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Diversifying portfolios of U.S. stocks with crude oil and natural gas: A regime-dependent optimization with several risk measures

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Abstract: We build a portfolio encompassing U.S. crude oil, natural gas and stocks to study the diversification power of energy commodities. Such diversification power depends on the joint dependence structure of the three types of assets. According to Gatfaoui (2016a), the dependence structure is time-varying because individual asset returns exhibit several variance regimes. We identify the corresponding regime-specific multivariate copulas, and incorporate them to well-chosen risk measures. Specifically, we minimize the portfolio's variance, semi-variance and tail risk, in the presence and the absence of constraints on the portfolio's expected return and/or stock investment. First, the return constraint reduces the performance of the optimal portfolio. Second, the regime-specific portfolio optimization implements an enhanced active management strategy over the whole sample period. Finally, the tail-risk optimal portfolio offers the most interesting risk-return tradeoff. However, variance and semi-variance optimal portfolios can also be considered in the absence of a return constraint.

Keywords: Copula, Energy commodity, Portfolio optimization, Stock market, Tail risk.

JEL codes: C16, C32, D81.

1. Introduction

Energy commodities are of interest to both the government and several parties acting in financial markets. From the viewpoint of countries and governments, energy prices are crucial to the economic development, and raise concerns about energy vulnerability (e.g. energy shortages and costs). Energy price uncertainty, such as price upsurges, can severely impair energy firms as well as energy consumers, and therefore economic activity. From the viewpoint of financial markets, commodity and commodity derivatives markets are useful to energy producers and consumers as well as portfolio managers. They play an important role in related risk-sharing processes. As a result, energy commodities have become an asset class, which is widely used for diversification, hedging or speculation prospects (de Roon et al., 2000; Gorton and Rouwenhorst, 2006; Jin and Jorion, 2006). According to the United States (U.S.) Commodity Futures Trading Commission,¹ the notional value of commodity index investments has evolved from \$39.6/\$11.4 Billion on December 31st, 2007 to \$33.9/\$14.3 Billion on January 31st, 2013 with respect to WTI crude oil and natural gas markets.² Moreover, energy and financial markets are known to interact to a large extent, and exhibit a joint dependency (Barsky and Kilian, 2004; Hamilton, 1985; Kilian, 2009).

The proposed research tackles the interaction between the U.S. stock, natural gas, and crude oil markets as well as its significance to portfolio management. Specifically, energy markets are known to interact with the stock market, so that crude oil and natural gas commodity assets play a diversifying role in stock portfolios. The incorporation of such commodity assets into stock portfolios contributes to mitigate the loss risk of the resulting portfolios. However, such diversifying role depends on the dependence structure between crude oil, natural gas and stocks. We aim at studying how the dependence structure between U.S. crude oil, natural gas and stocks evolves over time, and its impact on the diversifying role of commodity assets in stock portfolios. For this purpose, we consider a portfolio composed of the U.S. natural gas and crude oil as well as the Standard and Poor's 500 index (as a portfolio proxy for U.S. large-cap stocks), from January 1997 to October 2017. The Standard and Poor's 1500, Standard and Poor's MidCap 400 and Standard and Poor's SmallCap 600 indexes are also considered as a proxy of the U.S. global, mid-cap and small-cap stock markets respectively. Thus, we study the diversifying role of energy commodities in global, large-cap, mid-cap and small-cap stock portfolios. We build our study on the findings of Gatfaoui (2016a). The author studies the three-dimensional dependence structure between the U.S. stock, crude oil, and natural gas markets from January 1997 to January 2013. Her

¹ <http://www.cftc.gov>

² The notional value has recently fallen to \$27/\$8.9 Billion on June 30th, 2015.

findings highlight that the three markets exhibit structural changes/breaks over time, so that the U.S. stock, crude oil, and natural gas returns display all several variance regimes (e.g. alternating periods of low, medium and high variance of returns). Because of these multiple variance regimes, the resulting multivariate dependence structure is time-varying, and exhibits regime switches (i.e. individual variance regimes translate into numerous multivariate variance regimes). Therefore, the three-dimensional dependence structure between U.S. crude oil, natural gas and stocks is described by different copula functions across variance regimes. Such representative copulas exhibit numerous tail dependences (i.e. none, symmetric, upper and lower tail dependence across regimes), and various asset correlations (i.e. alternating positive and negative correlations over time). In this light, tail dependency is of huge significance to portfolio risk managers because it describes the dependence between the extreme returns of portfolio assets. For example, when the three-dimensional dependence structure exhibits positive lower tail dependence, the extreme negative returns of all asset components are positively correlated. Thus, the portfolio's risk of loss is substantial, and risk mitigation can become ineffective.

Our contributions extend previous study to portfolio management, and cover a larger time window. In particular, we scrutinize practical implications for portfolio optimization across variance regimes. First, we use the methodology of Gatfaoui (2016a) to identify the regime-switching dependence structure between U.S. crude oil, natural gas and Standard and Poor's 500 index returns over time. Such study yields optimal three-dimensional copula representations, which are specific to the detected variance regimes. Findings highlight successive upper, nonexistent, and symmetric tail dependence over the sample horizon (i.e. Gumbel, Gaussian and Student T copulas respectively). Tail dependency is of huge significance to portfolio risk managers since it highlights the joint dependence between the extreme returns of the portfolio's assets. In particular, the nature of the multivariate dependence structure drives the portfolio's performance over the investment horizon. Second, we build a portfolio composed of the U.S. crude oil, natural gas and Standard and Poor's 500 index (as a stock portfolio proxy). Using the regime-specific copulas, we determine the optimal composition of the three-asset portfolio over each variance regime. The optimal portfolio composition is obtained while minimizing various risk measures, such as variance, semi-variance and tail risk. Then, we compare the attributes and performance of the measure-specific optimal portfolios across regimes. We are thus able to gauge the diversifying role of commodities as a function of the dependence structure and the risk metrics in use (i.e. regime-specific diversification performance). Moreover, the same study is also undertaken three more times while replacing successively Standard and Poor's 500 index with Standard and Poor's 1500, MidCap 400 and SmallCap 600 indexes. In this light, our added value is threefold. First, we

propose a better way to identify a reduced number of variance regimes over a larger time window. Second, we employ the regime-switching dependence structure between crude oil, natural gas and Standard and Poor's 500 index (or Standard and Poor's 1500, MidCap 400, SmallCap 600 index) to study the diversifying role of commodities in stock portfolios. We assess the diversifying role of commodities in global market, large-cap, mid-cap and small-cap stock portfolios. Third, considering a portfolio composed of U.S. crude oil, natural gas and stocks, we use the time-varying dependence structure of the three asset components to determine the optimal portfolio composition across variance regimes. For this purpose, we consider several risk measures and various minimization constraints. The risk measures under consideration consist of the variance, semi-variance (e.g. downside variance) and tail risk (i.e. probability of underperforming a specified return threshold). All risk metrics are computed while incorporating the regime-specific multivariate copulas across the referenced variance regimes. And, portfolio optimization is implemented while minimizing these risk measures. Thus, we exploit the diversification power of energy commodities while minimizing the portfolio's (loss) risk. Such regime-specific analysis allows for implementing an active portfolio management strategy, which handles a potential dependence between the extreme returns of portfolio assets. An efficient strategy should rely on tail risk minimization. However, it can also rely on variance and semi-variance minimization in the absence of constraint on the portfolio's expected return. Lastly, the diversification power of energy commodities is often the most effective for SPMidCap 400 stock index portfolios.

The remainder of the paper is organized as follows. Section 2 summarizes the literature review. Section 3 introduces stock (i.e. SP500) and energy market data, and recalls data attributes. This section displays key statistics, and describes regime-specific multivariate dependencies (i.e. copula functions). Using the identified dependence structures, section 4 implements portfolio optimization under various risk management schemes. We focus on stock portfolios, which are diversified with holdings in both natural gas and crude oil assets. A performance diagnostic scrutinizes the risk and return tradeoff of corresponding optimal portfolios, and therefore the diversifying effectiveness of commodity assets. Section 6 extends the previous study to U.S. global market, mid-cap and small-cap stock portfolios. Finally, section 7 concludes and proposes possible future extensions.

2. Literature review

We introduce acknowledged links between the U.S. stock market, crude oil and natural gas. We also highlight the recent role of energy commodities as portfolio diversifiers.

Index commodity trading favors the integration of stock and commodity markets (Domanski and Heath, 2007; Silvennoinen and Thorp, 2013; Tang and Xiong, 2012). It also enforces comovements between stock and commodity prices. Such pattern supports the reported price/return correlation between these two asset classes, and within the commodity asset class (Eckaus, 2008; Falkowski, 2011; Kilian, 2009; Kilian and Park, 2009; Parsons, 2010; Pindyck, 2004).³ However, reported relationships are time-varying (Aloui et al., 2014; Brigida, 2014; Gatfaoui, 2016a). Crude oil and natural gas prices are cointegrated so that they exhibit a long-run equilibrium from which they depart in the short run (Brigida, 2014; Hartley and Medlock, 2014; Miller and Ratti, 2009; Villar and Joutz, 2006). Crude oil and natural gas prices do indeed decouple in the short term whereas they couple in the longer term (Brown and Yücel, 2008; Gatfaoui, 2016b; Hartley and Medlock, 2014; Ramberg and Parsons, 2012). As a result, crude oil-specific shocks need more or less time to contaminate natural gas prices. Such feature opens the door to potential market-timing strategies on the commodity market. Hence, incorporating crude oil and natural gas commodities into an investment portfolio can make sense. And, commodities can help mitigate the resulting portfolio's risk.

As regards risk mitigation prospects, commodity markets present some interests to market participants. Nowadays, their role as efficient portfolio diversifiers is strongly acknowledged (Goergiev, 2001; Jensen et al., 2000; Satyanarayan and Varangis, 1996). For example, futures on commodities help diversify equity portfolios. They contribute to reduce risk, and potentially improve portfolios' returns (Hensel and Ankrum, 1993; Lee and Leuthold, 1985). Incidentally, commodity markets exhibit a negative (and close to zero) correlation with stock markets (Harvey and Erb, 2006). Such negative or null correlation provides equity portfolios with diversification rewards (Conover et al., 2010). Commodities also contribute to improve the expected return of any conventional investment portfolio, which comprises stocks, bonds and cash (Bekkers et al., 2009). Thus, adding commodities to investment portfolios contributes to decrease (stabilize) portfolio volatility, and to stabilize (increase) corresponding returns (Conover et al., 2010; Christopherson et al., 2004; Harvey and Erb, 2006; Kim et al., 2011). In particular, they contribute to decrease the loss risk of portfolios (Daigler et al., 2017).

Including commodities into investment portfolios provides further benefits, specifically when equity markets are resilient and inflation is large. Commodity futures are negatively correlated with equity/bond returns, whereas they are positively correlated with changes in both expected inflation and unexpected inflation (Zvi and Rosanky, 1980; Gorton and Rouwenhorst, 2006). Thus, commodities offer a hedge against unexpected inflation (at least in the long run) because they are

³ For example, the speculative bubble has engendered oil price spikes during the financial crisis.

real assets (Greer, 2006; Hensel and Ankrim, 1993; Kaplan and Lummer, 1997). However, the diversification benefits of commodities vary over time. The diversification's effectiveness is sensitive to the financial market's disturbances (i.e. stressed market times; Büyüksahin et al., 2010; Cheung and Miu, 2010). Besides, the business cycle also influences such relationship (Chevallier and Ielpo, 2013; Chevallier et al., 2014). For example, the diversification power of commodities vanishes when the correlation between stock and commodity markets is strong (Silvennoinen and Thorp, 2013).

3. Energy and stock market data

We summarize the statistical features and key attributes of crude oil, natural gas, and Standard and Poor's index time series.

3.1. Dataset and properties

We consider daily logarithmic returns from January 8, 1997 to October 30, 2017 (i.e. 5249 observations per series), so that we extend the study of Gatfaoui (2016a) by four years and nine months. Returns are considered on a percentage basis. The data consist of Henry Hub Gulf Coast Natural Gas Spot Price (Gas) from the U.S. Energy Information Administration, the WTI crude oil Fixed Order Book price (Oil) from West Texas Intermediate exchange, and Standard and Poor's 500 index close (SP500) from Thomson Reuters.⁴ The selected sample window encompasses several market disturbances. Disturbances refer to the 1997 Asian crisis, the 1998 LTCM hedge fund default, the 2000-2002 dotcom bubble, the May 2005 U.S. credit crisis, the 2006 Amaranth hedge fund collapse, and the 2007-2009 subprime mortgage market crisis as well as the 2000s commodity bubbles. Such disturbances impact the causal connections and dependencies between

⁴ We compared crude oil and natural gas spot prices to the corresponding futures prices of contracts with the shortest maturities. To this end, we have implemented two tests. First, a test of mean equality checked if the log-returns of commodity spot prices and the log-returns of corresponding futures prices were equal over the studied time period. Specifically, it investigated if the differentials between commodity log-returns and futures log-returns were equal to zero, on an average basis. The null assumption stipulates that differentials of log-returns are equal to zero. Unreported results show that we fail to reject the null assumptions, so that the log-returns of commodity spot prices are equal to the log-returns of corresponding futures prices, on an average basis. Second, we focused on the time series behavior of spot prices and futures prices. To this end, we have implemented the linear regression of the differentials between commodity and futures log-returns on futures log-returns, while including an intercept. If commodity and futures log-returns are the same, both the intercept and the slope coefficients should be zero. At a five percent test level, unreported results show that the intercept and slope coefficients are not significantly different from zero for both crude oil and natural gas. Such pattern shows that commodity log-returns and futures log-returns behave in a similar way. Thus, using log-returns of either commodity spot prices or futures prices should yield the same results.

commodity and stock markets. And, they can potentially trigger structural changes in such relationships.

Analogously to Gatfaoui (2016a), we report skewed, leptokurtic, non-Gaussian, but mean-stationary⁵ returns (see appendix). Moreover, crude oil, natural gas and SP500 index returns exhibit structural changes.⁶ The F and WDmax(5) statistics of Bai and Perron (2003a, 2003b) and Qu and Perron (2007) support five structural breaks (see appendix). The reported five breakpoints allow for identifying six periods (see Table in appendix, and Table 1 below). First, mean percentage returns generally exhibit a unique regime since they are very close to zero.⁷ Moreover, an unreported paired t-test of mean equality shows that SP500, crude oil and natural gas return series exhibit the same mean return over each reported period. Second, the six reported periods summarize into various variance/volatility (i.e. standard deviation) regimes according to the robust variance test of Brown and Forsythe (1974), Conover et al. (1981), and Levene (1960).⁸ For example, the six periods summarize into four possible variance regimes for SP500 returns (see Figure 1). The four variances regimes consist of a very low (VL), low (L), medium (M) and high (H) regime. Variance regimes are specific to each return series (see Table 1), and are classified in, at least, three categories such as low, medium and high, according to the robust variance test. At most, five categories can be disentangled such as very low, low, medium, high and very high. Such classification depends on the estimated levels of period-specific standard deviations. When there are more than three variance regimes, we compare the period-specific levels to each other so as to match five possible variance classes. Obtained results confirm at most four variance regimes.

