A PROBABILISTIC APPROACH OF ULTRAVIOLET RENORMALISATION IN THE BOUNDARY SINE-GORDON MODEL

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Abstract. The Sine-Gordon model is obtained by tilting the law of a log-correlated Gaussian field X defined on a subset of $\mathbb{R}^d$ by the exponential of its cosine, namely $\exp(\alpha \int \cos(\beta X))$. It is an important model in quantum field theory or in statistic physics like in the study of log-gases. In spite of its relatively simple definition, the model has a very rich phenomenology. While the integral $\int \cos(\beta X)$ can properly be defined when $\beta^2 < d$ using the standard Wick normalisation of $\cos(\beta X)$, a more involved renormalization procedure is needed when $\beta^2 \in [d, 2d)$. In particular it exhibits a countable sequence of phase transition accumulating to the left of $\beta = \sqrt{2d}$, each transitions corresponding to the addition of an extra term in the renormalization scheme. The final threshold $\beta = \sqrt{2}$ corresponds to the Kosterlitz-Thouless (KT) phase transition of the log-gas. In this paper, we present a novel probabilistic approach to renormalization of the two-dimensional boundary (or 1-dimensional) Sine-Gordon model up to the KT threshold $\beta = \sqrt{2}$. The purpose of this approach is to propose a simple and flexible method to treat this problem which, unlike the existing renormalization group techniques, does not rely on translation invariance for the covariance kernel of X or the reference measure along which $\cos(\beta X)$ is integrated. To this purpose we establish by induction a general formula for the cumulants of a random variable defined on a filtered probability space expressed in terms of brackets of a family of martingales; to the best of our knowledge, the recursion formula is new and might have other applications. We apply this formula to study the cumulants of (approximations of) $\int \cos(\beta X)$. To control all terms produced by the induction proceedure, we prove a refinement of classical electrostatic inequalities, which allows to bound the energy of configurations in terms of the Wasserstein distance between $+$ and $-$ charges.

1. Introduction

1.1. **Log-Gases, $2D$-Yukawa Gas and $2D$-Coulomb Gas.** The d-dimensional *log-gas* is a model of statistical mechanics describing a gas of interacting charged particles, the interaction being given by a potential $V(x, y)$, where V is continuous outside of the diagonal and $V(x, y) \overset{|x-y|\to 0}{=} -\ln |x - y| + O(1)$.

Given two positive parameters $\alpha, \beta > 0$, α standing for particle activity and β^2 for inverse temperature (the reason for such an exotic parametrization appears below), its formal partition function over a bounded open set $\mathcal{O} \subset \mathbb{R}^d$ is given by

$$\mathcal{Z}^V_{\alpha,\beta}(\mathcal{O}) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \sum_{(\lambda_i)_{i=1}^n \in \{-1,1\}^n} \int_{\mathcal{O}^n} \exp\Big(- \beta^2 \sum_{1\le i<j\le n} \lambda_i\lambda_j V(x_i, x_j)\Big) \prod_{i=1}^n \mathrm{d}x_i. \tag{1.1}$$

The above integral diverges when $\beta^2 \geq d$. In this case, to have an interpretation of the above expression, we need to consider a reasonable sequence of bounded approximation V_ε of the potential V and consider the limit

$$\mathcal{Z}^V_{\alpha,\beta}(\mathcal{O}) = \lim_{\varepsilon\to 0} \mathcal{Z}^V_{\alpha,\beta,\varepsilon}(\mathcal{O}) \tag{1.2}$$

where

$$\mathcal{Z}^V_{\alpha,\beta,\varepsilon}(\mathcal{O}) := \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \sum_{(\lambda_i)_{i=1}^n \in \{-1,1\}^n} \int_{\mathcal{O}^n} \exp\Big(- \beta^2 \sum_{1\le i<j\le n} \lambda_i\lambda_j V_\varepsilon(x_i, x_j)\Big) \prod_{i=1}^n \mathrm{d}x_i. \tag{1.3}$$

While, of course, strictly speaking, $\mathcal{Z}^V_{\alpha,\beta,\varepsilon}$ also diverges when ε tends to 0 if $\beta^2 \geq d$, a non-trivial interpretation of the limit is obtained by dividing by an appropriate (and also diverging) counterterm. This

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renormalization scheme is the first step needed to define the Gibbs measure associated with the partition function $\mathcal{Z}_{\alpha,\beta}(\mathcal{O})$ and study its properties.

A prototypical example of log-gas is the $2D$-Yukawa gas (d=2) with mass $m>0$, corresponding to the potential

$$V_m(x,y)=\int_0^\infty \frac{1}{t}e^{-\frac{|x-y|^2}{2t}-mt}\mathrm{d}t.$$

Another example of central interest is the two-dimensional (free boundary) Coulomb gas, corresponding to taking the limit of the Yukawa potential when m goes to zero. This formally corresponds to taking $V(x,y)=-\ln|x-y|+\infty$, and has the effect of giving an infinite energy to any configuration with non-zero global charge. This corresponds to

$$\mathcal{Z}^{\mathrm{Coulomb}}_{\alpha,\beta}(\mathcal{O})=\sum_{n=0}^\infty \frac{\alpha^{2n}}{(n!)^2}\int_{\mathcal{O}^{2n}}\exp\Big(\beta^2\sum_{i<j}\lambda_i^{(n)}\lambda_j^{(n)}\ln|x_i-x_j|\Big)\prod_{i=1}^{2n}\mathrm{d}x_i, \tag{1.4}$$

where $\lambda_i^{(n)}=-1$ if $1\leq i\leq n$ and $\lambda_i^{(n)}=1$ if $n+1\leq i\leq 2n$. Again ln has to be replaced e.g. by $\ln_\varepsilon r:=\frac{1}{2}\ln(r^2+\varepsilon^2)$ for taking the limit when $\varepsilon\to 0$ and thus giving an appropriate renormalization scheme.

One could as well consider for V other types of Green functions with various boundary conditions, leading to Coulomb gases with various boundary conditions.

1.2. **The Sine-Gordon representation.** A well known tool to identify the proper renormalization for log-gases (including Coulomb and Yukawa) is their Sine-Gordon representation, which we briefly sketch now. To define it we need to assume that V is positive semidefinite in the sense that for any bounded continuous function on $\mathcal{O}$ we have

$$\int_{\mathcal{O}^2}V(x,y)g(x)g(y)\mathrm{d}x\mathrm{d}y\geq 0. \tag{1.5}$$

It is standard to check that the assumption is satisfied for the Yukawa potential. Concerning the Sine-Gordon representation of the (free boundary) Coulomb gas let us mention only that $V_R(x,y)=-\ln(|x-y|/R)$ is positive semidefinite if $2R$ is larger than the diameter of $\mathcal{O}$.

Consider a centered Gaussian Field X defined on $\mathcal{O}$ with covariance function

$$\mathbb{E}[X(x)X(y)]=V(x,y),\qquad \text{for } x,y\in\mathcal{O}.$$

The condition (1.5) guarantees that such a field X can be defined as a random distribution on $\mathcal{O}$: $X(x)$ does not make sense but $\int_{\mathcal{O}}X(x)g(x)\mathrm{d}x$ does for all sufficiently regular functions g.

The Sine-Gordon model is formally defined as the measure

$$\exp\big(2\alpha\int_{\mathcal{O}}\cos(\beta X(x))\,\mathrm{d}x\big)\mathbb{P}(\mathrm{d}X). \tag{1.6}$$

Since X is not a function but a random distribution, the mathematical interpretation of (1.6) requires further explanations. A proper definition requires renormalisation: more specifically, the field X can be smoothened by convolution with a reasonable (e.g. compactly supported and C^∞) isotropic kernel (in the paper we use another type of approximation of X which is not a convolution but this is not relevant for this part of the discussion). We call X_ε the regularized field and $V_\varepsilon(x,y)$ the corresponding covariance (we just write $V_\varepsilon(x)$ for the variance). The Sine-Gordon model is defined as the limit as ε goes to 0 of the following well-defined sequence of measures

$$\lim_{\varepsilon\to 0}\exp\big(2\alpha\int_{\mathcal{O}}\cos(\beta X_\varepsilon(x))e^{\frac{\beta^2}{2}V_\varepsilon(x)}\,\mathrm{d}x\big)\mathbb{P}(\mathrm{d}X). \tag{1.7}$$

This measure is directly related to the partition function of the d-dimensional log gas. Indeed, complex exponential moments of the Gaussian field are related to the partition function of the d-dimensional log gas with fixed number of particles through the following relation: for any charge distribution $(\lambda_i)_{i=1}^n$ with k positive charges and $n-k$ negative ones,

(1.8)
$$\mathbb{E}\Big[\Big(\int_{\mathcal{O}} e^{\mathbf{i}\beta X_\varepsilon(x)+\frac{\beta^2}{2}V_\varepsilon(x)}\,\mathrm{d}x\Big)^k\Big(\int_{\mathcal{O}} e^{-\mathbf{i}\beta X_\varepsilon(x)+\frac{\beta^2}{2}V_\varepsilon(x)}\,\mathrm{d}x\Big)^{n-k}\Big]=\int_{\mathcal{O}^n} e^{-\beta^2\sum_{1\le i<j\le n}\lambda_i\lambda_j V_\varepsilon(x_i,x_j)}\prod_{i=1}^n \mathrm{d}x_i.$$

As a consequence of (1.8) we have

$$(1.9)\quad \mathbb{E}\left[\exp\big(2\alpha\int_{\mathcal{O}}\cos(\beta X_\varepsilon(x))e^{\frac{\beta^2}{2}V_\varepsilon(x)}\,\mathrm{d}x\big)\right]=\mathbb{E}\left[e^{\alpha\int_{\mathcal{O}}(e^{\mathbf{i}\beta X_\varepsilon(x)+\frac{\beta^2}{2}V_\varepsilon(x)}+e^{-\mathbf{i}\beta X_\varepsilon(x)+\frac{\beta^2}{2}V_\varepsilon(x)})\,\mathrm{d}x}\right]$$
$$=\sum_{n=0}^\infty\frac{\alpha^n}{n!}\sum_{k=0}^n\binom{n}{k}\mathbb{E}\Big[\Big(\int_{\mathcal{O}}e^{\mathbf{i}\beta X_\varepsilon(x)}\,\mathrm{d}x\Big)^k\Big(\int_{\mathcal{O}}e^{-\mathbf{i}\beta X_\varepsilon(x)}\,\mathrm{d}x\Big)^{n-k}\Big]=\mathcal{Z}^V_{\alpha,\beta,\varepsilon}(\mathcal{O}).$$

For the representation of the (free boundary) Coulomb gas, we write $V_{R,\varepsilon}$ for the smoothened version of $V_R(x,y)=-\ln(|x-y|/R)$ and $X_{R,\varepsilon}$ for the associated Gaussian field. Let us fix R_0 such that $\mathcal{O}$ is included in a ball of radius R_0. Now observe that for $R\ge R_0$, $X_{R,\varepsilon}\overset{(d)}{=}X_{R_0,\varepsilon}+\sqrt{\ln(R/R_0)}Y$ where Y is an independent standard Gaussian, and that asymptotically $e^{\mathbf{i}\beta\sqrt{\ln(R/R_0)}Y}$ is uniformly distributed on the circle. Therefore we have

$$(1.10)\quad \mathcal{Z}^{\mathrm{Coulomb}}_{\alpha,\beta,\varepsilon}(\mathcal{O})=\lim_{R\to\infty}\mathbb{E}\left[\exp\big(2\alpha\int_{\mathcal{O}}e^{\frac{\beta^2}{2}V_{R_0,\varepsilon}(x)}\cos(\beta X_{R,\varepsilon}(x))\,\mathrm{d}x\big)\right]$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\mathbb{E}\Big[\exp\big(2\alpha\int_{\mathcal{O}}e^{\frac{\beta^2}{2}V_{R_0,\varepsilon}(x)}\cos(\theta+\beta X_{R_0,\varepsilon}(x))\,\mathrm{d}x\big)\Big]\,\mathrm{d}\theta.$$

We stress that it is important here not to include the dependence in R in the renormalization factor in order to obtain a non trivial limit, hence our choice of V_{R_0}. The Coulomb/Yukawa interaction being of log-type is a specificity of dimension 2. However, the phenomenology ruling the behavior of log-gases is independent of the dimension. Beyond its application to physically relevant $2D$-Coulomb/Yukawa cases, the problem of giving an interpretation to the limit (1.1) using the Sine-Gordon representation (1.7) is of interest from a mathematical perspective because it provides a testbed for renormalisation techniques, known as ultraviolet renormalisation of the Sine-Gordon model (as opposed to infrared renormalisation which is concerned with the large volume behaviour of the model).

1.3. **Renormalization and the multipole picture.** As soon as $\beta>0$, the definition of the limit (1.7) is non trivial since X_ε diverges pointwise. When $\beta^2<d$, however, for a large class of approximation schemes

$$\lim_{\varepsilon\to0}\exp\left(2\alpha\int_{\mathcal{O}}e^{\frac{\beta^2}{2}V_\varepsilon(z)}\cos(\beta X_\varepsilon(z))\,\mathrm{d}z\right)$$

exists and is integrable, and the obtained limit does not depend on the scheme we use for the approximation. This makes the limit (1.7) absolutely continuous with respect to $\mathbb{P}$, as was shown in [1]. As can be seen directly, the limit (1.1) is positive (and finite), and induces a distibution probability on the set of charged particles on $\mathcal{O}$ which only give weight to configurations with finitely many particles.

When β passes the threshold $\sqrt{d}$, the limit (1.1) becomes infinite and $\int_{\mathcal{O}}\cos(\beta X_\varepsilon(z))\,\mathrm{d}z$ does not converge as a random variable. When $\beta\in[\sqrt{d},\sqrt{2d})$ the divergency can be tamed by subtracting a number of field independent counterterms in the exponential (1.1), this corresponds to multiplying the sequence by $\exp(-\sum_{i=1}^k p_i(\varepsilon)\alpha^i)$, for a polynomial in α whose coefficients $p_i(\varepsilon)$ diverge as $\varepsilon\to0$.

The values of the counter terms and their number are obtained by an asymptotic analysis of the cumulants of the random variable

$$\int_{\mathcal{O}}e^{\frac{\beta^2}{2}V_\varepsilon(x)}cos(\beta X_\varepsilon(x))\,\mathrm{d}x.$$

When $\beta^2<2d$ only a finite number of cumulants diverge. This number increases with β and tends to infinity when β approaches $\sqrt{2d}$. Furthermore the analysis of cumulants allows us to define a non trivial limit for the distribution (1.7) which is singular with respect to $\mathbb{P}$.

