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# THE SINE KUMARASWAMY-G FAMILY OF DISTRIBUTIONS

CHRISTOPHE CHESNEAU <sup>1</sup>, FARRUKH JAMAL <sup>2</sup>

**ABSTRACT.** In this paper, we introduce a new trigonometric family of continuous distributions called the sine Kumaraswamy-G family of distributions. It can be presented as a natural extension of the well-established sine-G family of distributions, with new perspectives in terms of applicability. We investigate the main mathematical properties of the sine Kumaraswamy-G family of distributions, including asymptotes, quantile function, linear representations of the cumulative distribution and probability density functions, moments, skewness, kurtosis, incomplete moments, weighted moments and order statistics. Then, we focus our attention on a special member of this family called the sine Kumaraswamy exponential distribution. The statistical inference for the related parametric model is explored by using the maximum likelihood method. Among others, asymptotic confidence intervals and likelihood ratio test for the parameters are discussed. A simulation study is performed under varying sample size to assess the performance of the model. Finally, applications to two practical data sets are presented to illustrate its potentiality and robustness.

*Keywords:* Sine-G family of distributions; Kumaraswamy distribution; moments; practical data sets.

*AMS Subject:* 9A60; 62E15; 62H10.

## 1. INTRODUCTION

In recent years, much attention has been paid to the construction of trigonometric families of distributions. The advantage of these families are to keep a balance between a relative simplicity in their definitions, allowing a perfect comprehension of their mathematical properties, and a great applicability for modelling various kinds of practical data sets. These two points follows from an appropriate use of flexible trigonometric functions. To the best of our knowledge, the former trigonometric family of distributions is the sine-G family of distributions introduced by [10]. A brief description of this family is presented below. Let  $G(x)$  be the cumulative distribution function (cdf) of an univariate continuous distribution and  $g(x)$  be the corresponding probability density function (pdf). Then, the sine-G family of distributions is characterized by the cdf given by

$$F(x) = \sin\left(\frac{\pi}{2}G(x)\right), \quad x \in \mathbb{R}. \quad (1)$$

The corresponding pdf is given by

$$f(x) = \frac{\pi}{2}g(x) \cos\left(\frac{\pi}{2}G(x)\right), \quad x \in \mathbb{R}.$$

Thus, simple functions are involved and it is proved in [10] that the flexibility of  $G(x)$  can be significantly enriched by the sine transformations. The related parametric models take advantages of these properties for a nice fitting of various kinds of data sets. We refer to [10] and [17], with the consideration of different baseline cdfs  $G(x)$ . By exploiting the flexible nature of various trigonometric transformations, other trigonometric families of distributions have been developed. See, for instance, the cos-G family of distributions by [17, Chapter 4], the tan-G family of distributions by [17, Chapter 5], the sec-G family of distributions by [17, Chapter 6], the sec-G family of distributions by [17, Chapter 6], the CS-G family of distributions by [1] and the e TransSC-G family of distributions by [8].

In this paper, we propose a new trigonometric family of continuous distributions, called the sine Kumaraswamy-G family of distributions. It can be viewed as a "two power shape parameters generalization" of the former sin-G family of distributions. We describe it as follows. Let  $a > 0$ ,  $b > 0$ ,  $G(x)$  be the cdf of an univariate continuous distribution and  $g(x)$  be the corresponding pdf. Then, the sine Kumaraswamy-G family of distributions is characterized by the cdf given by

$$F(x) = \cos\left(\frac{\pi}{2}[1 - G(x)^a]^b\right), \quad x \in \mathbb{R}. \quad (2)$$

The corresponding pdf is given by

$$f(x) = \frac{\pi}{2}abg(x)G(x)^{a-1}[1 - G(x)^a]^{b-1} \sin\left(\frac{\pi}{2}[1 - G(x)^a]^b\right), \quad x \in \mathbb{R}. \quad (3)$$

As indicated by its name, by using a trigonometric formula, we can show that  $F(x)$  is obtained by the composition of the sine-G cdf given by (1) and the Kumaraswamy-G cdf given by  $H(x) = 1 - [1 - G(x)^a]^b$ ,  $x \in \mathbb{R}$ . Further details and applications on the Kumaraswamy-G family of distributions can be found in [2], [13], [5] and [16]. The roles of  $a$  and  $b$  are to add more flexibility to the former cdf  $G(x)$ , allowing the construction of models which take into account precise characteristics of various data sets. One can notice that, for  $b = 1$ ,  $F(x)$  becomes  $F(x) = \sin((\pi/2)G(x)^a)$ , which is the cdf of the sine exp-G family of distribution (new in the literature to the best of our knowledge, but very natural to consider) and for  $a = b = 1$ , we rediscover the cdf of the sine-G family of distributions. The idea of combining trigonometric and Kumaraswamy-G families of distributions finds trace in [17, Chapter 6], but for the sec-G family of distributions (not the sine-G one) and with the specific Kumaraswamy-Weibull distribution as baseline (not the general Kumaraswamy-G family of distributions, i.e., for any  $G(x)$ ). Thus, the sine Kumaraswamy-G family of distributions remains new in the literature and deserves a complete study, which is the aim of this paper. After providing a comprehensive treatment of its mathematical properties, we focus our attention on a special member of this family, defined with the exponential distribution as baseline. It is called the sine Kumaraswamy exponential distribution. Then, we consider it as a parametric statistical model, with the estimation of the unknown parameters via the maximum likelihood method. We take benefits of the existing convergence properties of this method to present a solid model for data analysis. This is illustrated by the means of two practical sets. In particular,

we show that the proposed model is better, in some sense, to well-recognized competitive models of the literature.

The rest of the paper is organized as follows. In Section 2, the main features of the sine Kumaraswamy-G family of distributions are explored. Then, the sine Kumaraswamy exponential distribution is studied in detail in Section 3. In Section 4, it is considered as parametric model, with a statistical inference study, including concrete applications. Conclusions are given in Section 5

## 2. MAIN FEATURES

In this section, we investigate the main features of the sine Kumaraswamy-G family of distributions. We recall that it is characterized by the cdf  $F(x)$  given by (2) and the corresponding pdf  $f(x)$  is given by (3).

**2.1. Main functions.** We now express the main functions of interest of the sine Kumaraswamy-G family of distributions. The corresponding survival function (sf) is given by

$$S(x) = 1 - F(x) = 2 \left[ \sin \left( \frac{\pi}{4} [1 - G(x)^a]^b \right) \right]^2, \quad x \in \mathbb{R}.$$

The corresponding hazard rate function (hrf) is given by

$$h(x) = \frac{f(x)}{S(x)} = \frac{\pi}{2} abg(x)G(x)^{a-1} [1 - G(x)^a]^{b-1} \cot \left( \frac{\pi}{4} [1 - G(x)^a]^b \right), \quad x \in \mathbb{R}.$$

The corresponding cumulative hazard rate function (chrf) is given by

$$\Omega(x) = -\log[S(x)] = -\log(2) - 2 \log \left[ \sin \left( \frac{\pi}{4} [1 - G(x)^a]^b \right) \right], \quad x \in \mathbb{R}.$$

Another central function of the sine Kumaraswamy-G family of distributions is the quantile function (qf) given by

$$Q(y) = Q_G \left( \left[ 1 - \left\{ \frac{2}{\pi} \arccos(y) \right\}^{1/b} \right]^{1/a} \right), \quad y \in (0, 1), \quad (4)$$

where  $Q_G(y)$  denotes the qf corresponding to  $G(x)$ . Let us recall that  $Q(y)$  is characterized by the non-linear equation  $F(Q(y)) = Q(F(y)) = y$ ,  $y \in (0, 1)$ . The median is given by

$$M = Q_G \left( \left[ 1 - \left\{ \frac{2}{\pi} \arccos(0.5) \right\}^{1/b} \right]^{1/a} \right),$$

with  $\arccos(0.5) \approx 1.04719755$ . The qf is also involved in the following key result: for a random variable  $U$  having the uniform distribution on the unit interval, the random variable  $X$  given by  $X = Q(U)$  has the cdf (2). Others uses of the qf will be developed in the next.

