function  $\lambda(t) = p_t(t)g(t)/G_t = g(t)/(1 - F(t))$ , for all  $t \geq 0$ .

Let us now consider the stochastic differential

$$d(G_t Z_t Y_t) = Y_t d(G_t Z_t) + (G_t Z_t) dY_t + d\langle GZ, Y \rangle_t \quad (5.8)$$

for which we first compute, setting  $\lambda(t) = g(t)/(1 - F(t))$ ,

$$\begin{aligned} d(G_t Z_t) &= Z_t dG_t + G_t dZ_t + d\langle G, Z \rangle_t = Z_t \left( -\lambda(t) G_t dt - \frac{\mu}{\sigma} G_t(1 - G_t) dW_t \right) \\ &\quad + G_t \left( \left(\frac{\mu}{\sigma}\right)^2 (1 - G_t) Z_t dt + \frac{\mu}{\sigma} Z_t dW_t \right) - \left(\frac{\mu}{\sigma}\right)^2 G_t(1 - G_t) Z_t dt \\ &= -\lambda(t) G_t Z_t dt + \frac{\mu}{\sigma} G_t Z_t [1 - 1 + G_t] dW_t \\ &= -\lambda(t) G_t Z_t dt + \frac{\mu}{\sigma} G_t^2 Z_t dW_t. \end{aligned} \quad (5.9)$$

Then, we can proceed with computing the expression (5.8) and obtain

$$\begin{aligned} d(G_t Z_t Y_t) &= Y_t \left( -\lambda(t) G_t Z_t dt + \frac{\mu}{\sigma} G_t^2 Z_t dW_t \right) \\ &\quad + G_t Z_t \left( \left(\frac{\mu}{\sigma}\right)^2 \frac{(1 - F(t))G_t}{Z_t} dt - \frac{\mu}{\sigma} \frac{(1 - F(t))}{Z_t} dW_t \right) - \left(\frac{\mu}{\sigma}\right)^2 (1 - F(t)) G_t^2 dt \\ &= -\lambda(t) G_t Z_t Y_t dt + \frac{\mu}{\sigma} G_t [G_t Z_t Y_t - (1 - F(t))] dW_t \end{aligned} \quad (5.10)$$

so that, by using the fact that  $d(1 - F(t)) = -g(t)dt$ , we get

$$\begin{aligned} d[G_t Z_t Y_t - (1 - F(t))] &= \left( -\lambda(t) G_t Y_t Z_t + g(t) \right) dt + \frac{\mu}{\sigma} G_t [G_t Z_t Y_t - (1 - F(t))] dW_t \\ &= -\frac{g(t)}{1 - F(t)} [G_t Z_t Y_t - (1 - F(t))] dt + \frac{\mu}{\sigma} G_t [G_t Z_t Y_t - (1 - F(t))] dW_t \end{aligned} \quad (5.11)$$

from where it is seen that the process  $V_t := G_t Z_t Y_t - (1 - F(t))$  is a solution of the stochastic differential equation  $dV_t = V_t(-\lambda(t) dt + (\mu/\sigma) G_t dW_t)$ , started at  $G_0 Z_0 Y_0 - (1 - F(0)) = 0$ , and thus, we have  $G_t Z_t Y_t - (1 - F(t)) = 0$ , for all  $t \geq 0$ .

Furthermore, by applying Itô's formula we have

$$dY_t = -\frac{\mu}{\sigma^2} \frac{1 - F(t)}{Z_t} dX_t + \left(\frac{\mu}{\sigma}\right)^2 \frac{1 - F(t)}{Z_t} dt \quad (5.12)$$

and thus

$$d\left(\frac{1}{Y_t}\right) = \frac{\mu}{\sigma} \frac{1}{Y_t^2} \frac{1 - F(t)}{Z_t} dW_t = \frac{\mu}{\sigma} \frac{G_t}{Y_t} dW_t \quad (5.13)$$

and

$$d\left(\frac{1}{Z_t Y_t}\right) = \frac{1}{Z_t} d\left(\frac{1}{Y_t}\right) + \frac{1}{Y_t} d\left(\frac{1}{Z_t}\right) + d\left\langle \frac{1}{Z}, \frac{1}{Y} \right\rangle_t = -\frac{\mu}{\sigma} \frac{1 - G_t}{Z_t Y_t} dW_t \quad (5.14)$$

and hence, the processes  $1/Y$  and  $1/(ZY)$  are  $(\mathcal{F}_t)_{t \geq 0}$ -martingales. We recall that  $G_t = \frac{1-F(t)}{L_t Y_t}$ . Then, we can construct a family of positive  $\mathbb{F}$ -martingales  $M(u) = (M_t(u))_{t \geq 0}$ , for  $u \geq 0$ , by

$$M_t(u) = \frac{1}{Y_t} \left( \int_u^{u \vee t} \frac{g(s)}{Z_s} ds + \frac{1}{Z_t} \int_{u \vee t}^\infty g(s) ds \right). \quad (5.15)$$