Table 1: Structural changes in return series and variance regimes

Period	Statistics	Gas	Oil	SP500
1	Mean	-0.0376	0.0074	0.0876*
	Std. Dev.	3.9817	2.5620	1.2267
	Regime	M	H	M
2	Mean	0.0941	0.0063	-0.0642
	Std. Dev.	5.8587	2.8193	1.4377
	Regime	H	H	H

⁵ Returns are stationary according to the five percent unit root test of Phillips-Perron.

⁶ The structural break test is detailed in the appendix.

⁷ Recall that returns are in percent so that, for example, a significant mean return of 0.1182% represents a very small value on a level basis (i.e. 0.001182).

⁸ Results are available upon request. The test is implemented at a five percent level.

3	Mean	0.0489	0.1182*	0.0235
	Std. Dev.	4.3625	2.0882	0.8516
	Regime	M	M	L
4	Mean	-0.1325	-0.0551	-0.0037
	Std. Dev.	4.3288	3.1446	1.8905
	Regime	M	H	H
5	Mean	-0.0020	0.0023	0.0556*
	Std. Dev.	3.9550	1.4889	0.8931
	Regime	L	L	L
6	Mean	-0.0349	-0.0585	0.0402
	Std. Dev.	3.5187	2.6156	0.7793
	Regime	L	H	VL

Note: Std. Dev. stands for standard deviation while Regime refers to the variance/volatility regime of asset returns. The variance regime can be very low (VL), low (L), medium (M) or high (H). Period 1 runs from 1/08/1997 to 3/24/2000, period 2 runs from 3/27/2000 to 5/20/2003, period 3 runs from 5/21/2003 to 7/16/2008, period 4 runs from 7/17/2008 to 8/29/2011, period 5 runs from 8/30/2011 to 10/13/2014, and period 6 runs from 10/14/2014 to 10/30/2017.

* Significant at a five percent Student t test level.

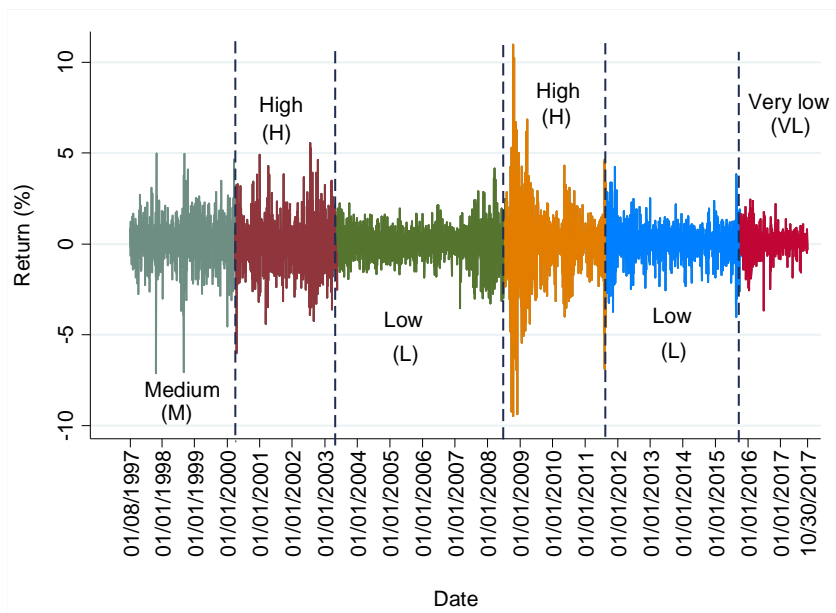


Figure 1: Variance regimes of SP500 index returns

Over the six periods, SP500, crude oil and natural gas return series exhibit different nonparametric correlations, as represented by Kendall's tau. Such concordance measure illustrates the joint dependence structure between asymmetric and fat-tailed time series (see Table 2). In particular, crude oil and natural gas returns are significantly and slightly positively correlated over periods 2 to 4, while crude oil and SP500 returns are significantly and positively correlated over periods 4 to 6. However, natural gas and SP500 returns are insignificantly correlated across the six periods. Thus, the relationship between U.S. energy commodity and stock market returns is time-varying, and its nature changes across periods, and therefore regimes.

Table 2: Kendall's tau

Period	Start	End	Oil and gas	Oil and SP500	Gas and SP500
1	1/08/1997	3/24/2000	0.0049	-0.0098	-0.0223
2	3/27/2000	5/20/2003	0.0908*	-0.0009	0.0025
3	5/21/2003	7/16/2008	0.0623*	-0.0234	-0.0058
4	7/17/2008	8/29/2011	0.0499*	0.3146*	0.0136
5	8/30/2011	10/13/2014	0.0330	0.2833*	-0.0058
6	10/14/2014	10/30/2017	-0.0137	0.1955*	-0.0152

* Significant at a five percent two-tailed test level.

3.2. Multivariate dependence structures across variance regimes

Consider a set of three real variables X , Y , and Z , which evolve randomly. In the present case, random variables X , Y and Z correspond to the returns of SP500 ($X=R_{SP500}$), crude oil ($Y=R_{Oil}$) and natural gas ($Z=R_{Gas}$) respectively. These three random variables can evolve jointly over time, and therefore exhibit a correlation risk within an asset portfolio composed of SP500, crude oil and natural gas. In particular, the positive correlation of extreme negative returns is of high significance for risk management and risk mitigation prospects (i.e. managing lower tail dependency). When the extreme negative returns of all the portfolio's assets are positively correlated, the resulting portfolio returns are negative (i.e. extreme investment losses, due to lower tail dependence). Alternatively, the portfolio's risk of loss can be mitigated when the extreme negative returns of the portfolio's assets are not all positively correlated. Therefore, such potential

correlation risk requires studying the dependence structure of these three assets' returns, which build a set of three real random variables $(X, Y, Z) = (R_{SP500}, R_{Oil}, R_{Gas})$. According to Sklar's theorem (1959), the dependence structure can be gauged with either the related multivariate cumulative distribution function or the corresponding copula function as follows:

$$F(x, y, z) = \Pr(X \leq x, Y \leq y, Z \leq z) = C(F_X(x), F_Y(y), F_Z(z)) = C(u, v, w) \quad (1)$$

where $F(\cdot)$ is the three-dimensional cumulative distribution function of SP500, crude oil and natural gas returns ; $C(\cdot)$ is the corresponding three-dimensional copula function ;⁹ $\Pr(\cdot)$ is the probability operator ; $x, y,$ and z are real numbers representing the realizations of random variables X, Y and Z ; $F_X(\cdot)=F_{SP500}(\cdot), F_Y(\cdot)=F_{Oil}(\cdot)$ and $F_Z(\cdot)=F_{Gas}(\cdot)$ are the univariate cumulative distribution functions of SP500, crude oil and natural gas returns respectively ; $u=F_X(x)=\Pr(X \leq x), v=F_Y(y)=\Pr(Y \leq y)$ and $w=F_Z(z)=\Pr(Z \leq z)$ are positive numbers which lie between 0 and 1. The interest of copula functions relies on the fact that they can avoid specifying univariate cumulative distribution functions. Copula functions can rather focus on the joint empirical dependence structure $C(u,v,w)$, which reduces the risk of model error.

At the univariate level, the reported six periods summarize into up to four variance regimes for each return series. Across these six periods, or equivalently, across corresponding variance regimes, the joint dependence structure between crude oil, natural gas and SP500 returns is investigated through copula representations. The set of possible representations consists of the Clayton, Frank, Gauss, Gumbel and Student T copulas, which are displayed in the appendix. These five copulas handle various tail behaviors (i.e. non existing, symmetric, lower or upper tail dependence; Cherubini *et al.*, 2004; Embrechts *et al.*, 2003; Gatfaoui, 2010; Malevergne and Sornette, 2003; McNeil *et al.*, 2005; Patton, 2009; Sklar, 1973). Thus, we consider three-dimensional dependence structures, which investigate the dependency between extreme return values (see Table 3).

Copula parameters are estimated with the maximum pseudo-likelihood procedure using a bootstrap method (Chen and Fan, 2006; Genest et al., 2009). The adequacy of the five copulas is gauged through a robust Cramer-von-Mises goodness-of-fit test (i.e. Rosenblatt's SnC statistics of Genest et al., 2009), which is performed at a five percent significance level. And, the optimal copula is selected based on the Akaike (AIC), Schwarz (SIC) and Hannan-Quinn (HQIC)

⁹ The copula function is unique provided that univariate cumulative distribution functions are continuous.

information criteria, when several copula representations are appropriate over a given period. When some discrepancies happen between all the criteria,¹⁰ we favor the AIC criterion (Brooks, 2008; Tsay, 2005). Table 3 displays the optimal copulas and their respective parameters. Obviously, the joint dependence structure of SP500, crude oil and natural gas returns exhibits six variance regimes.

Table 3: Optimal dependence structures across variance regimes

Period*	Regime** SP500/Oil/Gas	Copula	Tail	Parameter(s)***
1	M/H/H	Gumbel	Upper	$\theta = 1.5510$
2	H/H/H	Gaussian	None	$\rho = \begin{pmatrix} 1 & 0.0334 & -0.0133 \\ 0.0334 & 1 & 0.0906 \\ -0.0133 & 0.0906 & 1 \end{pmatrix}$
3	L/M/L	Gaussian	None	$\rho = \begin{pmatrix} 1 & -0.0478 & -0.0035 \\ -0.0478 & 1 & 0.0631 \\ -0.0035 & 0.0631 & 1 \end{pmatrix}$
4	H/H/M	T	Symmetric	$\rho = \begin{pmatrix} 1 & 0.3611 & 0.0820 \\ 0.3611 & 1 & 0.0987 \\ 0.0820 & 0.0987 & 1 \end{pmatrix} \nu = 6$
5	L/L/L	T	Symmetric	$\rho = \begin{pmatrix} 1 & 0.4206 & -0.0079 \\ 0.4206 & 1 & 0.0522 \\ -0.0079 & 0.0522 & 1 \end{pmatrix} \nu = 25$
6	VL/H/L	T	Symmetric	$\rho = \begin{pmatrix} 1 & 0.3388 & -0.0732 \\ 0.3388 & 1 & 0.0181 \\ -0.0732 & 0.0181 & 1 \end{pmatrix} \nu = 9$

* Period 1 runs from 1/08/1997 to 3/24/2000, period 2 runs from 3/27/2000 to 5/20/2003, period 3 runs from 5/21/2003 to 7/16/2008, period 4 runs from 7/17/2008 to 8/29/2011, period 5 runs from 8/30/2011 to 10/13/2014, and period 6 runs from 10/14/2014 to 10/30/2017.

** Variance regime of univariate returns can be very low (VL), low (L), medium (M) or high (H).

*** ρ is a 3×3 correlation matrix between SP500, crude oil and natural gas returns respectively. Correlation parameters exhibit various signs. Moreover, ν is the degree of freedom of the Student T copula and θ is the correlation parameter of the Gumbel copula.

During the first period, SP500 variance regime drives tail dependence. Over the remaining periods, the variance regimes of energy commodities drive the dependence structures. The stock

¹⁰ Discrepancies happen over period 2 where SIC and HQIC favor the Clayton copula while AIC favors the Gaussian copula.

market disturbances of late 90s and early 2000s coincide with upper tail dependence. Tail dependency challenges portfolio management because it refers to comovements between the low and/or high extreme returns of SP500 index, crude oil and natural gas. According to Gatfaoui (2016a), the balance between the variance regimes of crude oil, natural gas and SP500 drives tail dependence over time. As a result, the joint dependence structure between the three assets is regime-switching. Hence, a diversified portfolio composed of SP500 index, crude oil and natural gas assets needs to get reallocated across regimes to mitigate (tail) risk. Such feature favors a regime-specific portfolio management since portfolio rebalancing should operate over each regime. We will focus on an *in-sample analysis* in order to understand how the joint dynamics of SP500, crude oil and natural gas returns impacts active portfolio management across regimes.

Given the diversifying role of commodities, we will consider a stock portfolio, which we immunize with energy commodities, such as crude oil and natural gas. Such commodity investment allows for benefiting from a direct exposure to commodity price changes. Moreover, including crude oil and natural gas commodities benefits from the short-term decoupling and long-term coupling of corresponding prices. It allows for capturing the market-timing strategies, which result from the regime-switching nature of stock, crude oil and natural gas markets.

4. Portfolio optimization

The dynamic nature of natural gas, crude oil and U.S. stock markets' dependencies causes shifts in their joint risk across reported regimes. In particular, interdependencies between U.S. commodity and stock markets drive the risk-sharing process, such as volatility spillovers. Such feature has significant implications for risk mitigation and portfolio management (i.e. dynamic asset allocation process and related optimization practice; Buckley et al., 2008). The portfolio's risk exposure and performance are altered because the time variation in risk needs to be reckoned. As an example, we consider a portfolio composed of U.S. stocks, which are diversified with holdings in both natural gas and crude oil assets. The investment portfolio under consideration comprises the SP500 index as well as natural gas and crude oil commodities. Such portfolio strategy exploits the hedging role of energy commodities against the liquidity risk of equity markets, among others. The optimal portfolio composition, or equivalently, the portfolio allocation profile is determined while minimizing the portfolio's risk exposure (e.g. limiting the risk of underperformance) over a given investment horizon. The investment horizon consists of a specific

variance regime. In this light, we consider several risk measures, among which the variance of the portfolio's return, and the portfolio's downside risk such as semi-variance and tail risk (Mansini et al., 2014; Kolm et al., 2014). Corresponding portfolio optimization exploits previous copula representations as a robust tool (Kakouris and Rustem, 2014). In particular, we handle the asymmetry in returns, which is a significant issue to both portfolio selection (Harvey and Siddique, 2000; Kraus and Litzenberger, 1976; Samuelson, 1970), and portfolio decision making (Barberis and Huang, 2008; Mitton and Vorkink, 2007).

