



# Necessary players, Myerson fairness and the equal treatment of equals

Florian Navarro

## ► To cite this version:

Florian Navarro. Necessary players, Myerson fairness and the equal treatment of equals. *Annals of Operations Research*, 2019, 280, pp.111-119. 10.1007/s10479-018-3055-0 . hal-02118370

**HAL Id: hal-02118370**

**<https://hal.science/hal-02118370>**

Submitted on 11 Dec 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Necessary players, Myerson fairness and the equal treatment of equals\*

Florian NAVARRO<sup>†</sup>

Laboratoire LEM UMR 9221, Université de Lille, Maison de la Recherche  
Domaine universitaire du Pont de Bois, B.P. 60149,  
59653 Villeneuve d'Ascq Cedex, France

September 11, 2018

## Abstract

This article addresses linear sharing rules on transferable utility games (TU-games) with various structures, namely communication structures and conference structures as defined by Myerson in two papers (1977, 1980). Here, using matrix expressions, we rewrite those sharing rules. With this presentation we identify the close relationship between the fairness property and an equal treatment of necessary players axiom. Moreover, we show that the latter is implied by the equal treatment of equals, linking the fairness property to the notion of equality.

**Keywords** Game theory • Myerson value • fairness • graph • cooperative games

## 1 Introduction

In his article of 1977, Myerson ([10]) introduced games with communication structures. Since then, two generalizations of communication structures have been proposed. Myerson himself (1980, [11]) introduced conferences structure and Algaba *et al.* (2000, [1]) introduced union stable systems<sup>1</sup>. In this article we will only focus on the works of Myerson.

The two above mentioned articles by Myerson also extended the Shapley value (1953,[12]) to games with communication structures (1977, [10]) and games with conference structures (1980, [11]). Myerson proposed modifications of the original games to capture and project particularities of the concerned structures in a TU-game. The Myerson values are defined as the Shapley value of these modified games. The axiomatic characterization offered by Myerson in 1977 is based

---

\*This article is part of my Ph.D. dissertation that is conducted under the supervision of Pr. Gérard Hamiache. I am grateful for his guidance and precious advice.

<sup>†</sup>florinnavarro.unipro@gmail.com

<sup>1</sup>The propinquity between these frameworks is explored in Algaba *et al.* (2004, [3]).

heavily on the fairness property. This property states that when communication between two players is severed, both players are impacted exactly the same way. Is this requirement reasonable ? Indeed, let us imagine that communication is cut between two players with one left isolated whereas the other is still “well connected”. It seems far fetched to think that both of them will experience their lack of communication the same way and be impacted on the same level.

In this article, our aim is to investigate how the fairness property came to be satisfied by an extension of the Shapley value. We will proceed by showing the existing link between the fairness property and an equal treatment of necessary players axiom, similar in concept to the necessary players axiom presented by van den Brink and Gilles (1996, [5]). This equal treatment of necessary players axiom states that all players that are absolutely needed to produce worth should be rewarded the same amount. This axiom seems easier to accept as a requirement for a sharing rule and less controversial - for instance, we will show this property is implied by the equal treatment of equals. It is also much weaker than the fairness property. In this paper, we will show that the fairness property emerges from the equal treatment of necessary players property satisfied by the Shapley value. Furthermore, using the works of Hamiache (2012, [9]) and Xu (2008, [14]), we will express a general form of the Myerson method through a matrix approach. With this form we prove that the fairness and the equal treatment of necessary players properties are equivalent under the transformation the Myerson method operates on games with communications or conferences structures. This means extending any sharing rule for TU-games verifying the equal treatment of necessary players property using the Myerson method will result in a fair sharing rule.

In the recent years, the use of matrices in the literature on cooperative games has become much more common and our approach is consistent with that trend. Most of the works using the tools of matrix analysis deal with Hamiache’s associated consistency principle (2010, [8]) as seen for instance in Béal *et al.* (2016, [4]), Wang *et al.* (2017, [13]) and Xu *et al.* (2008, [15] ; 2015, [16]).