When $\beta^2 \geq 2d$, an infinite number of cumulants diverge. This makes renormalization impossible and there is thus no possible interpretation of (1.1) and (1.7) beyond this value. The origin of these divergencies can be understood in a simple way via the multipole picture originally described in [1]. Let us expose it in the framework of the $2D$-Coulomb gas[1] (but the phenomenology applies in every dimension and for other potentials)

- When $\beta^2 < 2$, the limit (1.4) exists and corresponds to a gas of a finite number of isolated particles.
- When $\beta^2 \in [2, 4)$, the partition function is dominated by the contribution of dipoles consisting of two nearby particles (within distance of order ε where ε is the scale at which V is smoothened) with opposite charges. Each dipole correspond to a negative energy $-|\log \varepsilon|$, so that a configuration of $2n$ particles forming n dipoles correspond to an energy $-n|\log \varepsilon|$. Such configurations occupy a volume of the state space $\mathcal{O}^{2n}$ which is of order ε^{2n} (if the position of positive charges are chosen freely, the negative charges have to be located in balls of volume ε^2 around the locations of positive ones). The total contribution of isolated dipoles to the partition function (1.4) restricted to configurations of size $2n$ is thus of order $\varepsilon^{(2-\beta^2)n}$, which diverges for $\beta^2 > 2$ (when $\beta^2 = 2$ a logarithmic divergence is obtained by summing over intermediate scales between ε and the macroscopic one). The gas is therefore dominated by configurations made up of a large number (of order $\varepsilon^{(2-\beta^2)n}$) of such dipoles. While the total number of charged particle diverges in the limit when ε goes to zero, local cancelations of charges on short distances allows us to define a limit of the charge distribution.
- When $\beta^2 \in [3, 4)$, the contribution of quadrupoles (that is a combination of two closely located dipoles in a configuration that makes the resulting energy smaller than the energy of the sum of two independent dipoles), while not dominant, comes to diverge also: the total weight of configurations formed of n-quadrupoles being of order $\varepsilon^{(6-2\beta^2)n}$ (when n is even). For this reason a second diverging counterterm is required in the renormalization.
- When $\beta^2 \in [10/3, 4)$ the contribution of sextupoles have to be taken into account, and in general $2p$-poles starts having a diverging contribution to the partition function when $\beta^2 \in [2(2 - p^{-1}), 4)$.
- When $\beta^2 \geq 4$, the contribution of $2p$-poles diverges for all p. Furthermore, the relative weight of $2p$-poles (considering energy and entropy) which is $\varepsilon^{(2(2p-1)-p\beta^2)n}$, becomes increasing in p, indicating a total collapse of the system.

This scenario is rather well understood [2, 4, 15, 16, 6], combining log gas/Sine Gordon approaches. Yet all the methods presented so far suffer from restrictions. The main reason for this is that they rely on renormalisation group (RG) techniques which, though powerful, are especially adapted to a translational invariant context, for which the RG map is most easily studied. It results that, in the papers mentioned just before, $2d$ (free boundary) Coulomb gas or Yukawa gas is often considered to recover translational invariance (other boundary conditions for the Coulomb gas are usually not translationally invariant) and existence of correlation functions of fractional charge densities is not studied because it consists in analyzing the model under a local change of background geometry (see Subsection (2.3) for a more precise statement), hence strongly breaking translational invariance too. Let us mention that correlation functions are studied in [4] (in a limited region of permitted parameters β) with a method which relies on Hamilton-Jacobi equations. Besides these restrictions, the methods above are also either limited in the region of permitted parameters β, or restricted to small α, or only provide non trivial bounds for the partition function (1.3) but do not establish existence of the limit. Therefore, it is certainly fair to say that the global understanding of the model is far from complete from the mathematical angle.

Finally, let us mention some further related results in dimension $d = 2$. A large deviation results for a space discretization of (1.4) in the case $\beta^2 < d$ is obtained in [14]. A dynamical approach of the model is studied in [11] for $\beta^2 \in [0, \frac{4}{3}d)$, and then extended to the whole subcritical regime $\beta^2 \in [0, 2d)$ in [5]. These papers construct the natural Langevin dynamics associated to the measure described by (1.4). The Berezinsky-Kosterlitz-Thouless phase transition occuring at $\beta^2 = 2d$ is studied in [10, 7].

In this paper we revisit the Sine-Gordon model in the case $d = 1$. This situation is also known as the *boundary Sine-Gordon model* in physics [8, 9] because it arises when constraining the $2D$-Coulomb gas particles to live on the boundary of a smooth planar domain and serves as a model for the Kondo effect,

[1] in which case the dimension is $d = 2$ and KT transition occurs at $\beta^2 = 2d = 4$.

resonant tunneling in quantum wires or between quantum Hall edge states. The main ingredient is a new probabilistic representation of the cumulants of a random variable on a filtered probability space which allows us to express the counterterms as brackets of martingales. Then we use standard Itô calculus and a refinement of the electrostatic inequalities (originally proved for the $3D$ Coulomb potential by Onsager see [17, 20] but that can also extend to other positive definite potentials) in order to control the cumulants inductively. In this context, we are able to produce a short proof, which is flexible in the model (Yukawa/log gas or even any reasonable perturbation of the ln in (1.4)), allows us to deal with any possible value of $\alpha \in \mathbb{R}$ and $\beta^2 < 2d$ and to treat renormalisation of fractional charge density correlation functions.

2. Setup and results

2.1. **Definitions and assumptions.** From from now on, we restrict the problem to the case of dimension $d = 1$. This restriction comes to play an important role in one crucial step of the proof. We choose however to keep writing d for the dimension in the statements in order to underline that large chunks of the proof can carry to the general case. We replace $\mathcal{O}$ by a bounded interval $I \subset \mathbb{R}$. Our method would also apply if one replaces I by a closed Jordan curve in $\mathbb{R}^2$ with distances measured by arclength.

Let $K(x, y)$ be a positive semidefinite kernel satisfying

$$K(x,y) := -\log|x-y| + h(x,y) \tag{2.1}$$

where h is bounded continuous on I^2.

The measure of integration we consider on I (which is simply Lebesgue measure in (1.4)) is a (positive) Borel measure of the form

$$\mu(\mathrm{d}x) = g(x)\mathrm{d}x, \qquad g \text{ bounded measurable.} \tag{2.2}$$

Choosing μ of this form with g bounded is not an artificial restriction since the presence of singularities can change the multipole picture presented above.

We assume that our kernel can be written in the form

$$K(x,y) := \lim_{t\to\infty} \int_0^t Q_u(x,y)\mathrm{d}u =: \lim_{t\to\infty} K_t(x,y), \tag{2.3}$$

where, for each $u \geqslant 0$, Q_u is a bounded symmetric positive semidefinite kernel for which we will specify some regularity assumptions (see Assumption 2.1 below). The bounded kernel K_t plays the role of V_ε in the introduction and we consider log-gas partition function of the following form

$$\mathcal{Z}_{\alpha,\beta,t}(I) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \sum_{(\lambda_i)_{i=1}^n \in \{-1,1\}^n} \int_{I^n} \exp\Big(-\beta^2 \sum_{1\le i<j\le n} \lambda_i\lambda_j K_t(x_i,x_j)\Big) \prod_{i=1}^n \mu(\mathrm{d}x_i). \tag{2.4}$$

The state space of particle configurations is the disjoint union $\Xi_I := \coprod_{n\geq 0}(I^n \times \{-1,1\}^n)$, namely the set $\{(n,x,\lambda) \,|\, n \in \mathbb{N}, x \in I^n, \lambda \in \{\pm 1\}^n\}$ equipped with its canonical sigma-algebra. A measurable function F on Ξ_I is thus a sequence $(F(n,\cdot,\cdot))_n$ of measurable functions on $I^n \times \{-1,1\}^n$. The partition function (2.4) induces a probability measure $\mathbf{P}_{\alpha,\beta,t}$ (with expectation $\mathbf{E}_{\alpha,\beta,t}$) on Ξ_I by setting for arbitrary bounded measurable function F on Ξ_I

$$\mathbf{E}_{\alpha,\beta,t}(F) = \frac{1}{\mathcal{Z}_{\alpha,\beta,t}(I)} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \sum_{(\lambda_i^n)_{i=1}^n \in \{-1,1\}^n} \int_{I^n} F(n,(x_i^n),(\lambda_i^n)) \exp\Big(-\beta^2 \sum_{1\le i<j\le n} \lambda_i\lambda_j K_t(x_i,x_j)\Big) \prod_{i=1}^n \mu(\mathrm{d}x_i). \tag{2.5}$$

Labels being irrelevant, the only physically relevant quantity under distribution $\mathbf{P}_{\alpha,\beta,t}$ is the charge distribution which we denote by ν. It is obtained as the pushforward (or image measure) of the probability measure $\mathbf{P}_{\alpha,\beta,t}$ by the map $\Pi : \Xi_I \mapsto \mathcal{M}(I)$ (the set of signed measures on I) which sums signed Dirac masses δ_{x_i} corresponding to particles' locations and charges

$$\nu := \Pi\,(n,(x_i)_{i=1}^n,(\lambda_i)_{i=1}^n) = \sum_{i=1}^n \lambda_i \delta_{x_i}.$$

Now we construct the Sine-Gordon representation for all values of $t \geq 0$ on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (this is important for our analysis). We consider thus a centered Gaussian field $(X_t(x))_{t \geqslant 0, x \in I}$ with covariance function

$$\mathbb{E}[X_t(x)X_s(y)] = K_{t\wedge s}(x,y), \tag{2.6}$$

which is almost surely continuous in both variables t, x: existence of such a process results both from Assumption (2.3), which ensures that the kernel K is positive semidefinite on $(\mathbb{R}_+ \times I)^2$, and Kolmogorov's continuity criterion, which can be applied thanks to the regularity assumptions on Q made in Assumption 2.1 below. Also, note that for a fixed t the process $(X_t(x))_{x\in I}$ is a centered continuous Gaussian field with covariance K_t. We denote by $\mathcal{F}_t$ $(t \geqslant 0)$ the filtration generated by $\{X_s(x);\, s \leqslant t, x \in I\}$, and by $\mathcal{F}_\infty$ the σ- algebra generated by $\cup_{s\geq 0}\mathcal{F}_s$.

Considering the martingale

$$M_t^{(\beta)} := \int_I \cos(\beta X_t(x)) e^{\frac{\beta^2}{2}K_t(x,x)}\mu(\mathrm{d}x) \tag{2.7}$$

we have $\mathbb{E}[e^{\alpha M_t^{(\beta)}}] = \mathcal{Z}_{\alpha/2,\beta,t}$. There is also a Sine-Gordon representation for the Fourier transform of the mass distribution: considering a bounded continuous function $\theta : I \to \mathbb{R}$ (recall that ν denotes the random charge distribution under $\mathbf{P}_{\alpha,\beta,t}$) we obtain by repeating the computation leading to (1.9)

$$\mathbf{E}_{\alpha/2,\beta,t}[e^{\mathbf{i}\langle\theta,\nu\rangle}] = \mathbf{P}_{\alpha/2,\beta,t}[e^{\mathbf{i}\sum_{i=1}^n \lambda_i\theta(z_i)}] = \frac{\mathbb{E}[e^{\alpha M_t^{(\beta,\theta)}}]}{\mathbb{E}[e^{\alpha M_t^{(\beta)}}]}, \tag{2.8}$$

where

$$M_t^{(\beta,\theta)} := \int_I \cos(\beta X_t(x) + \theta(x)) e^{\frac{\beta^2}{2}K_t(x,x)}\mu(\mathrm{d}x).$$

Our aim is thus to obtain results concerning the asymptotic behavior of the Laplace transform of $M_t^{(\beta,\theta)}$ for a large class of θ and deduce consequences concerning the charge distribution. We obtain such results under the following regularity assumption for the covariance function (recall $d = 1$).

Assumption 2.1. (Smooth white noise decomposition)

(1) For every u, $(x,y) \mapsto Q_u(x,y)$ is C^2 and there exists a constant C such that for any $x, y \in I$, $u > 0$

$$\begin{aligned} |Q_u(x,y)| &\leq C(1+e^u|x-y|)^{-(d+1)},\\ |\partial_x Q_u(x,y)| &\leq Ce^u(1+e^u|x-y|)^{-(d+1)},\\ |\partial_x\partial_y Q_u(x,y)| &\leq Ce^{2u}(1+e^u|x-y|)^{-(d+1)}. \end{aligned} \tag{2.9}$$

(2) $\sup_{t\geqslant 0}\sup_{x,y\in I} |K_t(x,y) + \ln(|x-y| \vee e^{-t})| < \infty$.

Remark 2.2. *The kernel Q_u in (2.3) is meant to account for the correlations present on scale e^{-u}. A prototypical example to have in mind is $Q_u(x,y) := Q(e^u x, e^u y)$ where Q is a fixed smooth translation invariant covariance function on $\mathbb{R}^d$ for which $Q(0,z)$ and its first two derivatives display fast decay. The Gaussian kernel $Q_u(x,y) = \exp(-\frac{(e^u|x-y|)^2}{2})$ which perfectly fits this framework is of peculiar importance as it is the basis for the construction of 2D Gaussian Free Fields (including Dirichlet/Neumann GFF or the massive GFF appearing in the Sine-Gordon representation of the Yukawa gas), which we can restrict on a one-dimensional manifold.*

Remark 2.3. *Note that Assumption 2.1 implies in particular that $K := \lim_{t\to\infty} K_t$ exists and is of the form given in Equation (2.1).*

We let $(\mathcal{C}_k^{(\beta,\theta)}(t))_k$ denote the successive cumulants of the martingale $M_t^{(\beta,\theta)}$ (a reminder of the definition of cumulants is given in Section 3). In the Sine-Gordon representation, the multipole picture presented in the introduction corresponds to an explosion of even order cumulants of the martingale when $t \to \infty$. This phenomenon is described by the following sequence of successive thresholds $(\beta_n)_{n\in\mathbb{N}^*}$ of the Sine-Gordon model (recall that here $d=1$)

$$\beta_0 = 0, \quad \forall n \geq 1, \qquad \beta_n = \sqrt{2d(1-\tfrac{1}{2n})}. \tag{2.10}$$

For $\beta \in [\beta_{n-1}, \beta_n)$, $\alpha \in \mathbb{R}$, and F a bounded ($\mathcal{F}_\infty$-)measurable function, we define the renormalized partition function which integrates F as

$$\bar{\mathcal{Z}}_{\alpha,\beta,t}(F) := \mathbb{E}[F(X)e^{\alpha M_t^{(\beta)}}]e^{-\sum_{i=1}^{n-1}\frac{\alpha^{2i}}{i!}\mathcal{C}_{2i}^{(\beta)}(t)}. \tag{2.11}$$

2.2. **Renormalization for the log-gas and its Sine-Gordon representation.** Our first result shows that the renormalized partition functions converge, and that the tilt by M_t induces a (non-Gaussian) limiting measure for the process X_t when t tends to infinity.

