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Abstract. We give formulas for phase recovering from appropriate monochromatic phaseless scattering data at $2n$ points in dimension $d = 3$ and in dimension $d = 2$. These formulas are recurrent and explicit and their precision is proportional to n. By this result we continue studies of [Novikov, Bull.Sci.Math. **139**, 923-936, 2015], where formulas of such a type were given for $n = 1$, $d \geq 2$.

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1. Introduction

We consider monochromatic wave propagation modelled by the equation:

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \ d \geq 1, \ E > 0, \tag{1.1}$$

where Δ is the Laplacian in x, v is a coefficient, e.g., such that

$$\begin{aligned} &v \in L^\infty(\mathbb{R}^d), \quad supp\, v \subset D, \\ &D \ \text{ is an open bounded domain in } \ \mathbb{R}^d. \end{aligned} \tag{1.2}$$

Equation (1.1) can be considered as the Schrödinger equation at fixed energy E that describes a quantum mechanical particle interacting with a macroscopic object contained in D. In this case v is the potential of this interaction.

Equation (1.1) can be also considered as the Helmholtz equation of electrodynamic or acoustic wave propagation at fixed frequency ω. In this case

$$v(x) = (1 - (n(x)))^2 \big(\frac{\omega}{c_0}\big)^2, \quad E = \big(\frac{\omega}{c_0}\big)^2, \tag{1.3}$$

where $n(x)$ is a scalar index of refraction, $n(x) \equiv 1$ on $\mathbb{R}^d \backslash D$, c_0 is a reference speed of wave propagation.

For equation (1.1) we consider the classical scattering solutions ψ^+ satisfying the Lippmann-Schwinger integral equation

$$\psi^+(x,k) = e^{ikx} + \int\limits_D G^+(x-y,k)v(y)\psi^+(y,k)dy, \tag{1.4}$$

$$G^+(x,k) \stackrel{\text{def}}{=} -(2\pi)^{-d} \int\limits_{\mathbb{R}^d} \frac{e^{i\xi x}d\xi}{\xi^2 - k^2 - i0} = G_0^+(|x|,|k|), \tag{1.5}$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{R}^d$, $k^2 = E$. Note that

$$G^+(x,k) = -\frac{i}{4} H_0^1(|x|\,|k|) \quad \text{for} \quad d=2, \quad G^+(x,k) = -\frac{e^{i|k||x|}}{4\pi|x|} \quad \text{for} \quad d=3, \tag{1.6}$$

where H_0^1 is the Hankel function of the first type.

Remark 1.1. In the present work, in addition to (1.2), we assume that

$$\text{equation (1.4) is uniquely solvable for } \psi^+(\cdot,k) \in L^\infty(\mathbb{R}^d) \text{ for fixed } E>0, \tag{1.7}$$

where $k \in \mathbb{R}^d$, $k^2 = E$. If, for example, v satisfies (1.2) and is real-valued, then (1.7) is fulfilled automatically.

In particular, the solutions ψ^+ have the following asymptotics as $|x| \to \infty$:

$$\psi^+(x,k) = e^{ikx} + \frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f_1(k, |k|\frac{x}{|x|}) + O\big(\frac{1}{|x|^{(d+1)/2}}\big), \tag{1.8}$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{R}^d$, $k^2 = E$. In addition,

$$f_1(k,l) = c(d,|k|) f(k,l), \tag{1.9}$$

$$c(d,|k|) = -\pi i (-2\pi i)^{(d-1)/2} |k|^{(d-3)/2}, \tag{1.10}$$

$$f(k,l) = (2\pi)^{-d} \int\limits_D e^{-ily} v(y) \psi^+(y,k)\, dy, \tag{1.11}$$

where $k \in \mathbb{R}^d$, $l \in \mathbb{R}^d$, $k^2 = l^2 = E$.

We recall that $\psi^+(x,k)$ at fixed k describes scattering of the incident plane wave described by e^{ikx} on the scatterer described by v. In addition, the second term on the right-hand side of (1.8) associates with the leading scattered spherical wave.

The function f arising in (1.8), (1.9) is the classical scattering amplitude for equation (1.1). This function is defined on

$$\Omega_E = \{k \in \mathbb{R}^d, \quad l \in \mathbb{R}^d, \quad k^2 = l^2 = E\} = \mathbb{S}^{d-1}_{\sqrt{E}} \times \mathbb{S}^{d-1}_{\sqrt{E}}, \tag{1.12}$$

where

$$\mathbb{S}^{d-1}_r = \{m \in \mathbb{R}^d : \quad |m| = r\}. \tag{1.13}$$

We recall that in quantum mechanics, by Born's principle, the values of the functions ψ^+ and f with phase have no direct physical sense, whereas the phaseless values of $|\psi^+|^2$ and $|f|^2$ have probabilistic interpretations and can be directly obtained in experiments; see [B], [FM]. On the other hand, in electrodynamic or acoustic experiments ψ^+ and f can be directly measured, at least, in principle. However, in many important cases of monochromatic electro-magnetic wave propagation (e.g., X-rays and lasers) the wave frequency is so great that only intensities like $|\psi^+|^2$ and $|f|^2$ can be measured in practice by modern technical devices; see [HN] and references therein.

In the present work we continue studies on the problem of finding f and ψ^+ from $|\psi^+|^2$, where ψ^+ and $|\psi^+|^2$ are considered outside of D.

In particular, in the present work, for fixed $(k,l) \in \Omega_E$, $k \neq l$, for $d=3$ and $d=2$, we give

$$\begin{aligned} &\text{formulas for finding } \ f(k,l) \ \text{ up to } \ O(s^{-n}) \ \text{ as } \ s \to +\infty \\ &\text{from } \ |\psi^+(x,k)|^2 \ \text{ given at } \ 2n \ \text{ points } \ x = x_1(s), \dots, x_{2n}(s), \end{aligned} \tag{1.14}$$

where

$$\begin{aligned} &x_i(s) = r_i(s)\hat{l}, \quad i = 1, \dots, 2n, \quad \hat{l} = l/|l|, \\ &r_{2j-1}(s) = \lambda_j s, \quad r_{2j}(s) = \lambda_j s + \tau, \quad j = 1, \dots, n, \\ &\lambda_1 = 1, \quad \lambda_{j_1} < \lambda_{j_2} \ \text{ for } \ j_1 < j_2, \quad \tau = \tau_{fixed} > 0. \end{aligned} \tag{1.15}$$

These formulas for f are recurrent and explicit and are presented in detail in Sections 3, 6, 7, 9, 10, 11. The precision of these formulas is $O(s^{-n})$, $s \to +\infty$, and in this sense is proportional to n.

In the present work, for fixed $(k,l) \in \Omega_E$, $k \neq l$, for $d=3$ and $d=2$, we also give

$$\begin{aligned} &\text{formulas for finding } \ \psi^+(s\hat{l},k) \ \text{ up to } \ O(s^{-n-(d-1)/2}) \ \text{ as } \ s \to +\infty \\ &\text{from } \ |\psi^+(x,k)|^2 \ \text{ given at } \quad x = x_1(s), \dots, x_{2n}(s), \end{aligned} \tag{1.16}$$

where $\hat{l}$ and $x_i(s)$, $i = 1, \dots, 2n$, are defined in (1.15). These formulas for ψ^+ are presented in detail in Sections 3, 12.

Actually, in the present work we continue studies of [N2], [N3], [N4]. In [N2], [N4] formulas of the type (1.14) were given for the 2-point case (i.e., for $n=1$) for $d \geq 2$ (but with less precise $O(s^{-1/2})$ in place of $O(s^{-1})$ for $d=2$); see Section 2. In turn, article [N3] gives exact versions (without error terms) of formulas (1.14), (1.16) for the 3-point case for $d=1$; see [N3], [N4] for details.

In addition, the present work continues studies on inverse wave propagation problems without phase information; see Chapter 10 of [ChS], [JL], [IK], [K1], [N2], [N3], [N4], [KR], [R], [K2], [AHN], [HN], [P] and references therein in connection with results given in the literature on the later problem. In particular, formulas (1.14) of the present work and known methods of inverse scattering from f (see [ChS], [N1], [N2] and references therein) yield new results on phaseless inverse scattering from $|\psi^+|^2$ measured outside of D for wave propagation modelled by equation (1.1).

Remark 1.2. The approach of the present work as well as of [N2] can be also used for phaseless inverse scattering for obstacles.

2. Two-point formulas for f for $d \geq 2$

We define

$$a(x,k) = |x|^{(d-1)/2}(|\psi^+(x,k)|^2 - 1), \quad x \in \mathbb{R}^d \backslash \{0\}, \quad k \in \mathbb{R}^d \backslash \{0\}, \tag{2.1}$$

where ψ^+ is the function of (1.4), (1.8).

The following formulas of [N2], [N4] hold:

$$\begin{pmatrix} Re\, f_1(k,l) \\ Im\, f_1(k,l) \end{pmatrix} = M \left(\begin{pmatrix} a(x_1,k) \\ a(x_2,k) \end{pmatrix} - \begin{pmatrix} \delta_1 a(x_1,k) \\ \delta_1 a(x_2,k) \end{pmatrix} \right), \tag{2.2}$$

$$M = \frac{1}{2\sin(\varphi_2-\varphi_1)} \begin{pmatrix} \sin\varphi_2 & -\sin\varphi_1 \\ -\cos\varphi_1 & \cos\varphi_2 \end{pmatrix}, \tag{2.3}$$

$$x_1 = s\hat{l}, \quad x_2 = (s+\tau)\hat{l}, \quad \hat{l} = l/|l|, \tag{2.4}$$

$$\varphi_j = |k||x_j| - kx_j, \quad j = 1,2, \tag{2.5}$$

$$\varphi_2 - \varphi_1 = \tau(|k| - k\hat{l}), \tag{2.6}$$

$$\delta_1 a(x_1,k) = O(s^{-\sigma}), \quad \delta_1 a(x_2,k) = O(s^{-\sigma}) \ \text{ as } \ s \to +\infty \tag{2.7}$$

uniformly in $\hat{k} = k/|k|$, $\hat{l} = l/|l|$ and τ at fixed $E > 0$,

$$\sigma = 1/2 \ \text{ for } \ d = 2, \quad \sigma = 1 \ \text{ for } \ d \geq 3, \tag{2.8}$$

where

$$\sin(\varphi_2 - \varphi_1) \neq 0, \tag{2.9}$$

$k \in \mathbb{R}^d$, $l \in \mathbb{R}^d$, $|k| = |l| = \sqrt{E}$, $s > 0$, $\tau > 0$.

In view of (1.9), formulas (2.1)-(2.8) are explicit asymptotic formulas for finding phased $f(k,l)$ at fixed $(k,l) \in \Omega_E$, $k \neq l$, from phaseless $|\psi^+(x,k)|^2$ at two points $x = x_1, x_2$ defined in (2.4), where $s \to +\infty$. For $d \geq 3$ these formulas can be considered as formulas (1.14) for $n = 1$. For $d = 2$ these formulas can be also considered as formulas of the type (1.14) but with less precise $O(s^{-1/2})$ in place of $O(s^{-1})$.

We recall that (see [N2]):

$$a(x,k) = a_1(x,k) + \delta_1 a(x,k), \tag{2.10}$$

where

$$a_1 = 2Re\, e^{i\varphi} f_1, \tag{2.11}$$

$$\delta_1 a = |x|^{-(d-1)/2}|f_1|^2 + 2|x|^{(d-1)/2} Re\,(\delta_1\psi^+ \bar{\psi}_1) + |x|^{(d-1)/2}|\delta_1\psi^+|^2 = \tag{2.12}$$

$$\delta_{1,1}a + \delta_{1,2}a + \delta_{1,3}a,$$

where $\varphi = |k||x| - kx$, $f_1 = f_1(k, |k|x/|x|)$ is the function of (1.8),

$$\psi_1^+ = e^{ikx} + \frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f_1(k, |k|\frac{x}{|x|}), \tag{2.13}$$

$$\delta_1\psi^+ = \psi^+ - \psi_1^+, \tag{2.14}$$

where ψ^+ is the function of (1.4), (1.8).

