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# CYCLIC COHOMOLOGY FOR GRADED $C^{*,\tau}$ -ALGEBRAS AND ITS PAIRINGS WITH VAN DAELE $K$ -THEORY

JOHANNES KELLENDONK

ABSTRACT. We consider cycles for graded  $C^{*,\tau}$ -algebras (Real  $C^*$ -algebras) which are compatible with the  $*$ -structure and the real structure. Their characters are cyclic cocycles. We define a Connes type pairing between such characters and elements of the van Daele  $K$ -groups of the  $C^{*,\tau}$ -algebra and its real subalgebra. This pairing vanishes on elements of finite order. We define a second type of pairing between characters and  $K$ -group elements which is derived from a unital inclusion of  $C^*$ -algebras. It is potentially non-trivial on elements of order two and torsion valued. Such torsion valued pairings yield topological invariants for insulators. The two-dimensional Kane-Mele and the three-dimensional Fu-Kane-Mele strong invariant are special cases of torsion valued pairings. We compute the pairings for a simple class of periodic models and establish structural results for two dimensional aperiodic models with odd time reversal invariance.

## 1. INTRODUCTION

Recent developments in solid state physics, notably the classification of topological phases [38, 29], underline the importance of real  $K$ - and  $KK$ -theory in physics. As part of this development, the  $C^*$ -algebraic approach to solid state systems [1] was extended to describe insulators of different types by including a grading or a real structure on the observable algebra [26] (see also [41] for a related proposal). In this article we discuss the cyclic cohomology (in the formulation as characters of cycles) of such algebras.

In the  $C^*$ -algebraic approach topological quantised transport coefficients are expressed as pairings of a  $K$ -group element either with a character of a cycle (a Connes pairing) or with a  $K$ -homology class (an index pairing). While the Connes pairing with a character is more directly related to the physical interpretation as a transport coefficient and yields a local formula, the index pairing proves integrality of the coefficient and can be extended to the strong disorder regime [2]. The two approaches are thus complementary. This theory of pairings has first been developed for the Integer Quantum Hall Effect [1, 14, 2, 8] and recently extended to topological insulators of complex type [34], and is now in active development for insulators of real type [9, 10]. The present work aims to contribute to this development.

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Insulators of real type are insulators which transform under an anti-linear automorphism of order 2, like complex conjugation, and so there is an additional ingredient to take into account: a real structure on the  $C^*$ -algebra. This leads to the consideration of  $K$ -groups of real  $C^*$ -algebras and brings in a feature which has not been of importance for complex insulators, namely the occurrence of elements of finite order (torsion elements) in the  $K$ -group. While also complex  $K$ -groups may contain torsion elements, for instance in the case of certain quasicrystals [18], the physical significance of those torsion elements remains unclear so far and so they don't play a big role. But for certain real topological insulators, like the Kane-Mele model [24], the most relevant  $K$ -group elements have order 2 and show up in experiments [30]. It is therefore a problem that the Connes pairing is trivial on elements of finite order.

To overcome this problem we define torsion-valued pairings, somewhat in the spirit of the determinant of de la Harpe-Skandalis. Such a torsion-valued pairing is defined on the kernel  $\ker \varphi_*$  of the map induced in  $K$ -theory by an inclusion  $B \xrightarrow{\varphi} \tilde{B}$  of  $C^*$ -algebras. We analyse it more closely in two cases which arise naturally when comparing a real  $C^*$ -algebra with its complexification. They are related to an exact sequence attributed to Wood-Karoubi. It turns out that, in these two cases, the  $K$ -class of the Hamiltonian of an insulator belongs to the above mentioned kernel if and only if that  $K$ -class admits a representative (possibly different from the Hamiltonian) which admits an extra symmetry: a spin symmetry, or an imaginary chiral symmetry. We refer to the first case as even, and to the second as odd. We provide explicit local formulae for the torsion valued pairing which involve the representative and its extra symmetry.

The results presented here will permit a formulation of the bulk boundary correspondence in the spirit of [27, 28], as an equality between torsion-valued pairings of characters of cycles with  $K$ -group elements. We intend to describe this in an upcoming publication. The complementary approach to the bulk boundary correspondence which is based on the index pairing has already been developed to quite some extent [20, 9, 10]. In its most powerful form it is based on  $KK$ -theory and the (unbounded) Kasparov product. These technics are very different from what we do here.

Our article is organised as follows. After recalling some preliminaries we explain briefly van Daele's formulation of  $K$ -theory for real or complex Banach algebras. In Section 4 we discuss the cyclic cohomology for graded algebras in the framework of cycles and their characters. We define a Connes-type pairing between characters and  $K$ -group elements. It reduces to the usual Connes pairing in case the grading on the algebra is trivial. We introduce the notion of the sign and the parity of a cycle and formulate necessary conditions under which the Connes-pairing and the torsion-valued pairing can be non-trivial. Section 4 ends with a proof that the pairing is compatible with the suspension construction also in the graded case.

In Section 5 we introduce torsion-valued pairings of characters of cycles with  $K$ -group elements. The basic construction is based on a unital inclusion of one  $C^*$ -algebra into another and we focus on two such inclusions, the inclusion of a real  $C^*$ -algebra  $B$  into its graded tensor product with the Clifford algebra  $Cl_{1,0}$ , and the inclusion of  $B$  into its complexification. We explain the relevance of extra symmetries and provide explicit formulas for the torsion valued pairings.

In Section 6 we compute the pairings for a class of simple periodic models which are used in the literature for modelling topological phases with various symmetries. These are closely related to the Bott element on the torus. Our approach is based on a systematic use of Clifford algebras, their real structures, and their graded and ungraded representations.

In Section 7 we provide structural results for aperiodic models, the main part here is restricted to two dimensional systems with odd time reversal invariance.

Our definition of the even torsion valued pairing was largely influenced by the recent work on periodically driven systems with odd time reversal invariance [12, 13]. We will explain the connection in more detail in the last section.

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## 2. PRELIMINARIES

We consider here real or complex associative algebras which mostly are equipped with a norm, a  $*$ -structure (an involution which is anti-linear in the complex case), a grading and sometimes also with a real structure.

Let  $G$  be an abelian group. Recall that a  $G$ -grading on an algebra  $A$  is a direct sum decomposition of  $A = \bigoplus_{g \in G} A_g$  such that the algebra product  $ab$  of  $a \in A_g$  with  $b \in A_h$  lies in  $A_{gh}$ . The elements of  $A_g$  have degree  $g$  and we denote that degree by  $|a|_G$ . If  $A$  is a  $*$ -algebra we also require that the subspaces  $A_g$  are invariant under the  $*$ -operation.

We are interested in the case that  $G = \mathbb{Z}_2$ ,  $G = \mathbb{Z}$ , or  $G = \mathbb{Z} \times \mathbb{Z}_2$ . Since it is the first case which arises most often in the formulas we simplify them by writing  $|a|$  for  $|a|_{\mathbb{Z}_2}$  and when we speak about a grading we mean a  $\mathbb{Z}_2$ -grading.

An alternative way to define a  $\mathbb{Z}_2$ -grading on a real or complex  $*$ -algebra is by means of an  $*$ -automorphism  $\gamma$  of order 2. Then  $A_+$ , the even elements, are those which satisfy  $\gamma(a) = a$  and  $A_-$ , the odd ones, are those which satisfy  $\gamma(a) = -a$ .

An *odd self-inverse* (OSI) of a graded algebra  $(A, \gamma)$  is an odd element of  $A$  which is its own inverse, and we denote by  $\mathfrak{F}(A, \gamma)$  the set of OSIs of  $A$

$$\mathfrak{F}(A, \gamma) = \{x \in A : -\gamma(x) = x = x^{-1}\}.$$

If  $A$  is a normed algebra we say that two OSIs of  $A$  are *osi-homotopic* if they are homotopic in  $\mathfrak{F}(A, \gamma)$ . Not all graded algebras contain an OSI and we say that the grading is *balanced* if  $A$  contains at least one. This requires that  $A$  is unital, and if  $A$  is not then we will have to add a unit. An *odd self-adjoint unitary* (OSU) of a graded  $*$ -algebra  $(A, \gamma)$  is a self-adjoint OSI, and we denote by  $\mathfrak{S}(A, \gamma)$  the set of OSUs of  $A$

$$\mathfrak{S}(A, \gamma) = \{x \in A : -\gamma(x) = x = x^* = x^{-1}\}.$$

If  $A$  is equipped with a norm then two OSUs of  $A$  are *osu-homotopic* if they are homotopic in  $\mathfrak{S}(A, \gamma)$ . For instance, any two anticommuting OSUs  $x, y$  of a normed  $*$ -algebra are *osu-homotopic*, a homotopy being given by  $c_t x + s_t y$ ,  $t \in [0, 1]$ , where  $c_t = \cos(\frac{\pi t}{2})$  and  $s_t = \sin(\frac{\pi t}{2})$ .

Examples of graded  $C^*$ -algebras are the complex Clifford algebras  $\mathcal{Cl}_k$ .  $\mathcal{Cl}_k$  is the  $C^*$ -algebra generated by  $k$  pairwise anticommuting OSUs  $\rho_1, \dots, \rho_k$ . We denote the grading of Clifford algebras always by *st*. Concretely,  $(\mathcal{Cl}_1, \text{st})$  is isomorphic (as graded algebra) to  $\mathbb{C} \oplus \mathbb{C}$  with grading given by exchange of the summands  $\text{st}(a, b) = (b, a)$  and  $(\mathcal{Cl}_2, \text{st})$  is isomorphic to  $M_2(\mathbb{C})$  with grading given by declaring diagonal matrices even and off-diagonal ones odd. As usual we will denote the generators also by  $\rho_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $\rho_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ . Then  $\sigma_z = -i\sigma_x\sigma_y$  is, of course, even.

The standard extension of a grading  $\gamma$  on an algebra  $A$  to the algebra of matrices  $M_m(A)$  is entrywise, we denote this extension by  $\gamma_m$ . If  $m = 2$  there is another grading which plays an important role, namely  $\gamma_{ev}$

$$\gamma_{ev} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \gamma(a) & -\gamma(b) \\ -\gamma(c) & \gamma(d) \end{pmatrix}.$$

A real structure on a complex  $C^*$ -algebra is an anti-linear  $*$ -automorphism  $\mathfrak{r}$  of order 2. If the algebra is graded then we require tacitly that the grading and the real structure commute  $\mathfrak{r} \circ \gamma = \gamma \circ \mathfrak{r}$ . We also call the data  $(\gamma, \mathfrak{r})$  a *graded real structure* on the algebra  $A$ . A  $C^{*,r}$ -algebra  $(A, \mathfrak{r})$  is a complex  $C^*$ -algebra equipped with a real structure. The subalgebra  $A^\mathfrak{r}$  of  $\mathfrak{r}$ -invariant elements is a real  $C^*$ -algebra and referred to as the *real subalgebra* of  $(A, \mathfrak{r})$ . Any real  $C^*$ -algebra arises as a sub-algebra of a complex  $C^{*,r}$ -algebra in such a way.  $C^{*,r}$ -algebras are elsewhere also called *Real  $C^*$ -algebras* (with capital R).

Examples of  $C^{*,r}$ -algebras are  $(\mathcal{Cl}_{r+s}, \mathfrak{l}_{r,s})$  where  $\mathfrak{l}_{r,s}$  is the real structure defined by  $\mathfrak{l}_{r,s}(\rho_i) = \rho_i$  for  $r$  generators, and  $\mathfrak{l}_{r,s}(\rho_j) = -\rho_j$  for the  $s$  other generators. The real subalgebra is thus the algebra generated by the  $r$  OSUs  $\rho_i$  and the  $s$  odd anti-selfadjoint unitaries  $i\rho_j$ . The latter square to  $-1$ . This real sub-algebra which we denote<sup>1</sup>  $\mathcal{Cl}_{r,s}$  is a real Clifford algebra.

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<sup>1</sup>the notation is not uniform, other authors use  $\mathcal{Cl}_{s,r}$  for this algebra

Define  $\mu(k)$  to be the greatest integer smaller or equal to  $\frac{k}{2}$ . Note that  $(-1)^{\mu(k)}$  is the sign of the permutation  $1 \cdots k \mapsto k \cdots 1$ . Define  $\Gamma_k \in \widehat{\mathcal{C}l}_k$  by

$$\Gamma_k = i^{-\mu(k)} \rho_1 \cdots \rho_k$$

where  $\rho_i$  are the generators. Then  $\Gamma_k^* = \Gamma_k$ .  $\Gamma_k$  depends on the choice of order of the generators, although only up to a minus sign, and we choose, for  $k = 2$ ,  $\rho_1 = \sigma_x$  and  $\rho_2 = \sigma_y$  so that  $\Gamma_2 = \sigma_z$ . If the context is clear we also simply write  $\Gamma$  for  $\Gamma_k$ .

We will consider graded tensor products of graded Banach or  $C^*$ -algebras with graded finite dimensional algebras where the grading may be a  $\mathbb{Z}_2$ -grading, in which case we denote the tensor product as  $A \hat{\otimes} B$ , or the  $\mathbb{Z}$ -grading, in which case we denote it as  $A \wedge B$ . The grading on the graded tensor product is the product grading. By definition,  $(1 \hat{\otimes} b)(a \hat{\otimes} 1) = (-1)^{|a||b|} a \hat{\otimes} b$ ,  $(a \hat{\otimes} b)^* = (-1)^{|a||b|} a^* \hat{\otimes} b^*$  and  $(1 \wedge b)(a \wedge 1) = (-1)^{|a||b|} a \wedge b$ ,  $(a \wedge b)^* = (-1)^{|a||b|} a^* \wedge b^*$ . In the case that one of the algebras is trivially  $\mathbb{Z}_2$ -graded, the graded tensor product  $\hat{\otimes}$  coincides with the ungraded one and for clarity we will simply write  $\otimes$  instead of  $\hat{\otimes}$  in that case.

The following simple result will be important.

**Lemma 2.1** ([15]). *Let  $(A, \gamma)$  be a complex balanced graded algebra and  $e$  an OSI in  $A$ . Then  $\psi_e : (A \hat{\otimes} \mathcal{C}l_2, \gamma \otimes \text{st}) \rightarrow (M_2(A), \gamma_2)$*

$$\psi_e(x \hat{\otimes} 1) = \begin{pmatrix} x & 0 \\ 0 & (-1)^{|x|} e x e^{-1} \end{pmatrix}, \quad \psi_e(1 \hat{\otimes} \sigma_x) = \begin{pmatrix} 0 & e^{-1} \\ e & 0 \end{pmatrix}, \quad \psi_e(1 \hat{\otimes} i \sigma_y) = \begin{pmatrix} 0 & e^{-1} \\ -e & 0 \end{pmatrix}$$

*is an isomorphism of graded algebras. If  $(A, \gamma)$  is a  $*$ -algebra and  $e$  an OSU then  $\psi_e$  is a  $*$ -isomorphism. If  $(A, \gamma)$  carries a real structure  $\tau$  (which commutes with  $\gamma$ ) and  $e$  is  $\tau$ -invariant then*

$$\begin{aligned} \psi_e \circ (\tau \otimes \mathfrak{l}_{1,1}) &= \tau_2 \circ \psi_e \\ \psi_e \circ (\tau \otimes \mathfrak{l}_{0,2}) &= \text{Ad}_Y \circ \tau_2 \circ \gamma_{ev} \circ \psi_e, \quad Y = \begin{pmatrix} 0 & e^{-1} \\ -e & 0 \end{pmatrix} \\ \psi_e \circ (\tau \otimes \mathfrak{l}_{2,0}) &= \text{Ad}_X \circ \tau_2 \circ \gamma_{ev} \circ \psi_e, \quad X = \begin{pmatrix} 0 & e^{-1} \\ e & 0 \end{pmatrix} \end{aligned}$$

*Proof.* This is a direct calculation. □

### 3. VAN DAELE $K$ -THEORY

In [26] we developed the point of view that an insulator corresponds to a self-adjoint invertible element in the observable  $C^*$ -algebra  $A$  and that the symmetry type of an insulator is described by a grading, or a real structure, or a graded real structure on  $A$ . Any self-adjoint invertible element of a  $C^*$ -algebra is homotopic (via a continuous path of invertible self-adjoint elements) to a self-adjoint unitary. By taking into account the grading (which has to be put in by tensoring with  $\mathcal{C}l_1$  if the insulator does not have chiral symmetry) the topological phases for a

given symmetry type and observable algebra  $A$  are classified by homotopy classes of OSUs in  $A$  (or  $A \otimes \mathbb{C}l_1$ ), and these define the van Daele  $K$ -group of the graded algebra  $C^*$ - or  $C^{*,r}$ -algebra  $A$ .

We recall the basic definitions of van Daele  $K$ -theory for graded Banach algebras referring the reader for details to the original articles by van Daele [15, 16]. Let  $(A, \gamma)$  be a balanced graded normed algebra. We choose an OSI  $e \in A$  and define the semigroup

$$V_e(A, \gamma) := \bigsqcup_{n \in \mathbb{N}} \mathfrak{F}(M_n(A), \gamma_n) / \sim_h^e$$

where  $M_m(A) \ni x \sim_h^e y \in M_l(A)$  if, for some  $n \in \mathbb{N}$ ,  $x \oplus e_{n-m}$  is osi-homotopic to  $y \oplus e_{n-l}$ . Here  $e_n = e \oplus \cdots \oplus e$  ( $n$  summands), and semigroup addition is given by the direct sum  $[x] + [y] = [x \oplus y]$ . The van Daele  $K$ -group  $DK_e(A, \gamma)$  is the Grothendieck group of  $V_e(A, \gamma)$ . We refer to the choice of  $e$  as a choice of base-point.  $DK_e(A, \gamma)$  depends on  $e$  only up to isomorphism. If  $e$  is osi-homotopic to its negative  $-e$  then  $V_e(A, \gamma)$  is already a group with neutral element  $[e]$  and inverse map  $[x] \mapsto [-exe]$ . In that case  $DK_e(A, \gamma) = V_e(A, \gamma)$ . It follows from Lemma 2.1 that  $DK_e(A, \gamma) \cong DK_{1 \hat{\otimes} \sigma_x}(A \hat{\otimes} \mathbb{C}l_{1,1}, \gamma \otimes \text{st})$  and so the r.h.s. may be taken as the definition of the van Daele  $K$ -group if  $A$  is unital but the grading on  $A$  not balanced or even trivial. For non-unital algebras the  $K$ -group is defined as usual as the kernel of the map induced by the epimorphism  $A^+ \rightarrow \mathbb{C}$  (or  $\mathbb{R}$ ) from the minimal unitization  $A^+$  to the one-dimensional algebra.

If  $A$  is a  $C^*$ -algebra then any element of  $\mathfrak{F}(A, \gamma)$  is osi-homotopic to an element of  $\mathfrak{S}(A, \gamma)$  and moreover, any two osi-homotopic OSUs are osu-homotopic [15]. For  $C^*$ -algebras one may therefore replace in the definition of  $V_e(A, \gamma)$  the sets  $\mathfrak{F}(M_n(A), \gamma_n)$  by  $\mathfrak{S}(M_n(A), \gamma_n)$  and osi-homotopy by osu-homotopy. The grading on a  $C^*$ -algebra is hence balanced if the algebra contains an OSU. This is what was done in [26] to describe topological insulators, but here the greater flexibility of working with OSIs will be convenient.

If  $(A, \gamma, \mathfrak{r})$  is a graded  $C^{*,r}$ -algebra then its van Daele  $K$ -group is by definition the van Daele  $K$ -group of its real subalgebra  $(A^{\mathfrak{r}}, \gamma)$ .

If the grading is clear then we simply write  $DK_e(A)$  for  $DK_e(A, \gamma)$ , or even only  $DK(A)$ , in case the dependence on  $e$  is not important.

The  $K$ -groups in other degrees are defined as

$$K_{1-i}(A, \gamma) := DK_{e \hat{\otimes} 1}(A \hat{\otimes} \mathbb{C}l_i, \gamma \otimes \text{st})$$

in the complex case, and

$$K_{1-r+s}(A, \gamma) := DK_{e \hat{\otimes} 1}(A \hat{\otimes} \mathbb{C}l_{r,s}, \gamma \otimes \text{st})$$

in the real case.

If  $A$  is trivially graded then the van Daele  $K$ -group  $K_i(A, \text{id})$  is isomorphic to the standard  $K$ -group of  $A$  which we denote, for complex  $A$  also by  $KU_i(A)$ , and for a real  $A$  by  $KO_i(A)$ . We recall also that if  $(A, \gamma)$  is inner graded, that is,  $\gamma$  is

given by conjugation with a self-adjoint unitary from  $A$ , then  $K_i(A, \gamma) \cong K_i(A, \text{id})$  [26]. Thus, under the assumption that chiral symmetry, when it appears, is inner, only standard real or complex  $K$ -theory is needed to describe topological phases. We find it however extremely useful in the following to use van Daele's formulation of  $K$ -theory also in the trivially graded case.

**3.1. Suspensions.** A grading  $\gamma$  on a Banach algebra  $A$  can be extended pointwise to the cone  $CA := \{f : [0, 1] \rightarrow A : f(0) = 0\}$  and the suspension  $SA = \{f : [0, 1] \rightarrow A : f(0) = f(1) = 0\}$  of  $A$ , and this is the extension we will always use. We thus have an exact sequence of graded algebras

$$(1) \quad 0 \rightarrow SA \rightarrow CA \rightarrow A \rightarrow 0.$$

There are two natural ways to extend a real structure  $\mathfrak{r}$  from  $A$  to  $SA$ : Pointwise, i.e.  $\tilde{\mathfrak{r}}(f)(t) = \mathfrak{r}(f(t))$ , or flip-pointwise  $\hat{\mathfrak{r}}(f)(t) = \mathfrak{r}(f(1-t))$ . However, only the pointwise extension also extends to the cone  $CA$  and thus turns (1) into an exact sequence of  $C^{*,r}$ -algebras. The standard suspension of the  $C^{*,r}$ -algebra  $(A, \mathfrak{r})$  is the  $C^{*,r}$ -algebra  $(SA, \tilde{\mathfrak{r}})$  and occurs as the ideal in (1). Its real subalgebra  $(SA)^{\tilde{\mathfrak{r}}} = SA^{\mathfrak{r}}$  is the ideal in the exact sequence of real  $C^*$ -algebras  $0 \rightarrow SA^{\mathfrak{r}} \rightarrow CA^{\mathfrak{r}} \rightarrow A^{\mathfrak{r}} \rightarrow 0$ . The boundary map in van Daele's formulation of  $K$ -theory, which we denote  $\beta$ , reads as follows. Denote  $c_t = \cos(\frac{\pi t}{2})$  and  $s_t = \sin(\frac{\pi t}{2})$ . For an OSU  $x \in M_m(A)$  define  $\nu(x) : [0, 1] \rightarrow M_m(A) \hat{\otimes} Cl_{1,0}$  to be the function

$$(2) \quad t \mapsto \nu_t(x) = c_t \hat{\otimes} 1 + s_t x \hat{\otimes} \rho$$

**Theorem 3.1** ([16]). *Let  $(A, \gamma)$  be a balanced graded Banach algebra and  $e$  a basepoint. The map  $\beta : DK_e(A) \rightarrow DK_{1 \hat{\otimes} \rho}(SA \hat{\otimes} Cl_{1,0})$ ,*

$$\beta[x] = [\nu(x)\nu^{-1}(e_m)(1 \hat{\otimes} \rho)_m \nu(e_m)\nu^{-1}(x)]$$

*is an isomorphism.*

For trivially graded  $C^*$ -algebras  $A$  this reduces to the standard isomorphisms [23]

$$\beta : KU_i(A) \rightarrow KU_{i-1}(SA), \quad \beta : KO_i(A^{\mathfrak{r}}) \rightarrow KO_{i-1}(SA^{\mathfrak{r}}).$$

For later use we recall the details in the case  $i = 0$  where the relevant algebra is  $(A \otimes Cl_1, \text{id} \otimes \text{st})$ , in the complex, and  $(A^{\mathfrak{r}} \otimes Cl_{1,0}, \text{id} \otimes \text{st})$  in the real case. We choose the basepoint to be  $e = 1 \otimes \rho$  and so to be an odd element which commutes with any other odd element of  $A \otimes Cl_1$ . Hence any OSU  $x$  of  $M_m(A) \hat{\otimes} Cl_{1,0}$  is of the form  $x = h e_m$  where  $h$  is a self-adjoint unitary in  $M_m(A)$  and thus has the form  $h = 2p - 1$  for a projection  $p$ . It follows that

$$\begin{aligned} \nu_t(x)\nu_t^{-1}(e_m)(1 \hat{\otimes} \rho)_m \nu_t(e_m)\nu_t^{-1}(x) &= \cos(\pi t(h-1))(1 \hat{\otimes} \rho) + \sin(\pi t(h-1))(e \hat{\otimes} 1) \\ &= \cos(-2\pi t p^\perp)(1 \hat{\otimes} \rho) + \sin(-2\pi t p^\perp)(e \hat{\otimes} 1) \end{aligned}$$



As  $1 \hat{\otimes} \rho$  anticommutes with  $e \hat{\otimes} 1$  the loop  $t \mapsto \cos(-2\pi t p^\perp)(1 \hat{\otimes} \rho) + \sin(-2\pi t p^\perp)(e \hat{\otimes} 1)$  is osu-homotopic to

$$(3) \quad Y(t) := \cos(-2\pi t p^\perp)(e \hat{\otimes} 1) - \sin(-2\pi t p^\perp)(1 \hat{\otimes} \rho)$$

(see also [23]). Furthermore, by identifying  $e \hat{\otimes} 1$  and  $1 \hat{\otimes} \rho$  with  $\sigma_x, \sigma_y \in M_2(\mathbb{C})$  we obtain the expression

$$(4) \quad \beta([x]) = [Y], \quad Y(t) = \begin{pmatrix} 0 & e^{-2\pi i t p^\perp} \\ e^{2\pi i t p^\perp} & 0 \end{pmatrix}.$$

The upper right corner corresponds to the usual formula for the Bott map applied to  $[p^\perp]$ .

#### 4. CYCLIC COHOMOLOGY FOR GRADED $C^{*,r}$ -ALGEBRAS

**4.1. Definition for graded algebras.** The definition of cyclic cohomology for a graded algebra  $(\mathcal{A}, \gamma)$  can be found in [25]. It is a straightforward generalisation of the ungraded case and follows naturally if one takes into account the sign rule for the transposition of two elements in the graded tensor product: the sign of the transposition  $a \hat{\otimes} b \mapsto b \hat{\otimes} a$  is  $(-1)^{1+|a||b|}$ . The cyclic permutation  $\lambda_{(n)} : \mathcal{A}^{\hat{\otimes} n} \rightarrow \mathcal{A}^{\hat{\otimes} n}$  must then be used with an extra sign coming from the transpositions:

$$\lambda_{(n)}(a_0 \hat{\otimes} \cdots \hat{\otimes} a_n) = (-1)^{n+|a_n|(\sum_{i=0}^{n-1} |a_i|)} (a_n \hat{\otimes} a_0 \hat{\otimes} \cdots \hat{\otimes} a_{n-1}).$$

The Hochschild complex  $(C^n, b)$  of  $\mathcal{A}$  consists of the modules  $C^n$  of linear maps  $\xi : \mathcal{A}^{\hat{\otimes} n+1} \rightarrow \mathbb{C}$  together with the boundary maps

$$b = \sum_{i=0}^{n+1} (-1)^i \delta_i$$

where  $\delta_i : C^n \rightarrow C^{n+1}$  is given by

$$\delta_i f(a_0 \hat{\otimes} \cdots \hat{\otimes} a_{n+1}) = \xi(a_0 \hat{\otimes} \cdots \hat{\otimes} a_i a_{i+1} \cdots \hat{\otimes} a_{n+1}), \quad \delta_{n+1} = (-1)^{n+1} \lambda \delta_0$$

with  $\lambda : C^n \rightarrow C^n$ ,  $\lambda \xi = \xi \circ \lambda_{(n)}$ . The cyclic cohomology is the cohomology of the subcomplex  $(C_\lambda^n, b)$  of the Hochschild complex of cyclic cochains, i.e.  $\xi \in C^n$  which satisfy  $\lambda \xi = \xi$ . A cyclic cochain of  $C^n$  which satisfies  $b\xi = 0$  is called a cyclic cocycle of dimension  $n$ .

**4.2. Cycles and their characters.** The cyclic cohomology of algebras can be described by means of characters of cycles. Recall from [14] that an  $n$ -dimensional chain over an ungraded algebra  $\mathcal{A}$  is a triple  $(\Omega, d, f)$ , a  $\mathbb{Z}$ -graded algebra  $\Omega = \bigoplus_{k \in \mathbb{Z}} \Omega_k$  with differential  $d : \Omega_k \rightarrow \Omega_{k+1}$ , an algebra homomorphism  $\varphi : \mathcal{A} \rightarrow \Omega_0$ , and a graded trace  $f : \Omega_n \rightarrow \mathbb{C}$  (or  $\mathbb{R}$ ). In the interest of clarity we denote this trace also by  $\int_{\mathcal{A}}$ . A chain is called a cycle if the trace is closed, that is, it vanishes on the image of  $d$ . We adapt this definition to graded algebras by

requiring in addition that  $\Omega$  is  $\mathbb{Z} \times \mathbb{Z}_2$ -graded, the differential  $d$  has degree  $(1, 0)$ ,  $\varphi$  preserves the  $\mathbb{Z}_2$ -grading and  $\int$  is graded cyclic in the sense that

$$(5) \quad \int \omega \omega' = (-1)^{|\omega|_{\mathbb{Z}}|\omega'|_{\mathbb{Z}}+|\omega||\omega'|} \int \omega' \omega$$

where  $|\omega|_{\mathbb{Z}}$  is the  $\mathbb{Z}$ -degree and  $|\omega|$  is the  $\mathbb{Z}_2$ -degree of  $\omega$ . With these additional requirements  $(\Omega, d, \int)$  is called a chain, or cycle resp., over the graded algebra  $\mathcal{A}$ . In the applications below the graded trace  $\int$  is non-trivial only on elements of a fixed  $\mathbb{Z}_2$ -degree  $\nu$ . We call this  $\nu$  the parity of the chain.