Theorem 2.4. *Assume $\beta^2 < 2d$, if F is bounded and measurable with respect to $\mathcal{F}_{t_0}$ for some $t_0 \in (0,\infty)$, then the following limit is well defined*

$$\mathcal{Z}_{\alpha,\beta}(F) := \lim_{t\to\infty} \bar{\mathcal{Z}}_{\alpha,\beta,t}(F).$$

Furthermore, the mapping

$$F \mapsto \langle F \rangle_{\alpha,\beta} := \frac{\mathcal{Z}_{\alpha,\beta}(F)}{\mathcal{Z}_{\alpha,\beta}(1)}$$

can be extended to all bounded ($\mathcal{F}_\infty$-)measurable functions F and defines a probability on $C(\mathbb{R}_+ \times I)$.

Remark 2.5. *In (2.11), the sum $\sum_{i=1}^{n-1}\frac{\alpha^{2i}}{2i!}\mathcal{C}_{2i}^{(\beta)}(t)$ in the exponential factor represents the diverging terms that have to be subtracted to $\mathbb{E}[Fe^{\alpha M_t}]$ in order to get a converging expression. We will show below that for $\beta^2 < 2d$ only cumulants of even order might diverge when $t \to \infty$ while those of odd order converge. The sum can hence be replaced by $e^{\sum_{i=1}^{2n-1}\frac{\alpha^i}{i!}\mathcal{C}_i^{(\beta)}(t)}$ without modifying the result.*

Remark 2.6. *When $\beta^2 < \beta_2$, the convergence result (2.11) is an instance of mod-Gaussian convergence for the variable $M_t^{(\beta)}$ (when t goes to infinity) in the terminology of [12]. When $\beta \geqslant \beta_2$, the convergence result (2.11) generalizes the notion of mod-ϕ convergence exposed in [13].*

While this is not an immediate consequence of the result, the proof of Theorem 2.4 provides also the key elements to establish the convergence in law of the charge distribution in the regime $\beta^2 < 2d$.

Theorem 2.7. *When $\beta \in (0, \sqrt{2d})$, for any 1/2-Hölder continuous function θ, the following limit exists*

$$\lim_{t\to\infty} \mathbf{E}_{\alpha,\beta,t}[e^{\mathbf{i}\langle\theta,\nu\rangle}] = \Psi(\theta) \tag{2.12}$$

Moreover Ψ is continuous for the Hölder norm

$$\|\theta\|_{1/2} := \max_{x\in I}|\theta(x)| + \max_{x,y\in I^2}\frac{|\theta(x)-\theta(y)|}{\sqrt{|x-y|}}.$$

In particular, the charge distribution ν converges in law under $\mathbf{P}_{\alpha,\beta,t}$ as $t\to\infty$ in the Schwartz space of tempered distributions.

The above convergence result for the charge distribution ν towards a non degenerate limit occurs for $\beta^2 \in [d, 2d)$ in spite of the fact that the total number of particule tends to infinity. This is due to the dipole picture mentioned in the introduction. While an infinite number of charged particle are present in the log-gas, short range cancelations makes the quantity $\langle\theta,\nu\rangle$ well defined for θ sufficiently regular.

2.3. **Renormalization for correlation functions.** Our last result concerns the asymptotics for the correlation function associated with fractional charge. The aim is, given $(\eta_1,\dots,\eta_k)\in\mathbb{R}^k$ and $(z_1,\dots,z_k)\in I^k$ distinct, to determine the asymptotics of the partition function of a system, where charges η_i have been placed on the site z_i for $i\in[\![1,k]\!]$, defined by

(2.13)
$$\widehat{\mathcal{Z}}_{\alpha,\beta,t}(\mathbf{z},\eta):=\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\sum_{(\lambda_i)_{i=1}^n\in\{-1,1\}^n}\int_{I^n}e^{-\beta^2\left(\sum_{1\le i<j\le n}\lambda_i\lambda_i K_t(x_i,x_j)+\sum_{i=1}^n\sum_{l=1}^k\lambda_i\eta_l K_t(x_i,z_l)\right)}\prod_{i=1}^n\mu(\mathrm{d}x_i).$$

More precisely, we are interested in the asymptotics of (recall (2.4)) $\widehat{\mathcal{Z}}_{\alpha,\beta,t}(\mathbf{z},\eta)/\mathcal{Z}_{\alpha,\beta,t}$. Using the Sine-Gordon representation, we have

$$\widehat{\mathcal{Z}}_{\alpha,\beta,t}(\mathbf{z},\eta)=\mathbf{E}_{\alpha,\beta,t}\left[e^{\alpha\widehat{M}_t^{(\mathbf{z},\eta)}}\right], \tag{2.14}$$

where (considering the analytic continuation of cos on $\mathbb{C}$)

$$\widehat{M}_t^{(\mathbf{z},\eta)}:=\int_I\cos\left(\beta X_t(x)+\mathbf{i}\beta^2\sum_{l=1}^k\eta_l K_t(z_l,x)\right)e^{\frac{\beta^2}{2}K_t(x,x)}\mathrm{d}\mu(x). \tag{2.15}$$

Note that we also have

$$\widehat{\mathcal{Z}}_{\alpha,\beta,t}(\mathbf{z},\eta)=\mathbb{E}\left[e^{\sum_{l=1}^k\left(\mathbf{i}\beta\eta_l X_t(z_l)+\frac{\beta^2\eta_l^2}{2}K_t(z_l,z_l)\right)}e^{\alpha M_t^{(\beta)}}\right],$$

with $M_t^{(\beta)}$ defined by (2.7) but the expression (2.14) is more useful in practice.

Theorem 2.8. *Setting $\|\eta\|_\infty:=\max|\eta_i|$, if $(1+2\|\eta\|_\infty)\beta^2<2$, then we have, for all $\alpha\in\mathbb{R}$ and all set of k distinct points*

$$\langle\prod_{l=1}^k e^{i\beta\eta_l X(z_l)}\rangle:=\lim_{t\to\infty}\frac{\widehat{\mathcal{Z}}_{\alpha,\beta,t}(\mathbf{z},\eta)}{\mathcal{Z}_{\alpha,\beta,t}} \tag{2.16}$$

exists and is continuous in η and $\mathbf{z}$.

Remark 2.9. *We expect that the result above remains valid under the less strict assumptions $\beta^2\max(\|\eta\|_\infty,1/2)<1$. However establishing such a result would require some refinement of our technique, which we leave for future work.*

3. Cumulants of continous martingales, a general approach

In this section we provide a general scheme which allows to compute the successive cumulants of an arbitrary continuous martingale, or equivalently of any variable defined on some probability space equipped with a continuous filtration. Let us start by recalling the definition of the cumulants of a random variable. If Z is a random variable such that $\mathbb{E}[e^{\varepsilon|Z|}]<\infty$ for some $\varepsilon>0$, then the log-Laplace transform of Z, $\alpha\mapsto\ln\mathbb{E}[\exp(\alpha Z)]$ is analytic in a neighborhood of zero and thus admits a power series expansion of the following form

$$\ln\mathbb{E}[\exp(\alpha Z)]=\sum_{i=1}^{\infty}\frac{\mathcal{C}_i(Z)}{i!}\alpha^i. \tag{3.1}$$

The expression of the coefficients $\mathcal{C}_i(Z)$ can be obtained by an extensive use of the Taylor formula for $x\mapsto\ln(1+x)$. In particular for $i\ge1$

$$\mathcal{C}_i(Z):=\mathbb{E}[Z^i]+q_i\left(\mathbb{E}[Z],\mathbb{E}[Z^2],\dots,\mathbb{E}[Z^{i-1}]\right). \tag{3.2}$$

where q_i is a polynomial in $i-1$ variables. In full generality, Equation (3.2) defines the i-th cumulant of Z under the less strict requirement $\mathbb{E}[|Z|^i]<\infty$.

We consider $(M_t)_{t\geq 0}$ a continuous martingale with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ which starts from $\mathcal{F}_0 = \{0,\Omega\}$ (this last assumption ensures that M_0 is almost surely constant). We define inductively a sequence of processes $A^{(i)}_t$ and $(M^{(i,t)}_s)_{s\in[0,t]}$ as follows. First set

$$A^{(1)}_t = M_t \quad \text{and} \quad (M^{(1,t)}_s)_{s\in[0,t]} := \mathbb{E}[A^{(1)}_t \mid \mathcal{F}_s] - \mathbb{E}[A^{(1)}_t] = M_s - M_0. \tag{3.3}$$

Then for $i \geq 2$ we define $A^{(i)}$ in terms of the quadratic variations of previous order martingales provided that they are well defined

$$A^{(i)}_t := \frac{1}{2}\sum_{j=1}^{i-1} \langle M^{(j,t)}, M^{(i-j,t)}\rangle_t \quad \text{and} \quad M^{(i,t)}_s := \mathbb{E}[A^{(i)}_t \mid \mathcal{F}_s] - \mathbb{E}[A^{(i)}_t]. \tag{3.4}$$

While our result might hold with greater generality, we assume for simplicity (and because this corresponds to the applications we have in mind) that all the quantities above are well defined and the quadratic variations above are essentially bounded in the sense that for every i and t

$$\|\langle M^{(i,t)}\rangle_t\|_\infty < \infty \quad \text{where} \quad \|Z\|_\infty := \inf\{u\geq 0 : \mathbb{P}[|Z|>u]=0\}. \tag{3.5}$$

Lemma 3.1. *Given $j\geq 1$, consider the martingale $(N^{(j,\alpha,t)}_s)_{s\in[0,t]}$ defined by*

$$N^{(i,\alpha,t)}_s := \sum_{i=1}^{j} \alpha^i M^{(i,t)}_s. \tag{3.6}$$

Under the assumption (3.5), we have the decomposition

$$\log \mathbb{E}\left[e^{\alpha M_t}\right] = \sum_{i=1}^{j} \alpha^i \mathbb{E}[A^{(i)}_t] + \log \mathbb{E}\left[e^{N^{(j,\alpha,t)}_t - \frac{1}{2}\langle N^{(j,\alpha,t)}\rangle_t} e^{Q^{(j,\alpha)}_t}\right]. \tag{3.7}$$

where

$$Q^{(j,\alpha)}_t := \frac{1}{2}\sum_{i=j+1}^{2j} \alpha^i \sum_{k=i-j}^{j} \langle M^{(k,t)}, M^{(i-k,t)}\rangle_t. \tag{3.8}$$

As a consequence the i-th cumulant of M_t is given by

$$\mathcal{C}_i(M_t) := i!\mathbb{E}[A^{(i)}_t]. \tag{3.9}$$

Proof. To illustrate the idea of the proof let us start with the case $j=1$ (which of course could be obtained with a simpler direct computation). We have

$$\mathbb{E}\left[e^{\alpha M_t}\right] = e^{\alpha\mathbb{E}[M_t]}\mathbb{E}\left[e^{\alpha M^{(1,t)}_t}\right] = e^{\alpha\mathbb{E}[M_t]}\mathbb{E}\left[e^{\alpha M^{(1)}_t - \frac{\alpha^2}{2}\langle M^{(1,t)}\rangle_t} e^{\frac{\alpha^2}{2}\langle M^{(1,t)}\rangle_t}\right]. \tag{3.10}$$

Considering

$$(\mathrm{d}\widetilde{\mathbb{P}}/\mathrm{d}\mathbb{P})(\omega) := e^{\alpha M^{(1,t)}_t - \frac{\alpha^2}{2}\langle M^{(1,t)}\rangle_t}$$

as probability density we have thus

$$\ln \mathbb{E}\left[e^{\alpha M_t}\right] = \alpha\mathbb{E}[M_t] + \ln \widetilde{\mathbb{E}}\left[e^{\frac{\alpha^2}{2}\langle M^{(1)}\rangle_t}\right]. \tag{3.11}$$

Using our assumption (3.5), we obtain that the last term is at most of order α^2 and we can conclude that the first cumulant is given by $\mathbb{E}[M_t]$. Our construction of $A^{(i)}$ and $M^{(i,t)}$ as been made so that we can iterate the above process, each $A^{(i)}_t$ being designed to cancel the quadratic variations of terms that have appeared on previous steps. The computation leading to (3.7) is the following

$$\begin{aligned}\langle N^{(j,\alpha,t)}\rangle_t &= \sum_{i=2}^{2j}\alpha^i\sum_{k=1}^{i-1}\mathbb{1}_{\{\max(k,i-k)\le j\}}\langle M^{(k,t)},M^{(i-k,t)}\rangle_t\\ &= 2\sum_{i=2}^{j}\alpha^i A_t^{(i)}+\sum_{i=j+1}^{2j}\alpha^i\sum_{k=i-j}^{j}\langle M^{(k,t)},M^{(i-k,t)}\rangle_t \\ &= 2\left[(N_t^{(j,\alpha,t)}-\alpha M_t)+\sum_{i=1}^{j}\alpha^i\mathbb{E}[A_t^{(i)}]+Q_t^{(j,\alpha)}\right].\end{aligned}\tag{3.12}$$

Interpreting $e^{N_t^{(j,\alpha,t)}-\frac12\langle N^{(j,\alpha,t)}\rangle_t}$ as a probability density, we can deduce from (3.12) that (recall (3.5))

$$\left|\log\mathbb{E}\left[e^{\alpha M_t}\right]-\sum_{i=1}^{j}\alpha^i\mathbb{E}[A_t^{(i)}]\right|\le\|Q_t^{(j,\alpha)}\|_\infty\le j^2\alpha^{j+1}\max_{i\le j}\|\langle M^{(i,t)}\rangle_t\|_\infty.\tag{3.13}$$

We conclude using the characterization of the cumulants (3.1). □

4. Proof of Theorem 2.4: Renormalization of the 1D- Sine-Gordon Model

Recalling Equation (2.11), an important step for the convergence in the case $F\equiv 1$ is to show that for every $n\ge 1$, we have $\lim_{t\to\infty}\mathcal{C}_{2n}^{(\beta)}(t)$ exists and is finite for $\beta<\beta_n$ and that $\lim_{t\to\infty}\mathcal{C}_{2n-1}(t)$ converges for all $\beta<\sqrt{2d}$. Hence our main effort in the proof will on the proof of convergence of these cumulants. The convergence for general F then mostly follows from the technique developped to control the cumulants.

4.1. **Computing the cumulants.** In this Section, we present our main technical results concerning the cumulants (Proposition 4.1) which allows us to ensure their convergence. We use an inductive approach to find an integral expression for cumulants of the martingale $M_t^{(\beta)}$ (2.7).