**2.2. Asymptotic properties.** Let us now investigate the asymptotic properties of the functions  $F(x)$ ,  $f(x)$  and  $h(x)$ . As  $G(x) \rightarrow 0$ , using the equivalence  $(1 - y^a)^b \sim 1 - by^a$  when  $y \rightarrow 0$ , we have

$$F(x) \sim \frac{\pi}{2}bG(x)^a, \quad f(x) \sim \frac{\pi}{2}abg(x)G(x)^{a-1}, \quad h(x) \sim \frac{\pi}{2}abg(x)G(x)^{a-1}.$$

As  $G(x) \rightarrow 1$ , using  $\cos(y) \sim 1 - y^2/2$  when  $y \rightarrow 0$ , we have

$$F(x) \sim 1 - \frac{\pi^2}{8}[1 - G(x)^a]^{2b}, \quad f(x) \sim \frac{\pi^2}{4}abg(x)[1 - G(x)^a]^{2b-1}, \quad h(x) \sim 2abg(x)[1 - G(x)^a]^{-1}.$$

The convergence and limits of  $f(x)$  and  $h(x)$  can not be determined in full generality; they depend on  $a$ ,  $b$  and the definition of  $G(x)$  (and  $g(x)$  a fortiori).

**2.3. Critical points.** The critical points of  $f(x)$ , say  $x_0$ , satisfy the following equation:  $[\log(f(x))]'|_{x=x_0} = 0$ , i.e.,

$$\begin{aligned} \frac{g'(x_0)}{g(x_0)} + (a-1)\frac{g(x_0)}{G(x_0)} - (b-1)\frac{ag(x_0)G(x_0)^{a-1}}{1-G(x_0)^a} \\ - \frac{\pi}{2}abg(x_0)G(x_0)^{a-1}[1-G(x_0)^a]^{b-1} \cot\left(\frac{\pi}{2}[1-G(x_0)^a]^b\right) = 0. \end{aligned} \quad (5)$$

By investigating the sign of  $\tau = [\log(f(x))]''|_{x=x_0}$ , we can determine the nature of  $x_0$ ; it corresponds to a local maximum if  $\tau < 0$ , a local minimum if  $\tau > 0$  and a point of inflection if  $\tau = 0$ .

Similarly, the critical points of  $h(x)$ , say  $x_*$ , satisfy the following equation:  $[\log(h(x))]'|_{x=x_*} = 0$ , i.e.,

$$\begin{aligned} \frac{g'(x_*)}{g(x_*)} + (a-1)\frac{g(x_*)}{G(x_*)} - (b-1)\frac{ag(x_*)G(x_*)^{a-1}}{1-G(x_*)^a} \\ + \frac{\pi}{2}abg(x_*)G(x_*)^{a-1}[1-G(x_*)^a]^{b-1} \left[ \cot\left(\frac{\pi}{4}[1-G(x_*)^a]^b\right) - \cot\left(\frac{\pi}{2}[1-G(x_*)^a]^b\right) \right] = 0. \end{aligned} \quad (6)$$

Also, the sign of  $\theta = [\log(h(x))]''|_{x=x_*}$  is informative concerning the nature of  $x_*$ .

**2.4. Linear representations.** Here, some linear representations for  $F(x)$  and  $f(x)$  are determined. It follows from the series expansion of the cosine function that

$$F(x) = \cos\left(\frac{\pi}{2}[1-G(x)^a]^b\right) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!} \frac{\pi^{2k}}{2^{2k}} [1-G(x)^a]^{2bk}.$$

Then, the generalized binomial formula gives

$$[1-G(x)^a]^{2bk} = \sum_{\ell=0}^{+\infty} \binom{2bk}{\ell} (-1)^\ell G(x)^{a\ell},$$

where  $\binom{2bk}{\ell} = 2bk(2bk-1)\dots(2bk-\ell+1)/\ell!$ . We immediately deduce the following linear representation for  $F(x)$ :

$$F(x) = \sum_{\ell=0}^{+\infty} a_{\ell} G(x)^{a\ell}, \quad a_{\ell} = (-1)^{\ell} \sum_{k=0}^{+\infty} \frac{(-1)^k \pi^{2k}}{(2k)! 2^{2k}} \binom{2bk}{\ell}. \quad (7)$$

By differentiation, we obtain the following linear representation for  $f(x)$ :

$$f(x) = \sum_{\ell=0}^{+\infty} a_{\ell} [a\ell g(x) G(x)^{a\ell-1}]. \quad (8)$$

Thus, some mathematical properties of the sine Kumaraswamy-G family of distributions can be derived from these expansions and those of the properties of the exp-G family of distribution.

Alternatively, one can investigate linear representations for  $F(x)$  and  $f(x)$  in terms of the survival function corresponding to  $G(x)$ , i.e.,  $S_G(x) = 1 - G(x)$ . This can be more useful if  $S_G(x)$  is more tractable than  $G(x)$ . By using the generalized binomial formula, we have

$$G(x)^{a\ell} = \sum_{m=0}^{+\infty} \binom{a\ell}{m} (-1)^m S_G(x)^m.$$

It follows from (7) that

$$F(x) = \sum_{m=0}^{+\infty} b_m S_G(x)^m, \quad b_m = (-1)^m \sum_{\ell=0}^{+\infty} \binom{a\ell}{m} a_{\ell}. \quad (9)$$

By differentiation, we obtain the following linear representation for  $f(x)$ :

$$f(x) = \sum_{m=0}^{+\infty} b_m^* [mg(x) S_G(x)^{m-1}], \quad b_m^* = -b_m. \quad (10)$$

Applications of (9) and (10) will be proposed in Section 3 for a given cdf  $G(x)$ .

**2.5. Moments.** Hereafter, it is supposed that all the presented quantities exist (integral, sum. . .).

Let  $r$  be an integer. Then, the  $r$ -th moment of the sine Kumaraswamy-G family of distributions is given by

$$\begin{aligned} \mu'_r &= \int_{-\infty}^{+\infty} x^r f(x) dx \\ &= \int_{-\infty}^{+\infty} x^r \frac{\pi}{2} abg(x) G(x)^{a-1} [1 - G(x)^a]^{b-1} \sin\left(\frac{\pi}{2} [1 - G(x)^a]^b\right) dx. \end{aligned}$$

By applying the change of variable  $x = Q(y)$ , where  $Q(y)$  denotes the qf given by (4), we get

$$\mu'_r = \int_0^1 Q(y)^r dy = \int_0^1 \left[ Q_G \left( \left[ 1 - \left\{ \frac{2}{\pi} \arccos(y) \right\}^{1/b} \right]^{1/a} \right) \right]^r dy.$$

This integral may be not expressed simply with standard integral techniques. However, in most of the cases, for given  $G(x)$ ,  $a$ ,  $b$  and  $r$ , it can be evaluated numerically by the use of a modern mathematical software.

Alternatively, linear representations of  $\mu'_r$  can be derived to (8) or (10), according to the definition of  $G(x)$ . Indeed, by using (8), we have

$$\mu'_r = \sum_{\ell=0}^{+\infty} a_\ell \int_{-\infty}^{+\infty} x^r [a_\ell g(x) G(x)^{a\ell-1}] dx = \sum_{\ell=0}^{+\infty} a_\ell \int_0^1 [a_\ell y^{a\ell-1} Q_G(y)^r] dy.$$

Similarly, by using (10), we obtain

$$\mu'_r = \sum_{m=0}^{+\infty} b_m^* \int_{-\infty}^{+\infty} x^r [m g(x) S_G(x)^{m-1}] dx = \sum_{m=0}^{+\infty} b_m^* \int_0^1 [m y^{m-1} Q_G(1-y)^r] dy. \quad (11)$$

Especially, the mean is given by  $\mu = \mu'_1$  and the variance is given by  $\sigma^2 = \mu'_2 - \mu^2$ . Also, the  $r$ -th central moment is given by

$$\mu_r = \int_{-\infty}^{+\infty} (x - \mu)^r f(x) dx = \sum_{k=0}^r \binom{r}{k} (-1)^k (\mu'_1)^k \mu'_{r-k}$$

and the  $r$ -th descending factorial moment is given by

$$\mu'_{(r)} = \int_{-\infty}^{+\infty} x(x-1)(x-r+1)f(x)dx = \sum_{k=0}^r s_{sti}(r, k) \mu'_k,$$

where  $s_{sti}(r, k)$  denotes the Stirling number of the first kind defined by  $s_{sti}(r, k) = (1/k!) [x(x-1) \dots (x-r+1)]^{(k)} |_{x=0}$ . We end this subsection by mentioning that the moment generating function can be obtained with arguments similar to those used for  $\mu'_r$ .