## 2 The formal framework

Let  $U$  be a finite set of players, we call a coalition a subset of  $U$ . We write  $(N, v)$  to represent a coalitional game with transferable utility (TU-game) where  $N \subseteq U$  is the set of players and  $v : 2^N \rightarrow \mathbb{R}$  (with  $v(\emptyset) = 0$ ) is the characteristic function of the game. Let  $\Gamma$  be the set of all such games.

We call a unanimity game, a game  $(N, u_R)$  with  $u_R(S) = 1$  if  $R \subseteq S \subseteq N$  and  $u_R(S) = 0$  otherwise. We call a Dirac game, a game  $(N, \delta_R)$  with  $\delta_R(S) = 1$  if  $S = R$  and  $\delta_R(S) = 0$  otherwise.

Let  $g_N = \{\{i, j\} | i \neq j, i, j \in N\}$  be the set of all possible links between two players in  $N$ . We denote as  $g \subseteq g_N$  a subset of those links. We call a graph a pair  $(N, g)$  where the vertices of the graph represent the players in  $N$  and the edges represent the ability for two players to communicate. For all  $S \subseteq N$  and all  $g \subseteq g_N$ , we define  $g(S) = \{\{i, j\} | i \in S, j \in S, \{i, j\} \in g\}$

the links of the subgraph  $(S, g(S))$  induced by coalition  $S$ . We call a path between two vertices  $i$  and  $j \in N$ , a string of vertices  $i = i_1, i_2, \dots, i_k = j$  such that,  $\forall q$  with  $1 \leq q \leq k-1$ , we have  $\{i_q, i_{q+1}\} \in g$ . If there exists a path between two vertices of  $N$ , we say they are connected by the graph  $(N, g)$ . We symbolize the connectedness of two players  $i, j \in N$  by the graph  $(N, g)$  with the following notation :  $i \xrightarrow{(N, g)} j$ . For all graphs  $(N, g)$ , we denote

$N/g = \left\{ \{i \mid i \xrightarrow{(N, g)} j, i \in N\} \cup \{j\} \mid j \in N \right\}$  as the set of the components of  $N$ . A graph is connected if  $|N/g| = 1$ . A subgraph  $(S, g(S))$  is connected if  $|S/(g(S))| = 1$ .

A game with communication structure is a triplet  $(N, v, g)$ , where  $(N, v)$  is a TU-game and  $(N, g)$  is a graph. We shall write as  $\Gamma_G$  the set of those games. We define the quotient game  $v/g$  as

$$(v/g)(S) = \sum_{R \in S/(g(S))}^R v(R), \quad \forall S \subseteq N. \quad (1)$$

The worth of a coalition in the quotient game is the sum of the worth of its components.

Given a set of players  $N$ , let  $\mathcal{H}^N = \{S \mid S \subseteq N, |S| \geq 2\}$  be the set of the non-singleton subsets of  $N$ . We call a conference structure a pair  $(N, \mathcal{H})$  where  $\mathcal{H} \subseteq \mathcal{H}^N$ . We note  $\mathcal{H}(S) = \{H \in \mathcal{H} \mid H \subseteq S\}$ . Two players  $i$  and  $j$  are connected by  $\mathcal{H}$  if there is a sequence  $\{H_1, H_2, \dots, H_k\}$  such that  $H_1, \dots, H_k \in \mathcal{H}$ ,  $i \in H_1$ ,  $j \in H_k$  and  $H_q \cap H_{q+1} \neq \emptyset$  for all  $q$  such that  $1 \leq q \leq k-1$ . We note this connectedness by  $i \xrightarrow{\mathcal{H}} j$ . We then write  $N/\mathcal{H} = \left\{ \{i \mid i \in N, i \xrightarrow{\mathcal{H}} j\} \cup \{j\} \mid j \in N \right\}$ , the set of the components of  $N$ .