4.1. Variance minimization

We consider an investor whose wealth is devoted to a portfolio composed of the SP500, crude oil and natural gas assets. We label w_1 , w_2 and w_3 those parts of wealth, which are respectively invested in each of the three assets. As formerly introduced by Markowitz (1952, 1959), the investor seeks a mean-variance efficient portfolio. Under a given return target, which represents the desired average portfolio performance over the investment horizon, the efficient portfolio's risk, which is measured by its return's variance, is minimized. Assuming that the investor dedicates all of his/her wealth to the portfolio, the portfolio's weights sum to unity. Hence, the investor needs to solve for the following optimization problem in order to determine his/her optimal portfolio (i.e. the best portfolio's allocation, subject to the performance constraint):

$$\begin{aligned} & \min \text{Var}(R_p) \\ \text{s.t.} & \begin{cases} E[R_p] = r \\ w_1 + w_2 + w_3 = 1 \end{cases} \end{aligned} \quad (2)$$

where $R_p = w_1 R_{SP500} + w_2 R_{Oil} + w_3 R_{Gas}$ is the portfolio's return, and $E[R_p] = w_1 E[R_{SP500}] + w_2 E[R_{Oil}] + w_3 E[R_{Gas}]$ is its average counterpart ($E[.]$ is the expectation operator), r is the targeted average return, and $\text{Var}(R_p)$ is the variance of the portfolio's return. When weights are positive (and, specifically, between 0 and 1), no short sale is allowed, while short selling is allowed under negative weights. When returns follow a Gaussian probability distribution, a simple solution to the optimization problem exists (Lintner, 1965; Markowitz, 1952; Sharpe, 1964). But, asset returns are far from being Gaussian because of their tail fatness and asymmetry properties. Moreover, their dynamic joint behaviors across reported regimes often deviate from the Gaussian setting.

Let w and R be the vectors of weights and asset returns respectively. The portfolio's return rewrites as $R_p = w' R$ where $w' = (w_1 \ w_2 \ w_3)$ is the transpose of vector w . Thus, the variance of the portfolio's return depends on the variance of the return vector R , and the joint dependence

structure of the three portfolio constituents. As a result, the variance of the portfolio's return can rewrite as a function of the optimal copula, which describes the joint dependence structure of SP500, crude oil and natural gas returns. And, we can solve for optimization problem (1) while rewriting the components of the portfolio's variance $Var(R_P) = E[R_P^2] - (E[R_P])^2$ as follows:

$$E[R_P] = \iiint_{[0,1]^3} (w_1 F_{SP500}^{-1}(u) + w_2 F_{Oil}^{-1}(v) + w_3 F_{Gas}^{-1}(w)) c(u, v, w) du dv dw \quad (3)$$

and

$$E[R_P^2] = \iiint_{[0,1]^3} (w_1 F_{SP500}^{-1}(u) + w_2 F_{Oil}^{-1}(v) + w_3 F_{Gas}^{-1}(w))^2 c(u, v, w) du dv dw \quad (4)$$

where $F_R(\cdot)$ is the empirical cumulative distribution function of random variable R (i.e. a given return series; see Deheuvels, 1979), and $F_R^{-1}(\cdot)$ is its inverse counterpart, and $c(u, v, w)$ is the relevant copula density function (i.e. optimal copula representation). By so doing, we handle potential departures from normality, when computing the variance risk measure. We consider four possible cases while solving for the optimization problem above-mentioned (see Table 4). Such cases are compared to a benchmark portfolio, which consists of a naïve portfolio, or equivalently, an equally-weighted portfolio.¹¹ First, cases 1 and 2 assume a fixed target return (i.e. the investor's objective of average portfolio performance), whereas cases 3 and 4 let the optimization procedure determine the target return. The two latter cases are useful to study the feasibility of the optimization problem, and help correct for a too ambitious, or unrealistic target of portfolio performance. Under cases 1 and 2, the target return is equal to the naïve portfolio's performance plus a proportion of its absolute performance. In particular, the target return is set to the naïve portfolio's average return plus 0.5 times (i.e. fifty percent of) the absolute value of the naïve portfolio's average return. Hence, the optimal portfolio is required to outperform the naïve portfolio. Under cases 1 and 2, the optimization process yields the optimal portfolio's weights under a fixed target return. As an extension, cases 3 and 4 yield both the optimal portfolio's weights and corresponding optimal target return. The latter optimization cases help study the portfolio's risk-return tradeoff under the chosen risk measure.

¹¹ The naïve portfolio is an equally-weighted portfolio, which comprises SP500 index, and crude oil and natural gas commodities (i.e. each weight is equal to one-third).

Table 4: Optimization constraints and portfolio parameters

Case #	Minimize subject to (un)constrained weights and target return			
1	w_1 is free	w_2 is free	w_3 is free	r is fixed
2	w_1 lies in $[0,1]$	w_2 is free	w_3 is free	r is fixed
3	w_1 is free	w_2 is free	w_3 is free	r is free
4	w_1 lies in $[0,1]$	w_2 is free	w_3 is free	r is free

Note: The weights w_1 , w_2 and w_3 are the parts of wealth, which are respectively invested in SP500, crude oil and natural gas.

Second, cases 2 and 4 constrain the investment (w_1) in SP500 to lie between 0 and 1, while w_2 and w_3 weights (i.e. commodity investments) are unconstrained. Under such setting, the investor necessarily holds the SP500 index in his/her portfolio, and diversifies the index with (short/long) positions in commodity investments. Allowing negative values for w_2 and w_3 weights exploits the hedging effectiveness of commodities with respect to SP500. Specifically, diversification possibilities vanish in the presence of positive dependencies, so that the portfolio requires short sales to hedge against such positive interlinkages (particularly, when the desired target return is high).

We also introduce the upside potential ratio (UPR) as a performance indicator, which writes as $UPR = E[\max(R_P - r, 0)] / (E[\min(R_P - r, 0)^2])^{1/2}$. It is the ratio of the upper partial moment of order one to the square root of the lower partial moment of order 2. Over a given investment horizon, the UPR is the ratio of all excess gains (with respect to the benchmark return r) to the downside risk (Sortino et al. 1999). In the UPR, lower partial moments measure the downside risk (Jarrow and Zhao, 2006; Unser, 2000).

4.2. Minimizing downside risk: Semi-variance and tail risk

So far, portfolio managers worry more about the downside risk (i.e. risk of negative returns, or risk of underperforming a return target) than the symmetric risk measure proposed by the variance. In this light, the square root of the variance, or equivalently, the standard deviation is an absolute risk measure, which lacks information or risk representativeness in the presence of asymmetric returns. As an extension, several downside risk measures such as the semi-variance and tail risk have been proposed (Alexander, 1998; Steinbach, 2001). Downside risk measures help set up an optimal hedge, which offers protection against exposures to the risk of loss (Conlon and Cotter, 2013). The semi-variance focuses on the portfolio's returns, which underperform a fixed target

return r . These underperforming returns lie below the targeted performance r (e.g. a benchmark return, which illustrates the minimum acceptable average return; Markowitz, 1959). In practice, the semi-variance measures the variance of the negative values of the difference ($R_p - r$) between the portfolio's return and its targeted performance. Thus, the semi-variance symbolizes a measure of regret because it focuses on the failure to reach the target return. Under such setting, the portfolio optimization process applies to a new risk measure, which consists of the semi-variance. The optimal portfolio's composition is then determined while minimizing the downside risk (Lari-Lavassani and Li, 2003).

Semi-variance optimization:

Assuming the targeted portfolio's performance to be r , the semi-variance writes as $SemiVar(R_p) = Var[Min(0, R_p - r)]$ and the new optimization problem rewrites:

$$\begin{aligned} \min SemiVar(R_p) = \min E \left[\left\{ Min(0, R_p - r) \right\}^2 \right] - \left(E \left[Min(0, R_p - r) \right] \right)^2 \\ s.t. \begin{cases} E[R_p] = r \\ w_1 + w_2 + w_3 = 1 \end{cases} \end{aligned} \quad (5)$$

with

$$E \left[Min(0, R_p - r) \right] = \iiint_{[0,1]^3} Min(0, w_1 F_{SP500}^{-1}(u) + w_2 F_{Oil}^{-1}(v) + w_3 F_{Gas}^{-1}(w) - r) c(u, v, w) du dv dw \quad (6)$$

and

$$E \left[\left\{ Min(0, R_p - r) \right\}^2 \right] = \iiint_{[0,1]^3} \left\{ Min(0, w_1 F_{SP500}^{-1}(u) + w_2 F_{Oil}^{-1}(v) + w_3 F_{Gas}^{-1}(w) - r) \right\}^2 c(u, v, w) du dv dw \quad (7)$$

where $F_R(\cdot)$ is the empirical cumulative distribution function of random variable R (i.e. a given return series; see Deheuvels, 1979), and $F_R^{-1}(\cdot)$ is its inverse counterpart, and $c(u, v, w)$ is the relevant copula density function (i.e. optimal copula representation). By so doing, we handle

unfavorable deviations of the optimal portfolio's return from the targeted performance level, while accounting for empirical return behaviors.

Tail risk optimization:

The tail risk focuses on the probability of underperforming a given return threshold (e.g. worst-case study or analysis of bad scenarios; Dowd and Blake, 2006). Under such setting, the investor is risk averse, and hates losses, specifically extreme losses. He/she focuses on more or less extreme scenarios, under which his/her portfolio performs very poorly. In particular, the optimal portfolio allocation seeks to minimize the likelihood that the portfolio's return underperforms the chosen threshold (i.e. minimize the probability of a more or less extreme bad scenario). We focus on extreme bad scenarios, under which the bad return target (i.e. worst envisaged critical threshold) consists of the five percent quantile of SP500 index return. We label such quantile $q_{5\%}$. Under this setting, the investor seeks to minimize the probability that the portfolio's return belongs to the lower tail of SP500 returns' distribution (i.e. the five percent lowest values of SP500 returns) over the investment horizon. Such view requires the optimal portfolio to outperform SP500 index during disturbed market times, so that energy market investments hedge extreme exposures to stock market risk (i.e. extreme negative SP500 returns). Assuming the targeted portfolio's performance to be r , tail risk writes as $\lambda(q_{5\%}) = \Pr(R_p \leq q_{5\%})$, and the optimization problem rewrites:

$$\begin{aligned} \min \lambda(q_{5\%}) &= \min \Pr(R_p \leq q_{5\%}) \\ \text{s.t.} \begin{cases} E[R_p] = r \\ w_1 + w_2 + w_3 = 1 \end{cases} \end{aligned} \quad (8)$$

with

$$\begin{aligned} \lambda(q_{5\%}) &= \Pr(R_p \leq q_{5\%}) = \Pr(w_1 R_{SP500} + w_2 R_{Oil} + w_3 R_{Gas} \leq q_{5\%}) \\ &= \Pr(w_1 F_{SP500}^{-1}(u) + w_2 F_{Oil}^{-1}(v) + w_3 F_{Gas}^{-1}(w) \leq q_{5\%}) \\ &= \iiint_I c(u, v, w) du dv dw \end{aligned} \quad (9)$$

where $\Pr(\cdot)$ is the probability operator, $I = \left\{ (u, v, w) \in [0, 1]^3 \mid w_1 F_{SP500}^{-1}(u) + w_2 F_{Oil}^{-1}(v) + w_3 F_{Gas}^{-1}(w) \leq q_{5\%} \right\}$, $F_R(\cdot)$ is the empirical cumulative distribution function of random variable R (i.e. a given return series; see Deheuvels, 1979), and $F_R^{-1}(\cdot)$ is its inverse counterpart, and $c(u, v, w)$ is the relevant

copula density function (i.e. optimal copula representation for u , v , and w between 0 and 1). By so doing, we handle the optimal portfolio's downside risk because we target a reduced probability of unfavorable scenarios $\lambda(q_{5\%}) = \Pr(R_P - q_{5\%} \leq 0)$, and we account for empirical return behaviors.

5. Optimal portfolios' attributes and performance

We introduce the results of portfolio optimization, among which portfolio allocations, optimal portfolios' performance, and related performance diagnostics. All optimization processes employ the Davidon-Fletcher-Powell method, with a 10^{-4} accuracy of gradient calculations (Davidon, 1991; Fletcher and Powell, 1963).

5.1. Variance optimization results

Table 6 displays all the optimization results while Table 5 displays relevant results for the naïve portfolio (i.e. benchmark portfolio).

Table 5: Annualized attributes of the naïve portfolio

Regime	Average return	Target return r	Standard Deviation	Skewness	Excess kurtosis	UPR
1	4.9770	7.5573	26.0663	0.3753	0.2499	53.3530
2	3.1131	4.7056	36.4667	0.3474	5.7293	49.9564
3	17.5000	27.3620	26.7941	0.1425	-1.6984	54.2171
4	-14.9532	-7.7780	33.4830	0.2667	0.6758	47.1784
5	4.4484	6.7460	23.9701	1.1925	-0.2358	50.8039
6	-4.4027	-2.2261	24.5136	0.0492	-1.6352	52.0842

Note: All data are displayed on a percentage basis, and $r = \text{Naïve average return} + 0.5 \times |\text{Naïve average return}|$. UPR stands for upside potential ratio. Period 1 runs from 1/08/1997 to 3/24/2000, period 2 runs from 3/27/2000 to 5/20/2003, period 3 runs from 5/21/2003 to 7/16/2008, period 4 runs from 7/17/2008 to 8/29/2011, period 5 runs from 8/30/2011 to 10/13/2014, and period 6 runs from 10/14/2014 to 10/30/2017.

With respect to case 1, the optimal portfolio allocation yields no short sales of any asset component over all regimes. Case 2 further constrains the holdings (w_I) in SP500 index to lie between zero and unity. Therefore, under case 2, short sales of SP500 become forbidden. Under such case, the optimal portfolio allocation yields also no short sales of any asset component over all regimes. Moreover, the optimal portfolio exhibits a higher degree of exposure to crude oil over regimes 1, 2 and 3, while it exhibits a higher degree of exposure to SP500 index over the remaining regimes. Such features result from the joint dependence structures of portfolio

constituents over previous regimes. Strikingly, cases 3 and 4 are alike because the optimal weights of SP500 are always positive and very close in the presence and the absence of short sales. The only exception consists of regime 1, over which, SP500 weight is above one for case 3 and decreases below one for case 4. The optimal portfolio attributes negative weights to both natural gas over regime 1 and crude oil over regime 6. And, the weight constraint of case 4 decreases the magnitude of the investment in natural gas. Moreover, the optimal portfolio exhibits a higher degree of exposure to SP500 over all regimes. Finally, when switching from case 1 to case 2, the optimal portfolio's performance often remains constant as sketched by the stability of the upside potential ratio (UPR). However, when switching from case 3 to case 4, the optimal portfolio's performance often experiences a very small drop. Adding a weight constraint to w_I lowers more or less the optimal portfolio's performance.

