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Path integral for quantum Mabuchi K-energy

Hubert Lacoïn ^{*}, Rémi Rhodes [†], Vincent Vargas [‡]

Monday 29th April, 2019

Abstract

We construct a path integral based on the coupling of the Liouville action and the Mabuchi K-energy on a one-dimensional complex manifold. To the best of our knowledge this is the first rigorous construction of such an object and this is done by means of probabilistic tools. Both functionals play an important role respectively in Riemannian geometry (in the case of surfaces) and Kähler geometry. As an output, we obtain a path integral whose Weyl anomaly displays the standard Liouville anomaly plus an additional K-energy term. Motivations come from theoretical physics where these type of path integrals arise as a model for fluctuating metrics on surfaces when coupling some non conformal matter fields to quantum gravity as advocated by A. Bilal, F. Ferrari, S. Klevtsov and S. Zelditch. Interestingly, our computations show that quantum corrections perturb the classical Mabuchi K-energy and produce a quantum Mabuchi K-energy: these type of corrections are reminiscent of the quantum Liouville theory. Our probabilistic construction relies on a variant of the theory of Gaussian multiplicative chaos (GMC) and Derivative GMC (DGMC for short). The technical backbone of our construction consists in two estimates on (derivative and standard) GMC which are of independent interest in probability theory. Firstly, we show that these DGMC random variables possess negative exponential moments and secondly we derive optimal small deviations estimates for the GMC associated with a recentered Gaussian Free Field.

Key words or phrases: *2d* Quantum Gravity, quantum field theory, Gaussian multiplicative chaos, random Kähler geometry, Mabuchi K-energy.

MSC 2000 subject classifications: 81T40, 81T20, 60D05.

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^{*}IMPA, Estrada Dona Castorina 110, Rio de Janeiro, RJ-22460-320, Brasil. H.L. acknowledges support from FAPERj (grant JCNE) and CPNq (grant Universal and productivity grant).

[†]Université Aix-Marseille, I2M, Marseille, France. Partially supported by grant ANR-15-CE40-0013 Liouville.

[‡]ENS Ulm, DMA, 45 rue d'Ulm, 75005 Paris, France. Partially supported by grant ANR-15-CE40-0013 Liouville.

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1 Introduction and motivations

The goal of this paper is to construct a new form of $2d$ random geometry¹ (a so-called **quantum Kähler geometry**) using probabilistic methods in view of applications to $2d$ quantum gravity. Given a manifold M , (Euclidean) quantum gravity is a geometrical prescription to pick at random a geometry (say a metric tensor) together with a matter field on M (a configuration of some model of statistical physics). This prescription is a non trivial coupling of these two objects in such a way that the nature of the matter field shapes the random geometry. Yet this mechanism is understood (at the level of physics rigor) only in the very specific situation when the matter field possesses conformal symmetries, namely is a Conformal Field Theory (CFT for short)². In the physics literature, Polyakov’s seminal work [Pol] and the DDK ansatz [Dav, Di-Ka] have paved the way towards a complete understanding of this case: the random geometry is then ruled by the Liouville CFT (LCFT for short), which can be seen as the natural probabilistic theory of Riemann surfaces. Giving a rigorous meaning to this picture is under active research in probability theory nowadays and we will not try in this introduction to give an account on all the recent developments on the topic: we refer to [Da-Ku-Rh-Va, Gui-Rh-Va] for rigorous constructions of LCFT and the works [Cur, Du-Mi-Sh] for the link between LCFT and the scaling limit of discrete planar maps weighted by a critical statistical physics model. In the specific case of pure gravity where the matter field is trivial, one can also equip LCFT with a distance function called the Brownian map: see [LeG, Mier] for the convergence (in the sense of Gromov-Hausdorff) of discrete maps to the Brownian map and [Mil-Sh1, Mil-Sh2, Mil-Sh3] for a construction of the Brownian map in the continuum setting of LCFT. Our paper is concerned with the study of random geometries that may arise when matter fields move (slightly) away from conformal symmetries. Our approach is inspired by the series of works [Bi-Fe-Kl, Fe-Kl-Ze, Fe-Kl-Ze2, La-Er-Sv] suggesting models deeply anchored in both Riemannian and Kähler geometries.

In the framework of Riemannian geometry, we are given a compact Riemann surface M , i.e. a one dimensional complex manifold³, equipped with a Riemannian metric g . In view of the classification of Riemann surfaces⁴, an important question which goes back to Picard and Poincaré is to find a metric \hat{g}

¹ $2d$ refers to the real dimension; equivalently, this corresponds to one dimension in the complex setting.

²Recall that many CFTs are expected to describe the scaling limit of discrete statistical physics model at criticality.

³We will restrict in this paper to the case of one-dimensional complex manifolds. Though Riemann and Kähler geometries make sense in higher dimensions, our approach does not.

⁴This is also related to solutions of Einstein field equations in general relativity.

with uniformized Ricci scalar curvature $K_{\widehat{g}} = -2\pi\mu$ in the conformal class $[g]$ of g defined by

$$[g] := \{e^\omega g; \omega \in C^\infty(M)\}.$$

Such metrics $\widehat{g} = e^\omega g$ can be found by searching for the critical points⁵ of the *classical* Liouville functional

$$S_L^{\text{cl},\mu}(\widehat{g}, g) := \int_M (|d\omega|_g^2 + 2K_g\omega + 4\pi\mu e^\omega) dv_g, \quad (1.1)$$

where K_g is the Ricci scalar curvature of the metric g ⁶, v_g its volume form and $d\omega$ the differential of ω . More generally, in arbitrary dimensions, metrics with Ricci tensor proportional to the metric are called Einstein metrics. The problem of finding Einstein metrics on $2d$ real (or one dimensional complex) manifolds is now well understood but turns out to be much harder in higher dimensions. This has certainly been a source of motivation to put some further structure on the manifold in order to make the search for Einstein metrics more tractable. This explains at least partly the success of Kähler geometry. Indeed, though Kähler geometry is a natural extension of Riemannian geometry in the sense that it is designed in the spirit of complex Euclidean geometry, it can also be seen as another parametrization of the set of metrics that allows one to reduce the problem of finding Einstein metrics to a complex Monge-Ampère equation, as illustrated by the works of Aubin [Aub] or the proof of the Calabi-Yau conjecture [Yau]. These works treat the cases of negatively curved or Ricci-flat manifolds. The case of positively curved manifolds, known as the Yau-Tian-Donaldson conjecture, has been more problematic and has been settled only recently by Chen-Donaldson-Sun-Tian in a series of works [Ch-Do-Su, Tian] in which the Mabuchi K-energy presented below has played an important role: for further details we refer to the review paper [Sze]. From now on, we come back to the simpler framework of one dimensional complex manifolds and filter as much as possible geometrical considerations.

In the Kähler framework, the Kähler potential ϕ of the metric $\widehat{g} = e^\omega g$ is defined by the relation

$$e^\omega = \frac{V_{\widehat{g}}}{V_g} + \frac{V_{\widehat{g}}}{2} \Delta_g \phi \quad (1.2)$$

where Δ_g is the (negative) Laplace-Beltrami operator, with expression in local real coordinates (x_1, x_2)

$$\Delta_g = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x_j} \right),$$

and $V_g := v_g(M)$ is the volume of M in the metric g (similarly, $V_{\widehat{g}}$ is the volume of M in the metric \widehat{g}). This equation can always be solved up to constant in ϕ . An important functional called the Mabuchi K-energy can be written in terms of ω and ϕ

$$S_M^{\text{cl}}(\widehat{g}, g) = \int_M \left(2\pi(1 - \mathbf{h})\phi \Delta_g \phi + \left(\frac{8\pi(1 - \mathbf{h})}{V_g} - K_g \right) \phi + \frac{2}{V_{\widehat{g}}} \omega e^\omega \right) dv_g \quad (1.3)$$

where \mathbf{h} is the genus of M . Extremal points of the Mabuchi K-energy are also metrics with uniformized scalar curvature, hence the connection with Kähler-Einstein metrics.

The concept of uniformization of Riemann surfaces has its probabilistic counterpart. Indeed, Feynman's approach of quantum mechanics prescribes to associate to the Liouville functional on the Riemannian manifold (M, g) a path integral (i.e. a measure on some functional space) formally defined by

$$\langle F \rangle_{L,g} = \int F(\varphi) e^{-S_L(\varphi,g)} \mathcal{D}\varphi \quad (1.4)$$

⁵In some cases, these critical points are also minimizers.

⁶In isothermal coordinates of the form $e^{\omega(z)}|dz|^2$, the curvature $K_g(z)$ is given by $-e^{-\omega(z)}\Delta_z\omega(z)$ where Δ_z is the standard flat Laplacian.

for suitable functionals F , where $\mathcal{D}\varphi$ is the putative Lebesgue measure⁷ on some functional space of maps $\varphi : M \rightarrow \mathbb{R}^8$ and \mathcal{S}_L is the *quantum* Liouville functional (in what follows, for practical purpose the quantum actions are written as functions of the conformal factor)

$$\mathcal{S}_L(\varphi, g) := \frac{1}{4\pi} \int_M (|d\varphi|_g^2 + QK_g\varphi + 4\pi\mu e^{\gamma\varphi}) dv_g \quad (1.5)$$

where γ is a positive parameter belonging to $(0, 2)$, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$ and $\mu > 0$ is a positive parameter called the cosmological constant (see subsection 2.4 for further details). This path integral turns out to be a CFT, hence is called Liouville CFT. Such a path integral has been constructed **non perturbatively** only very recently using probability theory: see [Da-Ku-Rh-Va, Gui-Rh-Va]. This is in sharp contrast with many approaches to quantum field theory which usually provide constructions that are perturbative, i.e. are defined by formal power series (in the case of Liouville CFT on compact Riemann surfaces, the work of Takhtajan-Teo [Tak-Teo] provides such a construction in terms of a formal power series in the parameter γ). Notice that the *quantum action* (1.5) differs from the *classical action* (1.1) evaluated at (\hat{g}, g) where $\hat{g} = e^{\gamma\omega}g$

$$\frac{1}{4\pi\gamma^2} \mathcal{S}_L^{\text{cl}, \mu\gamma^2}(\hat{g}, g) := \frac{1}{4\pi} \int_M (|d\omega|_g^2 + \frac{2}{\gamma}K_g\omega + 4\pi\mu e^{\gamma\omega}) dv_g \quad (1.6)$$

through the value of Q where an extra $\frac{\gamma}{2}$ term appears: this is due to what physicists call *quantum corrections* appearing in controlling ultraviolet divergencies, which can be understood in mathematical terms as the effects of removing diverging quantities.

The Weyl anomaly (also called Polyakov formula) describes the way a quantum field theory reacts to conformal changes of metrics. In the case of Liouville CFT, it can be expressed in terms of the classical Liouville action: consider a conformal metric $\hat{g} = e^{\omega}g$ then

$$\langle F \rangle_{L, \hat{g}} = \langle F(\cdot - \frac{Q}{2}\omega) \rangle_{L, g} \exp\left(\frac{\mathbf{c}_L}{96\pi} \mathcal{S}_L^{\text{cl}, 0}(\hat{g}, g)\right) \quad (1.7)$$

where $\mathcal{S}_L^{\text{cl}, 0}$ is the classical Liouville functional (with $\mu = 0$)

$$\mathcal{S}_L^{\text{cl}, 0}(\hat{g}, g) := \int_M (|d\omega|_g^2 + 2K_g\omega) dv_g, \quad (1.8)$$

and $\mathbf{c}_L = 1 + 6Q^2$ is the *central charge* of Liouville CFT. The fact that the Weyl anomaly is log-proportional to the Liouville action characterizes a CFT in general (up to regularity issues). Such a transformation rule encodes a great deal of information about the theory: in particular, conformal Ward identities come out of (1.7) (see [Gaw] for an argument for CFTs up to regularity issues and [Ku-Rh-Va1] for a proof in the case of Liouville CFT), which leads to exact formulae for the theory (see in particular the recent proof of the DOZZ formula in [Ku-Rh-Va2]).

It is natural to wonder whether classical Kähler geometry admits a probabilistic counterpart too. The purpose of this paper is to construct in the (complex) one-dimensional setting a path integral exhibiting a Mabuchi K-energy term in the Weyl anomaly based on the quantization of the Mabuchi K-energy. Motivations come from the need of understanding $2d$ quantum gravity coupled to non conformal quantum field theories (for the interested reader, this is translated in terms of random planar maps in appendix D). On the Riemann surface (M, g) , this corresponds naively to constructing a functional integration measure of the type (with $\beta > 0$ a coupling constant and F an arbitrary functional)

$$\int F(\varphi) e^{-\beta \mathcal{S}_M^{\text{cl}}(e^{\gamma\varphi}g, g) - \mathcal{S}_L(\varphi, g)} \mathcal{D}\varphi. \quad (1.9)$$

⁷This measure is called the free field measure in the physics literature and it is not defined mathematically (not to be confused with the **Gaussian** free field measure which can be defined mathematically via probability theory).

⁸For (log)-conformal factors, we use throughout the papers two different notations: ω when it is deterministic and φ when it serves as an integration variable.