In addition, we have that (see [N2]):

$$|\delta_1\psi^+(x,k)| \le |x|^{-(d+1)/2}C_1, \tag{2.15}$$
$$|\delta_{1,1}a(x,k)| \le |x|^{-(d-1)/2}(F_1)^2, \tag{2.16}$$
$$|\delta_{1,2}a(x,k)| \le 2|x|^{-1}C_1(1+|x|^{-(d-1)/2}F_1), \tag{2.17}$$
$$|\delta_{1,3}a(x,k)| \le |x|^{-(d+3)/2}(C_1)^2, \tag{2.18}$$
$$x \in \mathbb{R}^d\backslash\{0\}, \quad k \in \mathbb{R}^d, \quad k^2 = E > 0,$$

where $C_1 = C_1(v,E)$, $F_1 = F_1(v,E)$ are positive constants depending on v and E only, in addition $F_1 = \|f_1\|_{L^\infty(\Omega_E)}$.

Formulas (2.12)-(2.18) can be used for obtaining detailed versions of estimates (2.7), (2.8).

Note also that formulas (2.2), (2.3) for f_1 can be rewritten as follows:

$$\begin{pmatrix} f_1(k,l) \\ \bar{f}_1(k,l) \end{pmatrix} = M\left(\begin{pmatrix} a(x_1,k) \\ a(x_2,k) \end{pmatrix} - \begin{pmatrix} \delta_1 a(x_1,k) \\ \delta_1 a(x_2,k) \end{pmatrix}\right), \tag{2.19}$$
$$M = \frac{-1}{2i\sin(\varphi_2-\varphi_1)}\begin{pmatrix} e^{-i\varphi_2} & -e^{-i\varphi_1} \\ -e^{i\varphi_2} & e^{i\varphi_1} \end{pmatrix}. \tag{2.20}$$

Finally, note that formulas (1.8), (1.9), (2.2)-(2.8) also yield formulas (1.16) for $n = 1$, $d \ge 3$ and formulas of the type (1.16) but with less precise $O(s^{-1})$ in place of $O(s^{-3/2})$ for $n = 1$, $d = 2$.

3. General idea of 2n-point formulas for f and ψ^+ for $d \ge 2$

The main drawback of the two-point formulas (1.9), (2.2)-(2.9) for finding phased f from phaseless $|\psi^+|^2$ is a slow decay of the error as $s \to +\infty$; see (2.2), (2.7), (2.8). This drawback motivates our considerations given below.

For $d \ge 2$, to obtain rapidly convergent formulas for finding $f(k,l)$ from $|\psi^+(x,k)|^2$ at appropriate 2n points

$$x = x_1(s), \dots, x_{2n}(s), \tag{3.1}$$

where

$$\begin{aligned} &x_i(s) = r_i(s)\hat{l}, \quad i = 1,\dots,2n, \quad \hat{l} = l/|l|, \\ &r_{i_1}(s) < r_{i_2}(s) \quad \text{for} \quad i_1 < i_2, \quad r_1(s) = s \to +\infty, \end{aligned} \tag{3.2}$$

we suggest the following general scheme.

We use that, under our assumptions (1.2), (1.7), formula (1.8) admits the following much more precise version:

$$\psi^+(x,k) = e^{ikx} + \frac{e^{i|k||x|}}{|x|^{(d-1)/2}}\left(\sum_{j=1}^{n}\frac{f_j(k,|k|\frac{x}{|x|})}{|x|^{j-1}} + O\big(\frac{1}{|x|^n}\big)\right) \quad \text{as} \quad |x| \to \infty, \tag{3.3}$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{R}^d$, $k^2 = E > 0$, $n \in \mathbb{N}$. Actually, expansion (3.3) follows from assumptions (1.2), equation (1.4) and an extended version of Proposition 1.1 of [M]. Note also that formulas (1.9)-(1.11) for f_1 have analogs for f_j, $j \ge 2$.

We look for

$$\begin{aligned}&\text{formulas for finding } \ f_j(k,l) \ \text{ up to } \ O(s^{-(n-j+1)}) \ \text{ as } \ s\to+\infty,\ j=1,\dots,n,\\ &\text{from } \ |\psi^+(x,k)|^2 \ \text{ given at } \ 2n \ \text{ points } \ x=x_1(s),\dots,x_{2n}(s) \ \text{ of the form (3.2)},\end{aligned} \tag{3.4}$$

where $f_j = f_j(k,l)$, $j = 1,\dots,n$, are the functions arising in (3.3), k and l are fixed, $k, l \in \mathbb{R}^d$, $|k| = |l| > 0$, $l \neq k$.

In particular, in view of (1.9) and (3.4) for $j = 1$ this scheme yields 2n-point formulas for finding $f = f(k,l)$ from $|\psi^+|^2 = |\psi^+(x,k)|^2$ given at $x = x_1(s),\dots,x_{2n}(s)$ of the form (3.2) with the error $O(s^{-n})$ as $s \to +\infty$, that is with rapidly decaying error if n is sufficiently large.

Besides, in view of (3.3), (3.4) this scheme also yields

$$\begin{aligned}&\text{formulas for finding } \ \psi^+(s\hat{l},k) \ \text{ up to } \ O(s^{-n-(d-1)/2}) \ \text{ as } \ s\to+\infty\\ &\text{from } \ |\psi^+(x,k)|^2 \ \text{ given at } 2n \ \text{ points } \ x=x_1(s),\dots,x_{2n}(s) \ \text{ of the form } \ (3.2),\end{aligned} \tag{3.5}$$

where k and l are fixed, $k, l \in \mathbb{R}^d$, $|k| = |l| > 0$, $l \neq k$, $\hat{l} = l/|l|$.

The realization of the general scheme of this Section depends on the concrete choice of the functions $r_i = r_i(s)$, $i = 1,\dots,2n$, of (3.1), (3.2). In the present work we realize this scheme for the case when $r_i = r_i(s)$ are defined as in (1.15) for $d = 3$ and for $d = 2$. In particular, in this case $x_{2j}(s) - x_{2j-1}(s) = \tau\hat{l}$, $j = 1,\dots,n$, as well as $x_2(s) - x_1(s) = \tau\hat{l}$ for x_2, x_1 of the 2-point case considered in Section 2 (see formula (2.4)).

Note that, originally, the realization of the general scheme of this Section was started in [NG] for the case when $r_i(s) = \lambda_i s$, $i = 1,\dots,2n$, $1 = \lambda_1 < \lambda_2,\dots,< \lambda_{2n}$.

Finally, our results on formulas (1.14), (1.16), (3.4), (3.5), for $x_1(s),\dots,x_{2n}(s)$ defined by (1.15), can be summarized as follows.

For $d \geq 3$, $n = 1$, and $x_1(s)$, $x_2(s)$ defined according to (1.15) (or by (2.4)), formulas (1.14), (3.4) are realized (in [N2], [N4]) as formulas (2.2)-(2.9) of Section 2.

For $d = 3$, $n = 2$, and $x_1(s),\dots,x_4(s)$ defined according to (1.15) (or by (6.1)), formulas (1.14), (3.4) are realized as formulas (6.13)-(6.16), (6.5), (6.3) of Section 6.

For $d = 3$, $n > 2$, and $x_1(s),\dots,x_{2n}(s)$ defined by (1.15) (or by (7.1)), formulas (1.14), (3.4) are realized as inductive formulas (7.4), (7.5), (7.16),(7.12), (7.6)-(7.10) of Section 7.

For $d = 2$, $n = 1$, and $x_1(s)$, $x_2(s)$ defined according to (1.15) (or by (2.4)), formulas (1.14), (3.4) with $O(s^{-1})$ (and not with $O(s^{-1/2})$ as in (2.2)-(2.9) for $d = 2$) are realized as formulas (9.3), (9.4), (9.2) of Section 9.

For $d = 2$, $n = 2$, and $x_1(s),\dots,x_4(s)$ defined according to (1.15), formulas (1.14), (3.4) are realized as formulas (10.20)-(10.23), (10.13), (10.10), (10.11), (10.8), (10.9) of Section 10.

For $d = 2$, $n > 2$, and $x_1(s),\dots,x_{2n}(s)$ defined by (1.15), formulas (1.14), (3.4) are realized as inductive formulas (11.2), (11.6), (11.7) and (7.4) for $d = 2$, (11.11), (11.12) of Section 11.

Formulas (1.16), (3.5) are realized as formulas (12.1), (12.2) of Section 12.

Finally, note that detailed versions of $O(s^{-\sigma})$ in (2.7), $O(s^{-(n-j+1)})$ in (3.4) and $O(s^{-n-(d-1)/2})$ in (3.5) will be given in subsequent papers.

4. Presentations for a for $d = 3$

The function a defined by (2.1) can be presented as follows for $d = 3$:

$$a(x,k) = a_n(x,k) + \delta_n a(x,k), \tag{4.1}$$

$$a_n(x,k) = a_n^1(x,k) + a_n^2(x,k), \tag{4.2}$$

$$a_n^1(x,k) = \sum_{j=1}^{n} \frac{e^{i\varphi} f_j}{|x|^{j-1}} + \sum_{j=1}^{n} \frac{e^{-i\varphi} \bar{f}_j}{|x|^{j-1}}, \tag{4.3}$$

$$a_n^2(x,k) = \sum_{\substack{j\in\mathbb{N}\\ 2j\le n}} \frac{|f_j|^2}{|x|^{2j-1}} + \sum_{\substack{j_1,j_2\in\mathbb{N}\\ j_1\ne j_2, j_1+j_2\le n}} \frac{f_{j_1}\bar{f}_{j_2}}{|x|^{j_1+j_2-1}}, \tag{4.4}$$

$$\delta_n a(x,k) = O\Big(\frac{1}{|x|^n}\Big) \ \text{ as } \ |x|\to\infty, \tag{4.5}$$

where

$$\varphi = \varphi(x,k) = |x|(|k| - k\,\hat{x}), \tag{4.6}$$

$$f_j = f_j(k, |k|\hat{x}), \quad j = 1,\dots,n, \tag{4.7}$$

$x \in \mathbb{R}^3\backslash\{0\}$, $\hat{x} = x/|x|$, $k \in \mathbb{R}^3$, $k^2 = E > 0$, $n \in \mathbb{N}$.

Formulas (4.1)-(4.7) follow from (2.1), (3.3) for $d = 3$.