The character  $\xi$  of the chain  $(\Omega, d, \int)$  is defined to be

$$\xi(a_0 \hat{\otimes} \cdots \hat{\otimes} a_n) := \int \varphi(a_0) d\varphi(a_1) \cdots d\varphi(a_n).$$

As in the ungraded case [14][Chap. III.1.α, Prop. 4] one shows that the character of a cycle is a cyclic cocycle of dimension  $n$  and that conversely, any  $n$ -dimensional cyclic cocycle arises from an  $n$ -dimensional cycle in the above way.

The cyclic cohomology of  $C^*$ -algebras is too poor for the applications in solid state physics which we have in mind. One way to overcome this problem is to consider characters of cycles whose differential  $d$  and graded trace  $\int$  are perhaps only densely defined but which still have a well-defined pairing with the  $K$ -groups of  $A$ . Here the holomorphic functional calculus will play a role.

Recall that a unital normed complex algebra  $\mathcal{A}$  is closed under holomorphic functional calculus if for any  $x \in \mathcal{A}$  and any function  $f$  which is holomorphic in a neighbourhood of the spectrum of  $x$  (the set of  $\lambda \in \mathbb{C}$  such that  $x - \lambda 1$  is not invertible in the completion of  $\mathcal{A}$ ) we have  $f(x) \in \mathcal{A}$ . If  $\mathcal{A}$  is not unital then we say that it is closed under holomorphic functional calculus if this is the case for its unitization  $\mathcal{A}^+$ . Such algebras are called local Banach algebras [4]. This has the following consequences.

**Lemma 4.1.** *Let  $(\mathcal{A}, \gamma)$  be a unital normed complex graded algebra whose even part is closed under holomorphic functional calculus. If  $x, y \in \mathfrak{F}(\mathcal{A}, \gamma)$  satisfy  $\|x - y\| < 2\|x\|^{-1}$  then there exists a smooth path  $[0, 1] \ni t \mapsto x(t) \in \mathfrak{F}(\mathcal{A}, \gamma)$  with  $x(0) = x$  and  $x(1) = y$ . Moreover, if  $\mathcal{A}$  is a dense subalgebra of a  $C^*$ -algebra and  $x$  and  $y$  self-adjoint then  $x(t)$  can be chosen in  $\mathfrak{S}(\mathcal{A}, \gamma)$  and  $\|x(t) - x\| \leq C(\|x - y\|)$  for all  $t \in [0, 1]$  where  $C : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a continuous function with  $C(0) = 0$ .*

*Proof.* We follow [15][Prop. 3.2] to see that  $v = \frac{1}{2}(1 + yx)$  satisfies  $\|1 - v\| < 1$ , and hence is invertible in the completion of  $\mathcal{A}$ , and that  $v xv^{-1} = y$ . Furthermore  $\|1 - v\| < 1$  implies that the spectrum of  $v$  lies in the domain of the analytic extension to  $\{z \in \mathbb{C} : \Re z > 0\}$  of the natural logarithm and so  $v^t = \exp(t \log(\frac{1}{2}(1 + yx)))$  is an element of  $\mathcal{A}$  for all  $t$ . Hence  $x(t) = v^t x v^{-t}$  is a path in  $\mathcal{A}$  linking  $x = x(0)$  to  $y = x(1)$ . Clearly the path is arbitrarily many times differentiable in  $\mathcal{A}$ .

We now assume that  $x$  and  $y$  are self-adjoint. This implies that they have norm 1 and that  $v$  is a normal element. Since  $\mathcal{A}$  is closed under polar decomposition

[4]  $\hat{x}(t) = x(t)(x(t)^*x(t))^{-\frac{1}{2}}$  belongs to  $\mathcal{A}$  and so defines a smooth path in  $\mathfrak{S}(\mathcal{A}, \gamma)$  joining  $x$  with  $y$ . Let  $b = \|x - y\|$ . Then  $\|1 - v\| \leq \frac{b}{2}$  and hence  $\|v^{-1}\| \leq \frac{1}{1 - \frac{b}{2}}$ . Hence  $\|v^t x v^{-t} - x\| \leq 2\|v^t - 1\|\|v^{-t}\| \leq \frac{b}{1 - \frac{b}{2}}$  for all  $t \in [0, 1]$ . Using  $\|x^*(t)x(t) - 1\| = \|(x^*(t) - x(t))x(t)\| \leq \|(x^*(t) - x(t))\|\|x(t)\|$  we see that  $\|(x^*(t)x(t))^{\frac{1}{2}} - 1\|$  tends to 0 if  $b$  tends to 0. Thus  $\|\hat{x}(t) - x\|$  tends to 0 if  $b$  tends to 0.  $\square$

**Corollary 4.2.** *Let  $(A, \gamma)$  be a balanced complex graded  $C^*$ -algebra and  $\mathcal{A}$  a dense  $*$ -subalgebra whose even part is closed under holomorphic functional calculus. Assume furthermore that within distance  $\frac{1}{2}$  of any OSU of  $A$  there is an OSU of  $\mathcal{A}$ . If two OSUs  $x, y$  of  $\mathcal{A}$  are homotopic in  $\mathfrak{S}(A, \gamma)$  then there exists a continuous, piecewise continuously differentiable path  $x(t) \in \mathfrak{S}(\mathcal{A}, \gamma)$  with  $x(0) = x$  and  $x(1) = y$ .*

*Proof.* If two OSUs  $x, y$  of  $\mathcal{A}$  are homotopic in  $\mathfrak{S}(A, \gamma)$  then there is a finite collection  $(x_i)_{i=0, \dots, N}$  of OSUs in  $A$  such that  $x = x_0, y = x_N$  and  $\|x_i - x_{i+1}\| < 1$ . By the assumption we may move the  $x_i$  a bit so that they are OSUs of  $\mathcal{A}$  and  $\|x_i - x_{i+1}\| < 2$ . Now the result follows from the last lemma.  $\square$

**Definition 4.3.** *An  $n$ -dimensional chain  $(\Omega, d, \int)$  over a graded  $C^*$ -algebra  $A$  with domain algebra  $\mathcal{A}$  is a  $\mathbb{Z} \times \mathbb{Z}_2$ -graded algebra  $\Omega$  with a graded  $*$ -homomorphism  $\varphi : A \rightarrow \Omega_0$ , a densely defined differential  $d$  of degree  $(1, 0)$ , a densely defined linear functional  $\int : \Omega_n \rightarrow \mathbb{C}$  and a dense  $*$ -subalgebra  $\mathcal{A}$  of  $A$ , referred to as the domain algebra of the chain, such that*

- (C1)  $\varphi(\mathcal{A})$  lies in the domain of  $d$ .
- (C2)  $\varphi(\mathcal{A})(d\varphi(\mathcal{A}))^n$  lies in the domain of  $\int$  and  $\int$  is graded cyclic in the sense of (5) on  $\varphi(\mathcal{A})(d\varphi(\mathcal{A}))^n$ . If the chain is a cycle we require  $\int$  to vanish on  $(d\varphi(\mathcal{A}))^n$ .
- (C3) The even part of  $\mathcal{A} \hat{\otimes} \text{Cl}_k$  is closed under functional holomorphic calculus, for all  $k \geq 0$ .
- (C4) In any neighbourhood of an OSU of  $M_n(A \hat{\otimes} \text{Cl}_k)$  there is an OSU of  $M_n(\mathcal{A} \hat{\otimes} \text{Cl}_k)$ , for all  $n \geq 1$  and  $k \geq 0$ .

If  $A$  is balanced we require in addition that

- (C5)  $\mathcal{A}$  contains an OSU  $e$  such that  $de = 0$ .

The character of such a cycle defines a cyclic cocycle over the domain algebra  $\mathcal{A}$ . It is similar to what Connes calls a higher trace in [14] for ungraded algebras, but instead of requiring the norm estimates of [14][Chap. III.6.α, Def. 11] (see also [28]) we require directly closedness under functional holomorphic calculus.

Note that if  $A$  is balanced then by Lemma 2.1 the conditions (C3) and (C4) will hold for all positive  $k$  if they hold for  $k = 0, 1$ . If  $A$  is unital then  $A \hat{\otimes} \text{Cl}_1$  is balanced and hence conditions (C3) and (C4) will hold for all positive  $k$  if they hold for  $k = 0, 1, 2$ . We will see below that for trivially graded  $A$  condition (C4)

follows from condition (C3). It would be interesting to know whether this is also the case for general graded algebras.

We simplify our notation by surpressing the homomorphism  $\varphi$ , which should not create confusion, as for all our cycles below  $\varphi$  is injective.

**4.3. Extension of cycles to  $M_m(A) \hat{\otimes} \mathbb{C}l_k$ .** Kassel establishes a Künneth formula for the cyclic cohomology of graded algebras. It implies that  $HC^n(A \hat{\otimes} \mathbb{C}l_k)$  contains  $\bigoplus_{i+j=n} HC^i(A) \otimes HC^j(\mathbb{C}l_k)$ . He shows moreover that  $HC^0(\mathbb{C}l_k) \cong \mathbb{C}$  and that the (up to normalisation unique) linear map  $\kappa_k : \mathbb{C}l_k \rightarrow \mathbb{C}$  which is non-zero only on the product of all generators  $\rho_1 \cdots \rho_k$  provides a generator for  $HC^0(\mathbb{C}l_k)$ . All further elements in  $HC^{ev}(\mathbb{C}l_k)$  are images of  $\kappa$  under Connes'  $S$ -operator and  $HC^{odd}(\mathbb{C}l_k)$  vanishes. We use the cup product with the above generator on the level of chains to extend characters from  $A$  to  $A \hat{\otimes} \mathbb{C}l_k$ . We normalise the generator as follows. Let

$$\kappa = \sqrt{2} e^{\frac{\pi i}{4}},$$

a square root of  $2i$ , and

$$\kappa_k(\rho_1 \cdots \rho_k) = \kappa^k, \quad \kappa_k(\rho_{i_1} \cdots \rho_{i_j}) = 0 \text{ if } j < k.$$

Note that  $\kappa_k$  is a graded trace on  $\mathbb{C}l_k$ , i.e.  $\kappa(c_1 c_2) = (-1)^{|c_1||c_2|} \kappa(c_2 c_1)$ . Thus  $(\mathbb{C}l_k, 0, \kappa_k)$  is a cycle over  $\mathbb{C}$ . Furthermore,  $(M_m(\mathbb{C}), 0, \text{Tr}_m)$  is a cycle over  $M_m(\mathbb{C})$  where  $\text{Tr}_m$  is the standard trace on  $m \times m$  matrices. We extend chains over  $A$  to chains over  $M_m(A) \hat{\otimes} \mathbb{C}l_k$  by taking their product with the above cycles.

**Definition 4.4.** *Let  $\mathcal{A}$  be a graded algebra and  $(\Omega, d, f)$  a chain over  $\mathcal{A}$ . The extension of this chain to  $M_m(\mathcal{A}) \hat{\otimes} \mathbb{C}l_k$  is the chain  $(M_m(\Omega) \hat{\otimes} \mathbb{C}l_k, d \otimes \text{id}, \int \circ \text{Tr}_m \circ \kappa_k)$ . Here  $d$  is extended to  $M_m(\Omega)$  entrywise and  $\kappa_k(\omega \hat{\otimes} c) = \kappa_k(c)\omega$  for  $\omega \in M_m(\Omega)$  and  $c \in \mathbb{C}l_k$ .*

We denote the character of the extension by  $\xi \# \text{Tr}_m \# \kappa_k$  where  $\xi$  is the character of  $(\Omega, d, f)$ , or simply also by  $\xi \# \kappa_k$ , or even  $\xi$ , as its entries make clear what the values for  $m$  and  $k$  are. We have  $\kappa_k \# \kappa_l = \kappa_{k+l}$ .

Chains over a graded  $C^*$ -algebra  $A$  with domain algebra  $\mathcal{A}$  are extended similarly, the domain algebra for  $A \hat{\otimes} \mathbb{C}l_k$  being  $\mathcal{A} \hat{\otimes} \mathbb{C}l_k$ .

**4.4. Connes pairing with van Daele  $K$ -groups.** We start by considering cycles on a balanced graded normed algebra  $(\mathcal{A}, \gamma)$ . In later applications this algebra will be the domain algebra of a graded  $C^*$ -algebra.

**Lemma 4.5.** *Let  $(\Omega, d, f)$  be an  $n$ -dimensional cycle over a graded normed algebra  $(\mathcal{A}, \gamma)$ . Suppose that there exists an element  $e \in \mathfrak{F}(\mathcal{A}, \gamma)$  which satisfies  $de = 0$ . Let  $t \mapsto x(t)$  be a continuously differentiable path in  $\mathfrak{F}(\mathcal{A}, \gamma)$ . Then*

$$\int (x(t) - e)(dx(t))^n$$

*does not depend on  $t$ .*

*Proof.* Given any derivation  $\delta$  the identity  $x^2 = 1$  implies that  $(\delta x)^i x = (-1)^i x (\delta x)^i$  for all  $i \in \mathbb{N}$ . Let  $z = x - e$ . Clearly  $\int (x - e)(dx)^n = \int z(dx)^n$ . Using  $\dot{x} = -x\dot{x}$  and the graded cyclicity of  $\int$  we get

$$\int \dot{z}(dx)^n = - \int x\dot{x}(dx)^n = -(-1)^{|x||\dot{x}(dx)^n|} \int \dot{x}(dx)^n x = - \int \dot{z}(dx)^n$$

as  $|x|_{\mathbb{Z}} = 0$ . Hence  $\int \dot{z}(dx)^n$  must vanish. Furthermore,  $zdz + (dz)z$  is a total derivative. Hence

$$z(dz)^\nu dz = (-1)^\nu (dz)^\nu z dz + R_1 = (-1)^{\nu+1} (dz)^{\nu+1} \dot{z} + R_1 + R_2$$

where  $R_1$  and  $R_2$  are total derivatives and hence vanish under the trace  $\int$ . Thus

$$\int z(dz)^\nu dz (dz)^{n-\nu-1} = \epsilon \int (dz)^{\nu+1} \dot{z} (dz)^{n-\nu-1} = \epsilon' \int \dot{z} (dz)^n$$

for certain  $\epsilon, \epsilon' \in \{\pm 1\}$ . Hence the derivative of  $\int (x(t) - e)(dx(t))^n$  w.r.t.  $t$  vanishes.  $\square$

#### 4.4.1. Pairing for balanced graded $C^*$ -algebras.

**Definition 4.6.** Let  $(A, \gamma)$  be a unital balanced graded  $C^*$ -algebra and  $(\Omega, d, f)$  an  $n$ -dimensional cycle over  $A$  with character  $\xi$  and domain algebra  $\mathcal{A}$ . Let  $e \in \mathfrak{S}(\mathcal{A}, \gamma)$  satisfy  $de = 0$ . The pairing of  $\xi$  with an element  $[x] \in DK_e(A)$  is defined to be

$$\langle \xi, [x] \rangle = \int \text{Tr}_m(x - e_m)(dx)^n$$

where we take a representative  $x$  for  $[x]$  which lies in  $\mathfrak{S}(M_m(\mathcal{A}), \gamma_m)$ .

Cor. 4.2 and Lemma 4.5 guarantee that the pairing is well-defined. Indeed, the assumptions on  $\mathcal{A}$  assure that  $(\Omega, d, f)$  restricts to a bounded cycle over  $\mathcal{A}$ , that the hypothesis of Cor. 4.2 are satisfied and that  $[x]$  admits a representative in  $\mathfrak{S}(M_m(\mathcal{A}), \gamma_m)$ .

Since  $(x_1 \oplus x_2)(d(x_1 \oplus x_2))^n = x_1(dx_1)^n \oplus x_2(dx_2)^n$  the map  $V_e(A) \ni [x] \mapsto \langle \xi, [x] \rangle \in \mathbb{C}$  is a homomorphism of semi-groups. Since the pairing with the class of  $e$  is 0 the homomorphism induces a homomorphism of groups  $DK_e(A) \ni [x] \mapsto \langle \xi, [x] \rangle \in \mathbb{C}$ . Note that

$$\int \text{Tr}_m(x - e_m)(dx)^n = \int \text{Tr}_m x (dx)^n$$

provided  $n > 0$ . In the case  $n = 0$  the pairing is a priori only independent of the choice of basepoint  $e$  if it is  $\text{osu}$ -homotopic to its negative.

The above formulation of the pairing does not make reference to whether we work with complex or with real  $C^*$ -algebras. We find it very convenient, however, to work with  $C^{*,r}$ -algebras where real  $C^*$ -algebras appear as subalgebras of elements which are invariant under the real structure. The complex pairing defined for the

$C^{*,r}$ -algebra will then define by restriction to real homotopy classes a pairing with the subalgebra of real elements.

To define pairings with higher  $K$ -groups or for trivially graded algebras we consider tensor products with  $\mathcal{C}l_k$  and extend the cycle as in Def. 4.4. We remark that  $k$  must be strictly larger than 1 in case  $A$  is not balanced graded.

**Definition 4.7.** Let  $(A, \gamma)$  be a (possibly trivially) graded unital  $C^*$ -algebra and  $(\Omega, d, \int)$  an  $n$ -dimensional (possibly unbounded) cycle over  $A$  with character  $\xi$  and domain algebra  $\mathcal{A}$ . Let  $e \in \mathcal{A} \hat{\otimes} \mathcal{C}l_k$  be an OSU which satisfies  $de = 0$ . The pairing of  $\xi$  with an element  $[x] \in DK_e(\mathcal{A} \hat{\otimes} \mathcal{C}l_k) \cong K_{1-k}(A)$  is, by definition, the pairing of the extension  $\xi \# \kappa_k$  with  $[x]$ ,

$$\langle \xi, [x] \rangle = \int \mathrm{Tr}_m \kappa_k((x - e_m)(dx)^n)$$

for  $x \in \mathfrak{S}(M_m(\mathcal{A} \hat{\otimes} \mathcal{C}l_k), (\gamma \otimes \mathrm{st})_m)$ .

For convenience we extend this definition to invertible odd self-adjoint elements  $x$  by setting  $\langle \xi, [x] \rangle = \langle \xi, [\hat{x}] \rangle$  where  $\hat{x} = x|x|^{-1}$  is the spectral flattening of  $x$ .

If  $A$  is a balanced graded algebra then by Lemma 2.1  $(A \hat{\otimes} \mathcal{C}l_2, \gamma \otimes \mathrm{st})$  is isomorphic to  $(M_2(A), \gamma_2)$ . We therefore have *a priori* two ways to pair an element of the  $K$ -group with a character: one involves the formula with  $k$  and the other with  $k + 2$ . That these two ways yield the same answer is the following result.

**Lemma 4.8.** Let  $\mathcal{A}$  be a balanced graded algebra and  $e$  an OSI in  $\mathcal{A}$ . Let  $(\Omega, d, \int)$  be an  $n$ -dimensional cycle over  $\mathcal{A}$  and  $\omega \in \Omega_n \hat{\otimes} \mathcal{C}l_2$ . Then

$$\int \kappa_2(\omega) = \int \mathrm{Tr}_2(\psi_e(\omega))$$

where  $\psi_e : \Omega_n \hat{\otimes} \mathcal{C}l_2 \rightarrow M_2(\Omega)$  is defined as in Lemma 2.1 with  $A$  replaced by  $\Omega_n$  and  $\mathrm{Tr}_2$  is the matrix trace.

*Proof.* Expand  $\omega = \omega_0 \hat{\otimes} 1 + \omega_1 \hat{\otimes} \sigma_x + \omega_2 \hat{\otimes} \sigma_y + \omega_3 \hat{\otimes} \sigma_z$  with  $\omega_i \in \Omega_n$ . Then  $\kappa_2(\omega) = -i\kappa^2\omega_3 = 2\omega_3$ . On the other hand

$$\begin{aligned} \mathrm{Tr}_2(\psi_e(\omega)) &= \mathrm{Tr}_2 \left( \begin{array}{cc} \omega_0 + \omega_3 & (\omega_1 - i\omega_2)e \\ e((-1)^{|\omega_1|}\omega_1 + i(-1)^{|\omega_2|}\omega_2) & e((-1)^{|\omega_0|}\omega_0 - (-1)^{|\omega_3|}\omega_3)e \end{array} \right) \\ &= \omega_0 + (-1)^{|\omega_0|}e\omega_0e + \omega_3 - (-1)^{|\omega_3|}e\omega_3e \end{aligned}$$

Now  $\int ewe = (-1)^{|\omega|+1} \int \omega$ , by graded cyclicity. Hence

$$\int \mathrm{Tr}_2(\psi_e(\omega)) = 2 \int \omega_3.$$

□

**Corollary 4.9.** *Let  $\mathcal{A}$  be a balanced graded algebra and  $e$  an OSI in  $\mathcal{A} \hat{\otimes} \mathbb{C}l_k$ . Let  $\xi$  be the character of an  $n$ -dimensional cycle over  $\mathcal{A}$  and  $x \in M_m(\mathcal{A}) \hat{\otimes} \mathbb{C}l_2 \hat{\otimes} \mathbb{C}l_k$  an OSI. Then*

$$\int \mathrm{Tr}_m \kappa_{2+k}((x - \tilde{e}_m)(dx)^n) = \int \mathrm{Tr}_{m+2} \kappa_k((\psi_e(x) - e_{2m})(d\psi_e(x))^n).$$

where  $\tilde{e} \in \mathcal{A} \hat{\otimes} \mathbb{C}l_2 \hat{\otimes} \mathbb{C}l_k$  corresponds to  $e \hat{\otimes} \sigma_z \in \mathcal{A} \hat{\otimes} \mathbb{C}l_2 \hat{\otimes} \mathbb{C}l_k$  under the isomorphisms  $\mathbb{C}l_2 \hat{\otimes} \mathbb{C}l_k \rightarrow \mathbb{C}l_k \hat{\otimes} \mathbb{C}l_2$ .

*Proof.* We apply Lemma 4.8 to  $a = (x - \tilde{e}_m)(dx)^n \in \Omega_n \hat{\otimes} \mathbb{C}l_{2+k}$ . The result follows as  $\psi_e$  commutes with  $d$  and  $\psi_e(\tilde{e}) = e_2$ .  $\square$

4.4.2. *Pairing for nonunital algebras.* If  $A$  is non-unital then its van Daele  $K$ -group is a subgroup of the  $K$ -group of the unitization  $A^+$  of  $A$ . More precisely we consider the exact sequence of graded  $C^*$ -algebras

$$0 \rightarrow (A \hat{\otimes} \mathbb{C}l_2, \gamma \otimes \mathrm{st}) \rightarrow (A^+ \hat{\otimes} \mathbb{C}l_2, \gamma^+ \otimes \mathrm{st}) \xrightarrow{q} (\mathbb{C} \otimes \mathbb{C}l_2, \mathrm{id} \otimes \mathrm{st}) \rightarrow 0$$

and take  $e = 1 \hat{\otimes} \sigma_x$  as base point for  $A^+ \hat{\otimes} \mathbb{C}l_2$  and  $\mathbb{C} \otimes \mathbb{C}l_2$ . Then the elements of  $DK(A)$  are by definition the homotopy classes  $[x]$  of OSU's  $x \in M_m(A^+) \hat{\otimes} \mathbb{C}l_2$  such that  $q(x)$  is homotopic to  $e_m$ . Let now  $\xi$  be a character of a cycle over  $(A, \gamma)$ . If  $\xi$  extends to  $\mathcal{A}^+$ , the unitization of the domain algebra  $\mathcal{A}$ , then we may directly apply Def. 4.6 for its pairing. If  $\xi$  does not extend to  $\mathcal{A}^+$  then the formula makes only sense if  $x - e_m \in \ker q = M_m(\mathcal{A}) \hat{\otimes} \mathbb{C}l_2$ . Van Daele proves in [15][Prop. 3.7] that any  $[x] \in DK(A)$  has a representative  $x$  which satisfies  $x - e_m \in \ker q = M_m(\mathcal{A}) \hat{\otimes} \mathbb{C}l_2$ . A closer look at his proof (based again on the construction of a holomorphic logarithm as in the proof of Lemma 4.1) shows that any  $[x] \in DK(A)$  has a representative which even satisfies  $x - e_m \in M_m(\mathcal{A}) \hat{\otimes} \mathbb{C}l_2$ . We thus define the pairing of  $\xi$  with  $[x]$  by the same formula as in Def. 4.6  $\langle \xi, [x] \rangle = \int \mathrm{Tr}(x - e_m)(dx)^n$  but require that  $x$  is such that  $x - e_m \in M_m(\mathcal{A}) \hat{\otimes} \mathbb{C}l_2$ . The homotopy invariance of this pairing can be shown in two steps: For differentiable paths in  $\mathfrak{S}(M_m(\mathcal{A}) \hat{\otimes} \mathbb{C}l_2)$  of the form  $x(t) - e_m$  we conclude as in Lemma 4.5 that the derivative of  $\int \mathrm{Tr}(x(t) - e_m)(dx(t))^n$  w.r.t.  $t$  vanishes. Indeed, the argument using the graded cyclicity can be employed as  $\dot{x}(dx)^n$  and  $\dot{z}(dz)^{n-\nu-1}$  lie in  $M_m(\mathcal{A}) \hat{\otimes} \mathbb{C}l_2$ , and furthermore the total derivatives  $R_1, R_2$  are derivatives of elements from  $M_m(\mathcal{A}) \hat{\otimes} \mathbb{C}l_2$ . Now in a second step we need to make sure that if  $y$  is OSU-homotopic to  $x$  and also satisfies  $y - e \in M_m(\mathcal{A}) \hat{\otimes} \mathbb{C}l_2$  then we can even find a homotopy  $x(t)$  from  $x$  to  $y$  such that  $x(t) - e$  remains in  $M_m(\mathcal{A}) \hat{\otimes} \mathbb{C}l_2$  for all  $t$ . To see this, let  $x(t)$  be a continuously differentiable homotopy between  $x(0) = x$  and  $x(1) = y$  in  $\mathfrak{S}(M_m(\mathcal{A}^+) \hat{\otimes} \mathbb{C}l_2)$  and consider the path  $c(t) = q(x(t))$  in  $\mathfrak{S}(M_m(\mathbb{C}) \hat{\otimes} \mathbb{C}l_2)$ . The path  $c(t)$  can be represented as  $\begin{pmatrix} 0 & U(t)^* \\ U(t) & 0 \end{pmatrix}$  where  $U(t)$  is a continuous path of unitaries in  $M_m(\mathbb{C})$  which is 1 at  $t = 0$  and  $t = 1$ . Let  $W(t) = \begin{pmatrix} U(t)^{\frac{1}{2}} & 0 \\ 0 & U(t)^{-\frac{1}{2}} \end{pmatrix}$  where  $U(t)^{\frac{1}{2}}$  a square

root of  $U(t)$  which is continuous in  $t$  for  $0 \leq t \leq 1$  and equal to 1 at  $t = 0$ . It follows that  $W(t)c(t)W^*(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Moreover,  $W(1)$  commutes with  $c(1)$  and has eigenvalues  $\pm 1$ . Since  $U_m(\mathbb{C})$  is contractible we can find a path of unitaries  $V(t) \in M_{2m}(\mathbb{C})$ ,  $t \in [1, 2]$  which commutes with  $c(1)$  and connects  $W(1)$  to  $1 \in M_{2m}(\mathbb{C})$ . Let

$$\tilde{x}(t) = \begin{cases} s(W(t))x(t)s(W(t))^* & \text{for } 0 \leq t \leq 1 \\ s(V(t))x(1)s(V(t))^* & \text{for } 1 \leq t \leq 2 \end{cases}$$

where  $s : \mathbb{C} \otimes \mathcal{Cl}_2 \rightarrow A^+ \hat{\otimes} \mathcal{Cl}_2$  be a section, i.e.  $q \circ s = \text{id}$ . Then  $\tilde{x}(t)$  is an osu-homotopy between  $x$  and  $y$  which satisfies  $q(\tilde{x}(t)) = e_m$  for all  $t \in [0, 2]$ .

**4.4.3. Pairing for trivially graded  $C^*$ -algebras.** In the context of trivially graded  $A$  the above pairing is an adaptation of Connes pairing to van Daele's formulation of (complex)  $K$ -theory, as we shall see now. We begin with a couple of remarks.

If  $(\Omega, d, f)$  is a cycle over a trivially graded algebra then only the even part of  $\Omega$  enters into the definition of the character and so we may assume without loss of generality that  $\Omega$  is trivially graded.

If  $A$  is a graded  $C^*$ -algebra with domain algebra  $\mathcal{A}$  then the condition that  $\mathcal{A}$  is closed under holomorphic functional calculus implies that the even parts of  $A \otimes \mathcal{Cl}_k$  are for all positive  $k$  also closed under holomorphic functional calculus. This is trivially the case for  $k = 1$ , follows for  $k = 2$  from the description of the even part as the diagonal matrices of  $M_2(\mathcal{A})$ , and for  $k > 2$  from the fact that  $A \otimes \mathcal{Cl}_1$  (or  $A^+ \otimes \mathcal{Cl}_1$  if  $A$  is not unital) is balanced.

Furthermore, closedness under holomorphic functional calculus of  $\mathcal{A}$  implies (C4). Indeed, for  $k = 0$  the condition (C4) is empty. For  $k = 1$  any OSU of  $A \otimes \mathcal{Cl}_1$  has the form  $(2p - 1) \otimes \rho$  where  $p$  is a projection. Arbitrarily close to  $p$  we can find an element  $p' \in \mathcal{A}$  which is perhaps not a projection, but its spectrum lies in a small neighbourhood  $U$  of the set  $\{0, 1\}$ . There exists a holomorphic function  $f$  on  $U$  which is 1 near 1 and 0 near 0 and hence  $f(p')$  is a projection and  $f(p) = p$ . By continuity of  $f$ ,  $f(p')$  is close to  $f(p)$  and hence  $(2p - 1) \otimes \rho$  osu-homotopic to  $(2p' - 1) \otimes \rho$  in  $A \hat{\otimes} \mathcal{Cl}_1$ . Finally, in the case  $k = 2$  any OSU of  $A \otimes \mathcal{Cl}_2$  has the form  $\begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}$  for some unitary  $U \in A$ . Since the invertible elements of  $A$  are open we find an invertible element  $Q \in \mathcal{A}$  close to  $U$ . Since  $\mathcal{A}$  is closed under functional calculus we can polar decompose  $Q$  in  $\mathcal{A}$  to see that  $Q$  is close to a unitary  $U'$  in  $\mathcal{A}$ . It follows that  $U'$  is homotopic to  $U$  in the set of unitaries of  $A$ .