It turns out that the naive proposal (1.9) does not possess the expected Weyl anomaly because it overlooks renormalization effects. Quantum corrections in the Mabuchi K-energy, reminiscent to those arising in the Liouville functional, force to consider instead the **quantum** Mabuchi action

$$\mathcal{S}_M(\varphi, g) = \int_M \left(2\pi(1 - \mathbf{h})\phi\Delta_g\phi + \left(\frac{8\pi(1 - \mathbf{h})}{V_g} - K_g \right)\phi + \frac{2}{1 - \frac{\gamma^2}{4}} \frac{1}{V_{\hat{g}}} (\gamma\varphi)e^{\gamma\varphi} \right) dv_g. \quad (1.10)$$

where ϕ is the Kähler potential of the metric $e^{\gamma\varphi}$ (see Section 3 for precise definitions). Though the quantum versions of the Liouville and Mabuchi actions depend on γ , we most of the time do not stress the dependence in the notation. Compared to the classical Mabuchi K-energy (1.3), one can notice a quantum correction term $1 - \frac{\gamma^2}{4}$ in the entropic term. As far as we know, this is the first occurrence of the quantum version of the K-energy in the literature. Now, the main input of the paper is to construct the path integral

$$\langle F \rangle_{\text{ML},g} = \int F(\varphi) e^{-\beta\mathcal{S}_M(\varphi,g) - S_L(\varphi,g)} \mathcal{D}\varphi. \quad (1.11)$$

Compared to the Liouville path integral which corresponds to $\beta = 0$, there is a serious extra difficulty in defining (1.11) due to the potential term $(\gamma\varphi)e^{\gamma\varphi}$ in (1.10). Making sense of (1.11) requires controlling this term from below, a non trivial task due to renormalization effects. This situation is reminiscent from $P(\phi)_2$ -Euclidean quantum field theory (see [Sim]) but more difficult because it cannot be treated with Nelson's hypercontractive bounds as this term does not live in a fixed polynomial chaos. Also the path integral (1.11) has the expected Weyl anomaly (see Theorem 3.4 for a precise statement): let $\hat{g} = e^\omega g$ be a metric conformal to g and denote by ϕ its Kähler potential w.r.t. g . Then

$$\langle F \rangle_{\text{ML},\hat{g}} = \langle F(\cdot - \frac{Q}{2}\omega) \rangle_{\text{ML},g} \times \exp\left(\frac{1+6Q^2}{96\pi} S_L^{\text{cl},0}(\hat{g}, g) + \beta S_M^{\text{cl}}(\hat{g}, g)\right) \quad (1.12)$$

where $S_L^{\text{cl},0}$ and S_M^{cl} are respectively the classical Liouville functional (1.8) and the classical Mabuchi K-energy (1.3). This path integral is a way of giving sense to a random geometry of Kähler type. In particular, the volume of the manifold is then promoted to a random variable: we prove that it has a Gamma law $\Gamma(s, \mu)$ with an explicit formula for s in terms of γ, β : see subsection 3.1. This parameter s thus appears as an area scaling exponent: it plays an important role in physics where it is called *string susceptibility*⁹. Physicists do not have necessarily access to exact expressions for the string susceptibility so that they usually perform a *loop expansion*, which is simply an asymptotic expansion of s as $\gamma \rightarrow 0$. Our exact formula for the string susceptibility reproduces exactly the loop expansion found in Bilal-Ferrari-Klevtsov [Bi-Fe-Kl], see subsection 3.1. This is somewhat striking as our formula for s is shaped by the quantum corrections in the Mabuchi action, whereas the computations in [Bi-Fe-Kl] do not rely on the same path integral approach. More precisely, our approach differs from [Bi-Fe-Kl] as it is based on sampling the conformal factor φ according to the free field measure whereas the approach of [Bi-Fe-Kl] samples the Kähler potential ϕ according to the free field measure: the link between the two approaches is not mathematically proven and in particular would imply making sense of a change of variable $\phi \rightarrow \varphi$ with an ill-defined Jacobian.

Our construction is based on Gaussian Multiplicative Chaos (GMC for short) as well as a variant. GMC theory enables to define the exponential of the Gaussian Free Field (GFF). We have chosen, for the sake of presentation, to introduce the rigorous and technical definitions behind our construction of the path integral (1.11) only in Section 3, but let us just mention that the construction is based on interpreting $e^{-\frac{1}{4\pi} \int_M |d\varphi|_g^2 dv_g} \mathcal{D}\varphi$ as a GFF measure and expressing the other terms in the actions as functions of the GFF. With this in mind, the term $e^{\gamma\varphi}$ in the Liouville action (1.5) gives rise to GMC and the $(\gamma\varphi)e^{\gamma\varphi}$ term in the Mabuchi action (1.10) gives rise to a derivative (with respect to γ) of GMC.

More precisely, consider a GFF (see section 2.2) X with zero average when integrating with respect to the volume-form associated with g . A GMC measure is a random Radon (positive) measure M_γ of the form

$$M_\gamma(dx) := e^{\gamma X(x) - \frac{\gamma^2}{2} \mathbb{E}[X^2(x)]} \nu_g(dx). \quad (1.13)$$

⁹In fact, the string susceptibility is equal to $s + 2$.

This expression is only formal as the GFF is a random distribution (in the sense of Schwartz), hence it is not a fairly defined function; this can be seen at the level of the variance which satisfies $\mathbb{E}[X^2(x)] = \infty$. Renormalizing this into a meaningful expression is what GMC theory is aiming for and it was mainly developed by Kahane in the eighties [Kah] (or see also [Rh-Va2]). In our context, it asserts that the quantity (1.13) is well defined and non trivial for $\gamma \in (0, 2)$. Now consider what we call **derivative GMC**¹⁰ (DGMC for short)

$$M'_\gamma(dx) := (X(x) - \gamma\mathbb{E}[X^2(x)])e^{\gamma X(x) - \frac{\gamma^2}{2}\mathbb{E}[X^2(x)]} \nu_g(dx) \quad (1.14)$$

in order to make sense of the $(\gamma\varphi)e^{\gamma\varphi}$ term in the Mabuchi action (1.10).

On the technical side, our approach involves three ingredients (the last two are **new** results in probability theory) related to GMC or DGMC:

- (1) Universality of the meaning of (1.14) with respect to cut-off regularizations of the GFF X . Universality of GMC measures is well established and key ingredients for that are Kahane's convexity inequalities (see [Rh-Va2]) as well as positivity of GMC measures. For DGMC we lose both of these properties; as a matter of fact, DGMC is not even a signed measure almost surely (except for the limiting case $\gamma = 2$). Yet universality can be restored for $\gamma \in (0, \sqrt{2})$ by using L^2 -computations.
- (2) Concentration methods to bound the left tail of DGMC (Theorem 4.4): Given a ball B we show

$$\forall v \geq 0, \quad \mathbb{P}(M'_\gamma(B) \leq -v) \leq 2e^{-cv^2} \quad (1.15)$$

for some constant $c > 0$. The technical estimates we use restrict our statement to the values $\gamma \in (0, 1)$. We conjecture that (1.15) can be extended (and in fact improved) to a sharp statement valid for $\gamma \in (0, 2)$: see Section 4 for the detailed conjecture.

- (3) An **optimal** small deviation result for GMC: let us recenter the GFF so it has zero spatial average with respect to the measure ν_g over a set S , namely $\tilde{X} = X - \frac{1}{\nu_g(S)} \int_S X d\nu_g$ and denote by \tilde{M}_γ the GMC measure for the field \tilde{X} . Then for $\gamma \in (0, 2)$ and $s \geq 0$

$$\mathbb{P}(\tilde{M}_\gamma(S) \leq v) \leq c \exp\left(-cv^{-\frac{4}{\gamma^2}} |\ln v|^\kappa\right)$$

for some $\kappa, c > 0$: see Theorem 4.5 for a precise statement. Small deviations for GMC have received a lot of attention recently [Du-Sh, Nik, Ga-Ho-Se-Su] and are crucial estimates in many contexts. In all these works, the tail corresponds (at best) to lognormal random variables because the leading fluctuation term corresponds to that of the spatial average of the field. With the recentering procedure described above, we explain this mechanism and identify the lower order contribution. This result should be sharp (when ignoring the log-correction) as illustrated by exact density results obtained in [Rem, Rem-Zhu].

We stress here that the technical restrictions in items (1) and (2) above prevent our statements from covering the whole range of expected valid parameters $\gamma \in (0, 2)$. Note in particular that the degeneracy of the quantum Mabuchi K-energy (1.10) for $\gamma = 2$ (though Liouville CFT is well defined) is quite intriguing and generalizing our theory to the limiting case $\gamma = 2$ perhaps involves introducing a 2-nd order derivative GMC. Furthermore another restriction in our statements has a geometrical flavor: we only consider the case of hyperbolic surfaces, in which case $\mathbf{h} \geq 2$. This entails two simplifications: first we avoid this way having to introduce conical singularities in the surface M (recall for instance that this is the case for Liouville CFT on the Riemann sphere [Da-Ku-Rh-Va]) and, second, the sign in front of the term $\phi\Delta_g\phi$ in the quantum Mabuchi K-energy (2.5) goes the easy way. It is not hard to see that this term does not rise issues for small values of γ on surfaces with genus 0 or 1 but a full treatment can be more problematic. To keep the paper reasonably short, we restrict to hyperbolic surfaces but investigating the case of the Riemann sphere or torus seems definitely interesting and challenging.

¹⁰The name comes from the fact that (1.14) can be obtained from (1.13) by differentiating with respect to the parameter γ .

Organization of the paper

In Section 2, we introduce the necessary technical background required to construct the Liouville path integral including uniformization of Riemannian surfaces, Green Functions, Gaussian Free Field and Gaussian Multiplicative Chaos. In Section 3, we present our construction of the path integral with action given by the combination Mabuchi K-energy and Liouville Action (Definition-Theorem 3.4) and introduce the main technical result on which the construction relies (Theorem 3.2). Section 4 announces the technical backbone behind the proof of Theorem 3.2. More precisely it reduces the statement to left tail estimates for derivative GMC, and optimal small deviations for GMC measures proven respectively in Section 5 and 6.

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2 Background and notations

In this section, we list our notations and recall the precise definition of Liouville CFT (LCFT) as given in [Gui-Rh-Va].

2.1 Convention and notations.

Given a Riemannian manifold (M, g) we denote by dv_g the associated Riemannian measure, $V_g = v_g(M)$ the total volume, Δ_g the Laplace-Beltrami operator (negative definite Laplacian), K_g the scalar curvature, $B_g(x, r)$ the ball centered at x with radius r in the metric g .

We use standard notations for the spaces $C^\infty(M)$ of smooth (i.e. infinitely differentiable) functions on M and $L^p(M)$ for the (equivalence classes) of p -th power integrable functions. The notation dx stands for the Lebesgue measure and we write $\ln_+(x)$ for $\max(\ln x, 0)$.

Hyperbolic surfaces

Let M be a connected compact surface of genus $\mathbf{h} \geq 2$ (without boundary). The set of smooth metrics on M is a Fréchet manifold denoted by $\text{Met}(M)$. Let $g \in \text{Met}(M)$. The Gauss-Bonnet formula asserts that

$$\int_M K_g dv_g = 4\pi\chi(M) \quad (2.1)$$

where $\chi(M) = (2 - 2\mathbf{h})$ is the Euler characteristic.

The Fréchet space $C^\infty(M)$ acts on $\text{Met}(M)$ by conformal multiplication $(\omega, g) \mapsto e^\omega g$. The orbits of this action are called *conformal classes* and the conformal class of a metric g is denoted by $[g]$. For a metric $\hat{g} = e^\omega g$, one has the relation

$$K_{\hat{g}} = e^{-\omega}(K_g - \Delta_g \omega). \quad (2.2)$$

The uniformisation theorem says that in the conformal class $[g]$ of g , there exists a unique metric $\hat{g} = e^\omega g$ of scalar curvature $K_{\hat{g}} = -2$. Such metrics with uniformized negative curvature are called hyperbolic.

Green function

Each compact Riemann surface (M, g) has a Green function G_g defined to be the symmetric integral kernel for the linear operator $R_g : L^2(M) \rightarrow L^2(M)$ defined by $-\Delta_g R_g f = f$ for any f with zero mean ($\int f dv_g = 0$) and $R_g f = 0$ for constant functions. By integral kernel, we mean that for each $f \in L^2(M)$

$$R_g f(x) = \int_M G_g(x, x') f(x') v_g(dx').$$

Lemma 2.1 ([Gui-Rh-Va, Lemma 2.1]). *If g is a hyperbolic metric on the surface M , the Green function $G_g(x, x')$ for Δ_g has the following form near the diagonal*

$$G_g(x, x') = -\frac{1}{2\pi} \ln(d_g(x, x')) + m_g(x, x') \quad (2.3)$$

for some smooth function m_g on $M \times M$. Near each point $x_0 \in M$, there are isothermal coordinates z so that $g = |dz|^2/\text{Im}(z)^2$ and near x_0

$$G_g(z, z') = -\frac{1}{2\pi} \ln|z - z'| + F(z, z')$$

with F smooth. Finally, if \hat{g} is any metric conformal to g , (2.3) holds with \hat{g} replacing g but with $m_{\hat{g}}$ continuous.