To motivate this approach, we compute the first cumulant using the procedure proposed in Section 3. For practical purpuse we state a general formula for the quadratic variation established with basic Itô calculus. We use it repeatedly throughout our computations. For bounded measurable functions f and g we have

$$\begin{aligned}\int_{t_1}^{t_2}\int_{I^2}f(X_u(x),u)g(X_u(y),u)\mathrm{d}\langle X(x),X(y)\rangle_u\mu(\mathrm{d}x)\mu(\mathrm{d}y)\\ =\int_{t_1}^{t_2}\int_{I^2}f(X_u(x),u)g(X_u(y),u)Q_u(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y)\mathrm{d}u.\end{aligned}\tag{4.1}$$

Within computations we often omit the dependence in β in some notations for the sake of readability. We set first $M_s^{(1,t)}=M_s-\mu(I)$ for $s\leqslant t$. Then Itô's formula gives

$$\mathrm{d}M_t=-\int_I\Big(\beta\sin(\beta X_t(x))e^{\frac{\beta^2}{2}K_t(x,x)}\mathrm{d}X_t(x)\Big)\mu(\mathrm{d}x).\tag{4.2}$$

By transforming the product of sin in

$$2\sin(\beta X_1)\sin(\beta X_2)=\cos\left(\beta(X_1-X_2)\right)-\cos\left(\beta(X_1+X_2)\right),$$

Definition (3.3) and Equation (4.1) yield

$$\begin{aligned}A_t^{(2)}=\frac{\beta^2}{2}\int_0^t\int_{I^2}Q_u(x_1,x_2)e^{\frac{\beta^2}{2}(K_u(x_1,x_1)+K_u(x_2,x_2))}\\ \times\left[\cos\left(\beta(X_u(x_1)-X_u(x_2))\right)-\cos\left(\beta(X_u(x_1)+X_u(x_2))\right)\right]\mu(\mathrm{d}x_1)\mu(\mathrm{d}x_2)\mathrm{d}u.\end{aligned}\tag{4.3}$$

This suggests that $A_t^{(i)}$ can in general be written in the following form

$$A_t^{(i)}:=\sum_{p=0}^{\lfloor i/2\rfloor}\int_0^t\int_{I^i}F^{(i,p,\beta)}(\mathbf{x},u,t)e^{\frac{\beta^2}{2}\sum_{k=1}^{i}K_u(x_k,x_k)}\cos\left[\beta\left(\sum_{k=1}^{i}\lambda_k^{(p)}X_u(x_k)\right)\right]\mu(\mathrm{d}\mathbf{x})\mathrm{d}u.\tag{4.4}$$

where we use the shorthand notation $\mathbf{x} = (x_1, \dots, x_i)$ and $\mu(\mathrm{d}\mathbf{x}) = \prod_{k=1}^{i} \mu(dx_k)$. The sequence $(\lambda_k^{(p)})_k$ is defined by

$$\lambda_k^{(p)} = -1 \text{ for } k \le p \quad \lambda_k^{(p)} = 1 \text{ for } k \ge p+1, \tag{4.5}$$

and $\mathbf{x} \mapsto F^{(i,p,\beta)}(\mathbf{x}, u, t)$ is a continuous function on I^i. We are going to prove (4.4) by induction, but the more important part of our task is to establish relevant properties for $F^{(i,p,\beta)}$ which will imply convergence of the cumulants. We let $D_i := \{\mathbf{x} \in I^i : \exists k \neq l, x_k = x_l\}$ denote the set of on-diagonal points in I^i.

Proposition 4.1. *For every $i \ge 2$, $A_t^{(i)}$ can be written in the form (4.4), where $F^{(i,p,\beta)}(\mathbf{x}, u, t)$ is analytic in β and satisfies:*

(1) For every $\mathbf{x} \in I^i \setminus D_i$ there exists a positive constant $C(\mathbf{x}, \varepsilon)$ such that for every $0 < u < t$, $\beta \le \sqrt{2} - \varepsilon$, we have

$$|F^{(i,p,\beta)}(\mathbf{x}, u, t)| \le C(\mathbf{x}, \varepsilon) e^{-u}, \tag{4.6}$$

The following convergence occurs uniformy on any compact subset of $I^i \setminus D_i$ and uniformy in $\beta \in [0, \sqrt{2} - \varepsilon]$

$$\lim_{t \to \infty} F^{(i,p,\beta)}(\mathbf{x}, u, t) = \bar{F}^{(i,p,\beta)}(\mathbf{x}, u), \tag{4.7}$$

and give $\mathbf{x}$, the convergence is uniform in β.

(2) For every $\varepsilon > 0$, there exists a constant $C = C(i, \varepsilon) > 0$ such that for every $u \ge 0$, for any fixed k and $x_k \in I$ and

$$\int_{I^{i-1}} \sup_{t \geqslant u} \sup_{\beta \in [0, \sqrt{2} - \varepsilon]} |F^{(i,p,\beta)}(\mathbf{x}, u, t)| \mu(\mathrm{d}\mathbf{x}^{(k)}) \le C e^{-(i-1)u}, \tag{4.8}$$

where $\mu(\mathrm{d}\mathbf{x}^{(k)}) := \mu(\mathrm{d}x_1) \dots \mu(\mathrm{d}x_{k-1}) \mu(\mathrm{d}x_{k+1}) \dots \mu(\mathrm{d}x_i)$.

(3) There exists a martingale $(\bar{M}_s^{(i)})_{s \ge 0}$ with $\bar{M}_s^{(i)} = 0$ such that for any $s \ge 0$, we have almost surely

$$\lim_{t \to \infty} \langle M^{(i,t)} - \bar{M}^{(i)} \rangle_s = 0.$$

In particular $M^{(i,t)}$ converges to $\bar{M}^{(i)}$ in probability.

Remark 4.2. *To make the proof lighter, we do not provide details concerning the uniformity in $\beta \in [0, \sqrt{2} - \varepsilon]$ for the estimates (4.6) to (4.8). They can be check via a tedious verification procedure which present no particular technical difficulties.*

The proof of this result is carried out in the upcoming Section 4.2. As an immediate corollary, we obtain convergence and continuity of cumulants

Corollary 4.3. *For each $t > 0$ and $i \in \mathbb{N}$, we have*

- $\lim_{t \to \infty} \mathcal{C}_{2i-1}^{(\beta)}(t) =: \mathcal{C}_{2i-1}^{(\beta,\infty)}$*, uniformly on all compact subsets of $[0, \sqrt{2d})$.*
- $\lim_{t \to \infty} \mathcal{C}_{2i}^{(\beta)}(t) =: \mathcal{C}_{2i}^{(\beta,\infty)}$*, uniformly on all compact subsets of $[0, \beta_i)$.*

The limiting function are continuous and uniform in β.

Proof. Let us recall Lemma 3.1 which gives the relation $\mathcal{C}_i(t) = i! \mathbb{E}[A_t^{(i)}]$. We have

$$\mathbb{E}[A_t^{(i)}] := \sum_{p=0}^{\lfloor i/2 \rfloor} \int_0^t \int_{I^i} F^{(i,p,\beta)}(\mathbf{x}, u, t) e^{\frac{\beta^2}{2} V_{i,p}(\mathbf{x}, u)} \mu(\mathrm{d}\mathbf{x}) \mathrm{d}u. \tag{4.9}$$

where (recall (4.5))

$$U_{i,p}(\mathbf{x}, u) := 2 \sum_{1 \le k < l \le n} \lambda_k^{(p)} \lambda_l^{(p)} K_u(x_k, x_l). \tag{4.10}$$

Let us show that

$$\lim_{t \to \infty} \int_0^t \int_{I^i} F^{(i,p,\beta)}(\mathbf{x}, u, t) e^{-\frac{\beta^2}{2} U_{i,p}(\mathbf{x}, u)} \mu(\mathrm{d}\mathbf{x}) \mathrm{d}u = \int_0^\infty \int_{I^i} \bar{F}^{(i,p,\beta)}(\mathbf{x}, u) e^{-\frac{\beta^2}{2} U_{i,p}(\mathbf{x}, u)} \mu(\mathrm{d}\mathbf{x}) \mathrm{d}u \tag{4.11}$$

converges for $\beta^2 < 2\left(1-\frac{1}{i}\right)$ when $p = i/2 \in \mathbb{N}$ and for all $\beta^2 < 2$ when $p \neq i/2$ (this is automatic when i is odd), using dominate convergence. The continuity in β follows from our uniform domination estimates. Using (4.8) it is sufficient to show that

$$\int_0^\infty e^{-(i-1)u} \sup_{\mathbf{x}\in I^i} e^{\frac{\beta^2}{2}V_{i,p}(\mathbf{x},u)} \mathrm{d}u < \infty. \tag{4.12}$$

For the first case, (4.10) and Item (3) of Assumption 2.1 imply that there exists $C > 0$ such that for all u and $\mathbf{x}$

$$U_{i,p}(\mathbf{x},u) = \mathrm{Var}\Big(\sum_{k=1}^{n} \lambda_k^{(p)} X(x_k)\Big) - \sum_{k=1}^{i} K_u(x_k,x_k) \geq -iu - C. \tag{4.13}$$

Then we observe that our restriction on β implies $\int_0^\infty e^{-(i-1)u} e^{\frac{\beta^2}{2}iu}\mathrm{d}u < \infty$. If $p \neq i/2$, we use Lemma 4.4 below (Equation (4.23) with definition (4.17)) which yields for all $\mathbf{x} \in I^i$

$$U_{i,p}(\mathbf{x},u) \leq -(i-1)u - C \tag{4.14}$$

and allows us to conclude in the same manner. □

4.2. **Proof of Proposition 4.1.**

Presentation of the induction setup. We prove the result by induction on i and omit all dependences in β in the notation. For $i = 2$ and from (4.3), the functions $F^{(2,0)}, F^{(2,1)}$ are given by

$$F^{(i,1)}(x_1,x_2,u,t) = -F^{(i,0)}(x_1,x_2,u,t) = \frac{\beta^2}{4} Q_u(x_1,x_2).$$

In that special case there is no dependence in t, so (4.7) is trivial. The convergence (4.6) follows from our assumption on Q_u, namely Assumption 2.1 item 1. Equation (4.8) is also easy to check. For the induction step, we are going to prove that for every $i \geq 3$, $j \leq \lfloor i/2 \rfloor$, setting $Y_p(\mathbf{x},u) := \sum_{k=1}^{i} \lambda_k^{(p)} X_u(x_k)$ we have

$$\langle M^{(i-j,t)}, M^{(j,t)}\rangle_t = \sum_{p=0}^{\lfloor i/2\rfloor} \int_0^t \int_{I^i} F^{(i,j,p)}(\mathbf{x},u,t) e^{\sum_{k=1}^{i} \frac{\beta^2}{2} K_u(x_k,x_k)} \cos(\beta Y_p(\mathbf{x},u)) \mu(\mathrm{d}\mathbf{x})\,\mathrm{d}u, \tag{4.15}$$

for a function $F^{(i,j,p)}$ satisfying (4.6)-(4.8). The statement for $A_t^{(i)}$ immediately follows. The first task is to obtain an expression for $F^{(i,j,p)}(\mathbf{x},u,t)$ in terms of the function obtained in previous iterations. We set $V_p(\mathbf{x},u) := \mathbb{E}\Big[Y_p(\mathbf{x},u)^2\Big]$. As a consequence of our assumption (4.4), we can write $M^{(i-j,t)}$ as

$$\begin{aligned} M_s^{(i-j,t)} := \sum_{p=0}^{\lfloor (i-j)/2\rfloor} \int_0^t \int_{I^{i-j}} & F^{(i-j,p)}(\mathbf{x},u,t) e^{\frac{\beta^2}{2}\left(\sum_{k=1}^{i-j} K_u(x_k,x_k) - V_p(\mathbf{x},u)\right)} \\ & \times \Big(\cos\left(\beta Y_p(\mathbf{x},u\wedge s)\right) e^{\frac{\beta^2}{2}V_p(\mathbf{x},u\wedge s)} - 1\Big)\mu(\mathrm{d}\mathbf{x})\mathrm{d}u. \end{aligned} \tag{4.16}$$

The martingale $M_s^{(j,t)}$ can be written as a similar sum over $q \in [\![0, \lfloor j/2 \rfloor]\!]$ and thus $\langle M^{(i-j,t)}, M^{(j,t)}\rangle_t$ is the sum over p and q of the quadratic variations generated by the corresponding terms in the sum. We are going to prove that each of these quadratic variations can be written as (4.15) with functions that satisfy the properties specified in Proposition 4.1.

In what follows, for the sake of readability, we make a change of variable and replace the pair $(i-j, j)$ with (i, j) (hence $i + j$ now has the role formerly played by i). From now on, we also use the notation

$\mathbf{x}_1 = (x_1, \dots, x_i)$, $\mathbf{x}_2 = (x_{i+1}, \dots, x_{i+j})$ and $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$. We also introduce a notation for the following energy terms

$$\begin{aligned} U_{i,p}(\mathbf{x}_1, u, v) &:= 2 \sum_{1 \le k < l \le i} \lambda_k^{(p)} \lambda_l^{(p)} \int_u^v Q_r(x_k, x_l) \mathrm{d}r, \quad \text{and} \quad U_{i,p}(\mathbf{x}, v) = U_{i,p}(\mathbf{x}, 0, v) \\ U_{j,q}(\mathbf{x}_2, u, v) &:= 2 \sum_{i+1 \le k < l \le i+j} \lambda_k^{(q)} \lambda_l^{(q)} \int_u^v Q_r(x_k, x_l) \mathrm{d}r, \quad \text{and} \quad U_{j,q}(\mathbf{x}, v) = U_{j,q}(\mathbf{x}, 0, v). \end{aligned} \tag{4.17}$$