Table 6: Optimal portfolios' attributes under variance minimization

Regime	Optimization scheme											
	Case 1			Case 2								
	w_1	w_2	w_3	$r\ fixed$	UPR	$w_1 \in [0, 1]$	w_2	w_3	$r\ fixed$	UPR		
1	37.2351	43.7289	19.0360	4.9770	53.3530	37.2351	43.7289	19.0360	4.9770	53.3530		
2	22.2734	46.4076	31.3190	3.1131	49.9564	22.2734	46.4076	31.3190	3.1131	49.9564		
3	21.8896	74.9944	3.1160	17.5000	54.2171	21.8896	74.9944	3.1160	17.5000	54.2171		
4	69.0436	15.0690	15.8874	-14.9532	47.1784	69.0436	15.0690	15.8874	-14.9532	47.1784		
5	48.2544	40.9890	10.7566	4.4484	50.8039	48.2544	40.9890	10.7566	4.4484	50.8039		
6	46.1619	36.6339	17.2042	-4.4027	52.0842	46.1619	36.6339	17.2042	-4.4027	52.0842		
Regime	Case 3						Case 4					
	w_1	w_2	w_3	$Optimal\ r$	UPR	$w_1 \in [0, 1]$	w_2	w_3	$Optimal\ r$	UPR		
1	107.1391	1.9612	-9.1003	28.0626	51.3954	97.0138	6.4368	-3.4506	24.6431	50.8893		
2	75.0067	19.0179	5.9753	-9.9691	54.8351	73.6384	20.5325	5.8291	-9.7774	54.6916		
3	82.7173	14.4028	2.8799	10.0879	52.5433	82.1275	14.8826	2.9899	10.2228	52.6580		
4	70.4852	14.5205	14.9943	-7.4416	46.8992	70.2635	14.8415	14.8950	-7.4504	46.8876		
5	80.9683	12.0882	6.9434	12.0010	49.4845	79.1588	13.6869	7.1543	11.7037	49.5595		
6	95.1729	-0.9118	5.7389	9.7984	47.6769	95.1729	-0.9118	5.7389	9.7984	47.6769		

Note: All data are displayed on a percentage basis. The target return r corresponds to the average annual return of the optimal portfolio during each regime. UPR stands for upside potential ratio. Period 1 runs from 1/08/1997 to 3/24/2000, period 2 runs from 3/27/2000 to 5/20/2003, period 3 runs from 5/21/2003 to 7/16/2008, period 4 runs from 7/17/2008 to 8/29/2011, period 5 runs from 8/30/2011 to 10/13/2014, and period 6 runs from 10/14/2014 to 10/30/2017. The weights w_1 , w_2 and w_3 are the parts of wealth, which are respectively invested in SP500, crude oil and natural gas.

5.2. Downside risk optimization

Table 7 displays the results of semi-variance optimization while Table 8 introduces the results of tail risk optimization.

Semi-variance optimization results

With respect to cases 1 and 2, the optimal portfolio allocation yields no short sales of any asset component over all regimes. Optimal weights exhibit close values across both cases. Moreover, the optimal portfolio exhibits a higher degree of exposure to crude oil over regimes 1, 2 and 3, while it exhibits a higher degree of exposure to SP500 index over the remaining regimes. Such features result from the joint dependence structures of portfolio constituents over previous regimes. Amazingly, cases 3 and 4 are alike because the optimal weights of SP500 are always positive and similar in the presence and the absence of short sales. The optimal portfolio attributes negative weights to both natural gas over regime 1 and crude oil over regime 6. Furthermore, the optimal portfolio exhibits a higher degree of exposure to SP500 over all regimes. Lastly, when switching from case 1 to case 2, the optimal portfolio's performance often experiences a small increase as sketched by the noticeable rise of the upside potential ratio (UPR). Adding a weight constraint to w_I raises more or less the optimal portfolio's performance. However, when switching from case 3 to case 4, the optimal portfolio's performance often remains constant.

Table 7: Optimal portfolios' attributes under semi-variance minimization

Regime	Optimization scheme											
	Case 1			Case 2								
	w_1	w_2	w_3	r^{fixed}	UPR	$w_1 \in [0,1]$	w_2	w_3	r^{fixed}	UPR		
1	38.5997	39.9306	21.4698	4.9770	50.8893	38.7320	39.5622	21.7058	4.9770	53.2855		
2	20.4390	49.7168	29.8442	3.1131	54.6916	21.1408	48.4508	30.4084	3.1131	50.0969		
3	21.9608	75.0205	3.0187	17.5000	52.6580	21.8629	74.9845	3.1526	17.5000	54.2184		
4	68.4463	16.0630	15.4906	-14.9532	46.8876	68.8777	15.3450	15.7772	-14.9532	47.1619		
5	48.2560	41.2908	10.4531	4.4484	49.5595	48.2531	40.7498	10.9971	4.4484	50.7819		
6	46.2581	36.9408	16.8011	-4.4027	47.6769	45.9604	35.9909	18.0486	-4.4027	52.0636		
Regime	Case 3						Case 4					
	w_1	w_2	w_3	$Optimal\ r$	UPR	$w_1 \in [0,1]$	w_2	w_3	$Optimal\ r$	UPR		
1	97.2283	6.3275	-3.5558	24.7125	50.9020	97.2283	6.3276	-3.5558	24.7125	50.9020		
2	74.1240	20.3229	5.5531	-9.9114	54.7228	74.1239	20.3230	5.5531	-9.9114	54.7228		
3	81.7608	15.2660	2.9733	10.3232	52.7207	81.7608	15.2660	2.9733	10.3232	52.7207		
4	67.6903	16.3265	15.9832	-7.9583	47.2738	67.6899	16.3283	15.9819	-7.9581	47.2735		
5	79.1601	13.9791	6.8608	11.7036	49.5560	79.1601	13.9792	6.8608	11.7036	49.5560		
6	94.5320	-0.2629	5.7309	9.6216	47.6865	94.4778	-0.7655	6.2877	9.6432	47.8286		

Note: All data are displayed on a percentage basis. The target return r corresponds to the average annual return of the optimal portfolio during each regime. UPR stands for upside potential ratio. Period 1 runs from 1/08/1997 to 3/24/2000, period 2 runs from 3/27/2000 to 5/20/2003, period 3 runs from 5/21/2003 to 7/16/2008, period 4 runs from 7/17/2008 to 8/29/2011, period 5 runs from 8/30/2011 to 10/13/2014, and period 6 runs from 10/14/2014 to 10/30/2017. The weights w_1 , w_2 and w_3 are the parts of wealth, which are respectively invested in SP500, crude oil and natural gas.

Tail risk optimization results

Under cases 1 and 2, the optimal portfolio allocation yields no short sales of any asset component over all regimes. Besides, the optimal portfolio exhibits a higher degree of exposure to crude oil over regimes 1, 2 and 3, while it exhibits a higher degree of exposure to SP500 index over the remaining regimes. Such features result from the joint dependence structures of portfolio constituents over previous regimes. Outstandingly, cases 3 and 4 are alike because the optimal weights of SP500 are always positive and very close in the presence and the absence of short sales. The only exception consists of regime 1, over which, SP500 weight is slightly above one for case 3 and falls marginally below one for case 4. The optimal portfolio attributes negative weights to both natural gas over regime 1 and crude oil over regime 6. And, the weight constraint of case 4 lessens marginally the magnitude of the investment in natural gas. Additionally, the optimal portfolio exhibits a higher degree of exposure to SP500 over all regimes. Lastly, when switching from case 1 to case 2, the optimal portfolio's performance often remains steady as outlined by the constancy of the upside potential ratio (UPR). Nevertheless, when switching from case 3 to case 4, the optimal portfolio's performance often experiences a minor upsurge. Adding a weight constraint to w_1 strengthens moderately the optimal portfolio's performance.

Comparing weights' signs across risk measures and optimization constraints

When switching from case 1 to case 2 (i.e. under a return constraint coupled first with, and then, without SP500 short sales), the signs of optimal weights persist under all risk measures. Few negative signs appear when switching from case 1 to case 3, and from case 2 to case 4 (i.e. when relaxing the constraint on the expected return of the optimal portfolio). Negative weights prevail only for crude oil over regime 6, and natural gas over regime 1. Therefore, without a constraint on its expected return, the optimal portfolio allows few short sales. Under all risk measures and without the return constraint, the positive correlation between SP500 index and either crude oil or natural gas, as well as the high variance regime of crude oil (over period 6) and natural gas (over period 1) drive the persisting negative weights. Moreover, the Gumbel dependence structure (i.e. positive correlation, and upper tail dependence) yields stronger negative weights than the Student T dependence structure (positive correlation, and no tail dependence).

Table 8: Optimal portfolios' attributes under tail risk minimization

Regime	Optimization scheme											
	Case 1			Case 2								
	w_1	w_2	w_3	r fixed	UPR	$w_1 \in [0, 1]$	w_2	w_3	r fixed	UPR		
1	37.5780	42.7744	19.6476	4.9770	53.3577	37.5784	42.7732	19.6484	4.9770	53.3577		
2	26.6922	38.4362	34.8716	3.1131	49.2614	26.6918	38.4371	34.8712	3.1131	49.2615		
3	22.4035	75.1831	2.4133	17.5000	54.2006	22.4034	75.1831	2.4135	17.5000	54.2006		
4	69.9444	13.5696	16.4859	-14.9532	47.2637	69.9441	13.5701	16.4857	-14.9532	47.2637		
5	48.2561	41.3106	10.4333	4.4484	50.8310	48.2561	41.3106	10.4333	4.4484	50.8310		
6	44.7445	32.1112	23.1443	-4.4027	51.7256	44.7446	32.1114	23.1441	-4.4027	51.7257		
Regime	Case 3						Case 4					
	w_1	w_2	w_3	Optimal r	UPR	$w_1 \in [0, 1]$	w_2	w_3	Optimal r	UPR		
1	101.2054	3.6705	-4.8759	25.9161	51.0800	99.9999	4.5831	-4.5830	25.5655	51.0285		
2	74.4249	20.0048	5.5702	-9.9566	54.7550	73.1956	20.7048	6.0996	-9.6513	54.6653		
3	80.8393	16.2623	2.8984	10.5824	52.9028	80.1528	16.7465	3.1007	10.7256	53.0532		
4	77.2168	10.4179	12.3654	-6.1397	45.9563	74.9676	11.6353	13.3971	-6.6049	46.3109		
5	83.5798	12.1009	4.3193	12.4294	49.3941	81.6571	11.5057	6.8372	12.1143	49.4583		
6	97.8905	-2.5446	4.6541	10.4770	47.4107	96.8546	-2.3995	5.5448	10.2493	47.6478		

Note: All data are displayed on a percentage basis. The target return r corresponds to the average annual return of the optimal portfolio during each regime. UPR stands for upside potential ratio. Period 1 runs from 1/08/1997 to 3/24/2000, period 2 runs from 3/27/2000 to 5/20/2003, period 3 runs from 5/21/2003 to 7/16/2008, period 4 runs from 7/17/2008 to 8/29/2011, period 5 runs from 8/30/2011 to 10/13/2014, and period 6 runs from 10/14/2014 to 10/30/2017. The weights w_1 , w_2 and w_3 are the parts of wealth, which are respectively invested in SP500, crude oil and natural gas.

5.3. Comparing optimal portfolios and performance diagnostics

We first compare the compositions of optimal portfolios through a well-chosen distance measure. Then, we undertake a performance diagnostic based on cumulative returns and expected maximum drawdowns. The annualized cumulative return reflects the performance of optimal portfolios, whereas the expected maximum drawdown represents a measure for optimal portfolios' risk.

The potential heterogeneity, or equivalently, dissimilarity of obtained optimal portfolios is assessed through a well-chosen distance measure, across the 6 referenced periods of time. Analogously to Fulga (2016),¹² we consider the L₁-norm as a distance measure. We define the average distance measure as the following dissimilarity index (DI):

$$DI(j, k) = \frac{1}{n} \sqrt{\sum_{i=1}^n (w_i^j - w_i^k)^2} \quad (10)$$

where the number of constituent assets n is equal to 3, and j and k represent a pair of cases among the four possible cases (i.e. four different optimal portfolios). Setting j equal to k is equivalent to calculate the dissimilarity index of the same portfolio. In such situation, we have $w_i^j - w_i^k = 0$ whatever the considered asset i . As a result, the dissimilarity index is zero. Besides, the larger the dissimilarity index is, the more heterogeneous and dissimilar portfolios become. Since cases 1 and 2 are very close, we assimilate case 2 to case 1, so that finally we consider only 3 cases.¹³ As reported in Table 9, optimal portfolios are generally heterogeneous under all risk measures. However, the optimal portfolios of cases 3 and 4 become homogenous over regimes 1, 2, 3, 4 and 5 under the semi-variance measure. Such findings coincide with previous results because cases 3 and 4 exhibit the same optimal weights, except over regime 6. Over regime 6, the optimal portfolio's distribution across crude oil and natural gas is slightly different but still close enough under both cases. Such marginal discrepancy results from the difference in the joint dependence structure between regimes 1 (i.e. Gumbel copula) and 6 (i.e. Student T copula). Moreover, variance and tail-risk optimal portfolios are generally close under cases 3 and 4.

¹² The author defines a dissimilarity index for a two-asset portfolio whose weights lie only between 0 and 1. Hence, the dissimilarity index is bounded between 0 and 1.

¹³ The average distance between case 1 and case 2 is very small (when not equal to zero) under the variance, semi-variance and tail risk measures respectively.

Table 9: Dissimilarity index across periods and risk measures

Risk Measure	Pairs $j-k^{**}$	1	2	3	4	5	6
Variance	1/2*-3	28.7185	21.5337	28.6190	0.5941	14.6059	20.9317
	1/2*-4	24.6528	20.9699	28.3667	0.5297	13.7980	20.9317
	3-4	4.1429	0.6821	0.2561	0.1342	0.8079	0.0000
Semi-variance	1/2*-3	24.0203	21.9498	28.1793	0.3133	13.7997	20.6479
	1/2*-4	24.0203	21.9497	28.1793	0.3134	13.7997	20.7028
	3-4	0.0000	0.0000	0.0000	0.0008	0.0000	0.2507
Tail risk	1/2*-3	26.2021	19.6545	27.6619	2.9777	15.4141	22.0288
	1/2*-4	25.6952	19.1625	27.3867	2.0687	14.9699	21.6441
	3-4	0.5134	0.5035	0.2880	0.9193	1.0745	0.4580

* Cases 1 and 2 are similar so that they reduce to one case.

** Comparison between case j and case k .

Note: Period 1 runs from 1/08/1997 to 3/24/2000, period 2 runs from 3/27/2000 to 5/20/2003, period 3 runs from 5/21/2003 to 7/16/2008, period 4 runs from 7/17/2008 to 8/29/2011, period 5 runs from 8/30/2011 to 10/13/2014, and period 6 runs from 10/14/2014 to 10/30/2017.