Mabuchi and other classical actions

Let M be a connected compact surface of genus $\mathbf{h} \geq 2$ (without boundary). Let $g \in \text{Met}(M)$ and $\hat{g} \in [g]$, i.e. $\hat{g} = e^\omega g$ for some $\omega \in C^\infty(M)$. The Kähler potential $\phi := \phi_{\hat{g}, g}$ of the metric \hat{g} w.r.t to g is defined by the formula

$$\phi = -\frac{2}{V_{\hat{g}}} \int G_g(\cdot, y) v_{\hat{g}}(dy). \quad (2.4)$$

Another definition of ϕ is to define ϕ as the unique solution of the equation $e^\omega = \frac{V_{\hat{g}}}{V_g} + \frac{V_{\hat{g}}}{2} \Delta_g \phi$ satisfying $\int \phi dv_g = 0$. Using this potential, the (classical) **Mabuchi K-energy** can be defined as

$$S_M^{\text{cl}}(\hat{g}, g) = \int_M \left(2\pi(1 - \mathbf{h})\phi \Delta_g \phi + \left(\frac{8\pi(1 - \mathbf{h})}{V_g} - K_g \right) \phi + \frac{2}{V_{\hat{g}}} \omega e^\omega \right) dv_g. \quad (2.5)$$

We also introduce two classical actions that we use throughout this paper. With the above notation, the **Liouville functional**, $S_L^{\text{cl},0}(\hat{g}, g)$, and the **Aubin-Yau functional**, $S_{\text{AY}}^{\text{cl}}(\hat{g}, g)$, are respectively defined by

$$S_L^{\text{cl},0}(\hat{g}, g) := \int_M \left(|d\omega|_g^2 + 2K_g \omega \right) dv_g, \quad (2.6)$$

$$S_{\text{AY}}^{\text{cl}}(\hat{g}, g) := \int_M \left(\frac{1}{4} \phi \Delta_g \phi + \frac{\phi}{V_g} \right) dv_g. \quad (2.7)$$

(This is a slight abuse of notation as $S_{\text{AY}}^{\text{cl}}$ is rather a function of ϕ than of the metric). For later purposes, notice that both the Mabuchi and Liouville actions satisfy cocycle identities for conformal metrics g_1, g_2, g_3

$$S_L^{\text{cl},0}(g_3, g_1) = S_L^{\text{cl},0}(g_3, g_2) + S_L^{\text{cl},0}(g_2, g_1) \quad \text{and} \quad S_M^{\text{cl}}(g_3, g_1) = S_M^{\text{cl}}(g_3, g_2) + S_M^{\text{cl}}(g_2, g_1). \quad (2.8)$$

Also, recall the following change of metric formula for Green functions

$$G_{\hat{g}}(x, y) = G_g(x, y) + \frac{1}{2}(\phi(x) + \phi(y)) - S_{\text{AY}}^{\text{cl}}(\hat{g}, g). \quad (2.9)$$

Regularized determinant of Laplacian

Here we summarize results that can be found in [Os-Ph-Sa] for instance. For a Riemannian metric g on a connected oriented compact surface M , the nonnegative Laplacian $-\Delta_g$ has discrete spectrum $\text{Sp}(\Delta_g) = (\lambda_j)_{j \in \mathbb{N}_0}$ with $\lambda_0 = 0$ and $\lambda_j \rightarrow +\infty$ sorted in increasing order. We can define the regularized determinant of Δ_g by

$$\det'(-\Delta_g) = \exp(-\partial_s \zeta(s)|_{s=0}) \quad (2.10)$$

where $\zeta(s)$ is the spectral zeta function of $-\Delta_g$ defined as by meromorphic continuation of $\sum_{j=1}^{\infty} \lambda_j^{-s}$ which is well defined for $\text{Re}(s) > 1$ (using Weyl's law λ_j is of order j). This extension is defined on the full complex

plane and is holomorphic at $s = 0$. If $\widehat{g} = e^\omega g$ for some $\omega \in C^\infty(M)$, the variations of the regularized Laplacian with respect to conformal changes of metrics are determined by the so-called Polyakov formula (see [Os-Ph-Sa, eq. (1.13)])

$$\ln \left(\frac{\det'(\Delta_{\widehat{g}})}{V_{\widehat{g}}} \right) = \ln \left(\frac{\det'(\Delta_g)}{V_g} \right) - \frac{1}{48\pi} S_L^{\text{cl},0}(\widehat{g}, g). \quad (2.11)$$

where $S_L^{\text{cl},0}(\widehat{g}, g)$ is the Liouville action (2.6).

2.2 Gaussian Free Field

We refer to [Dub, Section 4.2] for references concerning this subsection. The Laplacian $-\Delta_g$ has an orthonormal basis of real valued eigenfunctions $(\varphi_j)_{j \in \mathbb{N}}$ in $L^2(M, g)$ with associated eigenvalues $\lambda_j \geq 0$ sorted in increasing order. On the Riemannian manifold (M, g) , we define the Sobolev spaces for $s \in \mathbb{R}$

$$H^s(M, g) := \left\{ f = \sum_{j \geq 0} f_j \varphi_j : (f_j)_j \in \mathbb{R}^{\mathbb{N}}, |f|_{H^s}^2 := \sum_{j \geq 1} |f_j|^2 \lambda_j^s < \infty \right\}.$$

Let us denote by $\langle \cdot, \cdot \rangle$ the duality bracket. We denote by $H_0^s(M, g)$ the subspace of $H^s(M, g)$ made up of elements f such that $\langle f, 1 \rangle = 0$.

The Gaussian Free Field (GFF) X_g on (M, g) is a random variable taking values in $\bigcap_{s>0} H_0^{-s}(M, g)$. It is characterized by its mean and covariance kernel for test functions $f, f' \in H^s(M, g)$, $s > 0$,

$$\mathbb{E}[\langle X_g, f \rangle] = 0 \quad \text{and} \quad \mathbb{E}[\langle X_g, f \rangle \langle X_g, f' \rangle] = 2\pi \iint_{M^2} f(x) G_g(x, y) f'(y) v_g(dx) v_g(dy).$$

In view of Lemma 2.1, the covariance Kernel associated with X_g , which according to the above equation is given by $2\pi G_g(x, y)$ displays a pure logarithmic divergence on the diagonal. With some slight abuse of notation, we use sometimes $\int X_g f dv_g$ for $\langle X_g, f \rangle$.

2.3 Gaussian multiplicative chaos

To make sense of quantities like $e^{\gamma X_g}$ at for some values of $\gamma \in \mathbb{R}$ we use a renormalization procedure after regularization of the field X_g . We describe the construction for g hyperbolic and we shall remark that in fact the construction works as well for any conformal metric $\widehat{g} = e^\omega g$ by using Lemma 2.1.

First, for $\varepsilon > 0$ sufficiently small, we define a regularization $X_{g,\varepsilon}$ of X_g by averaging on geodesic circles¹¹ of radius ε . Let $x \in M$ and let $\mathcal{C}(x, \varepsilon)$ be the geodesic circle of center x and radius $\varepsilon > 0$, and let $(f_{x,\varepsilon}^n)_{n \in \mathbb{N}} \in C^\infty(M)$ be a sequence which satisfies $\|f_{x,\varepsilon}^n\|_{L^1} = 1$ and can be written in the form $f_{x,\varepsilon}^n = \theta^n(d_g(x, \cdot)/\varepsilon)$ where $\theta^n(r) \in C_c^\infty(\mathbb{R})$ is non-negative and supported in $[1 - n^{-1}, 1 + n^{-1}]$. This implies in particular that, $n \rightarrow \infty$, $f_{x,\varepsilon}^n dv_g$ converges in $\mathcal{D}'(M)$, to the uniform probability measure $\mu_{x,\varepsilon}$ on $\mathcal{C}_g(x, \varepsilon)$ as (for ε sufficiently small, the geodesic circles have a 1d-manifold structure and the trace of g along this manifold gives rise to a finite measure, from which $\mu_{x,\varepsilon}$ is obtained after renormalization so as to have mass 1. It can also be defined in terms of 1-dimensional Hausdorff measure constructed with the volume form on M and restricted to this geodesic circle). The following statement allows to define the regularization $X_{g,\varepsilon}$.

Lemma 2.2 ([Gui-Rh-Va, Lemma 3.2]). *The random variable $\langle X_g, f_{x,\varepsilon}^n \rangle$ converges to a random variable as $n \rightarrow \infty$, which has a modification $X_{g,\varepsilon}(x)$ with continuous sample paths with respect to $(x, \varepsilon) \in M \times (0, \varepsilon_0)$, with covariance*

$$\mathbb{E}[X_{g,\varepsilon}(x) X_{g,\varepsilon}(x')] = 2\pi \int G_g(y, y') d\mu_{x,\varepsilon}(y) \mu_{x',\varepsilon}(dy')$$

and we have as $\varepsilon \rightarrow 0$

$$\mathbb{E}[X_{g,\varepsilon}(x)^2] = -\ln(\varepsilon) + W_g(x) + o(1) \quad (2.12)$$

¹¹It turns out that other types of regularizations by convolution could work as well but averaging along circles simplifies some computations.

where W_g is the smooth function on M given by $W_g(x) = 2\pi m_g(x, x)$ if m_g is the smooth function of Lemma 2.1.

Next from Lemma 2.2, we are able to define the Gaussian Multiplicative Chaos (GMC) first considered by Kahane [Kah] in the eighties and for which an elementary and self-contained construction is presented [Ber].

Proposition 2.3 ([Ber, Theorem 1.1]). *If $\gamma > 0$, the random measures $\mathcal{G}_{g,\varepsilon}^\gamma(dx) := \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma X_{g,\varepsilon}(x)} \nu_g(dx)$ converge in probability and weakly in the space of Radon measures towards a random measure $\mathcal{G}_g^\gamma(dx)$. The measure $\mathcal{G}_g^\gamma(dx)$ is non zero if and only if $\gamma \in (0, 2)$.*

The construction above is not particular to the hyperbolic metric and works for any field on \mathbb{R}^2 with logarithmic diverging covariance. For it to work one just need to show that the divergence of the covariance is not changed after a local isometric mapping of M to \mathbb{R}^2 . This relies only on two facts:

- (i) The covariance of X_g satisfies $2\pi G_g(x, x') = -\ln d_g(x, x') + F(x, x')$ with F continuous.
- (ii) In local isothermal coordinates one can write $g = e^{2f(z)} |dz|^2$

$$\ln d_g(z, z') = \ln |z - z'| + \mathcal{O}(1).$$

Remark 2.4. *For later purpose we will need to make the following observation related to changes in the choice of the metric used to regularized the GFF. If $\hat{g} = e^\omega g$, consider the GFF X_g with vanishing spatial average in the g metric regularized with circle averages in the \hat{g} metric*

$$\hat{X}_{g,\varepsilon}(x) := \lim_{n \rightarrow \infty} \langle X_g, \hat{f}_{x,\varepsilon}^n \rangle_{\hat{g}} \quad (2.13)$$

for each $x \in M$ where $\hat{f}_{x,\varepsilon}^n(\cdot) := \theta^n(d_{\hat{g}}(x, \cdot)/\varepsilon)$ with θ^n like above, so that $\hat{f}_{x,\varepsilon}^n d\nu_{\hat{g}}$ converge as $n \rightarrow \infty$ to the uniform probability measure $\hat{\mu}_{x,\varepsilon}$ on the geodesic circle $\mathcal{C}_{\hat{g}}(x, \varepsilon)$ of center x and radius ε with respect to \hat{g} . Using an isothermal chart of coordinate for which x is mapped to some point z in the upper half plane and the metric is $g = |dz|^2 / \text{Im}(z)^2$, the circle $\mathcal{C}_{\hat{g}}(x, \varepsilon)$ is parametrized by

$$\varepsilon e^{-\frac{1}{2}\omega(z) + \varepsilon h_\varepsilon(\alpha)} e^{i\alpha}, \alpha \in [0, 2\pi]$$

for some continuous function $h_\varepsilon(\alpha)$ uniformly bounded in ε . Then one has

$$\mathbb{E}(\hat{X}_{g,\varepsilon}(x) \hat{X}_{g,\varepsilon}(x')) = 2\pi \int G_g(y, y') \hat{\mu}_{x,\varepsilon}(dy) \hat{\mu}_{x',\varepsilon}(dy')$$

and by the arguments in the proof of Lemma 2.2, we have as $\varepsilon \rightarrow 0$

$$\mathbb{E}(\hat{X}_{g,\varepsilon}(x)^2) = -\ln(\varepsilon) + W_g(x) + \frac{1}{2}\omega(x) + o(1). \quad (2.14)$$

Then by the same arguments as for Proposition 2.3, the random measure

$$\hat{\mathcal{G}}_{g,\varepsilon}^\gamma(dx) := \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma \hat{X}_{g,\varepsilon}(x)} \nu_{\hat{g}}(dx) \quad (2.15)$$

converges weakly as $\varepsilon \rightarrow 0$ to some measure $\hat{\mathcal{G}}_g^\gamma$ which satisfies

$$\hat{\mathcal{G}}_g^\gamma(dx) = e^{\frac{\gamma^2}{2}\omega(x)} \mathcal{G}_g^\gamma(dx). \quad (2.16)$$

2.4 Liouville CFT

Fix $\gamma \in (0, 2)$, $\mu > 0$ and set $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$. For $F : H^{-s}(M) \rightarrow \mathbb{R}$ (with $s > 0$) a bounded continuous functional, set

$$\begin{aligned} \langle F \rangle_{L,g} &:= (\det'(\Delta_g) / \text{Vol}_g(M))^{-1/2} \\ &\times \int_{\mathbb{R}} \mathbb{E} \left[F(c + X_g) \exp \left(-\frac{Q}{4\pi} \int_M K_g(c + X_g) dv_g - \mu e^{\gamma c} \mathcal{G}_g^\gamma(M) \right) \right] dc. \end{aligned} \quad (2.17)$$

The expression above defines LCFT, namely it gives a mathematical interpretation to the formal functional integral

$$\int F(\varphi) e^{-\mathcal{S}_L(\varphi, g)} \mathcal{D}\varphi$$

where $\mathcal{S}_L(\varphi, g)$ is the quantum Liouville action appearing in (1.5). Properties which holds true almost surely with respect to the measure $\langle \cdot \rangle_{L,g}$ will be said to be true L-almost surely.

