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# Projections in enlargements of filtrations under Jacod's hypothesis and examples\*

Pavel V. Gapeev<sup>†</sup>      Monique Jeanblanc<sup>‡</sup>      Dongli Wu

In this paper, we consider two kinds of enlargements of a Brownian filtration  $\mathbb{F}$ : the initial enlargement with a random time  $\tau$ , denoted by  $\mathbb{F}^{(\tau)}$ , and the progressive enlargement with  $\tau$ , denoted by  $\mathbb{G}$ . We assume Jacod's equivalence hypothesis, that is, the existence of a positive  $\mathbb{F}$ -conditional density for  $\tau$ . Then, starting with the predictable representation of an  $\mathbb{F}^{(\tau)}$ -martingale  $Y(\tau)$  in terms of a standard  $\mathbb{F}^{(\tau)}$ -Brownian motion, we consider its projection on  $\mathbb{G}$ , denoted by  $Y^{\mathbb{G}}$ , and on  $\mathbb{F}$ , denoted by  $y$ . We show how to obtain the coefficients which appear in the predictable representation property for  $Y^{\mathbb{G}}$  (and  $y$ ) in terms of  $Y(\tau)$  and its predictable representation. In the last part, we give examples of conditional densities.

## 1 Introduction

In this paper, we consider two kinds of enlargement of a Brownian filtration  $\mathbb{F}$  generated by a Brownian motion  $W$ : the initial enlargement with a random time (a positive random variable)  $\tau$ , denoted by  $\mathbb{F}^{(\tau)}$ , and the progressive enlargement with  $\tau$ , denoted by  $\mathbb{G}$ . We assume Jacod's

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equivalence hypothesis, that is, the existence of a positive conditional density for  $\tau$  (see (2.1) below in Section 2 for a precise definition). Processes considered in the filtration  $\mathbb{F}^{(\tau)}$  will be denoted as  $Y(\tau)$  since, under Jacod's condition, any  $\mathcal{F}_t^{(\tau)}$ -measurable random variable is of the form  $Y_t(\omega, \tau(\omega))$  for some  $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$  measurable function  $(\omega, u) \rightarrow Y_t(\omega, u)$  (see [1; Proposition 4.22]), processes considered in  $\mathbb{G}$  will be indicated with the superscript  $\mathbb{G}$ , as  $Y^{\mathbb{G}}$ . Processes denoted without these symbols are  $\mathbb{F}$ -adapted as  $Y$  or  $y(u)$ .

In Section 2, we recall basic facts on enlargements of filtrations. In Section 3, we start with an  $\mathbb{F}^{(\tau)}$ -martingale  $Y(\tau)$ , which admits a decomposition

$$Y_t(\tau) = Y_0(\tau) + \int_0^t y_s(\tau) dW_s(\tau) \quad (1.1)$$

for all  $t \geq 0$ , where  $W(\tau)$  is an  $\mathbb{F}^{(\tau)}$ -Brownian motion and, for any  $u \geq 0$  the process  $y(u)$  is  $\mathbb{F}$ -predictable and for any  $t$  the map  $u \rightarrow y_t(u)$  is Borelian, and we study its  $\mathbb{G}$ -optional projection  $Y^{\mathbb{G}}$  and its  $\mathbb{F}$ -optional projection  $y$ . From the predictable representation properties [5], we obtain the existence of two  $\mathbb{G}$ -predictable processes  $\beta^{\mathbb{G}}$  and  $\gamma^{\mathbb{G}}$  such that

$$Y_t^{\mathbb{G}} = \mathbb{E}[Y_t(\tau) | \mathcal{G}_t] = Y_0^{\mathbb{G}} + \int_0^t \beta_s^{\mathbb{G}} dW_s^{\mathbb{G}} + \int_0^t \gamma_s^{\mathbb{G}} dM_s^{\mathbb{G}} \quad (1.2)$$

for all  $t \geq 0$ , where  $M^{\mathbb{G}}$  is the compensated  $\mathbb{G}$ -martingale of the default process  $\mathbb{1}_{\{\tau \leq t\}}$ , and the existence of an  $\mathbb{F}$ -predictable process  $\sigma$  such that

$$Y_t = \mathbb{E}[Y_t(\tau) | \mathcal{F}_t] = y_0 + \int_0^t \sigma_s dW_s.$$

We show how to compute  $\beta^{\mathbb{G}}, \gamma^{\mathbb{G}}$  and  $\sigma$  in terms of  $Y(\tau)$  and  $y(\tau)$ . We apply these computations to compare equivalent martingale measures for a financial market with price process following an  $\mathbb{F}$ -geometric Brownian motion considered in the two enlarged filtrations in Section 4. In the last section, we give a family of example of conditional densities.

In the whole paper, "positive" means "strictly positive".

## 2 Enlargement of filtrations

We recall that, if  $\mathbb{H} \subset \mathbb{K}$ , the  $\mathbb{H}$ -optional projection of a  $\mathbb{K}$ -martingale  $\mu$  is the  $\mathbb{H}$ -optional process  $\nu$  such that  $\mathbb{E}[\mu_{\vartheta} \mathbb{1}_{\{\vartheta < \infty\}} | \mathcal{H}_{\vartheta}] = \nu_{\vartheta} \mathbb{1}_{\{\vartheta < \infty\}}$  for any  $\mathbb{H}$ -stopping time  $\vartheta$ . This optional projection satisfies  $\mathbb{E}(\mu_t | \mathcal{H}_t) = \nu_t$ . By abuse of language, we shall call  $\mathbb{E}(\mu_t | \mathcal{H}_t)$  the  $\mathbb{H}$ -optional projection of  $\mu$ .

We consider a probability space  $(\Omega, \mathcal{G}, \mathbb{P})$  endowed with a standard Brownian motion  $W = (W_t)_{t \geq 0}$  and a positive random variable  $\tau$  with the positive density  $g$  on  $\mathbb{R}^+$  under  $\mathbb{P}$ . We denote by  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  the natural right-continuous and completed filtration of  $W$  and  $\mathbb{G}$  is the progressive enlargement of  $\mathbb{F}$  with  $\tau$ . We assume Jacod's equivalence hypothesis (see [1; Chapter IV; Section 5]), that is, the existence of a positive function  $(\omega, t, u) \rightarrow p_t(\omega, u)$  such that

- (i): the map  $u \mapsto p_t(u; \omega)$  is Borelian,  $(dt \times d\mathbb{P})$  a.s.
- (ii): for each  $u \geq 0$ , the process  $p(u) = (p_t(u))_{t \geq 0}$  is a continuous  $\mathbb{F}$ -martingale,
- (iii): for each  $t \geq 0$ , for every bounded Borel function  $f$ , we have

$$\mathbb{E}[f(\tau) \mid \mathcal{F}_t] = \int_0^\infty f(u) p_t(u) g(u) du. \quad (2.1)$$

Note that  $\int_0^\infty p_t(u) g(u) du = 1$ , for all  $t \geq 0$ , and  $p_0(u) = 1$ , for each  $u \geq 0$ . Moreover, it follows from the predictable representation theorem in a Brownian filtration that, for each  $u \geq 0$ , there exists an  $\mathbb{F}$ -predictable process  $\varphi(u) = (\varphi_t(u))_{t \geq 0}$  such that the positive martingale  $p(u)$  admits the representation

$$p_t(u) = p_0(u) + \int_0^t p_s(u) \varphi_s(u) dW_s \quad (2.2)$$

so that it takes the form of the Doléans-Dade stochastic exponential

$$p_t(u) = p_0(u) \exp \left( \int_0^t \varphi_s(u) dW_s - \frac{1}{2} \int_0^t \varphi_s^2(u) ds \right) \quad (2.3)$$

for all  $t \geq 0$ . It can be shown that  $u \mapsto \varphi_t(u)$  is a Borel function on  $[0, \infty)$ , for all  $t \geq 0$ .

We introduce the supermartingale  $G$  defined as  $G_t := \mathbb{P}(\tau > t \mid \mathcal{F}_t)$  which can be written in terms of the conditional density as

$$G_t = \int_t^\infty p_t(u) g(u) du, t \geq 0. \quad (2.4)$$

Note that  $G$  is a positive continuous  $\mathbb{F}$ -supermartingale called the Azéma supermartingale of  $\tau$  which admits a Doob-Meyer decomposition  $G_t = N_t - A_t$ , where  $N = (N_t)_{t \geq 0}$  is an  $\mathbb{F}$ -martingale under  $\mathbb{P}$  and  $A = (A_t)_{t \geq 0}$  is a nondecreasing  $\mathbb{F}$ -predictable process. Then, by means of Jacod's equivalence hypothesis, we have

$$A_t = \int_0^t p_u(u) g(u) du \quad \text{and} \quad N_t = 1 - \int_0^t (p_t(u) - p_u(u)) g(u) du \quad (2.5)$$

for all  $t \geq 0$  (see [7; Subsection 4.2.1]). Moreover, by means of Itô-Wentzell's lemma (see, e.g. Kunita [19], Wentzell [22], or [14; Theorem 1.5.3.2]), we have

$$dN_t = - \int_0^t d_t p_t(u) g(u) du = - \left( \int_0^t \varphi_t(u) p_t(u) g(u) du \right) dW_t \quad (2.6)$$

where  $d_t$  means that one makes use of the stochastic differential, and the associated predictable covariation processes have the form

$$d\langle W, N \rangle_t = - \left( \int_0^t \varphi_t(u) p_t(u) g(u) du \right) dt \quad \text{and} \quad d\langle W, p(u) \rangle_t = p_t(u) \varphi_t(u) dt \quad (2.7)$$

for all  $t, u \geq 0$ .

It follows from Jacod's theorem for initial enlargements of filtrations ([13; Corollaire 1.11] or [1; Chapter IV, Proposition 4.40]) that the process  $W(\tau) = (W_t(\tau))_{t \geq 0}$  defined by

$$W_t(\tau) := W_t - \int_0^t \frac{d\langle p(u), W \rangle_s |_{u=\tau}}{p_s(\tau)} = W_t - \int_0^t \varphi_s(\tau) ds \quad (2.8)$$

for all  $t \geq 0$ , is a  $\mathbb{P}$ -Brownian motion with respect to the initially enlarged filtration  $\mathbb{F}^{(\tau)}$  defined by  $\mathbb{F}^{(\tau)} = (\mathcal{F}_t \vee \sigma(\tau))_{t \geq 0}$ . Note that, according to the results of Fontana [9], the filtration  $\mathbb{F}^{(\tau)}$  is right-continuous.