As regards cumulative returns, Table 10 displays the best-performing optimal portfolios across risk measures, over each listed period, whereas Table 11 displays annualized cumulative returns across cases. Table 11 presents the cumulative returns over the whole sample period, under each case. Table 10 displays the best-performing portfolio, which presents the highest annualized cumulative return across the three risk measures, over a given period. The optimal portfolios offering the highest period-specific annualized cumulative returns result from variance, semi-variance and tail risk minimization in 8.33%, 37.50% and 54.17% of situations respectively. The optimal portfolios resulting from tail risk minimization often outperform, when the joint distribution of asset returns follows a Student T copula. However, they also outperform under case 4, when the optimal return and weights are jointly gauged over each regime. The optimal portfolios resulting from semi-variance minimization frequently offer the best annualized cumulative returns, when SP500, crude oil and natural gas returns jointly follow either a Gaussian or a Gumbel copula. In the remainder of situations, variance optimal portfolios outperform. Additionally, the cumulative returns in Table 11 reflect the performance of a period-specific portfolio rebalancing strategy, across the four

possible cases. Under any given case, portfolio optimization/rebalancing occurs over each regime within the sample period,¹⁴ and the same optimization criterion applies to each regime. Such regime-specific active portfolio management captures the portfolio’s structural changes over time, namely the time-varying dependence structure of its assets. Tail-risk optimal portfolios outperform under cases 3 and 4, when the expected return is unconstrained. Conversely, the optimal portfolios resulting from the semi-variance minimization outperform under cases 1 and 2, when the expected return is fixed (i.e. when a performance target is imposed). Finally, the positive weight constraint on SP500 index lowers the performance of variance and tail-risk optimal portfolios under case 4. Banning short sales prevents the hedging role of energy commodities, specifically over regimes 2 and 4 (see Figure 2). As an example, Figure 2 displays the annualized percentage cumulative returns of optimal portfolios across regimes. During periods 2 and 4, the SP500 index exhibits a stronger correlation with crude oil than with natural gas in absolute value. Moreover, all asset returns exhibit more or less high variance regimes. Such patterns alter the optimal portfolio’s performance.

Table 10: Best-performing portfolios with highest annualized cumulative returns

Period	Case 1	Case 2	Case 3	Case 4
1	SV	SV	V	TR
2	SV	SV	SV	TR
3	SV	SV	TR	TR
4	TR	TR	TR	TR
5	SV	TR	TR	TR
6	V	SV	TR	TR

Note: V, SV and TR stand for variance, semi-variance and tail risk. Period 1 runs from 1/08/1997 to 3/24/2000, period 2 runs from 3/27/2000 to 5/20/2003, period 3 runs from 5/21/2003 to 7/16/2008, period 4 runs from 7/17/2008 to 8/29/2011, period 5 runs from 8/30/2011 to 10/13/2014, and period 6 runs from 10/14/2014 to 10/30/2017.

¹⁴ We focus on an in-sample study to understand the dynamics of the three asset components, and its impact on the portfolio’s performance across regimes.

Table 11: Annualized cumulative returns of optimal portfolios across cases (%)

Measure/Case	1	2	3	4
Variance	4.0485	4.0485	5.7087	5.1362
Semi-variance	4.0760	4.0706	4.9593	4.9638
Tail Risk	3.9622	3.9622	6.2263	6.0248

Note: Highest cumulative returns are in bold.

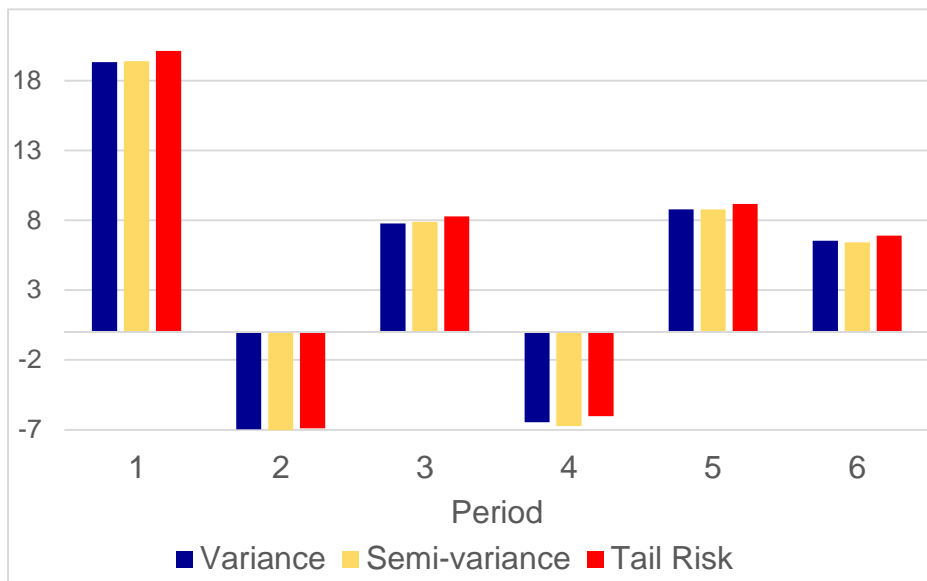


Figure 2: Cumulative returns of optimal portfolios under a positive SP500 weight constraint

Note: Annualized percentage cumulative returns.

As regards drawdowns, the maximum drawdown represents the maximum loss that a portfolio can experience in value or percentage (i.e. percentage drawdown), over a given period of time (e.g. investment horizon). It is measured as the difference between a peak and a nadir value of the portfolio. The maximum drawdown depends on the portfolio's average return and returns' fluctuations. The expected maximum drawdown represents the expected value of the maximum drawdown. Specifically, the expected maximum drawdown is computed while assuming the portfolio's cumulative returns to follow a Brownian motion (e.g. a random walk). Under such assumption, cumulative returns exhibit a constant drift over time with random deviations from such drift (Magdon-Ismail et al., 2004). The expected maximum drawdown depends positively on both the investment horizon and return volatility, but negatively on the portfolio's expected return. Table 12 displays the expected maximum drawdowns of optimal portfolios over the whole sample, and under the four considered cases. The expected maximum drawdowns reflect the risk of

optimally rebalanced portfolios under the considered cases. Under all cases, tail-risk optimal portfolios exhibit the lowest expected maximum drawdowns, whatever the constraints on the expected return or SP500 weight. The strongest loss scenario happens under cases 1 and 2, when both the expected return of the optimal portfolio is fixed, and the semi-variance is the risk measure. The weakest loss scenario happens under cases 3 and 4, when the expected return of the optimal portfolio is unconstrained, and tail risk is the risk measure. Thus, the constraint on the expected return reinforces the expected maximum drawdown. Such effect is slightly magnified when we add a positivity constraint to SP500 weight in tail-risk optimal portfolios. However, cumulative losses slightly lessen, when SP500 weight forbids short sales for remaining portfolios. When risk is measured by the expected maximum drawdown, and performance is measured by the annualized cumulative return, the tail-risk optimal portfolio offers therefore the most interesting risk-return tradeoff under cases 3 and 4. Analogously, the semi-variance optimal portfolio offers the most interesting risk-return tradeoff under case 3, while being close to the variance optimal portfolio under case 4 and semi-variance optimal portfolios under cases 3 and 4 (see Figure 3). The tradeoff focuses on higher cumulative returns with lower cumulative losses (i.e. increasing profits while reducing risk).

Table 12: Expected maximum drawdowns of optimal portfolios across cases (%)

Measure/Case	1	2	3	4
Variance	150.3851	150.3851	100.4680	94.0043
Semi-variance	158.5003	157.6532	95.1536	95.1503
Tail Risk	144.3803	144.3812	92.8681	93.6813

Note: Lowest expected maximum drawdowns are in bold.

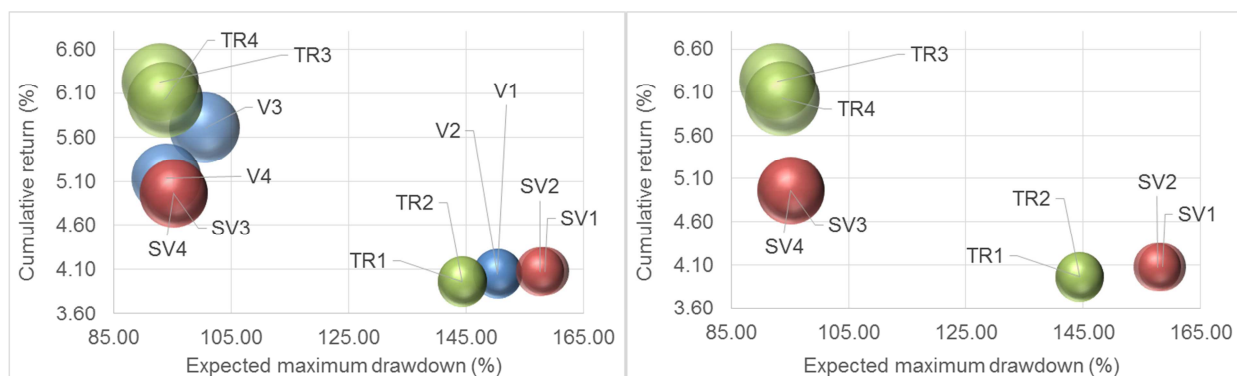


Figure 3: Risk-return tradeoff of optimal portfolios over the whole sample period

Note: Data are expressed in percent, and cumulative returns are annualized. The size of the bubbles represents the ratio of the cumulative return to the expected maximum drawdown. V, SV and TR stand for variance, semi-variance and tail risk. The number after the acronym represents the case under consideration. For example, TR4 stands for tail risk measure under case 4.

For comparison purposes, we also carry out the active management strategy, which results from Table 10. Over each regime, we build/rebalance the portfolio according to the most favorable risk minimization setting. Under each case and over each regime, we select the best-performing portfolio among the three measure-specific (i.e. variance, semi-variance or tail-risk) optimal portfolios. Such methodology allows for building superoptimal portfolios over the whole sample period. Superoptimal portfolios are thus optimally rebalanced across regimes and risk measures. Table 13 displays the risk and return attributes of the superoptimal portfolios. The annualized cumulative returns and expected maximum drawdowns of these superoptimal portfolios are close to the ones in Table 11 and Table 12 (see Figure 4). Due to the difficulty to forecast dependence structures and suitable risk minimization, investors can rely on previous findings since optimal and superoptimal portfolios offer close risk-return tradeoffs. As a result, active portfolio managers should build tail-risk optimal portfolios in the absence of a return constraint. Without such constraint, they can also build either variance or semi-variance optimal portfolios, depending on the joint dependence structure of assets. Under such scheme, the positivity constraint on SP500 weight slightly reduces the portfolios' performance. Finally, active portfolio managers should build tail-risk optimal portfolios in the presence of a return constraint.

Table 13: Attributes of superoptimal portfolios across cases (%)

Case	1	2	3	4
Annualized cumulative Return	4.0803	4.0746	6.6555	6.0248
Expected maximum drawdown	118.1443	118.2476	70.1571	69.6141

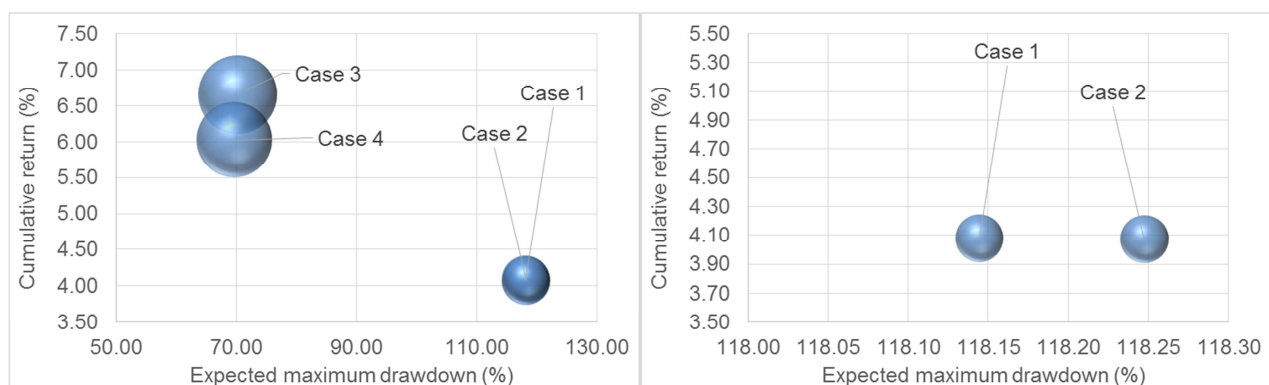


Figure 4: Risk-return tradeoff of superoptimal portfolios over the whole sample period

Note: Data are expressed in percent, and cumulative returns are annualized. The size of the bubbles represents the ratio of the cumulative return to the expected maximum drawdown.

6. Extension to global, mid-cap and small-cap stock portfolios

The study above-mentioned scrutinizes the diversification power of commodities in large-cap portfolios, as represented by the Standard and Poor's 500 index (SP500). Expanding on such a study, we replace successively SP500 index with Standard and Poor's 1500 (SP1500), Standard and Poor's MidCap 400 (SPMid400) and Standard and Poor's SmallCap600 (SPSmall600) indexes. Thus, we are able to gauge the diversifying role of crude oil and natural gas in global market (SP1500), mid-cap (SPMid400) and small-cap (SPSmall600) stock portfolios.

6.1. Variance regimes and optimal copulas

Unreported results exhibit five structural breaks for each series of index returns, so that we end with six periods. Strikingly, SP1500 index exhibits the same periods as SP500 index. As regards SPMid400 and SPSmall600 indexes, the first three periods are not identical to but exhibit a non-negligible overlapping fraction with the ones of SP500 index. Besides, the three last periods are the same as the ones of SP500 index. During the first three index-specific periods, the period-specific standard deviations of crude oil and natural gas returns are very close to their counterparts over the SP500-specific periods (see supplementary material). Moreover, over the six referenced periods of each stock index under consideration (i.e. across the index-specific periods), unreported zero-mean equality and robust variance equality tests highlight a unique zero-mean regime and various variance regimes for all stock and commodity return series. As a result, each set of a given stock index, crude oil and natural gas returns exhibits six different three-dimensional variance regimes. Such variance regimes are very close across stock indexes over corresponding periods (see Table 14, Table 15 and Table 16). Finally, Table 14, Table 15 and Table 16 display also the optimal copula representations across the six referenced variance regimes of each stock index. Such copulas are selected based on the Akaike (AIC) information criterion and the robust Cramer-von-Mises goodness-of-fit test (i.e. Rosenblatt's SnC statistics of Genest et al., 2009). Strikingly, SP1500 index portfolio exhibits the same behaviour as SP500 index portfolio. And, SPMid400 and SPSmall600 index portfolios exhibit the same dependence structures as SP500 index portfolio over their last four variance regimes (i.e. from period 3 to period 6). However, they exhibit different dependence structures over their first two variance regimes. Specifically, the dependence structure between SPMid400/SPSmall600 index, crude oil and natural gas exhibits symmetric tail dependence over the first variance regime with a Student T copula. Differently, the dependence structure between SPMid400/SPSmall600 index, crude oil and natural gas exhibits lower/no tail dependence over the second variance regime with a Clayton/Frank copula respectively. Results

emphasize therefore the regime-switching dependence structure of all three-asset portfolios (i.e. diversified global, mid-cap and small-cap stock portfolios) over the sample period.