In addition, we have that:

$$a(x+\tau\hat{x},k) = a_n^{1,\tau}(x,k) + (a_n^1(x+\tau\hat{x},k) - a_n^{1,\tau}(x,k)) + a_n^2(x+\tau\hat{x},k) + \delta_n a(x+\tau\hat{x},k), \tag{4.8}$$

$$a_n^1(x+\tau\hat{x},k) - a_n^{1,\tau}(x,k) = \Delta_n^{1,\tau}(x,k) + \rho_n^{1,\tau}(x,k), \tag{4.9}$$

where

$$a_n^{1,\tau}(x,k) = \sum_{j=1}^{n} \frac{e^{i\varphi_\tau} f_j}{|x|^{j-1}} + \sum_{j=1}^{n} \frac{e^{-i\varphi_\tau} \bar{f}_j}{|x|^{j-1}}, \tag{4.10}$$

$$\Delta_n^{1,\tau}(x,k) = \sum_{j=1}^{n-1} \frac{e^{i\varphi_\tau} f_j}{|x|^{j-1}} w_{j-1}(|x|,\tau) + \sum_{j=1}^{n-1} \frac{e^{-i\varphi_\tau} \bar{f}_j}{|x|^{j-1}} w_{j-1}(|x|,\tau), \tag{4.11}$$

$$\rho_n^{1,\tau}(x,k) = \frac{e^{i\varphi_\tau} f_n}{|x|^{n-1}} w_{n-1}(|x|,\tau) + \frac{e^{-i\varphi_\tau} \bar{f}_n}{|x|^{n-1}} w_{n-1}(|x|,\tau), \tag{4.12}$$

$$w_{j-1}(|x|,\tau) = \frac{1}{(1+\tau/|x|)^{j-1}} - 1 = O\Big(\frac{1}{|x|}\Big), \ |x|\to\infty, \tag{4.13}$$

at fixed τ, $j = 1,\dots,n$,

$$a_n^2(x+\tau\hat{x},k) = \sum_{\substack{j\in\mathbb{N}\\ 2j\le n}} \frac{|f_j|^2}{(|x|+\tau)^{2j-1}} + \sum_{\substack{j_1,j_2\in\mathbb{N}\\ j_1\ne j_2, j_1+j_2\le n}} \frac{f_{j_1}\bar{f}_{j_2}}{(|x|+\tau)^{j_1+j_2-1}}, \tag{4.14}$$

$$\rho_n^{1,\tau}(x,k) = O\Big(\frac{1}{|x|^n}\Big), \quad \delta_n a(x+\tau\hat{x},k) = O\Big(\frac{1}{|x|^n}\Big) \ \text{ as } \ |x|\to\infty, \tag{4.15}$$

where

$$\varphi_\tau = \varphi_\tau(x,k) = (|x|+\tau)(|k|-k\hat{x}), \tag{4.16}$$

f_j are the same that in (3.3), (4.3), (4.4), $x \in \mathbb{R}^3 \backslash \{0\}$, $\hat{x} = x/|x|$, $k \in \mathbb{R}^3$, $k^2 = E > 0$, $n \in \mathbb{N}$.

5. An advanced version of formulas (2.2)-(2.9) for $d = 3$

Let

$$z_n(k,l,s) = \sum_{j=1}^{n} \frac{f_j(k,l)}{s^{j-1}}, \quad (k,l) \in \Omega_E, \quad s > 0, \quad n \in \mathbb{N}, \tag{5.1}$$

where f_j are the coefficients of (3.3).

Proposition 5.1. *Under assumptions (1.2), (1.7), the following formulas hold, for $d = 3$:*

$$\begin{pmatrix} z_n(k,l,s) \\ \bar{z}_n(k,l,s) \end{pmatrix} = M \begin{pmatrix} a(s\hat{l},k) \\ a((s+\tau)\hat{l},k) \end{pmatrix} - \tag{5.2}$$
$$M\left(\begin{pmatrix} a_n^2(s\hat{l},k) \\ \Delta_n^{1,\tau}(s\hat{l},k) + a_n^2((s+\tau)\hat{l},k) \end{pmatrix} + \begin{pmatrix} \delta_n a(s\hat{l},k) \\ \rho_n^{1,\tau}(s\hat{l},k) + \delta_n a((s+\tau)\hat{l},k) \end{pmatrix} \right),$$

$$M = M_{k,l,\tau,s} = \frac{-1}{2i\sin(\varphi_2 - \varphi_1)} \begin{pmatrix} e^{-i\varphi_2} & -e^{-i\varphi_1} \\ -e^{i\varphi_2} & e^{i\varphi_1} \end{pmatrix}, \tag{5.3}$$

$$\varphi_1 = (|k| - k\hat{l})s, \quad \varphi_2 = (|k| - k\hat{l})(s+\tau), \tag{5.4}$$

$$\delta_n a(s\hat{l},k) = O(s^{-n}), \quad \delta_n a((s+\tau)\hat{l},k) = O(s^{-n}) \ \text{ as } s \to +\infty \tag{5.5}$$

uniformly in $\hat{k}$, $\hat{l}$ and τ at fixed $E > 0$,

$$\rho_n^{1,\tau}(s\hat{l},k) = O(s^{-n}) \ \text{ as } \ s \to +\infty \tag{5.6}$$

uniformly in $\hat{k}$, $\hat{l}$ at fixed $\tau > 0$ and $E > 0$, where

$$\sin(\varphi_2 - \varphi_1) \neq 0, \tag{5.7}$$

$k \in \mathbb{R}^3$, $l \in \mathbb{R}^3$, $|k| = |l| = \sqrt{E}$, $\hat{k} = k/|k|$, $\hat{l} = l/|l|$, $\tau > 0$, $n \in \mathbb{N}$.

Proof of Proposition 5.1. Using (4.1)-(4.3), (4.8)-(4.10), (5.1) we obtain that

$$a(s\hat{l},k) = e^{i\varphi_1} z_n(k,l,s) + e^{-i\varphi_1} \bar{z}_n(k,l,s) + a_n^2(s\hat{l},k) + \delta_n a(s\hat{l},k), \tag{5.8}$$

$$a((s+\tau)\hat{l},k) = e^{i\varphi_2} z_n(k,l,s) + e^{-i\varphi_2} \bar{z}_n(k,l,s) + \tag{5.9}$$
$$\Delta_n^{1,\tau}(s\hat{l},k) + a_n^2((s+\tau)\hat{l},k) + \rho_n^{1,\tau}(s\hat{l},k) + \delta_n a((s+\tau)\hat{l},k),$$

where φ_1, φ_2 are given by (5.4).

Considering (5.8), (5.9) as a linear system for z_n, $\bar{z}_n$ we obtain (5.2), (5.3).

Estimates (5.5), (5.6) follow from (4.5), (4.12), (4.13).

Proposition 5.1 is proved.

Actually, formulas (5.2)-(5.7) for $n = 1$ reduce to formulas (2.2)-(2.9) (with (2.2), (2.3) written as (2.19), (2.20)), $d = 3$.

6. Finding f_1, f_2 for the 4-point case for $d = 3$

In this section we realize the scheme of Section 3 for the case when $n = 2$, $d = 3$ and

$$x_1 = s\hat{l},\ x_2 = (s+\tau)\hat{l},\ x_3 = \lambda s\hat{l},\ x_4 = (\lambda s+\tau)\hat{l}, \tag{6.1}$$

where $\tau > 0$ and $\lambda > 1$ are fixed, $s \to +\infty$.

6.1. Considerations at x_1 and x_2. Due to (5.2)-(5.7) for $n = 1$ (or, by other words, due to (2.19), (2.20), (2.4)-(2.9) for $d = 3$), we have that

$$\begin{aligned} &f_1 = f_{1,1}(s) + O(s^{-1}) \ \text{ as } \ s \to +\infty \\ &\text{for fixed } \ \sin((|k| - k\hat{l})\tau) \neq 0 \ \text{ and } \ E, \end{aligned} \tag{6.2}$$

where $f_1 = f_1(k,l)$, $f_{1,1}(s) = f_{1,1}(k,l,\tau,s)$,

$$\begin{pmatrix} f_{1,1}(s) \\ \bar{f}_{1,1}(s) \end{pmatrix} = M \begin{pmatrix} a(x_1,k) \\ a(x_2,k) \end{pmatrix}, \tag{6.3}$$

$M = M_{k,l,\tau,s}$ is defined by (5.3), x_1, x_2 are defined in (6.1), $(k,l) \in \Omega_E$, $d = 3$, $E > 0$, $\hat{l} = l/|l|$, $\tau > 0$.

Proposition 6.1. *Under assumptions (1.2), (1.7) for $d = 3$, the following formulas hold:*

$$\begin{aligned} &z_2(s) = z_{2,1}(s) + O(s^{-2}) \ \text{ as } \ s \to +\infty \\ &\text{for fixed } \ \sin((|k| - k\hat{l})\tau) \neq 0,\ E \ \text{ and } \ \tau, \end{aligned} \tag{6.4}$$

where $z_2(s) = z_2(k,l,s)$ is defined by (5.1) for $n = 2$, $z_{2,1}(s) = z_{2,1}(k,l,\tau,s)$ is defined as follows:

$$\begin{pmatrix} z_{2,1}(s) \\ \bar{z}_{2,1}(s) \end{pmatrix} = \begin{pmatrix} f_{1,1}(s) \\ \bar{f}_{1,1}(s) \end{pmatrix} - \frac{|f_{1,1}(s)|^2}{s} M \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{6.5}$$

$f_{1,1}(s) = f_{1,1}(k,l,\tau,s)$ is defined by (6.3), $M = M_{k,l,\tau,s}$ is defined by (5.3), $(k,l) \in \Omega_E$, $d = 3$, $E > 0$, $\hat{l} = l/|l|$, $\tau > 0$.

Proof of Proposition 6.1. Proposition 6.1 follows from Proposition 5.1 for $n = 2$ and from formulas (6.2), (6.3). In more detail, the proof consists of the following.

Formulas (4.4), (4.11), (4.13), (4.14) for $n = 2$ imply that

$$a_2^2(s\hat{l},k) = \frac{|f_1(k,l)|^2}{s}, \tag{6.6}$$

$$\Delta_2^{1,\tau}(s\hat{l},k) = 0, \tag{6.7}$$

$$a_2^2((s+\tau)\hat{l},k) = \frac{|f_1(k,l)|^2}{s+\tau}, \tag{6.8}$$

$s>0$, $\tau>0$.

Formulas (5.2)-(5.7) for $n=2$ and formulas (6.6)-(6.8) imply that

$$\begin{pmatrix} z_2(s) \\ \bar{z}_2(s) \end{pmatrix} = M \begin{pmatrix} a(s\hat{l},k) \\ a((s+\tau)\hat{l},k) \end{pmatrix} - M \begin{pmatrix} s^{-1}|f_1|^2 \\ (s+\tau)^{-1}|f_1|^2 \end{pmatrix} + O(\frac{1}{s^2}) \ \text{ as } s \to +\infty \qquad (6.9)$$

for fixed $\sin((|k|-k\hat{l})\tau) \neq 0$, E and τ.

Formulas (6.2), (6.3), (6.9) imply (6.4), (6.5).

This completes the proof of Proposition 6.1.

6.2. Considerations at x_3 and x_4. In addition to (6.4), we have that

$$\begin{aligned} &z_2(\lambda s) = z_{2,1}(\lambda s) + O((\lambda s)^{-2}) \ \text{ as } \ s \to +\infty \\ &\text{for fixed } \ \sin((|k|-k\hat{l})\tau) \neq 0, \ E \ \text{ and } \ \tau. \end{aligned} \qquad (6.10)$$

The point is that $z_{2,1}(\lambda s)$ is given in terms of $a(x_3,k)$, $a(x_4,k)$ via formulas (6.3), (6.5) with x_3, x_4 in place of x_1, x_2 and λs in place of s, where x_1, x_2, x_3, x_4 are defined by (6.1).

6.3. Formulas for f_1, f_2. Due to definition (5.1), for $n=2$, and formulas (6.4), (6.10), we have that

$$f_1 + s^{-1}f_2 = z_{2,1}(s) + O(s^{-2}), \quad s \to +\infty, \qquad (6.11)$$

$$f_1 + (\lambda s)^{-1}f_2 = z_{2,1}(\lambda s) + O((\lambda s)^{-2}), \quad s \to +\infty, \qquad (6.12)$$

$$\text{for fixed } \ \sin((|k|-k\hat{l})\tau) \neq 0, \ E \ \text{ and } \ \tau,$$

where $f_1 = f_1(k,l)$, $f_2 = f_2(k,l)$,
$z_{2,1}(s) = z_{2,1}(k,l,\tau,s)$, $(k,l) \in \Omega_E$, $d=3$, $E>0$, $\hat{l} = l/|l|$, $\tau>0$.