It follows from the results of Grorud and Pontier [12] and also Amendinger [2], that the process  $(1/p_t(\tau))_{t \geq 0}$  forms an  $\mathbb{F}^{(\tau)}$ -martingale under  $\mathbb{P}$ , and thus, since  $p_0(\tau) = 1$  holds, we have  $\mathbb{E}[1/p_t(\tau)] = 1$ , for all  $t \geq 0$ . Therefore, we can define a probability measure  $\mathbb{P}^*$  on  $\mathcal{F}_t^{(\tau)}$  by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} \Big|_{\mathcal{F}_t^{(\tau)}} = \frac{1}{p_t(\tau)} = \exp \left( - \int_0^t \varphi_s(\tau) dW_s(\tau) - \frac{1}{2} \int_0^t \varphi_s^2(\tau) ds \right) \quad (2.9)$$

for all  $t \geq 0$ . It is proved in [12] (see also [1; Chapter IV, Proposition 4.37]) that the probability measure  $\mathbb{P}^*$  defined in (2.9) coincides with  $\mathbb{P}$  on  $\mathbb{F}$  and on  $\sigma(\tau)$ , and the random time  $\tau$  is independent of  $\mathbb{F}$  under  $\mathbb{P}^*$ . This fact particularly implies that  $\mathbb{P}^*(\tau > u | \mathcal{F}_t) = \mathbb{P}^*(\tau > u) = \mathbb{P}(\tau > u)$  holds, for all  $t, u \geq 0$ , as well as that the process  $W$  is a  $(\mathbb{P}, \mathbb{F})$  standard Brownian motion with respect to  $\mathbb{F}$  under  $\mathbb{P}^*$ . Note that, by Girsanov's theorem, following the arguments of Callegaro et al. [5], we can recover that the process  $W(\tau)$  from (2.8) is a  $(\mathbb{P}, \mathbb{F}^{(\tau)})$ -Brownian motion.

Let us now introduce the progressively enlarged filtration  $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$  which is the smallest right-continuous filtration containing  $\mathbb{F}$  and making  $\tau$  a stopping time.

Note that, according to the hypothesis that the positive random variable  $\tau$  has a positive density with support  $\mathbb{R}^+$ , the  $\sigma$ -algebra  $\mathcal{G}_0$  is trivial. It is known (see Jeulin [17; Chapter III] or Callegaro et al. [5; Chapter I, Section 2]) that any  $\mathbb{G}$ -predictable process  $Z^{\mathbb{G}}$  can be written as

$$Z_t^{\mathbb{G}} = Z_t^0 \mathbb{1}_{\{t \leq \tau\}} + Z_t^1(\tau) \mathbb{1}_{\{\tau < t\}} \quad (2.10)$$

for all  $t \geq 0$ , where the process  $Z^0$  is  $\mathbb{F}$ -predictable and, for any  $u \geq 0$ , the process  $Z^1(u)$  is  $\mathbb{F}$ -predictable. In that case, using the key lemma for the computation of  $Z^0$  (see, e.g. [1; Chapter II, Lemma 2.9]), we get

$$Z_t^1(u) = Z_t(u), \text{ for } t \geq u, \quad \text{and} \quad Z_t^0 = \frac{1}{G_t} \int_t^\infty Z_t(u) p_t(u) g(u) du. \quad (2.11)$$

We shall say that  $Z^0$  is the  $\mathbb{F}$ -predictable reduction of  $Z^{\mathbb{G}}$ .

The decomposition (2.10), proved for any progressive enlargement in Jeulin does not extend in general to optional processes (see Barlow [3] for a counter example). However, under Jacod's equivalence hypothesis, any  $\mathbb{G}$ -optional process  $U^{\mathbb{G}}$  can be written as

$$U_t^{\mathbb{G}} = U_t^0 \mathbb{1}_{\{t < \tau\}} + U_t^1(\tau) \mathbb{1}_{\{\tau \leq t\}} \quad (2.12)$$

for all  $t \geq 0$ , where  $U^0$  is  $\mathbb{F}$ -optional and, for each  $u \geq 0$ ,  $U^1(u)$  is  $\mathbb{F}$ -optional (see Song [21]). In particular, we shall use that result for the  $\mathbb{G}$ -optional projection of an integrable  $\mathbb{F}^{(\tau)}$ -optional process  $U(\tau)$ , i.e., for

$$U_t^{\mathbb{G}} = \mathbb{E}[U_t(\tau) \mid \mathcal{G}_t] \quad (2.13)$$

In that case, similarly to (2.11), we have

$$U_t^1(u) = U_t(u), \text{ for } t \geq u, \quad \text{and} \quad U_t^0 = \frac{1}{G_t} \int_t^\infty U_t(u) p_t(u) g(u) du \quad (2.14)$$

for all  $t \geq 0$ . The process  $U^0$  is called the optional reduction of  $U$ . Note that, working in a Brownian filtration under Jacod's equivalence hypothesis,  $U^0$  and  $U^1(u)$  being continuous are also  $\mathbb{F}$ -predictable.

Then, it follows from the results of Jeanblanc and Le Cam [16] that the process  $W^{\mathbb{G}} = (W_t^{\mathbb{G}})_{t \geq 0}$  defined by

$$\begin{aligned} W_t^{\mathbb{G}} &:= W_t - \int_0^{t \wedge \tau} \frac{d\langle W, N \rangle_s}{G_s} - \int_{t \wedge \tau}^t \frac{d\langle W, p(u) \rangle_s |_{u=\tau}}{p_s(\tau)} \\ &= W_t + \int_0^{t \wedge \tau} \frac{1}{G_s} \left( \int_0^s \varphi_s(u) p_s(u) g(u) du \right) ds - \int_{t \wedge \tau}^t \varphi_s(\tau) ds \end{aligned} \quad (2.15)$$

$$= W_t + \int_0^t \alpha_s^{\mathbb{G}} ds \quad (2.16)$$

is a  $(\mathbb{P}, \mathbb{G})$ -standard Brownian motion, where  $\alpha^\mathbb{G}$  is the  $\mathbb{G}$ -predictable process with the decomposition

$$\alpha_t^\mathbb{G} = \mathbb{1}_{\{t \leq \tau\}} \frac{1}{G_t} \int_0^t \varphi_t(u) p_t(u) g(u) du \mathbb{1}_{\{\tau < t\}} \varphi_t(u) := \mathbb{1}_{\{t \leq \tau\}} \alpha_t^0 + \mathbb{1}_{\{\tau < t\}} \alpha_t^1(\tau) \quad (2.17)$$

for all  $t \geq 0$ . Observe from (2.8) and (2.15) that

$$W_t(\tau) = W_t^\mathbb{G} - \int_0^{t \wedge \tau} (\alpha_s^0 + \varphi_s(\tau)) ds \quad (2.18)$$

for all  $t \geq 0$ .

We define the  $\mathbb{G}$ -martingale

$$M_t^\mathbb{G} := \mathbb{1}_{\{\tau \leq t\}} - \int_0^t \lambda_s^\mathbb{G} ds = \mathbb{1}_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \lambda_s^0 ds \quad (2.19)$$

for all  $t \geq 0$ , where  $\lambda_s^\mathbb{G} = \mathbb{1}_{\{s \leq \tau\}} \lambda_s^0$  and  $\lambda_s^0 = p_s(s)g(s)/G_s$  (see, e.g. [5; Subsection 1.2] or [1; Chapter II]).

We recall that the process  $W(\tau)$  enjoys the  $\mathbb{F}^\tau$ -predictable representation property and that the pair  $(W^\mathbb{G}, M^\mathbb{G})$  enjoys the  $\mathbb{G}$ -predictable representation property (see, e.g. [5; Proposition 4.3]):

Any  $\mathbb{F}^{(\tau)}$ -martingale  $Y(\tau)$  has the form

$$Y_t(\tau) = Y_0(\tau) + \int_0^t y_s(\tau) dW_s(\tau) \quad (2.20)$$

for all  $t \geq 0$ , where  $Y_0$  is a (deterministic) function and  $y(u)$ ,  $u \geq 0$ , is a family of  $\mathbb{F}$ -predictable processes. In particular, any  $(\mathbb{P}, \mathbb{F}^{(\tau)})$ -martingale is continuous, and if  $Y(\tau)$  is square integrable on  $[0, T]$ , then  $\mathbb{E}[\int_0^T (y_s(\tau))^2 ds] < \infty$ .

Any  $\mathbb{G}$ -martingale admits the representation

$$Y_t^\mathbb{G} = Y_0^\mathbb{G} + \int_0^t \beta_s^\mathbb{G} dW_s^\mathbb{G} + \int_0^t \gamma_s^\mathbb{G} dM_s^\mathbb{G} \quad (2.21)$$

where  $\beta^\mathbb{G}$  and  $\gamma^\mathbb{G}$  are  $\mathbb{G}$ -predictable processes.

### 3 Martingales and projections

#### 3.1 Projections of $\mathbb{F}^{(\tau)}$ martingales on $\mathbb{G}$

**Proposition 3.1** *Let  $Y(\tau)$  be an  $\mathbb{F}^{(\tau)}$ -martingale with representation (2.20). Its  $\mathbb{G}$ -optional projection  $Y^\mathbb{G}$  admits the following representation*

$$Y_t^\mathbb{G} = \mathbb{E}[Y_t(\tau) | \mathcal{G}_t] = Y_0^\mathbb{G} + \int_0^t \beta_s^\mathbb{G} dW_s^\mathbb{G} + \int_0^t \gamma_s^\mathbb{G} dM_s^\mathbb{G} \quad (3.1)$$

where  $Y_0^{\mathbb{G}} = \mathbb{E}[Y_0(\tau)]$ , and the  $\mathbb{G}$ -predictable processes  $\beta^{\mathbb{G}}$  and  $\gamma^{\mathbb{G}}$  admit, from (2.10), the decomposition

$$\beta_t^{\mathbb{G}} = \beta_t^0 \mathbb{1}_{\{t \leq \tau\}} + \beta_t^1(\tau) \mathbb{1}_{\{\tau < t\}} \quad (3.2)$$

$$\gamma_t^{\mathbb{G}} = \gamma_t^0 \mathbb{1}_{\{t \leq \tau\}} + \gamma_t^1(\tau) \mathbb{1}_{\{\tau < t\}}. \quad (3.3)$$

Then,

$$\beta_t^0 = \frac{1}{G_t} \int_t^\infty (y_t(u) + (\alpha_t^0 + \varphi_t(u)) Y_t(u)) p_t(u) g(u) du, \quad \beta_t^1(u) = y_t(u), \quad \text{for } t \geq u \quad (3.4)$$

$$\gamma_t^0 = Y_t(t) - Y_t^0, \quad (3.5)$$

for all  $t \geq 0$ , and  $Y^0$  is given by (2.11). Note that for any choice of  $\gamma^1$ , the property  $\int_0^t \gamma_s^{\mathbb{G}} dM_s^{\mathbb{G}} = \int_0^t \gamma_s^0 dM_s^{\mathbb{G}}$  holds.