Indeed, from

$$M_t(u) = \mathbb{1}_{\{t < u\}} \frac{1}{Y_t Z_t} \int_u^\infty g(s) ds + \mathbb{1}_{\{u < t\}} \frac{1}{Y_t} \left( \int_u^t \frac{g(s)}{Z_s} ds + \frac{1}{Z_t} \int_t^\infty g(s) ds \right) \quad (5.16)$$

one has:

on  $\{t \leq u\}$ ,  $M_t(u) = (1 - F(u))/(Y_t Z_t)$  is an  $\mathbb{F}$ -martingale

on  $\{u < t\}$ ,  $M_t(u) = (1/Y_t) \left( \int_u^t (g(s)/Z_s) ds + (1/Z_t) \int_t^\infty g(s) ds \right)$  which leads to

$$d_t M_t(u) = \left( \int_0^u \frac{g(s)}{Z_s} ds \right) d\left(\frac{1}{Y_t}\right) + (1 - F(t)) d\left(\frac{1}{Y_t Z_t}\right) \quad (5.17)$$

so that  $M(u)$  forms a martingale, because  $1/Y$  and  $1/(ZY)$  are martingales. Note that the process  $M(u)$  is valued in  $(0, 1)$ , and  $M_t(u)$  is decreasing with respect to  $u$ , for any  $t \geq 0$  fixed, with  $M_t(0) = 1$ . Furthermore, we have  $M_t(t) = G_t$ ,  $M_0(u) = \int_u^\infty g(s) ds$ , and  $M_t(u) = \int_u^\infty p_t(s) g(s) ds$  where  $p_t(u) = g(u)/(Z_{u \wedge t} Y_t)$ .

It is therefore possible to construct, on an extended probability space, a random time  $\tau$  and a probability measure  $\mathbb{Q}$  such that  $\mathbb{Q}$  and  $\mathbb{P}$  coincide on  $\mathbb{F}$  and  $\mathbb{Q}(\tau > u | \mathcal{F}_t) = M_t(u)$ , for  $t, u \geq 0$  (see [15]). It is straightforward to check the important property  $\int_0^\infty p_t(u) g(u) du = 1$ , for any  $t \geq 0$ .

Let us now thus compute the dynamics of  $p(u)$  by means of the integration by parts and thus

$$d_t p_t(u) = d_t \left( \frac{1}{Z_{u \wedge t} Y_t} \right) = \frac{\mu}{\sigma} \left( \mathbb{1}_{\{u \leq t\}} \frac{G_t}{Z_u Y_t} - \mathbb{1}_{\{u > t\}} \frac{1 - G_t}{Z_t Y_t} \right) dW_t \quad (5.18)$$

so that

$$d_t p_t(u) = p_t(u) \frac{\mu}{\sigma} \left( \mathbb{1}_{\{u \leq t\}} G_t - \mathbb{1}_{\{u > t\}} (1 - G_t) \right) dW_t \quad (5.19)$$

and

$$\varphi_t(u) = \frac{\mu}{\sigma} \left( \mathbb{1}_{\{u \leq t\}} G_t - \mathbb{1}_{\{u > t\}} (1 - G_t) \right) = \frac{\mu}{\sigma} \left( \mathbb{1}_{\{u \leq t\}} - (1 - G_t) \right). \quad (5.20)$$

In the progressively enlarged filtration,  $dW_t^{\mathbb{G}} = dW_t - \alpha_t^{\mathbb{G}} dt$ , where

$$\alpha_t^0 = \frac{\mu}{\sigma} \frac{1}{G_t} \int_0^t G_t \frac{g(u)}{Z_u Y_t} du = \frac{\mu}{\sigma} \frac{1}{Y_t} \int_0^t \frac{g(u)}{Z_u} du \quad (5.21)$$

$$\begin{aligned} &= \frac{\mu}{\sigma} \frac{1}{Y_t} \left( Y_t - \frac{1 - F(t)}{Z_t} \right) = \frac{\mu}{\sigma} \left( 1 - \frac{1 - F(t)}{Y_t Z_t} \right) = \frac{\mu}{\sigma} (1 - G_t) \\ \alpha_t^1(\tau) &= -\frac{\mu}{\sigma} G_t, \end{aligned} \quad (5.22)$$

and

$$dW_t^{\mathbb{G}} = dW_t - \frac{\mu}{\sigma} \left( (1 - G_t) \mathbb{1}_{\{t \leq \tau\}} - G_t \mathbb{1}_{\{\tau < t\}} \right) dt. \quad (5.23)$$