Proceeding from (6.11), (6.12) we obtain the following result.

Proposition 6.2. *Under assumptions (1.2), (1.7) for $d=3$, the following formulas hold:*

$$f_1 = f_{1,2}(s) + O(s^{-2}), \quad s \to +\infty, \qquad (6.13)$$

$$f_2 = f_{2,1}(s) + O(s^{-1}), \quad s \to +\infty, \qquad (6.14)$$

$$\text{for fixed } \ \sin((|k|-k\hat{l})\tau) \neq 0, \ E, \ \tau \ \text{ and } \ \lambda,$$

where $f_1 = f_1(k,l)$, $f_2 = f_2(k,l)$,
$f_{1,2}(s) = f_{1,2}(k,l,\tau,\lambda,s)$, $f_{2,1}(s) = f_{2,1}(k,l,\tau,\lambda,s)$,

$$f_{1,2}(s) = -\frac{z_{2,1}(s)}{\lambda-1} + \frac{\lambda z_{2,1}(\lambda s)}{\lambda-1}, \qquad (6.15)$$

$$f_{2,1}(s) = \frac{s\lambda}{\lambda-1}(z_{2,1}(s) - z_{2,1}(\lambda s)), \qquad (6.16)$$

where $z_{2,1}(s) = z_{2,1}(k,l,\tau,s)$ *is found via (6.3), (6.5) from* $|\psi^+(x,k)|^2$ *at* $x = x_1, x_2$ *of (6.1),* $z_{2,1}(\lambda s) = z_{2,1}(k,l,\tau,\lambda s)$ *is found via (6.3), (6.5) (where* s *and* x_1, x_2 *are replaced by* λs *and* x_3, x_4 *of (6.1)),* $(k,l) \in \Omega_E$, $d = 3$, $E > 0$, $\hat{l} = l/|l|$, $\tau > 0$, $\lambda > 1$.

Propositions 6.1 and 6.2 realize the scheme of Section 3, for $n = 2$, $d = 3$, for finding f_1, f_2 as in (3.3) from $|\psi^+(x,k)|^2$ given at $x = x_1, x_2, x_3, x_4$ defined in (6.1).

7. Inductive finding $f_1, \dots, f_n$ for the $2n$-point case for $d = 3$

In this section we realize the scheme of Section 3 for the case when $d = 3$ and $x_1, \dots, x_{2n}$ are defined as in (1.15).

7.1. Step of induction. Suppose that

$$\begin{aligned} &f_{1,n-1}(s), f_{2,n-2}(s), \dots, f_{n-1,1}(s) \ \text{ are functions found from} \\ &|\psi^+(x,k)|^2 \ \text{ given at } \ x = x_1(s), \dots, x_{2(n-1)}(s) \ \text{ defined in } \ (1.15) \end{aligned} \tag{7.1}$$

and such that

$$\begin{aligned} &f_j = f_{j,n-j}(s) + O(s^{-(n-j)}), \quad s \to +\infty, \quad j = 1, \dots, n-1, \\ &\text{for fixed } \ \sin((|k| - k\hat{l})\tau) \neq 0, \ E, \ \tau \ \text{ and } \ \lambda_1, \dots, \lambda_{n-1}, \end{aligned} \tag{7.2}$$

where $f_j = f_j(k,l)$, $f_{j,n-j}(s) = f_{j,n-j}(k,l,\tau,\lambda_1,\dots,\lambda_{n-1},s)$, $(k,l) \in \Omega_E$, $d = 3$, $E > 0$, $\hat{l} = l/|l|$, $\tau > 0$. For $n = 2$ and $n = 3$ such functions are found in Sections 2 and 6.

The step of induction consists in finding $f_{1,n}, f_{2,n-1}, \dots, f_{n,1}$ proceeding from $f_{1,n-1}, f_{2,n-2}, \dots, f_{n-1,1}$ of (7.1), (7.2), where

$$\begin{aligned} &f_{1,n}(s), f_{2,n-1}(s), \dots, f_{n,1}(s) \ \text{ are functions found from} \\ &|\psi^+(x,k)|^2 \ \text{ given at } \ x = x_1(s), \dots, x_{2n}(s) \ \text{ defined by } \ (1.15) \end{aligned} \tag{7.3}$$

and such that

$$\begin{aligned} &f_j = f_{j,n-j+1}(s) + O(s^{-(n-j+1)}), \quad s \to +\infty, \quad j = 1, \dots, n, \\ &\text{for fixed } \ \sin((|k| - k\hat{l})\tau) \neq 0, \ E, \ \tau \ \text{ and } \ \lambda_1, \dots, \lambda_n, \end{aligned} \tag{7.4}$$

where $f_j = f_j(k,l)$, $f_{j,n-j+1}(s) = f_{j,n-j+1}(k,l,\tau,\lambda_1,\dots,\lambda_n,s)$, $(k,l) \in \Omega_E$, $d = 3$, $E > 0$, $\hat{l} = l/|l|$, $\tau > 0$.

This step of induction is realized via Theorems 7.1 and 7.2 formulated in the next subsection.

7.2. Realization of the inductive step. Let $f_{1,1}(\kappa s) = f_{1,1}(k,l,\tau,\kappa s)$ be defined by

$$\begin{pmatrix} f_{1,1}(\kappa s) \\ \bar{f}_{1,1}(\kappa s) \end{pmatrix} = M \begin{pmatrix} a(y_1,k) \\ a(y_2,k) \end{pmatrix}, \tag{7.5}$$

where $M = M_{k,l,\tau,\kappa s}$ is defined according to (5.3), (5.4), (5.7), a is defined by (2.1) (for $d = 3$),

$$y_1 = \kappa s \hat{l}, \quad y_2 = (\kappa s + \tau)\hat{l}, \tag{7.6}$$

$(k,l) \in \Omega_E$, $d=3$, $E>0$, $\hat{l}=l/|l|$, $s>0$, $\kappa \geq 1$.

Let

$$a^2_{n,n-1}(\kappa,s) = \sum_{\substack{j\in\mathbb{N}\\ 2j\leq n}} \frac{|f_{j,n-j}(s)|^2}{(\kappa s)^{2j-1}} + \sum_{\substack{j_1,j_2\in\mathbb{N}\\ j_1\neq j_2, j_1+j_2\leq n}} \frac{f_{j_1,n-j_1}(s)\bar{f}_{j_2,n-j_2}(s)}{(\kappa s)^{j_1+j_2-1}}, \tag{7.7}$$

$$a^{2,\tau}_{n,n-1}(\kappa,s) = \sum_{\substack{j\in\mathbb{N}\\ 2j\leq n}} \frac{|f_{j,n-j}(s)|^2}{(\kappa s+\tau)^{2j-1}} + \sum_{\substack{j_1,j_2\in\mathbb{N}\\ j_1\neq j_2, j_1+j_2\leq n}} \frac{f_{j_1,n-j_1}(s)\bar{f}_{j_2,n-j_2}(s)}{(\kappa s+\tau)^{j_1+j_2-1}}, \tag{7.8}$$

$$\begin{aligned} \Delta^{1,\tau}_{n,n-1}(\kappa,s) = \sum_{j=1}^{n-1} \frac{e^{i\varphi_\tau(\kappa s)} f_{j,n-j}(s)}{(\kappa s)^{j-1}} w_{j-1}(\kappa s,\tau) + \\ \sum_{j=1}^{n-1} \frac{e^{-i\varphi_\tau(\kappa s)} \bar{f}_{j,n-j}(s)}{(\kappa s)^{j-1}} w_{j-1}(\kappa s,\tau), \end{aligned} \tag{7.9}$$

where $f_{j,n-j}(s)$ and $w_{j-1}(s,\tau)$, $j=1,\ldots,n-1$, are the functions of (7.1), (7.2) and (4.13), $\varphi_\tau(s)=\varphi_\tau(s\hat{l},k)$, where $\varphi_\tau(x,k)$ is defined by (4.16), $\hat{l}=l/|l|$.

Theorem 7.1. *Under assumptions (1.2), (1.7) for $d=3$, the following formulas hold:*

$$\begin{aligned} &z_n(\kappa s) = \zeta_{n,n-1}(\kappa,s) + O(s^{-n}) \ \text{ as } \ s\to+\infty \\ &\text{for fixed } \ \sin((|k|-k\hat{l})\tau)\neq 0, \ E,\tau \ \text{ and } \ \lambda_1,\ldots,\lambda_{n-1},\kappa, \end{aligned} \tag{7.10}$$

where $z_n(s)=z_n(k,l,s)$ is defined by (5.1),
$\zeta_{n,n-1}(\kappa,s)=\zeta_{n,n-1}(k,l,\tau,\lambda_1,\ldots,\lambda_{n-1},\kappa,s)$ is defined as follows:

$$\begin{pmatrix} \zeta_{n,n-1}(\kappa,s) \\ \bar{\zeta}_{n,n-1}(\kappa,s) \end{pmatrix} = \begin{pmatrix} f_{1,1}(\kappa s) \\ \bar{f}_{1,1}(\kappa s) \end{pmatrix} - M \begin{pmatrix} a^2_{n,n-1}(\kappa,s) \\ \Delta^{1,\tau}_{n,n-1}(\kappa,s) + a^{2,\tau}_{n,n-1}(\kappa,s) \end{pmatrix}, \tag{7.11}$$

$f_{1,1}(\kappa s)$ is defined by (7.5), $M=M_{k,l,\tau,\kappa s}$ is defined according to (5.3), (5.4), $a^2_{n,n-1}(\kappa,s)$, $a^{2,\tau}_{n,n-1}(\kappa,s)$ and $\Delta^{1,\tau}_{n,n-1}(\kappa,s)$ are defined by (7.7)-(7.9), $(k,l)\in\Omega_E$, $d=3$, $E>0$, $\hat{l}=l/|l|$, $\tau>0$, $\lambda_1,\ldots,\lambda_{n-1}$ are the numbers of (1.15), $\kappa\geq 1$, $n\geq 2$.

In Theorem 7.1 we have that:

(a) $\zeta_{n,n-1}(\kappa,s)$ appropriately approximates $z_n(\kappa s)$ as $s\to+\infty$;

(b) $\zeta_{n,n-1}(\kappa,s)$ is given in terms of $f_{1,1}(\kappa s)$ defined by (7.5) and $f_{j,n-j}(s)$, $j=1,\ldots,n-1$, of (7.1), (7.2) (in view of (7.11), (7.7)-(7.9)), and, as a corollary, $\zeta_{n,n-1}(\kappa,s)$ is found from $|\psi^+(x,k)|^2$ at $x=x_1(s),\ldots,x_{2(n-1)}(s)$, $y_1(s),y_2(s)$ of (1.15) and (7.6).

Theorem 7.1 for $\kappa=1$ is an extension of Proposition 6.1 to the case $n>2$. In addition for $n=2$, in general, $z_{2,1}(\kappa s)\neq\zeta_{2,1}(\kappa,s)$.

Let

$$\Lambda=(\Lambda_{j_1,j_2})=(\lambda_{j_1}^{1-j_2}), \quad j_1,j_2=1,\ldots,n, \tag{7.12}$$

where $\lambda_1, \dots, \lambda_n$ are the numbers of (1.15). One can see that Λ is a Vandermonde matrix and, in particular,

$$\det \Lambda = \prod_{1 \le i < j \le n} (\lambda_j^{-1} - \lambda_i^{-1}) \ne 0, \tag{7.13}$$

where we used our assumptions on λ_j of (1.15) in order to have that $\det \Lambda \ne 0$.

