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RIGIDITY OF SQUARE-TILED INTERVAL EXCHANGE TRANSFORMATIONS

SÉBASTIEN FERENCZI AND PASCAL HUBERT

ABSTRACT. We look at interval exchange transformations defined as first return maps on the set of diagonals of a flow of direction θ on a square-tiled surface: using a combinatorial approach, we show that, when the surface has at least one true singularity both the flow and the interval exchange are rigid if and only if $\tan \theta$ has bounded partial quotients. Moreover, if all vertices of the squares are singularities of the flat metric, and $\tan \theta$ has bounded partial quotients, the square-tiled interval exchange transformation T is not of rank one. Finally, for another class of surfaces, those defined by the unfolding of billiards in Veech triangles, we build an uncountable set of rigid directional flows and an uncountable set of rigid interval exchange transformations.

To the memory of William Veech whose mathematics were a constant source of inspiration for both authors, and who always showed great kindness to the members of the Marseille school, beginning with its founder Gérard Rauzy.

Interval exchange transformations were originally introduced by Oseledec [30], following an idea of Arnold [2], see also Katok and Stepin [24]; an exchange of k intervals is given by a positive vector of k lengths together with a permutation π on k letters; the unit interval is partitioned into k subintervals of lengths $\alpha_1, \dots, \alpha_k$ which are rearranged according to π .

The history of interval exchange transformations is made with big *generic* results: almost every interval exchange transformation is uniquely ergodic (Veech [36], Masur [28]), almost every interval exchange transformation is weakly mixing (Avila-Forni [6]), while other results like simplicity [37] or Sarnak's conjecture [34] are still partly in the future. In parallel with generic results, people have worked to build constructive examples, and, more interesting and more difficult, counter-examples. In the present paper we want to focus on two less-known but very important measure-theoretic generic results, both by Veech: almost every interval exchange transformation is *rigid* [37], meaning that for some sequence q_n the q_n -th powers of the transformation converge to the identity (see Definition 14 below); almost every interval exchange transformation is of *rank one* [38], meaning that there is a generating family of Rokhlin towers, see Definition 16 below.

These results are not true for every interval exchange transformation. The last result admits already a wide collection of examples and counter-examples, as indeed the first two papers ever written on interval exchange transformations provide counter-examples to a weaker property (Oseledec [30]) and examples of a stronger property (Katok-Stepin [24]); in more recent times, many examples were built, such as most of those in [19] [16], and also a surprisingly vast amount of counter-examples, as, following Oseledec, many great minds built interval exchange transformations with given spectral multiplicity functions, for example Robinson [33] or Ageev [1] and this contradicts rank one as soon as the latter is not constantly one (simple spectrum); let us just remark

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that these brilliant examples, built on purpose, are a little complicated and not very explicit as interval exchange transformations. We know of only one family of interval exchange transformations which have simple spectrum but not rank one, these were built in [8] but only for 3 intervals.

As for the question of rigidity, it has been solved completely for the case of 3-interval exchange transformations in [17], where a necessary and sufficient condition is given, see also [43] where a different approach is developed. For more than three intervals, examples of rigidity can again be found in [19] [16].

But of course, possibly the main appeal of interval exchange transformations is the fact that they are closely linked to linear flows on translation surfaces, which are studied using Teichmüller dynamics. Generic results are obtained applying the $SL(2, \mathbb{R})$ action on translation surfaces. After all the efforts made to classify $SL(2, \mathbb{R})$ orbit closures in the moduli spaces of abelian differentials, especially after the work of Eskin, Mirzakhani and Mohammadi [11, 12], it is quite natural to want to solve these ergodic questions on suborbifolds of moduli spaces. The celebrated Kerckhoff-Masur-Smillie Theorem [26] solved the unique ergodicity question for every translation surface and almost every direction. Except for this general result, very little is known on the ergodic properties of linear flows and interval exchange transformations obtained from suborbifolds. Avila and Delecroix recently proved that, on a non arithmetic Veech surface, in a generic direction, the linear flow is weakly mixing [7].

In the present paper, we shall study two families of Veech surfaces, the *square-tiled* surfaces, and the surfaces built by unfolding *billiards in Veech triangles*.

In Teichmüller dynamics, square-tiled surface play a special role since they are integer points in period coordinates. Moreover, the $SL(2, \mathbb{R})$ orbit of a square-tiled surface is closed in its moduli space. The main part of the present paper studies families of interval exchange transformations associated with square-tiled surfaces. Our main results are:

Theorem 1. ¹ *Let X be a square-tiled surface of genus at least 2. The linear flow in direction θ on X is rigid if and only if the slope $\tan \theta$ has unbounded partial quotients.*

Remark 1. *The new and more difficult statement in Theorem 1 is the non rigidity phenomenon when the slope has bounded partial quotients.*

Theorem 1 can be restated in terms of interval exchange transformations. Given a square-tiled surface and a direction with positive slope $\tan \theta$, defining $\alpha = \frac{1}{1+\tan \theta}$, there is very natural way to associate an interval exchange transformation T_α , namely the first return map on the union of the diagonals of slope -1 of the squares (the length of diagonals is normalized to be 1). It is a finite extension of a rotation of angle α , and an interval exchange transformation on a multi-interval. We call it a *square-tiled interval exchange transformation*.

Theorem 2. *Let X be a square-tiled surface of genus at least 2. The square-tiled interval exchange transformation T_α is rigid if and only if α has unbounded partial quotients².*

Remark 2. *To our knowledge, these examples are the first appearance of non rigid interval exchange transformations on more than 3 intervals, together with the examples defined simultaneously by Robertson [32], where a different class of interval exchanges is shown to have the stronger property of mild mixing (no rigid factor). Our examples are not weakly mixing, and therefore not*

¹Unpublished partial results based on geometric ideas were obtained by Forni.

² α has bounded partial quotients if and only if $\tan \theta$ has bounded partial quotients

mildly mixing. Note also that Franczek [20] proved that mildly mixing flows are dense in genus at least two, and that Kanigowski and Lemańczyk [23] proved that mild mixing is implied by Ratner's property, which thus our examples do not possess.

Thus we can ask

Question 1. *Do there exist interval exchanges which are weakly mixing, not rigid but not mildly mixing?*

Question 2. *Find interval exchanges satisfying Ratner's property (note that the examples of Robertson are likely candidates).*

Remark 3. *Also, as was pointed out by one of the referees, the examples of non-rigid interval exchanges in our Theorem 2 are all linearly recurrent, which corresponds to bounded Teichmüller geodesics; this is the case also for all known examples, either Robertson's or those on three intervals. Moreover, all known examples of rigid interval exchanges, except those which are rotations, are not linearly recurrent.*

Thus the referee suggests these two natural questions:

Question 3. *Does there exist a non-rigid and non-linearly recurrent interval exchange or translation flow?*

Question 4. ³ *An important subclass of linearly recurrent interval exchanges is the class of self-induced ones, which correspond to periodic Teichmüller geodesics, and to natural codings which are substitution dynamical systems (see Section 1.3 below). Thus one can ask whether there exist rigid self-induced interval exchange transformations, outside the trivial case of the rotation class.*

The Arnoux-Yoccoz interval exchange transformation is a self-induced linearly recurrent interval exchange transformation, which is semi-conjugate to a rotation of the two dimensional torus (see [5], [3]). The rotation is rigid, thus the Arnoux-Yoccoz interval exchange will be rigid if the semi-conjugacy gives a full conjugacy in the measure-theoretic sense. This property seems widely assumed to be satisfied, a fact which is stated without proof in [4], Section 3.1.1; a (far from trivial) proof was recently proposed by J. Cassaigne (private communication, 2018), and thus this example does answer the question.

Our Theorem 2, in the direction of non-rigidity, constitutes the main result of the paper; its proof relies on the word combinatorics of the natural coding of the interval exchange. Indeed, rigidity of a symbolic system translates, through the ergodic theorem, into a form of approximate periodicity on the words: the iterates by some sequence q_n of a very long word $x = x_1 \dots x_k$ should be words arbitrarily close to x in the Hamming distance \bar{d} (Definition 8 below); to deny this property, the known methods consist either in showing that there are many possible $T^{q_n}x$ (thus for example *strong mixing* contradicts rigidity), but this will not be the case here, or else, in showing that \bar{d} -neighbours are scarce, and thus our approximate periodicity forces periodicity, which is then easy to disprove.

Indeed, a notion we choose now to call \bar{d} -separation was introduced first (but not formalized) by del Junco [22], who showed that it is satisfied by the Thue - Morse subshift: it requires that two close enough \bar{d} -neighbours must actually coincide on a connected central part (Definition 9 below), and is used in [22] to contradict the rank one property; \bar{d} -separation is also known to hold for Chacon's map [13]. Then Lemańczyk and Mentzen [27] proved that \bar{d} -separation is equivalent to

³This question was asked by G. Forni

non-rigidity in the particular class of *substitution dynamical systems* when the substitution is of constant length (Definitions 10 and 12 below). It was not known whether this equivalence still holds in the larger class of all substitution dynamical systems or in other classes of non-strongly mixing systems such as interval exchanges. The systems of our Theorem 2 are good candidates for \bar{d} -separation, so it came as a surprise that in general they do *not* satisfy it; indeed, they do provide counter-examples to the above -mentioned equivalence in both classes, of interval exchanges and of substitutions, see Proposition 9 and Corollary 10 below. However, all the systems in Theorem 2 do satisfy a weaker property on scarcity of \bar{d} -neighbours, which we call *average \bar{d} -separation*, see Definition 18 and Proposition 8 below; this property seems completely new and is sufficient to complete the proof of non-rigidity.