Proof: In a first step, we assume that  $Y(\tau)$  (hence  $Y^{\mathbb{G}}$ ), are square integrable on  $[0, T]$ . We start to determine  $\beta^{\mathbb{G}}$  (which is square integrable). For this purpose, we observe that for any bounded  $\mathbb{G}$ -predictable process  $n^{\mathbb{G}}$ , using the tower property and the fact that  $W^{\mathbb{G}}$  is a  $\mathbb{G}$ -martingale orthogonal to  $M^{\mathbb{G}}$ , we have

$$\begin{aligned} \mathbb{E} \left[ Y_t(\tau) \int_0^t n_s^{\mathbb{G}} dW_s^{\mathbb{G}} \right] &= \mathbb{E} \left[ Y_t^{\mathbb{G}} \int_0^t n_s^{\mathbb{G}} dW_s^{\mathbb{G}} \right] \\ &= \mathbb{E} \left[ \int_0^t \beta_s^{\mathbb{G}} dW_s^{\mathbb{G}} \int_0^t n_s^{\mathbb{G}} dW_s^{\mathbb{G}} \right] = \mathbb{E} \left[ \int_0^t \beta_s^{\mathbb{G}} n_s^{\mathbb{G}} ds \right] \end{aligned} \quad (3.6)$$

for all  $t \geq 0$ . Recall that, from (2.18), we have

$$W_t^{\mathbb{G}} = W_t(\tau) + \int_0^{t \wedge \tau} (\alpha_s^0 + \varphi_s(\tau)) ds \quad (3.7)$$

for all  $t \geq 0$ . We introduce the continuous  $\mathbb{G}$ -martingale

$$V_t^{\mathbb{G}} = \int_0^t n_s^{\mathbb{G}} dW_s^{\mathbb{G}} = \int_0^t n_s^{\mathbb{G}} dW_s(\tau) + \int_0^t (\alpha_s^0 + \varphi_s(\tau)) \mathbb{1}_{\{s \leq \tau\}} n_s^{\mathbb{G}} ds \quad (3.8)$$

for all  $t \geq 0$ . By means of the integration by parts, we have

$$\mathbb{E} [ Y_t(\tau) V_t^{\mathbb{G}} ] = \mathbb{E} \left[ + \int_0^t Y_s(\tau) dV_s^{\mathbb{G}} + \langle Y(\tau), V_t^{\mathbb{G}} \rangle_t^{\mathbb{F}(\tau)} \int_0^t V_s^{\mathbb{G}} dY_s(\tau) \right] \quad (3.9)$$

for all  $t \geq 0$ .

---

A martingale  $X$  is square integrable if  $\sup_{t \leq T} \mathbb{E}(X_t^2) < \infty$ .



We recall that a local martingale  $M$  satisfying  $\mathbb{E}[(\langle M \rangle_T)^{1/2}] < \infty$  is a martingale. Let us prove that the  $\mathbb{F}^{(\tau)}$ -local martingale  $M_t = \int_0^t V_s^{\mathbb{G}} dY_s(\tau)$  is a martingale. One has

$$\begin{aligned} \mathbb{E}[(\langle M \rangle_T)^{1/2}] &= \mathbb{E} \left[ \left( \int_0^T (V_s^{\mathbb{G}})^2 (y_s(\tau))^2 ds \right)^{1/2} \right] \leq \mathbb{E} \left[ \sup_{s \leq T} |V_s^{\mathbb{G}}| \left( \int_0^T (y_s(\tau))^2 ds \right)^{1/2} \right] \\ &\leq \mathbb{E}[\sup_{s \leq T} |V_s^{\mathbb{G}}|^2] + \mathbb{E} \left[ \int_0^T y_s(\tau)^2 ds \right]. \end{aligned}$$

From Burkholder's inequality {footnote Burkholder's inequality states that for any  $p \geq 1$ ,  $\mathbb{E}[\sup_{t \leq T} |M_t|^p]} \leq C_p \mathbb{E}[\langle M \rangle_T^{p/2}]$  where  $C_p$  is a constant depending only on  $p$ . the quantity  $\mathbb{E}[\sup_{s \leq T} |V_s^{\mathbb{G}}|^2]$  is smaller than  $C_2 \mathbb{E}[\int_0^T (n_s^{\mathbb{G}})^2 ds]$  which is bounded. Furthermore,  $\mathbb{E}[\int_0^T y_s(\tau)^2 ds]$  is bounded. Hence,  $\int_0^t V_s^{\mathbb{G}} dY_s(\tau)$  is a martingale and its expectation is null. The boundeness of  $n^{\mathbb{G}}$  and the square integrability of  $Y(\tau)$  imply that  $\int_0^t Y_s(\tau) n_s^{\mathbb{G}} dW_s(\tau)$  is a martingale, so that

$$\mathbb{E} \left[ \int_0^t V_s^{\mathbb{G}} dY_s(\tau) + \int_0^t Y_s(\tau) dV_s^{\mathbb{G}} \right] = \mathbb{E} \left[ \int_0^t Y_s(\tau) (\alpha_s^0 + \varphi_s(\tau)) \mathbb{1}_{\{s \leq \tau\}} n_s^{\mathbb{G}} ds \right]. \quad (3.10)$$

Recall that  $\langle Y(\tau), V_t^{\mathbb{G}} \rangle_{\mathbb{F}^{(\tau)}} = \int_0^t y_s(\tau) n_s^{\mathbb{G}} ds$ , for all  $t \geq 0$ . Then, we have

$$\mathbb{E}[Y_t(\tau) V_t^{\mathbb{G}}] = \mathbb{E} \left[ \int_0^t \left( Y_s(\tau) (\alpha_s^0 + \varphi_s(\tau)) \mathbb{1}_{\{s \leq \tau\}} + y_s(\tau) \right) n_s^{\mathbb{G}} ds \right] \quad (3.11)$$

hence, using the tower property

$$\mathbb{E} \left[ \int_0^t \beta_s^{\mathbb{G}} n_s^{\mathbb{G}} ds \right] = \mathbb{E} \left[ \int_0^t \mathbb{E}[y_s(\tau) + Y_s(\tau) (\alpha_s^0 + \varphi_s(\tau)) \mathbb{1}_{\{s \leq \tau\}} | \mathcal{G}_s] n_s^{\mathbb{G}} ds \right] \quad (3.12)$$

for all  $t \geq 0$  and for any bounded  $n^{\mathbb{G}}$ . It follows that

$$\beta_t^{\mathbb{G}} = \mathbb{E}[y_t(\tau) + Y_t(\tau) (\alpha_t^0 + \varphi_t(\tau)) \mathbb{1}_{\{t \leq \tau\}} | \mathcal{G}_t], \quad (3.13)$$

and the  $\mathbb{F}$ -predictable reduction of  $\beta^{\mathbb{G}}$  is

$$\beta_t^0 = \frac{1}{G_t} \int_t^\infty \left( y_t(u) + (\alpha_t^0 + \varphi_t(u)) Y_t(u) \right) p_t(u) g(u) du \quad (3.14)$$

and on  $\{t > \tau\}$ , one has, as expected, that  $\beta_t^{\mathbb{G}} = \mathbb{E}[y_t(\tau) | \mathcal{G}_t] = y_t(\tau)$  so that  $\beta_t^1(u) = y_t(u)$ .

In the second step, we determine  $\gamma^{\mathbb{G}}$ . For this purpose, on the one hand, for any bounded  $\mathbb{G}$ -predictable process  $n^{\mathbb{G}}$ , using the fact that  $M^{\mathbb{G}}$  is a  $\mathbb{G}$ -martingale orthogonal to  $W^{\mathbb{G}}$  and

that  $M^{\mathbb{G}}$  is flat after  $\tau$  ( $M_t^{\mathbb{G}} = M_{t \wedge \tau}$ ), and that the equality  $d\langle M^{\mathbb{G}} \rangle_t = \lambda_t^0 \mathbb{1}_{\{t \leq \tau\}} dt$  holds, we have

$$\begin{aligned} \mathbb{E} \left[ Y_t(\tau) \int_0^t n_s^{\mathbb{G}} dM_s^{\mathbb{G}} \right] &= \mathbb{E} \left[ Y_t(\tau) \int_0^t n_s^0 dM_s^{\mathbb{G}} \right] = \mathbb{E} \left[ Y_t^{\mathbb{G}} \int_0^t n_s^0 dM_s^{\mathbb{G}} \right] \\ &= \mathbb{E} \left[ \int_0^t \gamma_s^0 dM_s^{\mathbb{G}} \int_0^t n_s^0 dM_s^{\mathbb{G}} \right] = \mathbb{E} \left[ \int_0^t \gamma_s^0 \lambda_s^0 \mathbb{1}_{\{s < \tau\}} n_s^0 ds \right] = \mathbb{E} \left[ \int_0^t \gamma_s^0 \lambda_s^0 G_s n_s^0 ds \right] \end{aligned} \quad (3.15)$$

for all  $t \geq 0$ , where  $n^0$  is the  $\mathbb{F}$ -predictable reduction of  $n^{\mathbb{G}}$  and where the last equality comes from the tower property.