To summarize, for trivially graded  $C^*$ -algebras we can replace conditions (C3) and (C4) in Def. 4.3 by the single condition that  $\mathcal{A}$  is closed under holomorphic functional calculus. It seems an interesting question to ask whether this is also true for non-trivially graded  $C^*$ -algebras.

The following result is specific to trivially graded algebras.



**Lemma 4.10.** *Let  $(\Omega, d, f)$  be an  $n$ -dimensional cycle over a trivially graded normed algebra  $A$ . Let  $e \in \mathfrak{F}(A \otimes \mathbb{C}l_k)$  with  $de = 0$ . If  $k + n$  is even then  $\int \text{Tr} \kappa((x - e_m)(dx)^n) = 0$  for all  $x \in \mathfrak{F}(A \otimes \mathbb{C}l_k, \text{id} \otimes \text{st})$ .*

*Proof.* We can write  $(x - e_m)(dx)^n = \omega \otimes \rho_1 \cdots \rho_k + R$  where  $R$  belongs to the kernel of  $\kappa_k$ . Now the  $\mathbb{Z}_2$ -degree of  $(x - e_m)(dx)^n$  is  $n + 1 \pmod{2}$  and that of  $\omega \otimes \rho_1 \cdots \rho_k$  equal to  $k$ , as  $\Omega$  is trivially  $\mathbb{Z}_2$ -graded (one can choose it that way). It follows that  $n + 1 + k$  must be even or  $\omega = 0$ .  $\square$

We now show that, for a trivially graded unital  $C^*$ -algebra  $A$ , the pairing of the character of an  $n = 2j + 1$ -dimensional cycle with  $KU_1(A) = K_1(A, \text{id}) \cong DK_{1 \otimes \sigma_x}(A \otimes \mathbb{C}l_2, \text{id} \otimes \text{st})$  corresponds to the usual pairing as defined by Connes [14]. Any OSU  $x \in M_2(M_m(A)) \cong M_m(A) \otimes \mathbb{C}l_2$  is of the form

$$x = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}$$

for some unitary  $U \in M_m(A)$ . Let  $e = 1 \otimes \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Then

$$\begin{aligned} (x - e_m)dx(dx)^{n-1} &= \begin{pmatrix} (U^* - 1)dU & 0 \\ 0 & (U - 1)dU^* \end{pmatrix} \begin{pmatrix} dU^*dU & 0 \\ 0 & dUdU^* \end{pmatrix}^j \\ &= \frac{1}{2}((U^* - 1)dU(dU^*dU)^j - (U - 1)dU^*(dUdU^*)^j) \otimes \sigma_z + R \end{aligned}$$

where  $R$  lies in the kernel of  $\kappa_2$ .  $U \mapsto \int \text{Tr}(U^* - 1)dU(dU^*dU)^j$  is the usual pairing between unitaries and the character of the cycle and known to induce a group homomorphism on  $KU_1(A)$  [14]. In particular  $\int \text{Tr}(U - 1)dU^*(dUdU^*)^j = -\int \text{Tr}(U^* - 1)dU(dU^*dU)^j$  and, since  $\kappa_2(\sigma_z) = 2$  we have

$$\int \text{Tr}_m \kappa_2((x - e_m)(dx)^n) = 2 \int \text{Tr}_m (U^* - 1)dU(dU^*dU)^j.$$

Next we show that the pairing of the character of an even dimensional cycle with  $KU_0(A) = K_0(A, \text{id})$  corresponds to the usual pairing as defined by Connes [14] if  $A$  is trivially graded unital. Recall that  $K_0(A, \text{id}) \cong DK_e(A \otimes \mathbb{C}l_1, \text{st})$  where we may take  $e = -1 \otimes \rho$  ( $\rho$  the generator of  $\mathbb{C}l_1$ ) as basepoint. Since  $e$  is not homotopic to its negative  $DK_e(A \otimes \mathbb{C}l_1, \text{st})$  is the Grothendieck group of the semigroup defined by homotopy classes of OSUs. Any OSU has the form  $x = h \otimes \rho$  where  $h \in M_m(A)$  is a selfadjoint unitary. The map  $h \mapsto p = \frac{h+1}{2}$  induces an isomorphism between  $DK_e(A \otimes \mathbb{C}l_1)$  and the standard picture for the  $K_0$ -group of  $A$ . Then  $(x - e_m)(dx)^n = (h + 1)(dh)^n \rho^{n+1}$  and hence  $\kappa_1((x - e_m)(dx)^n) = \kappa_2 p(2dp)^n$  provided  $n$  is even and 0 otherwise. Hence, if  $n$  is even,

$$\int \text{Tr} \kappa_1((x - e_m)(dx)^n) = \kappa_2^{n+1} \int \text{Tr} p(dp)^n.$$

**4.5. Compatibility with the  $*$ -structure and the real structure.** We now formulate compatibility conditions on chains or cycles of  $C^{*,r}$ -algebras which will, in the context of pairings with their real subalgebras, lead to criteria under which the pairing can be non-trivial.

A  $(*, \mathfrak{r})$ -algebra is a  $*$ -algebra with a real structure  $\mathfrak{r}$ , that is, an anti-linear  $*$ -automorphism of order 2. An  $\mathbb{R}$ -linear map  $\varphi$  on a  $(*, \mathfrak{r})$ -algebra is a  $(*, \mathfrak{r})$ -morphism if  $\varphi(a^*) = (\varphi(a))^*$  and  $\varphi(\mathfrak{r}(a)) = \mathfrak{r}(\varphi(a))$ .

**Definition 4.11.** A chain  $(\Omega, d, \int)$  over a  $(*, \mathfrak{r})$ -algebra  $A$  is called a  $(*, \mathfrak{r})$ -chain of sign  $\mathfrak{s} \in \{+1, -1\}$  if  $\Omega$  is a  $(*, \mathfrak{r})$ -algebra,  $\varphi : A \rightarrow \Omega_0$  a  $(*, \mathfrak{r})$ -homomorphism,  $d$  a  $(*, \mathfrak{r})$ -derivation and the graded trace satisfies

$$\int \tilde{\mathfrak{r}}(\omega)^* = \mathfrak{s} \int \omega$$

where  $\tilde{\mathfrak{r}}$  is the real structure on  $\Omega$ .

Recall that the chain has parity  $\nu$  if the graded trace vanishes on elements of parity  $\nu + 1$ .

**Proposition 4.12.** Let  $(\Omega, \tilde{\mathfrak{r}}, d, \int)$  be an  $n$ -dimensional  $(*, \mathfrak{r})$ -chain over a graded  $C^{*,r}$ -algebra  $(A, \mathfrak{r})$ . Let  $u, x, y, z \in A$  be  $\mathfrak{r}$ -invariant odd self-adjoint elements,  $u$  being in addition unitary. A necessary condition for  $\int z(dx)^n$  to be non-zero is that the sign of the cycle is  $(-1)^n$  and its parity is  $n + 1$ . A necessary condition for  $\int zu(du)^n$  and  $\sum_{k=0}^n (-1)^k \int z(dx)^k y(dx)^{n-k}$  to be non-zero is that the sign of the cycle is  $-1$  and its parity is  $n$ .

*Proof.* The  $\mathbb{Z}_2$ -degree of  $z(dx)^n$  is  $n + 1 \pmod{2}$  and thus, if the parity of the cycle is different from  $n + 1$  then  $\int z(dx)^n = 0$ . If the parity is  $n + 1$  then

$$\mathfrak{s} \int z(dx)^n = \int \tilde{\mathfrak{r}}(z(dx)^n)^* = \int (dx)^n z = (-1)^n \int z(dx)^n$$

from which we deduce the first claim.

The  $\mathbb{Z}_2$ -degree of  $zx(dx)^n$  is  $n$  and thus if the parity of the cycle is  $n$  then

$$\int \tilde{\mathfrak{r}}((zx(dx)^n)^*) = (-1)^n \int x(dx)^n z = - \int zx(dx)^n$$

where we have used  $xdx = -dxx$ .

The  $\mathbb{Z}_2$ -degree of  $\sum_{k=0}^n (-1)^k z(dx)^k y(dx)^{n-k}$  is  $n$  and if the cycle has parity  $n$  then

$$\begin{aligned} \sum_{k=0}^n (-1)^k \int \tilde{\mathfrak{r}}(z(dx)^k y(dx)^{n-k})^* &= \sum_{k=0}^n (-1)^k \int (dx)^{n-k} y(dx)^k z \\ &= \sum_{k=0}^n (-1)^{k+n+1} \int z(dx)^{n-k} y(dx)^k \\ &= \sum_{k=0}^n (-1)^{k+1} \int z(dx)^k y(dx)^{n-k} \end{aligned}$$

from which we deduce the last claim.  $\square$

Recall that  $\mu : \mathbb{Z} \rightarrow \mathbb{Z}$  is given by  $\mu(n) = \lfloor \frac{n}{2} \rfloor$ .

**Lemma 4.13.** *Consider a  $(*, \mathfrak{r})$ -chain of parity  $\nu$  and sign  $\mathfrak{s}$  over  $(A, \mathfrak{r})$ . The extension to  $(A \hat{\otimes} \mathcal{C}l_{r+s}, \mathfrak{r} \otimes \mathfrak{l}_{r,s})$  is a  $(*, \mathfrak{r})$ -cycle of parity  $\nu + r - s$  and sign  $(-1)^{\mu(r-s) + \nu(r-s)} \mathfrak{s}$ .*

*Proof.* The parity of  $\int \kappa_{r+s}$  is the sum of the parities of  $\int$  and  $\kappa_{r+s}$ , hence equal to  $\nu + r - s$ . By definition of the real structures  $\mathfrak{l}_{r,s}$  on  $\mathcal{C}l_{r+s}$  we find

$$\mathfrak{l}_{r,s}(\rho_1 \cdots \rho_{r+s})^* = (-1)^{\mu(r-s)} \rho_1 \cdots \rho_{r+s}.$$

If the following expression is non-zero then  $\omega$  must have parity  $\nu$  and hence

$$\begin{aligned} \int \kappa_{r+s}(\tilde{\mathfrak{r}}(\omega) \hat{\otimes} \mathfrak{l}_{r,s}(\rho_1 \cdots \rho_{r+s}))^* &= (-1)^{\nu(r+s)} \int \kappa_{r+s}(\tilde{\mathfrak{r}}(\omega)^* \hat{\otimes} \mathfrak{l}_{r,s}(\rho_1 \cdots \rho_{r+s})^*) \\ &= (-1)^{\mu(r-s) + \nu(r-s)} \mathfrak{s} \int \kappa_{r+s}(\omega \otimes \rho_1 \cdots \rho_{r+s}) \end{aligned}$$

$\square$

**Corollary 4.14.** *Let  $(\Omega, \tilde{\mathfrak{r}}, d, f)$  be an  $n$ -dimensional  $(*, \mathfrak{r})$ -cycle over a graded  $C^{*,r}$ -algebra  $(A, \mathfrak{r})$  with parity  $\nu$  and sign  $\mathfrak{s}$ . A necessary condition for its character to pair non-trivially with  $DK_e(A^\mathfrak{r} \hat{\otimes} \mathcal{C}l_{r,s})$  is that  $\mathfrak{s} = (-1)^{\mu(r-s) + n(r-s+1)}$  and  $\nu = n + 1 - r + s$ .*

*Proof.* By Lemma 4.13 the extension of the cycle to  $A^\mathfrak{r} \hat{\otimes} \mathcal{C}l_{r,s}$  has parity  $\nu + r - s$  and sign  $(-1)^{\mu(r-s) + \nu(r-s)} \mathfrak{s}$ . The result follows now from Prop. 4.12, taking into account that  $\nu(r-s) = n(r-s)$ .  $\square$

Note that  $(-1)^{\mu(i) + n(i+1)} = -(-1)^{\mu(i+2) + n(i+2+1)}$ . Hence if  $\langle \xi, DK(A^\mathfrak{r} \hat{\otimes} \mathcal{C}l_{r,s}) \rangle \neq 0$  then  $\langle \xi, DK(A^\mathfrak{r} \hat{\otimes} \mathcal{C}l_{r',s'}) \rangle = 0$  where  $r' + s' = r + s \pm 2$ . For trivially graded real algebras we get the following stronger conditions.

**Corollary 4.15.** *Let  $A$  be a trivially graded  $C^{*,r}$ -algebra. Necessary conditions for an  $n$ -dimensional  $(*, r)$ -cycle of parity  $\nu$  and sign  $\mathfrak{s}$  over  $A$  to pair non-trivially*

with  $KO_i(A^\mathfrak{r})$  are that  $\mathfrak{s} = (-1)^{\mu(1-i)+i}$  and  $n+i$  is even. In particular, when these conditions are met the cycle pairs trivially with  $KO_{i+k}(A^\mathfrak{r})$  for  $k \notin 4\mathbb{Z}$ .

**4.6. Examples of pairings.** We discuss a variety of simple examples.

**4.6.1. Trace on  $\mathbb{C}$ .** Consider the  $C^{*,r}$ -algebra  $(\mathbb{C}, \mathfrak{c})$  with trivial grading. One easily sees that  $(\mathbb{C}, 0, \text{id})$  is a 0-dimensional  $(*, \mathfrak{r})$ -cycle of even parity and sign  $+1$  over  $(\mathbb{C}, \mathfrak{c})$ . Since  $\mathbb{C}$  is trivially graded only pairings with even  $K$ -group elements may be non-zero.

We pair with  $KU_0(\mathbb{C}) = DK_e(\mathbb{C} \otimes \mathbb{C}l_1)$ . Let  $\rho$  be the generator of  $\mathbb{C}l_1$  and choose  $e = -1 \otimes \rho$  as base point. Any OSU is of the form  $x = h \otimes \rho$  for some self-adjoint unitary  $h \in M_m(\mathbb{C})$  which we can write as  $h = 2p - 1$  with some projection  $p$ . Thus the character  $\xi$  of  $(\mathbb{C}, 0, \text{id})$  pairs as  $\langle \xi, [x] \rangle = \kappa 2 \text{Tr}(p)$ . Absorbing the constant  $2\kappa$  we denote

$$\text{ch}_0 = \frac{1}{2\kappa} \xi$$

calling it the standard Chern character on  $\mathbb{C}$ . Then  $\langle \text{ch}_0, KO_0(\mathbb{R}) \rangle = \mathbb{Z}$ .

We consider pairings with the even  $KO$ -groups of the real subalgebra  $\mathbb{R}$ . As  $(-1)^{\mu(0)} = +1$  pairings with elements of  $KO_i(\mathbb{R})$  for  $i = 2$  and  $6$  have to vanish by Cor. 4.15. This is to be expected as  $KO_2(\mathbb{R})$  is pure torsion and  $KO_6(\mathbb{R}) = 0$ .

For pairings with elements of  $KO_0(\mathbb{R}) = DK_{1 \otimes \rho}(\mathbb{R} \otimes \mathbb{C}l_{1,0})$  the analysis is exactly as in the complex case  $\langle \text{ch}_0, [x] \rangle = \text{Tr}(p)$  where  $x = (2p - 1) \otimes \rho$ , the only difference being that  $p$  is a projection in  $M_m(\mathbb{R})$ . In particular,  $\langle \text{ch}_0, KO_0(\mathbb{R}) \rangle = \mathbb{Z}$ .

Finally we consider pairing with elements from  $KO_4(\mathbb{R}) \cong DK(\mathbb{R} \otimes \mathbb{C}l_{0,3})$ . Let  $e = 1 \hat{\otimes} \rho \in \mathbb{C}l_2 \hat{\otimes} \mathbb{C}l_1$ . By Lemma 2.1 the isomorphism  $\psi_e : (\mathbb{C}l_2 \hat{\otimes} \mathbb{C}l_1, \text{st} \otimes \text{st}) \rightarrow (M_2(\mathbb{C}) \otimes \mathbb{C}l_1, \text{id}_2 \otimes \text{st})$  intertwines the real structure  $\mathfrak{l}_{0,2} \otimes \mathfrak{l}_{0,1}$  with the real structure  $\mathfrak{h} \otimes \mathfrak{l}_{1,0}$  where  $\mathfrak{h} = \text{Ad}_{\sigma_y} \circ \mathfrak{c}$ . Thus  $KO_4(\mathbb{R}) \cong DK_{1 \otimes \rho}(\mathbb{H} \otimes \mathbb{C}l_{1,0})$  where  $\mathbb{H} = M_2(\mathbb{C})^{\mathfrak{h}}$  is the (trivially graded) algebra of quaternions. The pairing is given now by  $\langle \text{ch}_0, [x] \rangle = \text{Tr}(P)$  where  $P = \frac{1}{2}(\psi_e(x) + 1)$  is a projection in  $M_m(\mathbb{H})$ .  $M_m(\mathbb{H})$  contains only projections of even rank as any eigenvalue of a self-adjoint element in  $M_{2m}(\mathbb{C})$  which is invariant under  $\mathfrak{h}_m$  has to be evenly degenerate (this is Kramer's degeneracy in physics). Thus  $\langle \text{ch}_0, KO_4(\mathbb{R}) \rangle = 2\mathbb{Z}$ .

**4.6.2. Winding number cycle.** A simple (trivially graded) example of a chain is given by  $(\Omega([0, 1]), d_{[0,1]}, \int_{[0,1]})$ , where  $\Omega([0, 1])$  are the differential forms on the closed interval  $[0, 1]$ ,  $d_{[0,1]}$  the exterior derivative, and  $\int_{[0,1]}$  the integral over 1-forms, normalized so as  $\int_{[0,1]} dt = 1$ . It is a 1-dimensional chain over  $C([0, 1])$  with domain algebra  $C^1([0, 1])$ . On the subalgebra of functions which satisfy  $f(1) = f(0)$  the integral is closed and hence the chain restricts to a cycle over  $C(S^1)$ . It can pair non-trivially only with odd  $K$ -group elements.

We consider its pairing with  $KU_1(C(S^1)) = DK_{\sigma_x}(C(S^1) \otimes \mathbb{C}l_2)$ . Upon writing an OSU  $x \in C(S^1) \otimes \mathbb{C}l_2$  as  $x = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}$ , with some unitary  $U(t)$  in  $M_m(\mathbb{C})$ , we

obtain for the pairing of the character  $\xi$  of the above cycle with  $[x]$ ,

$$\langle \xi, [x] \rangle = \int_{[0,1]} \text{Tr}_m \kappa_2((x - e_m)dx) = 2 \int_0^1 \text{Tr}_m((U(t)^* - 1)U'(t))dt.$$

We recognise this as  $4\pi i$  times the winding number of the determinant of  $U$  which is why we normalize  $\mathcal{W} := \frac{1}{2\pi\kappa^2}\xi$  and call the above cycle the winding number cycle. As a result  $\langle \mathcal{W}, KU_1(C(S^1)) \rangle = \mathbb{Z}$ . Since  $C(S^1)$  is the unitization of the suspension  $S\mathbb{C} = C_0((0, 1), \mathbb{C})$  of  $\mathbb{C}$  the above cycle is also a cycle for  $S\mathbb{C}$ .

There are two important real structures on  $C(S^1)$ , point-wise complex conjugation  $\mathfrak{c}(f) = \overline{f}$  and the composition of the latter with the automorphism induced by a reflection  $t \mapsto 1 - t$ ,  $\hat{\mathfrak{c}}(f)(t) = \overline{f(1-t)}$ . They both restrict to the suspension  $S\mathbb{C}$  and the real subalgebra of  $(S\mathbb{C}, \mathfrak{c})$  is the (standard) suspension  $S\mathbb{R}$  of  $\mathbb{R}$ , while the real subalgebra of  $(S\mathbb{C}, \hat{\mathfrak{c}})$  is, by definition, the dual suspension  $\tilde{S}\mathbb{R}$  of  $\mathbb{R}$ .

Let us consider first the real structure  $\mathfrak{c}$  given by point-wise complex conjugation. It is rather straightforward to see that by putting on  $\Omega([0, 1])$  the real structure given by point-wise complex conjugation the cycle becomes a  $(*, r)$  cycle of sign  $+1$ . We consider its pairings with elements of the real  $K$ -groups  $KO_i(C(S^1)^\mathfrak{c})$ . There are four cases to consider, and Cor. 4.15 predicts that only two are potentially non-trivial, namely those for which  $(-1)^{\mu(1-i)} = -1$ , that is,  $i = -1$  and  $i = 3$ . This can be understood on a more elementary level if one considers the characterisation of the  $KO$ -group elements via constraints on the unitary  $U$  [26, 7]: If  $i = 1$  then  $U(t)$  must be real and hence its determinant must be constant and so cannot have a non-trivial winding number (a similar argument involving quaternions holds for  $i = 5$ ). If  $i = -1$  then  $\overline{U(t)} = U(t)^*$ . This condition does not prevent the determinant to have trivial winding number, it is, for instance, satisfied by  $U(t) = e^{2\pi i t}$  and hence we see that

$$\langle \mathcal{W}, KO_{-1}(C(S^1, \mathbb{R})) \rangle = \mathbb{Z}.$$

If  $i = 3$  then  $\overline{U(t)} = -U(t)^*$ . This implies that the dimension of  $U(t)$  has to be even and  $\det(U - \lambda 1) = \det(\overline{U} - \overline{\lambda} 1) = (-1)^{2m} \det(U^* + \overline{\lambda} 1) = \overline{\det(U + \lambda 1)}$ , and hence that with  $\lambda(t)$  also  $-\lambda(t)$  is an eigenvalue of  $U(t)$ . Since both have the same winding number we see that the winding number of  $U$  must be even. Thus

$$\langle \mathcal{W}, KO_3(C(S^1, \mathbb{R})) \rangle = 2\mathbb{Z}.$$

Let us now consider the real structure  $\hat{\mathfrak{c}}$  defining the twisted suspension  $\tilde{S}\mathbb{R}$ . As  $df = \partial_t f dt$  we have  $d\hat{\mathfrak{c}}(f) = -\hat{\mathfrak{c}}(\partial_t f)dt$ . In order for the differential to commute with the real structure  $\hat{\mathfrak{c}}$  we thus need to set

$$\tilde{\hat{\mathfrak{c}}}(f dt) = -\hat{\mathfrak{c}}(f)dt.$$

In other words,  $\tilde{\hat{\mathfrak{c}}}$ -invariant 1-forms are of the form  $if dt$  where  $f \in \tilde{S}\mathbb{R}$ . It follows that  $(\Omega([0, 1]), \tilde{\hat{\mathfrak{c}}}, d_{[0,1]}, \int_{[0,1]})$  is a  $(*, r)$ -cycle of sign  $\mathfrak{s} = -1$ .

Cor. 4.15 predicts now that the pairing is only non-trivial for odd  $i$  which satisfy  $(-1)^{\mu(1-i)} = 1$ , that is,  $i = 1$  and  $i = 5$ . The elements of  $KO_1(\tilde{S}\mathbb{R})$  correspond to

unitaries satisfying  $\overline{U(t)} = U(1-t)$ , a condition which is satisfied by  $U(t) = e^{2\pi it}$  and hence

$$\langle \mathcal{W}, KO_1(\tilde{S}\mathbb{R}) \rangle = \mathbb{Z}.$$

**4.7. Suspension of cycles.** Given a  $C^*$ -algebra  $A$ , the algebra  $SA = C_0((0, 1), A)$  is called the suspension of  $A$ . It is a subalgebra of the unital algebra  $C([0, 1], A)$ . Let  $(\Omega, d, f)$  be an  $n$ -dimensional chain of parity  $\nu$  over a graded  $C^*$ -algebra  $A$ . One can extend this chain to an  $n+1$ -dimensional chain of parity  $\nu$  over  $C([0, 1], A) \cong A \otimes C([0, 1])$ . For that we consider the (graded) product of  $(\Omega, d, f)$  with the chain discussed above to obtain  $(\Omega \wedge \Omega([0, 1]), d^s, \int_{C([0, 1], A)})$  where

$$\begin{aligned} d^s(\omega \wedge \mu) &= d\omega \wedge \mu + (-1)^{|\omega|z} \omega \wedge d_{[0, 1]}\mu \\ \int_{C([0, 1], A)} (\omega \wedge \mu) &= \int \omega \int_{[0, 1]} \mu. \end{aligned}$$

Here  $\Omega \wedge \Omega_{[0, 1]}$  is the graded tensor product w.r.t. the  $\mathbb{Z}$ -degree and hence  $(\omega \wedge \mu)^* = (-1)^{|\omega|z} \omega^* \wedge \mu^*$ . The character of the extended chain is denoted by  $\xi_{[0, 1]}$ . It is given by

$$(6) \quad \xi_{[0, 1]}(f_0, \dots, f_{n+1}) = \sum_{k=0}^n (-1)^k \int_0^1 \int f_0 df_1 \cdots df_k \partial_t f_{k+1} df_{k+2} \cdots df_{n+1} dt.$$

This expression can only be non-trivial if the dimension  $n$  of the cycle  $(\Omega, d, f)$  has the same parity as the cycle.

When  $(\Omega, d, f)$  is a cycle then the above extension chain restricts to a cycle on  $SA := C_0((0, 1), A)$ , the so-called suspension cycle.

Let  $C^\infty([0, 1], \mathcal{A})$  be the dense subalgebra of continuous, piecewise smooth functions from  $[0, 1]$  into  $\mathcal{A}$  which vanish at the end points. This is a domain algebra for the suspension of the original cycle. Indeed, (C1) and (C2) are directly shown and (C3) follows as  $C^\infty([0, 1], \mathcal{B})$  is, for any complex algebra  $\mathcal{B}$  which is closed under holomorphic functional calculus, also closed under holomorphic functional calculus. To see that it satisfies (C4) let  $f \in SA$  and so  $f$  is a continuous function with values in  $\mathcal{S}\mathcal{A}$ . Let  $b > 0$ . We can partition  $[0, 1]$  into intervals  $[t_i, t_{i+1}]$ ,  $0 = t_0 < t_1 \cdots t_N = 1$  such that for all  $t \in [t_i, t_{i+1}]$  we have  $\|f(t) - f(t_i)\| < b$ . Since  $\mathcal{A}$  is a domain algebra, there are  $\tilde{f}_i \in \mathfrak{S}(\mathcal{A}, \gamma)$  such that  $\|\tilde{f}_i - f(t_i)\| < b$ . By Lemma 4.1 there is a smooth path  $\tilde{f}_i(s)$  from  $\tilde{f}_i = \tilde{f}_i(0)$  to  $\tilde{f}_{i+1} = \tilde{f}_i(1)$  in  $\mathfrak{S}(\mathcal{A}, \gamma)$  such that  $\|\tilde{f}_i(s) - \tilde{f}_i\| \leq C(3b)$  for all  $s \in [0, 1]$  where  $C : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a continuous function with  $C(0) = 0$ . Concatenating the paths  $\tilde{f}_0, \dots, \tilde{f}_{N-1}$  to one path  $\tilde{f}$  yields a continuous, piecewise smooth loop in  $\mathfrak{S}(\mathcal{A}, \gamma)$  which has distance  $\sup_t \|\tilde{f}(t) - f(t)\| \leq 3b + C(3b)$  from  $f$ . This proves (C4).

If  $\xi$  is the character of the original cycle then we denote by  $\xi^S$  the character of its suspension.

If  $A$  is a  $C^{*,r}$ -algebra then the suspension of an  $n$ -dimensional  $(*, r)$ -cycle is an  $n + 1$ -dimensional  $(*, r)$ -cycle with the same sign and parity.

**Corollary 4.16.** *Let  $(\Omega, \tilde{\mathfrak{r}}, d, f)$  be an  $n$ -dimensional  $(*, \mathfrak{r})$ -chain over a balanced graded  $C^{*,r}$ -algebra  $(A, \mathfrak{r})$  with parity  $\nu$  and sign  $\mathfrak{s}$ . Let  $\xi$  be its character and  $\xi_{[0,1]}$  its extension. If*

$$\mathfrak{s} = (-1)^{n+1} \quad \text{or} \quad \nu = n \pmod{2}$$

then  $\int z(dx)^n = 0$  for all odd self-adjoint elements  $x, z \in A^\mathfrak{r}$ . If

$$\mathfrak{s} = +1$$

then  $\int zu(du)^n = 0$  for all odd self-adjoint elements  $u, z \in A^\mathfrak{r}$ ,  $u$  unitary, and  $\xi_{[0,1]}(f, \dots, f) = 0$  for any odd self-adjoint  $f \in C^1([0, 1], A^\mathfrak{r})$ .

*Proof.* This follows from (6) and Prop. 4.12 upon taking  $x = f(t)$ ,  $z = x - e_m$ , and  $y = \dot{f}(t)$ .  $\square$

The following lemma is a generalisation to graded  $C^*$ -algebras of Pimsner's formula [31].