For the expression in (2.17) to be well defined for every bounded continuous F and that $\langle \cdot \rangle_{L,g}$ defines indeed a probability measure, one must check that the partition function, that is, the total mass of this measure which is obtained by substituting F by 1 in the above expression, is finite. This is indeed the case and recall the following result of [Gui-Rh-Va] on the Weyl anomaly of the theory:

Theorem 2.5 ([Gui-Rh-Va, Theorem 1.1]). *Let $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$ with $\gamma \in (0, 2)$ and g be a smooth metric on M . For each bounded continuous functional $F : H^{-s}(M) \rightarrow \mathbb{R}$ (with $s > 0$) and each $\omega \in C^\infty(M)$, set $\hat{g} = e^\omega g$. Then $\langle F \rangle_{L,\hat{g}}$ is finite and satisfies the following Weyl anomaly formula:*

$$\langle F \rangle_{L,\hat{g}} = \langle F(\cdot - \frac{Q}{2}\omega) \rangle_{L,g} \exp \left(\frac{1+6Q^2}{96\pi} S_L^{\text{cl},0}(\hat{g}, g) \right).$$

where $S_L^{\text{cl},0}$ is the classical Liouville functional (2.6). Let g be any metric on M and $\psi : M \rightarrow M$ be an orientation preserving diffeomorphism, then we have for each bounded measurable $F : H^{-s}(M) \rightarrow \mathbb{R}$ with $s > 0$

$$\langle F \rangle_{L,\psi^*g} = \langle F(\cdot \circ \psi) \rangle_{L,g}.$$

3 Defining the path integral for Mabuchi+Liouville actions

In this section, we introduce the main results of our paper. In particular we first explain the construction of the path integral (1.11) and then state its main properties.

Recall that Liouville CFT defines formally a random metric $e^{\gamma(c+X_g)}g$ with the law of the random field $c + X_g$, called Liouville field, ruled by the path integral (2.17). The Liouville field being a distribution this metric tensor is not well defined mathematically. Yet one can make sense of all the corresponding terms appearing in the quantum Mabuchi action associated with $e^{\gamma(c+X_g)}g$ because the quantum Mabuchi action only involves the volume form or log-conformal factor, which are well defined under the Liouville path integral.

Recall the definition (2.4) of the Kähler potential ϕ of some metric \hat{g} conformal to g , i.e. $\hat{g} = e^\omega g$. This expression can be extended to the case when \hat{g} is the Liouville metric, hence leading to the following expression of the Kähler potential of the Liouville metric with respect to a background measure g

$$\Phi(x) := -\frac{2}{\mathcal{G}_g^\gamma(M)} \int G_g(x, y) \mathcal{G}_g^\gamma(dy). \quad (3.1)$$

Proposition 3.1. *For $\gamma \in (0, 2)$, the Kähler potential of the Liouville metric is well defined for L-almost all realization of X_g . It is continuous on M and satisfies L-almost surely*

$$\int_M \Phi \Delta_g \Phi dv_g \leq 0.$$

Proof. Multifractal analysis (see [Rh-Va2, Th 2.14] for instance) entails that for some constant C and for all $q \in (0, \frac{4}{\gamma^2})$

$$\forall x \in M, \forall r \in (0, 1), \quad \mathbb{E}[\mathcal{G}_g^\gamma(B_g(x, r))^q] \leq Cr^{\xi(q)}$$

with

$$\xi(q) = (2 + \frac{\gamma^2}{2})q - \frac{\gamma^2 q^2}{2}.$$

Borel-Cantelli's Lemma then entails uniform Hölder continuity of the measure \mathcal{G}_g^γ , namely that there exists $\alpha > 0$ and some random variable C such that

$$\forall x \in M, \forall r \in (0, 1), \quad \mathcal{G}_g^\gamma(B_g(x, r)) \leq Cr^\alpha.$$

As the singularity of the Green function on the diagonal is logarithmic (Lemma 2.1), it is then straightforward to check that Φ is well defined (i.e. \mathcal{G}_g^γ integrates a log-singularity) and is furthermore continuous on M . The last statement follows from the positive definiteness of the Green function. \square

Our next step is to construct the term corresponding to $(\gamma\varphi)e^{\gamma\varphi}$ in (1.10) for the Liouville metric and to check that it satisfies some adequate integrability property. The latter condition requires $\gamma \in (0, 1)$ instead of $\gamma < 2$ because our proof of existence of exponential moments (Theorem 4.4 below) includes this restriction. The obstruction, however, does not seem to be more than technical and it seems very plausible that it could be overcome by refining our technique to treat the case $\gamma \in [1, 2)$.

For $\varepsilon > 0$, we consider the random (signed) measures

$$\mathcal{D}_{g,\varepsilon}^\gamma(dx) := \varepsilon^{\frac{\gamma^2}{2}} (\gamma X_{g,\varepsilon}(x) + \gamma^2 \ln \varepsilon) e^{\gamma X_{g,\varepsilon}(x)} \mathbf{v}_g(dx).$$

Our result concerns the limit of this object when ε tends to zero.

Theorem 3.2. (i) *If $\gamma \in (0, \sqrt{2})$ and f is a continuous function f on M , the family of random variables $(\int_M f(x) \mathcal{D}_{g,\varepsilon}^\gamma(dx))_\varepsilon$ converges in quadratic mean towards a limiting random variables denoted $\int_M f(x) \mathcal{D}_g^\gamma(dx)$.*

(ii) *Setting $\mathcal{D}_g^\gamma(M) := \int_M \mathcal{D}_g^\gamma(dx)$. Then, for $\gamma < 1$ for any $\alpha \geq 0$,*

$$\mathbb{E} \left[\exp \left(-\alpha \frac{\mathcal{D}_g^\gamma(M)}{\mathcal{G}_g^\gamma(M)} \right) \right] < +\infty. \quad (3.2)$$

The first part of the theorem results from an elementary L^2 -computation (see below). The proof of the second statement is the main technical part of the paper and runs from Section 4 to 6. It relies on two distinct concentration estimates for the quantities $\mathcal{D}_g^\gamma(M)$ and $\mathcal{G}_g^\gamma(M)$ which are of independent interest.

Proof of Theorem 3.2 (i). Considering two values $\varepsilon, \varepsilon' > 0$, setting $Y_\varepsilon(x) := (X_{g,\varepsilon}(x) + \gamma \ln \varepsilon) e^{\gamma X_{g,\varepsilon}(x)}$ we have

$$\begin{aligned} & \gamma^{-2} \mathbb{E} \left[\left(\int_M f(x) \mathcal{D}_{g,\varepsilon}^\gamma(dx) - \int_M f(x) \mathcal{D}_{g,\varepsilon'}^\gamma(dx) \right)^2 \right] \\ &= (\varepsilon\varepsilon')^{\frac{\gamma^2}{2}} \int_{M \times M} \mathbb{E} [(Y_\varepsilon(x) - Y_{\varepsilon'}(x))(Y_\varepsilon(y) - Y_{\varepsilon'}(y))] f(x)f(y) \mathbf{v}_g(dx) \otimes \mathbf{v}_g(dy). \end{aligned} \quad (3.3)$$

Using the fact that the random variables involved are Gaussian we obtain that

$$\begin{aligned} \mathbb{E}[Y_\varepsilon(x)Y_{\varepsilon'}(y)] &= e^{\frac{\gamma^2}{2}\mathbb{E}[(X_{g,\varepsilon}(x)+X_{g,\varepsilon'}(y))^2]} [(\gamma\mathbb{E}[(X_{g,\varepsilon}(x)+X_{g,\varepsilon'}(y))X_{g,\varepsilon}(x)] + \gamma \ln \varepsilon) \\ &\quad \times (\mathbb{E}[(X_{g,\varepsilon}(x)+X_{g,\varepsilon'}(y))X_{g,\varepsilon'}(y)] + \gamma \ln \varepsilon) + \mathbb{E}[X_{g,\varepsilon}(x)X_{g,\varepsilon'}(x)]]. \end{aligned} \quad (3.4)$$

In particular we have for $x \neq y$

$$\begin{aligned} & \lim_{\varepsilon, \varepsilon' \rightarrow 0} (\varepsilon \varepsilon')^{\frac{\gamma^2}{2}} \mathbb{E}[Y_\varepsilon(x) Y_{\varepsilon'}(y)] \\ &= e^{\frac{\gamma^2}{2}(W_g(x)+W_g(y))+\frac{\gamma^2}{2}G_g(x,y)} [(W_g(x) + G_g(x, y))(W_g(y) + G_g(x, y)) + G_g(x, y)], \end{aligned} \quad (3.5)$$

and given $\delta > 0$ there exists a constant such that

$$(\varepsilon \varepsilon')^{\frac{\gamma^2}{2}} \mathbb{E}[Y_\varepsilon(x) Y_{\varepsilon'}(y)] \leq C_\delta d_g(x, y)^{-(\gamma^2 + \delta)}. \quad (3.6)$$

Using dominated convergence Theorem for the integral in (3.4) (choosing δ such that $\gamma^2 + \delta < 2$), the limit of the fourth term in the product cancel out and we conclude that $\int_M f(x) \mathcal{D}_{g, \varepsilon}^\gamma(dx)$ is a Cauchy sequence in L^2 (for ε tending to zero). \square

Remark 3.3. *In the same spirit as Remark (2.4) we need to study the role of the metric in regularizing the GFF involved in the construction of the random variable $\int_M f(x) \mathcal{D}_g^\gamma(dx)$. So we consider another metric $\widehat{g} = e^\omega g$ and consider the random variable*

$$\widehat{\mathcal{D}}_{g, \varepsilon}^\gamma(dx) := \varepsilon^{\frac{\gamma^2}{2}} (\gamma \widehat{X}_{g, \varepsilon}(x) + \gamma^2 \ln \varepsilon) e^{\gamma \widehat{X}_{g, \varepsilon}(x)} v_g(dx) \quad (3.7)$$

as well as the limit $\int f(x) \widehat{\mathcal{D}}_g^\gamma(dx) = \lim_{\varepsilon \rightarrow 0} \int_M f(x) \widehat{\mathcal{D}}_{g, \varepsilon}^\gamma(dx)$ for continuous functions $f \in C^0(M)$. Similarly to (2.16) we obtain the relation

$$\int f(x) \widehat{\mathcal{D}}_g^\gamma(dx) = \int_M f(x) e^{\frac{\gamma^2}{2} \omega(x)} \mathcal{D}_g^\gamma(dx) + \frac{\gamma^2}{2} \int_M f(x) \omega(x) e^{\frac{\gamma^2}{2} \omega(x)} \mathcal{G}_g^\gamma(dx). \quad (3.8)$$

At this stage, we can provide a mathematical interpretation for the Mabuchi K -energy given by Equation (1.10) when (formally) $\gamma\varphi = \gamma X_g$, which is well defined L -almost surely. We define the random variable

$$\begin{aligned} \mathcal{S}_M(c + X_g, g) &:= -8\pi(1 - \mathbf{h}) \frac{1}{\mathcal{G}_g^\gamma(M)^2} \iint_{M^2} G_g(x, x') \mathcal{G}_g^\gamma(dx) \mathcal{G}_g^\gamma(dx') \\ &\quad - \frac{2}{\mathcal{G}_g^\gamma(M)} \iint_{M^2} \left(\frac{8\pi(1 - \mathbf{h})}{V_g} - K_g(x) \right) G_g(x, x') v_g(dx) \mathcal{G}_g^\gamma(dx') \\ &\quad + \frac{2}{1 - \frac{\gamma^2}{4}} \frac{1}{\mathcal{G}_g^\gamma(M)} \mathcal{D}_g^\gamma(M) + \frac{2}{1 - \frac{\gamma^2}{4}} \gamma c \end{aligned} \quad (3.9)$$

We can check that each of the three terms above correspond to one term in (1.10). For the first one, notice that for a smooth metric $\widehat{g} = e^\omega g$ and ϕ its Kähler potential given by (2.4), we have the relation

$$\int_M \phi \Delta_g \phi dv_g = -\frac{4}{V_g^2} \iint_{M^2} G_g(x, x') v_{\widehat{g}}(dx) \otimes v_{\widehat{g}}(dx').$$

Replacing $v_{\widehat{g}}$ by the volume form \mathcal{G}_g^γ gives the first term in (3.9). The second and third terms in (3.9) then correspond respectively to the second and third term in (1.10) in an obvious way given the expression (3.1).