Theorem 7.2. *Under assumptions (1.2), (1.7) for $d = 3$, formulas (7.4) hold, where*

$$f_{i,n-i+1}(s) = s^{i-1} \sum_{j=1}^{n} \Lambda_{i,j}^{-1} \zeta_{n,n-1}(\lambda_j, s), \quad i = 1, \dots, n, \tag{7.14}$$

where $\Lambda^{-1} = (\Lambda_{i,j}^{-1})$ is the inverse of the matrix Λ defined by (7.12), $\zeta_{n,n-1}(\kappa, s)$ is defined by (7.11), $n \ge 2$.

Note that $f_{1,2}$, $f_{2,1}$ of Theorem 7.2 for $n = 2$ are different, in general, from $f_{1,2}$, $f_{2,1}$ of Proposition 6.2.

Theorems 7.1 and 7.2 realize the step of induction of Subsection 7.1.

7.3. Proofs of Theorems 7.1 and 7.2.

Proof of Theorem 7.1. Theorem 7.1 follows from Proposition 5.1, formula (7.5) and the formulas

$$a_n^2(\kappa s) = a_{n,n-1}^2(\kappa, s) + O(s^{-n}), \quad s \to +\infty, \tag{7.15}$$

$$a_n^2(\kappa s + \tau) = a_{n,n-1}^{2,\tau}(\kappa, s) + O(s^{-n}), \quad s \to +\infty, \tag{7.16}$$

$$\Delta_n^{1,\tau}(\kappa s) = \Delta_{n,n-1}^{1,\tau}(\kappa, s) + O(s^{-n}), \quad s \to +\infty, \tag{7.17}$$

$$\text{for fixed} \quad \sin((|k| - k\hat{l})\tau) \ne 0, \ E, \tau \ \text{ and } \ \lambda_1, \dots, \lambda_{n-1},$$

where $a_n^2(\kappa s) = a_n^2(\kappa s \hat{l}, k)$, $a_n^2(\kappa s + \tau) = a_n^2((\kappa s + \tau)\hat{l}, k)$, $\Delta_n^{1,\tau}(\kappa s) = \Delta_n^{1,\tau}(\kappa s \hat{l}, k)$ are the functions of (4.4), (4.14), (4.11), $a_{n,n-1}^2(\kappa, s)$, $a_{n,n-1}^{2,\tau}(\kappa, s)$, $\Delta_{n,n-1}^{1,\tau}(\kappa, s)$ are the functions of (7.7), (7.8), (7.9).

Formula (7.15) follows from (4.4), (7.7), (7.2) and the observation that the terms

$$O(s^{-(n+j-1)}), \ O(s^{-(n+j_1-1)}), \ O(s^{-(n+j_2-1)}),$$
$$j \in \mathbb{N}, \ 2j \le n, \ j_1, j_2 \in \mathbb{N}, \ j_1 \ne j_2, \ j_1 + j_2 \le n,$$

arising at comparison of the summands of $a_n^2(\kappa s)$ and $a_{n,n-1}^2(\kappa, s)$ are majorized by $O(s^{-n})$ as $s \to +\infty$.

Formula (7.16) is proved in a similar way with formula (7.15).

Formula (7.17) follows from(4.11), (4.13), (7.9), (7.2).

This completes the proof of Theorem 7.1.

Proof of Theorem 7.2. Formula (5.1) and formula (7.10) for $\lambda = \lambda_1, \dots, \lambda_n$ imply the following system of approximate equations for $f_1, \dots, f_n$:

$$\sum_{j=1}^{n} \frac{f_j}{(\lambda_i s)^{j-1}} = \zeta_{n,n-1}(\lambda_i, s) + O(s^{-n}) \ \text{ as } \ s \to +\infty, \quad i = 1, \dots, n, \tag{7.18}$$

where $\lambda_1, \dots, \lambda_n$ are the number of (1.15).

Let

$$g(s) = (g_1(s), \dots, g_n(s))^T, \quad g_j(s) = \frac{f_j}{s^{j-1}}, \quad j = 1, \dots, n, \tag{7.19}$$

$$h(s) = (h_1(s), \dots, h_n(s))^T, \quad h_j(s) = \zeta_{n,n-1}(\lambda_j, s), \ j = 1, \dots, n. \tag{7.20}$$

Then (7.18) can be rewritten as

$$\Lambda g(s) = h(s) + O(s^{-n}) \ \text{ as } \ s \to +\infty, \tag{7.21}$$

where $O(s^{-n})$ is considered as an n-dimensional vector with the components $O(s^{-n})$.

From (7.21) we obtain that

$$g(s) = \Lambda^{-1} h(s) + O(s^{-n}) \ \text{ as } \ s \to +\infty. \tag{7.22}$$

Formulas (7.4), (7.14) follow from (7.19), (7.20), (7.22).

Theorem 7.2 is proved.

8. Analogs for $d = 2$ of formulas of Sections 4 and 5

Formulas (4.1)-(4.16) of Section 4 and formulas (5.1)-(5.9) of Section 5 have direct analogs for $d = 2$. The difference consists in the following modifications in formulas (4.4), (4.5), (4.14), (4.15) and (5.5):

$$a_n^2(x,k) = \sum_{\substack{j \in \mathbb{N} \\ 2j \le n}} \frac{|f_j|^2}{|x|^{2j-1-1/2}} + \sum_{\substack{j_1, j_2 \in \mathbb{N} \\ j_1 \ne j_2, j_1 + j_2 \le n}} \frac{f_{j_1} \bar{f}_{j_2}}{|x|^{j_1 + j_2 - 1 - 1/2}} \tag{8.1}$$

(in place of (4.4));

$$\delta_n a(x,k) = O\Big(\frac{1}{|x|^{n-1/2}}\Big) \ \text{ as } \ |x| \to \infty \tag{8.2}$$

(in place of (4.5));

$$a_n^2(x + \tau\hat{x}, k) = \sum_{\substack{j \in \mathbb{N} \\ 2j \le n}} \frac{|f_j|^2}{(|x| + \tau)^{2j-1-1/2}} + \sum_{\substack{j_1, j_2 \in \mathbb{N} \\ j_1 \ne j_2, j_1 + j_2 \le n}} \frac{f_{j_1} \bar{f}_{j_2}}{(|x| + \tau)^{j_1 + j_2 - 1 - 1/2}} \tag{8.3}$$

(in place of (4.14));

$$\delta_n a(x + \tau\hat{x}, k) = O\Big(\frac{1}{|x|^{n-1/2}}\Big) \ \text{ as } \ |x| \to \infty \tag{8.4}$$

(in place of the second term of formula (4.15));

$$\delta_n a(s\hat{l}, k) = O(s^{-n+1/2}), \quad \delta_n a((s+\tau)\hat{l}, k) = O(s^{-n+1/2}) \ \text{ as } s \to +\infty \tag{8.5}$$

(in place of (5.5)).

Besides, for $d = 2$, in all formulas of Sections 4 and 5 with modifications summed up in (8.1)-(8.5) we have that $x \in \mathbb{R}^2 \backslash \{0\}$, $k \in \mathbb{R}^2$, $l \in \mathbb{R}^2$.

9. Improved finding f_1 for the 2-point case for $d = 2$

Due to (2.19)-(2.20), (2.4)-(2.9) for $d = 2$, we have that

$$\begin{aligned} &f_1 = f_{1,1/2}(s) + O(s^{-1/2}) \ \text{ as } \ s \to +\infty \\ &\text{for fixed } \ \sin((|k| - k\hat{l})\tau) \neq 0 \ \text{ and } \ E, \end{aligned} \tag{9.1}$$

where $f_1 = f_1(k, l)$, $f_{1,1/2}(s) = f_{1,1/2}(k, l, \tau, s)$,

$$\begin{pmatrix} f_{1,1/2}(s) \\ \bar{f}_{1,1/2}(s) \end{pmatrix} = M \begin{pmatrix} a(x_1, k) \\ a(x_2, k) \end{pmatrix}, \tag{9.2}$$

$M = M_{k,l,\tau,s}$ is defined by (5.3), x_1, x_2 are defined by (2.4), $(k, l) \in \Omega_E$, $d = 2$, $E > 0$, $\hat{l} = l/|l|$, $\tau > 0$.

Proposition 9.1 *Under assumptions (1.2), (1.7) for $d = 2$, the following formulas hold:*

$$\begin{aligned} &f_1 = f_{1,1}(s) + O(s^{-1}) \ \text{ as } \ s \to +\infty \\ &\text{for fixed } \ \sin((|k| - k\hat{l})\tau) \neq 0, \ E \ \text{ and } \ \tau, \end{aligned} \tag{9.3}$$

where $f_{1,1}(s) = f_{1,1}(k, l, \tau, s)$ is defined as follows:

$$\begin{pmatrix} f_{1,1}(s) \\ \bar{f}_{1,1}(s) \end{pmatrix} = \begin{pmatrix} f_{1,1/2}(s) \\ \bar{f}_{1,1/2}(s) \end{pmatrix} - \frac{|f_{1,1/2}(s)|^2}{s^{1/2}} M \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{9.4}$$

$f_{1,1/2}(s) = f_{1,1/2}(k, l, \tau, s)$ is defined by (9.2), $M = M_{k,l,\tau,s}$ is defined by (5.3), $(k, l) \in \Omega_E$, $d = 2$, $E > 0$, $\hat{l} = l/|l|$, $\tau > 0$.

Proof of Proposition 9.1. Proposition 9.1 follows from Proposition 5.1 for $n = 2$, $d = 2$ and formulas (9.1), (9.2). In more detail, the proof consists of the following.

Formulas (8.1), (4.11), (4.13), (8.3) for $n = 2$ imply that

$$a_2^2(s\hat{l}, k) = \frac{|f_1(k, l)|^2}{s^{1/2}}, \tag{9.5}$$

$$\Delta_2^{1,\tau}(s\hat{l}, k) = 0, \tag{9.6}$$

$$a_2^2((s + \tau)\hat{l}, k) = \frac{|f_1(k, l)|^2}{(s + \tau)^{1/2}}, \tag{9.7}$$

$s > 0$, $\tau > 0$.

Formulas (5.2)-(5.4), (8.5), (5.6), (5.7) for $n = 2$ and formulas (9.5)-(9.7) imply that

$$\begin{aligned} &\begin{pmatrix} z_2(s) \\ \bar{z}_2(s) \end{pmatrix} = M \begin{pmatrix} a(s\hat{l}, k) \\ a((s + \tau)\hat{l}, k) \end{pmatrix} - M \begin{pmatrix} s^{-1/2}|f_1|^2 \\ (s + \tau)^{-1/2}|f_1|^2 \end{pmatrix} + \\ &O\big(\frac{1}{s^{3/2}}\big) \ \text{ as } \ s \to +\infty \\ &\text{for fixed } \ \sin((|k| - k\hat{l})\tau) \neq 0, \ E \ \text{ and } \ \tau. \end{aligned} \tag{9.8}$$

Formula (5.1), for $n = 2$, and formulas (9.1), (9.2), (9.8) imply (9.3), (9.4).

This completes the proof of Proposition 9.1.

10. Finding f_1, f_2 for the 4-point case for $d = 2$

In this section we realize the scheme of Section 3 for the case when $n = 2$, $d = 2$ and x_1, x_2, x_3, x_4 are defined as in (6.1).

Proceeding from considerations of Section 9 at x_1 and x_2, we obtain the following result.