**Lemma 4.17.** *Let  $A$  be a graded complex  $C^*$ -algebra and  $\xi$  be the character of a  $n$ -dimensional cycle  $(\Omega, d, \int_A)$  over  $A$ . Then*

$$\langle \xi^S, \beta[x] \rangle = c_n \langle \xi, [x] \rangle$$

where

$$c_n = (-1)^{n+1} \kappa \pi (n+1) \alpha_{n,n}, \quad \alpha_{n,n} = \int_0^1 \sin^n(\pi t) dt$$

*Proof.* We first need the following observation: If  $e$  is an OSU in  $A$  and  $w$  a even unitary in  $A$  then  $[wew^{-1}] = -[w^{-1}ew]$  as elements in  $DK_e(A)$ . Indeed,  $w \oplus w^{-1}$  is homotopic to  $1 \oplus 1$  in the set of even unitaries of  $A$ , and therefore  $wew^{-1} \oplus w^{-1}ew$  osu-homotopic to  $e \oplus e$ , which represents the neutral element in  $DK_e(A)$ . Using this we obtain from Theorem 3.1

$$\beta[x] = -[Z], \quad Z = \nu(e_m) \nu^{-1}(x) (1 \hat{\otimes} \rho_m) \nu(x) \nu^{-1}(e_m).$$

We abbreviate  $w_t = \nu_t(x)$  and  $r_t = \nu_t(e_m)$  where  $\nu$  is given in (2). Taking into account that  $x \hat{\otimes} 1$  and  $1 \hat{\otimes} \rho_m$  anticommute,

$$b_t := w_t^{-1} (1 \hat{\otimes} \rho_m) w_t = 1 \hat{\otimes} \rho_m (c_t \hat{\otimes} 1 + x \hat{\otimes} \rho_m s_t)^2 = c_{2t} \hat{\otimes} \rho_m - x s_{2t} \hat{\otimes} 1.$$

A direct calculation shows that  $\dot{Z} = r_t (r_t^{-1} \dot{r}_t b_t + \dot{b}_t - b_t r_t^{-1} \dot{r}_t) r_t^{-1}$ . Hence

$$Z(d^s Z)^{n+1} = \sum_{k=0}^n r_t (b_t (db_t)^k (r_t^{-1} \dot{r}_t b_t + \dot{b}_t - b_t r_t^{-1} \dot{r}_t) dt (db)^{n-k}) r_t^{-1}$$

When applying the graded trace  $\int$  then, by graded cyclicity, the conjugation with  $r_t$  simply drops out. Furthermore  $r_t^{-1}\dot{r}_t = \frac{\pi}{2}e_m \hat{\otimes} \rho_m$  and  $\dot{b}_t = -\pi(xc_{2t} \hat{\otimes} 1 + s_{2t} \hat{\otimes} \rho)$  so that

$$(r_t^{-1}\dot{r}_t b_t + \dot{b}_t - b_t r_t^{-1}\dot{r}_t) = \pi(c_{2t}(e-x) \hat{\otimes} 1 + (\frac{xe+ex}{2} - 1)s_{2t} \hat{\otimes} \rho)$$

Now all terms which are even powers in  $\rho$  lie in the kernel of  $\kappa_1$ . Furthermore, not counting the derivatives  $dx$ , all terms with even powers of  $x$  yields terms which are total derivatives, because  $x^2 = 1$ , an identity which can be used after permuting  $x$  with  $dx$  or permuting  $x$  cyclicly. Thus the only terms from  $b_t(db_t)^k([r_t^{-1}\dot{r}_t, b_t] + \dot{b}_t)$  are not in the kernel of  $\int^s \kappa_1$  are

$$\pi(c_{2t} \hat{\otimes} \rho)(db_t)^k(-xc_{2t} \hat{\otimes} 1) - \pi(xs_{2t} \hat{\otimes} 1)(db_t)^k(-s_{2t} \hat{\otimes} \rho) = \pi(x \hat{\otimes} 1)(db_t)^k(1 \hat{\otimes} \rho)$$

where we have used  $(1 \hat{\otimes} \rho)(db_t)^k(x \hat{\otimes} 1) = (-1)^{k+1}(db_t)^k(x \hat{\otimes} 1)(1 \hat{\otimes} \rho) = -(x \hat{\otimes} 1)(db_t)^k(1 \hat{\otimes} \rho)$ . We thus get, using  $db_t = -s_{2t} dx \hat{\otimes} 1$ ,

$$\begin{aligned} \int_{C([0,1],A)} \kappa_1(Z(d^s Z)^{n+1}) &= \pi \sum_{k=0}^n \int_{C([0,1],A)} \kappa_1((x \hat{\otimes} 1)(db_t)^k(dt \hat{\otimes} \rho)(db_t)^{n-k}) \\ &= \pi \sum_{k=0}^n \int_{C([0,1],A)} \kappa_1((-s_{2t})^n x(dx)^n dt \hat{\otimes} \rho) \\ &= \kappa\pi(n+1) \int_0^1 \sin^n(\pi t) dt \int_A x(dx)^n \end{aligned}$$

For the second equality we used that  $dt$  anticommutes with  $dx$  as the latter has  $\mathbb{Z}$ -degree 1 while  $1 \hat{\otimes} \rho$  anticommutes with  $dx \hat{\otimes} 1$  because the latter has  $\mathbb{Z}_2$ -degree 1. Since  $\langle \xi, [x] \rangle = \int x(dx)^n$  and  $\langle \xi^S, \beta[x] \rangle = -\int_{C([0,1],A)} \kappa_1(Z(d^s Z)^{n+1})$  we arrive at

$$c_n = (-1)^{n+1} \kappa\pi(n+1) \int_0^1 \sin^n(\pi t) dt. \quad \square$$

The integral in the definition of  $c_n$  is given by

$$(7) \quad \alpha_{n,n} = \int_0^1 \sin^n(\pi t) dt = \begin{cases} \frac{1}{2^n} \binom{n}{\frac{n}{2}} & \text{if } n \text{ is even} \\ \frac{2^n}{n\pi} \binom{n-1}{\frac{n-1}{2}}^{-1} & \text{if } n \text{ is odd} \end{cases}.$$

## 5. A TORSION VALUED PAIRING WITH $K_n(A^r, \gamma)$

Connes pairing is additive and takes values in the linear space  $\mathbb{C}$ . In particular we have  $\langle \xi, [x] \rangle = \frac{1}{2} \langle \xi, 2[x] \rangle$  showing that the pairing must vanish on 2-torsion elements. We now provide a construction which is geared to see such torsion elements.



5.1. **General construction.** Consider a unital inclusion  $\varphi$  of (real or complex) balanced graded  $C^*$ -algebras

$$B \xrightarrow{\varphi} \tilde{B}.$$

The relative cone of the inclusion is the algebra

$$\mathcal{C}(\varphi) = \{f \in C([0, 1], \tilde{B}) \mid f(0) \in 0, f(1) \in \varphi(B)\}$$

It fits into the short exact sequence

$$0 \rightarrow S\tilde{B} \xrightarrow{i} \mathcal{C}(\varphi) \xrightarrow{ev_1} B \rightarrow 0$$

which gives rise to the long exact sequence (6 periodic in the complex and 24-periodic in the real case)

$$K_i(S\tilde{B}) \xrightarrow{i_*} K_i(\mathcal{C}(\varphi)) \xrightarrow{ev_{1*}} K_i(B) \xrightarrow{\delta} K_{i-1}(S\tilde{B}) \rightarrow \dots$$

and when followed by the inverse of the Bott isomorphism  $\beta : K_i(\tilde{B}) \rightarrow K_{i-1}(S\tilde{B})$  the boundary map  $\delta$  becomes the map induced on  $K$ -theory by  $\varphi$

$$\begin{array}{ccc} K_i(B) & \xrightarrow{\varphi_*} & K_i(\tilde{B}) \\ \parallel & & \downarrow \beta \\ K_i(B) & \xrightarrow{\delta} & K_{i-1}(S\tilde{B}) \end{array}$$

We formulate a vanishing condition for characters  $\tilde{\xi}$  of chains over  $\tilde{B}$ .

(V) For any  $m$  and differentiable function  $g : [0, 1] \rightarrow \mathfrak{S}(M_m(\varphi(B)))$

$$\tilde{\xi}_{[0,1]} \# \text{Tr}_m(g, \dots, g) = 0.$$

In the following we suppose that  $B$  contains a basepoint  $e$  which is homotopic to its negative in  $\mathfrak{S}(B)$ .

**Definition 5.1.** Let  $\varphi : B \rightarrow \tilde{B}$  be a unital inclusion. Let  $(\Omega, d, \int_{\tilde{B}})$  be an  $n$ -dimensional chain with character  $\tilde{\xi}$  over a graded  $C^*$ -algebra  $\tilde{B}$  which satisfies (V). Suppose that  $B$  contains a basepoint  $e$  which is homotopic to its negative in  $\mathfrak{S}(B)$  and satisfies  $d\varphi(e) = 0$ . The torsion valued pairing is given by the homomorphism  $\Delta_{\tilde{\xi}}^{\varphi} : \ker \varphi_* \cap DK_e(B) \rightarrow \mathbb{C} / \langle \tilde{\xi} \# \kappa_1, DK_{\varphi(e)}(\tilde{B} \hat{\otimes} Cl_{0,1}) \rangle$

$$\Delta_{\tilde{\xi}}^{\varphi}([x]) \equiv c_n^{-1} \tilde{\xi}_{[0,1]} \# \text{Tr}_m(F - \varphi(e)_m, \dots, F - \varphi(e)_m)$$

where  $x \in \mathfrak{S}(M_l(B))$  and  $F$  is a continuous path from  $\varphi(e)_m$  to  $\varphi(x) \oplus \varphi(e)_{m-l}$  in  $\mathfrak{S}(M_m(\tilde{B}))$ , for some  $m \geq l$ .

Here  $\equiv$  means equality in the quotient group, that is, the value is to be understood modulo the subgroup  $\langle \tilde{\xi} \# \kappa_1, DK_{\varphi(e)}(\tilde{B} \hat{\otimes} Cl_{0,1}) \rangle$ . We deduce from (6) that  $\Delta_{\tilde{\xi}}^{\varphi}$  can only be non-trivial if the dimension and the parity of the chain  $(\Omega, d, \int_{\tilde{B}})$  coincide.

We argue why  $\Delta_{\tilde{\xi}}^{\varphi}$  is well defined:  $\ker \varphi_*$  is generated by elements whose representatives  $x$  belong to  $\mathfrak{S}(M_l(B))$  for some  $l$  and such that  $\varphi(x) \oplus \varphi(e)_{m-l}$  is

homotopic to  $\varphi(e)_m$  in  $\mathfrak{S}(M_m(\tilde{B}))$ , for some  $m \geq l$ .  $F$  is such a homotopy. If we take a different homotopy  $F'$  then the composition of  $F$  with  $F'$  run backwards yields a loop  $L$  with values in  $\mathfrak{S}(M_m(\tilde{B}))$  starting at  $\varphi(e)_m$ , and hence its homotopy class  $[L]$  defines an element of  $DK_{\varphi(e)}(S\tilde{B})$ . The ambiguity is thus given by  $c_n^{-1}\tilde{\xi}_{[0,1]}\#\text{Tr}_m(L - e_m, \dots, L - e_m)$  which is an element of  $c_n^{-1}\langle\tilde{\xi}_{[0,1]}, DK_{\varphi(e)}(S\tilde{B})\rangle$ . Now

$$\begin{aligned}\langle\tilde{\xi}_{[0,1]}, DK_{\varphi(e)}(S\tilde{B})\rangle &= \langle\tilde{\xi}_{[0,1]}\#\kappa_2, DK_{\varphi(e)}(S\tilde{B}\hat{\otimes}Cl_{1,1})\rangle \\ &= \langle\tilde{\xi}_{[0,1]}\#\kappa_1\#\kappa_1, DK_{\varphi(e)}(S\tilde{B}\hat{\otimes}Cl_{0,1}\hat{\otimes}Cl_{1,0})\rangle \\ &= c_n\langle\tilde{\xi}\#\kappa_1, DK_{\varphi(e)}(\tilde{B}\hat{\otimes}Cl_{0,1})\rangle\end{aligned}$$

The ambiguity is thus taken care of by moding out  $\langle\tilde{\xi}\#\kappa_1, DK_{\varphi(e)}(\tilde{B}\hat{\otimes}Cl_{0,1})\rangle$ .

If we take another representative  $x'$  which is homotopic to  $x$  then we can prolong  $F$  in  $\mathfrak{S}(M_m(\varphi(B)))$  from  $x$  to  $x'$  and condition (V) insures that this does not change the value of  $\tilde{\xi}_{[0,1]}\#\text{Tr}_m(F - e_m, \dots, F - e_m)$ .

We apply the above to two particular cases, an even and an odd one. They naturally occur if we have a  $C^{*,r}$ -algebra.

**5.2. Even torsion valued pairing.** Let  $B$  be a real graded  $C^*$ -algebra and  $B_{\mathbb{C}} = B \otimes_{\mathbb{R}} \mathbb{C}$  its complexification. On  $B_{\mathbb{C}}$  we consider the real structure  $\mathfrak{c}$  given by complex conjugation on the second factor. Then, of course,  $B$  is the real subalgebra of  $(B_{\mathbb{C}}, \mathfrak{c})$ . Let

$$\tilde{B} = B\hat{\otimes}Cl_{0,1}$$

and  $j : B \hookrightarrow B\hat{\otimes}Cl_{0,1}$  be given by

$$j(b) = b\hat{\otimes}1.$$

**Theorem 5.2.** *Let  $B$  be a real graded  $C^*$ -algebra and  $(\Omega, d, \int_B)$  an  $n$ -dimensional  $(*, \mathfrak{r})$ -cycle over  $B_{\mathbb{C}}$  with character  $\xi$ . Let  $\tilde{\xi} = \xi\#\kappa_1$  be its extension to  $B\hat{\otimes}Cl_1$ . Suppose that  $B$  contains a basepoint  $e$  which is homotopic to its negative in  $\mathfrak{S}(B)$  and satisfies  $de = 0$ . Then  $\Delta_{\tilde{\xi}}^j : \ker j_* \cap DK_e(B) \rightarrow \mathbb{C}/\langle\xi\#\kappa_2, DK_e(B\hat{\otimes}Cl_{0,2})\rangle$*

$$\Delta_{\tilde{\xi}}^j([x]) \equiv c_n^{-1}\xi_{[0,1]}\#\kappa_1\#\text{Tr}_m(F - e_m\hat{\otimes}1, \dots, F - e_m\hat{\otimes}1)$$

where  $F$  is a OSU-homotopy in  $M_m(B)\hat{\otimes}Cl_{0,1}$  between  $e_m\hat{\otimes}1$  and  $x\hat{\otimes}1$ , is a well-defined homomorphism. Necessary conditions for  $\Delta_{\tilde{\xi}}^j$  to be non-trivial are that the dimension  $n$ , parity  $\nu$ , and sign  $\mathfrak{s}$  of  $(\Omega, d, \int)$  satisfy

$$(8) \quad \mathfrak{s} = (-1)^\nu \quad \text{and} \quad \nu = n + 1 \pmod{2}.$$

Moreover, under these conditions  $\langle\xi, DK_e(B)\rangle = 0$ .

*Proof.* We first show (V). Indeed,  $\kappa_1$  vanishes on the image of  $j$ . Therefore  $\tilde{\xi}$  vanishes on the image of  $j$  which implies (V).

By Lemma 4.13 the extension  $\tilde{\xi} = \xi \# \kappa_1$  has dimension  $n$ , parity  $\nu + 1$ , and sign  $(-1)^{\nu + \mu(-1)} \mathfrak{s}$  on  $B \hat{\otimes} Cl_{0,1}$ . We deduce from (6) that, if  $n$  has parity different from  $\nu + 1$  then the expression  $\tilde{\xi}_{[0,1]}(f, \dots, f)$  vanishes. It then follows from Cor. 4.16 (applied to  $\tilde{\xi}$ ) that if  $(-1)^\nu \mathfrak{s} = -1$  then  $\tilde{\xi}_{[0,1]}(f, \dots, f)$  vanishes.

Finally we conclude from Cor. 4.16 applied to  $\xi$  that under the above conditions  $\langle \xi, DK_e(B) \rangle = 0$ .  $\square$

**5.3. Odd torsion valued pairing.** Let again  $B$  be a real  $C^*$ -algebra but  $\tilde{B} = B_{\mathbb{C}}$  be its complexification. Viewed differently, we can start with a complex  $C^{*,r}$ -algebra  $(\tilde{B}, \mathfrak{r})$  and set  $B = \tilde{B}^{\mathfrak{r}}$ . Let  $c : B \rightarrow \tilde{B}$  be the complexification map  $c(b) = b$ .

**Theorem 5.3.** *Let  $B$  be a real graded  $C^*$ -algebra and  $(\Omega, d, f)$  an  $n$ -dimensional  $(*, \mathfrak{r})$ -cycle over  $B_{\mathbb{C}}$  with character  $\tilde{\xi}$ . Suppose that  $B$  contains a basepoint  $e$  which is homotopic to its negative in  $\mathfrak{S}(B)$  and satisfies  $de = 0$ . If the sign satisfies  $\mathfrak{s} = +1$ , then  $\Delta_{\tilde{\xi}}^c : \ker c_* \cap DK_e(B) \rightarrow \mathbb{C} / \langle \tilde{\xi} \# \kappa_1, DK_e(B_{\mathbb{C}} \hat{\otimes} Cl_1) \rangle$*

$$\Delta_{\tilde{\xi}}^c([x]) \equiv c_n^{-1} \tilde{\xi}_{[0,1]} \# \text{Tr}_m(F - e_m, \dots, F - e_m)$$

where  $F$  is an OSU-homotopy in  $M_m(B_{\mathbb{C}})$  between  $e_m$  and  $x$ , is a well-defined homomorphism. A necessary condition for  $\Delta_{\tilde{\xi}}^c$  to be non-trivial is that the dimension  $n$  and the parity  $\nu$  of the cycle satisfy

$$(9) \quad \nu = n \pmod{2}.$$

Moreover, under these conditions  $\langle \tilde{\xi}, DK(B) \rangle = 0$ .

*Proof.* By Cor. 4.16 the first condition  $\mathfrak{s} = +1$  implies that the vanishing condition (V) is satisfied for the inclusion  $B \xrightarrow{c} B_{\mathbb{C}}$ . Condition (9) is necessary for the non-vanishing of  $\tilde{\xi}_{[0,1]}(f, \dots, f)$  where  $f$  is an odd self adjoint differentiable function with values in  $B_{\mathbb{C}}$  ( $f$  is not necessarily real). It also implies that  $\langle \tilde{\xi}, DK_e(B) \rangle = 0$ .  $\square$

We discuss furtherdown that  $\ker c_* = \text{im } j_*$  and that  $2DK(B) \subset \ker j_*$ . Hence  $\ker c_*$  contains only 2-torsion elements.

**5.4. The torsion valued pairings for the algebra  $\mathbb{R}$ .** As a simple application we show that the torsion part of the  $K$ -theory of  $\mathbb{R}$  can be detected by the torsion valued pairings, notably the odd pairing is non-trivial on  $K_1(\mathbb{R})$  and the even pairing is non-trivial on  $K_2(\mathbb{R}) = \mathbb{Z}_2$ . For that we use the up to normalisation only non-trivial cycle  $(\Omega, d, f) = (\mathbb{C}, 0, \text{id})$  over  $\mathbb{C}$ , its character is  $\xi = \text{id}$ .

We have  $K_1(\mathbb{R}) = DK(M_2(\mathbb{R}) \otimes Cl_{1,1}) \cong \mathbb{Z}_2$  and  $e = \sigma_z \otimes \sigma_x$  is a basepoint which is homotopic to its negative. The generator of  $DK(M_2(\mathbb{R}) \otimes Cl_{1,1})$  is given by the class of  $x = 1_2 \otimes \sigma_x$ . Let  $\Sigma = 1_2 \otimes \sigma_y$ . Then the concatenation of the two

paths,  $F_1(t) = \cos(\frac{\pi}{2}t)e + \sin(\frac{\pi}{2}t)\Sigma$  with  $F_2(t) = \cos(\frac{\pi}{2}t)\Sigma + \sin(\frac{\pi}{2}t)x$ , provides a homotopy from  $e$  to  $x$  in  $M_2(\mathbb{C}) \otimes \mathbb{C}l_2$ . Thus  $[x]$  lies in the kernel of  $c_*$  and

$$\begin{aligned} \Delta_{\text{id}}^c([x]) &\equiv c_0^{-1} \int_0^1 \text{Tr}_2 \# \kappa_2 (F_1(t) \dot{F}_1(t) + F_2(t) \dot{F}_2(t)) dt \\ &= \frac{\kappa^{-1}}{2} \text{Tr}_2 \# \kappa_2 (\sigma_z \otimes \sigma_x \sigma_y + 1_2 \otimes \sigma_y \sigma_x) = -\kappa \end{aligned}$$

The subgroup to be divided out is given by the values of

$$\frac{\kappa^{-1}}{\pi} \int_0^1 \text{Tr}_2 \# \kappa_2 (F(t) \dot{F}(t)) dt$$

where  $F : [0, 1] \rightarrow M_2(\mathbb{C}) \otimes \mathbb{C}l_2$  is an osu-valued loop. In particular,  $F$  has the form

$$F(t) = \begin{pmatrix} 0 & Z(t) \\ Z^*(t) & 0 \end{pmatrix}$$

where  $Z(t) \in M_2(\mathbb{C})$  is a unitary loop. Then

$$\begin{aligned} \int_0^1 \text{Tr}_2 \# \kappa_2 (F(t) \dot{F}(t)) dt &= \int_0^1 \text{Tr}_2 \# \text{Tr}_2 \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} \begin{pmatrix} Z(t) \dot{Z}^*(t) & 0 \\ 0 & Z^*(t) \dot{Z}(t) \end{pmatrix} dt \\ &= 2 \int_0^1 \text{Tr}_2 Z(t) \dot{Z}^*(t) dt = -4\pi i \mathcal{W}(Z) \end{aligned}$$

where  $\mathcal{W}(Z)$  is the winding number of  $Z$ . Thus we mod out the group  $2\kappa\mathbb{Z}$  and the odd torsion valued pairing is injective.

We have  $K_2(\mathbb{R}) = DK(M_2(\mathbb{R}) \otimes M_2(\mathbb{R}) \otimes Cl_{0,1}) \cong \mathbb{Z}_2$  and take as basepoint  $e = \sigma_z \otimes \sigma_y \otimes \rho_1$  where  $\rho_1$  is the generator of  $Cl_1$  (hence  $\sigma_y \otimes \rho_1 \in M_2(\mathbb{R}) \otimes Cl_{0,1}$ ). Again  $e$  is homotopic to its negative. A generator of  $K_2(\mathbb{R})$  is given by the class of  $x = 1_2 \otimes \sigma_y \otimes \rho_1$ . Let  $\Sigma = 1_2 \otimes \sigma_y \otimes \rho_2$  where  $\rho_2$  is the second generator of  $Cl_2$ . Hence  $\Sigma \in M_2(\mathbb{R}) \otimes M_2(\mathbb{R}) \otimes Cl_{0,2}$ . Again the concatenation of  $F_1(t) = \cos(\frac{\pi}{2}t)e + \sin(\frac{\pi}{2}t)\Sigma$  with  $F_2(t) = \cos(\frac{\pi}{2}t)\Sigma + \sin(\frac{\pi}{2}t)x$  yields a homotopy from  $e$  to  $x$  in  $M_2(\mathbb{R}) \otimes M_2(\mathbb{R}) \otimes Cl_{0,2}$ . Thus  $[x] \in \ker j_*$  and, similar to the above,

$$\begin{aligned} \Delta_{\text{id}}^j([x]) &\equiv c_0^{-1} \int_0^1 \text{Tr}_2 \# \kappa_2 (F_1(t) \dot{F}_1(t) + F_2(t) \dot{F}_2(t)) dt \\ &\equiv \frac{\kappa^{-1}}{2} \text{Tr}_4 \# \kappa_2 (\sigma_z - 1) \otimes 1_2 \otimes \rho_1 \rho_2 = 2\kappa \end{aligned}$$

The subgroup to be divided out is generated by the values of  $\frac{\kappa^{-1}}{\pi} \int_0^1 \text{Tr}_2 \# \kappa_2 (F(t) \dot{F}(t)) dt$  where  $F : [0, 1] \rightarrow M_2(\mathbb{R}) \otimes Cl_{0,2}$  is an osu-value loop which also in this case must

have the form  $F(t) = \begin{pmatrix} 0 & Z(t) \\ Z^*(t) & 0 \end{pmatrix}$  so that these values are given by  $-4\pi i \mathcal{W}(Z)$ .

But now the reality condition  $F(t) \in M_2(\mathbb{R}) \otimes Cl_{0,2}$  implies that  $Z(t)$  has to be a

unitary in  $M_2(\mathbb{C})$  which satisfies

$$\overline{Z(t)} = -Z^*(t)$$

(entrywise complex conjugation). In particular, the spectrum of  $Z$  is invariant under multiplication with  $-1$ . Thus its winding number must be even. It follows that we mod out the group  $4\kappa\mathbb{Z}$  and the even torsion valued pairing is injective as well.

**5.5. Definition of Property  $Y$ .** The simplest way to construct an OSU-valued homotopy between two OSUs  $x$  and  $x'$  is to find a third OSU  $y$  which anti-commutes with  $x$  and with  $x'$ . Indeed, with  $c(t) = \cos(\frac{\pi}{2}t)$  and  $s(t) = \sin(\frac{\pi}{2}t)$ , the path  $F_{x,y}(t) := c(t)x + s(t)y$  is an OSU-homotopy between  $F_{x,y}(0) = x$  and  $F_{x,y}(1) = y$  and so its concatenation with  $F_{y,x'}$  provides a homotopy between  $x$  and  $x'$ . This will be the basis of the explicit formulae we develop below for the even and odd torsion valued pairing where, for abstract  $K$ -theoretic reasons, such an extra OSU  $y$  always exist. Moreover, in the context of insulators  $y$  can be interpreted as an extra symmetry.

**Definition 5.4.** *Let  $(A, \mathfrak{r})$  be a graded  $C^{*,r}$ -algebra. We say that an element  $x \in \mathfrak{S}(A^\mathfrak{r})$  satisfies property  $Y^{(\pm)}$ , if  $iA^\mathfrak{r}$  contains a self-adjoint unitary  $\Sigma^{(\pm)}$  of degree  $\pm$  such that*

$$x\Sigma^{(\pm)} = \pm\Sigma^{(\pm)}x.$$

*If  $e$  is a base point in  $A^\mathfrak{r}$  such that also  $e\Sigma^{(\pm)} = \pm\Sigma^{(\pm)}e$  we say that  $x$  satisfies  $Y_e^{(\pm)}$ .*

$\Sigma^{(+)}$  is an even imaginary self-adjoint unitary which commutes with  $x$ .  $\Sigma^{(-)}$  is an odd imaginary self-adjoint unitary which anti-commutes with  $x$ . We call  $\Sigma^{(\pm)}$  the generator of an extra symmetry of  $x$ . We emphasize that  $\Sigma^{(\pm)}$  will not commute or anti-commute with all elements of the homotopy class of  $x$ , an extra symmetry is thus not a protecting symmetry of the topological phase.

**Lemma 5.5.** *Let  $(A, \mathfrak{r})$  be a balanced graded  $C^{*,r}$ -algebra with base point  $e$ .*

- *If  $x \in \mathfrak{S}(A^\mathfrak{r})$  satisfies property  $Y_e^{(+)}$ , then  $x \otimes 1$  is homotopic to  $e \otimes 1$  in  $\mathfrak{S}(A^\mathfrak{r} \hat{\otimes} Cl_{0,1})$  and hence  $j_*([x]) = 0$ .*
- *If  $x \in \mathfrak{S}(A^\mathfrak{r})$  satisfies property  $Y_e^{(-)}$ , then  $x$  is homotopic to  $e$  in  $\mathfrak{S}(A)$  and hence  $c_*([x]) = 0$ .*

*Proof.*  $\Sigma^{(+)} \otimes \rho$  belongs to  $\mathfrak{S}(A^\mathfrak{r} \hat{\otimes} Cl_{0,1})$  and anticommutes with  $x \otimes 1$  and  $e \otimes 1$ . By the standard argument this means that  $\Sigma^{(+)} \otimes \rho$  is homotopic to  $x \otimes 1$  and  $e \otimes 1$  in  $\mathfrak{S}(A^\mathfrak{r} \hat{\otimes} Cl_{0,1})$ .

$\Sigma^{(-)}$  belongs to  $\mathfrak{S}(A)$  and anticommutes with  $x$  and  $y$ . By the standard argument this means that it is homotopic to  $x$  and  $e$  in  $\mathfrak{S}(A)$ .  $\square$

**Corollary 5.6.** *The kernel  $\ker j_*$  contains  $2DK(A^\mathfrak{r})$ . In particular  $\text{im } j_*$  contains only 2-torsion elements.*

*Proof.* Let  $[x] \in DK(A^\natural)$ . Then  $2[x]$  has a representative of the form  $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$  which commutes with the imaginary self adjoint unitary  $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$  and hence satisfies  $Y_{e_2}^{(+)}$ .  $\square$

**5.6. The exact sequence relating  $j_*$  and  $c_*$ .** Let  $B$  be a graded real  $C^*$ -algebra and  $r : B_{\mathbb{C}} = B + iB \rightarrow M_2(B)$  be given by

$$r(a + ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

$r$  is a unital homomorphism of graded real  $C^*$ -algebras referred to as realification (or forgetting the complex structure). It hence induces a homomorphism  $r_*$  on  $K$ -theory. Composed with the Bott isomorphism  $b = \beta^2$  and together with  $j_*$  and  $c_*$  it forms an exact sequence relating the  $K$ -theory of  $B$  to that of its complexification  $B_{\mathbb{C}}$ .

**Theorem 5.7.** *Let  $B$  be a graded real  $C^*$ -algebra. The following sequence is exact.*

$$\cdots \rightarrow K_{i+2}(B_{\mathbb{C}}) \xrightarrow{r_* \circ b^{-1}} K_i(B) \xrightarrow{j_*} K_{i+1}(B) \xrightarrow{c_*} K_{i+1}(B_{\mathbb{C}}) \rightarrow \cdots$$

This theorem, attributed to Wood and Karoubi in [39] (in the commutative case), has been proven for trivially graded  $C^*$ -algebras in [5, 6] and generalised to equivariant  $KK$ -theory in [40] where however the map in the middle is given by Kasparov multiplication with the generator of  $K_1(\mathbb{R})$ . Guerin provides a proof for graded  $C^*$ -algebras in which he identifies the map in the middle with  $j_*$  [21].