We are now ready to define the quantum field theory associated with the Liouville action and the Mabuchi K -energy similarly to what is done in Theorem 2.5.

Theorem 3.4. (Definition of Quantum Mabuchi-Liouville Theory) *We fix $\gamma \in (0, 1)$ and $\beta \geq 0$. Let g be a smooth metric on M . We define the Quantum Mabuchi+Liouville theory as the following functional integral: for $F : H^{-s}(M) \rightarrow \mathbb{R}$ (with $s > 0$) a bounded continuous functional, we set*

$$\begin{aligned} \langle F \rangle_{\text{ML}, g} &:= \left(\frac{\det'(\Delta_g)}{\text{Vol}_g(M)} \right)^{-1/2} \\ &\quad \times \int_{\mathbb{R}} \mathbb{E} \left[F(c + X_g) \exp \left(-\beta \mathcal{S}_M(c + X_g, g) - \frac{Q}{4\pi} \int_M K_g(c + X_g) dv_g - \mu e^{\gamma c} \mathcal{G}_g^\gamma(M) \right) \right] dc, \end{aligned} \quad (3.10)$$

where the random variable $\mathcal{S}_M(c + X_g, g)$ is defined in (3.9). Furthermore

1) **Finite mass:** The total mass of this measure, i.e. $\langle 1 \rangle_{\text{ML},g}$, is finite for all $\beta \in (0, \frac{\mathbf{h}-1}{2}(\frac{4}{\gamma^2} - \frac{\gamma^2}{4}))$.

2) **Weyl anomaly:** Let $\widehat{g} = e^\omega g$ be a metric conformal to g and denote by ϕ its Kähler potential w.r.t. g . Then the ML-path integral obeys the following Weyl anomaly formula

$$\langle F \rangle_{\text{ML},\widehat{g}} = \langle F(\cdot - \frac{Q}{2}\omega) \rangle_{\text{ML},g} \exp\left(\frac{1+6Q^2}{96\pi} S_L^{\text{cl},0}(\widehat{g}, g) + \beta S_M^{\text{cl}}(\widehat{g}, g)\right) \quad (3.11)$$

where $S_L^{\text{cl},0}$ is the classical Liouville functional (2.6) and S_M^{cl} the classical Mabuchi action 2.5.

Remark 3.5. The expectation in Equation (3.10) is well defined when F is positive (with ∞ being a possible value for $\langle F \rangle_{\text{ML},\widehat{g}}$). For general F , $\langle F \rangle_{\text{ML},\widehat{g}}$ is of course properly defined only if $\langle |F| \rangle_{\text{ML},\widehat{g}} < \infty$. Part 1) of the statement implies that the integral is well defined for all bounded F when $\beta \in (0, \frac{\mathbf{h}-1}{2}(\frac{4}{\gamma^2} - \frac{\gamma^2}{4}))$.

3.1 Conditioning on area/string susceptibility

Let us explain how we can condition the path integral (3.10) on having fixed volume. As a consequence, we are able to give the ‘‘string susceptibility’’ (scaling with respect to area) of our quantum field theory. We fix a hyperbolic background metric g on M , hence with uniformized scalar curvature. The path integral (3.10) defines a random geometry with formal metric tensor $e^{\gamma(c+X_g)}g$. The volume form of this metric tensor is thus the random measure

$$V_\gamma(dx) := e^{\gamma c} \mathcal{G}_\gamma(dx)$$

with Kähler potential (3.1). We are going to compute the law of the couple (V_γ, Φ) conditionally on $V_\gamma(M) = y$ under the probability law defined by the path integral (3.10).

For that, define the random variable

$$P_\gamma := -8\pi(1 - \mathbf{h}) \frac{1}{\mathcal{G}_g^\gamma(M)^2} \iint_{M^2} G_g(x, x') d\mathcal{G}_g^\gamma(dx) d\mathcal{G}_g^\gamma(dx') + \frac{2}{1 - \frac{\gamma^2}{4}} \frac{\mathcal{D}_g^\gamma(M)}{\mathcal{G}_g^\gamma(M)}. \quad (3.12)$$

and the exponent

$$s := \frac{2Q}{\gamma}(\mathbf{h} - 1) - \frac{2\beta}{1 - \frac{\gamma^2}{4}}. \quad (3.13)$$

Let $\mathcal{R}(M)$ be the space of Radon measures on M and $C(M)$ the space of continuous function. We claim

Proposition 3.6. (Fixed volume Quantum Mabuchi-Liouville Theory) We fix $\gamma \in (0, 1)$ and $\beta \geq 0$. Let g be a hyperbolic metric on M . Let $F : \mathbb{R} \times \mathcal{R}(M) \times C(M) \rightarrow \mathbb{R}$ a bounded continuous functional, we have

$$\begin{aligned} & \langle F(V_\gamma(M), V_\gamma(dx), \Phi) \rangle_{\text{ML},g} \\ & := \gamma^{-1} \left(\frac{\det'(\Delta_g)}{\text{Vol}_g(M)} \right)^{-1/2} \int_{\mathbb{R}} \mathbb{E} \left[F(y, y \frac{\mathcal{G}_\gamma(dx)}{\mathcal{G}_\gamma(M)}, \Phi) \exp(-\beta P_\gamma) \mathcal{G}_\gamma(M)^{-s} \right] y^{s-1} e^{-\mu y} dy. \end{aligned} \quad (3.14)$$

Proof. The relation follows from the simple change of variables $y = e^{\gamma c} \mathcal{G}_\gamma(M)$ in the c -integral in (3.10). \square

In particular if we define a probability law by dividing the path integral (3.10) by its total mass, this shows that the random volume of $V_\gamma(M)$ follows a Gamma law $\Gamma(s, \mu)$. The exponent $s + 2$ thus appears as the **string susceptibility** of our quantum field theory.

Remark 3.7. The loop expansion is the asymptotic expansion as $\kappa^2 \rightarrow \infty$ (i.e. $\gamma \rightarrow 0$) of the string susceptibility in terms of the parameter $\kappa^2 = \frac{1+6Q^2}{3}$ or equivalently $\gamma = \left(\frac{3\kappa^2-1}{6}\right)^{1/2} - \left(\frac{3\kappa^2-25}{6}\right)^{1/2}$. Taking our expression (3.13) we find at two loops

$$s + 2 = \frac{\kappa^2}{2}(\mathbf{h} - 1) + \frac{19-7\mathbf{h}}{6} - 2\beta + \frac{2}{\kappa^2}(\mathbf{h} - 1) - \frac{4\beta}{\kappa^2} + o(\kappa^{-2}).$$

This is exactly the expression found in [Bi-Fe-Kl, Equation (1.5)].

3.2 Proof of Theorem 3.4 using Theorem 3.2

We start by proving the result when $g = g_0$ is uniformized, i.e. has constant scalar curvature. The reason for that is that the second term in the right-hand side of (3.9) vanishes because of the Gauss-Bonnet theorem (2.1), which drastically simplifies our task. Indeed, in that case the total mass is then given by

$$\langle 1 \rangle_{\text{ML},g} = \int_{\mathbb{R}} \left(\frac{\det'(\Delta_g)}{\text{Vol}_g(M)} \right)^{-1/2} \mathbb{E} \left[\exp \left(-\beta P - \mu e^{\gamma c} \mathcal{G}_g^\gamma(M) \right) \right] e^{-\frac{2\beta\gamma}{1-\frac{\gamma^2}{4}}c - 2Q(1-\mathbf{h})c} dc$$

where P the random variable defined by

$$P := 8\pi(\mathbf{h} - 1) \frac{1}{\mathcal{G}_g^\gamma(M)^2} \iint_{M^2} G_g(x, x') d\mathcal{G}_g^\gamma(dx) \mathcal{G}_g^\gamma(dx') + \frac{2}{1 - \frac{\gamma^2}{4}} \frac{\mathcal{D}_g^\gamma(M)}{\mathcal{G}_g^\gamma(M)}.$$

The first term in P is positive (due to positive definiteness of G_g) and thus finiteness of the expectation inside the integral is ensured by the finiteness of negative exponential moments of $(\mathcal{D}_g^\gamma/\mathcal{G}_g^\gamma)(M)$ (Theorem 3.2).

Now to check that the integral c -integral converges when $s := -\frac{2\beta}{1-\frac{\gamma^2}{4}} - \frac{2Q(1-\mathbf{h})}{\gamma} > 0$ (this corresponds to our topological restriction on β) we observe that a simple change of variables in the c -integral $c' = \mu \mathcal{G}_g^\gamma(M) e^{\gamma c}$ yields

$$\langle 1 \rangle_{\text{ML},g} = \gamma^{-1} \mu^{-s} \left(\frac{\det'(\Delta_g)}{\text{Vol}_g(M)} \right)^{-1/2} \mathbb{E} \left[\exp \left(-\beta P \right) \mathcal{G}_g^\gamma(M)^{-s} \right] \Gamma(s)$$

where Γ is the standard Gamma function.

For a generic metric \hat{g} , we use the fact that it is conformal to a uniformized metric g , i.e. $\hat{g} = e^\omega g$ for some $\omega \in C^\infty(M)$. Then, for bounded nonnegative functional F , we have

$$\begin{aligned} \langle F \rangle_{\text{ML},\hat{g}} &= \left(\frac{\det'(\Delta_{\hat{g}})}{\text{Vol}_{\hat{g}}(M)} \right)^{-1/2} \\ &\times \int_{\mathbb{R}} \mathbb{E} \left[F(c + X_{\hat{g}}) e^{-\beta \mathcal{S}_M(c + X_{\hat{g}}, \hat{g})} \exp \left(-\frac{Q}{4\pi} \int_M K_{\hat{g}}(c + X_{\hat{g}}) dv_{\hat{g}} - \mu e^{\gamma c} \mathcal{G}_{\hat{g}}^\gamma(M) \right) \right] dc. \end{aligned}$$

We use the conformal anomaly of the Liouville measure (Theorem 2.5) to obtain

$$\begin{aligned} \langle F \rangle_{\text{ML},\hat{g}} &= e^{-\frac{1+6Q^2}{96\pi} S_L^0(\hat{g},g)} \left(\frac{\det'(\Delta_g)}{\text{Vol}_g(M)} \right)^{-1/2} \\ &\times \int_{\mathbb{R}} \mathbb{E} \left[F(c + X_g - \frac{Q}{2}) e^{-\beta \hat{\mathcal{S}}_M} \exp \left(-\frac{Q}{4\pi} \int_M K_g(c + X_g) dv_g - \mu e^{\gamma c} \mathcal{G}_g^\gamma(M) \right) \right] dc \end{aligned} \quad (3.15)$$

where the random variable $\hat{\mathcal{S}}_M$ is defined as (recall the definitions (2.15) and (3.7))

$$\begin{aligned} \hat{\mathcal{S}}_M &= \left(-8\pi(1-\mathbf{h}) \frac{1}{\left(\int_M e^{-\frac{\gamma Q}{2}\omega(x)} \hat{\mathcal{G}}_g^\gamma(dx) \right)^2} \iint_{M^2} G_{\hat{g}}(x, x') e^{-\frac{\gamma Q}{2}\omega(x) - \frac{\gamma Q}{2}\omega(x')} \hat{\mathcal{G}}_g^\gamma(dx) \hat{\mathcal{G}}_g^\gamma(dx') \right. \\ &\quad - \frac{2}{\int_M e^{-\frac{\gamma Q}{2}\omega(x)} \hat{\mathcal{G}}_g^\gamma(dx)} \iint_{M^2} \left(\frac{8\pi(1-\mathbf{h})}{V_{\hat{g}}} - K_{\hat{g}}(x) \right) G_{\hat{g}}(x, x') e^{-\frac{\gamma Q}{2}\omega(x')} v_{\hat{g}}(dx) \hat{\mathcal{G}}_g^\gamma(dx') \\ &\quad \left. + \frac{2}{1 - \frac{\gamma^2}{4}} \frac{1}{\int_M e^{-\frac{\gamma Q}{2}\omega(x)} \hat{\mathcal{G}}_g^\gamma(dx)} \int_M e^{-\frac{\gamma Q}{2}\omega(x)} (\hat{\mathcal{D}}_g^\gamma(dx) - \frac{\gamma Q}{2}\omega(x) \hat{\mathcal{G}}_g^\gamma(dx)) + \frac{2}{1 - \frac{\gamma^2}{4}} \gamma c \right) \\ &=: A_1 + A_2 + A_3 + \frac{2}{1 - \frac{\gamma^2}{4}} \gamma c. \end{aligned}$$