Recall that  $M^{\mathbb{G}}$  is a predictable bounded variation process in the initially enlarged filtration. Let  $U_t^{\mathbb{G}} = \int_0^t n_s^0 dM_s^{\mathbb{G}}$ . By integration by parts, using the fact that  $Y(\tau)$  is continuous, its bracket (in  $\mathbb{F}^{\mathbb{G}}$  with the  $\mathbb{F}^{(\tau)}$ -predictable bounded variation process  $U^{\mathbb{G}}$  is null

$$\mathbb{E}[Y_t(\tau)U_t^{\mathbb{G}}] = \mathbb{E} \left[ \int_0^t Y_s(\tau) n_s^0 dM_s^{\mathbb{G}} + \int_0^t U_s^{\mathbb{G}} dY_s(\tau) \right].$$

The same methodology than in the first part for  $\int V^{\mathbb{G}} dY_s(\tau)$  proves that the process  $\int_0^\cdot U_s^{\mathbb{G}} dY_s(\tau)$  is a martingale. Furthermore, setting  $H_t = \mathbb{1}_{\{t \leq \tau\}}$ , we have

$$\begin{aligned} \mathbb{E} \left[ \int_0^t Y_s(\tau) n_s^0 dM_s^{\mathbb{G}} \right] &= \mathbb{E} \left[ \int_0^t Y_s(\tau) n_s^0 dH_s - \int_0^t Y_s(\tau) n_s^0 \lambda_s^0 \mathbb{1}_{\{s < \tau\}} ds \right] \\ &= \mathbb{E} \left[ \mathbb{1}_{\{\tau \leq t\}} Y_\tau(\tau) n_\tau^0 - \int_0^t \mathbb{E}[Y_s(\tau) \mathcal{G}_s] n_s^0 \lambda_s^0 \mathbb{1}_{\{s < \tau\}} ds \right] \\ &= \mathbb{E} \left[ \int_0^t Y_s(\tau) n_u^0 p_t(s) g(s) ds - \int_0^t Y_s^0 n_s^0 \lambda_s^0 \mathbb{1}_{\{s < \tau\}} ds \right] \\ &= \mathbb{E} \left[ \int_0^t [Y_s(s) p_s(s) - Y_u^0 \lambda_u^0 G_u] n_u^0 du \right] \end{aligned}$$

where in the last equality, we used the fact that  $p(u)$  is an  $\mathbb{F}$ -martingale. To conclude, we have

$$\mathbb{E} \left[ \int_0^t \gamma_s^0 \lambda_s^0 G_s n_s^0 ds \right] = \mathbb{E} \left[ \int_0^t (Y_s(s) p_s(s) g(s) - \lambda_s^0 G_s Y_s^0) n_s^0 ds \right] \quad (3.16)$$

and using the fact that  $\lambda_s^0 G_s = p_s(s) g(s)$ , one has, for any  $\mathbb{F}$  adapted bounded process  $n^0$

$$\mathbb{E} \left[ \int_0^t (\gamma_s^0 - Y_s(s) + Y_s^0) \lambda_s^0 G_s n_s^0 ds \right] = 0 \quad (3.17)$$

for all  $t \geq 0$ , so that the expression in (3.5) holds. The result obtained for square integrable martingales  $Y(\tau)$  extends to all martingales.  $\square$

### 3.2 Projection of $\mathbb{F}^{(\tau)}$ -martingales on $\mathbb{F}$

**Proposition 3.2** *Let  $Y(\tau)$  be an  $\mathbb{F}^{(\tau)}$ -martingale of the form  $dY_t(\tau) = y_t(\tau)dW_t(\tau)$ . Its  $\mathbb{F}$ -optional projection is  $Y_t = \mathbb{E}[Y_t(\tau) | \mathcal{F}_t] = Y_0 + \int_0^t \sigma_s dW_s$ , where the  $\mathbb{F}$ -predictable process  $\sigma$  is given by*

$$\sigma_t = \int_0^\infty (y_t(u) + Y_t(u) \varphi_t(u)) p_t(u) g(u) du \quad (3.18)$$

for all  $t \geq 0$ .

Proof: It follows from the predictable representation property in the filtration  $\mathbb{F}$  that there exists an  $\mathbb{F}$ -predictable process  $\sigma$  such that

$$\mathbb{E}[Y_t(\tau) | \mathcal{F}_t] = Y_0 + \int_0^t \sigma_s dW_s \quad (3.19)$$

holds for all  $t \geq 0$ . On the one hand, for any bounded  $\mathbb{F}$ -adapted process  $n$ , we have

$$\mathbb{E}\left[Y_t(\tau) \int_0^t n_s dW_s\right] = \mathbb{E}\left[Y_t \int_0^t n_s dW_s\right] = \mathbb{E}\left[\int_0^t \sigma_s n_s ds\right] \quad (3.20)$$

for all  $t \geq 0$ . On the other hand, using (2.8) and an integration by parts on the left-hand side lead to, assuming that  $Y(\tau)$  is square integrable

$$\begin{aligned} \mathbb{E}\left[Y_t(\tau) \int_0^t n_s dW_s\right] &= \mathbb{E}\left[Y_t(\tau) \left(\int_0^t n_s dW_s(\tau) + \int_0^t n_s \varphi_s(\tau) ds\right)\right] \\ &= \mathbb{E}\left[\int_0^t (y_s(\tau) + Y_s(\tau) \varphi_s(\tau)) n_s ds\right] = \mathbb{E}\left[\int_0^t \mathbb{E}[y_s(\tau) + Y_s(\tau) \varphi_s(\tau) | \mathcal{F}_s] n_s ds\right] \end{aligned} \quad (3.21)$$

for all  $t \geq 0$ . Hence, we have

$$\sigma_t = \mathbb{E}[y_t(\tau) + Y_t(\tau) \varphi_t(\tau) | \mathcal{F}_t] = \int_0^\infty (y_t(u) + Y_t(u) \varphi_t(u)) p_t(u) g(u) du \quad (3.22)$$

for all  $t \geq 0$ . □

### 3.3 Projection of $\mathbb{G}$ -martingales on $\mathbb{F}$

**Proposition 3.3** *Consider the  $\mathbb{G}$ -martingale  $Y^\mathbb{G} = (Y_t^\mathbb{G})_{t \geq 0}$  defined by*

$$Y_t^\mathbb{G} = Y_0^\mathbb{G} + \int_0^t \beta_s^\mathbb{G} dW_s^\mathbb{G} + \int_0^t \gamma_s^\mathbb{G} dM_s^\mathbb{G} = Y_t^0 \mathbb{1}_{\{t < \tau\}} + Y_t^1(\tau) \mathbb{1}_{\{\tau \leq t\}}. \quad (3.23)$$

Then, its  $\mathbb{F}$ -optional projection is  $Y_t = \mathbb{E}[Y_t^{\mathbb{G}} | \mathcal{F}_t] = Y_0 + \int_0^t \eta_s dW_s$ , where the  $\mathbb{F}$ -predictable process  $\eta$  satisfies

$$\eta_t = \mathbb{E}[\beta_t^{\mathbb{G}} - Y_t^{\mathbb{G}} \alpha_t^{\mathbb{G}} | \mathcal{F}_t] = (\beta_t^0 - Y_t^0 \alpha_t^0) G_t + \int_0^t (\beta_t^1(u) - Y_t^1(u) \alpha_t^1(u)) p_t(u) g(u) du \quad (3.24)$$

for all  $t \geq 0$ .

Proof: We proceed as before and consider, for a bounded  $\mathbb{F}$ -adapted process  $n$ , the quantity  $\mathbb{E} \left[ Y_t^{\mathbb{G}} \int_0^t n_s dW_s \right]$ , for all  $t \geq 0$ . On the one hand, using the same procedure as in the analysis of (3.12), we get

$$\mathbb{E} \left[ Y_t^{\mathbb{G}} \int_0^t n_s dW_s \right] = \mathbb{E} \left[ \int_0^t \eta_s n_s ds \right] \quad (3.25)$$

for all  $t \geq 0$ . On the other hand, setting  $U_t = \int_0^t n_s dW_s$ , by means of integration by parts and (2.15), we obtain, using the same methodology as before

$$\begin{aligned} \mathbb{E} \left[ Y_t^{\mathbb{G}} \int_0^t n_s dW_s \right] &= \mathbb{E} [Y_t^{\mathbb{G}} U_t] = \mathbb{E} \left[ \int_0^t U_s dY_s^{\mathbb{G}} + \int_0^t Y_s^{\mathbb{G}} n_s dW_s + \int_0^t \beta_s^{\mathbb{G}} n_s ds \right] \\ &= \mathbb{E} \left[ \int_0^t (\beta_s^{\mathbb{G}} - Y_s^{\mathbb{G}} \alpha_s^{\mathbb{G}}) n_s ds \right] = \mathbb{E} \left[ \int_0^t \mathbb{E} [\beta_s^{\mathbb{G}} - Y_s^{\mathbb{G}} \alpha_s^{\mathbb{G}} | \mathcal{F}_s] n_s ds \right] \end{aligned} \quad (3.26)$$

for all  $t \geq 0$ . □

**Remark 3.4** Since  $\mathbb{E}[Y_t(\tau) | \mathcal{F}_t] = \mathbb{E}[\mathbb{E}[Y_t(\tau) | \mathcal{G}_t] | \mathcal{F}_t]$  we have  $\sigma = \eta$  where these quantities are defined in (3.18) and (3.24). Indeed, from (3.13),

$$\begin{aligned} \mathbb{E}[\beta_t^{\mathbb{G}} - \alpha_t^{\mathbb{G}} Y_t^{\mathbb{G}} | \mathcal{F}_t] &= \mathbb{E}[y_t(\tau) + Y_t(\tau)(\alpha_t^0 + \varphi_t(\tau)) \mathbb{1}_{\{t \leq \tau\}} - \alpha_t^0 Y_t^0 \mathbb{1}_{\{t \leq \tau\}} - \alpha^1(\tau) Y_t^1(\tau) \mathbb{1}_{\{\tau < t\}} | \mathcal{F}_t] \\ &= \mathbb{E}[y_t(\tau) + Y_t(\tau) \varphi_t(\tau) | \mathcal{F}_t]. \end{aligned} \quad (3.27)$$