Table 14: Period-specific variance regimes and optimal copulas for SP1500 index portfolio

Period	Start	End	Regime SP1500/Oil/Gas	Copula	Tail
1	01/08/1997	03/24/2000	LM/H/M	Gumbel	Upper
2	03/27/2000	05/20/2003	M/H/H	Gaussian	None
3	05/21/2003	07/16/2008	L/M/M	Gaussian	None
4	07/17/2008	08/29/2011	H/H/M	T	Symmetric
5	08/30/2011	10/13/2014	L/L/L	T	Symmetric
6	10/14/2014	10/30/2017	VL/H/L	T	Symmetric

Note: The variance regime of index returns can be very low (VL), low (L), low-medium (LM), medium (M), or high (H). The low-medium variance regime is an intermediate state, which lies between the low and the medium regimes. The very low variance regime is the lowest variance level, which is observed across the referenced periods.

Table 15: Period-specific variance regimes and optimal copulas for SPMid400 index portfolio

Period	Start	End	Regime SPMid400/Oil/Gas	Copula	Tail
1	01/08/1997	05/23/2001	M/H/M	T	Symmetric
2	05/24/2001	04/29/2005	M/H/H	Clayton	Lower
3	05/02/2005	07/16/2008	L/M/M	Gaussian	None
4	07/17/2008	08/29/2011	H/H/M	T	Symmetric
5	08/30/2011	10/13/2014	L/L/L	T	Symmetric
6	10/14/2014	10/30/2017	VL/H/L	T	Symmetric

Note: The variance regime of index returns can be very low (VL), low (L), medium (M), or high (H). The very low variance regime is the lowest variance level, which is observed across the referenced periods.

Table 16: Period-specific variance regimes and optimal copulas for SPSmall600 index portfolio

Period	Start	End	Regime SPSmall600/Oil/Gas	Copula	Tail
1	01/08/1997	09/21/2001	M/H/M	T	Symmetric
2	09/24/2001	02/02/2005	M/H/H	Frank	None
3	02/03/2005	07/16/2008	LM/M/M	Gaussian	None
4	07/17/2008	08/29/2011	H/H/M	T	Symmetric
5	08/30/2011	10/13/2014	LM/L/L	T	Symmetric
6	10/14/2014	10/30/2017	L/H/L	T	Symmetric

Note: The variance regime of index returns can be low (L), low-medium (LM), medium (M), or high (H). The low-medium variance regime is an intermediate state, which lies between the low and the medium regimes.

Expanding on the similarities and discrepancies between the dependence structures of stock index portfolios, we will assess the impact of such patterns on the optimization and performance of diversified stock portfolios. We expect that the commodity-based diversification process of SP1500 index, as a global stock portfolio proxy, will exhibit similar characteristics as the ones of SP500 index, as a large-cap stock portfolio proxy. Indeed, SP500 index is usually employed as a stock market proxy and its behavior is very close to the one of SP1500 index (see supplementary material). A priori, we also expect the diversification scheme of SPMid400 and SPSmall600 indexes to be close enough to the one of SP500 index from period 3 to period 6 (i.e. over the last four variance regimes). However, we expect more or less strong discrepancies over periods 1 and 2 (i.e. over the first two variance regimes).

6.2. Optimal portfolios' weights and attributes

We introduce the optimal portfolios' weights and their respective risk and performance attributes, under the three considered optimization schemes (i.e. variance, semi-variance and tail risk optimal portfolios). The performance metrics consists of the annualized cumulative return while the risk metrics consists of the expected maximum drawdown.

As regards optimal portfolios' allocations, Figure 5 to Figure 13 display variance, semi-variance and tail risk optimal weights across variance regimes, for each diversified stock portfolio (i.e. SP1500, SPMid400 and SPSmall600 indexes, which are diversified with holdings in crude oil and natural gas assets). With respect to SP1500 index, the optimal weights under cases 1 and 2 are close in general. The same comment applies to cases 3 and 4, except over period 1 under the variance and tail risk optimization schemes. Since the portfolio's allocation to the stock index goes beyond 100% under case 3, the constraint on SP1500 weight under case 4, which bounds the stock index allocation to 100%, contributes to increase the portfolio's allocation to crude oil. Moreover, whatever the optimization scheme, the optimal portfolio yields natural gas short sales over period 1, when the portfolio's expected return is jointly gauged with optimal weights. With respect to SPMid400 and SPSmall600 indexes, the optimal weights of cases 1 and 2 on the one hand, and of cases 3 and 4 on the other hand, are also close in general. The only exception applies to period 2 under cases 1 and 2, whatever the optimization scheme (except tail risk optimization for SPSmall600 index). Focusing on SPMid400 index, switching from case 1 to case 2 (i.e. bounding the stock index allocation between 0 and 1, under a fixed target return) reduces the stock index allocation of the portfolio to zero. Thus, diversification disappears since the optimal portfolio drops the stock investment and encompasses only (or mainly) energy commodities. Moreover, such scenario increases broadly the portfolio's allocation to crude oil, while yielding significant natural gas short sales. Focusing on SPSmall600 index, switching from case 1 to case 2 similarly reduces to zero (or almost zero) the stock index weight. However, the portfolio's investment in SPSmall600 index under case 1 is reallocated to crude oil under case 2. Furthermore, any optimization scheme yields both crude oil short sales and a large portfolio exposure to natural gas under case 2. Finally, apart from the scenarios above-mentioned, most of the optimal portfolios' weights are positive whatever the optimization scheme and stock index under consideration. Nonetheless, the optimal weights of SPMid400 and SPSmall600 index portfolios often differ non-negligibly from the ones of SP500 index portfolio. But, the optimal weights of SP1500 index portfolio are close to the ones of SP500 index portfolio.