The property of \bar{d} -separation is still satisfied in some particular cases, and we use it to prove

Theorem 3. *If all vertices of the squares are singularities of the flat metric, and α has bounded partial quotients, the square-tiled interval exchange transformation T_α is not rank one.*

Remark 4. *This condition is very restrictive and only holds for a finite number of square-tiled surfaces in each stratum.*

In the last part, we exhibit an uncountable set of rigid directional flows (see Proposition 15) and an uncountable set of rigid interval exchange transformations (see Proposition 14) associated with the unfolding of billiards in Veech triangles; in these examples, the directions are well approximated by periodic ones.

Remark 5. *The proof of Proposition 15 works mutatis mutandis for every Veech surface.*

Question 5. *On a primitive Veech surface, is the translation flow in a typical direction rigid?*

0.1. Organization of the paper. In Section 1 we recall the classical definitions about interval exchange transformations, coding, square-tiled surfaces and some facts in ergodic theory. Section 2 presents square-tiled interval exchange transformations and their symbolic coding. In Section 3, we give a proof of Theorem 2 using combinatorial methods; the main tool is Proposition 8. In Section 4, we deduce from Theorem 2 a proof of Theorem 1. We also prove Theorem 3. In Section 5, we tackle the case of billiards in Veech triangles.

1. DEFINITIONS

1.1. Interval exchange transformations. For any question about interval exchange transformations, we refer the reader to the surveys [41] [44]. Our intervals are always semi-open, as $[a, b[$.

Definition 1. *A k -interval exchange transformation T with vector $(\alpha_1, \alpha_2, \dots, \alpha_k)$, and permutation π is defined on $[0, \alpha_1 + \dots + \alpha_k[$ by*

$$\mathcal{I}x = x + \sum_{\pi^{-1}(j) < \pi^{-1}(i)} \alpha_j - \sum_{j < i} \alpha_j.$$

when x is in the interval

$$\left[\sum_{j < i} \alpha_j, \sum_{j \leq i} \alpha_j \right].$$

We put $\gamma_i = \sum_{j \leq i} \alpha_j$, and denote by D_i the interval $[\gamma_{i-1}, \gamma_i[$ if $2 \leq i \leq k-1$, while $D_1 = [0, \gamma_1[$ and $D_k = [\gamma_{k-1}, 1[$.

An exchange of 2 intervals, defined by the permutation $\pi(1, 2) = (2, 1)$ and the normalized vector $(1 - \alpha, \alpha)$ is identified with the rotation of angle α on the 1-torus. Its dynamical behavior is linked with the *Euclid continued fraction expansion of α* . We assume the reader is familiar with the notation $\alpha = [0, a_1, a_2, \dots]$, and recall

Definition 2. α has bounded partial quotients if the a_i are bounded.

A useful criterion for minimality of an interval exchange is the *infinite distinct orbit condition* (or i.d.o.c.) of [25]:

Definition 3. A k -interval exchange T satisfies the i.d.o.c. condition if the $k - 1$ negative orbits $\{T^{-n}\gamma_i\}_{n \geq 0}, 1 \leq i \leq k - 1$, of the discontinuities of T are infinite disjoint sets.

1.2. Word combinatorics. We begin with basic definitions. We look at finite words on a finite alphabet $\mathcal{A} = \{1, \dots, k\}$. A word $w_1 \dots w_t$ has length $|w| = t$ (not to be confused with the length of a corresponding interval). The *empty word* is the unique word of length 0. The *concatenation* of two words w and w' is denoted by ww' .

Definition 4. A word $w = w_1 \dots w_t$ occurs at place i in a word $v = v_1 \dots v_s$ or an infinite sequence $v = v_1 v_2 \dots$ if $w_1 = v_i, \dots, w_t = v_{i+t-1}$. We say that w is a factor of v . The empty word is a factor of any v . Prefixes and suffixes are defined in the usual way.

Definition 5. A language L over \mathcal{A} is a set of words such if w is in L , all its factors are in L , aw is in L for at least one letter a of \mathcal{A} , and wb is in L for at least one letter b of \mathcal{A} .

A language L is minimal if for each w in L there exists n such that w occurs in each word of L with n letters.

The language $L(u)$ of an infinite sequence u is the set of its finite factors.

Definition 6. A word w is called right special, resp. left special if there are at least two different letters x such that wx , resp. xw , is in L . If w is both right special and left special, then w is called bispecial.

Definition 7. A word Z is called a first return word of a word w if w occurs exactly twice in wZ , once as a prefix and once as a suffix.

We define now a distance on finite words, which is also called the *Hamming distance*:

Definition 8. For two words of equal length $w = w_1 \dots w_q$ and $w' = w'_1 \dots w'_q$, their \bar{d} -distance is $\bar{d}(w, w') = \frac{1}{q} \#\{i; w_i \neq w'_i\}$.

As mentioned in the introduction, we shall be interested in the scarcity of neighbours for the \bar{d} -distance; in a given nontrivial language, there will generally be arbitrarily long bispecial words such that awa' and bwb' are both in L , for letters of the alphabet \mathcal{A} , $a \neq a', b \neq b'$; thus $\bar{d}(awa', bwb')$ will be small. The following notion, whose history is told in the introduction, and which is defined here formally for the first time, states that \bar{d} -neighbours can only be of this or a slightly more general form.

Definition 9. A language L is \bar{d} -separated if there exists C such that for any two words w and w' of equal length q in L , if $\bar{d}(w, w') < C$, then $\{1, \dots, q\}$ is the disjoint union of three (possibly empty) integer intervals I_1, J, I_2 (in increasing order) such that

- $w_J = w'_J$
- for $j = 1, 2$, $\bar{d}(w_{I_j}, w'_{I_j}) = 1$ if I_j is nonempty,

where w_H denotes the word made with the h -th letters of the word w , $h \in H$.

Substitutions are mentioned in the introduction, and will appear in Corollary 10 below.

Definition 10. A substitution ψ is an application from an alphabet \mathcal{A} into the set \mathcal{A}^* of finite words on \mathcal{A} ; it extends naturally to a morphism of \mathcal{A}^* for the operation of concatenation.

ψ is of constant length if all the words ψa , $a \in \mathcal{A}$, have the same length.

A fixed point of ψ is an infinite sequence u with $\psi u = u$.

A language defined by ψ is any $L(u)$ where u is a fixed point of ψ .

1.3. Symbolic dynamics and codings.

Definition 11. The symbolic dynamical system associated to a language L is the one-sided shift $S(x_0x_1x_2\dots) = x_1x_2\dots$ on the subset X_L of $\mathcal{A}^{\mathbb{N}}$ made with the infinite sequences such that for every $t < s$, $x_t\dots x_s$ is in L .

For a word $w = w_1\dots w_s$ in L , the cylinder $[w]$ is the set $\{x \in X_L; x_0 = w_1, \dots, x_{s-1} = w_s\}$.

Note that the symbolic dynamical system (X_L, S) is minimal (in the usual sense, every orbit is dense) if and only if the language L is minimal,

Definition 12. A substitution dynamical system is a symbolic dynamical system associated to a language defined by a substitution

Definition 13. For a system (X, T) and a finite partition $Z = \{Z_1, \dots, Z_r\}$ of X , the trajectory of a point x in X is the infinite sequence $(x_n)_{n \in \mathbb{N}}$ defined by $x_n = i$ if $T^n x$ falls into Z_i , $1 \leq i \leq r$. Then $L(Z, T)$ is the language made of all the finite factors of all the trajectories, and $X_{L(Z, T)}$ is the coding of X by Z .

For an interval exchange transformation T , if we take for Z the partition made by the intervals D_i , $1 \leq i \leq k$, we denote $L(Z, T)$ by $L(T)$ and call $X_{L(T)}$ the natural coding of T .

1.4. Measure-theoretic properties. Let (X, T, μ) be a probability-preserving dynamical system.

Definition 14. (X, T, μ) is rigid if there exists a sequence $q_n \rightarrow \infty$ such that for any measurable set A $\mu(T^{q_n} A \Delta A) \rightarrow 0$.

Definition 15. In (X, T) , a Rokhlin tower is a collection of disjoint measurable sets called levels $F, TF, \dots, T^{h-1}F$. If X is equipped with a partition P such that each level $T^r F$ is contained in one atom $P_{w(r)}$, the name of the tower is the word $w(0) \dots w(h-1)$.

Definition 16. (X, T, μ) is of rank one if there exists a sequence of Rokhlin towers such that the whole σ -algebra is generated by the partitions $\{F_n, TF_n, \dots, T^{h_n-1}F_n, X \setminus \cup_{j=0}^{h_n-1} T^j F_n\}$.

1.5. Translation surfaces and square-tiled surfaces. A translation surface is an equivalence class of polygons in the plane with edge identifications: Each translation surface is a finite union of polygons in \mathbb{C} , together with a choice of pairing of parallel sides of equal length. Two such collections of polygons are considered to define the same translation surface if one can be cut into pieces along straight lines and these pieces can be translated and re-glued to form the other collection of polygons (see Zorich [46], Wright [42] for surveys on translation surfaces). For every direction θ , the linear flow in direction θ is well defined. The first return map to a transverse segment is an interval exchange.