**2.6. Skewness and kurtosis.** In the context of (probability) distributions, let us recall that the skewness corresponds to the asymmetry and the kurtosis corresponds to the tailedness. A skewness measure is given by

$$CS = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\mu'_3 - 3\mu'_2\mu + 2\mu^3}{\sigma^3}. \quad (12)$$

A kurtosis measure is given by

$$CK = \frac{\mu_4}{\mu_2^2} = \frac{\mu'_4 - 4\mu'_3\mu + 6\mu'_2\mu^2 - 3\mu^4}{\sigma^4}. \quad (13)$$

If the moments do not exist (mainly depending on the definition of  $G(x)$ ), we can envisage measures of skewness and kurtosis depending on the qf given by (4). For instance, for a skewness measure, we can use the Bowley skewness given by

$$B = \frac{Q(3/4) + Q(1/4) - 2Q(2/4)}{Q(3/4) - Q(1/4)}.$$

See [9]. For a kurtosis measure, we can use the Moors kurtosis given by

$$M = \frac{Q(3/8) - Q(1/8) + Q(7/8) - Q(5/8)}{Q(6/8) - Q(2/8)}.$$

Details and applications can be found in [12].

**2.7. Incomplete mean and consorts.** Let  $t \in \mathbb{R}$ . The incomplete mean of the sine Kumaraswamy-G family of distributions is given by

$$\mu^*(t) = \int_{-\infty}^t x f(x) dx = \int_{-\infty}^t x \frac{\pi}{2} abg(x)G(x)^{a-1}[1 - G(x)^a]^{b-1} \sin\left(\frac{\pi}{2}[1 - G(x)^a]^b\right) dx.$$

Equivalently, we have

$$\mu^*(t) = \int_0^{\cos(\frac{\pi}{2}[1-G(t)^a]^b)} \left[ Q_G \left( \left[ 1 - \left\{ \frac{2}{\pi} \arccos(y) \right\}^{1/b} \right]^{1/a} \right) \right]^r dy.$$

For given  $G(x)$ ,  $a$ ,  $b$  and  $t$ , this integral can be evaluated numerically.

Alternatively, we can use the linear representation given by (8) and (10). Indeed, by using (8), we have

$$\mu^*(t) = \sum_{\ell=0}^{+\infty} a_{\ell} \int_{-\infty}^t x \left[ a_{\ell} g(x) G(x)^{a_{\ell}-1} \right] dx = \sum_{\ell=0}^{+\infty} a_{\ell} \int_0^{G(t)} \left[ a_{\ell} y^{a_{\ell}-1} Q_G(y) \right] dy.$$

Similarly, by using (10), we obtain

$$\mu^*(t) = \sum_{m=0}^{+\infty} b_m^* \int_{-\infty}^t x \left[ m g(x) S_G(x)^{m-1} \right] dx = \sum_{m=0}^{+\infty} b_m^* \int_{S_G(t)}^1 \left[ m y^{m-1} Q_G(1-y) \right] dy.$$

From these expressions, several probabilistic quantities involving  $\mu^*(t)$  can be expressed. This is the case for the mean deviation about the mean given by

$$\delta_1 = \int_{-\infty}^{+\infty} |x - \mu| f(x) dx = 2\mu F(\mu) - 2\mu^*(\mu) = 2\mu \cos\left(\frac{\pi}{2}[1 - G(\mu)^a]^b\right) - 2\mu^*(\mu).$$

One can also mention the mean deviation about the median given by  $\delta_2 = \int_{-\infty}^{+\infty} |x - M| f(x) dx = \mu - 2\mu^*(M)$ , the mean residual life given by  $m(t) = [1 - \mu^*(t)]/S(t) - t$ , the mean waiting time given by  $M(t) = t - \mu^*(t)/F(t)$ , the Bonferroni curve given by  $B(y) = \mu^*(Q(y))/(y\mu)$  with  $y \in (0, 1)$  and the Lorenz curve given by  $L(y) = \mu^*(Q(y))/\mu$  with  $y \in (0, 1)$ .



**2.8. Weighted moments.** Let  $r$  and  $s$  be two integers. We now investigate the  $(r, s)$ -th weighted moment of the sine Kumaraswamy-G family of distributions defined by

$$\begin{aligned}\mu'_{r,s} &= \int_{-\infty}^{+\infty} x^r F(x)^s f(x) dx \\ &= \int_{-\infty}^{+\infty} x^r \left[ \cos \left( \frac{\pi}{2} [1 - G(x)^a]^b \right) \right]^s \frac{\pi}{2} abg(x)G(x)^{a-1}[1 - G(x)^a]^{b-1} \sin \left( \frac{\pi}{2} [1 - G(x)^a]^b \right) dx.\end{aligned}$$

Note that  $\mu'_{r,0} = \mu'_r$ . Such weighted moment naturally appears for the moment of the order statistics, as we will see later. Another expression of  $\mu'_{r,s}$  is given by

$$\mu'_{r,s} = \int_0^1 y^s \left[ Q_G \left( \left[ 1 - \left\{ \frac{2}{\pi} \arccos(y) \right\}^{1/b} \right]^{1/a} \right) \right]^r dy.$$

For given  $G(x)$ ,  $a$ ,  $b$ ,  $r$  and  $s$ , this integral can be evaluated numerically.

Alternatively, one can also investigate a linear representation for  $\mu'_{r,s}$  in terms of (raw) moments. Indeed, by applying a result established by [6, Section 0.314], we have

$$\begin{aligned}F(x)^{s+1} &= \left[ \cos \left( \frac{\pi}{2} [1 - G(x)^a]^b \right) \right]^{s+1} = \left[ \sum_{k=0}^{+\infty} \frac{(-1)^k \pi^{2k}}{(2k)! 2^{2k}} [1 - G(x)^a]^{2bk} \right]^{s+1} \\ &= \sum_{k=0}^{+\infty} c_{s,k} [1 - G(x)^a]^{2bk},\end{aligned}$$

where  $c_{s,0} = 1$  and, for any  $k \geq 1$ ,

$$c_{s,k} = \frac{1}{k} \sum_{\ell=1}^k [\ell(s+2) - k] \frac{(-1)^\ell \pi^{2\ell}}{(2\ell)! 2^{2\ell}} c_{s,k-\ell}.$$

The generalized binomial formula gives

$$[1 - G(x)^a]^{2bk} = \sum_{\ell=0}^{+\infty} \binom{2bk}{\ell} (-1)^\ell G(x)^{a\ell}.$$

So,

$$F(x)^{s+1} = \sum_{\ell=0}^{+\infty} d_{s,\ell} G(x)^{a\ell}, \quad d_{s,\ell} = (-1)^\ell \sum_{k=0}^{+\infty} c_{s,k} \binom{2bk}{\ell}.$$

Hence, by differentiation, we have

$$F(x)^s f(x) = \sum_{\ell=0}^{+\infty} d_{s,\ell}^* \left[ a\ell g(x)G(x)^{a\ell-1} \right], \quad d_{s,\ell}^* = \frac{d_{s,\ell}}{s+1}.$$

Therefore,

$$\mu'_{r,s} = \sum_{\ell=0}^{+\infty} d_{s,\ell}^* \int_{-\infty}^{+\infty} x^r [alg(x)G(x)^{a\ell-1}] dx = \sum_{\ell=0}^{+\infty} d_{s,\ell}^* \int_0^1 [aly^{a\ell-1}Q_G(y)^r] dy. \quad (14)$$

In terms of  $S_G(x)$ , by using the generalized binomial formula, we have

$$F(x)^{s+1} = \sum_{m=0}^{+\infty} e_{s,m} S_G(x)^m, \quad e_{s,m} = (-1)^m \sum_{\ell=0}^{+\infty} d_{s,\ell} \binom{\alpha\ell}{m}.$$

Hence, by differentiation, we have

$$F(x)^s f(x) = \sum_{m=0}^{+\infty} e_{s,m}^* [mg(x)S_G(x)^{m-1}], \quad e_{s,m}^* = -\frac{e_{s,\ell}}{s+1}.$$

So,

$$\mu'_{r,s} = \sum_{m=0}^{+\infty} e_{s,m}^* \int_{-\infty}^{+\infty} x^r [mg(x)S_G(x)^{m-1}] dx = \sum_{m=0}^{+\infty} e_{s,m}^* \int_0^1 [my^{m-1}Q_G(1-y)^r] dy. \quad (15)$$