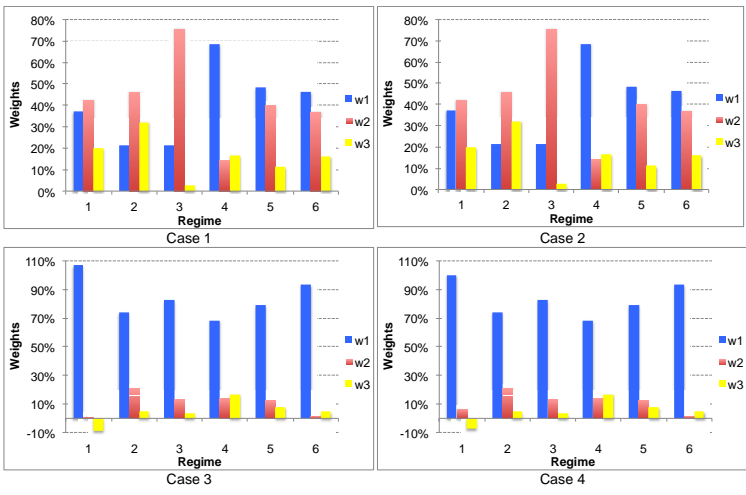


Figure 5: Variance optimal weights for SP1500 index portfolio

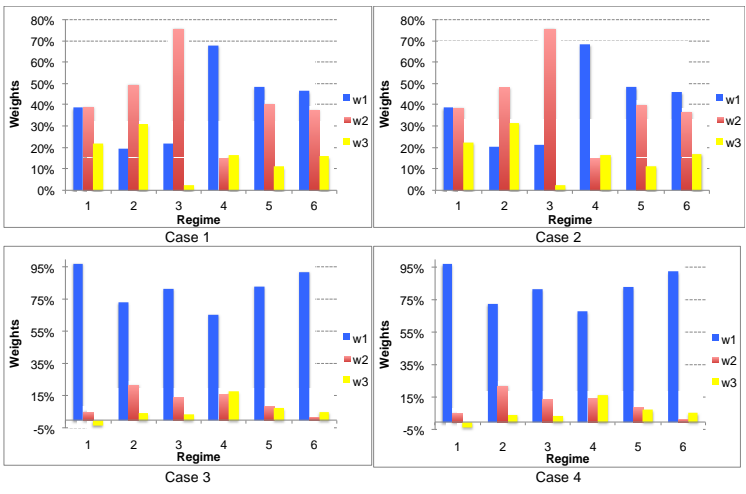


Figure 6: Semi-variance optimal weights for SP1500 index portfolio

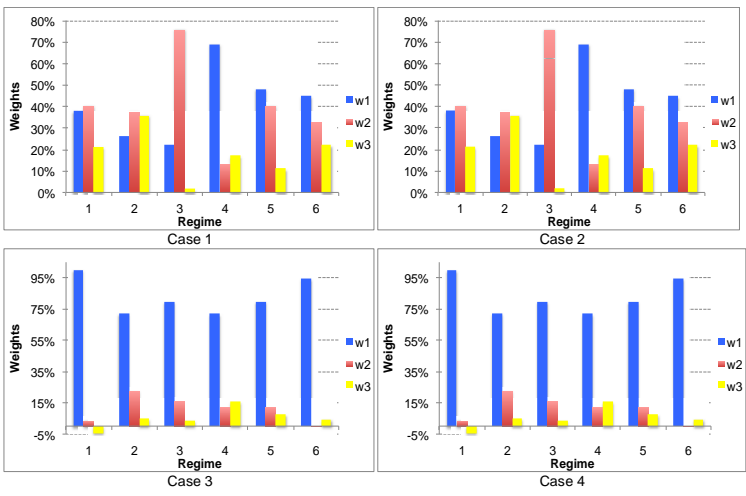


Figure 7: Tail risk optimal weights for SP1500 index portfolio

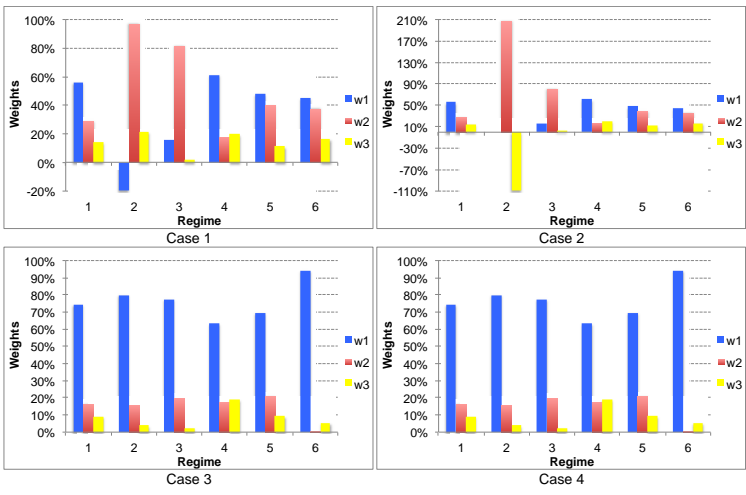


Figure 8: Variance optimal weights for SPMid400 index portfolio

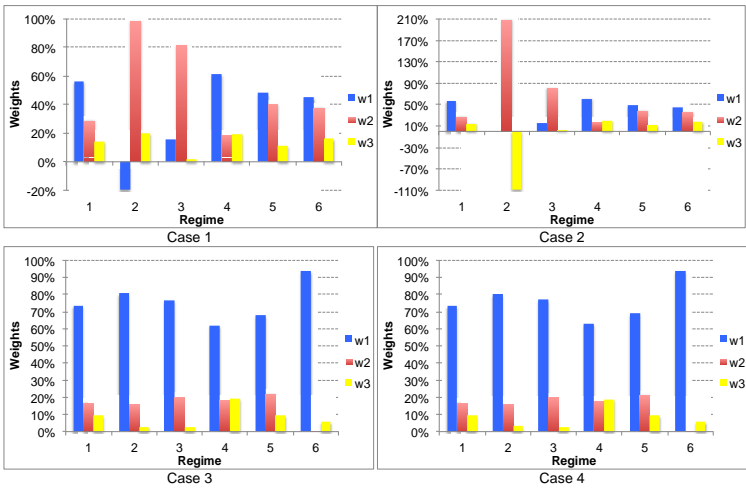


Figure 9: Semi-variance optimal weights for SPMid400 index portfolio

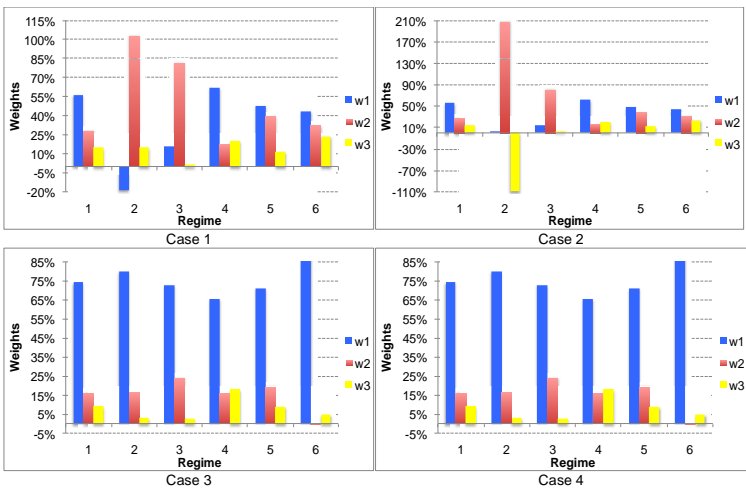


Figure 10: Tail risk optimal weights for SPMid400 index portfolio

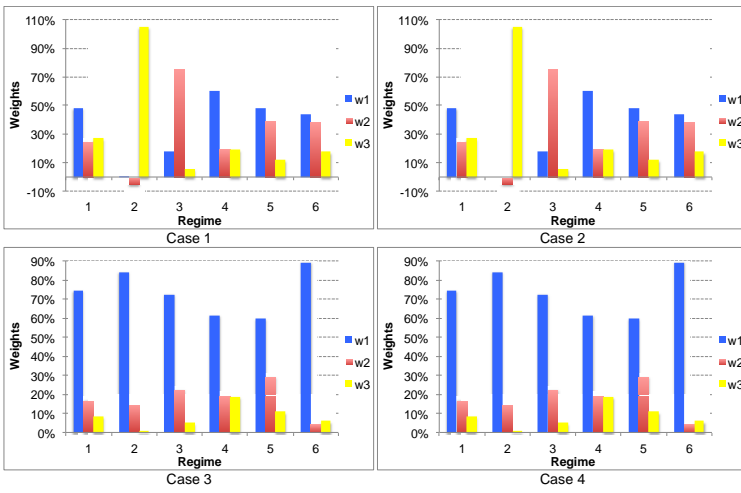


Figure 11: Variance optimal weights for SPSmall600 index portfolio

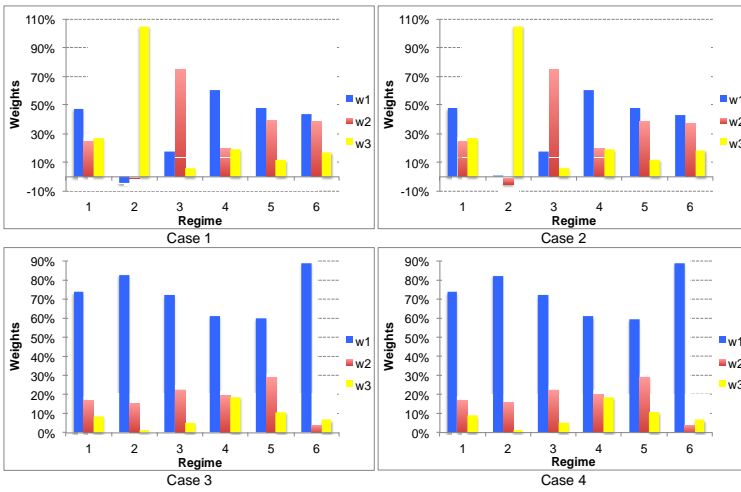


Figure 12: Semi-variance optimal weights for SPSmall600 index portfolio

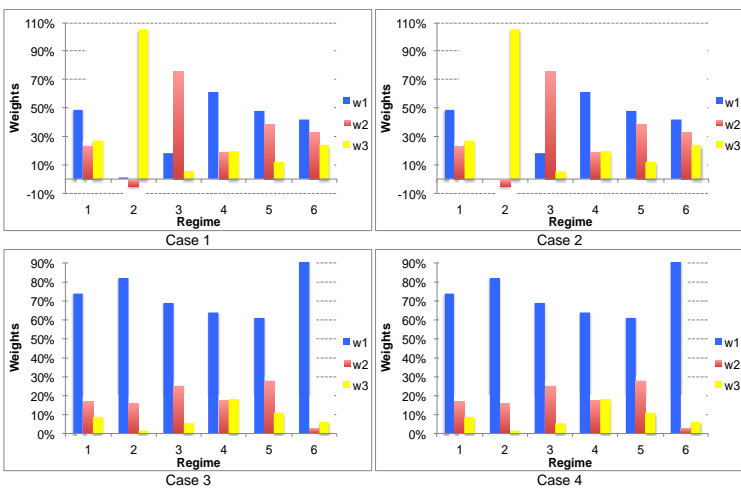


Figure 13: Tail risk optimal weights for SPSmall600 index portfolio

As regards optimal portfolios' attributes, Figure 14 to Figure 16 display the risk and return attributes of optimal portfolios over the whole sample period, and for all the stock indexes under consideration (see supplementary material for numerical details). The figures plot the annualized cumulative return against the expected maximum drawdown (as a risk measure). Obviously, general conclusions are similar to the ones of SP500 index study. Under all cases and across all stock indexes, tail-risk optimal portfolios exhibit the lowest expected maximum drawdowns and highest cumulative returns, whatever the constraints on the expected return or stock index weight. The strongest loss scenarios happen under cases 1 and 2, when the expected return of the optimal portfolio is fixed. In particular, the worst loss and cumulative return scenarios occur for SPMid400 index portfolio under case 2, the lowest values being reached when the risk measure is the tail risk. Strikingly, cumulative returns become negative under cases 1 and 2 for SPSmall600 index portfolio, and under case 2 for SPMid400 index portfolio. Such results emphasize thus the strong negative impact of the constraints on the stock index weight and the portfolio's expected return. The magnitude of the negative impact is the largest under case 2 for mid-cap portfolios. Therefore, portfolio optimization should avoid any return and, more specifically, stock weight constraints when it targets to diversify small/mid-cap stock portfolios with energy commodity assets. Conversely, the weakest loss scenarios happen under cases 3 and 4, when the expected return of the optimal portfolio is unconstrained, the lowest values being reached when tail risk is the risk measure. As a result, when risk is measured by the expected maximum drawdown, and performance is measured by the annualized cumulative return, tail-risk optimal portfolios offer the most interesting risk-return tradeoff under cases 3 and 4. However, the risk-return tradeoff of variance and semi-variance optimal portfolios is generally close to the one of tail risk optimal portfolios under cases 3 and 4. The best tradeoff focuses on higher cumulative returns, coupled with lower cumulative losses (i.e. increasing profits while reducing risk). Finally, focusing on the diversifying power of crude oil and natural gas in stock portfolios, optimally diversified mid-cap stock portfolios (i.e. SPMid400) outperform optimally diversified global, large-cap and small-cap stock portfolios under cases 1, 3 and 4. Differently, optimally diversified global stock portfolios (i.e. SP1500) outperform optimally diversified large-cap, mid-cap and small-cap stock portfolios under case 2. Such findings probably result from the balance between the volatility of constituting stocks and that one of crude oil and natural gas assets, in association with the portfolio's dependence structure. It is well known that the smaller the capitalization of stocks is, the larger the volatility of stock returns becomes (see supplementary material for example).

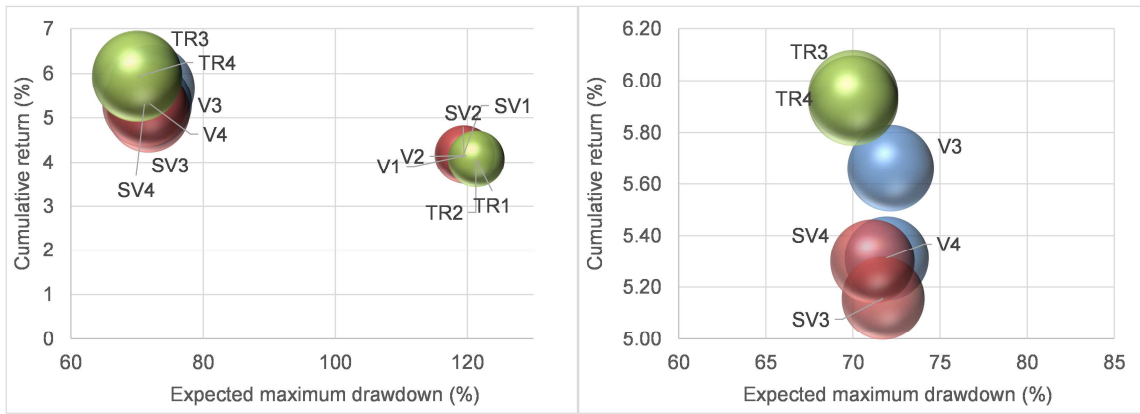


Figure 14: Risk-return tradeoff of optimal portfolios over the whole sample period for SP1500 index portfolio

Note: Data are expressed in percent, and cumulative returns are annualized. The size of the bubbles represents the ratio of the cumulative return to the expected maximum drawdown. V, SV and TR stand for variance, semi-variance and tail risk. The number after the acronym represents the case under consideration. For example, TR4 stands for tail risk measure under case 4.

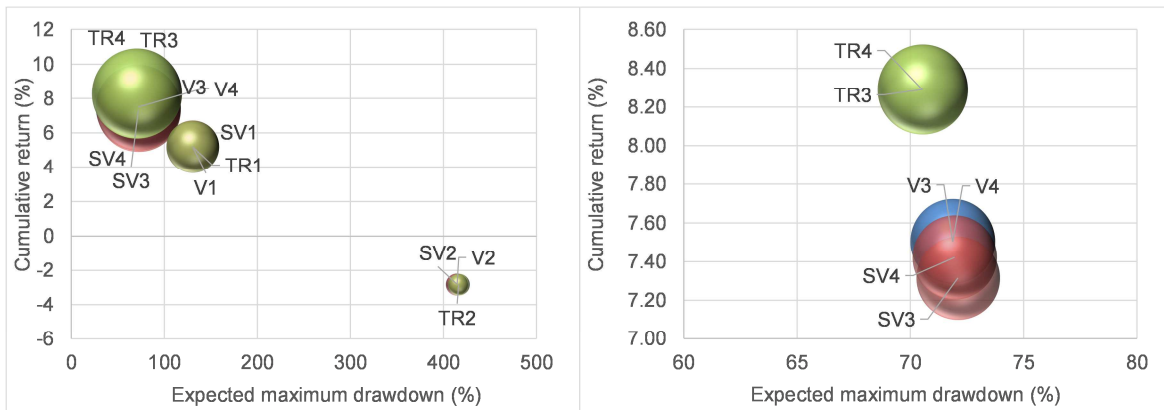


Figure 15: Risk-return tradeoff of optimal portfolios over the whole sample period for SPMid400 index portfolio

Note: Data are expressed in percent, and cumulative returns are annualized. The size of the bubbles represents the ratio of the absolute cumulative return to the expected maximum drawdown. V, SV and TR stand for variance, semi-variance and tail risk. The number after the acronym represents the case under consideration. For example, TR4 stands for tail risk measure under case 4.

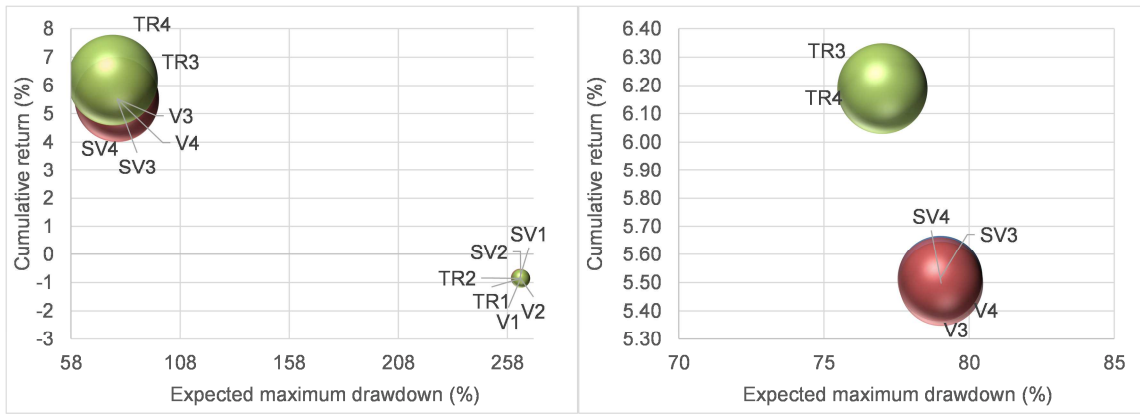


Figure 16: Risk-return tradeoff of optimal portfolios over the whole sample period for SPSmall600 index portfolio

Note: Data are expressed in percent, and cumulative returns are annualized. The size of the bubbles represents the ratio of the absolute cumulative return to the expected maximum drawdown. V, SV and TR stand for variance, semi-variance and tail risk. The number after the acronym represents the case under consideration. For example, TR4 stands for tail risk measure under case 4.

6.3. Building superoptimal portfolios

We carry out successively the following active management strategy for SP1500, SPMid400 and SPSmall600 stock index portfolios. Over each regime, we rebalance the portfolio according to the most favorable risk minimization setting (i.e. variance, semi-variance or tail-risk optimal weights). Specifically, over a given variance regime, we select the optimal portfolio with the highest cumulative return across the three considered risk measures. Such methodology allows for building superoptimal portfolios over the whole sample period. Hence, superoptimal portfolios are optimally rebalanced across regimes and risk measures (see supplementary material for details about the selected optimal portfolios across regimes and stock indexes).

Focusing on the risk-return profile of superoptimal portfolios, the general conclusions are the same as the ones for optimal portfolios (see Figure 17 to Figure 19). In particular, superoptimal portfolios under cases 3 and 4 outperform superoptimal portfolios under cases 1 and 2, whatever the stock index under consideration. Thus, the constraint on the portfolio's expected return impairs the performance of optimal portfolios. Such negative impact is stronger for SPMid400 index portfolio under case 2, and SPSmall600 index portfolio under cases 1 and 2. Under the cases above-mentioned, the respective optimal portfolios attempting to diversify SPMid400 and SPSmall600 indexes yield short sales in one of the energy commodities over regime 2. During such regime, optimal portfolios (and therefore superoptimal portfolios) experience significant negative annualized cumulative returns.

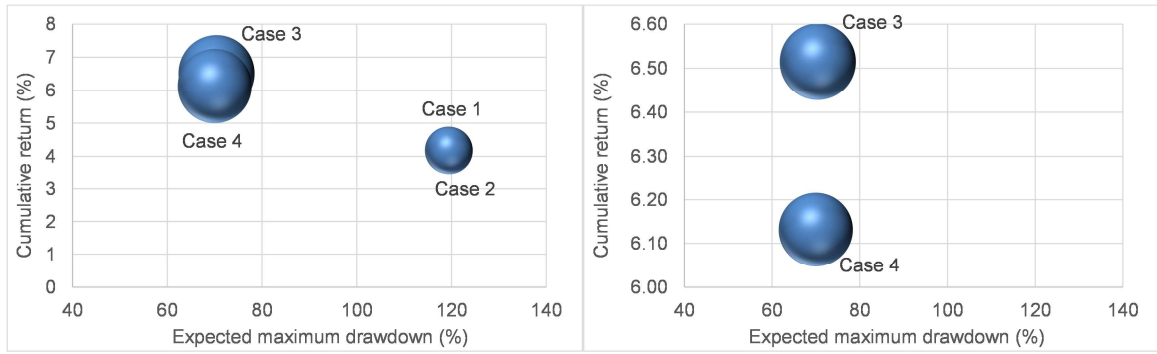


Figure 17: Risk-return tradeoff of superoptimal portfolios over the whole sample period, for SP1500 index portfolio

Note: Data are expressed in percent, and cumulative returns are annualized. The size of the bubbles represents the ratio of the cumulative return to the expected maximum drawdown.

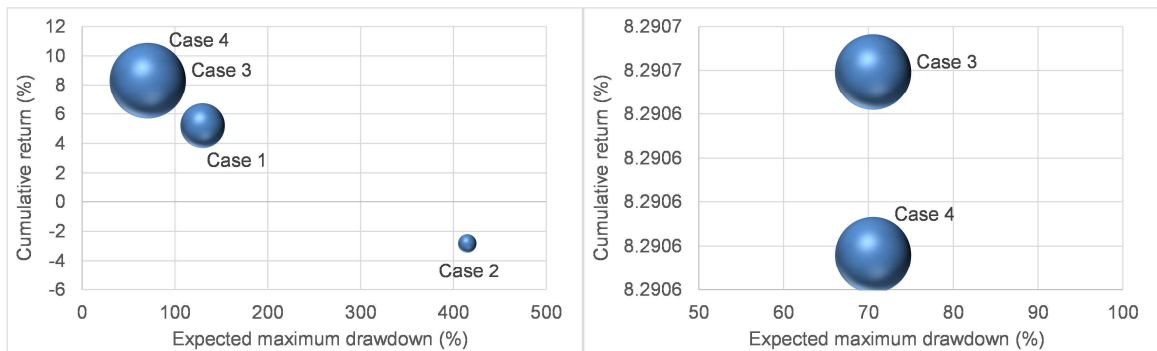


Figure 18: Risk-return tradeoff of superoptimal portfolios over the whole sample period, for SPMid400 index portfolio

Note: Data are expressed in percent, and cumulative returns are annualized. The size of the bubbles represents the ratio of the absolute cumulative return to the expected maximum drawdown.