Expanding all products, our task is reduced to the study of the bracket of martingales of the following type

$$\begin{aligned} N_s &= \int_0^t \int_{I^i} F(\mathbf{x}_1, u, t) e^{-\frac{\beta^2}{2} U_{i,p}(\mathbf{x}_1, u)} \Big(\cos\left(\beta Y_p(\mathbf{x}_1, u \wedge s)\right) e^{\frac{\beta^2}{2} V_p(\mathbf{x}_1, u \wedge s)} - 1 \Big) \mu(\mathrm{d}\mathbf{x}_1) \mathrm{d}u, \\ R_s &= \int_0^t \int_{I^j} G(\mathbf{x}_2, u, t) e^{-\frac{\beta^2}{2} U_{j,q}(\mathbf{x}_2, u)} \Big(\cos\left(\beta Y_q(\mathbf{x}_2, u \wedge s)\right) e^{\frac{\beta^2}{2} V_q(\mathbf{x}_2, u \wedge s)} - 1 \Big) \mu(\mathrm{d}\mathbf{x}_2) \mathrm{d}u, \end{aligned} \tag{4.18}$$

where $i, j \ge 2$ and functions F and G both satisfy (4.6)-(4.8) for i and j respectively. Note that, to cover the case $j = 1$, we also need to study the bracket $\langle N, M^{(1)} \rangle_t$, but this is comparatively simpler (the case is treated starting from Equation (4.37)). The bracket can be computed using (4.1)

$$\begin{aligned} \mathrm{d}\langle N, R \rangle_s = \int_{I^{i+j}} \beta^2 \left(\sum_{k=1}^{i} \sum_{l=i+1}^{i+j} \lambda_k^{(p)} \lambda_l^{(q)} Q_s(x_k, x_l) \right) \left(\int_s^t F(\mathbf{x}_1, u, t) e^{-\frac{\beta^2}{2} U_{i,p}(\mathbf{x}_1, s, u)} \mathrm{d}u \right) \\ \left(\int_s^t G(\mathbf{x}_2, v, t) e^{-\frac{\beta^2}{2} U_{j,q}(\mathbf{x}_1, s, v)} \mathrm{d}v \right) e^{\frac{\beta^2}{2} \sum_{k=1}^{i+j} K_s(x_k, x_k)} \\ \times \sin\Big(\beta Y_p(\mathbf{x}_1, s)\Big) \sin\Big(\beta Y_q(\mathbf{x}_2, s)\Big) \mu(\mathrm{d}\mathbf{x}_1) \mu(\mathrm{d}\mathbf{x}_2) \mathrm{d}s. \end{aligned} \tag{4.19}$$

Now we use trigonometry to change the product of sin into a sum which, after permutation of the coordinates, can be rewritten as the sum of two terms of the form $\cos(\beta Y_r(\mathbf{x}, s))$, one term with $r = p + q$ and the other with $r = \min(p + j - q, q + i - p)$. It remains to show that the function obtained in front of those terms, namely (we omit sign and a factor $\beta^2/2$ for obvious reasons)

$$\begin{aligned} H(\mathbf{x}_1, \mathbf{x}_2, s, t) := \left(\sum_{k=1}^{i} \sum_{l=i+1}^{i+j} \lambda_k^{(p)} \lambda_l^{(q)} Q_s(x_k, x_l) \right) \left(\int_s^t F(\mathbf{x}_1, u, t) e^{-\frac{\beta^2}{2} U_{i,p}(\mathbf{x}_1, s, u)} \mathrm{d}u \right) \\ \times \left(\int_s^t G(\mathbf{x}_2, v, t) e^{-\frac{\beta^2}{2} U_{i,q}(\mathbf{x}_2, s, v)} \mathrm{d}v \right), \end{aligned} \tag{4.20}$$

satisfies (4.6)-(4.8). We start with (4.6) and (4.7) which are easier to prove.

Proof of Item (1). It is sufficient to prove that

$$\int_s^t |F(\mathbf{x}_1, u, t)| e^{-\frac{\beta^2}{2} U_{i,p}(\mathbf{x}_1, s, u)} \mathrm{d}u \le C(\mathbf{x}_1) e^{-s} \tag{4.21}$$

and converges (because this also implies convergence without $|\cdot|$) when t tends to infinity (and the same for the term containing G). This is obtained by observing that for $\mathbf{x}_1$ fixed $U_{i,p}(\mathbf{x}_1, s, u)$ is uniformly bounded in u and s (e.g. by $2 \sum_{1 \le k < l \le r} \int_0^\infty |Q_r(x_k, x_l)| \mathrm{d}r$). Hence using Assumption (4.6) for F and dominated convergence, we prove both convergence and (4.21).

Proof of Item (2). Now let us move to (4.8). For this statement we are going to need estimates on $U_{i,p}(\mathbf{x}, s, u)$ (which depend on whether $p = i/2$). Our starting point is the observation that appart from the missing diagonal terms $U_{i,p}(\mathbf{x}, s, u)$ corresponds to the variance of $Y_p(\mathbf{x}, u) - Y_p(\mathbf{x}, s)$, and thus that

$$U_{i,p}(\mathbf{x}, s, u) \ge - \sum_{k=1}^{i} \int_u^s Q_u(x_k, x_k).$$

To obtain a better bound, we need to obtain a better estimate of $\mathrm{Var}(Y_p(\mathbf{x},u)-Y_p(\mathbf{x},s))$ instead of simply relying on positivity. This is the spirit of Onsager's electrostatic inequality (also known as Onsager's Lemma [17, 20], see also [18, Proposition 3.9]). The main upgrade w.r.t. to the previously mentionned inequality, is that we provide a lower bound, not in terms of the minimal interparticle distance, but in terms of a transport distance between the $+$ and $-$ charges.

In the case $i=2p$, we let $m(\mathbf{x}_1)$ denote the 1-Wasserstein distance between $\sum_{k=1}^p \delta_{x_k}$ and $\sum_{l=p+1}^{2p} \delta_{x_l}$ given by

$$m(\mathbf{x}_1)=\min_{\sigma\in\mathcal{S}_p}\sum_{k=1}^p |x_k-x_{p+\sigma(k)}|. \tag{4.22}$$

where $\mathcal{S}_p$ is the symmetric group on p elements. We state the following inequalities, whose proof is postponed to the next section.

Lemma 4.4. *For any choice of integers $i\geq 2$ and $p\leq i/2$ there exists a constant C only depending on i such that for every $\mathbf{x}\in I^i$:*

(1) If i is even and $p=i/2$

$$U_{i,p}(\mathbf{x},s,u)\geq -(i-1)(u-s)-(-\log m(\mathbf{x})-s)_+-C \tag{4.23}$$

(2) In all other cases

$$U_{i,p}(\mathbf{x},s,u)\geq -(i-1)(u-s)-C. \tag{4.24}$$

Now, we replace $F(\mathbf{x}_1,u,t)$ and $G(\mathbf{x}_2,v,t)$ respectively by $F(\mathbf{x}_1,u):=\sup_{t:t\geqslant u}|F(\mathbf{x}_1,u,t)|$ and $G(\mathbf{x}_2,v):=\sup_{t:t\geqslant v}|G(\mathbf{x}_2,v,t)|$ and replace t by ∞ in the integral bounds in (4.20). Of course, if we can prove appropriate bounds for the resulting quantity, this is enough for our claim. We start with (4.8), and the easier case, when $p\neq i/2$ nor $q\neq j/2$. In that case, using (4.24), we can conclude if we show that there exists a constant such that for any $k\in[\![1,i]\!]$ and $l\in[\![i+1,i+j]\!]$ and $m\in[\![1,i+j]\!]$ (not necessarily distinct of k or l) and any $x_m\in I$

$$\int_{I^{i+j-1}}|Q_s(x_k,x_l)|\left[\int_s^\infty F(\mathbf{x}_1,u)e^{\frac{\beta^2}{2}(i-1)(u-s)}\mathrm{d}u\right]\left[\int_s^\infty G(\mathbf{x}_2,v)e^{\frac{\beta^2}{2}(i-1)(v-s)}\mathrm{d}v\right]\mu(\mathrm{d}\mathbf{x}^{(m)})\leq Ce^{-(i+j-1)s}. \tag{4.25}$$

Now assume without loss of generality that $m\geq (i+1)$ (the two integrals have symmetric roles). Then we have from our induction hypothesis (for the first and last inequality) and Assumption (2.1) (for the second inequality) for every x_k, x_l and x_m respectively

$$\begin{aligned}\int_{I^{i-1}}F(\mathbf{x}_1,u)\mu(\mathrm{d}\mathbf{x}_1^{(k)})&\leq C_1e^{-(i-1)u},\\ \int_I|Q_s(x_k,x_l)|\mu(\mathrm{d}x_k)&\leq C_2e^{-s},\\ \int_{I^{j-1}}G(\mathbf{x}_2,v)\mu(\mathrm{d}\mathbf{x}_2^{(m)})&\leq C_3e^{-(j-1)v}.\end{aligned} \tag{4.26}$$

Combining the three inequalities, we get (4.25) by using using Fubini and integrating with respect to $\mathbf{x}_1^{(k)},x_k,\mathbf{x}_2^{(m)}$ (in that order). Therefore, we obtain that the l.h.s. of (4.20) is smaller than

$$C_4e^{-s}\int_s^\infty\int_s^\infty e^{-(i-1)u}e^{-(j-1)v}\mathrm{d}u\mathrm{d}v\leq C_5e^{-(i+j-1)s}. \tag{4.27}$$

We can now move to the proof of (4.8) when $i=2p$ or $j=2q$ (or both). Let us start with the case where $i=2p$, $j\neq 2q$ first. We have

$$\left(\sum_{k=1}^i\sum_{l=i+1}^{i+j}\lambda_k^{(p)}\lambda_l^{(q)}Q_s(x_k,x_l)\right)=-\sum_{l=i+1}^{i+j}\sum_{k=1}^p\lambda_l^{(q)}\left(Q_s(x_k,x_l)-Q_s(x_{\sigma(k)+p},x_l)\right), \tag{4.28}$$

where σ can be chosen as the permutation for which the minimum in (4.22) is attained. By symmetry we can assume that σ is the identity, meaning that all our integrals will be restricted to the set $A_{\mathrm{Id}} := \{\mathbf{x}_1 \,:\, \sigma = \mathrm{Id}\}$. This implies in particular that

$$\forall k \in [\![1,p]\!], \quad |x_k - x_{k+p}| \le m(\mathbf{x}_1). \tag{4.29}$$

Using the triangle inequality and Lemma 4.4 we can conclude provided we prove a uniform bound

$$\begin{aligned}\int_{I^{i+j-1}} &\frac{|Q_s(x_k,x_l)-Q_s(x_{k+p},x_l)|}{\min\left(m(\mathbf{x}_1)e^s,1\right)^{\frac{\beta^2}{2}}}\mathbb{1}_{\{x_1\in A_{\mathrm{Id}}\}} \\ &\times\left[\int_s^\infty F(\mathbf{x}_1,u)e^{\frac{\beta^2}{2}(i-1)(u-s)}\mathrm{d}u\right]\left[\int_s^\infty G(\mathbf{x}_2,v)e^{\frac{\beta^2}{2}(i-1)(v-s)}\mathrm{d}v\right]\mu(\mathrm{d}\mathbf{x}^{(m)}) \le Ce^{-(i+j-1)s}.\end{aligned} \tag{4.30}$$

valid uniformly in the choice of k, l and m. Now we claim that

$$|Q_s(x_k,x_l)-Q_s(x_{k+p},x_l)| \le \begin{cases} C[(1+e^s|x_k-x_l|)^{-2}+(1+e^s|x_{k+p}-x_l|)^{-2}] & \text{if } m(\mathbf{x}_1)\ge e^{-s},\\ Cm(\mathbf{x}_1)e^s(1+e^s|x_k-x_l|)^{-2} & \text{if } m(\mathbf{x}_1)\le e^{-s}.\end{cases} \tag{4.31}$$

Both inequalities comes from our assumption (2.9), the first line coming from the control on Q and the second from the control on its derivative

$$|Q_s(x_k,x_l)-Q_s(x_{k+p},x_l)| \le m(\mathbf{x}_1)\max_{a\in[x_k,x_{k+p}]}|\partial_x Q_s(a,x_l)| \le Cm(\mathbf{x}_1)e^s(1+e^s|x_k-x_l|)^{-2}$$

and noticing that if $m(\mathbf{x}_1)\le e^{-s}$ then $\max_{a\in[x_k,x_{k+p}]} e^s(1+e^s|a-x_l|)^{-2}$ is of the same order as $e^s(1+e^s|x_k-x_l|)^{-2}$. Because $\beta^2\le 2$ and for $m(\mathbf{x}_1)\le e^{-s}$, the contribution of the term $m(\mathbf{x}_1)e^s$ coming from (4.31) cancels out the one appearing in (4.30). As a consequence of the above considerations, (4.30) is proved if we can show that

$$\begin{aligned}\int_{I^{i+j-1}} (1+e^s|x_k-x_l|)^{-2}&\left[\int_s^\infty F(\mathbf{x}_1,u)e^{\frac{\beta^2}{2}(i-1)(u-s)}\mathrm{d}u\right] \\ &\times\left[\int_s^\infty G(\mathbf{x}_2,v)e^{\frac{\beta^2}{2}(i-1)(v-s)}\mathrm{d}v\right]\mu(\mathrm{d}\mathbf{x}^{(m)}) \le Ce^{-(i+j-1)s},\end{aligned} \tag{4.32}$$

which can be done exactly like (4.25) by observing that $\int_I (1+e^s|x_k-x_l|)^{-2}\mu(\mathrm{d}x_k)\le Ce^{-s}$.

Finally let us consider the case $i=2p$, $j=2q$. Again by symmetry, we can assume that both permutations involved in the definition of $m(\mathbf{x}_1)$ and $m(\mathbf{x}_2)$ are the identity, which ensures that

$$\forall k,l,\ |x_k-x_{k+p}|\le m(\mathbf{x}_1), \quad |x_l-x_{l+q}|\le m(\mathbf{x}_2). \tag{4.33}$$

In that case similarly to (4.28)

$$\left(\sum_{k=1}^{i}\sum_{l=i+1}^{i+j}\lambda_k^{(p)}\lambda_l^{(q)}Q_s(x_k,x_l)\right) = \sum_{l=i+1}^{i+q}\sum_{k=1}^{p}\left(Q_s(x_k,x_l)-Q_s(x_{k+p},x_l)-Q_s(x_k,x_{l+q})+Q_s(x_{k+p},x_{l+q})\right). \tag{4.34}$$

Hence the inequality we have to prove becomes

$$\begin{aligned}\int_{I^{i+j-1}} &\frac{|Q_s(x_k,x_l)-Q_s(x_{k+p},x_l)-Q_s(x_k,x_{l+q})+Q_s(x_{k+p},x_{l+q})|}{\left(\min\left(m(\mathbf{x}_1)e^s,1\right)\min\left(m(\mathbf{x}_2)e^s,1\right)\right)^{\beta^2/2}}\mathbb{1}_{\{\mathbf{x}_1\in A_{\mathrm{Id}}\ ;\ \mathbf{x}_2\in B_{\mathrm{Id}}\}} \\ &\times\left(\int_s^\infty F(\mathbf{x}_1,u)e^{-\frac{\beta^2(i-1)(u-s)}{2}}\mathrm{d}u\right)\left(\int_s^\infty G(\mathbf{x}_2,v)e^{-\frac{\beta^2(j-1)(v-s)}{2}}\mathrm{d}v\right)\mu(\mathrm{d}\mathbf{x}^{(m)}) \le Ce^{-(i+j-1)s}\end{aligned} \tag{4.35}$$

and has to be valid for every choice of k, l, and m. Here again we can conclude provided that we can replace the fraction by $C(1+e^s|x_k-x_l|)^{-2}$. Hence we need to show that the numerator in the fraction is smaller than $Cm(\mathbf{x}_1)m(\mathbf{x}_2)(1+e^s|x_k-x_l|)^{-2}$ when both $m(\mathbf{x}_1)\le e^{-s}$ and $m(\mathbf{x}_2)\le e^{-s}$ (other cases are similar but simpler). In this case, elementary differential calculus and our bound on the second derivative (2.9) show that the numerator in the fraction above is smaller than

$$\begin{aligned}(4.36)\quad &|Q_s(x_k,x_l)-Q_s(x_{k+p},x_l)-Q_s(x_k,x_{l+q})+Q_s(x_{k+p},x_{l+q})|\\ &\leq Cm(\mathbf{x}_1)m(\mathbf{x}_2)e^{2s}\max_{\substack{a\in[x_k,x_{k+p}]\\ b\in[x_l,x_{l+q}]}} C(1+e^{2s}|b-a|)^{-2}\leq Cm(\mathbf{x}_1)m(\mathbf{x}_2)(1+e^s|x_k-x_l|)^{-2},\end{aligned}$$

where the last inequality uses our assumption on $m(\mathbf{x}_1)$ and $m(\mathbf{x}_2)$ as well as (4.34). This concludes our proof of (4.8) except from the case $j=1$ left aside.