Recall that closed regular geodesics on a flat surface appear in families of parallel closed geodesics. Such families cover a cylinder filled with parallel closed geodesic of equal length. Each boundary

of such a cylinder contains a singularity of the flat metric.

A square-tiled surface is a finite collection of unit squares $\{1, \dots, z\}$, the left side of each square is glued by translation to the right side of another square. The top of a square is glued to the bottom of another square. A baby example is the flat torus $\mathbb{R}^2/\mathbb{Z}^2$. In fact every square-tiled surface is a covering of $\mathbb{R}^2/\mathbb{Z}^2$ ramified at most over the origin of the torus. A square-tiled surface is a translation surface, thus linear flows are well defined. Combinatorially, a square-tiled surface is defined by two permutations acting on the squares: τ encodes horizontal identifications, σ is responsible for the vertical identifications. For $1 \leq i \leq z$, $\tau(i)$ is the square on the right of i and $\sigma(i)$ is the square on top of i . The singularity of the flat metric are the projections of some vertices of the squares with angles $2k\pi$ with $k > 1$. The number k is explicit in terms of the permutations τ and σ . The lengths of the orbits of the commutator $[\tau, \sigma](i)$ give the angles at the singularities. Consequently τ and σ commute if and only if there is no singularity for the flat metric which means that the square-tiled surface is a torus. When τ and σ do not commute, only the i such that $\sigma\tau i \neq \tau\sigma i$ give rise to singularities. Moreover the surface is connected if and only if the group generated by τ and σ acts transitively on $\{1, 2, \dots, z\}$. A very good introduction to square-tiled surfaces can be found in Zmiaikou [45].

Example 1. *The simplest connected square-tiled surface is defined by $z = 3$, $\tau(1, 2, 3) = (2, 1, 3)$ and $\sigma(1, 2, 3) = (3, 2, 1)$; it is shown in Figure 1 below. Note that $\sigma \circ \tau i \neq \tau \circ \sigma i$ for all i .*

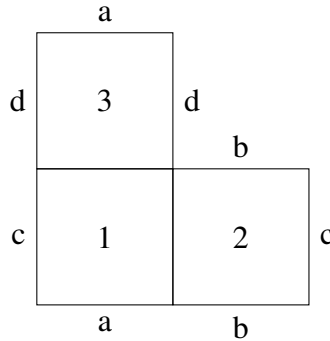


FIGURE 1. Square-tiled surface of Example 1
Letters a, b, c, d describe the sides identifications.

Example 2. *Another connected example we shall use is defined by $z = 4$, $\tau(1, 2, 3, 4) = (2, 1, 3, 4)$ and $\sigma(1, 2, 3, 4) = (3, 2, 4, 1)$. As $\sigma \circ \tau(1, 2, 3, 4) = (3, 1, 4, 2)$ and $\tau \circ \sigma(1, 2, 3, 4) = (2, 3, 4, 1)$, there is one “false” singularity as $\sigma \circ \tau 3 = \tau \circ \sigma 3$.*

2. INTERVAL EXCHANGE TRANSFORMATION ASSOCIATED TO SQUARE-TILED SURFACES

2.1. Generalities. As we already noticed in the introduction, a square-tiled interval exchange transformation is the first return map on the diagonals of slope -1 of the linear flow of direction θ on a square-tiled surface. Let $\alpha = \frac{1}{1+\tan\theta}$; this *square-tiled interval exchange* transformation $T = T_\alpha$ has the following combinatorial definition.

Definition 17. A square-tiled $2z$ -interval exchange transformation with angle α and permutations σ and τ is the exchange on $2z$ intervals defined by the positive vector $(1 - \alpha, \alpha, 1 - \alpha, \alpha, \dots, 1 - \alpha, \alpha)$ and permutation defined by $\pi(2i - 1) = 2\sigma(i)$, $\pi(2i) = 2\tau(i) - 1$, $1 \leq i \leq z$.

Note that everything in this paper remains true if we replace the $D_{2i-1} = [i - 1, i - \alpha[$ by $[a_i, a_i + 1 - \alpha[$ and the $D_{2i} = [i - \alpha, i[$ by $[a_i + 1 - \alpha, a_i + 1[$ for some sequence satisfying $a_i \leq a_{i+1} - 1$, and reorder the intervals in the same way. To avoid unnecessary complication, we shall always use $a_i = i - 1$ as in the definition.

Thus the discontinuities of T are some (not necessarily all, depending on the permutation) of the $\gamma_{2i-1} = i - \alpha$, $1 \leq i \leq z$, $\gamma_{2i} = i$, $1 \leq i \leq z - 1$, the discontinuities of T^{-1} are some of $\beta_{2i-1} = i - 1 + \alpha$, $1 \leq i \leq z$, $\beta_{2i} = i$, $1 \leq i \leq z - 1$.

Henceforth it will be convenient to change the usual notation: we denote by i_l the number $2i - 1$ and by i_r the number $2i$, $1 \leq i \leq z$. With this notation, Figures 2 and 3 show the interval exchange defined by Example 1 for a given $\alpha \leq \frac{1}{2}$.

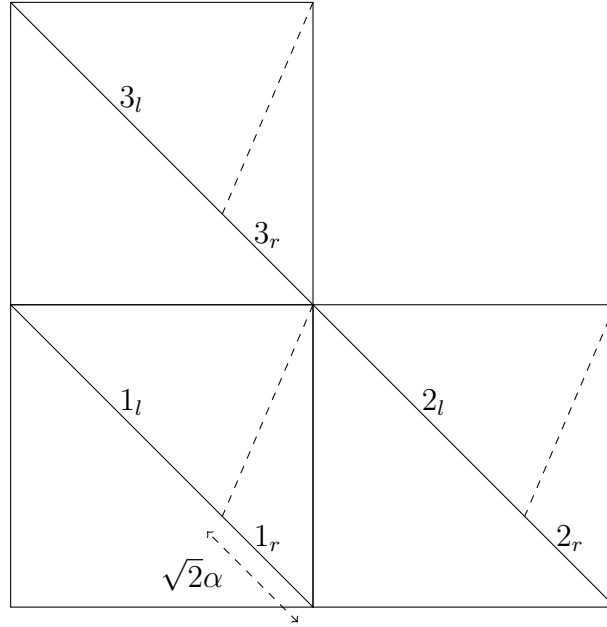


FIGURE 2. Building an interval exchange associated to the surface in Example 1

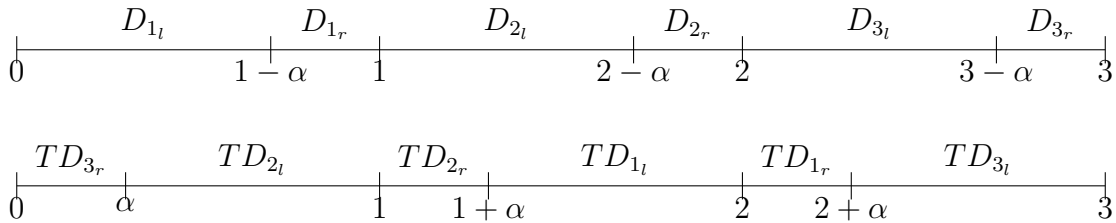


FIGURE 3. The square-tiled interval exchange we get in Example 1

We recall a classical result on minimality.

Proposition 4. *Let T be a square-tiled interval exchange transformation with irrational α ; T is aperiodic, and is minimal if and only if there is no strict subset of $\{1 \dots z\}$ invariant by σ and τ .*

Proof. Let X be the square-tiled surface corresponding to T . As we already remarked in Section 1.5, the hypothesis on the permutations means that the surface X is connected. A square-tiled surface satisfies the Veech dichotomy (see [40]). Thus the flow in direction θ is either periodic or minimal and uniquely ergodic. For square-tiled surfaces, periodic directions have rational slope. thus we get the result for the interval exchange transformation. \square

Remark 6. *For square-tiled surfaces the whole strength of the Veech dichotomy is not needed and the result is already contained in [39]. Also notice that for square-tiled interval exchange transformations minimality implies unique ergodicity by [9]; we denote by μ the unique invariant measure for T , namely the Lebesgue measure, and it is ergodic for T .*

Let T be a square-tiled interval exchange transformation. If we denote by (x, i) the point $i-1+x$, then the transformation T is defined on $[0, 1[\times\{1, \dots, z\}$ by $T(x, i) = (Rx, \phi_x(i))$ where $Rx = x + \alpha$ modulo 1, and $\phi_x = \sigma$ if $x \in [0, 1 - \alpha[$, $\phi_x = \tau$ if $x \in [1 - \alpha, 1[$. Thus T is also a z -point extension of a rotation. This implies that T has a rotation as a topological factor and thus a continuous eigenfunction, either for the topology of the interval or for the natural coding.