We call a game with conference structure a triplet  $(N, v, \mathcal{H})$  where  $(N, v)$  is a TU-game and  $(N, \mathcal{H})$  is a conference structure. We shall write as  $\Gamma_{\mathcal{H}}$  the set of those games. We define the conference game  $v/\mathcal{H}$  as

$$(v/\mathcal{H})(S) = \sum_{T \in S/(\mathcal{H}(S))}^T v(T), \quad \forall S \subseteq N. \quad (2)$$

We call a solution or a sharing rule on  $\Gamma$  a function  $\psi$  that associates to each game  $(N, v) \in \Gamma$  a vector  $\psi(N, v) \in \mathbb{R}^N$ . A sharing rule is called linear if  $\psi(N, \alpha v + \beta w) = \alpha \psi(N, v) + \beta \psi(N, w)$  for all games  $(N, v)$  and  $(N, w)$  and for all real numbers  $\alpha$  and  $\beta$ . This paper focuses on linear sharing rule.

In a similar way, we define a sharing rule for games with communication structure as a function  $\phi$  which associates with all games  $(N, v, g) \in \Gamma_G$  a vector  $\phi(N, v, g) \in \mathbb{R}^N$ .

We also define a sharing rule for games with a conference structure as a function  $\gamma$  which associates with all games with conference structure  $(N, v, \mathcal{H}) \in \Gamma_{\mathcal{H}}$  a vector  $\gamma(N, v, \mathcal{H}) \in \mathbb{R}^N$ .

Introduced by Shapley (1953, [12]), the Shapley value is a sharing rule for TU-

games. We denote it  $Sh$ , and we define,  $\forall i \in N$ ,

$$Sh_i(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S \setminus \{i\})],$$

with  $s = |S|$  as the cardinal of coalition  $S$  and  $n = |N|$ .

Two players  $i, j \in N$  are equal in the game  $(N, v)$  if and only if  $v(S \cup \{i\}) = v(S \cup \{j\})$ ,  $\forall S \subseteq N \setminus \{i, j\}$ .

**Axiom 1** (*Equal treatment of equals*) :

A sharing rule  $\psi$  for TU-games satisfies the equal treatment of equals if and only if, for all pairs of equal players  $i, j \in N$ , we have  $\psi_i(N, v) = \psi_j(N, v)$ .

**Axiom 2** (*Fairness G*) :

For all games with communication structure  $(N, v, g)$ , a sharing rule  $\phi$  is fair if and only if  $\phi_i(N, v, g) - \phi_i(N, v, g \setminus \{i, j\}) = \phi_j(N, v, g) - \phi_j(N, v, g \setminus \{i, j\})^2$  for all links  $\{i, j\} \in g$ .

**Axiom 3** (*Fairness C<sup>3</sup>*) :

For all games with conference structure  $(N, v, \mathcal{H})$ , a sharing rule  $\gamma$  is fair if and only if  $\gamma_i(N, v, \mathcal{H}) - \gamma_i(N, v, \mathcal{H} \setminus \{H\}) = \gamma_j(N, v, \mathcal{H}) - \gamma_j(N, v, \mathcal{H} \setminus \{H\})$  for all conferences  $H \in \mathcal{H}$  and all players  $i, j \in H$ .

We introduce a new axiom, with a concept similar to (albeit weaker than) the axiom introduced by van den Brink and Gilles ([5], 1996).

A player  $i \in N$  is a necessary player in a game  $(N, v)$  if and only if  $v(S) = 0$  for all  $S \subseteq N \setminus \{i\}$ .

**Axiom 4** (*Equal treatment of necessary players*) :

A sharing rule  $\psi$  for TU-games satisfies the equal treatment of necessary players if and only if, for all games  $(N, v) \in \Gamma$  such that  $\{i, j\} \subseteq N$  is a couple of necessary players, we have  $\psi_i(N, v) = \psi_j(N, v)$ .