**2.9. Order statistics.** Here, we focus on the order statistics related to the sine Kumaraswamy-G family of distributions. Let  $X_1, \dots, X_n$  be the random sample having the sine Kumaraswamy-G cdf given by (2) and  $X_{i:n}$  be the  $i$ -th order statistic, i.e., the  $i$ -th random variable such that, by arranging  $X_1, \dots, X_n$  in increasing order, we have  $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ . The complete theory about order statistics can be found in [4]. In particular, in our mathematical context, the cdf of  $X_{i:n}$  is given by

$$\begin{aligned} F_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} \frac{(-1)^k}{k+i} \binom{n-i}{k} F(x)^{k+i} \\ &= \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} \frac{(-1)^k}{k+i} \binom{n-i}{k} \left[ \cos\left(\frac{\pi}{2}[1-G(x)^a]^b\right) \right]^{k+i}, \quad x > 0. \end{aligned}$$

The corresponding pdf is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} f(x) F(x)^{k+i-1} \\ &= \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} S(x)^{n-i} \\ &= \frac{n!}{(i-1)!(n-i)!} 2^{n-i-1} \pi abg(x) G(x)^{a-1} [1-G(x)^a]^{b-1} \sin\left(\frac{\pi}{2}[1-G(x)^a]^b\right) \\ &\quad \left[ \cos\left(\frac{\pi}{2}[1-G(x)^a]^b\right) \right]^{i-1} \left[ \sin\left(\frac{\pi}{4}[1-G(x)^a]^b\right) \right]^{2(n-i)}. \end{aligned}$$

In particular, the pdf corresponding to the first order statistics, i.e.,  $X_{1:n} = \inf(X_1, X_2, \dots, X_n)$ , is given by

$$f_{1:n}(x) = n2^{n-2}\pi abg(x)G(x)^{a-1}[1 - G(x)^a]^{b-1} \sin\left(\frac{\pi}{2}[1 - G(x)^a]^b\right) \left[\sin\left(\frac{\pi}{4}[1 - G(x)^a]^b\right)\right]^{2(n-1)}$$

and the pdf corresponding to the last order statistics, i.e.,  $X_{n:n} = \sup(X_1, X_2, \dots, X_n)$ , is given by

$$f_{n:n}(x) = n\frac{\pi}{2}abg(x)G(x)^{a-1}[1 - G(x)^a]^{b-1} \sin\left(\frac{\pi}{2}[1 - G(x)^a]^b\right) \left[\cos\left(\frac{\pi}{2}[1 - G(x)^a]^b\right)\right]^{n-1}.$$

Several kinds of moments can be obtained from  $f_{i:n}(x)$ . In particular, the  $r$ -th moment of  $X_{i:n}$  is given by

$$\mu_r^o = \mathbb{E}(X_{i:n}^r) = \int_{-\infty}^{+\infty} x^r f_{i:n}(x) dx.$$

It can be calculated at least numerically for given  $G(x)$ ,  $a$ ,  $b$  and  $r$ . Alternatively, it can be expressed via the weighted moments given by (14). Indeed, we have

$$\begin{aligned} \mu_r^o &= \int_{-\infty}^{+\infty} x^r f_{i:n}(x) dx \\ &= \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} \int_{-\infty}^{\infty} x^r f(x) F(x)^{k+i-1} dx \\ &= \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} \mu'_{r,k+i-1}. \end{aligned} \quad (16)$$

Again, this integral can be evaluated numerically.

### 3. THE SINE KUMARASWAMY EXPONENTIAL DISTRIBUTION

This section is devoted to a special member of the sine Kumaraswamy-G family of distributions called the sine Kumaraswamy exponential (SKE) distribution.

**3.1. Definition and main functions.** As indicated by its name, the SKE distribution is the member of the sine Kumaraswamy-G family of distributions defined with the exponential distribution with parameter  $\lambda > 0$  as baseline. Hence, it is characterized by the cdf given by (2) with the cdf  $G(x) = 1 - e^{-\lambda x}$ ,  $x > 0$ , i.e.,

$$F(x) = \cos\left(\frac{\pi}{2}[1 - (1 - e^{-\lambda x})^a]^b\right), \quad x > 0. \quad (17)$$

One can remark that, for  $a = b = 1$ , we have  $F(x) = \cos((\pi/2)e^{-\lambda x})$ , the cdf of the SE distribution introduced by [10].

The pdf corresponding to (17) is given by

$$f(x) = \frac{\pi}{2} ab \lambda e^{-\lambda x} (1 - e^{-\lambda x})^{a-1} [1 - (1 - e^{-\lambda x})^a]^{b-1} \sin\left(\frac{\pi}{2} [1 - (1 - e^{-\lambda x})^a]^b\right), \quad x > 0. \quad (18)$$

The corresponding sf is given by

$$S(x) = 2 \left[ \sin\left(\frac{\pi}{4} [1 - (1 - e^{-\lambda x})^a]^b\right) \right]^2, \quad x \in \mathbb{R}.$$

The corresponding hrf is given by

$$h(x) = \frac{\pi}{2} ab \lambda e^{-\lambda x} (1 - e^{-\lambda x})^{a-1} [1 - (1 - e^{-\lambda x})^a]^{b-1} \cot\left(\frac{\pi}{4} [1 - (1 - e^{-\lambda x})^a]^b\right), \quad x > 0.$$

The corresponding chrf is given by

$$\Omega(x) = -\log(2) - 2 \log \left[ \sin\left(\frac{\pi}{4} [1 - (1 - e^{-\lambda x})^a]^b\right) \right], \quad x > 0.$$

The corresponding qf is given by

$$Q(y) = -\frac{1}{\lambda} \log \left( 1 - \left[ 1 - \left\{ \frac{2}{\pi} \arccos(y) \right\}^{1/b} \right]^{1/a} \right), \quad y \in (0, 1). \quad (19)$$

Median, quartiles and octiles can be derived, as well as other results.

**3.2. Some properties.** All the properties exhibited in Section 2 for the general sine Kumaraswamy-G family of distributions can be applied for the SKE distribution with the functions  $G(x) = 1 - e^{-\lambda x}$ ,  $x > 0$ ,  $g(x) = \lambda e^{-\lambda x}$  and  $Q_G(y) = -(1/\lambda) \log(1 - y)$ ,  $y \in (0, 1)$ . The most significant of them, with numerical illustrations, are presented below.

As  $x \rightarrow 0$ , we have

$$F(x) \sim \frac{\pi}{2} b \lambda^a x^a, \quad f(x) \sim \frac{\pi}{2} ab \lambda^a x^{a-1}, \quad h(x) \sim \frac{\pi}{2} ab \lambda^a x^{a-1}.$$

We can remark that, if  $a < 1$ , we have  $f(x) \rightarrow +\infty$ , if  $a = 1$ , we have  $f(x) \rightarrow (\pi/2)b\lambda$ , and if  $a > 1$ , we have  $f(x) \rightarrow 0$ . The same limits hold for  $h(x)$ .

As  $x \rightarrow +\infty$ , we have

$$F(x) \sim 1 - \frac{\pi^2}{8} a^{2b} e^{-2b\lambda x}, \quad f(x) \sim \frac{\pi^2}{4} b \lambda a^{2b} e^{-2b\lambda x}, \quad h(x) \sim 2ab\lambda.$$

Therefore, for all the values of the parameters, we have  $f(x) \rightarrow 0$  and  $h(x) \rightarrow 2ab\lambda$ .

By using (5) and (6), the critical points of  $f(x)$ , say  $x_0$ , satisfy the following equation:

$$\begin{aligned} & -\lambda^2 + (a-1) \frac{\lambda e^{-\lambda x_0}}{1 - e^{-\lambda x_0}} - (b-1) \frac{a \lambda e^{-\lambda x_0} (1 - e^{-\lambda x_0})^{a-1}}{1 - (1 - e^{-\lambda x_0})^a} \\ & - \frac{\pi}{2} ab \lambda e^{-\lambda x_0} (1 - e^{-\lambda x_0})^{a-1} [1 - (1 - e^{-\lambda x_0})^a]^{b-1} \cot\left(\frac{\pi}{2} [1 - (1 - e^{-\lambda x_0})^a]^b\right) = 0 \end{aligned}$$

and the critical points of  $h(x)$ , say  $x_*$ , satisfy the following equation:

$$\begin{aligned} & -\lambda^2 + (a-1)\frac{\lambda e^{-\lambda x_*}}{1-e^{-\lambda x_*}} - (b-1)\frac{a\lambda e^{-\lambda x_*}(1-e^{-\lambda x_*})^{a-1}}{1-(1-e^{-\lambda x_*})^a} \\ & + \frac{\pi}{2}ab\lambda e^{-\lambda x_*}(1-e^{-\lambda x_*})^{a-1}[1-(1-e^{-\lambda x_*})^a]^{b-1} \times \\ & \left[ \cot\left(\frac{\pi}{4}[1-(1-e^{-\lambda x_*})^a]^b\right) - \cot\left(\frac{\pi}{2}[1-(1-e^{-\lambda x_*})^a]^b\right) \right] = 0. \end{aligned}$$

They can be evaluated numerically. We illustrate the shapes of  $f(x)$  and  $h(x)$  in Figure 1 for selected values of  $a$ ,  $b$  and  $\lambda$ .