## 3.4 Change of probability measures

### 3.4.1 $\mathbb{F}^{(\tau)}$ versus $\mathbb{G}$

Note that, from PRP, and positive  $\mathbb{F}^{(\tau)}$ -martingale has the form

$$L_t(\tau) = L_0(\tau) \mathcal{E}(\zeta(\tau) \cdot W(\tau))_t = L_0(\tau) \exp \left( \int_0^t \zeta_s(\tau) dW_s(\tau) - \frac{1}{2} \int_0^t (\zeta_s(\tau))^2 ds \right), \quad t \geq 0 \quad (3.28)$$

where  $L_0$  is a positive function and  $\zeta(\tau)$  is  $\mathbb{F}^{(\tau)}$ -predictable, or in a closed form

$$L_t(\tau) = L_0(\tau) \exp \left( \int_0^t \zeta_s(\tau) dW_s(\tau) - \frac{1}{2} \int_0^t (\zeta_s(\tau))^2 ds \right)$$

and any positive  $\mathbb{G}$ -martingale has the form

$$C\mathcal{E}(\mu^{\mathbb{G}} \cdot W^{\mathbb{G}})_t \mathcal{E}(\psi^{\mathbb{G}} \cdot M^{\mathbb{G}})_t, \quad (3.29)$$

where  $C$  is a positive constant and  $\mu^{\mathbb{G}}$  and  $\psi^{\mathbb{G}}$  are  $\mathbb{G}$  predictable and  $\psi^{\mathbb{G}} > -1$ . We recall that

$$\mathcal{E}(\psi^{\mathbb{G}} \cdot M^{\mathbb{G}})_t = \exp\left(\int_0^{t \wedge \tau} \psi_s^{\mathbb{G}} \lambda_s^0 ds\right) (1 + \psi_{\tau}^{\mathbb{G}})^{H_t}.$$

**Proposition 3.5** *Let  $L(\tau) = (L_t(\tau))_{t \geq 0}$  be of the form (3.28). Then, its  $\mathbb{G}$ -optional projection  $L^{\mathbb{G}}$  satisfies*

$$L_t^{\mathbb{G}} = \mathbb{E}[L_t(\tau) | \mathcal{G}_t] = \mathbb{E}[L_0(\tau)] + \int_0^t L_{s-}^{\mathbb{G}} \mu_s^{\mathbb{G}} dW_s^{\mathbb{G}} + \int_0^t L_{s-}^{\mathbb{G}} \psi_s^{\mathbb{G}} dM_s^{\mathbb{G}} \quad (3.30)$$

where the  $\mathbb{G}$ -predictable processes  $\mu^{\mathbb{G}}$  and  $\psi^{\mathbb{G}}$  are given by

$$\mu_t^{\mathbb{G}} = \mathbb{1}_{\{t \leq \tau\}} \frac{1}{L_t^0 G_t} \int_t^{\infty} L_t(u) (\zeta_t(u) + \varphi_t(u) + \alpha_t^0) p_t(u) g(u) du + \mathbb{1}_{\{\tau < t\}} \zeta_t(\tau) \quad (3.31)$$

$$\psi_t^{\mathbb{G}} = \frac{L_t(t)}{L_t^0} - 1 \quad (3.32)$$

where  $L^0$  is the optional reduction of  $L^{\mathbb{G}}$ .

Proof: This is an application of Proposition 3.2.  $\square$

Note that  $\psi_t^{\mathbb{G}} > -1$ , as it must be. If  $\mathbb{E}[L_0(\tau)] = 1$ , and  $L(\tau)$  is a positive  $(\mathbb{P}, \mathbb{F}^{(\tau)})$ -martingale with initial value  $\ell(\tau)$ , one can associate to it the change of probability measure defined by  $d\tilde{\mathbb{P}} = L_t(\tau) d\mathbb{P}$  on  $\mathcal{F}_t^{(\tau)}$ , for any  $t \geq 0$ . The choice  $L_0 \equiv 1$  is equivalent to  $\tilde{\mathbb{P}}(\tau > u) = \mathbb{P}(\tau > u)$ , for each  $u \geq 0$ . Indeed, since  $\tau$  is  $\mathcal{F}_0 \vee \sigma(\tau)$ -measurable, we have, using tower property  $\tilde{\mathbb{P}}(\tau > u) = \mathbb{E}[\ell(\tau) \mathbb{1}_{\{\tau > u\}}]$  and the equality  $\mathbb{E}[L_0(\tau) \mathbb{1}_{\{\tau > u\}}] = \mathbb{E}[\mathbb{1}_{\{\tau > u\}}]$  holds, for each  $u \geq 0$ , which implies that  $L_0 \equiv 1$ .

In particular, in the case  $L_t(\tau) = p_0(\tau)/p_t(\tau)$ , which satisfies  $dL_t(\tau) = -L_t(\tau)\varphi_t(\tau)dW_t(\tau)$ , one obtains (recall that  $P^*$  is defined in (2.9))  $d\mathbb{P}^*|_{\mathcal{G}_t} = L_t^{\mathbb{G},*} d\mathbb{P}|_{\mathcal{G}_t}$  with

$$L_t^{\mathbb{G},*} = \mathbb{E}\left(\frac{p_0(\tau)}{p_t(\tau)} | \mathcal{G}_t\right) = \mathcal{E}(\mu^{\mathbb{G},*} \cdot W^{\mathbb{G}})_t \mathcal{E}(\psi^{\mathbb{G},*} \cdot M^{\mathbb{G}})_t \quad (3.33)$$

with

$$\mu_t^{\mathbb{G},*} = \mathbb{1}_{\{t \leq \tau\}} \alpha_t^0 - \mathbb{1}_{\{\tau < t\}} \varphi_t(\tau) \quad \text{and} \quad \psi_t^{\mathbb{G},*} = \mathbb{1}_{\{t \leq \tau\}} \left( \frac{G_t}{p_t(t)(1 - F(t))} - 1 \right) \quad (3.34)$$

where  $F(t) = \int_0^t g(u)du$ .

From the definition of  $L^{\mathbb{G},*}$  one has

$$L_t^{\mathbb{G},*} = \mathbb{1}_{\{t \leq \tau\}} \frac{1 - F(t)}{G_t} + \mathbb{1}_{\{\tau < t\}} L_t(\tau) \quad (3.35)$$

for all  $t \geq 0$ .

### 3.4.2 $\mathbb{G}$ versus $\mathbb{F}$

From PRP, any positive  $(\mathbb{P}, \mathbb{G})$ -martingale  $L^{\mathbb{G}} = (L_t^{\mathbb{G}})_{t \geq 0}$  can be written as

$$L_t^{\mathbb{G}} = L_0^{\mathbb{G}} + \int_0^t L_s^{\mathbb{G}} \mu_s^{\mathbb{G}} dW_s^{\mathbb{G}} + \int_0^t L_{s-}^{\mathbb{G}} \psi_s^{\mathbb{G}} dM_s^{\mathbb{G}} = L_t^0 \mathbb{1}_{\{t < \tau\}} + L_t^1(\tau) \mathbb{1}_{\{\tau \leq t\}}. \quad (3.36)$$

Let  $L = (L_t)_{t \geq 0}$  be its  $\mathbb{F}$ -optional projection  $L_t = \mathbb{E}[L_t^{\mathbb{G}} | \mathcal{F}_t]$ , for all  $t \geq 0$ . Then, we have  $L_t = L_0 + \int_0^t L_s \eta_s dW_s$ , where  $L_0 = \mathbb{E}[L_0(\tau)]$  and  $\eta$  is the  $\mathbb{F}$ -predictable process

$$\eta_t = \frac{L_t^0}{L_t} (\mu_t^0 - \alpha_t^0) G_t + \int_0^t L_t^1(u) (\mu_t^1(u) - \alpha_t^1(u)) p_t(u) g(u) du \quad (3.37)$$

for all  $t \geq 0$ .

### 3.4.3 $\mathbb{F}^{(\tau)}$ versus $\mathbb{F}$

Let  $L(\tau)$  be a positive  $\mathbb{F}^{(\tau)}$ -martingale of the form (3.28), then its  $\mathbb{F}$ -optional projection is  $L_t = \mathbb{E}[L_t(\tau) | \mathcal{F}_t] = L_0 + \int_0^t L_s \sigma_s dW_s$  where  $L_0 = \mathbb{E}[L_0(\tau)]$  and  $\sigma$  is the  $\mathbb{F}$ -predictable process

$$\sigma_t = \frac{1}{L_t} \int_0^\infty L_t(u) (\zeta_t(u) + \varphi_t(u)) p_t(u) g(u) du \quad (3.38)$$

for all  $t \geq 0$ .

## 3.5 Stability of the Brownian property

As we mentioned before, due to the fact that the processes  $W^{\mathbb{G}}$  and  $M^{\mathbb{G}}$  enjoy the predictable representation property in  $\mathbb{G}$ , any locally equivalent probability measure  $\mathbb{Q}$  which is equivalent to  $\mathbb{P}$  on  $\mathcal{G}_t$  is given by

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{G}_t} = \mathcal{E}(\mu^{\mathbb{G}} \cdot W^{\mathbb{G}})_t \mathcal{E}(\psi^{\mathbb{G}} \cdot M^{\mathbb{G}})_t \quad (3.39)$$

for all  $t \geq 0$ , where  $\mu^{\mathbb{G}}$  and  $\psi^{\mathbb{G}}$  are  $\mathbb{G}$  predictable and  $\psi^{\mathbb{G}} > -1$ . Note that the change of probability measure does not affect the intensity if and only if  $\psi^{\mathbb{G}} = 0$ .

For a probability measure  $\mathbb{Q}$  which is locally equivalent to  $\mathbb{P}$  on  $\mathbb{G}$ , let us denote by  $\mathbb{Q}^{\mathbb{F}}$  its restriction to  $\mathbb{F}$ , which is locally equivalent to  $\mathbb{P}$  on  $\mathbb{F}$ , with its density being of the form  $d\mathbb{Q}^{\mathbb{F}}/d\mathbb{P} = \mathcal{E}(\eta \cdot W)_t$ , for all  $t \geq 0$ . Under  $\mathbb{Q}^{\mathbb{F}}$ , the process  $W^{\mathbb{Q},\mathbb{F}}$  defined by  $W_t^{\mathbb{Q},\mathbb{F}} = W_t - \int_0^t \eta_s ds$  is a  $(\mathbb{Q}^{\mathbb{F}}, \mathbb{F})$ -standard Brownian motion. Let us provide a set of probability measures  $\mathbb{Q}$  which are equivalent to  $\mathbb{P}$  on  $\mathbb{G}$  such that the  $(\mathbb{Q}^{\mathbb{F}}, \mathbb{F})$ -standard Brownian motion  $W^{\mathbb{Q},\mathbb{F}}$  is a  $(\mathbb{Q}, \mathbb{G})$ -standard Brownian motion.