Remark 7. *For any irrational α , the rotation R is rigid. We do not know who proved this first, but it is immediate that Definition 14 is satisfied when the q_n are the denominators of the convergents of α for the Euclid continued fraction algorithm.*

Remark 8. *In general our square-tiled interval exchanges, even when they are minimal, do not satisfy the i.d.o.c. condition of Definition 3, because integer points may be discontinuities of both T and T^{-1} .*

However, it may happen that not all the γ_i and β_j are discontinuities, as in Figure 3 where D_{2_l} and D_{2_r} always remain adjacent, as well as D_{1_l} and D_{1_r} ; thus a square-tiled interval exchange on $2z$ intervals may indeed be an i.d.o.c. interval exchange on a smaller number of intervals; to our knowledge this was first remarked by Hmili [21], who uses some square-tiled interval exchanges (though they are not described as such) to provide examples of i.d.o.c. interval exchanges with continuous eigenfunctions; indeed, her simplest example is the one in Figure 2 above, with the surface as in Example 1 and any irrational $\alpha < \frac{1}{2}$, which is also an i.d.o.c. 4-interval exchange with permutation $\pi(1, 2, 3, 4) = (4, 2, 1, 3)$; as 3-interval exchanges are topologically weak mixing, this ranks among the counter-examples to that property with the smallest number of intervals.

To get new examples of non i.d.o.c. minimal interval exchanges, take the surface in Example 2 and any irrational α ; after deleting one “false” discontinuity, we get such an example on 7 intervals.

2.2. Coding of a square-tiled interval exchange transformation. We look now at the natural coding of T , which we denote again by (X, T) , . For any (finite or infinite) word u on the alphabet $\{i_l, i_r, 1 \leq i \leq z\}$, we denote by $\phi(u)$ the sequence deduced from u by replacing each i_l by l , each i_r by r . For a trajectory x for T under our version of the natural coding, $\phi(x)$ is a trajectory for R under the coding by the partition into two atoms $[l] = [0, 1 - \alpha[\times\{1, \dots, z\}$, $[r] = [1 - \alpha, 1[\times\{1, \dots, z\}$, thus it is a trajectory for R under its natural coding (as an exchange transformation of two intervals), and that is called a Sturmian sequence. The \bar{d} distance is defined in Definition 8 above.

Lemma 5. *For any word w in $L(T)$, there are exactly z words v such that $\phi(w) = \phi(v)$, and for each of these words either $v = w$ or $\bar{d}(w, v) = 1$.*

Proof. Using the definition of T , we identify the words of length 2 in $L(T)$:

- if $\alpha < \frac{1}{2}$, for $1 \leq i \leq z$, i_l can be followed by $(\tau^{-1}i)_l$ and $(\tau^{-1}i)_r$, i_r can be followed by $(\sigma^{-1}i)_l$;
- if $\alpha > \frac{1}{2}$, for $1 \leq i \leq z$, i_r can be followed by $(\sigma^{-1}i)_r$ and $(\sigma^{-1}i)_l$, i_l can be followed by $(\tau^{-1}i)_r$.

If $w = w_1 \dots w_t$, then $w_i = (u_i)_{s_i}$ with $u_i \in \{1, \dots, z\}$ and $s_i \in \{l, r\}$, and $u_{i+1} = \pi_i(u_i)$ with $\pi_i \in \{\sigma^{-1}, \tau^{-1}\}$. The above list of words of length 2 implies that π_i depends only on s_i ; thus two homologous (= having the same image by ϕ) words which have the same s_i , have also the same π_i . Thus the words $v = v_1 \dots v_t$ homologous to w are such that $v_1 = x_{s_1}$, $v_i = (\pi_{i-1} \dots \pi_1(x))_{s_i}$ for $i > 1$, thus there are as many such words as possible letters x , and if $x \neq u_1$ then $v_i \neq w_i$ for all i as $\pi_{i-1} \dots \pi_1$ are bijections. \square

Henceforth we shall make all computations for $\alpha < \frac{1}{2}$; the complementary case gives exactly the same results, mutatis mutandis.

To understand the coding of T , we need a complete knowledge of the Sturmian coding of R ; the one we quote here uses a different version of the classic Euclid algorithm, which is the self-dual induction of [18] in the particular case of two intervals; all what we need to know is contained in the following proposition, which can also be proved directly without difficulty.

Proposition 6. *Let $\alpha = [0, a_1 + 1, a_2, \dots]$, (see Section 1.1 above). We build inductively real numbers l_n and r_n and words w_n, M_n, P_n in the following way: $l_0 = \alpha, r_0 = 1 - \alpha, w_0$ is the empty word, $M_0 = l, P_0 = r$. Then*

- whenever $l_n > r_n$, $l_{n+1} = l_n - r_n, r_{n+1} = r_n, w_{n+1} = w_n P_n, P_{n+1} = P_n, M_{n+1} = M_n P_n$;
- whenever $r_n > l_n$, $l_{n+1} = l_n, r_{n+1} = r_n - l_n, w_{n+1} = w_n M_n, P_{n+1} = P_n M_n, M_{n+1} = M_n$.

Then $r_n > l_n$ for $0 \leq n \leq a_1 - 1$, $r_n < l_n$ for $a_1 \leq n \leq a_1 + a_2 - 1$, and so on. For $n \geq 1$, the w_n are all the nonempty bispecial words of $L(R)$, w_{n+1} being the shortest bispecial word beginning with w_n ; moreover, M_n and P_n constitute all the first return words of w_n (see Definition 7 above).

Example 3. Take $\alpha = \frac{3-\sqrt{5}}{2} = [0, 2, 1, 1, \dots]$. Because $L(R)$ is minimal, it can be described as $L(w)$, where w is the infinite word beginning by w_n for each n ; the successive w_n are $l, lrl, lrlrl, lrlrlrlrl, \dots$, and we recognize w as the Fibonacci word, fixed point of the substitution $l \rightarrow lr, r \rightarrow l$. However, in the present context it will be more useful to generate $L(R)$ with the M_n or P_n , given by the rules in Proposition 6; if we group these rules two by two, we generate M_{2n} and P_{2n} by repetition of the rule $M_{n+2} = M_n P_n M_n, P_{n+2} = P_n M_n$, thus $L(R)$ is (also) defined by the substitution $l \rightarrow lrl, r \rightarrow rl$.

Example 4. Take now $z = 3$ and the surface in Example 1, with $\alpha = \frac{3-\sqrt{5}}{2}$. We use Lemma 5 to build $L(T)$. The three words of length 2 in $L(R)$ are ll, lr, rl , and they lift to nine words in $L(T)$, namely $1_l 2_l, 2_l 1_l, 3_l 3_l, 1_l 2_r, 2_l 1_r, 3_l 3_r, 1_r 3_l, 2_r 2_l, 3_r 1_l$.

To generate $L(T)$, which is minimal, we can use any infinite sequence of the words $M_{n,i}$ and $P_{n,i}$, $n \geq 0, i = 1, 2, 3$, where $M_{n,i}$ is the word projecting by ϕ on M_n and beginning with i_l , $P_{n,i}$ is the word projecting by ϕ on P_n and beginning with i_r . These are given by rules which project on the rules of Proposition 6; these rules actually depend only on the last letters of the words $M_{n,i}$ and $P_{n,i}$, as for each one the knowledge of the words of length 2 gives a unique word which can be concatenated after it; as there is a finite number of possibilities, these rules must be eventually periodic. Indeed, hand computations show that these last letters are identical for

$n = 1$ and $n = 7$; the rules giving the $M_{7,i}$ and $P_{7,i}$ define a substitution ψ on the new alphabet $\{M_{1,i}, i = 1, 2, 3, P_{1,i}, i = 1, 2, 3\}$. Thus $L(T)$ is deduced from the language defined by ψ by replacing $M_{1,i}$ and $P_{1,i}$ by their actual expressions in the original alphabet, respectively $1_l, 2_l, 3_l, 1_r 3_l, 2_r 2_l, 3_r 1_l$; this is called a substitutive language..

Example 5. We take now $z = 4$ and the surface in Example 2, with $\alpha = \frac{3-\sqrt{5}}{2}$. We generate $L(T)$ as in Example 4, *mutatis mutandis*: the periodicity of the rules giving the $M_{n,i}$ and $P_{n,i}$, $i = 1, 2, 3, 4$, appears as the last letters of these words are the same for $n = 1$ and $n = 19$. The rules giving the $M_{19,i}$ and $P_{19,i}$ define a substitution ψ' on the new alphabet $\{i_g = M_{1,i}, i = 1, 2, 3, 4, i_d = P_{1,i}, i = 1, 2, 3, 4\}$; $L(T)$ is deduced from the language defined by ψ' by replacing i_g and i_d by their actual expressions in the original alphabet, namely $1_l, 2_l, 3_l, 4_l, 1_r 4_l, 2_r 2_l, 3_r 1_l, 4_r 3_l$. We shall use this example in Corollary 9 below.

The following lemma is also well known, but we did not find a proof in the existing literature; it says that in general two Sturmian trajectories, if they coincide along some letters then become different, will differ only along two letters, and then be the same again along a number of letters which, if α has bounded partial quotients, is comparable to the previous one.

Lemma 7. For all $n \geq 1$ the words defined in Proposition 6 satisfy

- $|P_n| + |M_n| = |w_n| + 2$,
- $P_n M_n$ and $M_n P_n$ are right extensions of $P_1 M_1$ and $M_1 P_1$ by the same word.

For $n \geq a_1 + 1$, w_n has exactly two extensions of length $|w_n| + \min(|P_n|, |M_n|)$, and these are $w_n l r V_n$ and $w_n r l V_n$ for the same word V_n .

If α has bounded partial quotients, there exists a constant C_1 such that $\min(|P_n|, |M_n|) > C_1 |w_n|$ for all n .