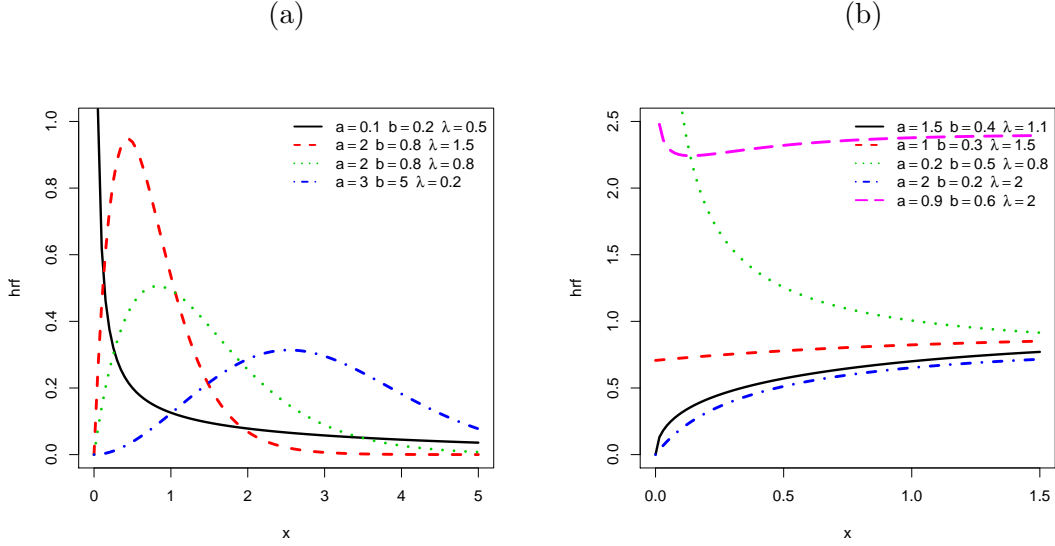


FIGURE 1. Plots of some (a) SKE pdfs and (b) SKE hrfs for selected values of  $a$ ,  $b$  and  $\lambda$ .

Also, by (10), we can express  $f(x)$  as an infinite linear combinations of exponential pdfs, i.e.,

$$f(x) = \sum_{m=0}^{+\infty} b_m^* [m\lambda e^{-\lambda m x}], \quad x > 0.$$

Let  $r$  be an integer. Then, the  $r$ -th moment of the SKE distribution exists. It can be expressed as integrals as in (11) or as the following linear representation:

$$\mu'_r = \sum_{m=0}^{+\infty} b_m^* \int_0^{+\infty} x^r [m\lambda e^{-\lambda m x}] dx = \lambda^{-r} \Gamma(r+1) \sum_{m=0}^{+\infty} b_m^* m^{-r},$$

where  $\Gamma(x) = \int_0^{+\infty} u^{x-1}e^{-u}du$  (the gamma function). Table 1 presents the numerical values of the moments of order 1, 2, 3 and 4, the variance  $\sigma^2$ , the coefficient of skewness  $CS$  and the coefficient of kurtosis  $CK$  defined by (12) and (13) respectively, for selected values of  $a$ ,  $b$  and  $\lambda$ .

TABLE 1. Some moments, skewness and kurtosis of  $X$  for SKE distribution for the following selected parameters values in order  $(a, b, \lambda)$ ; (i): (1, 2, 5), (ii): (3, 2, 5), (iii): (1.5, 1, 5) (iv): (1.5, 0.5, 0.5) (v): (5, 6, 0.5) and (vi): (30, 6, 0.5) .

|            | (i)     | (ii)    | (iii)  | (iv)      | (v)       | (vi)   |
|------------|---------|---------|--------|-----------|-----------|--------|
| $\mu'_1$   | 0.1984  | 0.0974  | 0.0742 | 0.2809    | 10.7531   | 1.3342 |
| $\mu'_2$   | 0.05637 | 0.0128  | 0.2613 | 2575.5120 | 126.3263  | 1.8944 |
| $\mu'_3$   | 0.02077 | 0.0020  | 0.4247 | 172252.4  | 1599.5430 | 2.8323 |
| $\mu'_4$   | 0.0094  | 0.0004  | 0.0005 | 0.9924    | 21625.63  | 4.4266 |
| $\sigma^2$ | 0.0169  | 0.0033  | 0.0049 | 0.1823    | 10.6944   | 0.1141 |
| $CS$       | 1.2845  | 1.00650 | 1.8457 | 3.1944    | 0.3182    | 0.0014 |
| $CK$       | 5.5791  | 4.4545  | 8.1281 | 18.6453   | 3.1172    | 2.8808 |

Other kinds of moments can be expressed. For instance, for  $t \geq 0$ , the incomplete  $r$ -th moment of the SKE distribution is given by

$$\mu_r^*(t) = \sum_{m=0}^{+\infty} b_m^* \int_0^t x^r [m\lambda e^{-\lambda mx}] dx = \lambda^{-r} \sum_{m=0}^{+\infty} b_m^* m^{-r} \gamma(r+1, \lambda mt),$$

where  $\gamma(x, t) = \int_0^t u^{x-1}e^{-u}du$  (the lower incomplete gamma function).

Similarly, using (15), the  $r$ -th weighted moment of the SKE distribution is given by

$$\mu'_{r,s} = \sum_{m=0}^{+\infty} e_{s,m}^* \int_0^{+\infty} x^r [m\lambda e^{-\lambda mx}] dx = \lambda^{-r} \Gamma(r+1) \sum_{m=0}^{+\infty} e_{s,m}^* m^{-r},$$

Finally, we mention that all the results on order statistics are presented in Subsection 2.9 can be applied, with the use of the weighted moments to express the (raw) moments of the  $i$ -th order statistic, as described in (16).

#### 4. ESTIMATION, SIMULATION AND APPLICATIONS

In this section, we investigate the SKE model governed by the cdf given by (17) (and the pdf given by (18)).

**4.1. Estimation.** We now investigate the estimation of the parameters  $a$ ,  $b$  and  $\lambda$  of the SKE model by using the maximum likelihood method, ensuring nice convergence properties of the obtained estimates called the maximum likelihood estimates (MLEs). Among others, they can be used to construct approximate confidence intervals for  $a$ ,  $b$  and  $\lambda$  and test statistics. The essential of the method adapted to the SKE distribution

is presented below. Let  $x_1, \dots, x_n$  be a sample of size  $n$  from the SKE distribution with parameters  $a, b$  and  $\lambda$ . The likelihood function for the vector of parameters  $\Theta = (a, b, \lambda)^\top$  is given by

$$\begin{aligned} L(\Theta) &= \prod_{i=1}^n f(x_i) \\ &= \left(\frac{\pi}{2} ab\lambda\right)^n \prod_{i=1}^n e^{-\lambda x_i} (1 - e^{-\lambda x_i})^{a-1} [1 - (1 - e^{-\lambda x_i})^a]^{b-1} \sin\left(\frac{\pi}{2} [1 - (1 - e^{-\lambda x_i})^a]^b\right). \end{aligned}$$

The corresponding log-likelihood function is given by

$$\begin{aligned} \ell(\Theta) &= \log [L(\Theta)] \\ &= n \log\left(\frac{\pi}{2}\right) + n \log(a) + n \log(b) + n \log(\lambda) - \lambda \sum_{i=1}^n x_i + (a-1) \sum_{i=1}^n \log(1 - e^{-\lambda x_i}) \\ &\quad + (b-1) \sum_{i=1}^n \log[1 - (1 - e^{-\lambda x_i})^a] + \sum_{i=1}^n \log\left[\sin\left(\frac{\pi}{2} [1 - (1 - e^{-\lambda x_i})^a]^b\right)\right]. \end{aligned}$$