Using the relation (2.16) the first in the above right-hand side A_1 becomes

$$A_1 = -8\pi(1 - \mathbf{h}) \frac{1}{\mathcal{G}_g^\gamma(M)^2} \iint_{M^2} G_{\widehat{g}}(x, x') \mathcal{G}_g^\gamma(dx) \mathcal{G}_g^\gamma(dx'),$$

and the relation (2.9) between Green functions leads to the expression

$$\begin{aligned} A_1 &= -8\pi(1 - \mathbf{h}) \frac{1}{\mathcal{G}_g^\gamma(M)^2} \iint_{M^2} \left(G_g(x, x') + \frac{1}{2}(\phi(x) + \phi(x')) - S_{\text{AY}}^{\text{cl}}(\widehat{g}, g) \right) \mathcal{G}_g^\gamma(dx) \mathcal{G}_g^\gamma(dx') \\ &= -8\pi(1 - \mathbf{h}) \frac{1}{\mathcal{G}_g^\gamma(M)^2} \iint_{M^2} G_g(x, x') \mathcal{G}_g^\gamma(dx) \mathcal{G}_g^\gamma(dx') + 8\pi(1 - \mathbf{h}) S_{\text{AY}}^{\text{cl}}(\widehat{g}, g) \\ &\quad - 8\pi(1 - \mathbf{h}) \frac{1}{\mathcal{G}_g^\gamma(M)} \int_M \phi(x) \mathcal{G}_g^\gamma(dx). \end{aligned} \quad (3.16)$$

Now we focus on the second term. Again, (2.16) produces the first simplification

$$A_2 = -\frac{2}{\mathcal{G}_g^\gamma(M)} \iint_{M^2} \left(\frac{8\pi(1 - \mathbf{h})}{V_{\widehat{g}}} - K_{\widehat{g}}(x) \right) G_{\widehat{g}}(x, x') v_{\widehat{g}}(dx) \mathcal{G}_g^\gamma(dx').$$

Recalling that the $v_{\widehat{g}}$ intergral of $G_{\widehat{g}}$ in either variable vanishes, the contribution of the term $\frac{8\pi(1 - \mathbf{h})}{V_{\widehat{g}}}$ reduces to 0. Then, using the Green relation (2.9) again and the curvature relation (2.2), we get

$$\begin{aligned} A_2 &= \frac{2}{\mathcal{G}_g^\gamma(M)} \iint_{M^2} K_{\widehat{g}}(x) G_{\widehat{g}}(x, x') v_{\widehat{g}}(dx) \mathcal{G}_g^\gamma(dx') \\ &= \frac{2}{\mathcal{G}_g^\gamma(M)} \iint_{M^2} (K_g(x) - \Delta_g \omega(x)) \left(G_g(x, x') + \frac{1}{2}(\phi(x) + \phi(x')) - S_{\text{AY}}^{\text{cl}}(\widehat{g}, g) \right) v_g(dx) \mathcal{G}_g^\gamma(dx'). \end{aligned}$$

Then we expand this expression and compute each term, using the Gauss-Bonnet formula (2.1) when necessary,

$$\begin{aligned} A_2 &= -\frac{2}{\mathcal{G}_g^\gamma(M)} \iint_{M^2} \left(\frac{8\pi(1 - \mathbf{h})}{V_g} - K_g(x) \right) G_g(x, x') v_g(dx) \mathcal{G}_g^\gamma(dx') + \int_M K_g \phi dv_g + 8\pi(1 - \mathbf{h}) \frac{1}{\mathcal{G}_g^\gamma(M)} \int_M \phi(x) \mathcal{G}_g^\gamma(dx) \\ &\quad - 16\pi(1 - \mathbf{h}) S_{\text{AY}}^{\text{cl}}(\widehat{g}, g) + \frac{2}{\mathcal{G}_g^\gamma(M)} \int_M \omega(x) \mathcal{G}_g^\gamma(dx) - \frac{2}{V_{\widehat{g}}} \int_M \omega(x) e^{\omega(x)} v_g(dx). \end{aligned}$$

Finally, the third term can be treated with (2.16) and (3.8)

$$A_3 = \frac{2}{1 - \frac{\gamma^2}{4}} \frac{\mathcal{D}_g^\gamma(M)}{\mathcal{G}_g^\gamma(M)} + \frac{\gamma^2 - \gamma Q}{(1 - \frac{\gamma^2}{4}) \mathcal{G}_g^\gamma(M)} \int_M \omega(x) \mathcal{G}_g^\gamma(dx). \quad (3.17)$$

Combining, we get

$$A_1 + A_2 + A_3 + \frac{2}{1 - \frac{\gamma^2}{4}} \gamma c = \mathcal{S}_M(c + X_g, g) - \mathcal{S}_M(\widehat{g}, g)$$

Hence (3.15) becomes

$$\langle F \rangle_{\text{ML}, \widehat{g}} = e^{\frac{1+6Q^2}{96\pi} S_L^{\text{cl},0}(\widehat{g}, g) + \beta S_M^{\text{cl}}(\widehat{g}, g)} \langle F(\cdot - \frac{Q}{2}\omega) \rangle_{\text{ML}, g}$$

when g is uniformized. Taking $F = 1$ this relation shows that the conditions for finiteness of the total mass does not depend on the choice of the background metric \widehat{g} . The general case (g not necessarily uniformized) then results from the above relation and cocycle identities (2.8) for (classical) Liouville and Mabuchi functionals. \square

4 Proof of Theorem 3.2 (ii): Negative exponential moments for GFF based \mathcal{D}/\mathcal{G}

In this section we first state general results for GMC/DGMC based on a field defined in the plane and satisfying some assumptions including the existence of a continuous scale decomposition. Our second task is to show that GFF on a manifold can be remapped to the plane using local charts in order to fit those assumptions.

4.1 Setup and result

In this section we consider a distributional Gaussian field X with covariance function K defined over an open neighborhood of the closure of a bounded open set $D \subset \mathbb{R}^2$, and we let μ be a finite mass Borel measure on D .

Assumption 4.1. (Smooth white noise decomposition)

1. The covariance kernel K can be written in the form

$$K(x, y) := \int_0^\infty Q_u(x, y) du, \quad (4.1)$$

where the above integral is convergent for all $x \neq y$ and Q_u is a bounded symmetric positive definite kernel for any fixed u .

2. The function $(x, y) \mapsto K(x, y) - \ln \frac{1}{|y-x|}$ can be extended on the diagonal to a bounded continuous function on $D \times D$. Setting $K_t := \int_0^t Q_u du$. There exists a positive constant C such that

$$\left| K_t(x, y) - (t \wedge \ln_+ \frac{1}{|x-y|}) \right| \leq C. \quad (4.2)$$

3. We have $\lim_{x \rightarrow \infty} Q_u(x, x) = 1$ with uniform convergence in $x \in D$.

4. For all $0 < \alpha < 2$, $\iint_{D^2} \int_0^\infty e^{\alpha u} |Q_u(x, y)| \mu(dx) \mu(dy) du < \infty$.

5. We have $\mu(dx) = h(x)dx$ for some positive bounded continuous function h , and there exists $v > 1$

$$\int_t^\infty |Q_u(x, y)| du \leq C e^{-e^{vt}|x-y|^v}. \quad (4.3)$$

For our application we consider $\mu(dx) = h(x)dx$ where h is a smooth positive function. Note that the assumption implies in particular that K has logarithmic divergence on the diagonal ($K(x, y) = -\ln|x-y| + O(1)$) so that the GMC and its derivative (cf. Proposition 2.3 and Theorem 3.2 (i)), can properly be defined.

Letting X_ε denote the circle average of the field X defined by $X_\varepsilon(x) := \int X(x-y) \widehat{\mu}_\varepsilon(dy)$ where $\widehat{\mu}_\varepsilon$ is the uniform measure on the circle of radius ε (the convolution with a singular measure can be obtained by functional approximation like in Lemma 2.2). Given w a positive continuous function on \bar{D} (in particular w is bounded away from 0 and ∞), we define

$$\begin{aligned} \mathcal{G}_\infty^w &:= \lim_{\varepsilon \rightarrow 0} \int_D e^{\gamma X_\varepsilon(x) - \frac{\gamma^2}{2} (\mathbb{E}[X_\varepsilon(x)^2] - w(x))} \mu(dx), \\ \mathcal{D}_\infty^w &:= \lim_{\varepsilon \rightarrow 0} \int_D [\gamma X_\varepsilon(x) - \gamma^2 (\mathbb{E}[X_\varepsilon(x)^2] - w(x))] e^{\gamma X_\varepsilon(x) - \frac{\gamma^2}{2} (\mathbb{E}[X_\varepsilon(x)^2] - w(x))} \mu(dx). \end{aligned} \quad (4.4)$$

Remark 4.2. Note that our convention here for renormalization is a bit different of the convention adopted in Section 2.3 since $\ln \varepsilon$ has been replaced by $\mathbb{E}[X_\varepsilon(x)^2]$. Yet, according to item 2 of Assumption 4.1, which implies $\lim_{\varepsilon \rightarrow 0} \mathbb{E}[X_\varepsilon(x)^2] + \ln \varepsilon$ is a continuous function, the effect of this change on \mathcal{G} is to multiply the integration measure μ by a multiplicative factor. Tuning w accordingly cancels this difference (same occurs for \mathcal{D}).

We are going to prove that Theorem 3.2 (ii) is a consequence of the following general statement.

Proposition 4.3. If Assumption 4.1 holds and $\gamma < 1$ then for any $\alpha > 0$ and any w we have

$$\mathbb{E}\left[e^{-\alpha \frac{\mathcal{D}_\infty^w}{\mathcal{G}_\infty^w}}\right] < \infty. \quad (4.5)$$

The proposition above is proved by combining two new results concerning Gaussian Multiplicative Chaos whose proof are given in Section 5 and 6 respectively. In the two next statements \mathcal{D}_∞ and \mathcal{G}_∞ stand for the limits obtained in the case $w = 0$. The first one entails that \mathcal{D}_∞ has a subgaussian negative tail.

Theorem 4.4. If Assumption 4.1 (1-4) holds then for $\gamma < 1$, there exists a constant $c := c(D, \gamma, Q) > 0$ such that for any $v > 0$

$$\mathbb{P}[\mathcal{D}_\infty < -v] \leq 2e^{-cv^2}. \quad (4.6)$$

The second one asserts that the probability of \mathcal{G}_∞ being smaller than s is subexponential in $1/s$ if the average value of the field is subtracted in the chaos expression. More precisely if we set

$$m_h(X) := \frac{1}{\mu(D)} \int_D X(x) \mu(dx),$$

we have the following.

Theorem 4.5. Given a field satisfying Assumption 4.1, then for all $\gamma \in (0, 2)$, for ζ sufficiently large there exists a constant $c > 0$ such that for all $s > 0$

$$\mathbb{P}\left(e^{-\gamma m_h(X)} \mathcal{G}_\infty \leq s\right) \leq 2e^{-c|\ln s|^{-\zeta}} s^{-\frac{4}{\gamma^2}}. \quad (4.7)$$

While we have chosen to present the result in a two-dimensional setup for simplicity, it is worth mentioning that the proof immediately extends to Gaussian fields in a domain of \mathbb{R}^d which satisfy Assumption 4.1. In that case the inequality (4.6) is valid for $\gamma < \sqrt{d}/2$, and (4.7) becomes

$$\mathbb{P}\left(e^{-\gamma m_h(X)} \mathcal{G}_\infty \leq s\right) \leq 2e^{-c|\ln s|^{-\zeta}} s^{-\frac{2d}{\gamma^2}}. \quad (4.8)$$

Let us mention furthermore that we believe that some extension of (4.6) should be valid beyond the case $\gamma < \sqrt{d}/2$. Let us mention this here as a conjecture:

Conjecture 1. For any Gaussian fields defined on $D \subset \mathbb{R}^d$ which satisfies assumption (4.1), we have for every $\gamma \in (0, \sqrt{2d})$

$$\log \mathbb{P}[\mathcal{D}_\infty < -v] \asymp -v^{\frac{2d}{\gamma^2}}. \quad (4.9)$$

where \asymp means that the quotient is asymptotically bounded when v tends to infinity.

To understand the heuristic behind (4.9) we need the Brownian decomposition of Section 5. We assume the best strategy to minimize \mathcal{D}_∞ would be first to make \mathcal{D}_t (defined in (5.4)) minimal making $(X_t(x) - \gamma t)$ close to -1 for all $x \in D$. This rare event has probability $e^{-e^{-dt}}$ (it does not depends on γ as this corresponds to the probability of X_t being roughly constant on D) and makes \mathcal{D}_t of order $-e^{-\frac{\gamma^2}{2}t}$, and we obtain the heuristic by taking $e^{-\frac{\gamma^2}{2}t} = v$.

Remark 4.6. Note that when $\gamma = \sqrt{2d}$, \mathcal{D}_∞ can also be defined (see [Du-Rh-Sh-Va, Du-Rh-Sh-Va1]), is almost surely negative and non-integrable. This means that the behavior of the tail changes abruptly at criticality. This is explained by the fact that the optimization strategy changes abruptly at that point.