Proof. The first two assertions come from the recursion formulas giving M_n and P_n in Proposition 6. These formulas ensure also that, when $n > a_1$, $|M_n|$ and $|P_n|$ are at least 2; hence, because by Proposition 6 M_n and P_n are first return words of w_n , two possible extensions of w_n of length $|w_n| + \min(|P_n|, |M_n|)$ are the prefixes of that length of $w_n M_n$ and $w_n P_n$, hence of $w_n M_n P_n$ and $w_n P_n M_n$, thus they are of the form $w_n l r V_n$ and $w_n r l V_n$. Moreover, as by Proposition 6 there are no right special words in $L(R)$ sandwiched between w_n and $w_n M_n$ or $w_n P_n$, there are only two extensions of that length of w_n , which proves the third assertion.

Let C_0 be the maximal value of the partial quotients of α ; because of the recursion formulas in Proposition 6, at the beginning of a string of n with $l_n < r_n$, we have $|P_n| < |M_n|$, then for every n in that string except the first one, and for the n just after the end of that string, $|M_n| < |P_n| < (C_0 + 1)|M_n|$, and *mutatis mutandis* for strings of n with $l_n > r_n$. Thus we get the last assertion from the first one. \square

3. PROOF OF THEOREM 2

As will be seen in Proposition 8, when α has bounded partial quotients, our interval exchanges satisfy the \bar{d} -separation property of Definition 9 if $\sigma\tau i \neq \tau\sigma i$ for all i , and a weaker property if $\sigma\tau \neq \tau\sigma$. We define now this property, which will be used in a forthcoming paper for another class of systems; it depends on an integer e , and is just \bar{d} -separation when $e = 1$.

Definition 18. A language L on an alphabet \mathcal{A} is average \bar{d} -separated for an integer $e \geq 1$ if there exists a language L' on an alphabet \mathcal{A}' , a K to one (for some $K \geq e$) map ϕ from \mathcal{A} to \mathcal{A}' ,

extended by concatenation to a map ϕ from L to L' , such that for any word w in L , there are exactly K words v such that $\phi(w) = \phi(v)$, and for each of these words either $v = w$ or $\bar{d}(w, v) = 1$, and a constant C , such that if v_i and v'_i , $1 \leq i \leq e$, are words in L , of equal length q , satisfying

- $\sum_{i=1}^e \bar{d}(v_i, v'_i) < C$,
- $\phi(v_i)$ is the same word u for all i ,
- $\phi(v'_i)$ is the same word u' for all i ,
- $v_i \neq v_j$ for $i \neq j$.

Then $\{1, \dots, q\}$ is the disjoint union of three (possibly empty) integer intervals I_1, J_1, I_2 (in increasing order) such that

- $v_{i, J_1} = v'_{i, J_1}$ for all i ,
- $\sum_{i=1}^e \bar{d}(v_{i, I_1}, v'_{i, I_1}) \geq 1$ if I_1 is nonempty,
- $\sum_{i=1}^e \bar{d}(v_{i, I_2}, v'_{i, I_2}) \geq 1$ if I_2 is nonempty,

where $w_{i, H}$ denotes the word made with the h -th letters of the word w_i for all h in H .

This implies in particular that $\#J_1 \geq q(1 - \sum_{i=1}^e \bar{d}(v_i, v'_i))$.

Proposition 8. *If α has bounded partial quotients, the language $L(T)$ is average \bar{d} -separated for any integer e with $1 + \#\{i; \sigma\tau i = \tau\sigma i\} \leq e \leq z$.*

Proof. In this case, the language L' of Definition 18 is $L(R)$ and ϕ is the map defined before Lemma 5, with $K = z$.

Let v_i and v'_i be as in Definition 18. The three integer intervals in the end will be built as follows: J_1 is the first (in increasing order) integer interval on which u and u' agree, I_1 and I_2 are then defined so that $\{1, \dots, q\}$ is the disjoint union of I_1, J_1, I_2 in increasing order. We shall prove that they do satisfy Definition 18.

We compare first u and u' ; note that if we see l , resp. r , in some word $\phi(\omega)$ we see some i_l , resp. j_r , at the same place in ω ; thus $\bar{d}(\omega, \omega') \geq \bar{d}(\phi(\omega), \phi(\omega'))$ for all ω, ω' ; in particular, if $\bar{d}(u, u') = 1$, then $\bar{d}(v_i, v'_i) = 1$ for all i and our assertion is proved.

Thus we can assume $\bar{d}(u, u') < 1$. We partition $\{1, \dots, q\}$ into successive integer intervals where u and u' agree or disagree: we get intervals $I_1, J_1, \dots, I_s, J_s, I_{s+1}$, where s is at least 1, the intervals are nonempty except possibly for I_1 or I_{s+1} , or both, and for all j , $u_{J_j} = u'_{J_j}$, and, except if I_j is empty, u_{I_j} and u'_{I_j} are completely different, i.e. their distance \bar{d} is one.

Then for $i \leq s - 1$, the word $u_{J_i} = u'_{J_i}$ is right special in the language $L(R)$ of the rotation, and this word is left special if $i \geq 2$.

(H0) *We suppose first that $u_{J_1} = u'_{J_1}$ is also left special and $u_{J_r} = u'_{J_r}$ is also right special.*

Then all the $u_{J_i} = u'_{J_i}$ are bispecial; thus, for a given i , $u_{J_i} = u'_{J_i}$ must be some w_n of Proposition 6; then Lemma 7 implies that either $\#J_j$ is smaller than a fixed m_1 , which is the length of w_{a_1} , or $\#I_{j+1} = 2$ and

$$\#I_{j+1} + \#J_{j+1} > C_1|w_n| \geq C_1\#J_j,$$

Similar considerations for R^{-1} imply that for $j > 1$ either $\#J_j < m_1$, or $\#I_j = 2$ and $\#J_{j-1} + \#I_j > C_1\#J_j$.

Note that this does not give any conclusion on $\bar{d}(u, u')$, and indeed by Remark 7 R is rigid, and thus admits a lot of \bar{d} -neighbours.

We look now at the words v_i and v'_i for some i ; by the remark above, v_{i, I_j} and v'_{i, I_j} are completely different if I_j is nonempty. As for v_{i, J_j} and v'_{i, J_j} , they have the same image by ϕ , thus by Lemma 5 they are equal if they begin by the same letter, completely different otherwise.

Moreover, suppose that J_j has length at least m_1 , and $v_{i, J_j} = v'_{i, J_j} = z_i$, ending with the letter s_i : because of Lemma 7 applied to $\phi(z_i)$. and taking into account the possible words of length 2 in $L(T)$, z_i has two extensions of length $|z_i|+3$ in $L(T)$, and they are $z_i(\tau^{-1}s_i)_l(\tau^{-1}\tau^{-1}s_i)_r(\sigma^{-1}\tau^{-1}\tau^{-1}s_i)_l$ and $z_i(\tau^{-1}s_i)_r(\sigma^{-1}\tau^{-1}s_i)_l(\tau^{-1}\sigma^{-1}\tau^{-1}s_i)_l$, which gives us the first letters of the two words $v_{i, J_{j+1}}$ and $v'_{i, J_{j+1}}$.

We estimate $c = \sum_{i=1}^e \bar{d}(v_i, v'_i)$, by looking at the indices in some set $G_j = J_j \cup I_{j+1} \cup J_{j+1}$, for any $1 \leq j \leq s-1$;

- if both $\#J_j$ and $\#J_{j+1}$ are smaller than m_1 the contribution of G_j to the sum c is at least $\frac{1}{2m_1+1}$ as I_{j+1} is nonempty by construction;
- if $\#J_j \geq m_1$, and for at least one i v_{i, J_j} and v'_{i, J_j} are completely different, then the contribution of G_j to c is bigger than $\min(\frac{1}{2}, \frac{C_1}{C_1+1})$ as either $\#J_{j+1} < m_1$ or $\#J_j + \#I_{j+1} > C_1\#J_{j+1}$;
- if $\#J_j \geq m_1$ and for all i , $v_{i, J_j} = v'_{i, J_j} = y_i$; then, because the v_i are all different and project by ϕ on the same word, the last letter (in the alphabet $\{1, \dots, z\}$) s_i of y_i takes e different values when i varies; thus $\tau^{-1}\sigma^{-1}\tau^{-1}s_i \neq \sigma^{-1}\tau^{-1}\tau^{-1}s_i$ for at least one i , and this ensures that for this i , $v_{i, J_{j+1}}$ and $v'_{i, J_{j+1}}$ are completely different. As $\#J_{j+1} + \#I_{j+1} > C_1\#J_j$, the contribution of G_j to c is bigger than $\frac{C_1}{C_1+1}$;
- if $\#J_{j+1} \geq m_1$, we imitate the last two items by looking in the other direction.

Now, if s is even, we can cover $\{1, \dots, q\}$ by sets G_j and some intermediate i_l , and get that c is at least a constant C_2 . If s is odd and at least 3, by deleting either I_1 and J_1 , or J_s and I_{s+1} , we cover at least half of $\{1, \dots, q\}$ by sets G_j and some intermediate i_l , and c is at least $\frac{C_2}{2}$.