Then, the corresponding score vector is given by  $U(\Theta) = (U_a(\Theta), U_b(\Theta), U_\lambda(\Theta))^\top$  with

$$\begin{aligned} U_a(\Theta) &= \frac{\partial}{\partial a} \ell(\Theta) = \frac{n}{a} + \sum_{i=1}^n \log(1 - e^{-\lambda x_i}) - (b-1) \sum_{i=1}^n \frac{(1 - e^{-\lambda x_i})^a \log(1 - e^{-\lambda x_i})}{1 - (1 - e^{-\lambda x_i})^a} \\ &\quad - \frac{\pi}{2} b \sum_{i=1}^n (1 - e^{-\lambda x_i})^a \log(1 - e^{-\lambda x_i}) [1 - (1 - e^{-\lambda x_i})^a]^{b-1} \cot\left(\frac{\pi}{2} [1 - (1 - e^{-\lambda x_i})^a]^b\right), \\ U_b(\Theta) &= \frac{\partial}{\partial b} \ell(\Theta) = \frac{n}{b} + \sum_{i=1}^n \log[1 - (1 - e^{-\lambda x_i})^a] \\ &\quad + \frac{\pi}{2} \sum_{i=1}^n [1 - (1 - e^{-\lambda x_i})^a]^b \log[1 - (1 - e^{-\lambda x_i})^a] \cot\left(\frac{\pi}{2} [1 - (1 - e^{-\lambda x_i})^a]^b\right), \\ U_\lambda(\Theta) &= \frac{\partial}{\partial \lambda} \ell(\Theta) = \frac{n}{\lambda} - \sum_{i=1}^n x_i + (a-1) \sum_{i=1}^n \frac{x_i e^{-\lambda x_i}}{1 - e^{-\lambda x_i}} - a(b-1) \sum_{i=1}^n \frac{x_i e^{-\lambda x_i} (1 - e^{-\lambda x_i})^{a-1}}{1 - (1 - e^{-\lambda x_i})^a} \\ &\quad - \frac{\pi}{2} ab \sum_{i=1}^n x_i e^{-\lambda x_i} (1 - e^{-\lambda x_i})^{a-1} [1 - (1 - e^{-\lambda x_i})^a]^{b-1} \cot\left(\frac{\pi}{2} [1 - (1 - e^{-\lambda x_i})^a]^b\right). \end{aligned}$$

The MLEs of  $a, b$  and  $\lambda$ , denoted by  $\hat{a}, \hat{b}$  and  $\hat{\lambda}$  respectively, satisfy the system  $U(\hat{\Theta}) = (0, 0, 0)^\top$ , with  $\hat{\Theta} = (\hat{a}, \hat{b}, \hat{\lambda})^\top$ . There are no closed form for these estimates. However, they can be obtained numerically with useful iterative algorithms (see [14]). Under regularity conditions, the subjacent distribution of  $\hat{\Theta}$  can be approximated by a 3 dimensional normal distribution with mean  $\Theta$  and covariance matrix given by  $J(\hat{\Theta})^{-1}$ , where

$J(\Theta) = -\partial^2 \ell(\Theta) / \partial \Theta \partial \Theta^T$ . Then, for  $h \in \{a, b, \lambda\}$ , an approximate confidence interval for  $h$  at the level  $100(1 - \omega)\%$  is given by

$$CI_h = [\hat{h} - z_\omega s_{\hat{h}}, \hat{h} + z_\omega s_{\hat{h}}], \quad (20)$$

where  $s_{\hat{h}}$  is the square-root of the diagonal element of  $J(\hat{\Theta})^{-1}$  at the same position as  $h$  and  $z_\omega = Q_Z(1 - \omega/2)$ , where  $Q_Z(x)$  is the quantile function of a standard normal random variable  $Z$ . Note that, for  $\omega = 0.05$ , we have  $z_\omega = 1.959964$  and for  $\omega = 0.01$ , we have  $z_\omega = 2.575829$ .

The likelihood ratio (LR) statistics for testing goodness-of-fit of the SKE model with its sub-models can also be described. Thus, we can consider hypotheses of the form:  $H_0 : \Theta = \Theta_0$  versus  $H_1 : \Theta \neq \Theta_0$ , where  $\Theta_0$  denotes a vector of 3 fixed values. In this case, the LR statistics is given by

$$LR = 2[\ell(\hat{\Theta}) - \ell(\Theta_0)], \quad (21)$$

where  $\hat{\Theta}_0$  contains the MLEs of  $a$ ,  $b$  and  $\lambda$  under  $H_0$ . Then, if  $H_0$  is assumed to be true, the subjacent distribution of  $LR$  converges in distribution to a random variable  $K$  following the chi square distribution with  $r$  degrees of freedom, where  $r$  is equal to the difference between the number of parameters estimated in the general case and the number of parameters estimated under  $H_0$ . The corresponding p-value is given by

$$p = \mathbb{P}(K > LR). \quad (22)$$

In our study, it is useful to check if the SKE model is superior to a fit using the SE model, i.e, with cdf  $F(x) = \cos((\pi/2)e^{-\lambda x})$ , for a given data set.

**4.2. Simulation.** The following result in distribution holds. For a random variable  $U$  following the uniform distribution on the unit interval, by using the qf given by (19), the random variable  $X$  defined by

$$X = Q(U) = -\frac{1}{\lambda} \log \left( 1 - \left[ 1 - \left\{ \frac{2}{\pi} \arccos(U) \right\}^{1/b} \right]^{1/a} \right)$$

follows the SKE distribution with parameters  $a$ ,  $b$  and  $\lambda$ . Based on this result, we can simulate data distributed following the SKE distribution. Here, we use this result to evaluate the performance of the MLEs of the SKE parameters via a graphical Monte Carlo simulation study. All the computations are done by using the software R. We generate  $N = 3000$  samples of size  $n = 20, 40, \dots, 500$  from the SKE distribution with true parameters values I:  $a = 2.5$ ,  $b = 5$ ,  $\lambda = 1.5$ , II:  $a = 2.5$ ,  $b = 3$ ,  $\lambda = 1.5$  and III:  $a = 2.5$ ,  $b = 5.5$ ,  $\lambda = 3$ . We also calculate the mean square error (MSE) of the MLEs empirically. For  $h \in \{a, b, \lambda\}$ , we consider the empirical MSE corresponding to  $h$  is given by

$$MSE_h = \frac{1}{N} \sum_{i=1}^N (\hat{h}_i - h)^2,$$



where  $\hat{h}_i$  denotes the MLE of  $h$  obtained at the  $i$ -th repetition of the simulation. The obtained results are given in Figures 2, 3 and 4.

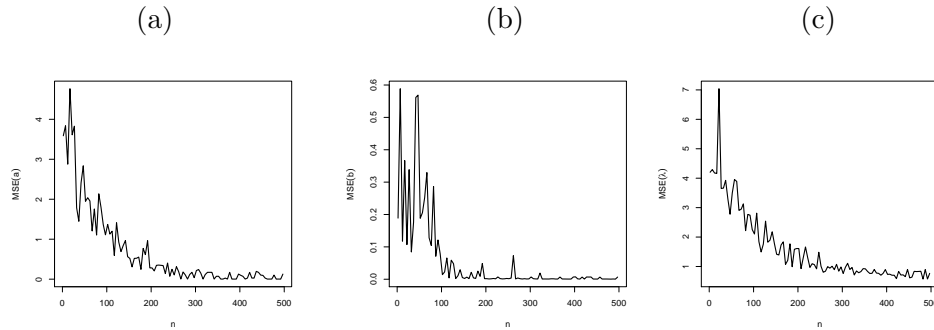


FIGURE 2. The MSE plots for the selected parameter values I for the SKE distribution, i.e.,  $a = 2.5$ ,  $b = 5$ ,  $\lambda = 1.5$ .