Proposition 10.1 *Under assumptions (1.2), (1.7) for $d = 2$, the following formulas hold:*

$$\begin{aligned} &z_2(s) = z_{2,1}(s) + O(s^{-3/2}) \ \text{ as } \ s \to +\infty \\ &\text{for fixed } \ \sin((|k| - k\hat{l})\tau) \neq 0, \ E \ \text{ and } \ \tau, \end{aligned} \tag{10.1}$$

where $z_2(s) = z_2(k, l, s)$ is defined by (5.1) for $n = 2$, $z_{2,1}(s) = z_{2,1}(k, l, \tau, s)$ is defined as follows:

$$\begin{pmatrix} z_{2,1}(s) \\ \bar{z}_{2,1}(s) \end{pmatrix} = \begin{pmatrix} f_{1,1/2}(s) \\ \bar{f}_{1,1/2}(s) \end{pmatrix} - \frac{|f_{1,1}(s)|^2}{s^{1/2}} M \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{10.2}$$

$f_{1,1/2}(s) = f_{1,1/2}(k, l, \tau, s)$ is defined by (9.2), $f_{1,1}(s) = f_{1,1}(k, l, \tau, s)$ is defined by (9.4), $M = M_{k,l,\tau,s}$ is defined by (5.3), $(k, l) \in \Omega_E$, $d = 2$, $E > 0$, $\hat{l} = l/|l|$, $\tau > 0$.

Proposition 10.1 follows from formulas (9.2), (9.3), (9.8).

Due to similar considerations at x_3 and x_4, we also have that

$$\begin{aligned} &z_2(\lambda s) = z_{2,1}(\lambda s) + O((\lambda s)^{-3/2}) \ \text{ as } \ s \to +\infty \\ &\text{for fixed } \ \sin((|k| - k\hat{l})\tau) \neq 0, \ E \ \text{ and } \ \tau, \end{aligned} \tag{10.3}$$

where $z_{2,1}(\lambda s)$ is given in terms of $a(x_3, k)$, $a(x_4, k)$ via formulas (9.2), (9.4), (10.2) with λs and x_3, x_4 in place of s and x_1, x_2.

Next, due to definition (5.1), for $n = 2$, and formulas (10.1), (10.3), we have that

$$f_1 + s^{-1} f_2 = z_{2,1}(s) + O(s^{-3/2}) \ \text{ as } \ s \to +\infty, \tag{10.4}$$

$$f_1 + (\lambda s)^{-1} f_2 = z_{2,1}(\lambda s) + O((\lambda s)^{-3/2}) \ \text{ as } \ s \to +\infty, \tag{10.5}$$

$$\text{for fixed } \ \sin((|k| - k\hat{l})\tau) \neq 0, \ E \ \text{ and } \ \tau,$$

where $f_1 = f_1(k, l)$, $f_2 = f_2(k, l)$, $z_{2,1}(s) = z_{2,1}(k, l, \tau, s)$, $(k, l) \in \Omega_E$, $d = 2$, $E > 0$, $\hat{l} = l/|l|$, $\tau > 0$.

Proceeding from (10.4), (10.5), we obtain the following result.

Proposition 10.2. Under assumptions (1.2), (1.7) for $d = 2$, the following formulas hold:

$$f_1 = f_{1,3/2}(s) + O(s^{-3/2}) \ \text{ as } \ s \to +\infty, \tag{10.6}$$

$$f_2 = f_{2,1/2}(s) + O(s^{-1/2}) \ \text{ as } \ s \to +\infty, \tag{10.7}$$

$$\text{for fixed } \ \sin((|k| - k\hat{l})\tau) \neq 0, \quad E, \tau \ \text{ and } \ \lambda,$$

where $f_1 = f_1(k,l)$, $f_2 = f_2(k,l)$, $f_{1,3/2}(s) = f_{1,3/2}(k,l,\tau,\lambda,s)$, $f_{2,1/2}(s) = f_{2,1/2}(k,l,\tau,\lambda,s)$,

$$f_{1,3/2}(s) = -\frac{z_{2,1}(s)}{\lambda-1} + \frac{\lambda z_{2,1}(\lambda s)}{\lambda-1}, \tag{10.8}$$

$$f_{2,1/2}(s) = \frac{s\lambda}{\lambda-1}(z_{2,1}(s) - z_{2,1}(\lambda s)), \tag{10.9}$$

where $z_{2,1}(s) = z_{2,1}(k,l,\tau,s)$ is found via (9.2), (9.4), (10.2) from $|\psi^+(x,k)|^2$ at $x = x_1, x_2$ of (6.1), $z_{2,1}(\lambda s) = z_{2,1}(k,l,\tau,\lambda s)$ is found via (9.2), (9.4), (10.2) (where s and x_1, x_2 are replaced by λs and x_3, x_4 of (7.1) for $n = 2$), $(k,l) \in \Omega_E$, $d = 2$, $E > 0$, $\hat{l} = l/|l|$, $\tau > 0$, $\lambda > 1$.

Let $f_{1,1/2}(\kappa s) = f_{1,1/2}(k,l,\tau,\kappa s)$ be defined by

$$\begin{pmatrix} f_{1,1/2}(\kappa s) \\ \bar{f}_{1,1/2}(\kappa s) \end{pmatrix} = M \begin{pmatrix} a(y_1,k) \\ a(y_2,k), \end{pmatrix} \tag{10.10}$$

$$y_1 = \kappa s\hat{l}, \quad y_2 = (\kappa s + \tau)\hat{l}, \tag{10.11}$$

where $M = M_{k,l,\tau,\kappa s}$ is defined according to (5.3), (5.4), (5.7), a is defined by (2.1), $(k,l) \in \Omega_E$, $d = 2$, $E > 0$, $\hat{l} = l/|l|$, $s > 0$, $\kappa \geq 1$.

Proposition 10.3 *Under assumptions (1.2), (1.7) for $d = 2$, the following formulas hold:*

$$\begin{aligned} &z_2(\kappa s) = \zeta_{2,3/2}(\kappa, s) + O(s^{-2}) \ \text{ as } \ s \to +\infty \\ &\text{for fixed } \ \sin((|k| - k\hat{l})\tau) \neq 0, \ E, \ \tau, \ \lambda \ \text{ and } \ \kappa, \end{aligned} \tag{10.12}$$

where $z_2(s) = z_2(k,l,s)$ is defined by (5.1) for $n = 2$, $\zeta_{2,3/2}(\kappa,s) = \zeta_{2,3/2}(k,l,\tau,\lambda,\kappa,s)$ is defined as follows:

$$\begin{aligned} &\begin{pmatrix} \zeta_{2,3/2}(\kappa,s) \\ \bar{\zeta}_{2,3/2}(\kappa,s) \end{pmatrix} = \begin{pmatrix} f_{1,1/2}(\kappa s) \\ \bar{f}_{1,1/2}(\kappa s) \end{pmatrix} - |f_{1,3/2}(s)|^2 M \begin{pmatrix} (\kappa s)^{-1/2} \\ (\kappa s + \tau)^{-1/2} \end{pmatrix} - \\ &\frac{(f_{1,3/2}(s)\bar{f}_{2,1/2}(s) + \bar{f}_{1,3/2}(s) f_{2,1/2}(s))}{(\kappa s)^{3/2}} M \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \end{aligned} \tag{10.13}$$

$f_{1,1/2}(\kappa s)$ is defined by (10.10), $M = M_{k,l,\tau,\kappa s}$ is defined according to (5.3), (5.4), $(k,l) \in \Omega_E$, $d = 2$, $E > 0$, $\hat{l} = l/|l|$, $\tau > 0$, $\kappa \geq 1$.

Proof of Proposition 10.3. Proposition 10.3 follows from Proposition 5.1 for $n = 3$, $d = 2$ and formulas (10.6), (10.7). In more detail, the proof consists of the following.

Formulas (8.1), (4.11), (4.13), (8.3) for $n = 3$ imply that

$$a_3^2(s\hat{l}, k) = \frac{|f_1|^2}{s^{1/2}} + \frac{f_1\bar{f}_2 + \bar{f}_1 f_2}{s^{3/2}}, \tag{10.14}$$

$$\Delta_3^{1,\tau}(s\hat{l}, k) = \frac{e^{i\varphi_\tau} f_2}{s} w_1(s,\tau) = O\big(\frac{1}{s^2}\big), \quad s \to +\infty, \tag{10.15}$$

$$a_3^2((s+\tau)\hat{l}, k) = \frac{|f_1|^2}{(s+\tau)^{1/2}} + \frac{f_1\bar{f}_2 + \bar{f}_1 f_2}{(s+\tau)^{3/2}}, \tag{10.16}$$

where $f_1 = f_1(k,l)$, $f_2 = f_2(k,l)$, $s > 0$, $\tau > 0$.

Formulas (5.2)-(5.4), (8.5), (5.6), (5.7) for $n = 3$ and formulas (10.14)-(10.16) imply that

$$\begin{pmatrix} z_3(\kappa s) \\ \bar{z}_3(\kappa s) \end{pmatrix} = M \begin{pmatrix} a(\kappa s\hat{l}, k) \\ a((\kappa s + \tau)\hat{l}, k) \end{pmatrix} - |f_1|^2 M \begin{pmatrix} (\kappa s)^{-1/2} \\ (\kappa s + \tau)^{-1/2} \end{pmatrix} -$$
$$(f_1\bar{f}_2 + \bar{f}_1 f_2) M \begin{pmatrix} (\kappa s)^{-3/2} \\ (\kappa s + \tau)^{-3/2} \end{pmatrix} + O\big(\frac{1}{(\kappa s)^{5/2}}\big) \quad \text{as } s \to +\infty \tag{10.17}$$
$$\text{for fixed } \ \sin((|k| - k\hat{l})\tau) \neq 0, \ E \ \text{ and } \ \tau,$$

where $M = M_{k,l,\tau,\kappa s}$.

Formula (5.1), for $n = 2$ and for $n = 3$, and formulas (10.10), (10.11), (10.6), (10.7), (10.17) imply (10.12), (10.13).

This completes the proof of Proposition 10.3.

Next, due to definition (5.1), for $n = 2$, and formula (10.12), for $\kappa = 1$ and for $\kappa = \lambda$, we have that

$$f_1 + s^{-1} f_2 = \zeta_{2,3/2}(1,s) + O(s^{-2}) \ \text{ as } \ s \to +\infty, \tag{10.18}$$
$$f_1 + (\lambda s)^{-1} f_2 = \zeta_{2,3/2}(\lambda,s) + O(s^{-2}) \ \text{ as } \ s \to +\infty, \tag{10.19}$$
$$\text{for fixed } \ \sin((|k| - k\hat{l})\tau) \neq 0, \ E, \ \tau, \ \text{ and } \ \lambda,$$

where $f_1 = f_1(k,l)$, $f_2 = f_2(k,l)$, $\zeta_{2,3/2}(\kappa,s) = \zeta_{2,3/2}(k,l,\tau,\lambda,\kappa,s)$, $(k,l) \in \Omega_E$, $d = 2$, $E > 0$, $\hat{l} = l/|l|$, $\tau > 0$.

Proceeding from (10.18), (10.19), we obtain the following result.