Recall that  $K_i(A)$  is defined as  $DK_e(A \hat{\otimes} Cl_{r,s})$  where  $r - s = 1 - i$ . We can always choose  $r$  large enough so that  $A \hat{\otimes} Cl_{r,s}$  contains an OSU  $e$  which is homotopic to its negative. Recall that a differential  $d$  of a cycle over  $A$  extends to a differential on  $M_m(A) \hat{\otimes} Cl_{r,s}$  using the identity on the right factor and entrywise extension to matrices. In the same way the maps  $r$ ,  $j$ , and  $c$  extend to  $M_m(A) \hat{\otimes} Cl_{r,s}$ . Note that  $d$  commutes with  $r$ ,  $j$ , and  $c$ .

**Proposition 5.8.** *Let  $A$  be a graded real  $C^*$ -algebra. Suppose that  $A \hat{\otimes} Cl_{r,s}$  contains a basepoint  $e$  which is homotopic to its negative. Let  $d$  be a differential of a cycle over  $A$ .*

- (1) *Any element of  $\ker j_* \cap DK_e(A \hat{\otimes} Cl_{r,s})$  admits a representative which satisfies property  $Y_e^{(+)}$ . Moreover, the corresponding generator  $\Sigma^{(+)}$  can be chosen to satisfy  $d\Sigma^{(+)} = 0$ .*
- (2) *Any element of  $\ker c_* \cap DK_e(A \hat{\otimes} Cl_{r,s})$  admits a representative which satisfies property  $Y_e^{(-)}$ . The corresponding generator  $\Sigma^{(-)}$  can be chosen to satisfy  $d\Sigma^{(-)} = 0$ . If moreover  $e = e' \hat{\otimes} 1$  for some  $e' \in A \hat{\otimes} Cl_{r,s-1}$  then the representative satisfies property  $Y_e^{(-)}$ .*

*Proof.* By Theorem 5.7  $\ker j_* = \text{im} r_*$ . Since  $e$  is homotopic to its negative any element of  $\ker j_*$  is represented by an OSU  $r(z)$ , for some OSU  $z$  in  $M_m(A_{\mathbb{C}}) \hat{\otimes} Cl_{r,s}$ . Furthermore, viewing  $e$  as an element of  $A_{\mathbb{C}}$  we have  $r(e) = e_2$ . Clearly  $z$  and  $e_m$  commute with  $i1_m \otimes 1 \in M_m(A_{\mathbb{C}}) \hat{\otimes} Cl_{r,s}$ , where  $1_m$  is the unit matrix in  $M_m(\mathbb{C})$ . Thus  $\Sigma^{(+)} := ir(i1_m)$  is an imaginary self-adjoint unitary which commutes with  $r(z)$  and  $r(e_m) = e_{2m}$ . Furthermore,  $d\Sigma^{(+)} = ir(id1_m) = 0$ .

By Theorem 5.7  $\ker c_* = \text{im} j_*$ . Therefore any element of  $\ker c_*$  is represented by an OSU  $j(z) = z \hat{\otimes} 1$ , for some OSU  $z$  in  $M_m(A) \hat{\otimes} Cl_{r,s-1}$ . Clearly  $z \hat{\otimes} 1$  anti-commutes with the imaginary odd self-adjoint unitary  $\Sigma^{(-)} := i1_m \hat{\otimes} \rho$  ( $\rho$  is the generator of  $\mathbb{C}l_1$ ) and so satisfies property  $Y^{(-)}$ . We have  $d\Sigma^{(-)} := id1_m \hat{\otimes} \rho = 0$ . If  $e_m = e'_m \hat{\otimes} 1$  then also  $e_m$  commutes with  $i1_m \hat{\otimes} \rho$ .  $\square$

**5.7. Formulae for torsion valued pairing.** We derive now formulae for the torsion valued pairings related to the character  $\tilde{\xi}$  of a cycle  $(\Omega, d, f)$  over a unital  $C^*$ -algebra  $\tilde{B}$ .

Let  $F : [0, 1] \rightarrow \mathfrak{G}(\tilde{B})$  be a continuous path from  $F(0) = E$  to  $F(1) = X$  where we suppose that  $dE = 0$ . We want to compute

$$\tilde{\xi}_{[0,1]}(F - E, \dots, F - E) = \int_{C([0,1], \tilde{B})} (F - E)(d^s F)^{n+1}.$$

The integral can be split into two summands, because  $\tilde{\xi}$  is densely defined on  $\tilde{B}$ . Using  $FdF = -(dF)F$  we obtain for the first summand

$$\int_{C([0,1], \tilde{B})} F(d^s F)^{n+1} = (n+1) \int_0^1 \int_{\tilde{B}} F \dot{F} (dF)^n dt$$

where  $\dot{F}$  is derivative w.r.t.  $t$ . Using  $dE = 0$  the second summand is the boundary term

(10)

$$\int_{C([0,1], \tilde{B})} E(d^s F)^{n+1} = \int_{C([0,1], \tilde{B})} d^s(EF(d^s F)^n) = \int_{\tilde{B}} EF(dF)^n \Big|_0^1 = \int_{\tilde{B}} EX(dX)^n.$$

Suppose that  $F$  has the form  $F_{x,y}(t) = c(t)x + s(t)y$  for two anti-commuting OSUs. Then  $F_{x,y} \dot{F}_{x,y} = \frac{\pi}{2}yx$  and

$$\begin{aligned} \int_{C([0,1], \tilde{B})} F_{x,y}(\tilde{d}F_{x,y})^{n+1} &= \frac{(n+1)\pi}{2} \int_0^1 \int yx(c(t)dx + s(t)dy)^n dt \\ &= \frac{(n+1)\pi}{2} \sum_{k=0}^n \alpha_{k,n} \int yx P_k(dx, dy) \end{aligned}$$

where

$$P_k(a, b) = \sum_{\pi \in S_{n,k}} c_1^\pi \cdots c_n^\pi, \quad c_i^\pi = \begin{cases} a & \text{if } \pi^{-1}(i) \leq k \\ b & \text{if } \pi^{-1}(i) > k \end{cases}$$

$S_{n,k}$  being the subgroup of permutations of  $n$  elements which preserve the order of the first  $k$  and of the last  $n - k$  elements. ( $P_k(dx, dy)$  is the sum over all products of  $k$  factors  $dx$  with  $n - k$  factors  $dy$  in all possible orders) and

$$\alpha_{k,n} = \int_0^1 c^k(t) s^{n-k}(t) dt$$

We have  $\alpha_{k,n} = \alpha_{n-k,n}$  and  $\alpha_{k,n} = \alpha_{k-2,n-2} - \alpha_{k-2,n}$ .

Suppose now that  $E$  and  $X$  anticommute with an OSU  $\Sigma \in \tilde{B}$ . Then we can take the concatenation of  $F_1 := F_{E,\Sigma}$  with  $F_2 := F_{\Sigma,X}$  for  $F$  to compute the first above summand,

$$c_n^{-1} \int_{C([0,1], \tilde{B})} F(d^s F)^{n+1} = \frac{\kappa^{-1}}{2} \left( \sum_{k=0}^n \beta_{k,n} \int_B X \Sigma P_k(dX, d\Sigma) + \int_B \Sigma E (d\Sigma)^n \right)$$

where  $\beta_{k,n} = \frac{\alpha_{k,n}}{\alpha_{n,n}}$ .

We apply this to the even and the odd torsion valued pairing.

5.7.1. *Even torsion valued pairing.* Recall that the even torsion valued pairing is defined for the inclusion  $B \xrightarrow{j} B \hat{\otimes} Cl_{0,1}$  for a real graded  $C^*$ -algebra  $B$ . Here  $\xi$  is the character of an  $n$ -dimensional cycle over  $B$  and  $\tilde{\xi} = \xi \# \kappa_1$  its extension to  $\tilde{B} = B \hat{\otimes} Cl_{0,1}$ . To evaluate the pairing on a class  $[x] \in DK(B)$  we choose a representative  $x$  which satisfies property  $Y_e^{(+)}$  with generator  $\Sigma^{(+)}$  and set

$$E = e \hat{\otimes} 1, \quad \Sigma = \Sigma^{(+)} \hat{\otimes} \rho, \quad X = x \hat{\otimes} 1.$$

The boundary term  $\frac{1}{2} \int EX(dX)^n$  vanishes as  $\int \circ \kappa_1$  vanishes when evaluated on elements of the image of  $j$ . We have  $X d\Sigma P_k(dX, d\Sigma) = x d\Sigma^{(+)} \tilde{P}_k(dx, d\Sigma^{(+)}) \hat{\otimes} \rho^{n-k+1}$  where

$$\tilde{P}_k(a, b) = (-1)^k \sum_{\pi \in S_{n,k}} \text{sign}(\pi) c_1^\pi \cdots c_n^\pi, \quad c_i^\pi = \begin{cases} a & \text{if } \pi^{-1}(i) \leq k \\ b & \text{if } \pi^{-1}(i) > k \end{cases}$$

Now

$$\kappa_1(X \Sigma P_k(dX, d\Sigma)) = \begin{cases} \kappa x \Sigma^{(+)} \tilde{P}_k(dx, d\Sigma^{(+)}) & \text{if } n - k \text{ is even} \\ 0 & \text{otherwise} \end{cases}.$$

from which we deduce that, if  $e$  is homotopic to its negative and  $de = 0$ ,

$$\Delta_{\xi \# \kappa_1}^j : \ker j_* \cap DK(B) \rightarrow \mathbb{C} / \langle \xi \# \kappa_2, DK(B \hat{\otimes} Cl_{0,2}) \rangle$$

is given by

(11)

$$\Delta_{\xi \# \kappa_1}^j([x]) \equiv \frac{1}{2} \left( \sum_{\substack{k=0 \\ n-k \text{ even}}}^n \beta_{k,n} \int_B x \Sigma^{(+)} \tilde{P}_k(dx, d\Sigma^{(+)}) - \delta_n^{ev} \int_B e \Sigma^{(+)} (d\Sigma^{(+)})^n \right)$$



where  $\delta_n^{ev} = 1$  if  $n$  is even and 0 otherwise. Note that the above expression changes sign if one replaces  $\Sigma^{(+)}$  by  $-\Sigma^{(+)}$ . Hence  $2\Delta_\xi^j([x]) \equiv 0$  showing that  $\text{im}\Delta_\xi^j$  contains only 2-torsion elements. If  $d\Sigma^{(+)} = 0$  the formula simplifies enormously,

$$(12) \quad \Delta_{\xi\#\kappa_1}^j([x]) \equiv \frac{1}{2} \int_B \Sigma^{(+)} (x(dx)^n - \delta_{n0}e).$$

The group which is quotiented out is  $\langle \xi\#\kappa_2, DK(B\hat{\otimes}Cl_{0,2}) \rangle = \langle \xi, K_2(B) \rangle$ , is it generated by the elements of the form

$$\int_B (x'(dx')^n - \delta_{n0}e)$$

where  $x'$  is an osu in  $B\hat{\otimes}Cl_{0,2}$ .

**5.7.2. Odd torsion valued pairing.** We come back to the situation where the inclusion map is given by the complexification,  $\varphi = c$ ,  $\tilde{B}$  the complexification of a real graded  $C^*$ -algebra  $B$ . Now  $\xi$  is simply  $\xi$ , the character of an  $n$ -dimensional cycle over  $B_{\mathbb{C}}$  whose sign is  $\mathfrak{s} = 1$ . We choose a representative  $x$  which satisfies property  $Y_e^{(-)}$  with generator  $\Sigma^{(-)}$  and set

$$E = e, \quad \Sigma = \Sigma^{(-)}, \quad X = x.$$

Note that  $P_k(dx, d\Sigma^{(-)})$  is self-adjoint and  $\tilde{\mathfrak{t}}(P_k(dx, d\Sigma^{(-)})) = (-1)^{n-k} P_k(dx, d\Sigma^{(-)})$ . Hence

$$\begin{aligned} \int \tilde{\mathfrak{t}}(x\Sigma^{(-)} P_k(dx, d\Sigma^{(-)}))^* &= (-1)^{n-k+1} \int (x\Sigma^{(-)} P_k(dx, d\Sigma^{(-)}))^* \\ &= (-1)^{n-k} \int x\Sigma^{(-)} P_k(dx, d\Sigma^{(-)}) \end{aligned}$$

from which we conclude that the expression vanishes if  $\mathfrak{s} \neq (-1)^{n-k}$ , that is,  $n-k$  is odd. It follows that  $\Delta_\xi^c : \ker c_* \cap DK(B) \rightarrow \mathbb{C}/\langle \xi\#\kappa_1, DK(B\hat{\otimes}Cl_1) \rangle$  is given by

$$(13) \quad \Delta_\xi^c([x]) \equiv \frac{\kappa^{-1}}{2} \left( \sum_{\substack{k=0 \\ n-k \text{ even}}}^n \beta_{k,n} \int_B x\Sigma^{(-)} P_k(dx, d\Sigma^{(-)}) - \delta_n^{ev} \int_B e\Sigma^{(-)} (d\Sigma^{(-)})^n \right)$$

Indeed, the boundary term  $\frac{1}{2} \int ex(dx)^n$  vanishes as the sign is  $+1$  (see Cor. 4.16). As above we see that  $2\Delta_\xi^c([x]) \equiv 0$ , that is, the elements of  $\text{im}\Delta_\xi^c$  are 2-torsion. If  $d\Sigma^{(-)} = 0$  the formula simplifies enormously,

$$(14) \quad \Delta_\xi^c([x]) \equiv \frac{\kappa^{-1}}{2} \int_B (x\Sigma^{(-)}(dx)^n - \delta_{n0}e).$$

The group which is quotiented out is  $\langle \xi \# \kappa_1, DK(B_{\mathbb{C}} \hat{\otimes} Cl_1) \rangle = \langle \xi, K_0(B_{\mathbb{C}}) \rangle$ , is it generated by the elements of the form

$$\int_{B_{\mathbb{C}}} \kappa_1 (x' (dx')^n - \delta_{n0} e)$$

where  $x'$  is an osu in  $B_{\mathbb{C}} \hat{\otimes} Cl_1$ .

## 6. EXPLICIT CALCULATIONS FOR A SIMPLE CLASS OF PERIODIC TIGHT BINDING MODELS

We consider here a class of periodic models for which the pairings can be computed explicitly. This allows us also to demonstrate that our formulae reproduce the known formulae in the literature for these models and hence the Kane-Mele and the Fu-Kane-Mele invariant.

After a Bloch transformation a periodic  $d$ -dimensional tight binding model (without external magnetic field) is described by a self-adjoint Hamiltonian on  $L^2(\mathbb{T}^d, \mathbb{C}^N)$ . Here  $\mathbb{T}^d$  is the Brillouin zone and  $\mathbb{C}^N$  an internal Hilbert space for the degrees of freedom at the lattice sites (including spin). We consider here Hamiltonians of the form

$$(15) \quad h = \sum_{i=0}^d \gamma_i h_i$$

where  $\gamma_i \in M_N(\mathbb{C})$ ,  $i = 0, \dots, d$  are pairwise anticommuting self-adjoint unitary matrices, the  $h_i \in C(\mathbb{T}^d, \mathbb{R})$  act as multiplication operators by

$$h_i(k) = \sin(k_i) \text{ for } i > 0, \quad h_0(k) = m(k),$$

and  $m \in C(\mathbb{T}^d, \mathbb{R})$  is an even real function of class  $C^1$ . The spectrum of such an operator is rather simple as  $h^2 = \sum_{i=0}^d h_i^2 1_N$ . Hence  $h$  is invertible if and only if  $m(k) \neq 0$  for all  $k \in \text{TRI} := \{k \in \mathbb{T}^d \mid \forall i : k_i \in \{0, \pi\}\}$ . Under this condition  $h$  defines a topological phase w.r.t. a observable algebra  $A$ . The choice of  $A$  has to be made on physical grounds and we adopt here the point of view that the topological phase is protected by the lattice symmetry which means that deformation is only allowed in the algebra of periodic operators. The observable algebra is therefore

$$(16) \quad A = C(\mathbb{T}^d, M_N(\mathbb{C})).$$

It will be fruitful to consider the matrices  $\gamma_i$  as the images of the generators  $\rho_i$  of the (complex) Clifford algebra of  $d+1$  generators in a representation on  $\mathbb{C}^N$ , i.e.  $\gamma_i = \varphi(\rho_i)$  for some algebra morphism  $\varphi : Cl_{d+1} \rightarrow M_N(\mathbb{C})$ .

A protecting symmetry playing the role of chiral symmetry can be introduced by specifying a grading on  $M_N(\mathbb{C})$  which then extends pointwise to  $A$ . For the Hamiltonians of the form (15) we choose the grading in such a way that the  $\gamma_i$  are odd. In other words the representation morphism  $\varphi$  becomes a morphism of graded algebras.

When further protected by a real symmetry and/or a chiral symmetry the above models are simple examples of insulators with topologically non-trivial phases without external magnetic field. They have been studied for instance in [19, 37, 34, 35], the first two in a field theory context. But also from the purely mathematical perspective the above models are interesting, as with a particular choice for  $m$  they correspond to the Bott element on  $\mathbb{T}^d$ . The Bott element on the sphere  $S^d = \{y \in \mathbb{R}^{d+1} : \sum_{i=0}^d y_i^2 = 1\}$  is the van Daele class of the OSU  $b_{S^d} \in C(S^d, \mathbb{C}l_{d+1})$ ,

$$b_{S^d}(y) = \sum_{i=0}^d \rho_i y_i$$

where  $\rho_i$  are the self-adjoint generators of the graded Clifford algebra  $\mathbb{C}l_{d+1}$ . It is known to be the generator of  $DK(C_0(\mathbb{R}^d, \mathbb{C}l_{d+1}))$ , which is  $KU_0(C_0(\mathbb{R}^d))$  for even and  $KU_1(C_0(\mathbb{R}^d))$  for odd  $d$ . The Bott element on  $\mathbb{T}^d$  is the van Daele class of the pull-back of  $b_{S^d}$  under the composition  $f = f_2 \circ f_1$  of maps  $\mathbb{T}^d = ([-\pi, \pi]/\sim)^d \xrightarrow{f_1} \mathbb{R}^d \cup \{\infty\} \xrightarrow{f_2} S^d \subset \mathbb{R}^{d+1}$ , where  $f_1$  stretches the fundamental domain  $[-\pi, \pi]^d$  and identifies the  $d - 1$ -skeleton of the torus with the point at infinity  $\{\infty\}$ ,

$$f_1(k_1, \dots, k_d) = \left( \tan \frac{k_1}{2}, \dots, \tan \frac{k_d}{2} \right)$$

and  $f_2$  is a version of the inverse stereographic projection,

$$f_2(x_1, \dots, x_d) = \left( \frac{1 - x^2}{1 + x^2}, \frac{2x_1}{1 + x^2}, \dots, \frac{2x_d}{1 + x^2} \right).$$

The pull back of  $b_{S^d}$  is by definition the OSU  $f^*(b_{S^d}) = \sum_{i=0}^d \rho_i f_i$ , and, for  $i \geq 1$ ,

$$\begin{aligned} f_i(k_1, \dots, k_d) &= \frac{A_i(k)}{(1-d)A(k) + \sum_{j=1}^d A_j(k)} \sin k_i \\ f_0(k_1, \dots, k_d) &= \frac{(1+d)A(k) - \sum_{j=1}^d A_j(k)}{(1-d)A(k) + \sum_{j=1}^d A_j(k)} \end{aligned}$$

where  $A_i = \sum_{j \neq i} \cos^2 \frac{k_j}{2}$  and  $A = \sum_{j=1}^d \cos^2 \frac{k_j}{2}$ . For  $i \geq 1$  the expression  $f_i(k)$  vanishes only at  $k = 0$  and on the boundary  $\partial[-\pi, \pi]^d$ . On the other hand  $f_0(0) = 1$  while on the boundary  $f_0(k)$  is  $-1$ . Note that  $1 - d + \sum_{i=1}^d \cos k_i$  takes the value 1 at  $k = 0$  and is strictly negative at all other points of TRI. We can therefore deform the factor  $\frac{A_i(k)}{(1-d)A(k) + \sum_{j=1}^d A_j(k)}$  in  $f_i(k)$  along a straight line homotopy to 1 and then  $\frac{(1+d)A(k) - \sum_{j=1}^d A_j(k)}{(1-d)A(k) + \sum_{j=1}^d A_j(k)}$  along a straight line homotopy to  $1 - d + \sum_{i=1}^d \cos k_i$  without closing the gap in the spectrum of  $f^*(b_{S^d})$ . Hence  $f^*(b_{S^d})$  is homotopic in

the set of invertible odd self-adjoint elements of  $C(\mathbb{T}^d, \mathbb{C}l_{d+1})$  to

$$(17) \quad b_{\mathbb{T}^d}(k) := \rho_0 \left( 1 - d + \sum_{i=1}^d \cos k_i \right) + \sum_{i=1}^d \rho_i \sin k_i.$$

The van Daele class of its spectral flattening  $\hat{b}_{\mathbb{T}^d} = b_{\mathbb{T}^d} |b_{\mathbb{T}^d}|^{-1}$  is the Bott element on the torus  $\mathbb{T}^d$ .

We associate to  $b_{\mathbb{T}^d}$  a Hamiltonian of the type (15) by considering two types of representations for  $\mathbb{C}l_{d+1}$ . If  $d$  is odd then we consider the Clifford algebra as graded (the grading is given by the grading operator  $\Gamma_{d+1}$ ) and define  $h_{Bott}^{(d)} = \varphi_{odd}(b_{\mathbb{T}^d})$  using the bijective representation  $\varphi_{odd} : \mathbb{C}l_{d+1} \rightarrow M_N(\mathbb{C})$  where  $N = 2^{\frac{d+1}{2}}$ . Interpreting the grading operator as the generator of a chiral symmetry  $h_{Bott}^{(d)}$  is thus anti-invariant under chiral symmetry. Its topological phase is classified by its class in  $DK(C(\mathbb{T}^d) \otimes \mathbb{C}l_{d+1})$  [26].

If  $d$  is even then we use the bijective representation  $\mathbb{C}l_d^u \rightarrow M_N(\mathbb{C})$  of the ungraded Clifford algebra  $\mathbb{C}l_d^u$  with  $N = 2^{\frac{d}{2}}$  to obtain an ungraded representation  $\varphi_{ev}$  of  $\mathbb{C}l_{d+1}$  on  $\mathbb{C}^N$ , namely  $\varphi_{ev}(\rho_i) = \gamma_i$  where  $\gamma_1, \dots, \gamma_d$  are the representatives of the generators of  $\mathbb{C}l_d^u$  and

$$\gamma_0 = (-i)^{\frac{d}{2}} \gamma_1 \cdots \gamma_d.$$

Again we define  $h_{Bott}^{(d)} = \varphi_{ev}(b_{\mathbb{T}^d})$  in this representation, but this time it has no chiral symmetry. In both cases

$$h_{Bott}^{(d)} = \gamma_0 \left( 1 - d + \sum_{i=1}^d \cos k_i \right) + \sum_{i=1}^d \gamma_i \sin k_i.$$

We call  $h_{Bott}^{(d)}$  the Bott Hamiltonian. It can be interpreted as a tight binding operator with nearest neighbor interaction. For odd  $d$  it has chiral symmetry. These models have been studied in a more general context (with external magnetic field and contracting disorder) in [34], for  $d = 2$  the model is also referred to as the half BHZ model or the Qui-Wu-Zhang-model<sup>2</sup> [3].

Note that the above can also be formulated as follows: If  $d$  is odd then  $\mathbb{C}l_{d+1}$  is isomorphic to  $M_{2^{\frac{d+1}{2}}}(\mathbb{C})$  with standard even grading and under the isomorphism  $b_{\mathbb{T}^d}$  becomes  $h_{Bott}^{(d)}$ . If  $d$  is even then

$$\rho_i \mapsto \gamma_i \otimes \rho, \quad \text{with } \gamma_0 = (-i)^{\frac{d}{2}} \gamma_1 \cdots \gamma_d$$

defines a graded isomorphism between  $\mathbb{C}l_{d+1}$  and the tensor product  $\mathbb{C}l_d^u \otimes \mathbb{C}l_1$  and  $b_{\mathbb{T}^d}$  becomes  $h_{Bott}^{(d)} \otimes \rho$ .

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<sup>2</sup>at a particular value of its parameter  $u$

**6.1. Real protecting symmetries.** A real protecting symmetry playing the role of TRS or PHS can be introduced as follows. Choose a real structure  $\mathfrak{r}'$  on  $M_N(\mathbb{C})$  and then define the real structure on  $A = C(\mathbb{T}^d, M_N(\mathbb{C}))$  through

$$\mathfrak{r}(a)(k) := \mathfrak{r}'(a)(-k),$$

for  $a \in A$ .  $h$  has TRS if  $\mathfrak{r}(h) = h$  and PHS if  $\mathfrak{r}(h) = -h$ . Up to conjugacy, and if  $N = 2K$  is even, there are only two distinct choices of real structures  $\mathfrak{r}'$  on  $M_N(\mathbb{C})$  which lead to a simple real subalgebra. These are, entrywise complex conjugation which we denote  $\mathfrak{c}$ , and  $\mathfrak{h} = \text{Ad}_{\sigma_2 \otimes \text{id}_K} \circ \mathfrak{c}$ . In the first case the real subalgebra is  $M_N(\mathbb{R})$  and the second it is  $M_K(\mathbb{H})$  where  $\mathbb{H}$  is the algebra of quaternions. The algebra  $M_N(\mathbb{R})$  is Morita equivalent to  $\mathbb{R}$  and we call the algebra  $A^\mathfrak{r}$  of real type and the symmetry even<sup>3</sup>. Likewise  $M_K(\mathbb{H})$  is Morita equivalent to  $\mathbb{H}$  and we say that  $A^\mathfrak{r}$  is of quaternionic type and the symmetry odd.

For Hamiltonians of the type (15) we will more specifically define (or constrain) the real structure by means of a real structure on  $\mathbb{C}l_{d+1}$  demanding that the representation preserves the real structure. There are two useful real structures for this purpose. The first one declares  $\rho_0$  to be real and all  $\rho_i$  with  $1 \leq i \leq d$  to be imaginary. We denote this real structure by  $\mathfrak{l}_{1,d}$ . Hence  $\gamma_0$  is real and all  $\gamma_i$ ,  $i \geq 1$  imaginary and  $h$  is invariant under this real structure ( $h$  has TRS). The second option is to declare  $\rho_0$  to be imaginary and all  $\rho_i$  with  $1 \leq i \leq d$  to be real. This real structure is denoted  $\mathfrak{l}_{d,1}$ . Then  $h$  is anti-invariant (it has PHS). It should be noted that these prescriptions do not fix the real structure on  $M_N(\mathbb{C})$  in case the representation is not surjective. We now analyse these options for the two representations used so far.

If  $d$  is odd the graded representation  $\varphi_{\text{odd}} : \mathbb{C}l_{d+1} \rightarrow M_N(\mathbb{C})$ ,  $N = 2^{\frac{d+1}{2}}$  is bijective. Therefore the real structure on  $M_N(\mathbb{C})$  is uniquely determined by the real structure on  $\mathbb{C}l_{d+1}$ , and the real subalgebra  $M_N(\mathbb{C})^\mathfrak{r}$  is simply isomorphic to  $Cl_{1,d}$ , in the TRS case, or to  $Cl_{d,1}$  in the PHS case. Since  $\mathfrak{l}_{d,1} \circ \mathfrak{l}_{1,d}$  is the grading automorphism, the two structures are not independent and for the classification of the topological phases with two protecting symmetries it suffices to take into account only one real structure, let's say  $\mathfrak{l}_{1,d}$  (TRS), and whether the grading operator  $\Gamma_{d+1}$  is real or imaginary under  $\mathfrak{l}_{1,d}$  [26]. Clearly  $\Gamma_{d+1}$  is real if  $d + \mu(d+1)$  is even, which is the case for  $d = 1 \pmod{4}$ . Likewise,  $\Gamma_{d+1}$  is imaginary if  $d = 3 \pmod{4}$ . The real subalgebras  $M_N(\mathbb{C})^\mathfrak{r}$  are,  $Cl_{1,1} \cong M_2(\mathbb{R})$ ,  $Cl_{1,3} \cong M_2(\mathbb{H})$ ,  $Cl_{1,5} \cong M_4(\mathbb{H})$ ,  $Cl_{1,7} \cong M_{16}(\mathbb{R})$ . From this we can conclude that the Bott-Hamiltonian in  $d = 1$  has even TRS with real chiral symmetry, in  $d = 3$  odd TRS with imaginary chiral symmetry, in  $d = 5$  odd TRS with even chiral symmetry, and in  $d = 7$  even TRS with imaginary chiral symmetry.

In the case that  $d$  is even we have considered the representation  $\varphi_{\text{ev}} : \mathbb{C}l_{d+1}^u \rightarrow M_N(\mathbb{C})$  with  $N = 2^{\frac{d}{2}}$ . In particular there is no grading. This representation is surjective but not faithful so that we have constraints on the dimension in which

<sup>3</sup>Complex conjugation induces the reference real structure  $\mathfrak{f}$  on  $A$  in the sense of [26].

$\varphi_{ev}$  can preserve  $\mathfrak{l}_{1,d}$  or  $\mathfrak{l}_{d,1}$ . Indeed, since  $\gamma_0 = (-i)^{\frac{d}{2}}\gamma_1 \cdots \gamma_d$  we can only have  $\mathfrak{l}_{1,d}$  (TRS) if  $d + \frac{d}{2}$  is even, which is the case for  $d = 0 \pmod{4}$ . Likewise we can only have  $\mathfrak{l}_{d,1}$  (PHS) if  $d = 2 \pmod{4}$ . The real subalgebras  $M_N(\mathbb{C})^{\mathfrak{r}}$  are,  $Cl_{0,0}^u \cong \mathbb{R}$ ,  $Cl_{0,4}^u \cong M_2(\mathbb{H})$ , for TRS, and  $Cl_{2,0}^u \cong M_2(\mathbb{R})$ ,  $Cl_{6,0}^u \cong M_4(\mathbb{H})$ , for PHS. From this we can conclude that the Bott-Hamiltonian in  $d = 0$  has even TRS, in  $d = 2$  even PHS, in  $d = 4$  odd TRS, , and in  $d = 6$  odd PHS.