We can check that the property  $\int_0^\infty \mathbb{E}(\alpha_t^{\mathbb{G}} | \mathcal{F}_t) dt = 1$  holds. Moreover, from

$$dW_t^{\mathbb{G}} = dW_t - \frac{\mu}{\sigma} \left( \mathbb{1}_{\{\tau \leq t\}} - (1 - G_t) \right) dt \quad (5.24)$$

we deduce that

$$dX_t = \mu \mathbb{1}_{\{\tau \leq t\}} dt + \sigma dW_t^{\mathbb{G}}. \quad (5.25)$$

Acknowledgment : Monique Jeanblanc thanks Marie Claire Quenez for a fruitfull discussion

## References

- [1] AKSAMIT, A. and JEANBLANC, M. (2017). *Enlargement of filtration with finance in view*, Springer.
- [2] AMENDINGER, J. (1999). *Initial Enlargement of Filtrations and Additional Information in Financial Markets*. PhD thesis, Technical University Berlin.
- [3] Barlow, M. T. (1978) *Study of filtration expanded to include an honest time*, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete, 44, 307-323.
- [4] CRÉPEY, S., JEANBLANC, M. and WU, D. (2013). Informationally dynamized Gaussian copula. International Journal of Theoretical and Applied Finance **16**(2) 1350008 (29 pp.).
- [5] CALLEGARO, G., JEANBLANC, M. and ZARGARI, B. (2013). Carthagian enlargement of filtrations, *ESAIM: Probability and Statistics* **17** (550–566).
- [6] DELLACHERIE, C. (1972). *Capacités et Processus Stochastiques*. Springer, Berlin.
- [7] EL KAROUI, N., JEANBLANC, M. and JIAO, Y. (2010). What happens after a default: the conditional density approach. *Stochastic Processes and their Applications*, **120**(7) (1011–1032).
- [8] EL KAROUI, N., JEANBLANC, M., JIAO, Y. and ZARGARI, B. (2014). Conditional default probability and density. In the Volume *Inspired by Finance*, Kabanov, Yu. M., Rutkowski, M., and Zariphopoulou, T. eds. Springer (201–219).
- [9] FONTANA, C. (2018). The strong predictable representation property in initially enlarged filtrations under the density hypothesis. *Stochastic Processes and their Applications* **128**(3) (1007–1033).
- [10] GAPEEV, P. V., JEANBLANC, M., LI, L. and RUTKOWSKI, M. (2010). Constructing random times with given survival process and applications to valuation of credit derivatives. In the Volume *Contemporary Quantitative Finance*, Chiarella, C. and Novikov, A. eds. Springer (255-280).

- [11] GAPEEV, P. V. and JEANBLANC, M. (2009). Pricing of contingent claims in a two-dimensional model with random dividends. *International Journal of Theoretical and Applied Finance* **12** (1091–1104).
- [12] GRORUD, A. and PONTIER, M. (1998). Insider trading in a continuous time market model. *International Journal of Theoretical and Applied Finance* **1** (331–347).
- [13] JACOD, J. (1985). Grossissement initial, hypothèse (H') et théorème de Girsanov. *Lecture Notes in Mathematics* **1118** (15–35).
- [14] JEANBLANC, M., YOR, M. and CHESNEY, M. (2009). *Mathematical Methods for financial Markets*. Springer.
- [15] JEANBLANC, M. and SONG, S. (2011). Explicit model of default time with given survival probability. *Stochastic Processes and their Applications* **121** (1678–1704).
- [16] JEANBLANC, M. and LE CAM, Y. (2009). Progressive enlargement of filtrations with initial times. *Stochastic Processes and their Applications* **119** (2523–2543).
- [17] JEULIN, TH. (1980), *Semi-martingales et Grossissement de Filtration*. Lecture Notes in Mathematics **833**, Springer.
- [18] JEULIN, TH. and YOR, M. (1979). Inégalité de Hardy, semimartingales et faux-amis. *Séminaire de Probabilités XIII*, Lecture Notes in Mathematics **721**, Springer (332–359).
- [19] KUNITA, H. (1981). Some extensions of Itô's formula. *Séminaire de Probabilités XV*, Lecture Notes in Mathematics **850**, Springer (118–141).
- [20] LIPTSER, R. S. and SHIRYAEV, A. N. (2001). *Statistics of Random Processes I*. (Second Edition, First Edition 1977) Springer, Berlin.
- [21] SONG, S. (2014). Optional splitting formula in a progressively enlarged filtration. *ESAIM Probability and Statistics* **18** (829–853).
- [22] VENTZEL, A. D. (1965). On equations of the theory of conditional Markov processes, *Theory of Probability and its Applications* **10** (357–361).