In the case $j=1$ we need to compute a bracket of the type $\langle N, M^{(1)}\rangle_t$ where N is as above and $M_t^{(1)}$ is our original martingale (minus its mean). We have

$$\begin{aligned}(4.37)\quad \mathrm{d}\langle N, M^{(1)}\rangle_s &= \int_{I^{i+1}}\beta^2\left(\sum_{k=1}^{i}\lambda_k^{(p)}Q_s(x_k,x_{i+1})\right)\left(\int_s^t F(\mathbf{x}_1,u,t)e^{-\frac{\beta^2}{2}U_{i,p}(\mathbf{x}_1,s,u)}\mathrm{d}u\right)\\ &\qquad e^{\frac{\beta^2}{2}\sum_{k=1}^{i+1}K_s(x_k,x_k)}\sin\left(\beta Y_p(\mathbf{x}_1,s)\right)\sin\left(\beta X_s(x_{i+1})\right)\mu(\mathrm{d}\mathbf{x}_1)\mu(\mathrm{d}x_{i+1})\mathrm{d}s.\end{aligned}$$

Using the notation $\mathbf{x}=(\mathbf{x}_1,x_{i+1})$, the function for which convergence and domination have to be proved is

$$(4.38)\qquad H(\mathbf{x},s,t):=\left(\sum_{k=1}^{i}\lambda_k^{(p)}Q_s(x_k,x_{i+1})\right)\left(\int_s^t F(\mathbf{x}_1,u,t)e^{-\frac{\beta^2}{2}U_{i,p}(\mathbf{x}_1,s,u)}\mathrm{d}u\right).$$

The proof of (4.6) and (4.7) for this H works as previously. Now concerning (4.8), we prove the bound for

$$(4.39)\qquad H(\mathbf{x},s)=\left|\sum_{k=1}^{i}\lambda_k^{(p)}Q_s(x_k,x_{i+1})\right|\left(\int_s^\infty F(\mathbf{x}_1,u)e^{-\frac{\beta^2}{2}U_{i,p}(\mathbf{x}_1,s,u)}\mathrm{d}u\right).$$

with $F(\mathbf{x}_1,u):=\sup_{t\geq u}F(\mathbf{x}_1,u,t)$. The proof is the same as for the $j\geq 2$ case.

Proof of Item (3). For this, the starting point is again to observe that the validity of (4.4) entails that each martingale $M^{(i,t)}$ takes the form (4.16), and thus and thus the candidate limit is the martingale

$$(4.40)\qquad \bar{M}_s^{(i)}:=\sum_{p=0}^{\lfloor (i)/2\rfloor}\int_0^\infty\int_{I^i}\bar{F}^{(i,p)}(\mathbf{x},u)e^{-\frac{\beta^2}{2}U_{i,p}(\mathbf{x},u)}\Big(\cos\left(\beta Y_p(\mathbf{x},u\wedge s)\right)e^{\frac{\beta^2}{2}V_p(\mathbf{x},u\wedge s)}-1\Big)\mu(\mathrm{d}\mathbf{x})\mathrm{d}u.$$

The bracket derivative $\partial_s\langle \bar{M}^{(i)}-\bar{M}^{(i,t)}\rangle$ is given by a sum of terms of the form (recall (4.19), we adopt the convention $F^{(i,p)}(\mathbf{x},u,t)=0$ for $u>t$)

$$\begin{aligned}(4.41)\quad &\int_{I^{2i}}\left(\sum_{k=1}^{i}\sum_{l=i+1}^{2i}\lambda_k^{(p)}\lambda_l^{(q)}Q_s(x_k,x_l)\right)\left(\int_s^\infty(\bar{F}^{(i,p)}(\mathbf{x}_1,u)-F^{(i,p)}(\mathbf{x}_1,u,t))e^{-\frac{\beta^2}{2}U_{i,p}(\mathbf{x}_1,s,u)}\mathrm{d}u\right)\\ &\qquad\left(\int_s^\infty(\bar{F}^{(i,q)}(\mathbf{x}_2,u)-F^{(i,q)}(\mathbf{x}_2,u,t))e^{-\frac{\beta^2}{2}U_{i,q}(\mathbf{x}_2,s,u)}\mathrm{d}u\right)e^{\frac{\beta^2}{2}\sum_{k=1}^{i+j}K_s(x_k,x_k)}\\ &\qquad\qquad\times\sin\Big(\beta Y_p(\mathbf{x}_1,s)\Big)\sin\Big(\beta Y_q(\mathbf{x}_2,s)\Big)\mu(\mathrm{d}\mathbf{x}_1)\mu(\mathrm{d}\mathbf{x}_2).\end{aligned}$$

We show that each term converges to zero and remains uniformly bounded for $s\in[0,K]$. By dominated convergence it is sufficient to show that (recall $F^{(i,p)}(\mathbf{x},u):=\sup_{t>u}|F^{(i,p)}(\mathbf{x},u,t)|$)

$$\begin{aligned}(4.42)\quad &\int_{I^{2i}}\left(\sum_{k=1}^{i}\sum_{l=i+1}^{2i}\lambda_k^{(p)}\lambda_l^{(q)}Q_s(x_k,x_l)\right)\left(\int_s^\infty F^{(i,p)}(\mathbf{x},u)e^{-\frac{\beta^2}{2}U_{i,p}(\mathbf{x}_1,s,u)}\mathrm{d}u\right)\\ &\qquad\qquad\left(\int_s^\infty F^{(i,q)}(\mathbf{x},u)e^{-\frac{\beta^2}{2}U_{i,p}(\mathbf{x}_1,s,u)}\mathrm{d}u\right)\mu(\mathrm{d}\mathbf{x}_1)\mu(\mathrm{d}\mathbf{x}_2)<\infty.\end{aligned}$$

This can be performed by repeating the compution of the induction step in the proof of Item (2). $\square$

4.3. **Proof of Lemma 4.4.** Let us first show that we can reduce the proof to the case where $Q_u(x,y) = Q(e^u|x-y|)$ with Q a C^2 positive definite function support in the segment $[-1,1]$. Indeed we have

$$\sup_{t\geqslant 0}\sup_{x,y\in I}\Big|\int_0^t Q(e^u|x-y|)\,du - \ln(|x-y|\vee e^{-t})\Big| < +\infty. \tag{4.43}$$

Therefore, Assumption 2.1 Item (3) entails that

$$\sup_{s,t\geqslant 0}\sup_{x,y\in I}\Big|\int_s^t \big(Q_u(x,y) - Q(e^u|x-y|)\big)\,\mathrm{d}u\Big| < +\infty, \tag{4.44}$$

and hence that for some constant $C>0$ we have

$$\sum_{1\le k<l\le i}\varepsilon_k\varepsilon_l\int_s^u Q_r(x_k,x_l)\mathrm{d}r \geqslant \sum_{1\le k<l\le i}\varepsilon_k\varepsilon_l\int_s^u Q(e^r|x_k-x_l|)\mathrm{d}r - i(i-1)C. \tag{4.45}$$

Then considering $Z^{(r)}$ a centered field of covariance $Q(e^r|x-y|)$ and $Y_r := \sum_{k=1}^i \varepsilon_k Z^{(r)}(x_k)$ we have

$$\mathrm{Var}(Y^{(r)}) = i + 2\sum_{1\le k<l\le i}\varepsilon_k\varepsilon_l Q(e^r|x_k-x_l|).$$

Thus we have in any case

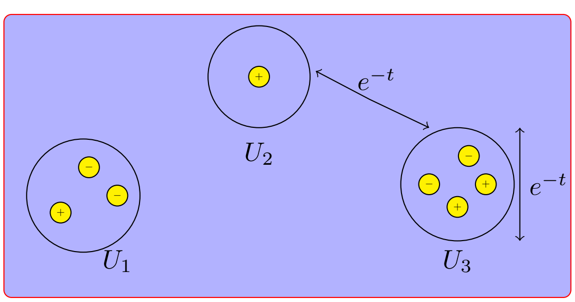
$$2\sum_{1\le k<l\le i}\varepsilon_k\varepsilon_l Q(e^r|x_k-x_l|) \ge -i. \tag{4.46}$$

In order to obtain a better estimate, we need a good lower bound on the variance of $Y^{(r)}$. We need to show that there exists $C>0$ such that

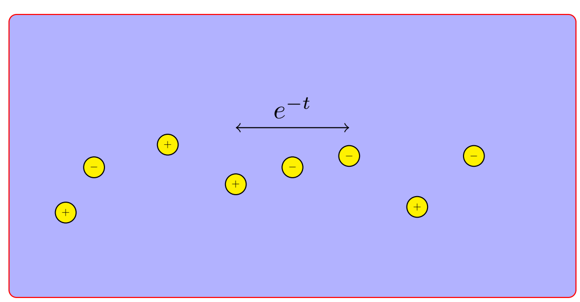
$$\int_s^u \mathrm{Var}(Y^{(r)})\,dr \ge \begin{cases} (u-s) - C & \text{when } \sum\varepsilon_k \neq 0,\\ (u-s) - (-\log m(\mathbf{x}) - s)_+ - C & \text{when } \sum\varepsilon_k = 0.\end{cases} \tag{4.47}$$

Let us partition $[\![1,i]\!]$ in disjoint groups of indices $(U_n)_{n\ge 1}$ where the mutual distance within group is smaller than e^{-r} and the distance between groups is larger than e^{-r}

$$\begin{aligned}\min_{n\neq q}\min_{k\in U_n, l\in U_q}|x_k-x_l| &\ge e^{-r},\\ \forall n,\quad \max_{k,l\in U_n}|x_k-x_l| &< e^{-r}.\end{aligned} \tag{4.48}$$



(A) Configuration with existing partition



(B) Configuration with no possible partition

FIGURE 1. Partition into clusters

Such subdivision may not exist but is unique when it does, see Figure 1. A sufficient condition for existence is that e^{-r} is not an approximation of any of the distances between points

$$\forall k,l\in[\![1,i]\!],\quad e^r|x_k-x_l| \in \mathbb{R}_+ \setminus \left[\frac{1}{i-1},1\right). \tag{4.49}$$

Thus in particular the set of r for which the partition is not defined has Lebesgue measure bounded above by $\frac{i(i-1)}{2}\log(i-1)$. We let $\mathcal{A}$ be the set of $r\in\mathbb{R}$ for which the partition exists. We write

$$Y^{(r)} := \sum_{n\geq 1} Y_n^{(r)} \text{ where } Y_n^{(r)} := \sum_{k\in U_n} \varepsilon_k Z^{(r)}(x_k). \tag{4.50}$$

As Q is supported on $[-1,1]$, $\mathrm{Cov}(Y_{n_1}^{(r)}, Y_{n_2}^{(r)})=0$ for $n_1\neq n_2$ and using the fact that Q is Lipschitz on the unit ball, we have

$$\begin{aligned}\mathrm{Var}(Y_n^{(r)}) &= \sum_{k,l\in U_n} \varepsilon_k\varepsilon_l Q(e^r|x_k-x_l|)\\ &=\Big(\sum_{k\in U_n}\varepsilon_k\Big)^2 + \sum_{k,l\in U_n}\varepsilon_k\varepsilon_l\Big(Q(e^r|x_k-x_l|)-1\Big)\\ &\geq\Big(\sum_{k\in U_n}\varepsilon_k\Big)^2 - C\sum_{k,l\in U_n} e^r|x_k-x_l|.\end{aligned}$$

The term $\sum_{k,l\in U_n} e^r|x_k-x_l|$ can be neglected. More precisely we have

$$\int_{\mathcal{A}}\Big(\sum_n\sum_{k,l\in U_n} e^r|x_k-x_l|\mathrm{d}r \leq \int_{\mathbb{R}_+}\sum_{1\leq k,l\leq i} e^r|x_k-x_l|\mathbb{1}_{\{|x_k-x_l|<e^{-r}\}}\mathrm{d}r \leq Ci(i-1). \tag{4.51}$$

Thus for any value of u and s we have (with a constant that depends on i)

$$\int_s^u \mathrm{Var}(Y^{(r)})\,\mathrm{d}r \geq \int_{[s,u]\cap\mathcal{A}}\sum_{n\geq 1}\left(\sum_{k\in U_n}\varepsilon_k\right)^2\mathrm{d}r - C. \tag{4.52}$$

Now if $\sum\varepsilon_k\neq 0$ we have necessarily a group with non-cancelling signs and thus $\sum_{n\geq 1}\left(\sum_{k\in U_n}\varepsilon_k\right)^2\geq 1$, for all $r\in\mathcal{A}$. This is sufficient to prove the first line of (4.47) using the fact that the measure of $[u,s]\setminus\mathcal{A}$ is bounded by a constant. If $\sum\varepsilon_k=0$, $r\in\mathcal{A}$ and $e^{-r}<2m(\mathbf{x})/i$ then necessarily there must be a group with non zero sign ($\sum_{k\in U_n}\varepsilon_k\neq 0$). Indeed, if not, each group U_n contains the same number of x_k's with positive ε_k or negative ε_k. We can then pair these x_k inside each U_n so as to obtain $\sigma\in\mathcal{S}_p$ (where $i=2p$) such that

$$\sum_{k=1}^p |x_k - x_{p+\sigma(k)}| \leqslant pe^{-r}.$$

Thus, if all groups have zero sign then $m(\mathbf{x})\leqslant ie^{-r}/2$. On the complement of this event, $\sum_{n\geq 1}\left(\sum_{k\in U_n}\varepsilon_k\right)^2\geq 1$, from which we deduce the second line of (4.47). ☐