Imposing TRS via a real structure  $\mathfrak{r}$ , the topological phase of  $h$  is classified by the van Daele class  $[h \otimes \rho] \in DK(A^{\mathfrak{r}} \otimes Cl_{1,0})$  if the system has no chiral symmetry, by  $[h] \in DK(A^{\mathfrak{r}} \otimes Cl_{1,1})$  if the generator of chiral symmetry is real, and by  $[h] \in DK(A^{\mathfrak{r}} \otimes Cl_{0,2})$  if the generator is imaginary. Imposing PHS by a real structure  $\mathfrak{r}$  the topological phase is classified by  $[h \otimes \rho] \in DK(A^{\mathfrak{r}} \otimes Cl_{0,1})$  if there is no chiral symmetry. PHS with chiral symmetry is related to TRS with chiral symmetry as we mentioned above. For these results see [26].

**6.2. Top dimensional Chern characters.** We will pair the above van Daele classes with the  $d$ -dimensional Chern character over the Brillouin zone  $\mathbb{T}^d$ . It is defined as follows.

Consider the complexified exterior differential algebra over  $\mathbb{T}^d$  which we denote by  $(\Omega(\mathbb{T}^d), d)$ . It is  $\mathbb{Z}$ -graded, but trivially  $\mathbb{Z}_2$ -graded. Integration of  $d$ -forms  $\int_{\mathbb{T}^d}$  is a graded trace on  $\Omega(\mathbb{T}^d)$ .  $(\Omega(\mathbb{T}^d), d, \int_{\mathbb{T}^d})$  is thus a cycle over  $C(\mathbb{T}^d)$ , the algebra of smooth functions over  $\mathbb{T}^d$  is a domain algebra for the cycle.

The  $*$ -structure on  $C(\mathbb{T}^d)$  which is given by complex conjugation extends uniquely to a  $*$ -structure on  $\Omega(\mathbb{T}^d)$ . The  $*$ -map flips the order of a product of differential forms. We thus have  $(dk_1 \cdots dk_d)^* = (-1)^{\mu(d)} dk_1 \cdots dk_d$ . The real structure  $\mathfrak{r}$  flips the sign of the coordinates  $k_i$  and hence we have  $\mathfrak{r}(dk_i) = -dk_i$ . It follows that

$$\int_{\mathbb{T}^d} \mathfrak{r}(h_1 dh_2 \cdots dh_d)^* = (-1)^{d+\mu(d)} \int_{\mathbb{T}^d} h_1 dh_2 \cdots dh_d$$

and hence the cycle has sign  $\mathfrak{s} = (-1)^{d+\mu(d)}$ . Since  $\Omega(\mathbb{T}^d)$  is trivially  $\mathbb{Z}_2$ -graded the cycle has even parity. It follows from Cor. 4.15 that this cycle can pair non-trivially with  $KO_i(A^{\mathfrak{r}})$  only if  $\mu(1-i) - \mu(d)$  and  $d-i$  are even. These conditions are equivalent to  $d = i \pmod{4}$ .

We denote the character of  $(\Omega(\mathbb{T}^d), d, \int_{\mathbb{T}^d})$  by  $\text{ch}'_d$  and extend it to  $A \otimes Cl_k$  in the way we introduced above, notably by  $\text{ch}'_d \# \kappa_k$ . We discuss the two kind of pairings introduced above, the Connes pairing and the torsion valued pairing.

**6.3. Connes pairing with  $\text{ch}_d$ .** We compute the Connes pairing of  $\text{ch}'_d$  with the van Daele class  $[x] \in DK(C(\mathbb{T}^d) \otimes Cl_{d+1})$  where  $x = \sum_{i=0}^d \rho_i \hat{h}_i$  with  $h_i$  as for (15). We consider  $d > 0$ , as the case  $d = 0$  has already been looked at above. The pairing is given by Def. 4.7

$$\langle \text{ch}'_d, [x] \rangle = \int_{\mathbb{T}^d} \kappa_{d+1} x (dx)^d = \kappa^{d+1} \int_{\mathbb{T}^d} \epsilon^{i_0 \cdots i_d} \hat{h}_{i_0} d\hat{h}_{i_1} \cdots d\hat{h}_{i_d}$$

where  $\epsilon^{i_0 \dots i_d}$  is the totally antisymmetric  $\epsilon$ -tensor and we have used the sum convention. As before,  $\hat{h} = h|h|^{-1}$  is the spectrally flattened Hamiltonian and  $\hat{h}_i = h_i|h|^{-1}$ .

If  $d$  is odd then  $\mathbb{C}l_{d+1}$  is isomorphic to  $M_N(\mathbb{C})$  with  $N = 2^{\frac{d+1}{2}}$  and grading defined by the grading operator  $\Gamma_d$ . We thus have  $x = \hat{h}$  with  $h$  as in (15) and  $\gamma_i = \rho_i$  so that, by Lemma 4.8

$$\langle \text{ch}'_d, [x] \rangle = \int_{\mathbb{T}^d} \text{Tr}_N \hat{h} (d\hat{h})^d.$$

On the other hand if  $d$  is even, then  $\mathbb{C}l_d^u$  is isomorphic to  $M_N(\mathbb{C})$  with  $N = 2^{\frac{d}{2}}$  (with trivial grading). We then have  $x = \hat{h} \otimes \rho$  with  $h$  as in (15) and  $\rho_i = \gamma_i \otimes \rho$  where and  $\gamma_0 = (-i)^{\frac{d}{2}} \gamma_1 \dots \gamma_d$ . This leads to

$$\langle \text{ch}'_d, [x] \rangle = \kappa \int_{\mathbb{T}^d} \text{Tr}_N \hat{h} (d\hat{h})^d.$$

**Lemma 6.1.** *Let  $d$  be any strictly positive integer. For  $i = 0, \dots, d$  let  $h_i$  be differentiable functions on a manifold and  $|h| := \sqrt{\sum_{i=0}^d |h_i|^2}$ . Suppose that  $|h|$  is invertible and let  $\hat{h}_i = h_i|h|^{-1}$ . Then*

$$\epsilon^{i_0 \dots i_d} \hat{h}_{i_0} d\hat{h}_{i_1} \dots d\hat{h}_{i_d} = |h|^{-d-1} \epsilon^{i_0 \dots i_d} h_{i_0} dh_{i_1} \dots dh_{i_d}.$$

*Proof.* Let  $x = dh_{i_1} \dots dh_{i_j}$  for some  $0 \leq j < d$ . Since  $h_i$  commutes with  $h_j$  and  $dh_j$  we have  $\hat{h}_1 x h_2 = \hat{h}_2 x h_1$ . Therefore

$$\begin{aligned} \hat{h}_1 x d\hat{h}_2 - \hat{h}_2 x d\hat{h}_1 &= (\hat{h}_1 x dh_2 - \hat{h}_2 x dh_1) |b|^{-1} + (\hat{h}_1 x h_2 - \hat{h}_2 x h_1) d|b|^{-1} \\ &= |b|^{-1} (\hat{h}_1 x dh_2 - \hat{h}_2 x dh_1). \end{aligned}$$

Hence  $\epsilon^{i_0 \dots i_d} \hat{h}_{i_0} d\hat{h}_{i_1} \dots d\hat{h}_{i_d} = |b|^{-1} \epsilon^{i_0 \dots i_d} \hat{h}_{i_0} d\hat{h}_{i_1} \dots d\hat{h}_{i_{d-1}} dh_{i_d}$  and the statement follows iteratively.  $\square$

The following result can be found in literature (see, for instance, [19, 35, 34]). For the convenience of the reader we provide the details of the calculation.

**Proposition 6.2.** *Let  $d$  be any positive integer, and  $h_i : \mathbb{T}^d \rightarrow \mathbb{R}$  be given by  $h_i(k) = \sin(k_i)$  if  $1 \leq i \leq d$  and  $h_0(k) = m(k)$  for an even real function which does not vanish on the discrete set  $\text{TRI} := \{k \in \mathbb{T}^d \mid \forall i : k_i \in \{0, \pi\}\}$ . Then*

$$\int_{\mathbb{T}^d} \epsilon^{i_0 \dots i_d} \hat{h}_{i_0} d\hat{h}_{i_1} \dots d\hat{h}_{i_d} = d! a_d a'_d \sum_{k' \in \text{TRI}} \text{sign}(m(k')) \prod_{i=1}^d \cos(k'_i)$$

where  $a_d$  is the surface of the  $d$ -ball of radius 1 in  $\mathbb{R}^d$  and  $a'_d = \frac{\pi}{2} \alpha_{d-1, d-1}$  (c.f. (7)).

We abbreviate

$$I(m) = \frac{1}{2} \sum_{k' \in \text{TRI}} \text{sign}(m(k')) \prod_{i=1}^d \cos(k'_i)$$

and note that  $I(m)$  must be integer, as it is one half of a sum of an even number of  $\pm 1$ 's.

*Proof.* We know from the general theory that the integral on the left side is homotopy invariant, as long as  $|h|$  remains invertible. We may therefore include a parameter  $t > 0$ , consider  $h_0 = tm$  instead and perform the calculation in the limit  $t \rightarrow 0$ . Let  $\delta > 0$ . Then, away from all  $\delta$ -balls with center in TRI we get

$$\lim_{t \rightarrow 0} \int_{\mathbb{T}^d \setminus B_\delta(\text{TRI})} |h|^{-d-1} \epsilon^{i_0 \dots i_d} h_{i_0} dh_{i_1} \dots dh_{i_d} = 0$$

as  $\lim_{t \rightarrow 0} h_0 = \lim_{t \rightarrow 0} dh_0 = 0$  and  $\lim_{t \rightarrow 0} |h| \geq \frac{1}{2}\delta$  for small enough  $\delta$ . On  $B_\delta(k')$  with  $k' \in \text{TRI}$  we approximate up to order  $\delta$

$$\sin(k_i) \cong k_i - k'_i, \quad m(k) \cong m(k').$$

Indeed, since  $m$  is even, the first order term in the Taylor expansion of  $m$  at points of TRI vanishes. Since  $h_i$  is of order  $\delta$  if  $i > 0$  we have, up to order  $\delta$  on  $B_\delta(k')$

$$\begin{aligned} \epsilon^{i_0 \dots i_d} h_{i_0} dh_{i_1} \dots dh_{i_d} &\cong \epsilon^{0i_1 \dots i_d} h_0 dh_{i_1} \dots dh_{i_d} \\ &\cong d! tm(k') \left( \prod_{i=1}^d \cos(k'_i) \right) dk_1 \dots dk_d \end{aligned}$$

and hence, up to order  $\delta$ ,

$$\begin{aligned} \int_{B_\delta(k')} |h|^{-d-1} \epsilon^{i_0 \dots i_d} h_{i_0} dh_{i_1} \dots dh_{i_d} &\cong d! \int_{B_\delta(k')} \frac{tm(k') \prod_{i=1}^d \cos(k'_i)}{(t^2 m^2(k') + (k - k')^2)^{\frac{d+1}{2}}} dk_1 \dots dk_d \\ &= d! \text{sign}(m(k')) \left( \prod_{i=1}^d \cos(k'_i) \right) \int_0^\delta \frac{\tilde{t} a_d r^{d-1}}{(\tilde{t}^2 + r^2)^{\frac{d+1}{2}}} dr \end{aligned}$$

where  $a_d$  is the surface of the  $d$ -ball of radius 1 and  $\tilde{t} = t|m(k')|$ . We evaluate the integral in the limit  $t \rightarrow 0$

$$\int_0^\delta \frac{tr^{d-1}}{(t^2 + r^2)^{\frac{d+1}{2}}} dr \xrightarrow{t \rightarrow 0} \int_0^{+\infty} \frac{r^{d-1}}{(1 + r^2)^{\frac{d+1}{2}}} dr = \int_0^{\frac{\pi}{2}} \sin^{d-1}(\theta) d\theta = \frac{\pi}{2} \alpha_{d-1, d-1}.$$

Putting everything together we obtain with Lemma 6.1 the result.  $\square$

For the Bott element on the torus we get  $I(m) = 1$ , that is,  $\langle \text{ch}'_d, b_{\mathbb{T}^d} \rangle = \kappa^{d+1} d! a_d a'_d$ . We therefore normalise

$$(18) \quad \text{ch}_d := (2\kappa^{d+1} d! a_d a'_d)^{-1} \text{ch}'_d$$

calling  $\text{ch}_d$  the (top) standard Chern character of the  $d$  dimensional torus. We thus see that, for a Hamiltonian of the form (15)  $h = \sum_{i=0}^d \varphi(\rho_i) h_i$  ( $\varphi = \varphi_{\text{odd}}$  or  $\varphi_{\text{ev}}$  depending on whether  $d$  is odd or even), the Connes pairing of  $\text{ch}_d$  with the corresponding van Daele class is equal to  $I(m)$ . Furthermore



**Corollary 6.3.** *We have*

$$\langle \text{ch}_d, KU_i(C(\mathbb{T}^d)) \rangle = \begin{cases} \mathbb{Z} & \text{if } i = d \bmod 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\langle \text{ch}_d, KO_i(C(\mathbb{T}^d)^{\mathfrak{f}}) \rangle = \begin{cases} \mathbb{Z} & \text{if } i = d \bmod 8 \\ 2\mathbb{Z} & \text{if } i = d + 4 \bmod 8 \\ 0 & \text{otherwise} \end{cases}$$

where  $\mathfrak{f}(f)(k) = \overline{f(-k)}$ .

*Proof.* The corollary is a special case of Lemma 7.1 which we prove below.  $\square$

**6.4. Torsion valued expressions with  $\text{ch}_d$ .** Here we consider the even and odd torsion valued pairings of  $\text{ch}_d$  with the  $K$ -theory class defined by a Hamiltonian  $h \in A$  of the form (15) and  $A$  as in (16). Again we see  $h$  as the representative of the element  $\sum_{i=0}^d \rho_i h_i \in C(\mathbb{T}^d) \otimes \mathbb{C}l_{d+1}$  in a representation of  $\mathbb{C}l_{d+1}$  on  $\mathbb{C}^N$ , however the representations and hence the Hamiltonians will be different from those of the last section; indeed this must be the case since the torsion valued pairing can only be non-trivial if the Connes pairing is trivial.

**6.4.1. TRS but no chiral symmetry.** Topological phases of hamiltonians which are invariant under a real structure  $\mathfrak{r}$  (have TRS) but not protected by a chiral symmetry are classified by the  $K$ -group  $DK(A^{\mathfrak{r}} \otimes Cl_{1,0})$ . As we have no protecting chiral symmetry we disregard the grading on  $\mathbb{C}l_{d+1}$  and denote the latter by  $\mathbb{C}l_{d+1}^u$ . This algebra can be faithfully represented on  $\mathbb{C}^N$  where  $N = 2^{\mu(d+2)}$ . We denote this representation by  $\varphi^u : \mathbb{C}l_{d+1}^u \rightarrow M_N(\mathbb{C})$ . It is, of course, only unique up to conjugation. As in Section 6.1 we obtain a TRS invariant model by considering on  $M_N(\mathbb{C})$  a real structure  $\mathfrak{r}'$  which extends the real structure  $\mathfrak{l}_{1,d}$  on  $\mathbb{C}l_{d+1}^u$ . More precisely we consider  $\gamma_0$  to be real and all  $\gamma_i$  with  $i > 0$  to be imaginary.

Given an  $d$ -dimensional cycle over  $A$  of sign  $\mathfrak{s}$ , we extend it to a cycle over  $A \otimes \mathbb{C}l_1$  and conclude from Lemma 4.13 and conditions (8) of Theorem 5.2 that the even torsion valued expression can only be non-trivial if  $\mathfrak{s} = -1$  and  $d$  is even. Applied to our cycle on the Brillouin zone this means that  $(-1)^{\mu(d)+d} = -1$  and  $d$  is even. This is the case for  $d = 2 \bmod 4$ . Likewise with conditions (9) of Theorem 5.3 we find that the odd torsion valued expression can only be non-trivial if  $(-1)^{\mu(d)+d} = 1$  and  $d$  is odd. This is the case for  $d = 3 \bmod 4$ .

Let  $d = 2 \bmod 4$ . Then  $\Gamma_{d+1} = (-i)^{\mu(d+1)} \gamma_0 \cdots \gamma_d$  commutes with  $h$  and is imaginary for the real structure  $\mathfrak{l}_{1,d}$ . Choosing as trivial insulator  $h_0 = \gamma_0$  we thus see that  $h \otimes \rho$  and  $h_0 \otimes \rho$  commute with  $\Sigma^{(+)} = \Gamma_{d+1} \otimes 1$  and hence satisfy property  $Y_{\gamma_0 \otimes \rho}^{(+)}$ . It follows that the torsion-valued expression  $\Delta_{\text{ch}_d \# \kappa_2}^j([h \otimes \rho])$  is well-defined

on van Daele classes of  $B = A^\mathfrak{r} \otimes Cl_{1,0}$  and given by (12) (note that  $\tilde{B} = A^\mathfrak{r} \otimes Cl_{1,1}$  so that  $\tilde{\xi} = \text{ch}'_2 \# \kappa_2$ )

$$\Delta_{\text{ch}'_d \# \kappa_2}^j([h \otimes \rho]) \equiv \frac{1}{2} \int_{A^\mathfrak{r}} \kappa_1 \text{Tr}_N \hat{h} \Gamma_{d+1} (d\hat{h})^d \otimes \rho^{d+1} \equiv \frac{\kappa_1 N}{2i} \int_{\mathbb{T}^d} \epsilon^{i_0 \dots i_d} \hat{h}_{i_0} d\hat{h}_{i_1} \dots d\hat{h}_{i_d}$$

where  $N = 2^{\frac{d+1}{2}}$ . As  $\frac{\kappa_1 N}{2i \kappa^{d+1}} = 1$  we obtain with our normalisation (18)

$$\Delta_{\text{ch}_d}^{(+)}([h]) := \Delta_{\text{ch}_d \# \kappa_2}^j([h \otimes \rho]) \equiv I(m).$$

Note that  $I(m) = 1$  if we take  $h = \varphi^u(b_{\mathbb{T}^d})$  ( $h$  is not the Bott hamiltonian  $h_{\text{Bott}}^{(d)} = \varphi_{ev}(b_{\mathbb{T}^2})$  as the representations  $\varphi^u$  and  $\varphi_{ev}$  are not isomorphic). The above expression for  $\Delta_{\text{ch}_d}^{(+)}([h])$  has to be taken modulo the subgroup

$$\langle \text{ch}_d \# \kappa_3, DK(A^\mathfrak{r} \otimes Cl_{1,2}) \rangle = \langle \text{ch}_d, KO_2(A^\mathfrak{r}) \rangle.$$

This subgroup depends on the real structure  $\mathfrak{r}$  which, in turn, depends on the real structure  $\mathfrak{r}'$  on  $M_N(\mathbb{C})$ . As  $d$  is even,  $\mathfrak{r}'$  is not uniquely determined by the real structure  $\mathfrak{l}_{1,d}$  on the Clifford algebra and there are two inequivalent ways to extend it. If  $M_N(\mathbb{C})^{\mathfrak{r}'} = M_N(\mathbb{R})$  (which means that TRS is even) then  $KO_2(A^\mathfrak{r}) = KO_2(C(\mathbb{T}^d)^f)$  and, by Cor. 6.3,  $\langle \text{ch}_2, KO_2(C(\mathbb{T}^2)^f) \rangle = \mathbb{Z}$  and  $\langle \text{ch}_6, KO_2(C(\mathbb{T}^6)^f) \rangle = 2\mathbb{Z}$ . In the first case the torsion value pairing is trivial, in the second it is surjective onto  $\mathbb{Z}_2$ . On the other hand, if  $M_N(\mathbb{C})^{\mathfrak{r}'} = M_{\frac{N}{2}}(\mathbb{H})$  (TRS is odd) then  $KO_2(A^\mathfrak{r}) \cong KO_6(C(\mathbb{T}^d)^f)$  and Cor. 6.3 implies that  $\langle \text{ch}_2, KO_2(A^\mathfrak{r}) \rangle = 2\mathbb{Z}$  whereas  $\langle \text{ch}_6, KO_2(A^\mathfrak{r}) \rangle = \mathbb{Z}$ . Now the torsion value pairing is surjective onto  $\mathbb{Z}_2$  in the first case and trivial in the second.

To compare the above with the invariant of Kane-Mele we consider now  $d = 2$  and use the representation of [3],  $\varphi^u : Cl_3^u \rightarrow M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ ,  $\varphi^u(\rho_0) = 1 \otimes \sigma_z$ ,  $\varphi^u(\rho_1) = \sigma_z \otimes \sigma_x$ , and  $\varphi^u(\rho_2) = 1 \otimes \sigma_y$ . Furthermore we take the real structure  $\mathfrak{r} = \text{Ad}_{i\sigma_y \otimes 1} \circ \mathfrak{c}$ . Then  $h = \varphi^u(b_{\mathbb{T}^2})$  is exactly what is referred to as the inversion symmetric Hamiltonian of HgTe in [3][Chap. 11.3]. This shows that the Kane-Mele invariant is a special case of the torsion valued-pairing, simply because there are only two distinct strong topological phases in the Kitaev classification table [29] and the inversion symmetric Hamiltonian of HgTe is known to have non-trivial Kane-Mele invariant.

Also the representation  $\varphi_{ev} : Cl_3^u \rightarrow M_2(\mathbb{C})$  is only unique up to conjugation, and when using  $\varphi_{ev}(\rho_0) = \sigma_z$ ,  $\varphi_{ev}(\rho_1) = \sigma_x$ ,  $\varphi_{ev}(\rho_2) = \sigma_y$  we see how  $h = \varphi^u(b_{\mathbb{T}^2})$  is related to the Bott hamiltonian  $h_{\text{Bott}}^{(2)} = \varphi_{ev}(b_{\mathbb{T}^2})$ , namely

$$\varphi^u(b_{\mathbb{T}^2}) = \begin{pmatrix} \varphi_{ev}(b_{\mathbb{T}^2}) & 0 \\ 0 & \mathfrak{f}(\varphi_{ev}(b_{\mathbb{T}^2})) \end{pmatrix}.$$

Furthermore, the generator of  $K_2(A^\mathfrak{r})$  is  $\begin{pmatrix} \varphi_{ev}(b_{\mathbb{T}^2}) & 0 \\ 0 & -\mathfrak{f}(\varphi_{ev}(b_{\mathbb{T}^2})) \end{pmatrix}$ . Here  $\mathfrak{f}(f)(k) = \mathfrak{c}(f(-k))$  for a  $2 \times 2$  matrix valued function over  $\mathbb{T}^2$ .

Let  $d = 3 \pmod 4$ . As  $d$  the representation  $\varphi^u$  is bijective and hence  $\mathfrak{r}'$  uniquely determined by  $\mathfrak{l}_{1,d}$ ; the model has thus odd TRS if  $d = 3$  and even TRS if  $d = 7$ . Now  $\Gamma_{d+1}$  anti-commutes with  $h$  of the form (15) and is imaginary for the real structure  $\mathfrak{l}_{1,d}$ . We thus see that  $h \otimes \rho$  and  $h_0 \otimes \rho$  anti-commute with  $\Sigma^{(-)} = \Gamma_{d+1} \otimes \rho$ . Again the relevant algebra is  $B = A^{\mathfrak{r}} \otimes Cl_{1,0}$ . Hence  $h \otimes \rho$  satisfies property  $Y_{\gamma_0 \otimes \rho}^{(-)}$  and the torsion-valued pairing  $\Delta_{\text{ch}_d \# \kappa_1}^c([h \otimes \rho])$  is well-defined. We obtain by (14)

$$\Delta_{\text{ch}_d \# \kappa_1}^c([h \otimes \rho]) \equiv \frac{\kappa^{-1}}{2} \int_{\mathbb{T}^d} \kappa_1 \text{Tr}_N \hat{h} \Gamma_{d+1} (d\hat{h})^d \otimes \rho^{d+2} \equiv \frac{N}{2} \int_{\mathbb{T}^d} \epsilon^{i_0 \dots i_d} \hat{h}_{i_0} d\hat{h}_{i_1} \dots d\hat{h}_{i_d}$$

where  $N = 2^{\mu(d+2)} = 2^{\frac{d+1}{2}}$ . Hence  $\frac{N}{\kappa^{d+1}} = 1$  and we obtain from (18)

$$\Delta_{\text{ch}_d}^{(-)}([h]) := \Delta_{\text{ch}_d \# \kappa_1}^c([h]) \equiv \frac{1}{2} I(m)$$

and, again,  $I(m) = 1$  is obtained by using the Hamiltonian  $h = \varphi^u(b_{\mathbb{T}^d})$ . These values are understood modulo  $\langle \text{ch}_d \# \kappa_2, DK(A \otimes Cl_2) \rangle = \langle \text{ch}_d, KU_1(A) \rangle = \mathbb{Z}$ . The torsion valued pairing in  $d = 3 \pmod 4$  is thus surjective onto  $\mathbb{Z}_2$ . If  $d = 3 \pmod 8$  the algebra  $A^{\mathfrak{r}}$  is of quaternionic type whereas if  $d = 7 \pmod 8$  it is of real type. With similar arguments as in the 2-dimensional case we find that the 3-dimensional strong Fu-Kane-Mele invariant is a special case of the odd torsion valued pairing. Indeed, when using the representation  $\varphi^u : \mathbb{C}l_4^u \rightarrow M_N(\mathbb{C})$  of [3] (extending the above by  $\varphi^u(\rho_3) = \sigma_x \otimes \sigma_x$ ) we find that  $h = \varphi^u(b_{\mathbb{T}^3})$  is the Hamiltonian used in [37] as an exemple of a Hamiltonian with non-trivial strong Fu-Kane-Mele invariant, and there are only two distinct strong phases for 3-dimensional odd TRS invariant Hamiltonians.

**6.4.2. PHS but no chiral symmetry.** Topological phases with this protecting symmetry are classified by  $DK(A^{\mathfrak{r}} \otimes Cl_{0,1})$ . We use again the ungraded representation  $\varphi^u : \mathbb{C}l_{d+1}^u \rightarrow M_N(\mathbb{C})$  with  $N = 2^{\mu(d+2)}$ , but now with real structure  $\mathfrak{l}_{d,1}$  on  $\mathbb{C}l_{d+1}^u$  for which  $\gamma_0$  is imaginary and all other  $\gamma_i$  real. For odd  $d$  this determines uniquely the real structure  $\mathfrak{r}'$  on  $M_N(\mathbb{C})$  whereas for  $d$  even we have a choice of extension.

A similar analysis as in the last section can be performed to determine the dimensions  $d$  for which the torsion valued pairing can be non-trivial. The different real structure which one has for PHS enters into Lemma 4.13 and leads with the conditions (8) and (9) now to the result that the even torsion valued expression can only be non-trivial if  $d = 0 \pmod 4$  and the odd one if  $d = 1 \pmod 4$ .

We consider first the case  $d = 0 \pmod 4$ .  $\Gamma_{d+1}$  commutes with  $h$  and is imaginary. We choose  $h_0 = -\gamma_0$  and observe that  $h \otimes \rho$  and  $h_0 \otimes \rho$  commute with the imaginary  $\Sigma^{(+)} = \Gamma_{d+1} \otimes 1$ . Thus  $h \otimes \rho$  satisfies thus  $Y_{\gamma_0 \otimes \rho}^{(+)}$  but the relevant algebra is now  $B = A^{\mathfrak{r}} \otimes Cl_{0,1}$ . It follows that  $\Delta_{\text{ch}_d \# \kappa_1}^j([h \otimes \rho])$  is well-defined and given by (12)

$$\Delta_{\text{ch}_d \# \kappa_2}^j([h \otimes \rho]) \equiv \frac{1}{2} \int_{\mathbb{T}^d} \kappa_1 \text{Tr}_N (\hat{h} - \hat{h}_0) \Gamma_{d+1} (d\hat{h})^d \otimes \rho^{d+1}$$

If  $d = 0$  then  $N = 2$ ,  $\Gamma_1 = \gamma_0$ ,  $h = m\gamma_0$  so that

$$\Delta_{\text{ch}'_0\#\kappa_2}^j([h \otimes \rho]) \equiv \frac{\kappa}{2} \text{Tr}_2((\text{sign}(m) + 1)\gamma_0^2) \equiv \kappa(\text{sign}(m) + 1)$$

If  $d > 0$  then

$$\Delta_{\text{ch}'_d\#\kappa_2}^j([h \otimes \rho]) \equiv \frac{N\kappa}{2} \int_{\mathbb{T}^d} \epsilon^{i_0 \dots i_d} \hat{h}_{i_0} d\hat{h}_{i_1} \dots d\hat{h}_{i_d}$$

As  $N\kappa = \pm 2\kappa^{d+1}$  we obtain with our normalisation

$$\Delta_{\text{ch}_d}^{(+)}([h]) := \Delta_{\text{ch}'_d\#\kappa_2}^j([h \otimes \rho]) \equiv I(m)$$

where  $I(m) = \frac{1}{2}(\text{sign}(m) + 1)$  if  $d = 0$ . The value  $I(m) \equiv 1$  is obtained for  $h = \varphi^u(b_{\mathbb{T}^d})$ . We have to quotient out  $\langle \text{ch}_d\#\kappa_3, DK(A^\mathfrak{r} \otimes Cl_{0,3}) \rangle = \langle \text{ch}_d, KO_4(A^\mathfrak{r}) \rangle$  which depends on the extension  $\mathfrak{r}'$  of  $\mathfrak{l}_{d,1}$  to  $M_N(\mathbb{C})$ . If  $d = 4 \pmod 8$  and  $M_N(\mathbb{C})^{\mathfrak{r}'} = M_N(\mathbb{R})$  (PHS is even), or,  $d = 0 \pmod 8$  and  $M_N(\mathbb{C})^{\mathfrak{r}'} = M_{\frac{N}{2}}(\mathbb{H})$  (PHS is odd) then  $\langle \text{ch}_d, KO_4(A^\mathfrak{r}) \rangle = \mathbb{Z}$  whereas for the other two combinations  $\langle \text{ch}_d, KO_4(A^\mathfrak{r}) \rangle = 2\mathbb{Z}$ . Hence the torsion valued pairing is surjective onto  $\mathbb{Z}_2$  if  $d = 4 \pmod 8$  and PHS is odd, or,  $d = 0 \pmod 8$  and PHS is even. Note that the result for  $d = 0$  corresponds to the results of Section 4.6.1 where the torsion valued pairing on  $KO_2(\mathbb{R})$  was computed.