This axiom states that if some players are absolutely needed to produce worth,

<sup>2</sup>The correct notation for  $g \setminus \{i, j\}$  would be  $g \setminus \{\{i, j\}\}$ . However, for ease of reading, we decided to simplify the notation.

<sup>3</sup>A third appearance of the fairness axiom is also defined for union stable systems by Algaba *et al.* (2001, [2]) to define the Myerson value for union stable systems.

they should receive the same payment.

**Proposition 1**

*Let  $\psi$  be a sharing rule for the set of TU-games  $\Gamma$ ; if  $\psi$  satisfies the equal treatment of equals, then  $\psi$  satisfies the equal treatment of necessary players.*

Since necessary players in a game  $(N, v)$  are also equals in  $(N, v)$ , this proposition is obviously true.

### 3 Fairness and the Myerson method for games with communication structure

The Myerson value for games with communication structure ([10] Myerson, 1977), denoted in this paper by  $MY^G$ , is defined as follows,  $\forall i \in N$ :

$$MY_i^G(N, v, g) = Sh_i(N, v/g) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(s-1)!(n-s)!}{n!} [v/g(S) - v/g(S \setminus \{i\})].$$

For all linear sharing rules  $\psi$  for TU-games, we can define a sharing rule  $\phi$  for games with communication structure that is an extension of  $\psi$  using the Myerson method. We write this extension as

$$\phi(N, v, g) = \psi(N, v/g),$$

for all games  $(N, v, g)$ .

In his PhD dissertation, Xu (2008, [14]) used the set of Dirac games as a basis for the set of TU-games. He introduced a matrix  $\mathbf{M}^\psi$  with  $[\mathbf{M}^\psi]_{i,S} = \psi_i(N, \delta_S)$ ,  $\forall i \in N$ ,  $\forall S \subseteq N$  and proved that it is the only matrix such that  $\psi(N, v) = \mathbf{M}^\psi \times v^4$  for all games  $(N, v)$  and all linear sharing rules  $\psi$ . Applying his work to our specific case, we can write

$$\psi(N, v/g) = \mathbf{M}^\psi \times (v/g).$$

From here, we develop a matrix approach to the Myerson method. We define, as did Hamiache (2012, [9]), a matrix  $\mathbf{C}^g$  of the components such that

$$[\mathbf{C}^g]_{S \subseteq N, T \subseteq N} = \begin{cases} 1 & \text{if } T \in S/(g(S)), \\ 0 & \text{otherwise,} \end{cases}$$

---

<sup>4</sup>Since we deal with a finite number of coalitions, the characteristic function  $v$  can also be written in a finite vector form. Each element of the vector is the worth of the corresponding coalition. In the following we use  $v$  to denote the function or the vector. Further down we also use  $v/g$  the same way.

Every row  $S$  of this matrix is a vector of zeros and ones that decomposes  $S$  into its components. Using this matrix, we write

$$(v/g)(S) = \sum_{\substack{R \\ R \in S/(g(S))}} v(R) = \sum_{\substack{R \\ R \subseteq N}} [\mathbf{C}^g]_{S,R} v(R),$$

or simply

$$v/g = \mathbf{C}^g \times v.$$

Therefore, we can express any sharing rule  $\phi(N, v, g) = \psi(N, v/g)$  as

$$\phi(N, v, g) = \mathbf{M}^\psi \times \mathbf{C}^g \times v,$$

or

$$\phi_i(N, v, g) = \sum_{\substack{T \\ T \subseteq N}} [\mathbf{M}^\psi \mathbf{C}^g]_{i,T} v(T), \forall i \in N.$$

Using this form, we uncover the two following results :