**Proposition 3.6** *Let  $L^{\mathbb{G}}$  be a positive  $(\mathbb{P}, \mathbb{G})$ -martingale of the form  $\mathcal{E}(\mu^{\mathbb{G}} \cdot W^{\mathbb{G}})$ , and define the probability measure  $\mathbb{Q}$  by*

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{G}_t} = L_t^{\mathbb{G}} \quad (3.40)$$

for all  $t \geq 0$ . The process  $W^{\mathbb{Q},\mathbb{F}}$  is a  $(\mathbb{Q}, \mathbb{G})$ -standard Brownian motion if and only if  $\mu^{\mathbb{G}} - \alpha^{\mathbb{G}}$  is  $\mathbb{F}$ -adapted. Then, we have  $\eta_t = \mu_t^{\mathbb{G}} - \alpha_t^{\mathbb{G}}$ , for all  $t \geq 0$ .

Proof: On the one hand, we have seen in (2.15) that  $W_t^{\mathbb{G}} = W_t + \int_0^t \alpha_s^{\mathbb{G}} ds$ , for all  $t \geq 0$ . Hence, we have  $W_t^{\mathbb{G}} = W_t^{\mathbb{Q},\mathbb{F}} + \int_0^t \sigma_s ds + \int_0^t \alpha_s^{\mathbb{G}} ds$ , for all  $t \geq 0$ . On the other hand, we see that  $W_t^{\mathbb{G}} - \int_0^t \mu_s^{\mathbb{G}} ds$  is a  $(\mathbb{Q}, \mathbb{G})$ -standard Brownian motion. It follows that  $W_t^{\mathbb{Q},\mathbb{F}} + \int_0^t \sigma_s ds + \int_0^t \alpha_s^{\mathbb{G}} ds - \int_0^t \mu_s^{\mathbb{G}} ds$  is a  $\mathbb{G}$ -standard Brownian motion. Therefore,  $W^{\mathbb{Q},\mathbb{F}}$  is a  $(\mathbb{Q}, \mathbb{G})$ -standard Brownian motion if and only if  $\sigma_t + \alpha_t^{\mathbb{G}} - \mu_t^{\mathbb{G}} = 0$ , for all  $t \geq 0$ . Note that indeed, if  $\alpha_t^{\mathbb{G}} - \mu_t^{\mathbb{G}}$  is  $\mathbb{F}$  adapted, then, from (3.24)  $L_t \sigma_t = \mathbb{E}(L_t^{\mathbb{G}}(\alpha_t^{\mathbb{G}} - \mu_t^{\mathbb{G}}) | \mathcal{F}_t) = L_t(\alpha_t^{\mathbb{G}} - \mu_t^{\mathbb{G}})$  where  $L_t = \mathbb{E}(L_t^{\mathbb{G}} | \mathcal{F}_t)$ .  $\square$

**Examples 3.7** (i) If the process  $\mu^{\mathbb{G}}$  is such that

$$\mu_t^0 = 0 \quad (3.41)$$

then  $\mathbb{Q}$  is equal to  $\mathbb{P}$  on  $\mathcal{G}_\tau$  and the immersion holds for  $\mathbb{F}$  and  $\mathbb{G}$  under  $\mathbb{Q}$ . The choice  $\mu_t^0 = 0$  leads to  $\eta = \alpha^0$ , hence  $\mu_t^1(u) = \alpha_t^1(u) - \alpha_t^0 = -\varphi_t(u) - \alpha_t^0$ , for all  $t \geq u$ . Therefore, we have

$$\begin{aligned} & \mathcal{E}(\mu^1 \cdot W^{\mathbb{G}})_t \quad (3.42) \\ &= \mathbb{1}_{\{t < \tau\}} + \mathbb{1}_{\{\tau \leq t\}} \exp \left( \int_\tau^t \alpha_s^1(\tau) dW_s^{\mathbb{G}} - \frac{1}{2} \int_\tau^t (\alpha_s^1)^2 ds \right) \exp \left( - \int_\tau^t \alpha_s^0 dW_s^{\mathbb{G}} - \frac{1}{2} \int_\tau^t (\alpha_s^0)^2 ds \right) \\ &= \mathbb{1}_{\{t < \tau\}} + \mathbb{1}_{\{\tau \leq t\}} \frac{p_\tau(\tau)}{p_t(\tau)} \frac{Z_t}{Z_\tau} \end{aligned}$$

where we have used the fact that, on  $\{\tau < t\}$ , one has  $p_\tau(\tau)/p_t(\tau) = \exp\left(\int_\tau^t \alpha_s^1 dW_s + \frac{1}{2} \int_\tau^t (\alpha_s^1)^2 ds\right)$  and  $Z_t = \mathcal{E}(-\alpha^0 \cdot W)_t = G_t e^{\Lambda t}$ , for all  $t \geq 0$ . More precisely, we have

$$L_t^{\mathbb{G}} = \mathbb{1}_{\{t < \tau\}} + \mathbb{1}_{\{\tau \leq t\}} \frac{p_\tau(\tau) e^{\Lambda t} G_t}{p_t(\tau) e^{\Lambda \tau} G_\tau} \quad (3.43)$$

and

$$\eta_t = \frac{G_t}{1 - F(t)} \quad (3.44)$$

This example was presented in [1; Chapter V].

(ii) the case  $\mu^0 = \alpha^0$  leads to  $\eta = 0$  and  $\mu^1 = \alpha^1$ , that is, this is the case in which  $\mathbb{Q} = \mathbb{P}^*$ .

## 4 Equivalent Martingale Measures

consider the case where  $\nu$  and  $\sigma$  are processes Let us consider a model of a financial market in which the stock price process  $S = (S_t)_{t \geq 0}$  started at  $S_0 = 1$  satisfies the stochastic differential equations (written in various filtrations)

$$dS_t = S_t (\nu dt + \sigma dW_t) = S_t (\nu_t^{\mathbb{G}} dt + \sigma dW_t^{\mathbb{G}}) = S_t ((\nu + \sigma \varphi_t(\tau)) dt + \sigma dW_t(\tau)) \quad (4.1)$$

for some  $\sigma > 0$  fixed, where, from (2.15), we have

$$\nu_t^{\mathbb{G}} = \nu + \sigma \alpha_t^{\mathbb{G}} = \nu + \sigma \alpha_t^0 \mathbb{1}_{\{t \leq \tau\}} - \varphi_t(\tau) \mathbb{1}_{\{\tau < t\}}. \quad (4.2)$$

We assume that a riskless asset is traded with a null interest rate. It is straightforward to show that, for positive function  $\ell$  satisfying  $\mathbb{E}[\ell(\tau)] = 1$ , the positive martingale

$$Y_t(\tau) = \ell(\tau) \exp\left(\int_0^t y_s(\tau) dW_s(\tau) - \frac{1}{2} \int_0^t y_s^2(\tau) ds\right) \quad (4.3)$$

is a Radon-Nikodým density of an equivalent martingale measure on  $\mathbb{F}^{(\tau)}$  (i.e. such that  $SY(\tau)$  is an  $\mathbb{F}^{(\tau)}$ -martingale) if and only if

$$y_t(\tau) = -\varphi_t(\tau) - \frac{\nu}{\sigma} \quad (4.4)$$

for all  $t \geq 0$ . We denote by  $L(\tau) = (L_t(\tau))_{t \geq 0}$  such a positive martingale defined by

$$L_t(\tau) = L_0(\tau) \exp\left(-\int_0^t \left(\varphi_s(\tau) + \frac{\nu}{\sigma}\right) dW_s(\tau) - \frac{1}{2} \int_0^t \left(\varphi_s(\tau) + \frac{\nu}{\sigma}\right)^2 ds\right) \quad (4.5)$$



for all  $t \geq 0$ . Then, we define a new probability measure  $\tilde{\mathbb{P}}$  which is locally equivalent to  $\mathbb{P}$  on  $\mathbb{F}^{(\tau)}$  by

$$\left. \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{F}_t^{(\tau)}} = L_t(\tau) \quad (4.6)$$

for all  $t \geq 0$ . Actually, there exists infinitely many such probabilities, which differ from each other by the choice of the initial value  $L_0$ , that is, by the choice of the law of  $\tau$  (under  $\tilde{\mathbb{P}}$ ):  $\tilde{\mathbb{P}}(\tau > u) = \mathbb{E} [L_0(\tau) \mathbb{1}_{\{\tau > u\}}] = \int_u^\infty L_0(v)g(v)dv$ , for all  $u \geq 0$ . The case  $L_0 \equiv 1$  corresponds to the situation in which  $\tilde{\mathbb{P}}(\tau > u) = \mathbb{P}(\tau > u)$ , for any  $u \geq 0$ . Note that, by virtue of Girsanov's theorem, the process  $\tilde{W}(\tau)$  defined as

$$\tilde{W}_t(\tau) = W_t(\tau) + \int_0^t \left( \varphi_s(\tau) + \frac{\nu}{\sigma} \right) ds \quad (4.7)$$

is a  $(\tilde{\mathbb{P}}, \mathbb{F}^{(\tau)})$ -standard Brownian motion.

Working in the filtration  $\mathbb{G}$ , it easily seen that the set of  $\mathbb{G}$ -equivalent martingale measures (i.e. the set of positive  $\mathbb{G}$ -martingales  $Y^\mathbb{G}$  such that  $SY^\mathbb{G}$  is a  $\mathbb{G}$ -martingale) is

$$dY_t^\mathbb{G} = Y_{t-}^\mathbb{G} \left( -\frac{\nu_t^\mathbb{G}}{\sigma} dW_t^\mathbb{G} + \gamma_t^\mathbb{G} dM_t^\mathbb{G} \right) \quad (4.8)$$

so that

$$Y_t^\mathbb{G} = \mathcal{E}(-(\nu^\mathbb{G}/\sigma) \cdot W^\mathbb{G})_t \mathcal{E}(\gamma^\mathbb{G} \cdot M^\mathbb{G})_t \quad (4.9)$$

where  $\gamma$  is any  $\mathbb{G}$ -predictable process, with  $\gamma > -1$ .