Proposition 10.4. Under assumptions (1.2), (1.7) for $d = 2$, the following formulas hold:

$$f_1 = f_{1,2}(s) + O(s^{-2}) \ \text{ as } \ s \to +\infty, \tag{10.20}$$
$$f_2 = f_{2,1}(s) + O(s^{-1}) \ \text{ as } \ s \to +\infty, \tag{10.21}$$
$$\text{for fixed } \ \sin((|k| - k\hat{l})\tau) \neq 0, \ \ E, \tau \ \text{ and } \ \lambda,$$

where $f_1 = f_1(k,l)$, $f_2 = f_2(k,l)$, $f_{1,2}(s) = f_{1,2}(k,l,\tau,\lambda,s)$,
$f_{2,1}(s) = f_{2,1}(k,l,\tau,\lambda,s)$,

$$f_{1,2}(s) = -\frac{\zeta_{2,3/2}(1,s)}{\lambda - 1} + \frac{\lambda\zeta_{2,3/2}(\lambda,s)}{\lambda - 1}, \tag{10.22}$$
$$f_{2,1}(s) = \frac{s\lambda}{\lambda - 1}(\zeta_{2,3/2}(1,s) - \zeta_{2,3/2}(\lambda,s)), \tag{10.23}$$

where $\zeta_{2,3/2}(1,s) = \zeta_{2,3/2}(k,l,\tau,\lambda,1,s)$, $\zeta_{2,3/2}(\lambda,s) = \zeta_{2,3/2}(k,l,\tau,\lambda,\lambda,s)$ are found via (10.13), (10.10), (10.11), (10.8), (10.9) from $|\psi^+(x,k)|^2$ at $x = x_1, x_2, x_3, x_4$ of (6.1) for $n = 2$, $(k,l) \in \Omega_E$, $d = 2$, $E > 0$, $\hat{l} = l/|l|$, $\tau > 0$, $\lambda > 1$.

11. Inductive finding $f_1, \ldots, f_n$ for the 2n-point case for $d = 2$

In this section we realize the scheme of Section 3 for the case when $d = 2$ and $x_1, \dots, x_{2n}$ are defined as in (1.15).

11.1 Step of induction. Formulas (7.1)-(7.4) of Subsection 7.1 with $d = 2$ in place of $d = 3$ can be considered as the step of induction for $d = 2$. However, for $d = 2$ this step of induction consists of the following two substeps.

The first substep consists in finding $f_{1,n-1/2}, f_{2,n-1-1/2}, \dots, f_{n,1/2}$ proceeding from $f_{1,n-1}, f_{2,n-2}, \dots, f_{n-1,1}$ of (7.1), (7.2) for $d = 2$, where

$$\begin{aligned} &f_{1,n-1/2}(s), f_{2,n-1-1/2}(s), \dots, f_{n,1/2}(s) \ \text{ are functions found from} \\ &|\psi^+(x,k)|^2 \ \text{ given at } \ x = x_1(s), \dots, x_{2n}(s) \ \text{ defined by } \ (1.15) \end{aligned} \tag{11.1}$$

and such that

$$\begin{aligned} &f_j = f_{j,n-j+1/2}(s) + O(s^{-(n-j+1/2)}), \quad s \to +\infty, \quad j = 1, \dots, n, \\ &\text{for fixed } \ sin((|k| - k\hat{l})\tau) \neq 0, \quad E, \quad \tau \ \text{ and } \ \lambda_1, \dots, \lambda_n, \end{aligned} \tag{11.2}$$

where $f_j = f_j(k,l)$, $f_{j,n-j+1/2}(s) = f_{j,n-j+1/2}(k,l,\tau,\lambda_1,\dots,\lambda_n,s)$, $(k,l) \in \Omega_E$, $d = 2$, $E > 0$, $\hat{l} = l/|l|$, $\tau > 0$. For $n = 2$ and $n = 3$ the aforementioned functions $f_{1,n-1}, f_{2,n-2}, \dots, f_{n-1,1}$ of (7.1), (7.2) for $d = 2$ are found in Sections 9 and 10.

This substep is realized via Theorems 11.1 and 11.2 formulated in the next subsection.

The second substep consists in finding $f_{1,n}, f_{2,n-1}, \dots, f_{n,1}$ proceeding from $f_{1,n-1/2}, f_{2,n-1-1/2}, \dots, f_{n,1/2}$ of (11.1), (11.2), where $f_{1,n}, f_{2,n-1}, \dots, f_{n,1}$ are the functions of (7.3), (7.4) for $d = 2$.

This substep is realized via Theorems 11.3 and 11.4 formulated in the next subsection.

11.2 Realization of the inductive step. Let $f_{1,1/2}(\kappa s) = f_{1,1/2}(k,l,\tau,\kappa s)$ be defined by (10.10), (10.11).

Let

$$a^2_{n,n-1}(\kappa,s) = \sum_{\substack{j\in\mathbb{N}\\ 2j\le n}} \frac{|f_{j,n-j}(s)|^2}{(\kappa s)^{2j-1-1/2}} + \sum_{\substack{j_1,j_2\in\mathbb{N}\\ j_1\neq j_2, j_1+j_2\le n}} \frac{f_{j_1,n-j_1}(s)\bar{f}_{j_2,n-j_2}(s)}{(\kappa s)^{j_1+j_2-1-1/2}}, \tag{11.3}$$

$$a^{2,\tau}_{n,n-1}(\kappa,s) = \sum_{\substack{j\in\mathbb{N}\\ 2j\le n}} \frac{|f_{j,n-j}(s)|^2}{(\kappa s+\tau)^{2j-1-1/2}} + \sum_{\substack{j_1,j_2\in\mathbb{N}\\ j_1\neq j_2, j_1+j_2\le n}} \frac{f_{j_1,n-j_1}(s)\bar{f}_{j_2,n-j_2}(s)}{(\kappa s+\tau)^{j_1+j_2-1-1/2}}, \tag{11.4}$$

where $f_{j,n-j}(s)$, $j = 1, \dots, n-1$, are the functions of (7.1), (7.2) for $d = 2$.

Let $\Delta^{1,\tau}_{n,n-1}(\kappa,s)$ be defined by (7.9) for $d = 2$.

Theorem 11.1. *Under assumptions (1.2), (1.7) for $d = 2$, the following formulas hold:*

$$\begin{aligned} &z_n(\kappa s) = \zeta_{n,n-1}(\kappa,s) + O(s^{-(n-1/2)}) \ \text{ as } \ s \to +\infty \\ &\text{for fixed } \ \sin((|k| - k\hat{l})\tau) \neq 0, \ E, \tau \ \text{ and } \ \lambda_1, \dots, \lambda_{n-1}, \kappa, \end{aligned} \tag{11.5}$$

where $z_n(s) = z_n(k,l,s)$ is defined by (5.1),

$\zeta_{n,n-1}(\kappa,s)=\zeta_{n,n-1}(k,l,\tau,\lambda_1,\dots,\lambda_{n-1},\kappa,s)$ *is defined as follows:*

$$\begin{pmatrix}\zeta_{n,n-1}(\kappa,s)\\ \bar\zeta_{n,n-1}(\kappa,s)\end{pmatrix}=\begin{pmatrix}f_{1,1/2}(\kappa s)\\ \bar f_{1,1/2}(\kappa s)\end{pmatrix}-M\begin{pmatrix}a^2_{n,n-1}(\kappa,s)\\ \Delta^{1,\tau}_{n,n-1}(\kappa,s)+a^{2,\tau}_{n,n-1}(\kappa,s)\end{pmatrix},\tag{11.6}$$

$f_{1,1/2}(\kappa s)$ *is defined by (10.10),* $M=M_{k,l,\tau,\kappa s}$ *is defined according to (5.3), (5.4),* $a^2_{n,n-1}(\kappa,s)$, $a^{2,\tau}_{n,n-1}(\kappa,s)$ *are defined by (11.3), (11.4),* $\Delta^{1,\tau}_{n,n-1}(\kappa,s)$ *is defined according to (7.9),* $(k,l)\in\Omega_E$, $d=2$, $E>0$, $\hat l=l/|l|$, $\tau>0$, $\lambda_1,\dots,\lambda_{n-1}$ *are the numbers of (1.15),* $\kappa\geq 1$, $n\geq 2$.

Theorem 11.2. *Under assumptions (1.2), (1.7) for* $d=2$, *formulas (11.2) hold, where*

$$f_{i,n-i+1/2}(s)=s^{i-1}\sum_{j=1}^{n}\Lambda^{-1}_{i,j}\zeta_{n,n-1}(\lambda_j,s),\quad i=1,\dots,n,\tag{11.7}$$

where $\Lambda^{-1}=(\Lambda^{-1}_{i,j})$ *is the inverse of the matrix* Λ *defined by (7.12),* $\zeta_{n,n-1}(\kappa,s)$ *is defined by (11.6),* $n\geq 2$.

Note that $f_{1,3/2}$, $f_{2,1/2}$ of Theorem 11.2 for $n=2$ are different, in general, from $f_{1,3/2}$, $f_{2,1/2}$ of Proposition 10.2.

Theorems 11.1 and 11.2 realize the first substep of induction of Subsection 11.1.

Let

$$a^2_{n+1,n-1/2}(\kappa,s)=\sum_{\substack{j\in\mathbb{N}\\ 2j\leq n+1}}\frac{|f_{j,n-j+1/2}(s)|^2}{(\kappa s)^{2j-1-1/2}}+\sum_{\substack{j_1,j_2\in\mathbb{N}\\ j_1\neq j_2,j_1+j_2\leq n+1}}\frac{f_{j_1,n-j_1+1/2}(s)\bar f_{j_2,n-j_2+1/2}(s)}{(\kappa s)^{j_1+j_2-1-1/2}},\tag{11.8}$$

$$a^{2,\tau}_{n+1,n-1/2}(\kappa,s)=\sum_{\substack{j\in\mathbb{N}\\ 2j\leq n+1}}\frac{|f_{j,n-j+1/2}(s)|^2}{(\kappa s+\tau)^{2j-1-1/2}}+\sum_{\substack{j_1,j_2\in\mathbb{N}\\ j_1\neq j_2,j_1+j_2\leq n+1}}\frac{f_{j_1,n-j_1+1/2}(s)\bar f_{j_2,n-j_2+1/2}(s)}{(\kappa s+\tau)^{j_1+j_2-1-1/2}},\tag{11.9}$$

where $f_{j,n-j+1/2}(s)$, $j=1,\dots,n$, are the functions of (11.1), (11.2), (11.7).

Theorem 11.3. *Under assumptions (1.2), (1.7) for* $d=2$, *the following formulas hold:*

$$\begin{aligned}&z_n(\kappa s)=\zeta_{n,n-1/2}(\kappa,s)+O(s^{-n})\ \text{ as }\ s\to+\infty\\ &\text{for fixed }\ \sin((|k|-k\hat l)\tau)\neq 0,\ E,\tau\ \text{ and }\ \lambda_1,\dots,\lambda_n,\kappa,\end{aligned}\tag{11.10}$$

where $z_n(s)=z_n(k,l,s)$ *is defined by (5.1),*
$\zeta_{n,n-1/2}(\kappa,s)=\zeta_{n,n-1}(k,l,\tau,\lambda_1,\dots,\lambda_n,\kappa,s)$ *is defined as follows:*

$$\begin{pmatrix}\zeta_{n,n-1/2}(\kappa,s)\\ \bar\zeta_{n,n-1/2}(\kappa,s)\end{pmatrix}=\begin{pmatrix}f_{1,1/2}(\kappa s)\\ \bar f_{1,1/2}(\kappa s)\end{pmatrix}-M\begin{pmatrix}a^2_{n+1,n-1/2}(\kappa,s)\\ \Delta^{1,\tau}_{n,n-1}(\kappa,s)+a^{2,\tau}_{n+1,n-1/2}(\kappa,s)\end{pmatrix},\tag{11.11}$$

$f_{1,1/2}(\kappa s)$ is defined by (10.10), $M = M_{k,l,\tau,\kappa s}$ is defined according to (5.3), (5.4), $a^2_{n+1,n-1/2}(\kappa, s)$, $a^{2,\tau}_{n+1,n-1/2}(\kappa, s)$ are defined by (11.8), (11.9), $\Delta^{1,\tau}_{n,n-1}(\kappa, s)$ is defined according to (7.9), $(k, l) \in \Omega_E$, $d = 2$, $E > 0$, $\hat{l} = l/|l|$, $\tau > 0$, $\lambda_1, \dots, \lambda_n$ are the numbers of (1.15), $\kappa \geq 1$, $n \geq 2$.