**Lemma 1**

*Let  $\psi$  be a linear sharing rule on the set of TU-games  $\Gamma$ ;  $\psi$  satisfies the equal treatment of necessary players if and only if  $[\mathbf{M}^\psi]_{i,T} = [\mathbf{M}^\psi]_{j,T}$  for all  $T$ ,  $\{i, j\} \subseteq T \subseteq N$ , that is to say, if and only if  $\psi$  satisfies the equal treatment of necessary players on Dirac games.*

**Proof :** It is obvious that if  $\psi$  satisfies the equal treatment of necessary players, then  $\psi$  satisfies this axiom on Dirac games. Therefore, let us show that if a linear sharing rule  $\psi$  satisfies the equal treatment of necessary players on Dirac games, it is satisfied on all games. By linearity, we can write

$$\psi_i(N, v) = \sum_{\substack{T \\ T \subseteq N}} [\mathbf{M}^\psi]_{i,T} v(T).$$

By detailing, we obtain

$$\psi_i(N, v) = \sum_{\substack{T \\ \{i,j\} \subseteq T \subseteq N}} [\mathbf{M}^\psi]_{i,T} v(T) + \sum_{\substack{T \subseteq N \\ |T \cap \{i,j\}| < 2}} [\mathbf{M}^\psi]_{i,T} v(T).$$

The equal treatment of necessary players axiom states that two necessary players must receive the same payment. Therefore, let us assume that  $i$  and  $j$  are necessary players in the game  $(N, v)$ . This means that  $v(T) = 0$  when  $|T \cap \{i, j\}| < 2$ , and thus

$$\psi_i(N, v) = \sum_{\substack{T \\ \{i,j\} \subseteq T \subseteq N}} [\mathbf{M}^\psi]_{i,T} v(T).$$

Players  $i$  and  $j$  are necessary players in a Dirac game  $\delta_T$  if  $i, j \in T$ . If  $\psi$  satisfies the equal treatment of necessary players on Dirac games, then  $[\mathbf{M}^\psi]_{i,T} = [\mathbf{M}^\psi]_{j,T}$ ,  $\forall T$ ,  $\{i, j\} \subseteq T \subseteq N$ , which implies that  $\psi_i(N, v) = \psi_j(N, v)$  when players  $i$  and  $j$  are necessary players in the game  $(N, v)$ .  $\square$

**Proposition 2**

Let  $\psi$  be a linear sharing rule on the set of TU-games  $\Gamma$ . Let  $\phi$  be a sharing rule for games with communication structure such that  $\phi(N, v, g) = \mathbf{M}^\psi \times \mathbf{C}^g \times v$ . The sharing rule  $\phi$  is fair if and only if  $\psi$  satisfies the equal treatment of necessary players.

**Proof :** If  $\phi(N, v, g) = \psi(N, v/g)$ , then we can write

$$\begin{aligned} & \phi_i(N, v, g) - \phi_i(N, v, g \setminus \{i, j\}) \\ &= \sum_{\substack{T \\ T \subseteq N}} [\mathbf{M}^\psi \mathbf{C}^g]_{i,T} v(T) - \sum_{\substack{T \\ T \subseteq N}} [\mathbf{M}^\psi \mathbf{C}^{g \setminus \{i, j\}}]_{i,T} v(T) \\ &= \sum_{\substack{T \\ T \subseteq N}} \left[ \sum_{\substack{S \\ S \subseteq N}} [\mathbf{M}^\psi]_{i,S} [\mathbf{C}^g]_{S,T} - \sum_{\substack{S \\ S \subseteq N}} [\mathbf{M}^\psi]_{i,S} [\mathbf{C}^{g \setminus \{i, j\}}]_{S,T} \right] v(T) \\ &= \sum_{\substack{T \\ T \subseteq N}} \left[ \sum_{\substack{S \\ S \subseteq N}} [\mathbf{M}^\psi]_{i,S} \left[ [\mathbf{C}^g]_{S,T} - [\mathbf{C}^{g \setminus \{i, j\}}]_{S,T} \right] \right] v(T). \end{aligned}$$

Let us distinguish among the three following cases :  $\{i, j\} \subseteq N \setminus S$ ,  $|\{i, j\} \cap S| = 1$  and  $\{i, j\} \subseteq S$ .