Thus if $\sum_{i=1}^e \bar{d}(v_i, v'_i)$ is smaller than a constant C_3 , we must have $s = 1$; then if $\sum_{i=1}^e \bar{d}(v_i, v'_i) < 1$, $v_{i, J_1} = v'_{i, J_1}$. Thus if $c < C = \min(C_3, 1)$, we get our conclusion under the extra hypothesis (H0), with I_1 , J_1 , and $I_2 = I_{s+1}$ as defined above, by partitioning $\{1, \dots, q\}$ into successive integer intervals where u and u' agree or disagree.

If (H0) is not satisfied, we modify the v_i and v'_i to \tilde{v}_i and \tilde{v}'_i to get it.

Note that if $u_{J_1} = u'_{J_1}$ is not left special, then I_1 is empty, and u and u' are uniquely extendable to the left, and by the same letter; we continue to extend uniquely to the left as long as the extension of $u_{J_1} = u'_{J_1}$ remains not left special, and this will happen until we have extended u and u' (by the same letters) to a length q_0 . As for v_{i, J_1} and v'_{i, J_1} , they are either equal or completely different; then

- if for at least one i v_{i, J_1} and v'_{i, J_1} are completely different, we delete the prefix v_{i, J_1} from every v_i , the prefix v'_{i, J_1} from every v'_i ;

- if for all i $v_{i,J_1} = v'_{i,J_1}$; then v_i and v'_i are uniquely extendable to the left, and by the same letter, as long as u and u' are; then for all i , we take the unique left extensions of length q_0 of v_i and v'_i .

If $u_{J_s} = u'_{J_s}$ is not right special, we do the same operation on the right; thus we get new pairs of words \tilde{v}_i and \tilde{v}'_i , of length \tilde{q} . In building them, we have added no difference (in the sense of counting \bar{d}) between v_i and v'_i , but have possibly deleted a set of q_1 indices which gave a contribution at least one to the sum c , while when we extend the words we can only decrease the distances \bar{d} ; thus if $c < C \leq 1$, $\sum_{i=1}^c \bar{d}(\tilde{v}_i, \tilde{v}'_i) \leq \frac{qc - q_1}{q - q_1} \leq c$. Then our pairs satisfy all the conditions of the part we have already proved (the \tilde{v}_i are all different because they are different on at least one letter and have the same image by ϕ).

Thus $\{1, \dots, \tilde{q}\}$ is partitioned into $\tilde{I}_1, \tilde{J}_1, \tilde{I}_2$, with the properties in the end of Definition 18.

We go back now to the original v_i and v'_i .

- Suppose first that to get the new words we have either shortened or not modified the v_i on the left, and either shortened and not modified the v_i on the right: then we get our conclusion with J_1 a translate of \tilde{J}_1 , I_1 the union of a translate of \tilde{I}_1 and an interval I_0 corresponding to a part we have cut, I_2 the union of a translate of \tilde{I}_2 and an interval I_3 corresponding to a part we have cut.
- Suppose that to get the new words we have either shortened or not modified the v_i on the left, and lengthened the v_i on the right: then we get our conclusion with J_1 a translate of a nonempty subset of \tilde{J}_1 , I_1 the union of a translate of \tilde{I}_1 and an interval I_0 corresponding to a part we have cut, I_2 empty as \tilde{I}_2 .
- A symmetric reasoning applies if to get the new words we have either shortened or not modified the v_i on the right, and lengthened the v_i on the left.
- Suppose that to get the new words we have lengthened the v_i on the right and on the left: then we get our conclusion with J_1 a translate of a nonempty subset of \tilde{J}_1 , I_1 empty as \tilde{I}_1 , I_2 empty as \tilde{I}_2 .

□

For the surface in Example 1 and any α with bounded partial quotients, Proposition 8 holds for $e = 1$ and thus $L(T)$ is average \bar{d} -separated for the integer $e = 1$, which means $L(T)$ is \bar{d} -separated; but indeed, in most cases this will not be true.

Proposition 9. *When α has bounded partial quotients, $L(T)$ is not average \bar{d} -separated for the integers $e \leq \#\{i; \sigma\tau i = \tau\sigma i\}$, and thus not \bar{d} -separated if $\#\{i; \sigma\tau i = \tau\sigma i\} > 0$.*

Proof. if we take v_i and v'_i such that $\phi(v_i) = w_n l r y_n$, $\phi(v'_i) = w_n r l y_n$, and that the $|w_n|$ -th letter of v_i and v'_i is s_i where $\tau^{-1}\sigma^{-1}\tau^{-1}s_i = \sigma^{-1}\tau^{-1}\tau^{-1}s_i$, then the v_i and v'_i do not satisfy the conclusion if y_n and w_n are of comparable lengths, though $\sum \bar{d}(v_i, v'_i) \leq z \frac{2}{|w_n| + |y_n| + 2}$ may be arbitrarily small. □

Corollary 10. *\bar{d} -separation is not equivalent to non-rigidity in the class of substitutions or in the class of interval exchanges.*

Proof. The surface in Example 2 provides an example where $\#\{i; \sigma\tau i = \tau\sigma i\} = 1$. Thus for any α with bounded partial quotients, T is a non-rigid interval exchange where $L(T)$ is average \bar{d} -separated (for $e \geq 2$) but not \bar{d} -separated.

For substitutions, the symbolic dynamical system defined by ψ' of Example 5 above provides the required counter-example. Note that ψ' , on the alphabet $\{i_g, i_d, i = 1, 2, 3, 4\}$, is explicit, but, as the images of the letters have lengths 2584 and 4181, we shall not write it here.

By construction, the natural projections on sequences on $\{g, d\}$ of infinite sequences in this system are Sturmian sequences, corresponding to a rotation by an α' with bounded partial quotients. Then we can apply the reasonings of Propositions 8 and 9: the only point to check are the transitions from the last letter s_i of a word $v_{i,J_j} = v'_{i,J_j}$ to the first letters t_i and t'_i of $v_{i,J_{j+1}}$ and $v'_{i,J_{j+1}}$. In the proof of Proposition 8, what is between s_i and t_i , resp. t'_i , projects on lr , resp. rl ; in the present case, lr and rl are replaced by dg and gd , which, through the explicit value of the words i_d and i_g , correspond to lrl and rll . Thus the fact that $\tau^{-1}\tau^{-1}\sigma^{-1}\tau^{-1}i = \tau^{-1}\sigma^{-1}\tau^{-1}\tau^{-1}i$ for exactly one i ensures average \bar{d} -separation for $e = 2$, giving non-rigidity, but not for $e = 1$. \square

Note that the above counter-example could be made with any square-tiled surface on z squares with $1 \leq \#\{i; \sigma\tau i = \tau\sigma i\} < z$, and any α with ultimately periodic continued fraction expansion.

We now prove the hard part of Theorem 2 from Proposition 8.

Proof. We look at the $2z$ intervals D_i giving the natural coding.

Assume that α has bounded partial quotients but (X, T) is rigid; then there exists a sequence q_k tending to infinity such that $\mu(D_i \Delta T^{q_k} D_i)$ tends to zero for $1 \leq i \leq 2z$.

We fix $\epsilon < \frac{C}{2z^2}$, for the C from Definition 18 for $L(T)$ (with $e = z$), and k such that for all i

$$\mu(D_i \Delta T^{q_k} D_i) < \epsilon.$$

Let $A_i = D_i \Delta T^{q_k} D_i$; by the ergodic theorem, $\frac{1}{m} \sum_{j=0}^{m-1} 1_{T^j A_i}(x)$ tends to $\mu(A_i)$, for almost all x (indeed for all x because (X, T) is uniquely ergodic). Thus for all x , there exists m_0 such that for all m larger than some m_0 and all i ,

$$\frac{1}{m} \sum_{j=0}^{m-1} 1_{T^j A_i}(x) < \epsilon.$$

By summing these $2z$ inequalities, we get that

$$\bar{d}(x_0 \dots x_{m-1}, x_{q_k} \dots x_{q_k+m-1}) < 2z\epsilon$$

for all $m > m_0$. Moreover, given an x , we can choose m_0 such that for all $m > m_0$ these inequalities are satisfied if we replace x by any of the z different points x^i such that $\phi(x^i) = \phi(x)$.

We choose such an x , and, through Proposition 8, apply Definition 18 to $e = z$ and the words $v_i = (x^i)_0, \dots, (x^i)_{m-1}$, $v'_i = (x^i)_{q_k}, \dots, (x^i)_{q_k+m-1}$. As we know that C is smaller than $2z^2\epsilon$, we get that for any $m > m_0$, the words $(x_0 \dots x_{m-1})$ and $(x_{q_k} \dots x_{q_k+m-1})$ must coincide on a connected part larger than m multiplied by a constant; thus $x_l \dots x_{p-1}$ and $x_{q_k+l} \dots x_{q_k+p-1}$ coincide for some fixed l and all p large enough, but this implies that there is a periodic point, which has been disproved in Proposition 4. \square

The other direction of Theorem 2 is already known, but we include it with a short proof using our combinatorial methods.