Let us now consider  $L^\mathbb{G}$ , the  $\mathbb{G}$ -optional projection of  $L_t(\tau)$ . The process  $L^\mathbb{G}$  defines an equivalent martingale measure for  $S$  and can be written as in Section 3.4

$$L_t^\mathbb{G} = 1 + \int_0^t L_{s-}^\mathbb{G} \mu_s^\mathbb{G} dW_s^\mathbb{G} + \int_0^t L_{s-}^\mathbb{G} \psi_s^\mathbb{G} dM_s^\mathbb{G} = \mathcal{E}(\mu^\mathbb{G} \cdot W^\mathbb{G})_t \mathcal{E}(\psi^\mathbb{G} \cdot M^\mathbb{G})_t \quad (4.10)$$

where the processes  $\mu^\mathbb{G}$  and  $\psi^\mathbb{G}$  are given in (3.31). Using the fact that  $L_t^0 = \frac{1}{G_t} \int_0^t L_t(u)p_t(u)g(u)du$ , one has

$$\mu_t^\mathbb{G} = \mathbb{1}_{\{t \leq \tau\}} \frac{1}{G_t L_t^0} \left( \int_0^t \left( -\frac{\nu}{\sigma} - \alpha_t^0 \right) L_t(u) p_t(u) g(u) du + \mathbb{1}_{\{\tau < t\}} \varphi_t(\tau) \right) \quad (4.11)$$

$$\begin{aligned} &= -\mathbb{1}_{\{t \leq \tau\}} \left( \frac{\nu^\mathbb{G}}{\sigma} + \alpha_t^0 \right) + \mathbb{1}_{\{\tau < t\}} \varphi_t(\tau) = -\mathbb{1}_{\{t \leq \tau\}} \frac{\nu_t^\mathbb{G}}{\sigma} + \mathbb{1}_{\{\tau < t\}} \varphi_t(\tau) \\ \psi_t^\mathbb{G} &= \mathbb{1}_{\{t \leq \tau\}} \left( \frac{L_t(t)}{L_t^0} - 1 \right). \end{aligned} \quad (4.12)$$

We obtain here an equivalent martingale measures.

However, the set of all equivalent martingale measures is larger, since it is the family  $\mathcal{E}(\mu^{\mathbb{G}} \cdot W^{\mathbb{G}})\mathcal{E}(\gamma^{\mathbb{G}} \cdot M^{\mathbb{G}})$ , for any  $\mathbb{G}$ -predictable process  $\gamma^{\mathbb{G}} = (\gamma_t^{\mathbb{G}})_{t \geq 0}$  such that  $\gamma_t^{\mathbb{G}} > -1$ , for all  $t \geq 0$ . The compensated  $(\mathbb{P}, \mathbb{G})$ -martingale of  $H = (H_t)_{t \geq 0}$  with  $H_t = \mathbb{1}_{\{\tau \leq t\}}$ , for all  $t \geq 0$  is the (uniformly integrable)  $(\mathbb{P}, \mathbb{G})$ -martingale

$$M_t^{\mathbb{G}} = H_t - \int_0^{\tau \wedge t} \frac{p_s(s)g(s)}{G_s} ds = H_t - \int_0^{t \wedge \tau} \lambda_s^0 ds \quad (4.13)$$

(see, e.g., [5; Subsection 1.2] or [1]). We introduce  $L^{\mathbb{G}}$  as

$$L_t^{\mathbb{G}} = \mathcal{E}(\mu^{\mathbb{G}} \cdot W^{\mathbb{G}})\mathcal{E}(\gamma^{\mathbb{G}} \cdot M^{\mathbb{G}})$$

and the measure  $\widehat{\mathbb{P}}$  on  $\mathbb{G}$  with

$$\left. \frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{G}_t} = L_t^{\mathbb{G}} \quad (4.14)$$

By virtue of Girsanov's theorem, we have that the process  $\widehat{W}$  defines as

$$\widehat{W}_t^{\mathbb{G}} = W_t^{\mathbb{G}} - \int_0^t \mu_s^{\mathbb{G}} ds \quad (4.15)$$

is a standard Brownian motion under  $\widehat{\mathbb{P}}$  with respect to  $\mathbb{G}$ . Moreover, the process

$$\widehat{M}_t^{\mathbb{G}} = M_t^{\mathbb{G}} - \int_0^{t \wedge \tau} \gamma_s^0 \lambda_s^0 ds = H_t - \int_0^{t \wedge \tau} (1 + \gamma_s^0) \lambda_s^0 ds \quad (4.16)$$

is a (uniformly integrable)  $(\widehat{\mathbb{P}}, \mathbb{G})$ -martingale. The above change of probability changes the Brownian motion and the intensity of the default. The specific choice  $\gamma = 0$  leads to a change of probability measure which does not affect the intensity.

**Remark 4.1** Assume for simplicity that  $S$  is a martingale under  $\mathbb{P}$ . By definition  $S$  is also a  $\mathbb{Q}^{\mathbb{F}}$  martingale, hence  $W = W^{\mathbb{G}}$ . Then, for any  $\zeta_T \in \mathcal{F}_T$  and any equivalent martingale measure  $\mathbb{Q}^{\mathbb{G}}$  on  $\mathbb{G}$ , one has  $\mathbb{E}_{\mathbb{Q}^{\mathbb{G}}}[\zeta_T | \mathcal{G}_t] = \mathbb{E}_{\mathbb{P}}[\zeta_T | \mathcal{F}_t]$ , for  $0 \leq t \leq T$ . The first reason is that  $\zeta_T$  is hedgeable in  $\mathbb{F}$ , and thus in  $\mathbb{G}$ . The second reason is that from  $\zeta_T = x + \int_0^T x_s dW_s + \int_0^T x_s dW_s^{\mathbb{G}}$  and using the orthogonality of  $W^{\mathbb{G}}$  and  $M^{\mathbb{G}}$ , setting  $\zeta_t = x + \int_0^t x_s dW_s^{\mathbb{G}}$ , the product  $\mathcal{E}(\gamma \cdot M^{\mathbb{G}})_t \zeta_t$  is a martingale, hence, by Bayes' formula

$$\mathbb{E}_{\mathbb{Q}^{\mathbb{G}}}[\zeta_T | \mathcal{G}_t] = \frac{1}{\mathcal{E}(\gamma \cdot M^{\mathbb{G}})_t} \mathbb{E}_{\mathbb{P}}[\mathcal{E}(\gamma \cdot M^{\mathbb{G}})_T \zeta_T | \mathcal{G}_t] = \zeta_t \quad (4.17)$$

for  $0 \leq t \leq T$ .

**Remark 4.2** We can extend easily the study to the case where the interest rate is  $\mathbb{G}$ -adapted with  $r_t(\tau) = r_t^0 \mathbb{1}_{\{t < \tau\}} + r_t^1(\tau) \mathbb{1}_{\{\tau \leq t\}}$ , where the processes  $r^0 = (r_t^0)_{t \geq 0}$  and  $r^1(u) = (r_t^1(u))_{t \geq 0}$  are  $\mathbb{F}$ -adapted, for each  $u \geq 0$  fixed, and we define the discounted process  $\tilde{S}(\tau) = (\tilde{S}_t(\tau))_{t \geq 0}$  by

$$\tilde{S}_t(\tau) = \exp\left(-\int_0^t r_s(\tau) ds\right) S_t \quad (4.18)$$

for all  $t \geq 0$ .

## 5 Examples of conditional density

In the literature, there are some explicit examples of random times satisfying Jacod's Hypothesis, among them the Gaussian ones given in Crépey et al. [4] and the one in [10]. Here, we shall work in the same framework as in [10], with the difference that we do not make use of a change of probability measure, and we are working under the given probability  $\mathbb{P}$ .

Let  $W = (W_t)_{t \geq 0}$  be a standard Brownian motion with its natural filtration  $(\mathcal{F}_t)_{t \geq 0}$ . Let  $F$  be the cumulative distribution function defined by  $F(t) = \int_0^t g(s) ds$ , with the given probability density function  $g$ . For some constants  $\mu$  and  $\sigma > 0$ , we define  $G = (G_t)_{t \geq 0}$  as the unique strong solution of the stochastic differential equation

$$dG_t = -G_t \frac{g(t)}{1 - F(t)} dt - \frac{\mu}{\sigma} G_t (1 - G_t) dW_t \quad (G_0 = 1). \quad (5.1)$$

for all  $t \geq 0$ . It is easily seen from the expressions in (5.1) that  $G$  is a supermartingale valued in  $[0, 1]$ . We also define the process  $X = (X_t)_{t \geq 0}$  by

$$X_t = \int_0^t \mu (1 - G_s) ds + \int_0^t \sigma dW_s \quad (5.2)$$

so that, we have  $\mathcal{F}_t^X \subseteq \mathcal{F}_t$  with  $\mathcal{F}_t^X = \sigma(X_s | 0 \leq s \leq t)$ , for all  $t \geq 0$ .

Let us now provide the multiplicative decomposition for the supermartingale  $G$ . For this purpose, we define the process  $Z = (Z_t)_{t \geq 0}$  by:

$$Z_t = \exp\left(\frac{\mu}{\sigma^2} X_t - \frac{\mu^2}{2\sigma^2} t\right) \quad (5.3)$$

and the process  $Y = (Y_t)_{t \geq 0}$  by

$$Y_t = \int_0^t \frac{g(s)}{Z_s} ds + \frac{1 - F(t)}{Z_t} \quad (5.4)$$

for all  $t \geq 0$ . Then, it is shown by means of standard arguments based on Itô's formula that the process  $Z$  solves the stochastic differential equation

$$dZ_t = Z_t \frac{\mu}{\sigma^2} dX_t = \left(\frac{\mu}{\sigma}\right)^2 (1 - G_t) Z_t dt + \frac{\mu}{\sigma} Z_t dW_t \quad (Z_0 = 1) \quad (5.5)$$

so that the process  $1/Z$  takes the expression

$$d\left(\frac{1}{Z_t}\right) = -\frac{1}{Z_t} \frac{\mu}{\sigma^2} dX_t + \frac{1}{Z_t} \left(\frac{\mu}{\sigma}\right)^2 dt \quad \left(\frac{1}{Z_0} = 1\right) \quad (5.6)$$

and thus, the process  $Y$  admits the representation

$$dY_t = \left(\frac{\mu}{\sigma}\right)^2 \frac{(1 - F(t))G_t}{Z_t} dt - \frac{\mu}{\sigma} \frac{(1 - F(t))}{Z_t} dW_t \quad (Y_0 = 1). \quad (5.7)$$

Observe from the expression in (5.1) that  $G_t = N_t - A_t$  with  $dN_t = -(\mu/\sigma)G_t(1 - G_t)dW_t$  and that  $p_t(t) = G_t/(1 - F(t))$ , so that the intensity of  $\tau$  is a deterministic function  $\lambda(t) = p_t(t)g(t)/G_t = g(t)/(1 - F(t))$ , for all  $t \geq 0$ .