Case 1 :

If  $\{i, j\} \subseteq N \setminus S$ , then  $\{i, j\} \notin g(S)$ , which implies that  $[\mathbf{C}^g]_{S,T} = [\mathbf{C}^{g \setminus \{i, j\}}]_{S,T}$  as  $S/(g(S)) = S/((g \setminus \{i, j\})(S))$ . This cancels the equation  $\forall T \subseteq N$ .

Case 2 :

If  $|\{i, j\} \cap S| = 1$ , we obtain the same results as before because the link  $\{i, j\}$  is not contained in  $g(S)$ . Its deletion does not affect the components of  $S$ .

Case 3 :

We are left with the last case,  $\{i, j\} \subseteq S$ . We can write the previous equation



as

$$\begin{aligned} & \phi_i(N, v, g) - \phi_i(N, v, g \setminus \{i, j\}) \\ &= \sum_{\substack{T \subseteq N \\ T \subseteq N}} \left[ \sum_{\substack{S \subseteq N \\ \{i, j\} \subseteq S \subseteq N}} [\mathbf{M}^\psi]_{i,S} \left[ [\mathbf{C}^g]_{S,T} - [\mathbf{C}^{g \setminus \{i, j\}}]_{S,T} \right] \right] v(T). \quad (3) \end{aligned}$$

Therefore, if  $\psi$  satisfies  $[\mathbf{M}^\psi]_{i,S} = [\mathbf{M}^\psi]_{j,S}$  for all coalitions  $S$  such that  $\{i, j\} \subseteq S \subseteq N$ , then the condition  $\phi_i(N, v, g) - \phi_i(N, v, g \setminus \{i, j\}) = \phi_j(N, v, g) - \phi_j(N, v, g \setminus \{i, j\})$  holds. By the use of lemma (1), we conclude that if  $\psi$  satisfies the weak necessary player axiom, then  $\phi$  is fair.

Let us now consider the other side of the equivalence. We assume that  $\phi$  satisfies  $\phi_i(N, v, g) - \phi_i(N, v, g \setminus \{i, j\}) = \phi_j(N, v, g) - \phi_j(N, v, g \setminus \{i, j\})$ ,  $\forall i, j \in N$ ,  $\{i, j\} \in g$ . Which we can write, from equation (3), as

$$\begin{aligned} & \sum_{\substack{T \subseteq N \\ T \subseteq N}} \left[ \sum_{\substack{S \subseteq N \\ \{i, j\} \subseteq S \subseteq N}} [\mathbf{M}^\psi]_{i,S} \left[ [\mathbf{C}^g]_{S,T} - [\mathbf{C}^{g \setminus \{i, j\}}]_{S,T} \right] \right] v(T) \\ &= \sum_{\substack{T \subseteq N \\ T \subseteq N}} \left[ \sum_{\substack{S \subseteq N \\ \{i, j\} \subseteq S \subseteq N}} [\mathbf{M}^\psi]_{j,S} \left[ [\mathbf{C}^g]_{S,T} - [\mathbf{C}^{g \setminus \{i, j\}}]_{S,T} \right] \right] v(T). \end{aligned}$$

As this condition must be satisfied for any characteristic function  $v$ , in particular for Dirac games over  $T$ , we obtain, for all coalitions  $T \subseteq N$

$$\begin{aligned} & \sum_{\substack{S \subseteq N \\ \{i, j\} \subseteq S \subseteq N}} [\mathbf{M}^\psi]_{i,S} \left[ [\mathbf{C}^g]_{S,T} - [\mathbf{C}^{g \setminus \{i, j\}}]_{S,T} \right] \\ &= \sum_{\substack{S \subseteq N \\ \{i, j\} \subseteq S \subseteq N}} [\mathbf{M}^\psi]_{j,S} \left[ [\mathbf{C}^g]_{S,T} - [\mathbf{C}^{g \setminus \{i, j\}}]_{S,T} \right]. \quad (4) \end{aligned}$$