Proposition 11. *Let T be a minimal square-tiled interval exchange transformation such that α is irrational and has unbounded partial quotients; then (X, T, μ) is rigid.*

Proof. For all n , the trajectories of the rotation are covered by disjoint occurrences of M_n and P_n (of Proposition 6) as these are the first return words of w_n . Suppose for example $l_m > r_m$ for $b_n \leq m \leq b_n + a_n - 1$; then because of the previous step $|P_{b_n}| > |M_{b_n}|$; then $P_{b_n+a_n} = P_{b_n}$, $M_{b_n+a_n} = M_{b_n}P_{b_n}^{a_n}$, $P_{b_n+a_n+1} = P_{b_n}M_{b_n}P_{b_n}^{a_n}$, $M_{b_n+a_n+1} = M_{b_n}P_{b_n}^{a_n}$. Hence disjoint occurrences of the word $P_{b_n}^{a_n}$ fill a proportion at least $\frac{a_n}{a_n+2}$ of the length of both M_N and P_N for each $N \geq a_n + b_n + 1$. The trajectories for T are covered by the $2z$ words $P_{N,i}$ and $M_{N,i}$ which project on P_N and M_N by ϕ , and a proportion at least $\frac{a_n}{a_n+2}$ of them are covered by disjoint occurrences of the z words which project by ϕ on $P_{b_n}^{a_n}$. For each N , each $P_{N,i}$ can be followed by exactly one $P_{N,j}$, and thus the $P_{N,i}$, $1 \leq i \leq z$, are grouped into $z'_r \leq z$ cycles $P_{N,i_{N,j,1}} \cdots P_{N,i_{N,j,c_{N,j}}}$, $1 \leq j \leq z'_N$, $1 \leq c_{N,j} \leq z$, where for a given N all the possible $P_{N,i_{N,j,l}}$ are different and the only $P_{N,h}$ which can follow $P_{N,i_{N,j,c_{N,j}}}$ is $P_{N,i_{N,j,1}}$. Let $s_n \leq z^z$ the least common multiple of all the $c_{b_n,j}$, $1 \leq j \leq z'_{b_n}$; then if we move by $T^{s_n|P_{b_n}|}$ inside one of the words which project on $P_{b_n}^{a_n}$, we see the same letter. Thus, if E is a fixed cylinder of length L , $\mu(E\Delta T^{s_n|P_{b_n}|}E)$ is at most $\frac{2}{a_n} + \frac{s_n}{a_n} + \frac{L}{|P_{b_n}|}$. Thus, possibly replacing P by M for the cases $l_m < r_m$, we get that if the a_n are unbounded T is rigid, as the cylinders for the natural coding generate the whole σ -algebra. \square

4. PROOF OF THEOREM 1 AND THEOREM 3

4.1. Rigidity of the flow.

Proposition 12. *Let X be a square-tiled surface and θ a direction, S_t the linear flow in direction θ and $T = T_\alpha$ the associated interval exchange transformation. The flow S_t is rigid whenever T is rigid.*

Proof. The key point is that the flow S_t is a suspension flow over T with constant roof function. Denote by I the union of the diagonals of slope -1 . In fact, if a point belongs to I , the return time ρ to I is independent of the point since diagonals are parallel (see for instance Figure 2).

Now, suppose T is rigid; if q_n is a rigidity sequence for T , then ρq_n is a rigidity sequence for the flow S_t , and thus S_t is rigid.

Suppose the flow is rigid, with rigidity sequence Q_n ; let $Q_n = \rho Q'_n$. Denote by q_n the nearest integer to Q'_n . Since the return time ρ is constant, Q'_n is close to the integer q_n : looking at the projection in the torus $\mathbb{R}^2/\mathbb{Z}^2$, a point in I cannot be close to I otherwise. Thus, as Q_n is a rigidity time for the flow, q_n is a rigidity time for T . \square

4.2. Rank. We now prove Theorem 3.

Proof. If T is of rank one, its natural coding satisfies the non-constructive symbolic definition of rank one, see the survey [14]: for every positive ϵ , for every natural integer l , there exists a word B of length $|B|$ bigger or equal to l such that, for all n large enough, on a subset of X of measure at least $1 - \epsilon$, the prefixes of length n of the trajectories are of the form $Z_1 B_1 \cdots Z_p B_p Z_{p+1}$, with $|Z_1| + \cdots + |Z_p| < \epsilon n$ and $\bar{d}(B_i, B) < \epsilon$ for all i . But then the \bar{d} -separation of $L(T)$ implies, possibly after shortening B by a prefix and a suffix of total relative length at most ϵ , and lengthening the Z_i accordingly, that the same is true with $B_i = B$ for all i . By projecting by ϕ , we get a similar structure for the trajectories of the rotation R . Such a structure for R implies that the quantity F

defined in Definition 4 of [10] is equal to 1, and by Proposition 5 of that paper this is impossible when α has bounded partial quotients. \square

Thus we can prove that T is not of rank one for the surface in Example 1 and any α with bounded partial quotients, but we do not know how to prove it for the surface in Example 2.

5. INTERVAL EXCHANGE TRANSFORMATIONS ASSOCIATED TO BILLIARDS IN VEECH TRIANGLES

We consider the famous examples of [40]: unfolding the billiard in the right-angled triangle with angles $(\pi/2z, \pi/2, (z-1)\pi/2z)$, one gets a regular double $2z$ -gon. A path, which starts in the interior of the polygon, moves with constant velocity until it hits the boundary, then it re-enters the polygon at the corresponding point of the parallel side, and continues travelling with the same velocity.

We follow the presentation of [35]. The sides of the $2z$ -gon are labelled A_1, \dots, A_z from top to bottom on the right, and two parallel sides have the same label. We draw the diagonal from the right end of the side labelled A_i on the right to the left end of the side labelled A_i on the left. There always exists i such that the angle θ between the billiard direction and the orthogonal of this diagonal is between $\frac{-\pi}{2z}$ and $\frac{\pi}{2z}$ (see Figure 4).

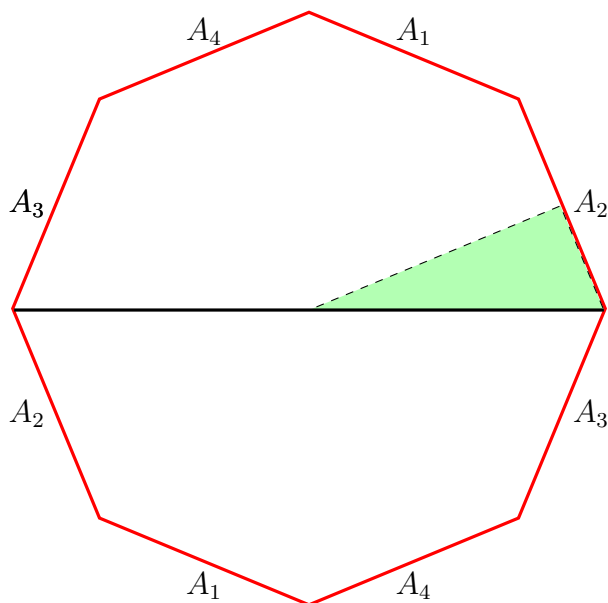


FIGURE 4. Regular Octagon

We put on the circle the points $-ie^{\frac{ij\pi}{z}}$ from $j = 0$ to $j = z$, which are the vertices of the $2z$ -gon; our diagonal is the vertical line from $-i$ to i , we project on it the sides of the polygon which are to the right of the diagonal, partitioning it into intervals I_1, \dots, I_z , and the sides of the polygon which are to the left of the diagonal, partitioning it into intervals J_1, \dots, J_z . The transformation which exchanges the intervals (I_1, \dots, I_z) with the (J_1, \dots, J_z) is identified with the interval exchange transformation \mathcal{I} on $[-1, 1[$ whose discontinuities are $\gamma_j = -\cos \frac{j\pi}{z} + \tan \theta \sin \frac{j\pi}{z}$, $1 \leq j \leq z-1$, while the discontinuities of \mathcal{I}^{-1} are $\beta_j = -\gamma_{z-j}$, composed with the map $x \rightarrow -x$ if $\theta < 0$. \mathcal{I} is a z -interval exchange transformation with permutation p defined by $p(j) = z - j + 1$ (see Figure 5).

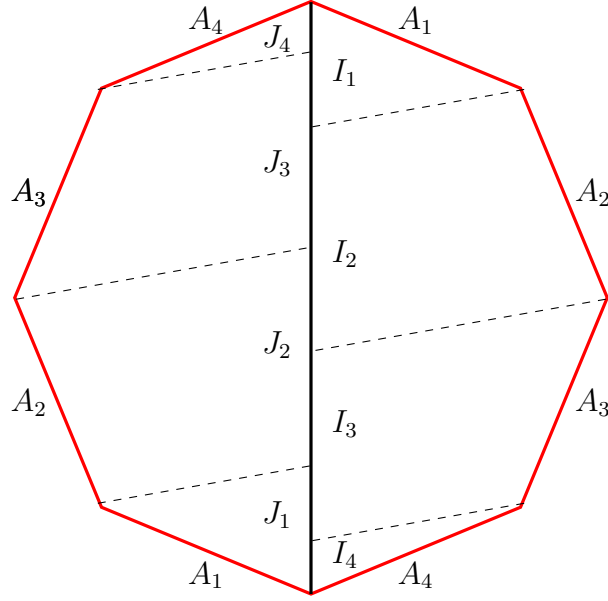


FIGURE 5. Interval exchange transformation in the regular octagon

Thus we consider the one-parameter family of interval exchange transformations \mathcal{I} , which depend on the parameter θ , $\frac{-\pi}{2z} < \theta < \frac{\pi}{2z}$ or equivalently on the parameter

$$y = \frac{1}{2} \left(\frac{\sin \frac{\pi}{z}}{|\tan \theta|} - \left(1 + \cos \frac{\pi}{z} \right) \right) > 0.$$

5.1. A rigid subfamily of interval exchange transformations. Let $\lambda = 2 \cos^2 \frac{\pi}{2z} = 1 + \cos \frac{\pi}{z}$. We define an application g by $g(y) = y - \lambda$ if $y > \lambda$, $g(y) = \frac{y}{1-2y}$ if $0 < y < \frac{1}{2}$ (the value of g on other sets is irrelevant).