Theorem 11.4. *Under assumptions (1.2), (1.7) for $d = 2$, formulas (7.4) for $d = 2$ hold, where*

$$f_{i,n-i+1}(s) = s^{i-1} \sum_{j=1}^{n} \Lambda^{-1}_{i,j} \zeta_{n,n-1/2}(\lambda_j, s), \quad i = 1, \dots, n, \tag{11.12}$$

where $\Lambda^{-1} = (\Lambda^{-1}_{i,j})$ is the inverse of the matrix Λ defined by (7.12), $\zeta_{n,n-1/2}(\kappa, s)$ is defined by (11.11), $n \geq 2$.

Note that $\zeta_{2,3/2}$ and $f_{1,2}$, $f_{2,1}$ of Theorems 11.3 and 11.4 for $n = 2$ are different, in general, from $\zeta_{2,3/2}$ and $f_{1,2}$, $f_{2,1}$ of Propositions 10.3 and 10.4.

Theorems 11.3 and 11.4 realize the second substep of induction of Subsection 11.1.

11.3. Proofs of Theorems 11.1-11.4.

Proof of Theorem 11.1. Theorem 11.1 follows from Proposition 5.1 for $d = 2$ with formulas (8.5) in place of (5.5), formula (10.10) and the formulas

$$a^2_n(\kappa s) = a^2_{n,n-1}(\kappa, s) + O(s^{-(n-1/2)}), \quad s \to +\infty, \tag{11.13}$$

$$a^2_n(\kappa s + \tau) = a^{2,\tau}_{n,n-1}(\kappa, s) + O(s^{-(n-1/2)}), \quad s \to +\infty, \tag{11.14}$$

$$\Delta^{1,\tau}_n(\kappa s) = \Delta^{1,\tau}_{n,n-1}(\kappa, s) + O(s^{-n}), \quad s \to +\infty, \tag{11.15}$$

$$\text{for fixed} \quad \sin((|k| - k\hat{l})\tau) \neq 0, \ E, \tau \ \text{ and } \ \lambda_1, \dots, \lambda_{n-1}, \kappa,$$

where $a^2_n(\kappa s) = a^2_n(\kappa s\hat{l}, k)$, $a^2_n(\kappa s + \tau) = a^2_n((\kappa s + \tau)\hat{l}, k)$ are the functions of (8.1), (8.3), $\Delta^{1,\tau}_n(\kappa s) = \Delta^{1,\tau}_n(\kappa s\hat{l}, k)$ is the function of (4.11) for $d = 2$, $a^2_{n,n-1}(\kappa, s)$, $a^{2,\tau}_{n,n-1}(\kappa, s)$ are the functions of (11.3), (11.4), $\Delta^{1,\tau}_{n,n-1}(\kappa, s)$ is the function of (7.9) for $d = 2$.

Formula (11.13) follows from (8.1), (11.3), (7.2) for $d = 2$, and the observation that the terms

$$O(s^{-(n+j-1-1/2)}), \ O(s^{-(n+j_1-1-1/2)}), \ O(s^{-(n+j_2-1-1/2)}),$$
$$j \in \mathbb{N}, \ 2j \leq n, \ j_1, j_2 \in \mathbb{N}, \ j_1 \neq j_2, \ j_1 + j_2 \leq n,$$

arising at comparison of the summands of $a^2_n(\kappa s)$ and $a^2_{n,n-1}(\kappa, s)$ are majorized by $O(s^{-(n-1/2)})$ as $s \to +\infty$.

Formula (11.14) is proved in a similar way with formula (11.13).

Formula (11.15) for $d = 2$ is completely similar to formula (7.17) for $d = 3$.

This completes the proof of Theorem 11.1.

Proof of Theorem 11.2. Using formula (5.1) for $d = 2$ and formula (11.5) we obtain (7.18)-(7.22) for $d = 2$, where $O(s^{-n})$ is replaced by $O(s^{-(n-1/2)})$ in (7.18), (7.21), (7.22).

Formulas (11.2), (11.7) follow from (7.19), (7.20), (7.22) for $d = 2$, where $O(s^{-n})$ is replaced by $O(s^{-(n-1/2)})$ in (7.22).

Theorem 11.2 is proved.

Proof of Theorem 11.3. Due to formulas (5.1)-(5.4), (5.6), (5.7) for $d = 2$ and formulas (8.5), (10.10), (10.11) we have that

$$\begin{pmatrix} z_{n+1}(\kappa s) \\ \bar{z}_{n+1}(\kappa s) \end{pmatrix} = \begin{pmatrix} f_{1,1/2}(\kappa s) \\ \bar{f}_{1,1/2}(\kappa s) \end{pmatrix} - \tag{11.16}$$
$$M\left(\begin{pmatrix} a^2_{n+1}(\kappa s) \\ \Delta^{1,\tau}_{n+1}(\kappa s) + a^2_{n+1}(\kappa s + \tau) \end{pmatrix} + \begin{pmatrix} \delta_{n+1}a(\kappa s) \\ \rho^{1,\tau}_{n+1}(\kappa s) + \delta_{n+1}a(\kappa s + \tau) \end{pmatrix}\right),$$
$$\delta_{n+1}a(\kappa s) = O(s^{-n-1/2}), \quad \delta_{n+1}a(\kappa s + \tau) = O(s^{-n-1/2}) \ \text{ as } s \to +\infty, \tag{11.17}$$
$$\rho^{1,\tau}_{n+1}(\kappa s) = O(s^{-n-1}) \ \text{ as } \ s \to +\infty \tag{11.18}$$

uniformly in $\hat{k}$, $\hat{l}$ at fixed $\tau > 0$ and $E > 0$, where $z_{n+1}(\kappa s) = z_{n+1}(k, l, \kappa s)$, $f_{1,1/2}(\kappa s) = f_{1,1/2}(k, l, \tau, \kappa s)$, $M = M_{k,l,\tau,\kappa s}$, $a^2_{n+1}(\kappa s) = a^2_{n+1}(\kappa s\hat{l}, k)$, $a^2_{n+1}(\kappa s + \tau) = a^2_{n+1}((\kappa s + \tau)\hat{l}, k)$, $\Delta^{1,\tau}_{n+1}(\kappa s) = \Delta^{1,\tau}_{n+1}(\kappa s\hat{l}, k)$, $\delta_{n+1}a(\kappa s) = \delta_{n+1}a(\kappa s\hat{l}, k)$, $\rho^{1,\tau}_{n+1}(\kappa s) = \rho^{1,\tau}_{n+1}(\kappa s\hat{l}, k)$, $\delta_{n+1}a(\kappa s + \tau) = \delta_{n+1}a((\kappa s + \tau)\hat{l}, k)$, $(k, l) \in \Omega_E$, $d = 2$, $E > 0$, $\hat{k} = k/|k|$, $\hat{l} = l/|l|$, $\tau > 0$, $n \in \mathbb{N}$, $\kappa \geq 1$, $\sin((|k| - k\hat{l})\tau) \neq 0$.

In addition, using (4.11), (4.13), (5.1) for $d = 2$, one can see that

$$z_{n+1}(\kappa s) = z_n(\kappa s) + O(s^{-n}), \quad s \to +\infty, \tag{11.19}$$
$$\Delta^{1,\tau}_{n+1}(\kappa s) = \Delta^{1,\tau}_n(\kappa s) + O(s^{-n}), \quad s \to +\infty, \tag{11.20}$$

uniformly in $\hat{k}$, $\hat{l}$ and $\kappa \geq 1$ at fixed $\tau > 0$ and $E > 0$.

Besides, the following formulas hold:

$$a^2_{n+1}(\kappa s) = a^2_{n+1,n-1/2}(\kappa, s) + O(s^{-n}), \quad s \to +\infty, \tag{11.21}$$
$$a^2_{n+1}(\kappa s + \tau) = a^{2,\tau}_{n+1,n-1/2}(\kappa, s) + O(s^{-n}), \quad s \to +\infty, \tag{11.22}$$

for fixed $\sin((|k| - k\hat{l})\tau) \neq 0$, E, τ and $\lambda_1, \ldots, \lambda_n, \kappa$, where $a^2_{n+1}(\kappa s)$, $a^2_{n+1}(\kappa s + \tau)$ are the functions arising in (11.16), $a^2_{n+1,n-1/2}(\kappa, s)$, $a^{2,\tau}_{n+1,n-1/2}(\kappa, s)$ are defined by (11.8), (11.9).

Formula (11.21) follows from (8.1) with $n + 1$ in place of n, (11.8), (11.2) and the observation that the terms

$$O(s^{-(n+j-1)}), \quad O(s^{-(n+j_1-1)}), \quad O(s^{-(n+j_2-1)}),$$
$$j \in \mathbb{N}, \quad 2j \leq n + 1, \quad j_1, j_2 \in \mathbb{N}, \quad j_1 \neq j_2, \quad j_1 + j_2 \leq n + 1,$$

arising at comparison of the summands of $a^2_{n+1}(\kappa s)$ and $a^2_{n+1,n-1/2}(\kappa, s)$ are majorized by $O(s^{-n})$ as $s \to +\infty$.

Formulas (11.10), (11.11) follow from (11.16), (11.19), (11.17), (11.18), (11.21), (11.22) and (11.20), (11.15).

This completes the proof of Theorem 11.3.

Proof of Theorem 11.4. Using formula (5.1) for $d = 2$ and formula (11.10) we obtain (7.18)-(7.22) for $d = 2$, where $\zeta_{n,n-1}$ is replaced by $\zeta_{n,n-1/2}$.

Formulas (7.4) for $d = 2$, (11.12) follow from (7.19), (7.20), (7.22), where $\zeta_{n,n-1}$ is replaced by $\zeta_{n,n-1/2}$ in (7.20).

Theorem 11.4 is proved.

12. Formulas for ψ^+

Theorem 12.1. *Under assumptions (1.2), (1.7) for $d = 3$ and for $d = 2$, the following formulas hold:*

$$\begin{aligned} &\psi^+(s\hat{l}, k) = \tilde{\psi}_n^+(s) + O(s^{-n-(d-1)/2}) \ \text{ as } \ s \to +\infty \\ &\text{for fixed } \ \sin((|k| - k\hat{l})\tau) \neq 0, \ E, \tau \ \text{ and } \ \lambda_1, \dots, \lambda_n, \end{aligned} \tag{12.1}$$

where ψ^+ is the function of (1.4), (3.3), $\tilde{\psi}_n^+(s) = \tilde{\psi}_n^+(k, l, \tau, \lambda_1, \dots, \lambda_n, s)$ is found from $|\psi^+(x, k)|^2$ given at $x = x_1(s), \dots, x_{2n}(s)$ defined by (1.15),

$$\tilde{\psi}_n^+(s) = e^{isk\hat{l}} + \frac{e^{i|k|s}}{s^{(d-1)/2}} \left(\sum_{j=1}^{n} \frac{f_{j,n-j+1}(s)}{s^{j-1}} \right), \tag{12.2}$$

where $f_{j,n-j+1}$, $j = 1, \dots, n$, are the functions of (7.3), (7.4) (see Section 7 for $d = 3$ and Section 11 for $d = 2$), $(k, l) \in \Omega_E$, $E > 0$, $\hat{l} = l/|l|$, $n \in \mathbb{N}$.

Formulas (12.1) follow from expansion (3.3), definition (12.2) and the asymptotic formulas (7.4).

Theorem 12.1 realizes formulas (1.16), (3.5).

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