We come to  $d = 1 \pmod 4$ . As the representation  $\varphi^u$  is surjective in this case the real structure on  $A$  is uniquely determined by  $\mathfrak{l}_{d,1}$ . If  $d = 1 \pmod 8$  the algebra  $A^\mathfrak{r}$  is of real type (even PHS) and if  $d = 5 \pmod 8$  it is of quaternionic type (PHS is odd).  $\Gamma_{d+1}$  anti-commutes with  $h$  and is real for the real structure  $\mathfrak{l}_{d,1}$ . Hence  $\Sigma^{(-)} = \Gamma_{d+1} \otimes \rho$  anti-commutes with  $h \otimes \rho$  and  $h_0 \otimes \rho$  and is imaginary, as  $\rho$  is imaginary (the relevant algebra is  $B = A^\mathfrak{r} \otimes Cl_{0,1}$ ). We thus find that  $h \otimes \rho$  satisfies  $Y_{\gamma_0 \otimes \rho}^{(-)}$  so that  $\Delta_{\text{ch}'_d\#\kappa_1}^c([h \otimes \rho])$  is well-defined and, by (14),

$$\Delta_{\text{ch}'_d\#\kappa_1}^c([h]) \equiv \frac{\kappa^{-1}}{2} \int_{\mathbb{T}^d} \kappa_1 \text{Tr}_N \hat{h} \Gamma_{d+1} d\hat{h} \otimes \rho^{d+2} \equiv \frac{N}{2i} \int_{\mathbb{T}^1} \epsilon^{i_0 \dots i_d} h_{i_0} \dots dh_{i_d}$$

As  $\frac{N}{\kappa^{d+1}} = (-i)^{\frac{d+1}{2}}$  we get

$$\Delta_{\text{ch}_d}^{(-)}([h]) := \Delta_{\text{ch}'_d\#\kappa_1}^c([h]) \equiv \frac{1}{2} I(m).$$

The group which is quotiented out is  $\langle \text{ch}_d\#\kappa_2, DK(A \otimes Cl_2) \rangle = \langle \text{ch}_d, KU_1(A) \rangle = \mathbb{Z}$  so that the torsion valued pairing is thus surjective onto  $\mathbb{Z}_2$ .

**6.4.3. TRS and chiral symmetry.** Topological phases with this protecting symmetry are classified by  $DK(A^\mathfrak{r})$ . We implement a chiral symmetry on  $A$  by considering the natural grading on  $Cl_{d+1}$  and define  $h = \varphi_{+1}(\sum_{i=0}^d \rho_i h_i)$  using the graded representation  $\varphi_{+1} : Cl_{d+1} \rightarrow M_N(\mathbb{C})$  with  $N = 2^{\mu(d+3)}$  which is the composition of the inclusion  $Cl_{d+1} \hookrightarrow Cl_{d+2}$  followed by the faithful representation

$\varphi : \mathbb{C}l_{d+2} \rightarrow M_N(\mathbb{C})$  considered already above ( $M_N(\mathbb{C})$  with standard even grading). We implement TRS on  $h$  by declaring  $\rho_0$  to be real and  $\rho_i$  for  $i = 1, \dots, d$  to be imaginary. This does not determine uniquely the real structure on  $M_N(\mathbb{C})$  and we now distinguish between the two cases  $d$  even and  $d$  odd.

Let  $d$  be even. The grading operator on  $A = C(\mathbb{T}^d, M_N(\mathbb{C}))$  is  $\Gamma_{d+2}$ . Moreover  $\Sigma^{(-)} := \gamma_{d+1}$  anticommutes with  $h$  and  $\Gamma_{d+2}$ . Thus if  $\Sigma^{(-)}$  is imaginary it fulfills the requirements of Theorem 5.3. We therefore extend the real structure to  $\mathfrak{l}_{1,d+1}$  on  $\mathbb{C}l_{d+2}$  so that  $\gamma_{d+1}$  is imaginary. This fixes the real structure on  $A$  as  $\varphi$  is surjective. It follows that  $DK(A^\mathfrak{r}) = DK(C(\mathbb{T}^d)^\mathfrak{f} \otimes Cl_{1,d+1}) \cong KO_{d+1}(C(\mathbb{T}^d)^\mathfrak{f})$ . Then the grading operator is real if  $d = 0 \pmod{4}$  and imaginary if  $d = 2 \pmod{4}$ . Furthermore, the real subalgebra  $A^\mathfrak{r}$  is of real type if  $d \in \{0, 6\} \pmod{8}$ . For the other values  $d \in \{2, 4\} \pmod{8}$  the algebra is of quaternionic type. We infer from Theorem 5.3 that  $\Delta_{\text{ch}'_d}^c([h])$  is well-defined and potentially non-trivial on  $DK(C(\mathbb{T}^d)^\mathfrak{f} \otimes Cl_{1,d+1})$  and there given by

$$\Delta_{\text{ch}'_d}^c([h]) \equiv \frac{\kappa^{-1}}{2} \int_{\mathbb{T}^d} \kappa_{d+2} \gamma_{d+1} \hat{h}(d\hat{h})^d \equiv \frac{\kappa^{d+1}}{2} \int_{\mathbb{T}^d} \epsilon^{i_0 \dots i_d} \hat{h}_{i_0} d\hat{h}_{i_1} \dots d\hat{h}_{i_d}$$

and hence

$$\Delta_{\text{ch}_d}^{(-)}([h]) := \Delta_{\text{ch}_d}^c([h \otimes \rho]) \equiv \frac{1}{2} I(m)$$

The group which is quotiented out is  $\langle \text{ch}_d \# \kappa_{d+3}, DK(A \hat{\otimes} \mathbb{C}l_{d+3}) \rangle$ . As  $A = C(\mathbb{T}^d) \otimes \mathbb{C}l_{d+2}$  we have  $\langle \text{ch}_d \# \kappa_{d+3}, DK(A \hat{\otimes} \mathbb{C}l_{d+3}) \rangle = \langle \text{ch}_d, KU_0(\mathbb{T}^d) \rangle = \mathbb{Z}$ . The torsion valued pairing is thus surjective onto  $\mathbb{Z}_2$ .

Let now  $d$  be odd. Then  $M_N(\mathbb{C})$  is isomorphic to  $\mathbb{C}l_{d+3}$  via the representation  $\varphi$  and the grading operator is  $\Gamma_{d+3}$ . Furthermore  $\Sigma^{(+)} = i\gamma_{d+1}\gamma_{d+2}$  is an even self-adjoint unitary which commutes with  $h$  and  $\Gamma_{d+3}$  and hence fulfills the requirements of Theorem 5.2 provided it is imaginary. There are two real structures on  $\mathbb{C}l_{d+3}$  extending  $\mathfrak{l}_{1,d}$  which achieve this, namely  $\mathfrak{l}_{3,d}$  and  $\mathfrak{l}_{1,d+2}$ . With the first choice,  $\mathfrak{r}' = \mathfrak{l}_{3,d}$ , we have  $DK(A^\mathfrak{r}) = DK(C(\mathbb{T}^d)^\mathfrak{f} \otimes Cl_{3,d}) \cong KO_{d-2}(C(\mathbb{T}^d)^\mathfrak{f})$  and with the second  $DK(A^\mathfrak{r}) = DK(C(\mathbb{T}^d)^\mathfrak{f} \otimes Cl_{1,d+2}) \cong KO_{d+2}(C(\mathbb{T}^d)^\mathfrak{f})$ . Furthermore, the grading operator is real if  $d = 3 \pmod{4}$  and imaginary if  $d = 1 \pmod{4}$ . We now infer from Theorem 5.2 that  $\Delta_{\text{ch}_d \# \kappa_1}^j([h \otimes \rho])$  is well-defined and potentially non-trivial on  $DK(A^\mathfrak{r})$  and given by ( $d$  is odd)

$$\Delta_{\text{ch}_d \# \kappa_1}^j([h \otimes \rho]) \equiv \frac{1}{2} \int_{\mathbb{T}^d} \kappa_{d+4} i \gamma_{d+1} \gamma_{d+2} \hat{h}(d\hat{h})^d \otimes \rho^{d+2} \equiv \frac{i\kappa^{d+3}}{2} \int_{\mathbb{T}^d} \epsilon^{i_0 \dots i_d} \hat{h}_{i_0} d\hat{h}_{i_1} \dots d\hat{h}_{i_d}$$

As  $i\kappa^2 = -2$  we have

$$\Delta_{\text{ch}_d}^{(+)}([h]) := \Delta_{\text{ch}_d \# \kappa_{d+3}}^j([h \otimes \rho]) = I(m).$$

The group which we have to quotiented out is

$$\langle \text{ch}_d \# \kappa_2, DK(A^\mathfrak{r} \hat{\otimes} Cl_{0,2}) \rangle = \langle \text{ch}_d, KO_d(C(\mathbb{T}^d)^\mathfrak{f}) \rangle = \mathbb{Z}$$

if we use  $\mathfrak{r}' = \mathfrak{l}_{3,d}$  and

$$\langle \text{ch}_d \# \kappa_2, DK(A^r \hat{\otimes} Cl_{0,2}) \rangle = \langle \text{ch}_d, KO_{d+4}(C(\mathbb{T}^d)^f) \rangle = 2\mathbb{Z}$$

if we use  $\mathfrak{r}' = \mathfrak{l}_{1,d+2}$  as real structure. In the first case we thus get a trivial pairing, whereas in the second case the torsion value pairing is surjective onto  $\mathbb{Z}_2$ . In this second case, i.e.  $\mathfrak{r}' = \mathfrak{l}_{1,d+2}$ , the algebra  $A^r$  is quaternionic for  $d \in \{1, 3\} \bmod 8$  and real if  $d \in \{5, 7\} \bmod 8$ , and the grading operator  $\Gamma_{d+3}$  is imaginary if  $d = 1 \bmod 4$  and real if  $d = 3 \bmod 4$ .

**6.5. Tabular summary.** We summarize the various possibilities we have discussed above. The Hamiltonian is given by  $h = \sum_{i=0}^d \varphi(\rho_i) h_i \in C(\mathbb{T}^d, M_N(\mathbb{C}))$  where  $d$  is the dimension and  $\varphi : Cl_{d+1} \rightarrow M_N(\mathbb{C})$  is a representation of the Clifford algebra  $Cl_{d+1}$ . Which representation we take depends on which symmetry class we want to realise. The latter is determined by a graded real structure on  $Cl_{d+1}$  which is pushed forward to  $M_N(\mathbb{C})$  and possibly extended. In the presence of chiral symmetry the grading is the standard grading on the Clifford algebra and the standard even grading on the matrix algebra, and the representation  $\varphi$  is graded. Otherwise the algebras and the representation are ungraded.

The topological phase of  $h$  is classified by its van Daele class,  $[h]$  if there is CS,  $[h \otimes \rho]$  if not, in the relevant  $K$ -group. This  $K$ -group is isomorphic to  $KO_i(C(\mathbb{T}^d)^f)$  (or  $KU_i(C(\mathbb{T}^d))$ , if there is no real protecting symmetry) the index  $i$  depending on the symmetry class as explained in [26].

We present two tables. For Table 1 the real structures and the representation are chosen such that the van Daele class associated to  $h$  can have non-trivial Connes pairing with  $\text{ch}_d$ . In particular the pairing of the  $d$ -dimensional Bott Hamiltonian with  $\text{ch}_d$  is 1. The range of the pairing of  $\text{ch}_d$  with the relevant  $K$ -group is  $\mathbb{Z}$ . The torsion valued pairing is trivial (or undefined). The representation is the (up to conjugation unique) graded bijective representation  $\varphi^{odd} : Cl_{d+1} \rightarrow M_{\mu(d+2)}(\mathbb{C})$  if  $d$  is odd, or the ungraded surjective representation  $\varphi^{ev} : Cl_{d+1}^u \rightarrow M_{\mu(d+1)}(\mathbb{C})$  if  $d$  is even.

In Table 2 the real structure and the representation are chosen such that  $\Delta_d^{(+)}(h)$  or  $\Delta_d^{(-)}(h)$  can be non-trivial. This requires the Connes pairing with the van Daele class associated to  $h$  to be 0. The range of the pairing on the relevant  $K$ -group is  $\mathbb{Z}_2$ . The representation is the (up to conjugation unique) graded injective representation  $\varphi_{+1} : Cl_{d+1} \rightarrow M_{\mu(d+3)}(\mathbb{C})$  if we have CS, or the ungraded injective representation  $\varphi^u : Cl_{d+1}^u \rightarrow M_{\mu(d+2)}(\mathbb{C})$  if there is no CS.

## 7. TWO DIMENSIONAL APERIODIC TIGHT BINDING MODELS WITH ODD TRS

The methods developed in this paper apply also to aperiodic tight binding models for which there is no underlying Brillouin zone so that we cannot perform a Bloch transformation and the algebra  $A$  is non-commutative. In this context a couple of structural questions have to be solved, in particular the question about

TABLE 1. **Parameters for non-trivial Connes pairing with standard top Chern character.** The table presents the dimension  $d$ , the type of the algebra  $A$  or  $A^\natural$ , the symmetry type, and the  $K$ -group classifying the topological phase of the Hamiltonian.

$d$	type of alg.	CS	TRS	PHS	$K$ -group
0 mod 2	complex	-	-	-	$KU_0(C(\mathbb{T}^d))$
1 mod 2	complex	yes	-	-	$KU_1(C(\mathbb{T}^d))$
0 mod 8	real	-	even	-	$KO_0(C(\mathbb{T}^d)^\natural)$
1 mod 8	real	real	even	even	$KO_1(C(\mathbb{T}^d)^\natural)$
2 mod 8	real	-	-	even	$KO_2(C(\mathbb{T}^d)^\natural)$
3 mod 8	quaternionic	imag.	odd	even	$KO_3(C(\mathbb{T}^d)^\natural)$
4 mod 8	quaternionic	-	odd	-	$KO_4(C(\mathbb{T}^d)^\natural)$
5 mod 8	quaternionic	real	odd	odd	$KO_5(C(\mathbb{T}^d)^\natural)$
6 mod 8	quaternionic	-	-	odd	$KO_6(C(\mathbb{T}^d)^\natural)$
7 mod 8	real	imag.	even	odd	$KO_7(C(\mathbb{T}^d)^\natural)$

the domain of the torsion valued pairings, that is, the size of  $\ker j_*$  and  $\ker c_*$ . This is partly addressed in this section where we mainly restrict ourselves to two dimensional models which have an odd time reversal invariance, and hence consider only  $\ker j_*$ . A more comprehensive description of tight binding models of any dimension and with all types of protecting symmetries will be given elsewhere.

**7.1. Observable algebra for aperiodic solids.** The natural generalisation of the observable algebra for tight binding models describing aperiodic solids without external magnetic field is the crossed product algebra

$$A := C(\Xi, M_N(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^d$$

where  $\Xi$  is the space of microscopic configurations on  $\mathbb{Z}^d$  which can be realisations of the material, for instance by associating to a point in  $\mathbb{Z}^d$  its atomic orbital type, and  $\alpha$  the action of  $\mathbb{Z}^d$  by shift of the configuration<sup>4</sup> [1]. Depending on the circumstances one wants to describe, the elements of  $\Xi$  are disorder configurations, or quasiperiodic configurations, or even periodic configurations as for a periodic crystal. In the last case  $\Xi$  may be taken to be a single point and  $\alpha = \text{id}$ , as  $M_N(\mathbb{C}) \rtimes \mathbb{Z}^d \cong C(\mathbb{T}^d, M_N(\mathbb{C}))$ . We allow only finitely many possibilities of atomic orbitals at a point and in this case it is most natural to equip  $\Xi$  with a compact totally disconnected topology. The reason for this is that  $\Xi$  can be understood as an inverse limit of finite sets, namely the sets of configurations of finite size of which there are, for any given size, only finitely many. It is usually assumed that  $\Xi$  carries an ergodic invariant probability measure  $\mathbb{P}$  and contains a dense  $\mathbb{Z}^d$ -orbit

<sup>4</sup>we do not distinguish notationally the action of  $\mathbb{Z}^d$  on  $\Xi$  by homeomorphisms from its pull back action on  $C(\Xi)$

TABLE 2. **Parameters for a non-trivial torsion value pairing with the standard top Chern character.** We present the dimension  $d$ , the type of the real subalgebra  $A^\tau$ , the symmetry type, the type of the torsion valued pairing, and  $K$ -group classifying the topological phase of the Hamiltonian.

$d \bmod 8$	type of $A^\tau$	CS	TRS	PHS	$\Delta_d^{(\pm)}$	$K$ -group
0	real	-	-	even	even	$KO_2(C(\mathbb{T}^d)^f)$
1	real	-	-	even	odd	$KO_2(C(\mathbb{T}^d)^f)$
2	quaternionic	-	odd	-	even	$KO_4(C(\mathbb{T}^d)^f)$
3	quaternionic	-	odd	-	odd	$KO_4(C(\mathbb{T}^d)^f)$
4	quaternionic	-	-	odd	even	$KO_6(C(\mathbb{T}^d)^f)$
5	quaternionic	-	-	odd	odd	$KO_6(C(\mathbb{T}^d)^f)$
6	real	-	even	-	even	$KO_0(C(\mathbb{T}^d)^f)$
7	real	-	even	-	odd	$KO_0(C(\mathbb{T}^d)^f)$
0	real	real	even	even	odd	$KO_1(C(\mathbb{T}^d)^f)$
1	quaternionic	imag.	odd	even	even	$KO_3(C(\mathbb{T}^d)^f)$
2	quaternionic	imag.	odd	even	odd	$KO_3(C(\mathbb{T}^d)^f)$
3	quaternionic	real	odd	odd	even	$KO_5(C(\mathbb{T}^d)^f)$
4	quaternionic	real	odd	odd	odd	$KO_5(C(\mathbb{T}^d)^f)$
5	real	imag.	even	odd	even	$KO_7(C(\mathbb{T}^d)^f)$
6	real	imag.	even	odd	odd	$KO_7(C(\mathbb{T}^d)^f)$
7	real	real	even	even	even	$KO_1(C(\mathbb{T}^d)^f)$

which lies in the support of the measure. Physically this may be justified by saying that we consider deformations only in a fixed thermodynamic phase. This measure gives rise to a trace on  $C(\Xi)$  which extends to a positive trace  $\text{Tr}_A$  on  $A$ .

Elements of  $A$  can be approximated by finite sums of the form  $\sum_{n \in \mathbb{Z}^d} a_n u^n$  where  $a_n \in C(\Xi, M_N(\mathbb{C}))$ ,  $u^n = u_1^{n_1} \cdots u_d^{n_d}$  with  $d$  commuting unitaries  $u_1, \dots, u_d$ , and multiplication and  $*$ -structure are given by

$$(19) \quad a_n u^n a_m u^m = a_n \alpha_n(a_m) u^{n+m}, \quad (a_n u^n)^* = \alpha_{-n}(a_n^*) u^{-n}.$$

In particular the action  $\alpha$  on  $C(\Xi, M_N(\mathbb{C}))$  is induced by conjugation with the unitaries  $u_1, \dots, u_d$ . The trace  $\text{Tr}_A$  is given by  $\text{Tr}_A(a_n u^n) = \delta_{0n} \int_{\Xi} \text{Tr}_N(a_0) d\mathbb{P}$  and is thus invariant under the action  $\alpha$ .

The finite dimensional algebra  $M_N(\mathbb{C})$  is used to describe the internal degrees of freedom. Protecting symmetries are defined by real structures  $\tau'$  and/or a grading  $\gamma$  on  $M_N(\mathbb{C})$  as in the periodic case (last section) and then extended to the crossed product by acting trivially on the unitaries  $u_1, \dots, u_d$ ,  $\tau'(a_n u^n) = \tau'(a_n) u^n$ .



The Chern character  $\text{ch}_d$  can be generalised to the aperiodic case as follows. Consider the derivations  $\partial_1, \dots, \partial_d$ ,

$$\partial_j \left( \sum_{n \in \mathbb{Z}^n} a_n u^n \right) = \sum_{n \in \mathbb{Z}^n} i n_j a_n u^n.$$

They commute among each other and satisfy  $\text{Tr}_A \circ \partial_j = 0$ . Let  $\Omega = A \otimes \Lambda \mathbb{C}^d$  with  $\mathbb{Z}_2$ -grading  $\gamma \otimes \text{id}$  and  $\mathbb{Z}$ -grading corresponding to the usual grading of the Grassmann algebra  $\Lambda \mathbb{C}^d$ . Define the differential  $d : A \rightarrow A \otimes \Lambda^1 \mathbb{C}^d$  by  $da = \sum_{i=1}^d \partial_i(a) \otimes \lambda_i$  where  $\{\lambda_i\}_i$  is a base of  $\mathbb{C}^d$ . As usual the differential extends uniquely to all of  $\Omega$ . We define a  $*$ -structure on  $\Lambda \mathbb{C}^n$  by declaring the elements  $\lambda_i$  to be self-adjoint. Let  $\iota : A \otimes \Lambda \mathbb{C}^n \rightarrow A$  be given by

$$\iota(a \otimes \lambda_1 \wedge \dots \wedge \lambda_n) = a$$

and  $\iota(a \otimes w) = 0$  for all  $w \in \Lambda \mathbb{C}^d$  of degree less than  $d$ . Then  $(A \otimes \Lambda \mathbb{C}^n, d, \text{Tr}_A \circ \iota)$  is a  $d$ -dimensional cycle over  $A$ . The Fréchet algebra  $\mathcal{A} \subset A$  of infinite sums  $\sum_{n \in \mathbb{Z}^2} a_n u^n$  for which  $n \mapsto \|a_n\|$  is rapidly decreasing is a dense subalgebra which is closed under holomorphic functional calculus [36, 34] and thus a domain algebra for this cycle.

Under the Fourier-Bloch transformation and with the correct normalisation, the above cycle corresponds to the de Rham cycle over the Brillouin zone considered in the last section if  $\Xi$  is a single point.

Suppose now that the observable algebra  $A$  carries a real structure  $\mathfrak{r}$  of the form above,  $\mathfrak{r}(a_n u^n) = \mathfrak{r}'(a_n) u^n$ . Then

$$\partial_j(\mathfrak{r}(a)) = -\mathfrak{r}(\partial_j(a))$$

and we define a real structure on the Grassmann algebra by

$$\mathfrak{r}''(\lambda_j) = -\lambda_j$$

so as to guarantee that  $d$  commutes with the real structure  $\tilde{\mathfrak{r}} = \mathfrak{r} \otimes \mathfrak{r}''$  on  $A \otimes \Lambda \mathbb{C}^n$ . Note that the trace  $\text{Tr}_A$  satisfies  $\text{Tr}_A(\mathfrak{r}(a)^*) = \text{Tr}_A(a)$ . It follows that  $(A \otimes \Lambda \mathbb{C}^n, \tilde{\mathfrak{r}}, d, \text{Tr}_A \circ \iota)$  is a  $(*, \mathfrak{r})$ -cycle of sign

$$\mathfrak{s} = (-1)^{\mu(d)+d}.$$

We denote its character by  $\text{ch}'_d$ .

The observable algebra  $A$  contains the element

$$\tilde{b}_{\mathbb{T}^d} := \rho_0 \left( 1 + \frac{1}{2} \sum_{i=1}^d (u_i + u_i^* - 2) \right) + \frac{1}{2i} \sum_{i=1}^d \rho_i (u_i - u_i^*)$$

for any choice of configuration space  $\Xi$ . If the latter is reduced to one point, so that  $A \cong C(\mathbb{T}^d) \otimes \mathcal{Cl}_{d+1}$ , then  $\tilde{b}_{\mathbb{T}^d}$  becomes  $b_{\mathbb{T}^d}$  from (17), the element defining the Bott element on the torus. We therefore normalise the Chern character again as

$$\text{ch}_d := \langle \text{ch}'_d, [\tilde{b}_{\mathbb{T}^d}] \rangle^{-1} \text{ch}'_d$$

**7.2.  $K$ -theory of the observable algebra.** The tool to compute the  $K$ -theory of  $C(\Xi) \rtimes_{\alpha} \mathbb{Z}^d$  and of its real subalgebra  $C(\Xi, \mathbb{R}) \rtimes_{\alpha} \mathbb{Z}^d$  is the Pimsner Voiculescu exact sequence [32, 39]. We recall some details. Associated to an action  $\alpha$  on a (trivially graded)  $C^*$ -algebra  $B$  there is a short exact sequence, the so-called Toeplitz extension

$$(20) \quad 0 \rightarrow B \otimes \mathcal{K} \rightarrow \mathcal{T}(B, \alpha) \xrightarrow{q} B \rtimes_{\alpha} \mathbb{Z} \rightarrow 0.$$

$\mathcal{T}(B, \alpha)$  is the universal  $C^*$ -algebra generated by  $B$  and a coisometry  $S$ , i.e. an element  $S$  satisfying  $SS^* = 1$  and  $S^*S = 1 - p$ , such that  $SaS^* = \alpha(a)$ ,  $a \in B$ , and  $p$  is a nonzero projection commuting with  $B$ . Moreover  $q(aS) = au_1$  with  $u_1$  as in (19) (but for a  $\mathbb{Z}$ -action). Any real structure  $\mathfrak{r}$  on  $B$  which commutes with the action  $\alpha$  can be extended to the crossed product algebra  $B \rtimes_{\alpha} \mathbb{Z}$  by  $\tilde{\mathfrak{r}}(b \otimes u_1) = \mathfrak{r}(b) \otimes u_1$  for  $b \in B$ , and similarly it can be extended to the Toeplitz algebra  $\mathcal{T}(B, \alpha)$  by  $\tilde{\mathfrak{r}}(b \otimes S) = \mathfrak{r}(b) \otimes S$ . The above short exact sequence (20) is then equivariant w.r.t the real structure  $\tilde{\mathfrak{r}}$  and restricting to  $\tilde{\mathfrak{r}}$ -invariant elements we obtain the Toeplitz extension of the real crossed product  $B^{\mathfrak{r}} \rtimes_{\alpha} \mathbb{Z}$ . The important result of [32] which has been adapted to the real case in [39] is that the above short exact sequence gives rise to an exact sequence in complex or real  $K$ -theory (the Pimsner-Voiculescu exact sequence) which can be cut into short exact sequences, for each degree one. In the real case the short exact sequence in degree  $i$  is<sup>5</sup>

$$(21) \quad 0 \rightarrow C_{\alpha}KO_i(B^{\mathfrak{r}}) \xrightarrow{i_*} KO_i(B^{\mathfrak{r}} \rtimes_{\alpha} \mathbb{Z}) \xrightarrow{\delta} I_{\alpha}KO_{i-1}(B^{\mathfrak{r}}) \rightarrow 0.$$

Here  $C_{\alpha}KO_i(B^{\mathfrak{r}}) := KO_i(B^{\mathfrak{r}}) / \sim_{\alpha}$  is the quotient module of coinvariant elements, that is the module  $KO_i(B^{\mathfrak{r}})$  modulo elements of the form  $[x] - [\alpha(x)]$ ,  $[x] \in KO_i(B^{\mathfrak{r}})$ , and  $I_{\alpha}KO_{i-1}(B^{\mathfrak{r}}) := \{[x] \in KO_{i-1}(B^{\mathfrak{r}}) : [\alpha(x)] = [x]\}$  is the submodule of invariant elements. The quotient map  $\delta$  in (21) is the boundary map coming from the short exact sequence (20). The other map  $i_*$  of (21) is induced by the inclusion  $i : B \rightarrow B \rtimes_{\alpha} \mathbb{Z}$ .

Since crossed products with  $\mathbb{Z}^d$  can be seen as iterated crossed products with  $\mathbb{Z}$  the above can be iterated to compute in principle the  $K$ -theory of the observable algebra, however the final result becomes more and more complicated with higher  $d$  and can be given in closed form only if the short exact sequences (21) at each stage split. On the other hand, the pairing with  $\text{ch}_d$  can be simply expressed.

**Lemma 7.1.** *Let  $\Xi$  be a compact metrisable totally disconnected space with a continuous  $\mathbb{Z}^d$ -action  $\alpha$ . Suppose that the action has a dense orbit. Let  $\text{ch}_d$  be the character of the cycle over  $C(\Xi) \rtimes_{\alpha} \mathbb{Z}^d$  introduced above. We have*

$$\langle \text{ch}_d, KU_i(C(\Xi) \rtimes_{\alpha} \mathbb{Z}^d) \rangle = \begin{cases} \mathbb{Z} & \text{if } i = d \bmod 2 \\ 0 & \text{otherwise} \end{cases}$$

<sup>5</sup>as is customary, we use also the notation  $K_i = KU_i$  if  $\mathbb{F} = \mathbb{C}$  and  $K_i = KO_i$  if  $\mathbb{F} = \mathbb{H}$ .

and

$$\langle \text{ch}_d, KO_i(C(\Xi, \mathbb{R}) \rtimes_{\alpha} \mathbb{Z}^d) \rangle = \begin{cases} \mathbb{Z} & \text{if } i = d \pmod{8} \\ 2\mathbb{Z} & \text{if } i = d + 4 \pmod{8} \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* Let  $\mathbb{F}$  be  $\mathbb{C}$  or  $\mathbb{R}$ . Applying  $d$  times iteratively the the Pimsner Voiculescu exact sequence [32, 39]) one obtains the exact sequence

$$0 \rightarrow \ker \delta^{(d)} \rightarrow K_i(C(\Xi, \mathbb{F}) \rtimes_{\alpha} \mathbb{Z}^d) \xrightarrow{\delta^{(d)}} I_{\alpha} K_{i-d}(C(\Xi, \mathbb{F})) \rightarrow 0$$

where  $\delta^{(d)}$  is the composition of the  $d$  boundary maps of the individual Pimsner Voiculescu exact sequences and we have written  $I_{\alpha} = I_{\alpha_d} \cdots I_{\alpha_1}$ . Since  $\Xi$  is totally disconnected  $C(\Xi)$  is the direct limit of finite dimensional commutative algebras. By continuity of the  $K$ -functor we thus have  $K_{i-d}(C(\Xi, \mathbb{F})) \cong C(\Xi, K_{i-d}(\mathbb{F}))$  and under this isomorphism the action  $\alpha_*$  on  $K_{i-d}(C(\Xi, \mathbb{F}))$  becomes the usual pull back action on functions over  $\Xi$ . Since the action is transitive only constant functions are  $\alpha$ -invariant and thus  $I_{\alpha} K_{i-d}(C(\Xi, \mathbb{F})) \cong K_{i-d}(\mathbb{F})$ .