From Theorem 11 of [15], in the particular case of Theorem 13 of the same paper, we deduce the following result.

Proposition 13. *If y is such that there exist two sequences m_n and q_n , with $m_0 = q_0 = 0$, and the iterates $g^{(n)}(y)$ satisfy*

- $\lambda < g^{(n)}(y)$ if $m_0 + q_0 + m_1 + q_1 + \dots + m_k + q_k \leq n \leq m_0 + q_0 + m_1 + q_1 + \dots + m_k + q_k + m_{k+1} - 1$ for some k ,
- $0 < g^{(n)}(y) \leq \frac{1}{2}$ if $m_0 + q_0 + m_1 + q_1 + \dots + m_k + q_k + m_{k+1} \leq n \leq m_0 + q_0 + m_1 + q_1 + \dots + m_k + q_k + m_{k+1} + q_{k+1} - 1$ for some k ,

then for all n , the trajectories of \mathcal{I} are covered by disjoint occurrences of words $M_{n,i}$ and $P_{n,i}$, $1 \leq i \leq z - 1$, built inductively in the following way:

- $M_{0,i} = i$, $1 \leq i \leq z - 1$, $P_{0,1} = z1$, $P_{0,i} = i$, $2 \leq i \leq z - 1$;
- if $m_0 + q_0 + m_1 + q_1 + \dots + m_k + q_k \leq n \leq m_0 + q_0 + m_1 + q_1 + \dots + m_k + q_k + m_{k+1} - 1$ for some k ,

$$\begin{aligned} P_{n+1,i} &= P_{n,i} \quad \text{for } 1 \leq i \leq z - 1, \\ M_{n+1,i} &= M_{n,i} P_{n,z-i+1} P_{n,i} \quad \text{for } 2 \leq i \leq z, \\ M_{n+1,1} &= M_{n,1} P_{n,1}; \end{aligned}$$

- if $m_0 + q_0 + m_1 + q_1 + \dots + m_k + q_k + m_{k+1} \leq n \leq m_0 + q_0 + m_1 + q_1 + \dots + m_k + q_k + m_{k+1} + q_{k+1} - 1$ for some k ,

$$M_{n+1,i} = M_{n,i} \quad \text{for } 1 \leq i \leq z-1,$$

$$P_{n+1,i} = P_{n,i} M_{n+1,d-i} M_{n+1,i} \quad \text{for } 1 \leq i \leq z-1.$$

We can now state

Proposition 14. *There exists two sequences of functions F_n , from \mathbb{N}^{2n-2} to \mathbb{N} , and G_n from \mathbb{N}^{2n-1} to \mathbb{N} such that, if for infinitely many n either $m_n > F(m_0, q_0, m_1, q_1, \dots, m_{n-1}, q_{n-1})$ or $q_n > G(m_0, q_0, m_1, q_1, \dots, m_{n-1}, q_{n-1}, m_n)$, and y is as in Proposition 13, then \mathcal{I} is rigid.*

Proof. If m_n is large, as in Proposition 11 we cover most of the trajectories by words $(P_{n,z-i+1} P_{n,i})^{m_n}, 2 \leq i \leq z-1$, and $P_{n,1}^{m_n}$. Let s_n be the least common multiple of $|P_{n,z-i+1}| + |P_{n,i}|, 2 \leq i \leq z-1$, and $|P_{n,1}|$; when we move by s_n inside these words, we see the same letter; thus s_n will give a rigidity sequence for \mathcal{I} if all the $m_n(|P_{n,z-i+1}| + |P_{n,i}|), 2 \leq i \leq z-1$, and $m_n|P_{n,1}|$ are much larger than s_n , which gives a condition as in the hypothesis; and similarly with the M words if q_n is large. \square

5.2. Rigidity of the flow. Let X_z be the surface obtained from the regular $2d$ -gon by identifying parallel sides together. X is a translation surface thus the linear flow is defined in every direction.

Proposition 15. *There exists a dense G_δ set of directions θ , of positive Hausdorff dimension, for which the linear flow on X_z in direction θ is rigid.*

Proof. We recall that in every non minimal direction, the linear flow is periodic (see [40]). In a periodic direction, the surface is decomposed into parallel cylinders of commensurable moduli. Up to normalization, the vectors of the heights of the cylinders form a finite set. More precisely, the periodic directions correspond to cusps of a lattice in $\text{SL}(2, \mathbb{R})$ (see [40]).

We give a detailed proof in the case $z = 4$ since one can make explicit computations. We recall that in a periodic direction, the octagon is decomposed into cylinders. The ratio of the lengths of these cylinders is $\sqrt{2}$.

Let us fix a direction θ . We approximate θ by periodic directions θ_n . We denote by l_n the length of the shortest cylinder in direction θ_n . We say that θ is approximable by (θ_n) at speed a if $|\theta - \theta_n| < \frac{1}{l_n^{2+a}}$. Assume that this property holds. Denote by $C_{1,n}$ the cylinder of length l_n and $C_{2,n}$ the cylinder of length $l_n\sqrt{2}$. We approximate $\sqrt{2}$ by $\frac{p_n}{q_n}$, with $|\sqrt{2} - \frac{p_n}{q_n}| < \frac{1}{q_n^2}$.

Our rigidity sequence will be $p_n l_n$. As in Figure 5, flowing in direction θ , the subinterval B of the interval J of the cylinder $C_{1,n}$ that escapes the cylinder $C_{1,n}$ after time l_n has length $l_n|\theta - \theta_n|$. Thus the area of the sub rectangle that does not run along the cylinder has measure $l_n^2|\theta - \theta_n|$. After time $p_n l_n$, the part that escapes has measure $p_n l_n^2|\theta - \theta_n| < \frac{p_n}{l_n^a}$. This measure tends to zero as n tends to infinity if $p_n \ll l_n^a$.

When we move by the time $p_n l_n$ of the flow inside $C_{1,n}$, there is no vertical translation by construction; inside $C_{2,n}$, we move by $p_n l_n$ modulo $l_n\sqrt{2}$; but $p_n l_n = l_n(q_n\sqrt{2} + \frac{x_n}{q_n})$ with $|x_n| < 1$, so we move by less than $\frac{l_n}{q_n}$. Thus rigidity holds if $l_n \ll q_n$ or equivalently $l_n \ll p_n$.

Our two conditions $l_n \ll p_n \ll l_n^a$ are compatible if $a > 1$. Moreover, since the periodic directions correspond to the cusps of a lattice in $\text{SL}(2, \mathbb{R})$, the set of θ approximable at speed a has positive Hausdorff dimension (see [29]) and is a dense G_δ set of the unit circle. Nevertheless it has 0 measure.

For general z , we have $z - 2$ cylinders $C_{n,i}$ of lengths $l_n \tau_i$ with $\tau_1 = 1$. By Dirichlet, we find p_n and $q_{n,i}$, such that $|\frac{1}{\tau_i} - \frac{q_{n,i}}{p_n}| < \frac{1}{p_n^{1+b}}$ for all $i > 1$ where $b = \frac{1}{z-3}$. Thus $p_n l_n = l_n (q_{n,i} \tau_i + \frac{x_{n,i} \tau_i}{p_n^b})$ with $|x_{n,i}| < 1$, thus $p_n l_n$ is a rigidity sequence if both $p_n \ll l_n^a$ and $l_n \ll p_n^b$ which is possible if $ab > 1$ which means that $a > z - 3$. \square

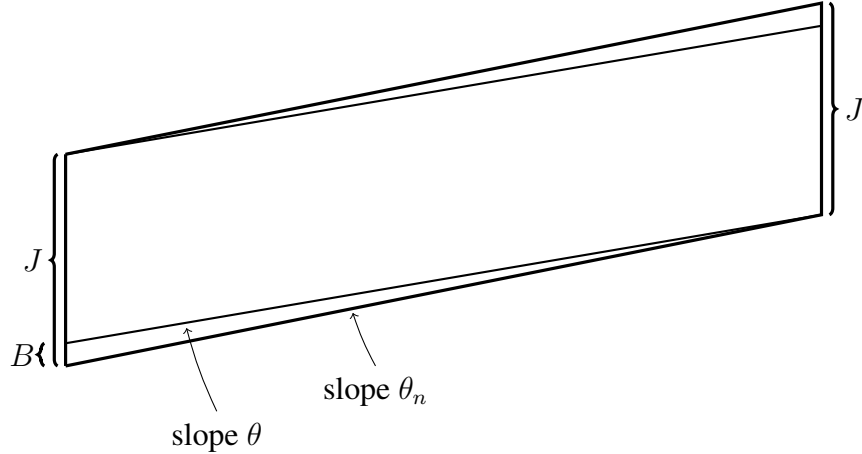


FIGURE 6. Trajectories in direction θ run along the cylinder from J in direction θ_n once unless they are in the subinterval B .

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