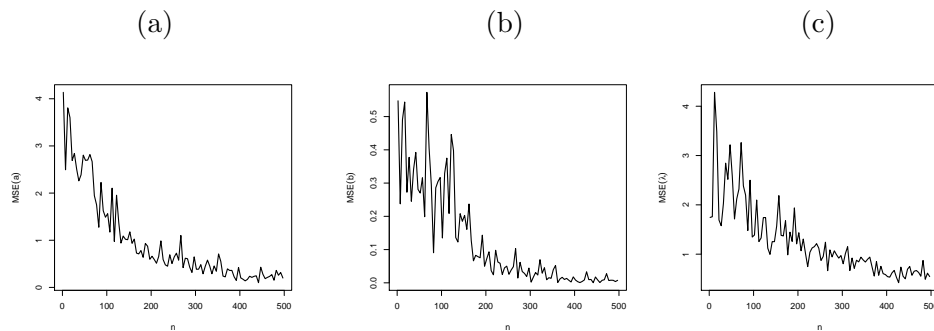


FIGURE 3. The MSE plots for the selected parameter values II for the SKE distribution, i.e.,  $a = 2.5$ ,  $b = 3$ ,  $\lambda = 1.5$ .

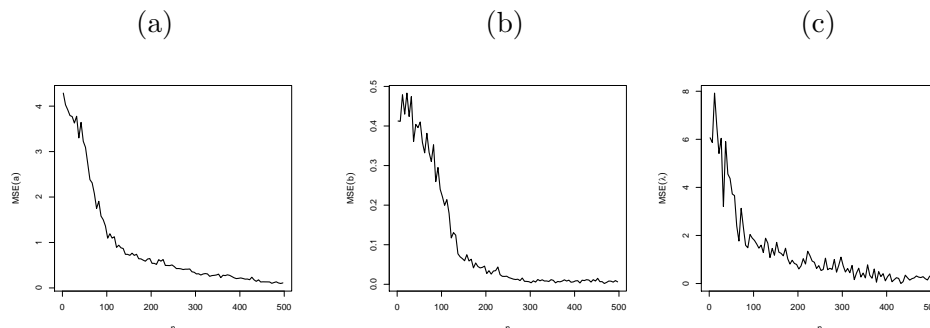


FIGURE 4. The MSE plots for the selected parameter values III for the SKE distribution, i.e.,  $a = 2.5$ ,  $b = 5.5$ ,  $\lambda = 3$ .

In each figure, we observe that, when the sample size increases, the empirical MSEs tend to zero in all cases. This is consistent with the subjacent theory of the MLEs ensuring the theoretical  $\mathbb{L}^2$  convergence.

**4.3. Applications.** In this subsection, the flexibility of the SKE model is shown by means of two real data sets. Also, the SKE model is compared with the four competitive models listed in Table 2. The following standard statistics are used:  $-\hat{\ell}$  (where  $\hat{\ell}$  denotes the maximized log-likelihood), AIC (Akaike information criterion), BIC (Bayesian information criterion), CVM (Cramér-Von Mises), AD (Anderson-Darling) and KS (Kolmogorov Smirnov), consistent Akaike information criterion (CAIC), and Hannan-Quinn information criterion (HQIC). All the computations are done by using the software R.

TABLE 2. The considered competitive models of the SKE model.

| Model  | Reference |
|--|-----------|
| Kumaraswamy Weibull (KW)                     | [3]       |
| Beta Weibull (BW)                            | [11]      |
| CS transformation of exponential ( $CS1_E$ ) | [1]       |
| Exponential (E)                              | Standard  |

The first application is a real data set given by [7]. It consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul. The data are: 0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

The second data set represent the tensile strength data measured in GPa for single carbon fibers. It is from [15]. The data are: 0.312, 0.314, 0.479, 0.552, 0.700, 0.803, 0.861,

0.865, 0.944, 0.958, 0.966, 0.997, 1.006, 1.021, 1.027, 1.055, 1.063, 1.098, 1.140, 1.179, 1.224, 1.240, 1.253, 1.270, 1.272, 1.274, 1.301, 1.301, 1.359, 1.382, 1.382, 1.426, 1.434, 1.435, 1.478, 1.490, 1.511, 1.514, 1.535, 1.554, 1.566, 1.570, 1.586, 1.629, 1.633, 1.642, 1.648, 1.684, 1.697, 1.726, 1.770, 1.773, 1.800, 1.809, 1.818, 1.821, 1.848, 1.880, 1.954, 2.012, 2.067, 2.084, 2.090, 2.096, 2.128, 2.233, 2.433, 2.585, 2.585.

**Analysis of data set 1.** For data set 1, descriptive statistics are given in Table 3. In particular, we see that the subjacent distribution of data set 1 is left-skewed (skewness estimated to 1.0866) with a non-negligible tail (kurtosis estimated to 1.2068). Table 4 provides the values of goodness-of-fit measures for the SKE model and other fitted models. We see that the SKE model has the lowest statistics, indicating that it provides a better fit to the considered competitors. The MLEs and their corresponding standard errors (SEs) (in parentheses) are listed in Table 5. The probability-probability (P-P), quantile-quantile (Q-Q), empirical probability density function (epdf) and empirical cumulative density function (ecdf) plots of the SKE are shown in Figure 5. In each cases, a nice fit is observed, indicating that the SKE model is appropriate for the analysis of data set 1. To complete this analysis, we provide in Table 6 the approximation confidence intervals of the parameters of the SKE model (see (20)). The levels 95% and 99% are considered. Finally, a LR test with the hypotheses:  $H_0 : a = b = 1$  versus  $H_1 : a \neq 1$  or  $b \neq 1$ , is performed in Table 7 (the formulas (21) and (22) are used). The p-value, using the chi square distribution with 2 degree of freedom, satisfies p-value  $< 0.0001$ . This shows the importance of the parameters  $a$  and  $b$  in terms of fit for data set 1 in comparison to the former SE model.

TABLE 3. Some descriptive statistics for data set 1.

| Statistics | N  | Mean   | Median | Variance | skewness | kurtosis |
|------------|----|--------|--------|----------|----------|----------|
| Data set 1 | 30 | 1.6750 | 1.4700 | 1.0012   | 1.0866   | 1.2068   |

TABLE 4. Goodness-of-fit measures for data set 1.

| Model   | $-\widehat{\ell}$ | AIC     | BIC     | CAIC    | HQIC    | KS     | CVM    | AD     |
|---------|-------------------|---------|---------|---------|---------|--------|--------|--------|
| SKE     | 36.8774           | 81.1549 | 85.3585 | 82.0780 | 82.4997 | 0.0635 | 0.0112 | 0.1041 |
| KW      | 37.9766           | 83.9533 | 89.5581 | 85.5533 | 85.7463 | 0.0681 | 0.0148 | 0.1065 |
| BW      | 38.0700           | 84.1400 | 89.7448 | 85.7400 | 85.9330 | 0.0631 | 0.0144 | 0.1045 |
| $CS1_E$ | 41.8522           | 89.7045 | 93.9081 | 90.6276 | 91.0493 | 0.0964 | 0.0702 | 0.4887 |
| E       | 45.4743           | 92.9480 | 94.3499 | 93.0916 | 93.3970 | 0.2351 | 0.0195 | 0.1086 |

TABLE 5. MLEs and SEs (in parentheses) for data set 1.

| Model                       | Estimates |          |          |          |
|-----------------------------|-----------|----------|----------|----------|
| SKE                         | 3.7201    | 0.3802   | 1.5250   |          |
| $(a, b, \lambda)$           | (0.6010)  | (0.1086) | (0.2959) |          |
| KW                          | 2.8788    | 0.1685   | 2.9571   | 1.4502   |
| $(a, b, \alpha, \beta)$     | (1.4350)  | (0.0467) | (0.1595) | (0.1688) |
| BW                          | 0.3536    | 0.8078   | 4.4861   | 5.5074   |
| $(a, b, \alpha, \beta)$     | (2.7762)  | (0.9862) | (9.9203) | (2.1934) |
| $CS1_E$                     | 0.8412    | 9.7350   | 0.5383   |          |
| $(\alpha, \theta, \lambda)$ | (1.3128)  | (1.5192) | (0.0865) |          |
| E                           | 0.5969    |          |          |          |
| $(\lambda)$                 | (0.1089)  |          |          |          |

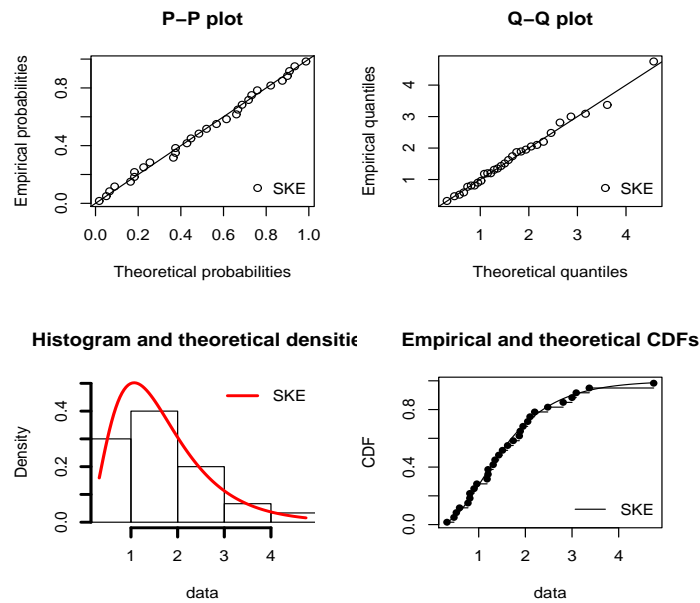


FIGURE 5. P-P, Q-Q, epdf and ecdf plots of the BCG-HC distribution for data set 1.