We have  $\langle \text{ch}_d, \ker \delta^{(d)} \rangle = 0$  as the inclusion  $\ker \delta^{(d)} \hookrightarrow K_i(C(\Xi, \mathbb{F}) \rtimes_{\alpha} \mathbb{Z}^d)$  is composed of the inclusion maps  $C(\Xi, \mathbb{F}) \rtimes \mathbb{Z}^n \hookrightarrow C(\Xi, \mathbb{F}) \rtimes \mathbb{Z}^{n+1}$  and hence the elements in its image vanish under  $\partial_d$ . There must therefore be a morphism  $\varphi : K_{i-d}(\mathbb{F}) \rightarrow \mathbb{C}$  such that, for all  $[x] \in K_i(C(\Xi, \mathbb{F}) \rtimes_{\alpha} \mathbb{Z}^d)$  we have  $\langle \text{ch}_d, [x] \rangle = \langle \varphi, \delta^{(d)}([x]) \rangle$ . Since  $K_{i-d}(\mathbb{F})$  has at most one generator, there is only one morphism up to normalisation. Clearly if  $K_{i-d}(\mathbb{F})$  is pure torsion, or trivial, we have  $\varphi = 0$ . Otherwise  $\varphi = c \text{ch}_0$  for some  $c \in \mathbb{C}$ . Up to a sign, the value of  $c$  can be obtained as follows.

Consider first the case that  $\mathbb{F} = \mathbb{C}$  and  $\Xi$  is a single point so that  $C(\Xi, \mathbb{F}) \rtimes_{\alpha} \mathbb{Z}^d \cong C(\mathbb{T}^d)$ . We determined in the last section that the smallest non-vanishing absolute value for  $\langle \text{ch}_d, K_d(C(\mathbb{T}^d)) \rangle$  is given by  $|\langle \text{ch}_d, [b_{\mathbb{T}^d}] \rangle| = 1$  and the calculations made in Section 4.6 show that  $\langle \text{ch}_0, K_0(\mathbb{C}) \rangle = \mathbb{Z}$ . Hence in this case  $\delta^{(d)}([\tilde{b}_{\mathbb{T}^d}])$  is a generator of  $K_0(\mathbb{C})$  and  $c = \pm 1$ . Now since  $C(\Xi, M_2(\mathbb{C})) \rtimes_{\alpha} \mathbb{Z}^d$  contains  $\tilde{b}_{\mathbb{T}^d}$ ,  $\delta^{(d)}([\tilde{b}_{\mathbb{T}^d}])$  must also be a generator of  $I_{\alpha} K_0(C(\Xi, \mathbb{Z})) = K_0(\mathbb{C})$  for general  $\Xi$ . Thus for  $\mathbb{F} = \mathbb{C}$  we have  $\langle \text{ch}_d, \cdot \rangle = \pm \langle \text{ch}_0, \delta^{(d)}(\cdot) \rangle$ . Since the Toeplitz sequence is equivariant w.r.t. the real structures the result remains true if we restrict it to the classes of the elements of the real sub-algebras. In particular,  $\langle \text{ch}_d, KO_d(C(\Xi, \mathbb{R}) \rtimes_{\alpha} \mathbb{Z}^d) \rangle = \langle \text{ch}_0, KO_0(\mathbb{R}) \rangle = \mathbb{Z}$  and  $\langle \text{ch}_d, KO_{d+4}(C(\Xi, \mathbb{R}) \rtimes_{\alpha} \mathbb{Z}^d) \rangle = \langle \text{ch}_d, KO_d(C(\Xi, \mathbb{H}) \rtimes_{\alpha} \mathbb{Z}^d) \rangle = \langle \text{ch}_0, KO_0(\mathbb{H}) \rangle = 2\mathbb{Z}$ .  $\square$

Corollary 6.3 is obtained from the above result upon taking  $\Xi$  to be a single point, as  $\mathbb{R} \rtimes \mathbb{Z}^d \cong C(\mathbb{T}^d)^{\dagger}$ .

**7.3. Two dimensional systems with odd TRS.** We now restrict our analysis to two dimensional (possibly aperiodic) systems with odd time reversal invariance. Our aim is to determine the domain of the torsion valued pairing  $\Delta_{\text{ch}_2}^{(+)} = \Delta_{\text{ch}_2 \# \kappa_1}^j$ , that is, the kernel of  $j_*$  on the relevant  $K$ -group.

Following [26] we implement odd time reversal on  $A = C(\Xi, M_N(\mathbb{C})) \rtimes_{\alpha} \mathbb{Z}^d$  through a real structure  $\tau$  of the form  $\tau = \text{Ad}_{\Theta} \circ \mathfrak{f}$  where  $\Theta \in A$  is a unitary which satisfies  $\Theta \mathfrak{f}(\Theta) = -1$  and  $\mathfrak{f}$  is the reference real structure  $\mathfrak{f}(a_n u^n) = \mathfrak{c}(a_n) u^n$ . Here  $\mathfrak{c}(a_n)$  is complex conjugation of the matrix elements in  $a_n \in M_N(\mathbb{C})$ . We know from the general theory of [26] that, under mild conditions on the spectrum of  $\Theta$  (for instance if the spectrum is finite),  $(M_2(A), \tau_2)$  is conjugate to  $(M_2(\mathbb{C}) \otimes A, \mathfrak{h} \otimes \mathfrak{f})$  where  $\mathfrak{h} = \text{Ad}_{i\sigma_y} \circ \mathfrak{c}$  is the quaternionic real structure on  $M_2(\mathbb{C})$  whose real subalgebra is the algebra of quaternions  $M_2(\mathbb{C})^{\mathfrak{h}} = \mathbb{H}$ . We may thus work from the beginning with the observable algebra  $A = M_2(\mathbb{C}) \otimes C(\Xi, M_n(\mathbb{C})) \rtimes_{\alpha} \mathbb{Z}^2$  equipped with the real structure  $\tau = \mathfrak{h} \otimes \mathfrak{f}$ . Its real subalgebra is  $A^{\tau} = \mathbb{H} \otimes C(\Xi, M_n(\mathbb{R})) \rtimes_{\alpha} \mathbb{Z}^2$ .

A Hamiltonian describing an insulator with odd TRS corresponds to a self-adjoint invertible element  $h \in A^{\tau}$ . It can thus be expressed as a  $2 \times 2$  matrix

$$h = \begin{pmatrix} h_1 & R \\ R^* & h_2 \end{pmatrix}$$

whose entries belong to  $C(\Xi, M_n(\mathbb{R})) \rtimes_{\alpha} \mathbb{Z}^2$  and satisfy  $\mathfrak{f}(h_1) = h_2$  and  $\mathfrak{f}(R) = -R^*$ . The models discussed in Section 6.4.1 (for  $d = 2$ ) and the Kane-Mele model are of the above type if one chooses  $\Xi$  to be one point and  $n = 2$ .

For the calculation of the  $K$ -theory of  $A$  and  $A^{\tau}$ , the value of  $n$  is not important and we will set it to 1.

**Proposition 7.2.** *Let  $\Xi$  be a compact metrisable totally disconnected space with a continuous  $\mathbb{Z}^2$ -action  $\alpha$ . Let  $\mathbb{F} = \mathbb{C}$  or  $\mathbb{F} = \mathbb{H}$ . For  $i = 0$  and  $i = 2$  we have the exact sequences*

$$0 \rightarrow C_{\alpha} C(\Xi, K_i(\mathbb{F})) \rightarrow K_i(C(\Xi, \mathbb{F}) \rtimes_{\alpha} \mathbb{Z}^2) \xrightarrow{\delta^{(2)}} I_{\alpha} C(\Xi, K_{i-2}(\mathbb{F})) \rightarrow 0.$$

Furthermore,  $KO_1(C(\Xi, \mathbb{H}) \rtimes_{\alpha} \mathbb{Z}^2)$  is torsion free.

Before giving the proof we remark that, if  $\mathbb{F} = \mathbb{C}$  then there is no distinction between  $i = 0$  and  $i = 2$ . Furthermore, the fact that  $K_2(\mathbb{H}) = 0$  simplifies the exact sequence in the case  $\mathbb{F} = \mathbb{H}$  and  $i = 2$ .

*Proof.* We view the  $C(\Xi, \mathbb{F}) \rtimes_{\alpha} \mathbb{Z}^2$  as a double crossed product  $(C(\Xi, \mathbb{F}) \rtimes_{\alpha_1} \mathbb{Z}) \rtimes_{\alpha_2} \mathbb{Z}$  and apply (21) twice. For  $\mathbb{F} = \mathbb{C}$  this calculation can be found in [17]. We consider here the case  $\mathbb{F} = \mathbb{H}$ . We obtain first from the action  $\alpha_1$  on  $C(\Xi, \mathbb{H})$

$$(22) \quad 0 \rightarrow C_{\alpha_1} KO_i(C(\Xi, \mathbb{H})) \rightarrow KO_i(C(\Xi, \mathbb{H}) \rtimes_{\alpha_1} \mathbb{Z}) \rightarrow I_{\alpha_1} KO_{i-1}(C(\Xi, \mathbb{H})) \rightarrow 0.$$

Since  $\Xi$  is totally disconnected we have  $KO_i(C(\Xi, \mathbb{H})) \cong C(\Xi, KO_i(\mathbb{H}))$ . It is known that  $KO_i(\mathbb{H})$  is  $\mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0$  in degrees  $i = 0, 1, \dots, 7$ . By (22)

$$\begin{aligned} KO_{-1}(C(\Xi, \mathbb{H}) \rtimes_{\alpha_1} \mathbb{Z}) &\stackrel{\delta_1}{\cong} I_{\alpha_1} C(\Xi, KO_{-2}(\mathbb{H})) \\ KO_0(C(\Xi, \mathbb{H}) \rtimes_{\alpha_1} \mathbb{Z}) &\cong C_{\alpha_1} C(\Xi, KO_0(\mathbb{H})) \\ KO_1(C(\Xi, \mathbb{H}) \rtimes_{\alpha_1} \mathbb{Z}) &\stackrel{\delta_1}{\cong} I_{\alpha_1} C(\Xi, KO_0(\mathbb{H})) \\ KO_2(C(\Xi, \mathbb{H}) \rtimes_{\alpha_1} \mathbb{Z}) &= 0 \end{aligned}$$

where  $\delta_1$  is the boundary map for the action  $\alpha_1$ . We insert this into the exact sequence (21) for the second action  $\alpha_2$  on  $B^r = C(\Xi, \mathbb{H}) \rtimes_{\alpha_1} \mathbb{Z}$ . For  $i = 0, 2$  we obtain precisely the exact sequences stated in the proposition. The quotient map  $\delta^{(2)}$  is thus the composition of the boundary maps  $\delta_1$  and  $\delta_2$  for the two actions  $\alpha_1$  and  $\alpha_2$ . For  $i = 1$  we obtain

$$0 \rightarrow C_{\alpha_2} I_{\alpha_1} C(\Xi, K_0(\mathbb{H})) \rightarrow K_1(C(\Xi, \mathbb{H}) \rtimes_{\alpha} \mathbb{Z}^2) \xrightarrow{\delta_2 \circ \delta_1} I_{\alpha_2} C_{\alpha_1} C(\Xi, K_0(\mathbb{H})) \rightarrow 0.$$

Since  $I_{\alpha_2} C_{\alpha_1} C(\Xi, \mathbb{Z})$  and  $C_{\alpha_2} I_{\alpha_1} C(\Xi, \mathbb{Z})$  are always torsion free  $K_1(C(\Xi, \mathbb{H}) \rtimes_{\alpha} \mathbb{Z}^2)$  is also torsion free.  $\square$

**Corollary 7.3.** *Let  $\Xi$  be a compact metrisable totally disconnected space with a continuous  $\mathbb{Z}^2$ -action  $\alpha$ . Suppose that the action has a dense orbit. Then  $KO_0(C(\Xi, \mathbb{H}) \rtimes_{\alpha} \mathbb{Z}^2)$  is determined by the exact sequence*

$$0 \rightarrow C_{\alpha} C(\Xi, \mathbb{Z}) \rightarrow KO_0(C(\Xi, \mathbb{H}) \rtimes_{\alpha} \mathbb{Z}^2) \xrightarrow{\delta^{(2)}} \mathbb{Z}_2 \rightarrow 0$$

and  $j_* : KO_0(C(\Xi, \mathbb{H}) \rtimes_{\alpha} \mathbb{Z}^2) \rightarrow KO_1(C(\Xi, \mathbb{H}) \rtimes_{\alpha} \mathbb{Z}^2)$  is the 0-map. In particular,  $\Delta_{\text{ch}_2}^{(+)}$  is defined on all of  $KO_0(C(\Xi, \mathbb{H}) \rtimes_{\alpha} \mathbb{Z}^2)$ . Furthermore,

$$KO_2(C(\Xi, \mathbb{H}) \rtimes_{\alpha} \mathbb{Z}^2) \cong \mathbb{Z}$$

with generator given by  $\begin{pmatrix} \varphi_{ev}(b_{\mathbb{T}^2}) & 0 \\ 0 & -\mathfrak{f}(\varphi_{ev}(b_{\mathbb{T}^2})) \end{pmatrix}$ .

*Proof.* The exact sequence is just the specialisation of that of Prop. 7.2 to  $i = 0$  taking into account that  $I_{\alpha} C(\Xi, K_{-2}(\mathbb{H})) \cong KO_2(\mathbb{R}) = \mathbb{Z}_2$ , as the action has a dense orbit. We have seen that the image of  $j_*$  is pure torsion. But  $KO_1(C(\Xi, \mathbb{H}) \rtimes_{\alpha} \mathbb{Z}^2)$  is torsion free and hence the image of  $j_*$  on the  $K_0$ -group trivial. Since  $KO_2(\mathbb{H}) = 0$  Prop. 7.2 yields that  $\delta^{(2)} : KO_2(C(\Xi, \mathbb{H}) \rtimes_{\alpha} \mathbb{Z}^2) \rightarrow I_{\alpha} C(\Xi, KO_0(\mathbb{H}))$  is an isomorphism. As  $I_{\alpha} C(\Xi, KO_0(\mathbb{H})) \cong KO_0(\mathbb{H}) \cong \mathbb{Z}$  the generator of  $KO_2(C(\Xi, \mathbb{H}) \rtimes_{\alpha} \mathbb{Z}^2)$  must be the same as in the periodic case where it was determined in the last section.  $\square$

**Corollary 7.4.** *Let  $\Xi$  be a compact metrisable totally disconnected space with a continuous  $\mathbb{Z}^2$ -action  $\alpha$  which has a dense orbit. Then*

$$\Delta_{\text{ch}_2 \# \kappa_1}^j : KO_0(C(\Xi, \mathbb{H})) \rightarrow \langle \text{ch}_2, KO_0(C(\Xi, \mathbb{H}) \rtimes_{\alpha} \mathbb{Z}^2) \rangle / \langle \text{ch}_2, KO_2(C(\Xi, \mathbb{H}) \rtimes_{\alpha} \mathbb{Z}^2) \rangle$$

is surjective onto  $\mathbb{Z}_2$ .

*Proof.* By Lemma 7.1  $\langle \text{ch}_2, KO_2(C(\Xi, \mathbb{H}) \rtimes_{\alpha} \mathbb{Z}^2) \rangle = 2\mathbb{Z}$  and we already computed that  $\Delta_{\text{ch}_2 \# \kappa_1}^j([\varphi^u(b_{\mathbb{T}^2})]) = 1 \pmod{2}$ .  $\square$

Theorem 5.7 in combination with the above corollary tells us that any element of  $KO_0(C(\Xi, \mathbb{H}) \rtimes_{\alpha} \mathbb{Z}^2)$  admits a representative which has an extra spin symmetry  $\Sigma^{(+)}$  and formula (11) tells us how to express the torsion-valued pairing with the help of that symmetry. If this spin symmetry is internal, that is, does not depend on the unitaries  $u_j$  then (11) simplifies to (12) and  $\Delta_{\text{ch}_2}^{(+)}([h])$  is one half of the difference of the Chern number of the projection of  $h$  onto the spin up sector minus the Chern number of its projection onto the spin down sector. We refer the reader to [33] for a thorough discussion of this interpretation as a so-called spin Chern number. It does not really need exact commutation and also the effect of strong disorder can be included [33].

**7.4. Periodically driven models.** We provide an application to periodically driven insulators (Floquet insulators) with observable algebra  $A = C(\Xi, M_N(\mathbb{C})) \rtimes_{\alpha} \mathbb{Z}^2$  and odd time reversal invariance. In the case that these are crystalline, that is,  $\Xi$  equal to a single point, they have been discussed by Carpentier *et al.*. Their work [13] was actually the source of inspiration for our construction in Section 5.

Let  $H(t) \in A$  be a continuously differentiable function of self-adjoint elements.  $H(t)$  should be thought of as a time dependent Hamiltonian describing the material subject to an external force. We suppose that the time dependence is periodic, of period  $T > 0$ . Let  $U(t)$  be the unitary time evolution operator, which is the solution of the initial value problem

$$i\dot{U}(t) = H(t)U(t), \quad U(0) = 1.$$

It follows that  $U(t)$  is  $T$ -periodic up to multiplication by  $U(T)$ ,

$$U(t + T) = U(T)U(t).$$

For simplicity we now set  $T = 1$ . We are interested in the topological properties of the spectral projections of  $U(1)$  which belong to  $A$ . The spectrum of  $U(1)$  is a subset of the circle  $S^1$  of complex numbers of modulus 1. We need to assume that it is not all of  $S^1$  but has at least two gaps so that we can define the spectral projection onto the spectral part between the gaps. More precisely, if  $z_0, z_1$  are two distinct points in  $S^1$  we denote by  $[z_0, z_1]$  the subset of points in  $S^1$  which are counter-clockwise to the left of  $z_0$  and to the right of  $z_1$ , and define  $P_{z_0, z_1}$  to be the spectral projection of  $U(1)$  onto  $[z_0, z_1]$ . If  $z_0$  and  $z_1$  do not belong to the spectrum of  $U(1)$  then  $P_{z_0, z_1}$  is a continuous function of  $U(1)$  and thus lies in  $A$ . Consequently it defines an element in  $KU_0(A)$  or, equivalently,  $x_{z_0, z_1} := (2P_{z_0, z_1} - 1) \otimes \rho$  an element of the van Daele  $K$ -group  $DK_0(A \otimes \mathbb{C}l_1)$ . We are interested in pairings of this element with Chern characters, in particular, in dimension two, with the standard Chern character  $\text{ch}_2$  described above. But note that we are considering spectral

projections of the time 1 evolution operator  $U(1)$  and not spectral projections of the Hamiltonian itself.

The spectral projection  $P_{z_0, z_1}$  can be computed as follows: Fix a point  $z \in S^1$  to define a domain for a complex logarithm  $\log : \mathbb{C} \setminus \mathbb{R}^+ z \rightarrow \mathbb{C}$ . Choose a branch of that logarithm, that is, a real number  $\epsilon$  such that  $e^{i\epsilon} = z$  so that  $\log_\epsilon(e^{i\varphi})$  is the imaginary number  $i\varphi_\epsilon$  which satisfies  $\epsilon \leq \varphi_\epsilon < \epsilon + 2\pi$  and  $\varphi_\epsilon - \varphi \in 2\pi\mathbb{Z}$ . Then

$$H_\epsilon^{\text{eff}} := i \log_\epsilon U(1)$$

is a selfadjoint operator which can be seen as an effective time independent Hamiltonian, because at times  $t = n$ ,  $n \in \mathbb{Z}$ , the time evolution of  $H(t)$  and of  $H_\epsilon^{\text{eff}}$  coincide,  $U(n) = e^{-inH_\epsilon^{\text{eff}}}$ . In between these times the difference of their time evolution is given by the periodized time evolution operator

$$V_\epsilon(t) = U(t)e^{itH_\epsilon^{\text{eff}}}$$

which satisfies  $V_\epsilon(t+1) = V_\epsilon(t)$ . If  $z$  does not belong to the spectrum of  $U(1)$  then  $H_\epsilon^{\text{eff}}$  is a continuous function of  $U(1)$  and hence an element of  $A$ .

Now suppose that  $z_0$  and  $z_1$  do not belong to the spectrum of  $U(1)$ . Choose  $\epsilon_i$  so that  $e^{i\epsilon_i} = z_i$  and  $0 \leq (\epsilon_1 - \epsilon_0) < 2\pi$ . Then, for  $z \neq z_1, z_2$  we have  $\log_{\epsilon_1}(z) - \log_{\epsilon_0}(z) = 2\pi i$  if  $z \in [z_1, z_2]$  while this expression is 0 otherwise. It follows from functional calculus that

$$2\pi i P_{z_0, z_1} = \log_{\epsilon_1}(U(1)) - \log_{\epsilon_0}(U(1)) = -i(H_{\epsilon_1}^{\text{eff}} - H_{\epsilon_0}^{\text{eff}}).$$

The last expression may be interpreted in  $K$ -theory as follows. Let  $\beta : KU_0(A) \rightarrow KU_1(SA)$  be the Bott map from (4). Then, using  $\exp(-2\pi i s P_{z_0, z_1}) = e^{is(H_{\epsilon_1}^{\text{eff}} - H_{\epsilon_0}^{\text{eff}})} = V_{\epsilon_1}(s)V_{\epsilon_0}^*(s)$  we find  $\beta([x_{z_0, z_1}]) = [Y]$  where  $Y$  is the loop in  $M_2(A)$ ,

$$Y(s) = \begin{pmatrix} 0 & V_{\epsilon_1}(s)V_{\epsilon_0}^*(s) \\ V_{\epsilon_0}(s)V_{\epsilon_1}^*(s) & 0 \end{pmatrix}.$$

**7.4.1. Odd time reversal symmetry.** We now consider the effect of time reversal symmetry. Recall that time reversal symmetry is implemented by a real structure  $\mathfrak{r}$  on  $A = M_2(\mathbb{C}) \otimes C(\Xi, M_n(\mathbb{C})) \rtimes_\alpha \mathbb{Z}^d$ . A time dependent Hamiltonian  $H(t)$  has time reversal symmetry if

$$\mathfrak{r}(H(t)) = H(-t).$$

This implies that  $\mathfrak{r}(U(t)) = U(-t)$ ,  $\mathfrak{r}(H_\epsilon^{\text{eff}}) = H_\epsilon^{\text{eff}}$ ,  $\mathfrak{r}(P_{z_0, z_1}) = P_{z_0, z_1}$ , and  $\mathfrak{r}(V_\epsilon(t)) = V_\epsilon(-t)$ . In particular,  $P_{z_0, z_1}$  defines an element of  $KO_0(A^\mathfrak{r})$ .

For periodic models with odd time reversal symmetry Carpentier *et al.* [12, 13] proposed to associate to  $P_{z_0, z_1}$  an invariant which we wish to describe in our framework. Indeed, we will show that their invariant corresponds to the torsion valued pairing of  $\text{ch}_2$  with  $[x_{z_0, z_1}]$  and thus generalises to aperiodic models. For periodic models  $A = M_{2n}(\mathbb{C}) \rtimes_\alpha \mathbb{Z}^2$  with trivial action  $\alpha$  and this algebra is isomorphic to  $M_{2n}(\mathbb{C}) \otimes C(\mathbb{T}^2)$ . Under the isomorphism the real structure becomes

$\mathfrak{r}(f)(k) = \mathfrak{h}(f(-k))$  for  $f : \mathbb{T}^2 \rightarrow M_2(\mathbb{C}) \otimes M_n(\mathbb{C})$ . This Fourier transformation understood, we view now  $P_{z_0, z_1} = P_{z_0, z_1}(k)$  and  $V_\epsilon = V_\epsilon(t, k)$  as continuous functions

$$P_{z_0, z_1} : \mathbb{T}^2 \rightarrow M_2(\mathbb{C}) \otimes M_n(\mathbb{C}), \quad V_\epsilon : \mathbb{T}^3 \rightarrow M_2(\mathbb{C}) \otimes M_n(\mathbb{C})$$

which satisfy  $\text{Ad}_{\sigma_y \otimes 1} P_{z_0, z_1}(k) = \overline{P_{z_0, z_1}(-k)}$  and

$$\text{Ad}_{\sigma_y \otimes 1} V_\epsilon(t, k) = \overline{V_\epsilon(-t, -k)}.$$

The invariant associated to  $P_{z_0, z_1}$  in [13] is obtained by first modifying  $V_\epsilon$ . Let

$$(23) \quad \hat{V}_\epsilon(t, k) = \begin{cases} V_\epsilon(t, k) & \text{for } t \in [0, \frac{1}{2}] \\ \hat{V}_\epsilon(t, k) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$$

where for  $[\frac{1}{2}, 1] \times \mathbb{T}^2 \ni (t, k) \mapsto \hat{V}_\epsilon(t, k) \in U_N(\mathbb{C})$  is any continuously differentiable function which satisfies the symmetry constraint

$$\text{Ad}_{\sigma_y \otimes 1} \hat{V}_\epsilon(t, k) = \overline{\hat{V}_\epsilon(t, -k)}$$

and the boundary conditions

$$\hat{V}_\epsilon(\frac{1}{2}, k) = V_\epsilon(\frac{1}{2}, k), \quad \hat{V}_\epsilon(1, k) = 1.$$

A substantial part of [13] is devoted to the proof that such a function  $\hat{V}_\epsilon$  exists. The degree of  $\hat{V}_\epsilon$  is defined to be

$$\text{deg} \hat{V}_\epsilon := \frac{1}{24\pi^2} \int_{\mathbb{T}^3} \text{Tr}(\hat{V}_\epsilon^* d\hat{V}_\epsilon)^3$$

where  $d$  is the exterior derivative and  $\int_{\mathbb{T}^3}$  the standard integral over  $\mathbb{T}^3$ . Finally, the invariant associated to  $P_{z_0, z_1}$  is the difference modulo 2

$$K(P_{z_0, z_1}) := \text{deg} \hat{V}_{\epsilon_1} - \text{deg} \hat{V}_{\epsilon_0} \pmod{2}.$$

Carpentier *et al.* then show that  $K(P_{z_0, z_1})$  coincides with the Kane-Mele invariant of  $P_{z_0, z_1}$ . Here we point out that  $K(P_{z_0, z_1})$  corresponds exactly to  $\Delta_{\text{ch}_2 \# \kappa_1}^j([x_{z_0, z_1}])$ . Indeed, we can split the integral

$$\frac{1}{24\pi^2} \int_{\mathbb{T}^3} \text{Tr}(\hat{V}_\epsilon^* d\hat{V}_\epsilon)^3 = \frac{1}{24\pi^2} \int_{[0, \frac{1}{2}] \times \mathbb{T}^2} \text{Tr}(V_\epsilon^* dV_\epsilon)^3 + \frac{1}{24\pi^2} \int_{[\frac{1}{2}, 1] \times \mathbb{T}^2} \text{Tr}(\hat{V}_\epsilon^* d\hat{V}_\epsilon)^3.$$

Integrating out the time variable  $t \in [0, \frac{1}{2}]$  one finds that

$$\frac{1}{24\pi^2} \int_{[0, \frac{1}{2}] \times \mathbb{T}^2} \text{Tr}(V_{\epsilon_1}^* dV_{\epsilon_1})^3 - \frac{1}{24\pi^2} \int_{[0, \frac{1}{2}] \times \mathbb{T}^2} \text{Tr}(V_{\epsilon_0}^* dV_{\epsilon_0})^3$$

is proportional to  $\langle \text{ch}_2, [x_{z_0, z_1}] \rangle$  which vanishes, as follows from the last statement of Thm. 5.2 (the dimension, parity, and sign of  $\text{ch}_2$  satisfy (8)). Let

$$F(t) = \begin{pmatrix} 0 & \hat{V}_{\epsilon_1}(\frac{t+1}{2}) \hat{V}_{\epsilon_0}^*(\frac{t+1}{2}) \\ \hat{V}_{\epsilon_0}(\frac{t+1}{2}) \hat{V}_{\epsilon_1}^*(\frac{t+1}{2}) & 0 \end{pmatrix}.$$



$F$  is a homotopy between  $\begin{pmatrix} 0 & P_{z_0, z_1} - P_{z_0, z_1}^\perp \\ P_{z_0, z_1} - P_{z_0, z_1}^\perp & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Identifying  $\rho$  with  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  we thus find that

$$\begin{aligned} \Delta_{\text{ch}_2 \# \kappa_1}^j([x_{z_0, z_1}]) &\equiv c_2^{-1} (8\pi\kappa^3)^{-1} \text{ch}'_{2[0,1]} \# \kappa_2(F, F, F) \\ &\equiv \frac{1}{24\pi^2} \int_{[0,1] \times \mathbb{T}^2} \text{Tr}(FdF)^3 \\ &\equiv \frac{1}{24\pi^2} \int_{[\frac{1}{2}, 1] \times \mathbb{T}^2} \text{Tr}(\hat{V}_{\epsilon_1} d\hat{V}_{\epsilon_1})^3 - \frac{1}{24\pi^2} \int_{[\frac{1}{2}, 1] \times \mathbb{T}^2} \text{Tr}(\hat{V}_{\epsilon_0} d\hat{V}_{\epsilon_0})^3 \end{aligned}$$

( $\equiv$  means equality modulo 2). This shows the result.

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