Let us now consider the stochastic differential

$$d(G_t Z_t Y_t) = Y_t d(G_t Z_t) + (G_t Z_t) dY_t + d\langle GZ, Y \rangle_t \quad (5.8)$$

for which we first compute, setting  $\lambda(t) = g(t)/(1 - F(t))$ ,

$$\begin{aligned} d(G_t Z_t) &= Z_t dG_t + G_t dZ_t + d\langle G, Z \rangle_t = Z_t \left( -\lambda(t) G_t dt - \frac{\mu}{\sigma} G_t (1 - G_t) dW_t \right) \\ &\quad + G_t \left( \left(\frac{\mu}{\sigma}\right)^2 (1 - G_t) Z_t dt + \frac{\mu}{\sigma} Z_t dW_t \right) - \left(\frac{\mu}{\sigma}\right)^2 G_t (1 - G_t) Z_t dt \\ &= -\lambda(t) G_t Z_t dt + \frac{\mu}{\sigma} G_t Z_t [1 - 1 + G_t] dW_t \\ &= -\lambda(t) G_t Z_t dt + \frac{\mu}{\sigma} G_t^2 Z_t dW_t. \end{aligned} \quad (5.9)$$

Then, we can proceed with computing the expression (5.8) and obtain

$$\begin{aligned} d(G_t Z_t Y_t) &= Y_t \left( -\lambda(t) G_t Z_t dt + \frac{\mu}{\sigma} G_t^2 Z_t dW_t \right) \\ &\quad + G_t Z_t \left( \left(\frac{\mu}{\sigma}\right)^2 \frac{(1 - F(t))G_t}{Z_t} dt - \frac{\mu}{\sigma} \frac{(1 - F(t))}{Z_t} dW_t \right) - \left(\frac{\mu}{\sigma}\right)^2 (1 - F(t)) G_t^2 dt \\ &= -\lambda(t) G_t Z_t Y_t dt + \frac{\mu}{\sigma} G_t [G_t Z_t Y_t - (1 - F(t))] dW_t \end{aligned} \quad (5.10)$$

so that, by using the fact that  $d(1 - F(t)) = -g(t)dt$ , we get

$$\begin{aligned} d[G_t Z_t Y_t - (1 - F(t))] &= \left( -\lambda(t) G_t Y_t Z_t + g(t) \right) dt + \frac{\mu}{\sigma} G_t [G_t Z_t Y_t - (1 - F(t))] dW_t \\ &= -\frac{g(t)}{1 - F(t)} [G_t Z_t Y_t - (1 - F(t))] dt + \frac{\mu}{\sigma} G_t [G_t Z_t Y_t - (1 - F(t))] dW_t \end{aligned} \quad (5.11)$$

from where it is seen that the process  $V_t := G_t Z_t Y_t - (1 - F(t))$  is a solution of the stochastic differential equation  $dV_t = V_t(-\lambda(t) dt + (\mu/\sigma) G_t dW_t)$ , started at  $G_0 Z_0 Y_0 - (1 - F(0)) = 0$ , and thus, we have  $G_t Z_t Y_t - (1 - F(t)) = 0$ , for all  $t \geq 0$ .

Furthermore, by applying Itô's formula we have

$$dY_t = -\frac{\mu}{\sigma^2} \frac{1 - F(t)}{Z_t} dX_t + \left(\frac{\mu}{\sigma}\right)^2 \frac{1 - F(t)}{Z_t} dt \quad (5.12)$$

and thus

$$d\left(\frac{1}{Y_t}\right) = \frac{\mu}{\sigma} \frac{1}{Y_t^2} \frac{1 - F(t)}{Z_t} dW_t = \frac{\mu}{\sigma} \frac{G_t}{Y_t} dW_t \quad (5.13)$$

and

$$d\left(\frac{1}{Z_t Y_t}\right) = \frac{1}{Z_t} d\left(\frac{1}{Y_t}\right) + \frac{1}{Y_t} d\left(\frac{1}{Z_t}\right) + d\left\langle \frac{1}{Z}, \frac{1}{Y} \right\rangle_t = -\frac{\mu}{\sigma} \frac{1 - G_t}{Z_t Y_t} dW_t \quad (5.14)$$

and hence, the processes  $1/Y$  and  $1/(ZY)$  are  $(\mathcal{F}_t)_{t \geq 0}$ -martingales. We recall that  $G_t = \frac{1 - F(t)}{L_t Y_t}$ . Then, we can construct a family of positive  $\mathbb{F}$ -martingales  $M(u) = (M_t(u))_{t \geq 0}$ , for  $u \geq 0$ , by

$$M_t(u) = \frac{1}{Y_t} \left( \int_u^{u \vee t} \frac{g(s)}{Z_s} ds + \frac{1}{Z_t} \int_{u \vee t}^\infty g(s) ds \right). \quad (5.15)$$

Indeed, from

$$M_t(u) = \mathbb{1}_{\{t < u\}} \frac{1}{Y_t Z_t} \int_u^\infty g(s) ds + \mathbb{1}_{\{u < t\}} \frac{1}{Y_t} \left( \int_u^t \frac{g(s)}{Z_s} ds + \frac{1}{Z_t} \int_t^\infty g(s) ds \right) \quad (5.16)$$

one has:

on  $\{t \leq u\}$ ,  $M_t(u) = (1 - F(u))/(Y_t Z_t)$  is an  $\mathbb{F}$ -martingale

on  $\{u < t\}$ ,  $M_t(u) = (1/Y_t) \left( \int_u^t (g(s)/Z_s) ds + (1/Z_t) \int_t^\infty g(s) ds \right)$  which leads to

$$d_t M_t(u) = \left( \int_0^u \frac{g(s)}{Z_s} ds \right) d\left(\frac{1}{Y_t}\right) + (1 - F(t)) d\left(\frac{1}{Y_t Z_t}\right) \quad (5.17)$$

so that  $M(u)$  forms a martingale, because  $1/Y$  and  $1/(ZY)$  are martingales. Note that the process  $M(u)$  is valued in  $(0, 1)$ , and  $M_t(u)$  is decreasing with respect to  $u$ , for any  $t \geq 0$  fixed, with  $M_t(0) = 1$ . Furthermore, we have  $M_t(t) = G_t$ ,  $M_0(u) = \int_u^\infty g(s) ds$ , and  $M_t(u) = \int_u^\infty p_t(s) g(s) ds$  where  $p_t(u) = g(u)/(Z_{u \wedge t} Y_t)$ .

It is therefore possible to construct, on an extended probability space, a random time  $\tau$  and a probability measure  $\mathbb{Q}$  such that  $\mathbb{Q}$  and  $\mathbb{P}$  coincide on  $\mathbb{F}$  and  $\mathbb{Q}(\tau > u | \mathcal{F}_t) = M_t(u)$ , for  $t, u \geq 0$  (see [15]). It is straightforward to check the important property  $\int_0^\infty p_t(u) g(u) du = 1$ , for any  $t \geq 0$ .

Let us now thus compute the dynamics of  $p(u)$  by means of the integration by parts and thus

$$d_t p_t(u) = d_t \left( \frac{1}{Z_{u \wedge t} Y_t} \right) = \frac{\mu}{\sigma} \left( \mathbb{1}_{\{u \leq t\}} \frac{G_t}{Z_u Y_t} - \mathbb{1}_{\{u > t\}} \frac{1 - G_t}{Z_t Y_t} \right) dW_t \quad (5.18)$$

so that

$$d_t p_t(u) = p_t(u) \frac{\mu}{\sigma} \left( \mathbb{1}_{\{u \leq t\}} G_t - \mathbb{1}_{\{u > t\}} (1 - G_t) \right) dW_t \quad (5.19)$$

and

$$\varphi_t(u) = \frac{\mu}{\sigma} \left( \mathbb{1}_{\{u \leq t\}} G_t - \mathbb{1}_{\{u > t\}} (1 - G_t) \right) = \frac{\mu}{\sigma} \left( \mathbb{1}_{\{u \leq t\}} - (1 - G_t) \right). \quad (5.20)$$

In the progressively enlarged filtration,  $dW_t^{\mathbb{G}} = dW_t - \alpha_t^{\mathbb{G}} dt$ , where

$$\begin{aligned} \alpha_t^0 &= \frac{\mu}{\sigma} \frac{1}{G_t} \int_0^t G_t \frac{g(u)}{Z_u Y_t} du = \frac{\mu}{\sigma} \frac{1}{Y_t} \int_0^t \frac{g(u)}{Z_u} du \\ &= \frac{\mu}{\sigma} \frac{1}{Y_t} \left( Y_t - \frac{1 - F(t)}{Z_t} \right) = \frac{\mu}{\sigma} \left( 1 - \frac{1 - F(t)}{Y_t Z_t} \right) = \frac{\mu}{\sigma} (1 - G_t) \end{aligned} \quad (5.21)$$

$$\alpha_t^1(\tau) = -\frac{\mu}{\sigma} G_t, \quad (5.22)$$

and

$$dW_t^{\mathbb{G}} = dW_t - \frac{\mu}{\sigma} \left( (1 - G_t) \mathbb{1}_{\{t \leq \tau\}} - G_t \mathbb{1}_{\{\tau < t\}} \right) dt. \quad (5.23)$$

We can check that the property  $\int_0^\infty \mathbb{E}(\alpha_t^{\mathbb{G}} | \mathcal{F}_t) dt = 1$  holds. Moreover, from

$$dW_t^{\mathbb{G}} = dW_t - \frac{\mu}{\sigma} \left( \mathbb{1}_{\{\tau \leq t\}} - (1 - G_t) \right) dt \quad (5.24)$$

we deduce that

$$dX_t = \mu \mathbb{1}_{\{\tau \leq t\}} dt + \sigma dW_t^{\mathbb{G}}. \quad (5.25)$$

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