4.4. **Convergence of the partition function.** Fix n and $\beta^2<2-\frac{1}{n}$. Let us consider some bounded measurable functional F which we further assume to be measurable with respect to $\mathcal{F}_{s_0}$ for some $s_0>0$. Using Lemma 3.1 we have

$$\mathcal{Z}_t^\beta(F) := \mathbb{E}[Fe^{\alpha M_t}]e^{-\sum_{i=1}^{2n-1}\frac{\alpha^i}{i!}\mathcal{C}_i(M_t)} = \mathbb{E}\left[Fe^{N_t^{(2n-1,\alpha,t)}-\frac{1}{2}\langle N^{(2n-1,\alpha,t)}\rangle_t}e^{Q_t^{(2n-1,\alpha)}}\right]$$

where $N_t^{(2n-1,\alpha,t)}$ and $Q_t^{(2n-1,\alpha)}$ are defined by (3.6) and (3.8). Let us define for $s\leqslant t$

$$Q_{s,t}^{(2n-1,\alpha)} := \frac{1}{2}\sum_{k=2n}^{4n-2}\alpha^k\sum_{l=k-2n+1}^{2n-1}\langle M^{(l,t)}, M^{(k-l,t)}\rangle_s. \tag{4.53}$$

where $Q_{t,t}^{(2n-1,\alpha)} = Q_t^{(2n-1,\alpha)}$. We can assume that $F\geqslant 1$ by adding a large constant to it. Now, because of our assumption on F of $\mathcal{F}_{s_0}$-measurability and the martingale property of $(N_s^{(2n-1,\alpha,t)})_{s\leqslant t}$, we have for

$s_0 \leqslant s \leqslant t$

$$\ln \left| \frac{\mathbb{E}\left[F e^{N_t^{(2n-1,\alpha,t)} - \frac{1}{2}\langle N^{(2n-1,\alpha,t)}\rangle_t} e^{Q_{t,t}^{(2n-1,\alpha)}} \right]}{\mathbb{E}\left[F e^{N_s^{(2n-1,\alpha,t)} - \frac{1}{2}\langle N^{(2n-1,\alpha,t)}\rangle_s} e^{Q_{s,t}^{(2n-1,\alpha)}} \right]} \right| \leq \|Q_{t,t}^{(2n-1,\alpha)} - Q_{s,t}^{(2n-1,\alpha)}\|_\infty. \tag{4.54}$$

Hence we can rewrite

$$\ln \mathbb{E}[F e^{\alpha M_t}] - \sum_{i=1}^{2n-1} \frac{\alpha^i}{i!} \mathcal{C}_i(M_t) = \ln \mathbb{E}\left[F e^{N_s^{(2n-1,\alpha,t)} - \frac{1}{2}\langle N^{(2n-1,\alpha,t)}\rangle_s} e^{Q_{s,t}^{(2n-1,\alpha)}} \right] + A_2(s,t). \tag{4.55}$$

with $|A_2(s,t)| \leq \|Q_{t,t}^{(2n-1,\alpha)} - Q_{s,t}^{(2n-1,\alpha)}\|_\infty$. We are going to show that

$$\lim_{t\to\infty} A_1(s,t) := \lim_{t\to\infty} \ln \mathbb{E}\left[F e^{N_s^{(2n-1,\alpha,t)} - \frac{1}{2}\langle N^{(2n-1,\alpha,t)}\rangle_s} e^{Q_{s,t}^{(2n-1,\alpha)}} \right] =: A_1(s) \tag{4.56}$$

exists for every s and that

$$\lim_{s\to\infty} \sup_{t\geq s} \|Q_{t,t}^{(2n-1,\alpha)} - Q_{s,t}^{(2n-1,\alpha)}\|_\infty = 0. \tag{4.57}$$

This is sufficient to conclude because repeating the argument in (4.54) we have

$$|A_1(s_1,t) - A_1(s_2,t)| \leq \|Q_{s_1,t}^{(2n-1,\alpha)} - Q_{s_2,t}^{(2n-1,\alpha)}\|_\infty. \tag{4.58}$$

As a consequence of (4.57) the sequence $A_1(s)$ is Cauchy and from (4.55) and elementary calculus we deduce that

$$\lim_{t\to\infty} \log(\mathbb{E}[F(X)e^{\alpha M_t}]) - \sum_{i=1}^{2n-1} \frac{\alpha^i}{i!} \mathcal{C}_i(M_t) = \lim_{s\to\infty} A_1(s). \tag{4.59}$$

The convergence (4.56) follows from the fact that for fixed s, the martingale $N^{(2n-1,\alpha,t)}$ can be written as a sum $N^{(2n-1,\alpha,t)} = \sum_{i=1}^{2n-1} \alpha^i M^{(i,t)}$ (see (3.6)). Proposition 4.1 Item (3) then ensures the convergence towards some limit denoted N_s as well as almost sure convergence of its quadratic variations. Furthermore the quadratic variations of N_s are essentially bounded, which ensures that N_s has exponential moments of all orders. Proposition 4.1 item 3 also asserts almost sure convergence (as t tends to infinity) of $Q_{s,t}^{(2n-1,\alpha)}$ towards an essentially bounded random variable.

Finally we show (4.57). From (4.53), it is enough to prove the claim for $\langle M^{(l,t)}, M^{(k-l,t)}\rangle_s$ with $k \in [\![2n, 4n-2]\!]$ and $l \in [\![k-2n+1, 2n-1]\!]$. Similarly to (4.4), such a bracket takes the form of a sum

$$\langle M^{(l,t)}, M^{(k-l,t)}\rangle_s = \sum_{p=0}^{\lfloor k/2\rfloor} \int_0^s \int_{I^k} F^{(k,p)}(\mathbf{x},u,t) e^{\frac{\beta^2}{2}\sum_{l=1}^k K_u(x_l,x_l)} \cos\left[\beta \left(\sum_{j=1}^k \lambda_j^{(p)} X_u(x_j) \right) \right] \mu(\mathrm{d}\mathbf{x})\mathrm{d}u. \tag{4.60}$$

in such a way that Proposition 4.1 item 2 provides the bound

$$\|\langle M^{(l,t)}, M^{(k-l,t)}\rangle_t - \langle M^{(l,t)}, M^{(k-l,t)}\rangle_s\|_\infty \leqslant C \int_s^\infty e^{\frac{\beta^2}{2}ku - (k-1)u}\, \mathrm{d}u.$$

This is enough for our claim because $k \geqslant 2n$ and $\beta^2 < 2 - \frac{1}{n}$. $\square$

5. Convergence of the charge distribution: Proof of Theorem 2.7

To implement this proof, let us move back to (2.8). We need to prove the convergence and continuity in θ of $\mathbb{E}[e^{\alpha M_t^{(\beta,\theta)}}]/\mathbb{E}[e^{\alpha M_t^{(\beta)}}]$. The proof presented in the previous section ensures that $\mathbb{E}[e^{\alpha M_t^{(\beta,\theta)}}]e^{-\sum_{i=1}^{n-1} \frac{\alpha^{2i}}{2i!}\mathcal{C}_{2i}^{(\theta,\beta)}(t)}$ converges. In order to ensure convergence and continuity of our quotient, we need to show for $i \leq n-1$, the different of the diverging cumulants converges

$$\lim_{t\to\infty} [\mathcal{C}_{2i}^{(\theta,\beta)}(t) - \mathcal{C}_{2i}^{(\beta)}(t)] \tag{5.1}$$

We are going to prove in fact that the convergence holds for all cumulants and that the corresponding limit are continuous in θ.

Using our reprentation (4.4) for both cumulants (the computation leading to the expression of $F^{(i,p)}$ is unchanged when a function θ is added), we obtain that for $A_t^{(i,\theta)}$ being defined as (3.4) for the martingale $M^{(\theta,\beta)}$ (recall $Y_p(\mathbf{x}) := \sum_{k=1}^n \lambda_k^{(p)} X(x_k)$)

$$
\begin{aligned}
(5.2)\quad & i!(\mathcal{C}_i^{(\theta,\beta)}(t) - \mathcal{C}_i^{(\beta)}(t)) = \mathbb{E}[A_t^{(i,\theta)} - A_t^{(i,\theta)}] \\
:= & \sum_{p=0}^{\lfloor i/2 \rfloor} \int_0^t \int_{I^i} F^{(i,p)}(\mathbf{x},u,t) e^{\frac{\beta^2}{2}\sum_{k=1}^i K_u(x_k,x_k)} \mathbb{E}\left[\cos\left[\beta\left(Y_p(\mathbf{x}) + \sum_{k=1}^i \lambda_k^{(p)}\theta(x_k)\right)\right] - \cos(\beta Y_p(\mathbf{x}))\right]\mu(\mathrm{d}\mathbf{x})\mathrm{d}u.
\end{aligned}
$$

By using that

$$
\mathbb{E}\left[\cos\left[\beta\left(Y_p(\mathbf{x}) + \sum_{k=1}^i \lambda_k^{(p)}\theta(x_k)\right)\right] - \cos(\beta Y_p(\mathbf{x}))\right] = \left[\cos\left(\beta\sum_{k=1}^i \lambda_k^{(p)}\theta(x_k)\right) - 1\right] e^{-\frac{\beta^2}{2}\mathrm{Var}(Y_p(\mathbf{x}))},
$$

we can reduce our problem to studying the convergence and continuity of

$$
\int_0^t \int_{I^i} F^{(i,p)}(\mathbf{x},u,t) e^{-\frac{\beta^2}{2}U_{i,p}(\mathbf{x})} \left[\cos\left(\sum_{k=1}^i \lambda_k^{(p)}\theta(x_k)\right) - 1\right]\mu(\mathrm{d}\mathbf{x})\mathrm{d}u. \tag{5.3}
$$

As we have convergence of $F^{(i,p)}(\mathbf{x},u,t)$ towards a limit, we want to use dominated convergence. Let us observe that

$$
\left|\cos\left(\sum_{k=1}^i \lambda_k^{(p)}\theta(x_k)\right) - 1\right| \le \frac{1}{2}\left|\sum_{k=1}^i \lambda_k^{(p)}\theta(x_k)\right|^2 \le \frac{i^2}{2}\|\theta\|_{1/2}^2. \tag{5.4}
$$

Hence from the computation of the previous section, we can deduce that we have convergence when $p \neq i/2$ (continuity in θ can be proved in the same manner). For the case $p = i/2$, recalling the definition of $m(\mathbf{x})$ (4.22) and that of the Hölder norm, we observe that for the optimal permutation σ we have

$$
\left|\sum_{k=1}^p \theta(x_k) - \theta(x_{p+\sigma(k)})\right|^2 \le p\sum_{k=1}^p |\theta(x_k) - \theta(x_{p+\sigma(k)})|^2 \le p\|\theta\|_{1/2}^2 m(\mathbf{x}), \tag{5.5}
$$

and to conclude we only need to check that (recall $F^{(i,p)}(\mathbf{x},u) := \sup_{t>u} F^{(i,p)}(\mathbf{x},u,t)$)

$$
\int_0^\infty \int_{I^i} F^{(i,p)}(\mathbf{x},u) e^{-\frac{\beta^2}{2}U_{i,p}(\mathbf{x},u)} m(\mathbf{x})\mu(\mathrm{d}\mathbf{x})\mathrm{d}u < \infty. \tag{5.6}
$$

but this follows immediately from (4.23) and (4.4).

Finally, convergence in the Schwartz space results from Levy-Schwartz Theorem [3, Th. 2.3]. □

6. Correlation functions

To prove of Theorem 2.8, let us (for the sake of aligning with the notations of previous sections) denote by n the number of singularities in order to use the variable k in other context. Then omitting the dependence in $(\mathbf{z},\eta)$ in the martingale notation, the quantity defined in (2.15) is the terminal value of the martingale $(\widehat{M}_s^{(t)})_{s\in[0,t]}$ defined by

$$
\widehat{M}_s^{(t)} := \int_I e^{\frac{\beta^2}{2}K_s(x,x)} \cos\left(\beta X_s(x) + \mathbf{i}\sum_{k=1}^n \beta^2\eta_k K_t(x,z_k)\right)\mathrm{d}x. \tag{6.1}
$$

For notational simplicity we set for the rest of this section ((z_k,η_k) being fixed)

$$
\psi_t(x) := \beta^2\sum_{k=1}^n \eta_k K_t(x,z_k),
$$

and $\psi = \lim_{t\to\infty} \psi_t$. Now we can see that the cumulant of $\widehat{M}_s^{(t)}$ can be obtained using the computation made for that of $M_t^{(\beta)}$. If $\widehat{\mathcal{C}}_i(t)$ denotes the i-th cumulant of $\widehat{M}_s^{(t)}$, starting with (4.4) and repeating the computation (5.2)-(5.3), we obtain

$$i!(\widehat{\mathcal{C}}_i(t) - \mathcal{C}_i(t)) = \sum_{p=0}^{\lfloor i/2 \rfloor} \int_0^t \int_{I^i} F^{(i,p)}(\mathbf{x},u,t) e^{-\frac{\beta^2}{2} U_{i,p}(\mathbf{x},u)} \left[\cosh\left(\sum_{k=1}^i \lambda_k^{(p)} \psi_t(x_k)\right) - 1\right] \mu(\mathrm{d}\mathbf{x})\mathrm{d}u. \tag{6.2}$$

Theorem 2.8 is proved if we can show that $i!(\widehat{\mathcal{C}}_i(t) - \mathcal{C}_i(t))$ converges for every i that for some value of n,

$$\lim_{t\to\infty} \mathbb{E}\left[e^{\alpha \widehat{M}_t^{(t)}}\right] e^{-\sum_{i=1}^n \frac{\alpha^i}{i!} \widehat{\mathcal{C}}_i(t)} < \infty \tag{6.3}$$

Let us first focus on the cumulant difference. The proof of (6.3), very similar to what is done in Section 4.4. As the term $\left[\cosh\left(\sum_{k=1}^i \lambda_k^{(p)} \psi_t(x_k)\right) - 1\right]$ is not bounded uniformly in t, we require an extension of Proposition 4.1 to treat it Let us introduce the quantity W on I and its extension to I^i as

$$W(x) = \max_{k\in[\![0,n]\!]} (-\log|x - z_k|) \quad \text{ and } \quad W(\mathbf{x}) := \sum_{k=1}^i W(x_k).$$

Note that it is easy to check that

$$\cosh\left(\sum_{k=1}^i \lambda_k^{(p)} \psi_t(x_k)\right) \le C e^{\beta^2 \|\eta\|_\infty W(\mathbf{x})}. \tag{6.4}$$

Hence we need an adapted version of Proposition 4.1, which allows us for the integration of the bound above, in order to proceed along the same proof as in Section 4.