This must be true for all graphs, in particular for star-graphs  $(N, g)$  with  $g = \{\{1, i\}, \{2, i\}, \dots, \{i-1, i\}, \{i+1, i\}, \dots, \{n, i\}\}$ . Note that, in this case, the graph  $(S, g(S))$  with  $\{i, j\} \subseteq S \subseteq N$  is connected.

We said that this must be true for all coalitions  $T \subseteq N$ ; let us consider every coalition  $T$  such that  $\{i, j\} \subseteq T$ . We have then  $[\mathbf{C}^{g \setminus \{i, j\}}]_{S,T} = 0$ ,  $\forall S \subseteq N$  because coalition  $T$  is no longer connected ( $j$  is isolated) and thus cannot be a component of  $S$ . We also have

$$[\mathbf{C}^g]_{S,T} = \begin{cases} 1 & \text{if } S = T, \\ 0 & \text{if } S \neq T, \end{cases}$$

because  $S$  is always connected. Equation (4) is then reduced to

$$[\mathbf{M}^\psi]_{i,S} = [\mathbf{M}^\psi]_{j,S}.$$

This means that if  $\phi$  satisfies the fairness property, it implies that the sharing rule  $\psi$  satisfies

$$[\mathbf{M}^\psi]_{i,S} = [\mathbf{M}^\psi]_{j,S}, \quad (5)$$

for all  $S$  such that  $\{i, j\} \subseteq S \subseteq N$ . As  $\psi$  is linear, by lemma (1), we conclude this side of the equivalence.

□

## 4 Fairness and the Myerson method for games with conference structure

The Myerson value for games with conference structure proposed by Myerson in 1980 ([11]), denoted by  $MY^C$ , is defined as,  $\forall i \in N$ :

$$MY_i^C(N, v, \mathcal{H}) = Sh_i(N, v/\mathcal{H}).$$

We deduce a general form for all linear sharing rules  $\psi$  for TU-games:

$$\gamma_i(N, v, \mathcal{H}) = \psi_i(N, v/\mathcal{H}).$$

Introducing the following matrix,

$$[\mathbf{K}]_{S \subseteq N, T \subseteq N} = \begin{cases} 1 & \text{if } T \in S/(\mathcal{H}(S)), \\ 0 & \text{otherwise,} \end{cases}$$

we can now express any sharing rule  $\gamma(N, v, \mathcal{H}) = \psi(N, v/\mathcal{H})$  as

$$\gamma_i(N, v, \mathcal{H}) = \sum_{\substack{T \\ T \subseteq N}} [\mathbf{M}^\psi \mathbf{K}]_{i,T} v(T), \quad \forall i \in N.$$

Using this form, we prove the following proposition:

### Proposition 3

*Let  $\psi$  be a linear sharing rule on TU-games. Let  $\gamma$  be a sharing rule for games with conference structure such that  $\gamma(N, v, g) = \mathbf{M}^\psi \times \mathbf{K} \times v$ . The sharing rule  $\gamma$  is fair if and only if  $\psi$  satisfies the equal treatment of necessary players.*

### Proof :

The proof is similar to the proof of proposition 2.

□

## 5 Conclusions

In this paper, we have constructed a matrix approach to games with communication and conference structure based on works by Hamiache and Xu. We have also expressed a general form that has helped define, in a similar way, the Myerson value for both communication and conference structures. In future works, we hope to determine the set of sharing rules that can be expressed in this form.