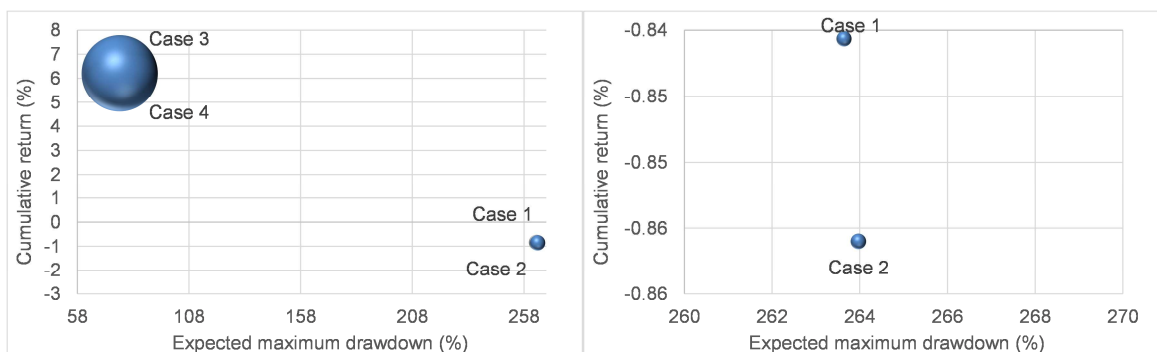


Figure 19: Risk-return tradeoff of superoptimal portfolios over the whole sample period, for SPSmall600 index portfolio

Note: Data are expressed in percent, and cumulative returns are annualized. The size of the bubbles represents the ratio of the absolute cumulative return to the expected maximum drawdown.

In order to rank superoptimal portfolios, we build a risk adjusted performance measure that we call the adjusted annualized cumulative return (AACR). The AACR is the ratio of the annualized cumulative return (ACR) to the expected maximum drawdown (EMDD), which is expressed in absolute value. Figure 20 illustrates such risk adjusted performance measure for all the stock indexes under consideration (see also supplementary material to get corresponding numerical values). Obviously, the superoptimal portfolio diversifying SPMid400 index outperforms all other superoptimal portfolios under cases 1, 2 and 3. However, the superoptimal portfolio diversifying SP1500 index outperforms all other superoptimal portfolios under case 2 (though the AACR of SP500 index portfolio is close to that one of SP1500 index portfolio).

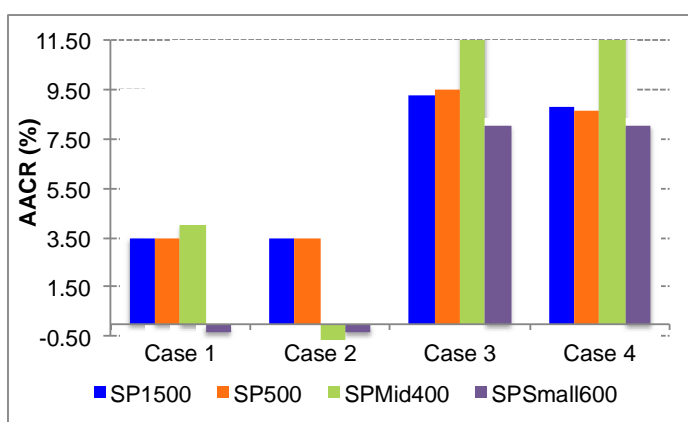


Figure 20: Risk adjusted annualized cumulative returns (AACR) of superoptimal portfolios

7. Conclusion

We consider the joint behavior of energy commodity prices and the U.S. stock market index over time. The SP1500/SP500/SPMid400/SPSmall600 indexes as well as crude oil and natural gas prices exhibit structural changes with various variance regimes. As a result, the joint dependence structure of any U.S. stock market index and the two energy commodities is unstable over time. Such feature has significant implications for investors building portfolios with these three types of assets, and requires a regime-specific analysis. In this light, we examine the implications of such regime-dependency for portfolio optimization. We minimize various risk measures after accounting for regime-specific dependence structures. The dependence structures handle the asymmetry in asset returns and tail dependency. The risk criteria consist of variance, and two downside risk measures such as semi-variance and tail risk. Moreover, we consider several minimization schemes while setting or discarding constraints on the portfolio's expected

return and stock index weight. We examine optimal portfolios while analyzing their degrees of similarity, their annualized cumulative returns as a performance indicator, and their expected maximum drawdowns as a risk measure.

The positive weight constraint on the stock index reduces the performance of the optimal portfolio over given regimes. Besides, optimal portfolios are heterogeneous and dissimilar across regimes and risk minimization schemes. We also build optimal portfolios, which we rebalance across regimes, and analyze their risk-return tradeoff over the whole sample period. Under each risk minimization scheme, the expected maximum drawdown serves as a risk measure while the annualized cumulative return gives insights about performance. Under a constraint on the expected return of the optimal portfolio, the tail-risk optimal portfolio offers the most interesting risk-return tradeoff (i.e. lowest risk and highest return). Analogously, the tail-risk optimal portfolio offers the most interesting risk-return tradeoff without constraint on the expected return. However, variance optimal portfolios, and then semi-variance optimal portfolios can also be considered in the absence of a return constraint. Finally, the diversification power of energy commodities is often the most effective for SPMidCap 400 index (i.e. mid-cap stock portfolios' diversification), since it generally reduces the expected maximum drawdown and increases the annualized cumulative return of corresponding portfolios. As a consequence, optimization results handle regime-dependency since they depict the joint risk structure of portfolio constituents over time. They capture the price uncertainty of both constituent assets and resulting portfolio over time, and highlight the efficiency of an active portfolio management strategy. Future research should exploit such findings, and attempt to forecast the upcoming portfolio scenarios. Potential research extensions rely on a scenario analysis, which faces two major challenges. First, the analysis needs to predict the upcoming variance regimes of constituent assets, and then those of the portfolio. Second, such approach needs to identify the plausible dependence structures over the forthcoming variance regimes. For example, such scenario analysis can exploit the plurality of past variance regimes and corresponding dependence structures. It could also rely on the link between the stock market and the business cycle, which can be forecasted to a large extent. Such link is also specific to both the capitalizations of stocks and their corresponding industry.

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Appendix A.

We display relevant theoretical background and statistical results in this section.

A.1. Descriptive statistics of returns

We display the descriptive statistics of return data over the whole sample and its six referenced periods. The Phillips-Perron unit root test (without trend and without intercept) is also provided.

Table A.1: Descriptive statistics about U.S. energy commodity and stock market returns

Statistics	Gas	Oil	SP500
Mean	-0.0050	0.0138	0.0234
Median	0.0000	0.0621	0.0559
Std. Dev.	4.3911	2.4733	1.2116
Skewness	0.5843	-0.0928	-0.2311
Kurtosis	23.6372	7.4910	11.0103
Jarque-Bera	93,445.0960	4,418.7093	14,080.2268
Nobs	5249	5249	5249

Note: Std. Dev. stands for standard deviation and Nobs stands for the number of observations (i.e. sample size).

Table A.2: Unit root test of energy commodity and stock index returns

Statistics	Gas	Oil	SP500
PP	-70.0377	-74.3348	-78.5123
Probability	0.0001	0.0001	0.0001

Note: PP stands for the Phillips-Perron statistics, and the probability reports corresponding p-value.

A.2. Structural break test

We focus on the joint dynamics of SP500, crude oil and natural gas returns, and look for structural changes, or equivalently, breaks in such joint dynamics. Given that we target to diversify a stock portfolio, as represented by SP500, with two energy commodities such as crude oil and natural gas, we consider the linear regression of SP500 returns on both crude oil

and natural gas returns. The regression is undertaken with the Heteroskedastic and Autocorrelation Consistent covariance (HAC) method¹⁵ as well as residuals' prewhitening with automatic lag selection (Hannan-Quinn criterion).

Investigating sudden changes in such relationship, we scrutinize changes in all regression parameters (i.e. intercept and slopes) as well as in the residuals' distribution across possible regimes. In particular, residuals follow a Gaussian distribution with a zero mean and different standard deviations across regimes. Since we allow for shifts in the regression's intercept, it is equivalent to consider structural breaks in mean, while shifts in standard deviations account for structural breaks in variance.

In line with Bai and Perron (2003a, 2003b), and Qu and Perron (2007), we consider the following specification:¹⁶

$$R_{SP500,t} = \alpha_i + \beta_{1,i} \times R_{Oil,t} + \beta_{2,i} \times R_{Gas,t} + \varepsilon_{t,i}$$

where $R_{SP500,t}$, $R_{Oil,t}$ and $R_{Gas,t}$ are the returns of SP500, crude oil and natural gas respectively at time t , α_i is the regression's intercept while $\beta_{1,i}$ and $\beta_{2,i}$ are the regression's slopes over a given regime i , and $\varepsilon_{t,i}$ are regression residuals at time t under a regime i .

We assume a number m of breaks so that the data sample splits into $(m+1)$ periods, which are defined as follows:

$$\text{Period } 1 = \{1, \dots, T_1\}$$

$$\text{Period } i = \{T_{i-1}+1, \dots, T_i\}$$

$$\text{Period } m+1 = \{T_m+1, \dots, T\}$$

where the index i lies between 2 and m , and T_i is a breakpoint indicating the end of period i . The standard deviation σ_i of residuals' distribution can change across any period i . Thus, we test for possible changes in mean returns α_i and corresponding standard deviations σ_i over the whole sample.

We investigate the existence of up to $m=5$ breaks so as to end with a maximum of $m+1=6$ periods or regimes. At a five percent test level, the reported F and WDmax(5) statistics

¹⁵ It is robust to heteroskedasticity and autocorrelation in regression residuals.

¹⁶ A pure structural change model.

advocate five breaks so that we need to consider 6 periods (see Table A.3: Structural break statistics for $m=5$ breaks and Table A.4: Identified periods from structural break test).

Table A.3: Structural break statistics for $m=5$ breaks

Statistic	Computed value	Critical value
WDMax(5)	74.6452	15.5900
F	39.8322	7.4600

Table A.4: Identified periods from structural break test

Period	Start	End
1	1/8/1997	3/24/2000
2	3/27/2000	5/20/2003
3	5/21/2003	7/16/2008
4	7/17/2008	8/29/2011
5	8/30/2011	10/13/2014
6	10/14/2014	10/30/2017

Once the six periods are identified, we investigate further period-specific statistics of SP500 and energy commodity returns over each period (see Table A.5: Further period-specific statistics).

Table A.5: Further period-specific statistics

Period	Statistics	Gas	Oil	SP500
1	Skewness	0.1574	0.9613	-0.3957
	Kurtosis	13.1504	8.0607	6.5488
2	Skewness	0.5327	-0.6511	0.2434
	Kurtosis	30.9248	7.2352	3.9993
3	Skewness	0.1982	-0.2106	-0.1810
	Kurtosis	7.5935	4.5715	5.0162
4	Skewness	0.8392	0.1286	-0.2654
	Kurtosis	13.1245	6.8222	9.1034
5	Skewness	1.6010	0.0105	-0.1832
	Kurtosis	33.3775	6.1391	5.3221
6	Skewness	0.2999	0.2469	-0.3910
	Kurtosis	9.5860	4.7214	6.3684

A.3. Copula functions

The copula functions under consideration are mentioned below as in Gatfaoui (2016a), and exhibit the characteristics displayed in Table A.6: Three-dimension copulas and characteristics. Recall that the continuous copula $C(\cdot)$ has a density-based representation $c(\cdot)$, so that $c(u, v, w) = \frac{\partial^3 C(u, v, w)}{\partial u \partial v \partial w}$ for u, v and w in $[0, 1]$. The Gaussian and Student T copulas write in their density forms while the remaining copula functions write in their classic forms (i.e. cumulative distribution forms).

Table A.6: Three-dimension copulas and characteristics

Copula	Type	Parameter(s)	Tail
Clayton	Archimedean	Correlation θ	Lower
Frank			None
Gumbel			Upper
Gaussian	Elliptical	Correlation matrix ρ	None
Student T		Degree of freedom ν , Correlation matrix ρ	Symmetric

Let $\theta > 0$ be a positive correlation parameter and u, v, w in $[0, 1]$, the **Clayton** copula writes:

$$C(u, v, w; \theta) = \frac{1}{(u^{-\theta} + v^{-\theta} + w^{-\theta} - 2)^{1/\theta}} \quad (11)$$

Let ρ be a correlation matrix and u, v, w in $[0, 1]$, the **Gaussian** copula density writes:

$$c(u, v, w; \rho) = \frac{1}{|\rho|^{1/2}} \exp \left\{ -\frac{1}{2} \xi^t (\rho^{-1} - \mathbf{I}) \xi \right\} \quad (12)$$

Where $\exp(\cdot)$ is the exponential function, ρ and ρ^{-1} are a three-dimension matrix and its inverse respectively, $|\rho|$ is the determinant of the correlation matrix, ξ is the vector of the inverse standard univariate Gaussian cumulative distribution function, which is applied to each element u, v, w , and finally ξ^t is the transposed vector of ξ . A three-dimension identity matrix \mathbf{I} (i.e. with unit diagonal terms and zero elsewhere) is also employed.

Let $\theta > 0$ be a positive correlation parameter and u, v, w in $[0, 1]$, the **Frank** copula writes:

$$C(u, v, w; \theta) = -\frac{1}{\theta} \ln \left\{ 1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)(e^{-\theta w} - 1)}{(e^{-\theta} - 1)^2} \right\} \quad (13)$$

Let $\theta > 1$ be a positive correlation parameter and u, v, w in $[0, 1]$, the **Gumbel** copula writes:

$$C(u, v, w; \theta) = \exp\left(-\{(-\ln u)^\theta + (-\ln v)^\theta + (-\ln w)^\theta\}^{1/\theta}\right) \quad (14)$$

Let ρ be a correlation matrix, ν a degree of freedom and u, v, w in $[0,1]$, the *Student T* copula density writes:

$$c(u, v, w; \rho, \nu) = \frac{1}{|\rho|^{1/2}} \frac{\Gamma(\frac{\nu+3}{2}) \{\Gamma(\frac{\nu}{2})\}^3 \left(1 + \frac{1}{\nu} \xi^t \rho^{-1} \xi\right)^{-\frac{\nu+3}{2}}}{\{\Gamma(\frac{\nu+1}{2})\}^3 \Gamma(\frac{\nu}{2}) \prod_{n=1}^3 \left(1 + \frac{\xi_n^2}{\nu}\right)^{-\frac{\nu+1}{2}}} \quad (15)$$

where ρ and ρ^{-1} are a three-dimension matrix and its inverse respectively, $|\rho|$ is the determinant of the correlation matrix, $\Gamma(\cdot)$ is the Gamma function, ξ is the vector (ξ_1, ξ_2, ξ_3) of the inverse univariate Student¹⁷ cumulative distribution function, which applies to each element u, v, w , and finally ξ^t is the transposed vector of ξ .

Appendix B. – Supplementary material (Online)

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¹⁷ The Student T distribution has ν degree(s) of freedom.

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