Remark 4.7. In the presentation of the result, we have assumed that the measure μ in the definition of $m_\mu(X)$ and the one used for the GMC (4.4) are the same, but this assumption can be relaxed. For instance the result still holds true with $m_h(X)$ replaced by $m_{h'}(X) := \frac{1}{\int_D h'(x)dx} \int_D X(x)h'(x)dx$, for any bounded continuous function h' (to see this it is sufficient to check that the GMC associated with h' is larger than $\frac{\min|h'|}{\max|h|}\mathcal{G}_\infty$).

Moreover most of the proof could work on more general assumption on the measure μ and one could almost replace $h(x)dx$ by a much more singular measure, say concentrated on a fractal set (the exponent $\frac{4}{\gamma^2}$ though would be altered and depend on the fractal dimension of the measure). The only part of our proof which uses the fact that the measure is nice is the estimate of the L^p moment in Lemma C.1. Let us mention that the problem of small deviations for GMC have been an object of particular interest recently, with [Ga-Ho-Se-Su] investigating the problem on fractal sets.

Proof of Proposition 4.3 from Theorem 4.4 and 4.5. For readability we use the notation $Z = m_h(X)$ and assume first that $w \equiv 0$. Let us consider

$$q(x) := -2 \int_D K(x, y)\mu(dx) + \int_D K(x, y)\mu(dx) \otimes \mu(dy) = -2\mathbb{E}[X(x)Z] + E[Z^2]$$

We observe that $e^{-\gamma Z}\mathcal{G}_\infty$ is the GMC associated with the field $X - Z$ and integrated w.r.t to measure $\tilde{\mu}(dx) := e^{\frac{\gamma^2 q(x)}{2}}\mu(dx)$. The covariance of $(X - Z)$ can be written in the form

$$\tilde{K}(x, y) = \int_0^\infty \tilde{Q}_u(x, y)du$$

where, setting $\bar{h} = (\mu(D))^{-1}h$ as the renormalized density,

$$\tilde{Q}_u(x, y) := Q_u(x, y) - \int_D (Q_u(x, z) + Q_u(y, z))\bar{h}(z)dz \int_D Q_u(z, z')\bar{h}(z)\bar{h}(z')dzdz'. \quad (4.10)$$

It is a tedious but straightforward computation to check that \tilde{Q} (and $\tilde{\mu}$) satisfy Assumption 4.1. The associated derivative GMC is given by

$$\tilde{\mathcal{D}}_\infty = e^{-\gamma Z}\mathcal{D}_\infty - \gamma Z e^{-\gamma Z}\mathcal{G}_\infty - \gamma^2 e^{-\gamma Z}\mathcal{G}_\infty^q, \quad (4.11)$$

where \mathcal{G}_∞^q is the GMC associated with X and the (non-necessary positive) measure $q(x)\mu(dx)$. As a consequence of the definition of GMC we have $|\mathcal{G}_\infty^q/\mathcal{G}_\infty| \leq \|q\|_\infty$, and thus using Cauchy-Schwartz, we have

$$\begin{aligned} \mathbb{E}\left[e^{-\alpha\frac{\tilde{\mathcal{D}}_\infty}{\mathcal{G}_\infty}}\right] &\leq e^{\alpha\gamma^2\|q\|_\infty}\mathbb{E}\left[\exp\left(-\frac{\alpha\tilde{\mathcal{D}}_\infty}{e^{-\gamma Z}\mathcal{G}_\infty} - \alpha\gamma Z\right)\right] \\ &\leq e^{\alpha\gamma^2\|q\|_\infty}\left(\mathbb{E}\left[\exp\left(-\frac{2\alpha\tilde{\mathcal{D}}_\infty}{e^{-\gamma Z}\mathcal{G}_\infty}\right)\right]\mathbb{E}[e^{-2\alpha\gamma Z}]\right)^{1/2}, \end{aligned} \quad (4.12)$$

and we only need to show that the first factor in the square root is finite. Now we use the fact if $a > 0$ and $b \in \mathbb{R}$ we have

$$ab \leq \frac{1}{3}a^3 + \frac{2}{3}(b_+)^{3/2}$$

for $a = e^{\gamma Z}(\mathcal{G}_\infty)^{-1}$ and $b = -\tilde{\mathcal{D}}_\infty$. We obtain using Cauchy-Schwarz again

$$\begin{aligned} \mathbb{E}\left[\exp\left(-\frac{2\alpha\tilde{\mathcal{D}}_\infty}{e^{-\gamma Z}\mathcal{G}_\infty}\right)\right]^2 &\leq \mathbb{E}\left[e^{2\alpha((e^{-\gamma Z}\mathcal{G}_\infty)^{-3}+|\mathcal{D}_\infty|^{3/2}\mathbb{1}_{\{\mathcal{D}_\infty < 0\}})}\right]^2 \\ &\leq \mathbb{E}\left[e^{4\alpha(e^{-\gamma Z}\mathcal{G}_\infty)^{-3}}\right]\mathbb{E}\left[e^{4\alpha|\mathcal{D}_\infty|^{3/2}\mathbb{1}_{\{\mathcal{D}_\infty < 0\}}}\right]. \end{aligned} \quad (4.13)$$

And finally we can use Theorems 4.5 and 4.4 (recall that $\gamma < 1$) to conclude. The proof remains valid if we have to consider $w + q$ rather than q for $\tilde{\mu}$, which yields the general statement. \square

4.2 Fitting the GFF on M to Assumption 4.1

To prove Theorem 3.2(ii) from Proposition 4.3, we need to use the local charts to map the field onto some domain of the plane, and prove the right estimate for the corresponding covariance function. Thus we first show that we can restrict to prove Equation 3.2 only for GMC associated with local neighborhoods of M . To control the covariance function on the neighborhoods after mapping them to the complex plane, we compare it to that of the Dirichlet GFF on a disk.

We consider the Gaussian Free Field X_g on a compact Riemann surface M without boundary endowed with a metric g . Given $\delta > 0$, we let $\mathcal{B}(\delta, M)$ be the set of smooth domains of small diameter on M

$$\mathcal{B}(\delta, M) := \{S \subset M : S \text{ open, } \partial S \text{ is a piecewise } C^1 \text{ Jordan curve and } \text{diam}_g(S) \leq \delta\}.$$

Lemma 4.8. *There exists $\delta > 0$ such that we have for all $S \in \mathcal{B}(\delta, M)$ and for all $\alpha > 0$*

$$\mathbb{E}\left[e^{-\alpha \mathcal{D}_g^\gamma(S)/\mathcal{G}_g^\gamma(S)}\right] < \infty. \quad (4.14)$$

Proof of Theorem 3.2(ii) from Lemma 4.8. Given a subset $A \subset M$, we denote by \bar{A} its closure. As M is a compact manifold, it can be covered by a finite union $\bigcup_{i \in I} \bar{S}_i$, where $S_i \in \mathcal{B}(\delta, M)$ for all i together with $S_i \cap S_j = \emptyset$ for $i \neq j$. As almost surely $\mathcal{D}_g^\gamma(\bar{S}_i) = \mathcal{D}_g^\gamma(S_i)$ and $\mathcal{G}_g^\gamma(\bar{S}_i) = \mathcal{G}_g^\gamma(S_i)$ for all i , we have

$$\mathbb{E}\left[e^{-\alpha \mathcal{D}_g^\gamma(M)/\mathcal{G}_g^\gamma(M)}\right] = \mathbb{E}\left[e^{-\alpha \sum_{i \in I} \lambda_i \frac{\mathcal{D}_g^\gamma(S_i)}{\mathcal{G}_g^\gamma(S_i)}}\right]$$

with $\lambda_i = \frac{\mathcal{G}_g^\gamma(S_i)}{\mathcal{G}_g^\gamma(M)}$. Observe that the λ_i 's are positive and sum up to 1. By Jensen,

$$\mathbb{E}\left[e^{-\alpha \mathcal{D}_g^\gamma(M)/\mathcal{G}_g^\gamma(M)}\right] \leq \mathbb{E}\left[\sum_{i \in I} \lambda_i e^{-\alpha \frac{\mathcal{D}_g^\gamma(S_i)}{\mathcal{G}_g^\gamma(S_i)}}\right] \leq \sum_{i \in I} \mathbb{E}\left[e^{-\alpha \frac{\mathcal{D}_g^\gamma(S_i)}{\mathcal{G}_g^\gamma(S_i)}}\right] < +\infty.$$

Hence our claim. \square

Our next step is to now to replace $S \in \mathcal{B}(\delta, M)$ by a subset of \mathbb{C} using local charts. By a compactness argument and choosing δ small enough, we can assume that every $S \in \mathcal{B}(\delta, M)$ can be included within the image of an open disc (or two dimensional ball) for a given chart. More precisely we can assume $\bar{S} \subset \psi(B)$ with $\text{dist}(\bar{S}, \psi(B)^c) > 0$, where B is an open ball with an isothermal coordinate chart $\psi : B \subset \mathbb{C} \rightarrow M$. It is then equivalent to prove (4.14) for the GMC (and derivative) associated with the Gaussian field $X_g \circ \psi$ on B endowed with a conformal metric $g_\psi(z)|dz|^2$ for some function $g_\psi(z) := e^{\omega(z)}$ with ω smooth, integrated over the set $\psi^{-1}(S)$. Note that without lack of generality one can assume that B is the unit disc centered at 0.

The next step is to write $X_g \circ \psi$ as the sum of a Dirichlet Gaussian Free Field (see appendix A for the definition).

The first correction term we want to discard is the spatial average of the field. We let $\hat{\mu}_{\partial B}$ denote the uniform probability measure over the circle ∂B and define $m_B := \int_{\partial B} X_g \circ \psi d\hat{\mu}_{\partial B}$ the spatial average of the field over the circle ∂B .

We set $\tilde{X}_B := X_g \circ \psi - m_B$. Let us define $\mathcal{D}_B(dx)$ and $\mathcal{G}_B(dx)$ using the same procedure as in Theorem 3.2 and Proposition 2.3 but replacing X_g by \tilde{X}_B and v_g by $g_\psi(z)|dz|^2$. One can easily check that

$$\mathcal{D}_B(\psi^{-1}(S)) = e^{-\gamma m_B} (\mathcal{D}_g^\gamma(S) - \gamma m_B \mathcal{G}_g^\gamma(S)) \quad \text{and} \quad \mathcal{G}_B(\psi^{-1}(S)) = e^{-\gamma m_B} \mathcal{G}_g^\gamma(S),$$

and hence

$$\frac{\mathcal{D}_g^\gamma(S)}{\mathcal{G}_g^\gamma(S)} = \frac{\mathcal{D}_B(\psi^{-1}(S))}{\mathcal{G}_B(\psi^{-1}(S))} + \gamma m_B.$$

We have thus for any $\alpha > 0$ by Cauchy-Schwartz inequality

$$\mathbb{E}[e^{-\alpha \mathcal{D}_g^\gamma(S)/\mathcal{G}_g^\gamma(S)}] \leq \mathbb{E}[e^{-2\alpha \mathcal{D}_B(\psi^{-1}(S))/\mathcal{G}_B(\psi^{-1}(S))}]^{1/2} \mathbb{E}[e^{-2\alpha \gamma m_B}]^{1/2}. \quad (4.15)$$

The variable m_B being Gaussian, the second factor is finite for every $\alpha > 0$. The proof of Lemma 4.8 is reduced to a statement about fields defined on subsets of the unit ball whose closure does not intersect the boundary (we need to prove (4.14) with $\mathcal{D}_g^\gamma(S)/\mathcal{G}_g^\gamma(S)$ replaced by $\mathcal{D}_B(\psi^{-1}(S))/\mathcal{G}_B(\psi^{-1}(S))$). The situation is more comfortable with \tilde{X}_B than with the field $X_g \circ \psi$ (which is defined on the same set) because the domain Markov property (see Appendix A) ensures that it can be written as an independent sum $X + H$ where X is a Dirichlet GFF inside B and H is the harmonic extension of the boundary values of \tilde{X}_B .

In the remainder of the proof, we must show that $\mathcal{D}_B(\psi^{-1}(S))$ and $\mathcal{G}_B(\psi^{-1}(S))$ are obtained as limits in (4.4), and that the covariance of \tilde{X}_B satisfies Assumption 4.1.

For the first point, it can be checked via second moment computation that taking average over Euclidean circles and subtracting a variance term (like in (4.4)), or taking average over circles induced by $g_\psi(z)|dz|^2$ and using the renormalization convention of Section 2.3 which amounts to subtract a $\ln \varepsilon$ term (which is what is done for $\mathcal{D}_B(\psi^{-1}(S))$ and $\mathcal{G}_B(\psi^{-1}(S))$), amounts to the same result provided that one chooses $w = W_g \circ \psi^{-1}$ (recall (2.12)).

Concerning the second point, if we let G_B denote the Dirichlet Green function on the unit disc (let us stress that G_B does not depend on the tensor $g_\psi(z)$ and is thus completely explicit) and K_H the covariance of the field H , the covariance of \tilde{X}_B is given by $2\pi G_B + K_H$.