TABLE 6. Confidence intervals for the parameters of the KSE model for data set 1.

| CI  | $a$             | $b$             | $\lambda$       |
|-----|-----------------|-----------------|-----------------|
| 95% | [2.5421 4.4989] | [0.1673 0.5930] | [0.9450 2.1049] |
| 99% | [2.1695 5.2706] | [0.1000 0.6603] | [0.7615 2.2884] |

TABLE 7. LR test for data set 1.

| Idea               | $H_0$       | $LR$    | p-value       |
|--------------------|-------------|---------|---------------|
| SKE versus SE [10] | $a = b = 1$ | 17.1938 | < 0.001 (***) |

**Analysis of data set 2.** For data set 2, we adopt the same methodology to the one used for data set 1. Thus, some descriptive statistics are presented in Table 8. Since the estimated skewness is close to zero, the subjacent distribution is near symmetric around its mean. The values of the goodness-of-fit measures for the SKE model and other fitted models are collected in Table 9, whereas the MLEs and their corresponding SEs are listed in Table 10. Again, we see that the SKE model has the lowest statistics, indicating that it is statistically superior to the competitor. The P-P, Q-Q, epdf and ecdf plots of the SKE are presented in Figure 6. We see nice fits, indicating that the SKE model is good choice for the analysis of data set 2. Then, we provide the approximation confidence intervals of the parameters of the SKE model in Table 11, for the levels 95% and 99%. Finally, a LR test with the hypotheses:  $H_0 : a = b = 1$  versus  $H_1 : a \neq 1$  or  $b \neq 1$ , is performed in Table 12. The p-value satisfies  $p\text{-value} < 0.0001$ , indicating that the SKE model is again preferable to the SE model.

TABLE 8. Some descriptive statistics for data set 2.

| Statistics | N  | Mean   | Median | Variance | skewness | kurtosis |
|------------|----|--------|--------|----------|----------|----------|
| Data set 2 | 69 | 1.4513 | 1.4780 | 0.2451   | -0.02821 | -0.05927 |

TABLE 9. Goodness-of-fit measures for data set 2.

| Model   | $-\widehat{\ell}$ | AIC      | BIC      | CAIC     | HQIC     | KS     | CVM    | AD     |
|---------|-------------------|----------|----------|----------|----------|--------|--------|--------|
| SKE     | 48.1311           | 104.2624 | 110.9647 | 104.6316 | 106.9214 | 0.0455 | 0.0211 | 0.1977 |
| KW      | 48.7684           | 105.5368 | 114.4733 | 106.1618 | 109.0822 | 0.0475 | 0.0226 | 0.1984 |
| BW      | 48.8954           | 105.7908 | 114.7272 | 106.4158 | 109.3362 | 0.0480 | 0.0256 | 0.2217 |
| $CS1_E$ | 49.5405           | 105.0810 | 111.7833 | 105.4502 | 107.7400 | 0.0487 | 0.0279 | 0.1989 |
| E       | 94.7013           | 191.4026 | 193.6367 | 191.4623 | 192.2890 | 0.3622 | 0.1238 | 0.8712 |

TABLE 10. MLEs and SEs (in parentheses) for data set 2.

| Model                       | Estimates |           |          |          |
|-----------------------------|-----------|-----------|----------|----------|
| SKE                         | 3.5848    | 50.6984   | 0.2100   |          |
| $(a, b, \lambda)$           | (0.5853)  | (4.0734)  | (0.1626) |          |
| KW                          | 0.7268    | 0.1621    | 1.0308   | 3.5369   |
| $(a, b, \alpha, \beta)$     | (0.0052)  | (0.0186)  | (0.0218) | (0.0086) |
| BW                          | 0.3585    | 3.7827    | 0.7813   | 5.7953   |
| $(a, b, \alpha, \beta)$     | (2.0367)  | (1.2916)  | (0.4105) | (2.5127) |
| $CS1_E$                     | 0.0916    | 10.7578   | 0.2785   |          |
| $(\alpha, \theta, \lambda)$ | (1.0176)  | (11.6449) | (0.0276) |          |
| E                           | 0.5969    |           |          |          |
| $(\lambda)$                 | (0.1089)  |           |          |          |

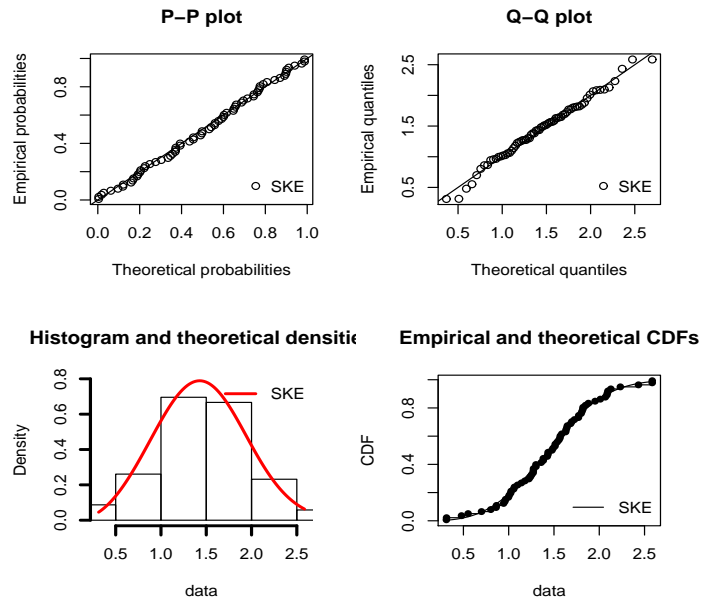


FIGURE 6. P-P, Q-Q, epdf and ecdf plots of the SKE distribution for data set 2.

TABLE 11. Confidence intervals for the parameters of the KSE model for data set 2.

| CI  | $a$             | $b$               | $\lambda$  |
|-----|-----------------|-------------------|------------|
| 95% | [2.4376 4.3433] | [42.7146 58.6822] | [0 0.5286] |
| 99% | [2.0747 5.0948] | [40.1890 61.2077] | [0 0.6295] |

TABLE 12. LR test for data set 2.

| Idea               | $H_0$       | $LR$    | p-value         |
|--------------------|-------------|---------|-----------------|
| SKE versus SE [10] | $a = b = 1$ | 93.1404 | $< 0.001$ (***) |



## 5. CONCLUSIONS

In the last decade, the trigonometric families of distributions have received a lot of attention, mainly thanks to their flexible properties in term of fitting a wide variety of real data sets. In this study, we explore a natural extension of the sine-G family of distributions, called the sine Kumaraswamy-G family of distributions. We investigate its main mathematical properties, including asymptotes, quantile function, linear representations of the cumulative distribution and probability density functions, moments, skewness and kurtosis, incomplete moments, weighted moments and order statistics. Then, a special focus is done on the sine Kumaraswamy exponential distribution, notable member of this family. After presenting its mathematical features, we study the ability of the related model in the fitting of data sets. The maximum likelihood method is used in to estimate the unknown parameters and a simulation study give numerical guarantees of their performances. Applications to two practical data sets are presented in details, showing that the proposed model outperformed some strong well-established competitors in the literature. We hope that the sine Kumaraswamy-G family of distributions and the related perspective of models may attract wider applications in statistics in various areas.

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