Proposition 6.1. *Given for every $i \ge 2$, $p \le i/2$, then whenever $2\alpha + \beta^2 < 2$ there exists a constant $C = C(i, \mathbf{z}, \alpha, \beta) > 0$ such that for any fixed k we have*

$$\sup_{x_k\in I} \int_{I^{i-1}} e^{\alpha(W(\mathbf{x}) - W(x_k))} \sup_{t\geqslant u} |F^{(i,p)}(\mathbf{x},u,t)| \mu(\mathrm{d}\mathbf{x}^{(k)}) \le C e^{(i-1)(\alpha-1)u}. \tag{6.5}$$

We wish to use Proposition 6.1 to prove that whenever $\beta^2(2\|\eta\|_\infty + 1) < 2$ we have for all i (recall $\bar{F}^{(i,p)}(\mathbf{x},u) = \lim_{t\to\infty} F^{(i,p)}(\mathbf{x},u,t)$)

$$\lim_{t\to\infty} i!(\widehat{\mathcal{C}}_i(t) - \mathcal{C}_i(t)) = \sum_{p=0}^{\lfloor i/2 \rfloor} \int_0^\infty \int_{I^i} \bar{F}^{(i,p)}(\mathbf{x},u) e^{-\frac{\beta^2}{2} U_{i,p}(\mathbf{x},u)} \left[\cosh\left(\sum_{k=1}^i \lambda_k^{(p)} \psi(x)\right) - 1\right] \mu(\mathrm{d}\mathbf{x})\mathrm{d}u < \infty. \tag{6.6}$$

Both the convergence and finiteness of the integrals are proved using dominate convergence starting with (6.2). We proceed separately for each i and p and start with the easier case $i \neq 2p$. Setting $\alpha = \beta^2\|\eta\|_\infty$ and as usual $F^{(i,p)}(\mathbf{x},u) := \sup_{t\geqslant u} |F^{(i,p)}(\mathbf{x},u,t)|$ and using (4.24) and (6.4), we obtain that the integrand in (6.2) is bounded uniformly in t by

$$C F^{(i,p)}(\mathbf{x},u) e^{\frac{\beta^2}{2}(i-1)u} e^{\alpha W(\mathbf{x})}.$$

According to (6.5) the integral of the above is (up to a constant factor) smaller than

$$\left(\int_I e^{\alpha W(x_1)} \mu(\mathrm{d}x_1)\right) \left(\int_0^\infty e^{(i-1)[\alpha + (\beta^2/2) - 1]u} \mathrm{d}u\right) < \infty, \tag{6.7}$$

where the finiteness of the above is implied by $(2\alpha + \beta^2) < 2$. When $i = 2p$ we need to be substantially more accurate in our approximation. Instead of (6.4) we observe (e.g. considering separately the case of $|\sum_{k=1}^i \lambda_k^{(p)} \psi_t(x_k)| \le 1$ and $|\sum_{k=1}^i \lambda_k^{(p)} \psi_t(x_k)| \ge 1$) that

$$\left[\cosh\left(\sum_{k=1}^i \lambda_k^{(p)} \psi_t(x)\right) - 1\right] \le C e^{\alpha W(\mathbf{x})} \min\left[1, \left(\sum_{k=1}^i \lambda_k^{(p)} \psi_t(x_k)\right)\right]^2. \tag{6.8}$$

To obtain a bound which is uniform in t we use the fact that there exists C such that for every $t \geq 0$

$$\sum_{k=1}^{i} \lambda_k^{(p)} \psi_t(x_k) \leq C \frac{m(\mathbf{x})}{D(\mathbf{x})}, \tag{6.9}$$

where $m(\mathbf{x})$ is the Wasserstein distance defined in (4.22) and $D(\mathbf{x}) = \min_{\substack{i\in[\![1,k]\!]\\ j\in[\![1,n]\!]}} |x_i - z_j|$. We can also replace the exponent 2 in (6.8) by $\beta^2/2$ (which is smaller than 1 in any case) and ignore the min. Using (4.23), the bound we obtain for the integrand is then

$$F^{(i,p)}(\mathbf{x},u) e^{\frac{\beta^2}{2}(i-1)u} e^{\alpha W(\mathbf{x})} \left(\min(m(\mathbf{x})^{-1}, e^u) \left(\frac{m(\mathbf{x})}{D(\mathbf{x})} \right) \right)^{\beta^2/2}.$$

Now observe that the last term (between parentheses) is smaller than

$$\frac{1}{D(\mathbf{x})^{\beta^2/2}} \leq \sum_{k=1}^{i} e^{\frac{\beta^2}{2} W(x_k)}. \tag{6.10}$$

To conclude we notice that for each $k \in [\![1,i]\!]$ we have from (6.5)

$$\begin{aligned} &\int_0^\infty \int_{I^i} F^{(i,p)}(\mathbf{x},u) e^{\frac{\beta^2}{2}(i-1)u} e^{\alpha W(\mathbf{x}) + \frac{\beta^2}{2} W(x_k)} \mu(\mathrm{d}\mathbf{x}) \mathrm{d}u \\ &\qquad \leq C \left(\int_I e^{\left(\alpha + \frac{\beta^2}{2}\right) W(x_k)} \mu(\mathrm{d}x_k) \right) \left(\int_0^\infty e^{(i-1)[\alpha + \beta^2/2 - 1]u} \mathrm{d}u \right), \end{aligned} \tag{6.11}$$

where the finiteness of the first and second integrals are both consequences of our assumption $2\alpha + \beta^2 < 2$.

Proof of Proposition 6.1. The proof goes by induction and follows the step of that of Proposition 4.1. The key observation is that there exists a constant C_1 (depending on α and $\mathbf{z}$) such that for every $t \geq u$ and $x \in I$

$$\int_I (1 + e^u |x-y|)^2 e^{\alpha W(y)} \mu(\mathrm{d}y) \leq C_1 e^{-(\alpha-1)u}. \tag{6.12}$$

We prove (6.5) by induction. Note that (6.12) and Assumption (2.1) allows us to check that the statement is valid when $i = 2$. Then the proof of the induction statement follows the steps of Proposition 4.1 item (2) until (4.26) where we have to figure out how to replace the three inequalities. We assume that $p \neq i/2$ and $q \geq j/2$ for simplicity (the adaptation needed for the case $2p = i$ and/or $2q = j$ are exactly identical to that in the proof of Proposition 4.1, and the same observation is valid for the case $j = 1$). Instead of (4.25) we need to prove

$$\begin{aligned} &\int_{I^{i+j-1}} e^{\alpha W(x_k)} |Q_s(x_k, x_l)| \left[\int_s^\infty F(\mathbf{x}_1, u) e^{\alpha W(\mathbf{x}_1^{(k)}) + \frac{\beta^2}{2}(i-1)(u-s)} \mathrm{d}u \right] \\ &\qquad \times \left[\int_s^\infty G(\mathbf{x}_2, v) e^{\alpha W(\mathbf{x}_2^{(k)}) + \frac{\beta^2}{2}(i-1)(v-s)} \mathrm{d}v \right] \mu(\mathrm{d}\mathbf{x}^{(k)}) \leq C e^{(i+j-1)(\alpha-1)s} \end{aligned} \tag{6.13}$$

where $\mathbf{x}_1^{(k)}$ stands for the vector $\mathbf{x}_1$ with k-th component removed (in particular it does not change if $\mathbf{x}_1$ has no k-th component) and similarly for $\mathbf{x}_2^{(k)}$. Now considering $u, v > s$, we observe (using the induction hypothesis for the first and last line and (6.12) for the middle one)

$$\begin{aligned} \int_{I^{i-1}} F(\mathbf{x}_1, u) e^{\alpha W(\mathbf{x}_1^{(k)})} \mu(\mathrm{d}\mathbf{x}_1^{(k)}) &\leq C_2 e^{(i-1)(\alpha-1)u}, \\ \int_I |Q_s(x_k, x_l)| e^{\alpha W(x_k)} \mu(\mathrm{d}x_k) &\leq C_3 e^{(\alpha-1)s}, \\ \int_{I^{j-1}} G(\mathbf{x}_2, v) e^{\alpha W(\mathbf{x}_2^{(m)})} \mu(\mathrm{d}\mathbf{x}_2^{(m)}) &\leq C_4 e^{(j-1)(\alpha-1)v}. \end{aligned} \tag{6.14}$$

To prove (6.13), we just need to combine these three inequalites in the l.h.s. and integrate over $u, v > s$ (using that $\alpha + \frac{\beta^2}{2} < 1$).

Now to prove (6.3), following the proof structure of Section 4.4, it is sufficient to prove that

$$\lim_{s\to\infty} \sup_{t\geq s} \|\widehat{Q}^{(n,\alpha)}_{t,t} - Q^{(n,\alpha)}_{s,t}\|_\infty = 0. \tag{6.15}$$

where

$$\widehat{Q}^{(n,\alpha)}_{s,t} := \frac{1}{2}\sum_{k=n+1}^{2n} \alpha^k \sum_{l=k-n}^{n} \langle \widehat{M}^{(l,t)}, \widehat{M}^{(k-l,t)}\rangle_s. \tag{6.16}$$

and $\widehat{M}^{(i,t)}$ are the martingales constructed from $\widehat{M}$ using the iteration 3.4. These type of martingale bracket can be written in the form

$$\begin{aligned} &\langle \widehat{M}^{(l,t)}, \widehat{M}^{(k-l,t)}\rangle_s \\ &\quad= \sum_{p=0}^{\lfloor k/2\rfloor} \int_0^s \int_{I^k} F^{(k,p)}(\mathbf{x},u,t) e^{\frac{\beta^2}{2}\sum_{l=1}^k K_u(x_l,x_l)} \cos\left[\left(\sum_{j=1}^k \beta\lambda_j^{(p)} X_u(x_j) + \mathbf{i}\psi_t(x_j)\right)\right]\mu(\mathrm{d}\mathbf{x})\mathrm{d}u, \end{aligned} \tag{6.17}$$

where $F^{(k,p)}$ verifies the properties of Propositions 4.1-6.1. In order to prove the convergence statement, we mostly need to prove domination. Hence we only need to observe that the cosinus term is bounded is absolute value by $Ce^{\beta^2\|\eta\|_\infty W(\mathbf{x})}$, and we conclude by showing that

$$\int_0^\infty \int_{I^k} \sup_{t\geq u} |F^{(k,p)}(\mathbf{x},u,t)| e^{\frac{\beta^2}{2}\sum_{l=1}^k K_u(x_l,x_l)} e^{\beta^2\|\eta\|_\infty W(\mathbf{x})}\mathrm{d}u < \infty \tag{6.18}$$

Setting $\alpha = \|\eta_\infty\|\beta^2$, as a consequence of Proposition 6.1, the above integral is smaller than

$$C\int_0^\infty e^{\frac{\beta^2}{2}ku} e^{(k-1)(\alpha-1)u}\mathrm{d}u, \tag{6.19}$$

and the above integral is finite as soon as $(k-1)(\alpha-1)u + \frac{\beta^2}{2}ku < 0$ which is valid for n sufficiently large (recall that we consider only $k \geq n+1$) as soon as $\beta^2(1+2\|\eta\|_\infty) < 2$. □

References

[1] J. Fröhlich: Classical and quantum statistical mechanics in one and two dimensions: two component Yukawa and Coulomb systems, Commun. Math. Phys. 47 (1976), p.233-268.

[2] G. Benfatto, G. Gallavotti, F. Nicolò: On the massive Sine-Gordon equation in the first few regions of collapse, Commun. Math. Phys. 83 no 3 (1982), p. 387-410.

[3] H. Biermé, O. Durieu, Y. Wang: Generalized random fields and Lévy's continuity theorem on the space of tempered distributions, arXiv:1706.09326.

[4] D. Brydges, T. Kennedy: Mayer expansions and the Hamilton-Jacobi equations, J. Stat. Phys. 48 issue 1-2 (1987), p. 19-49.

[5] A. Chandra, M. Hairer: The dynamical sine-Gordon model in the full subcritical regime, arXiv:1808.02594.

[6] J. Dimock , T.R. Hurd: Sine-Gordon revisited, Ann. Henri Poincaré 1, no. 3, (2000), 499Ð541.

[7] P. Falco: Kosterlitz-Thouless Transition Line for the Two Dimensional Coulomb Gas, Commun. Math. Phys. (2012), vol 312, p. 559-609.

[8] V. Fateev, S. Lukyanov, A. Zamolodchikov, A. Zamolodchikov: Expectation values of boundary fields in the boundary sine-Gordon model, Physics Letters B 406, 83-88, 1997.

[9] J. Fendley, F. Lesage, H. Saleur: Solving 1d plasmas and 2d boundary problems using Jack polynomials and functional relations, Journal of Statistical Physics, 1995, Volume 79, Issue 5-6, pp 799-819.

[10] J. Fröhlich, T. Spencer: The Kosterlitz-Thouless transition in two-dimensional Abelian spin systems and Coulomb gas, Commun. Math. Phys. 81, p.527- 602, 1981.

[11] M. Hairer, H. Shen: The dynamical Sine-Gordon model, Commun. Math. Phys. (2016), vol 341, p. 341-933.

[12] J. Jacod, E. Kowalski, A. Nikeghbali: Mod-Gaussian convergence: new limit theorems in Probability and Number Theory *Forum Mathematicum* 23 (4), p. 835-873 (2011).

[13] V. Feray, P-L. Meliot, A. Nikeghbali: Mod-ϕ Convergence Normality Zones and Precise Deviations, Springer Briefs in Probability and Mathematical Statistics (2016).

[14] T. Leblé, S. Serfaty, and O. Zeitouni. Large deviations for the two- dimensional two-component plasma. Comm. Math. Phys., 350(1):301–360, 2017. MR3606477.

[15] F. Nicolò: On the massive sine-Gordon equation in the higher regions of collapse. Comm. Math. Phys. 88, no. 4, (1983), p.581-600.

[16] F. Nicolò, J. Renn, A. Steinmann: On the massive sine-Gordon equation in all regions of collapse. Comm. Math. Phys. 105, no. 2, (1986), 291-326.

[17] L. Onsager: Electrostatic interaction between molecules. J. Phys. Chem. 43 (1939), 189-196.

[18] J. Junnila, E. Saksman and Christian Webb: Imaginary multiplicative chaos: moments, regularity and connections to the Ising model, arXiv:1806.02118.

[19] Rhodes R., Vargas, V.: Multidimensional multifractal random measures, *Electronic Journal of Probability*, **15** (2010) 241-258.

[20] J.P. Solovej: Some simple counterexamples to the Onsager lemma on metric spaces. Rep. Math.Phys. 28 (1989), no. 3, 335–338.

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