Using this new representation, we have proven that the fairness property is directly related to an equal treatment of necessary players axiom under the Myerson method. However, what is more important is that this necessary players axiom is implied by the equal treatment of equals. This sheds new light on the Myerson method and the use of the quotient game since any sharing rule satisfying the equal treatment of equals will satisfy the fairness property if extended using the Myerson method. However, the fairness property does not benefit from the same consensus as the equal treatment of equals. Therefore, if we want to extend a sharing rule satisfying the equal treatment of equals to games with communication structure, we have to either accept the fairness property despite its flaws, or find another method than the one proposed by Myerson. For instance, the sharing rule defined by the Banzhaf index of the quotient game would also be fair.

Furthermore, the equal treatment of necessary players axiom identifies precisely the concept of equality specific to sharing rules for communication structures that are sensitive only to the connectedness of coalitions. We can also draw a parallel between this axiom and the fairness axiom introduced by van den Brink (2001, [6]) and, therefore, with the differential marginality axiom presented by Casajus (2011, [7]). As a result, we believe this axiom may be useful for further research on symmetric allocation rules.

## References

- [1] E. Algaba, J. M. Bilbao, P. Borm, J. J. López. The position value for union stable systems. *Mathematical Methods of Operations Research*, **52**, pp. 221–236, 2000.
- [2] E. Algaba, J. M. Bilbao, P. Borm, J. J. López. The Myerson value for union stable systems. *Mathematical Methods of Operations Research*, **54**, pp. 359–371, 2001
- [3] E. Algaba, J. M. Bilbao, J. J. López. The position value in communication structures. *Mathematical Methods of Operations Research*, **59**, pp. 465–477, 2004.
- [4] S. Béal, E. Rémila, P. Solal. Characterizations of three linear values for TU games by associated consistency : simple proofs using the Jordan normal form. *International Game Theory Review*, **18**, pp. 1–18, 2016.

- [5] R. van den Brink and R.P. Gilles. Axiomatizations of the conjunctive permission value for games with permission structures. *Games and Economic Behaviour*, **12**, pp. 113–126, 1996.
- [6] R. van den Brink. An axiomatization of the Shapley Value using a fairness property. *International Journal of Game Theory*, **30**, pp. 309–319, 2001.
- [7] A. Casajus. Differential marginality, van den Brink fairness, and the Shapley value. *Theory of Decision*, **71**, pp. 163–174, 2011.
- [8] G. Hamiache. A Matrix Approach to the Associated Consistency with an Application to the Shapley Value. *International Game Theory Review*, **12**, pp. 175. 2010.
- [9] G. Hamiache. A Matrix Approach to TU-Games with Coalition and Communication Structures. *Social Choice Welfare*, **38**, pp. 85–100, 2012.
- [10] R. B. Myerson. Graph and cooperation in games. *Math. Oper. Res.*, **2**, pp. 225–229, 1977.
- [11] R. B. Myerson. Conference Structures and fair allocation rules. *International Journal of Game Theory*, **9**, pp. 169–182, 1980.
- [12] L.S. Shapley. A value for  $n$ -person games. In Contributions to the Theory of Games II, *Annals of Mathematics Studies*, pp. 307–317, Princeton University Press, Princeton 1953.
- [13] W. Wang, H. Sun, G. Xu, D. Hou. Procedural interpretation and associated consistency for the egalitarian Shapley values. *Operations Research Letters*, **45**, pp. 164–169, 2017.
- [14] G. Xu. Matrix approach to cooperative game theory. PhD Thesis, 2008.
- [15] G. Xu, T. S. H. Driessen, H. Sun. Matrix analysis for associated consistency in cooperative game theory. *Linear Algebra and its Applications*, **428**, pp. 1571–1586, 2008.
- [16] G. Xu, R. van den Brink, G. van der Laan, H. Sun. Associated consistency characterization of two linear values for TU games by matrix approach. *Linear Algebra and its Applications*, **471**, pp. 224–240, 2015.