If we let p_B denote the heat-Kernel associated with the (flat) Dirichlet Laplacian on the unit ball we have

$$G_B(x, x') = \int_0^\infty p_B(t, x, x') dt. \quad (4.16)$$

To fit our decomposition, and setting

$$\bar{G}(x, x') := \int_1^\infty p_B(t, x, x') dt,$$

we can rewrite it as

$$G_B(x, x') = 2 \int_0^\infty p_B(e^{-2u}, x, x') e^{-2u} du + \bar{G}(x, x'). \quad (4.17)$$

Now we have

$$2\pi G_B + K_H = \int_0^\infty Q_u du, \quad (4.18)$$

where

$$Q_u(x, x') = 4\pi e^{-2u} p_B(e^{-2u}, x, x') + (K_H + 2\pi \bar{G})(x, x') \mathbb{1}_{\{u \in [0, 1]\}}.$$

To check that Q satisfies Assumption 4.1 we only need to care about the first term (as the second one is clearly bounded and measurable). Now the heat-kernel on the unit ball can be expressed as

$$p_B(t, x, y) = \frac{1}{4\pi t} e^{-\frac{|x-y|^2}{4t}} - R(t, x, y).$$

where the first term corresponds to the heat-kernel on the full plane and $R(t, x, y) \geq 0$ corresponds to a correction term which accounts for the fact that the diffusion is killed at the boundary and is small when t is small and x and y are away from the boundary. Note that $Q_u^0(x, y) := e^{-\left(\frac{|x-y|}{2e^u}\right)^2}$ satisfies Assumption 4.1 trivially and we must thus only check that the correction term induced by R is not relevant. It is a classical

estimate (see e.g [Du-Rh-Sh-Va1, section D.2]) that given $\delta > 0$, there exists a constant $c > 0$ such that for all $x, y \in B(0, 1 - \delta)$ and $t \leq 1$

$$R(t, x, y) \leq \frac{1}{c} e^{-c/t}. \quad (4.19)$$

As a consequence we have for $u \geq 1$, $x, y \in \psi^{-1}(S) \subset B(0, 1 - \delta)$,

$$|Q_u(x, y) - Q_u^0(x, y)| = 4\pi e^{-2u} R(e^{-2u}, x, y) \leq e^{-ce^{2u}},$$

which is sufficient to prove that Q_u also satisfies Assumption 4.1.

5 Exponential moments for the DGMC: proof of Theorem 4.4

5.1 Brownian decomposition and martingales

Let us first explain the importance of the integral representation of the covariance function K . As the result depends only on the distribution on X we may construct the process as we wish. We choose to think of it as a limit of a continuous martingale. We define $(X_t(x))_{x \in D, t \geq 0}$ to be the jointly continuous process in x and t with covariance kernel is given by

$$\mathbb{E}[X_t(x)X_s(y)] := \int_0^{t \wedge s} Q_u(x, y) du. \quad (5.1)$$

Note that given $x \in D$, the process $(X_t(x))_{t \geq 0}$ is a Brownian Motion with a deterministic time change given by $K_t(x, x) = \int_0^t Q_u(x, x) dt$. According to our assumptions (4.2) and $Q_u \leq 1$ we have

$$|K_t(x, x) - t| \leq C \quad \text{and} \quad \limsup_{t \rightarrow \infty} \sup_{x \in D} |\partial_t K_t(x, x) - 1| = 0, \quad (5.2)$$

which makes the process very similar to a standard Brownian Motion. For $\gamma \in (0, 2)$, we define the random measure on D

$$\mathcal{G}_t(dx) := e^{\gamma X_t(x) - \frac{\gamma^2}{2} K_t(x, x)} \mu(dx) \quad (5.3)$$

and for $\gamma \in (0, \sqrt{2})$ the random distribution on D

$$\mathcal{D}_t(dx) := (X_t(x) - \gamma t) e^{\gamma X_t(x) - \frac{\gamma^2}{2} K_t(x, x)} \mu(dx). \quad (5.4)$$

With some harmless abuse of notation we set $\mathcal{G}_t := \int_D \mathcal{G}_t(dx)$ and $\mathcal{D}_t := \int_D \mathcal{D}_t(dx)$. By construction $(\mathcal{D}_t)_{t \geq 0}$ is a martingale for the filtration $(\mathcal{F}_t)_{t \geq 0}$. An explicit computation of the variance shows that \mathcal{D}_t is uniformly bounded in L^2 when $\gamma^2 < 2$. Let us call \mathcal{D}_∞ and \mathcal{G}_∞ the limit. Standard L^2 computations (similar the one performed in the proof of Theorem 3.2) allow us to show that these limits coincide with that obtained in 4.4 when $\gamma < \sqrt{2}$. In this framework we can use stochastic calculus to prove concentration-type results for \mathcal{D}_∞ .

5.2 Decomposition of the proof

We assume, for better readability that $Q_u(x, x) = 1$ so that $K_t(x, x) = t$ for all t . The reader can check that all the computation can be adapted when we only have (5.2).

The core idea of the proof is to obtain a bound on the negative exponential moment of \mathcal{D}_∞ by using its predictable bracket (which for continuous martingales coincides with the quadratic variation). Indeed if $(M_t)_{t \geq 0}$ is a continuous martingale with initial condition 0, we have

$$\mathbb{E} \left[e^{-\alpha M_t - \frac{\alpha^2}{2} \langle M \rangle_t} \right] = 1, \quad (5.5)$$

as the expression inside the expectation is also a martingale. This implies that if $\langle M \rangle_\infty = \lim \langle M \rangle_t$ is uniformly bounded ($\text{ess sup} \langle M \rangle_\infty < \infty$) then the limit of M_t displays Gaussian concentration.

Note that there is no hope to prove directly that $\langle \mathcal{D} \rangle_\infty$ is uniformly bounded: indeed \mathcal{D}_∞ does not display Gaussian concentration as we expect the same right tail as GMC, i.e. $\mathbb{E}[|\mathcal{D}_\infty|^p] = \infty$ whenever $p > \frac{4}{\gamma^2}$, see [Rh-Va3].

An explicit computation of \mathcal{D}_t can however give us some extra intuition on the problem. Setting

$$W_t^0(x) := (X_t(x) - \gamma t) e^{\gamma X_t(x) - \frac{\gamma^2}{2} t},$$

which is a martingale in t for all x , and following the rule of Itô calculus we obtain the following expression

$$\begin{aligned} \langle \mathcal{D} \rangle_t &= \int_{[0,t] \times D^2} d\langle W^0(x), W^0(y) \rangle_u \mu(dx) \mu(dy) \\ &= \int_{[0,t] \times D^2} Q_u(x, y) (1 + \gamma(X_u(x) - \gamma u)) (1 + \gamma(X_u(y) - \gamma u)) e^{\gamma X_t(x)} e^{\gamma X_u(x) + X_u(y) - \gamma^2 u} du. \end{aligned} \quad (5.6)$$

We see from the above expression that most of the contribution to $\langle \mathcal{D} \rangle_t$ is given by high values of X_u , $u \in [0, t]$. However, these high-values should not contribute much to the negative tail of \mathcal{D}_t since they tend to yield high positive values of the integrand $W_t^0(x)$.

Hence our idea is to compare \mathcal{D}_t with a martingale obtained by replacing $W_t^0(x)$ by an alternative martingale W_t which does not sum the variation of W_t^0 when the value of X_t is too large. Our definition of W_t has to be carefully chosen so that it compares well with \mathcal{D}_t . Let us fix $A > 0$ sufficiently large and η which satisfies

$$2\eta > \gamma \quad \text{and} \quad \gamma^2 + 2\eta\gamma < 2. \quad (5.7)$$

(Note here that with our assumption $\gamma < 1$, $\eta = 1/2$ satisfies (5.7), we felt however that keeping a parameter would make the computation more readable. The choice $A = 100$ is also amply sufficient for all our computation).

Then we define the following sequence of stopping times (with the convention that $R_0^x := 0$) for $k \geq 1$

$$T_k^x := \inf\{t \geq R_{k-1}^x : X_t(x) = (\gamma + \eta)t + A\} \quad (5.8)$$

$$R_k^x := \inf\{t \geq T_k^x : X_t(x) = \gamma t\}. \quad (5.9)$$

Introducing the notation $\mathcal{R}^x := \bigcup_{k=1}^\infty [R_{k-1}^x, T_k^x]$, we define (the second equality being derived from Itô's formula)

$$W_t(x) := \int_{[0,t] \cap \mathcal{R}^x} dW_t(x) = \int_0^t \left(\gamma(X_s(x) - \gamma s) + 1 \right) e^{\gamma X_s(x) - \frac{\gamma^2}{2} s} \mathbf{1}_{\{s \in \mathcal{R}^x\}} dX_s(x) \quad (5.10)$$

Now we reader can check the correctness of the following alternative expression for $W_t(x)$ the fact that $W_t^0(x)$ cancels at times R_i^x ,

$$W_t(x) = \begin{cases} (X_t(x) - \gamma t) e^{\gamma X_t(x) - \frac{\gamma^2}{2} t} + \sum_{i=1}^k (A + \eta T_i^x) e^{\gamma A + (\gamma\eta + \frac{\gamma^2}{2}) T_i^x} & \text{if } t \in (R_k^x, T_{k+1}^x) \\ \sum_{i=1}^k (A + \eta T_i^x) e^{\gamma A + (\gamma\eta + \frac{\gamma^2}{2}) T_i^x} & \text{if } t \in (T_k^x, R_k^x) \end{cases} \quad (5.11)$$

By construction $(W_t(x))_{t \geq 0}$ is a \mathcal{F}_t -martingale and so is

$$\tilde{\mathcal{D}}_t := \int_D W_t(x) \mu(dx). \quad (5.12)$$

Furthermore, repeating the computation from (5.6), we obtain

$$\langle \tilde{\mathcal{D}} \rangle_t := \int_{[0,t] \times D^2} Q_u(x, y) (1 + \gamma(X_u(x) - \gamma u)) (1 + \gamma(X_u(y) - \gamma u)) \mathbf{1}_{\{u \in \mathcal{R}^x \cap \mathcal{R}^y\}} \mathcal{G}_u(dx) \mathcal{G}_u(dy) ds. \quad (5.13)$$

This martingale is bounded in L^2 (its bracket is smaller than that of $(\mathcal{D}_t)_{t \geq 0}$), and thus converges in L^2 . We call $\tilde{\mathcal{D}}_\infty$ its limit. We can compare \mathcal{D}_∞ and $\tilde{\mathcal{D}}_\infty$: as a consequence of (5.11) (when $t \in (T_k^x, R_k^x)$ we have $X_t(x) - \gamma t \geq 0$)

$$W_t^0(x) \geq W_t(x) - \sum_{i=1}^{I_t^x} (A + \eta T_i^x) e^{\gamma A + (\gamma\eta + \frac{\gamma^2}{2}) T_i^x} \quad (5.14)$$

where $I_t^x = \sup\{i : T_i^x \leq t\}$. Hence if one sets

$$Q := \int_D \left(\sum_{i=1}^{\infty} \mathbf{1}_{\{T_i^x < +\infty\}} (A + \eta T_i^x) e^{(\gamma\eta + \frac{\gamma^2}{2}) T_i^x} \right) \mu(dx),$$

we have

$$\mathcal{D}_\infty \geq \tilde{\mathcal{D}}_\infty - e^{\gamma A} Q. \quad (5.15)$$

To prove Theorem 4.4, it is sufficient to show that both terms in the r.h.s. display Gaussian concentration.

Lemma 5.1. *There exists $C(A, \eta, \gamma)$ such that*

$$\forall \alpha \in \mathbb{R}, \quad \mathbb{E}[e^{\alpha \tilde{\mathcal{D}}_\infty}] \leq e^{C(A, \eta, \gamma) \alpha^2},$$

where

$$C(A, \eta, \gamma) := \int_{[0, t] \times D^2} [\gamma(A + \eta u) + 1]^2 e^{2\gamma A + (2\gamma\eta + \gamma^2)u} Q_u(x, y) \mu(dx) \mu(dy). \quad (5.16)$$

Lemma 5.2. *There exists $C(A, \eta, \gamma)$ such that*

$$\forall \alpha \in \mathbb{R}, \quad \mathbb{E}[e^{\alpha Q}] \leq e^{C(A, \eta, \gamma) \alpha^2}.$$

Lemma 5.1 is by far the easier of the two results as the construction of the martingale $\tilde{\mathcal{D}}$ has been tailored so that its quadratic variation is bounded. Lemma 5.2 requires more work, but the main idea is to control the total variation of the Doob Martingale associated with Q and \mathcal{F}_t (which is continuous), the details are provided in the next session

Proof of Lemma 5.1. From (5.5) it is sufficient to show that $\langle \tilde{\mathcal{D}} \rangle_\infty$ is bounded. Using the definition of \mathcal{R}^x and the fact (which can be verified by checking that the largest possible negative value is smaller in absolute value than the r.h.s. below) that

$$\max_{u \leq A + (\gamma + \eta)t} \left| [\gamma(u - \gamma t) + 1] e^{\gamma u - \frac{\gamma^2 t}{2}} \right| = [\gamma(\eta t + A) + 1] e^{\frac{\gamma^2 t}{2} + \gamma[\eta t + A]}, \quad (5.17)$$

we obtain that the bracket (5.13) of $\tilde{\mathcal{D}}_t$ satisfies

$$\langle \tilde{\mathcal{D}} \rangle_t \leq \int_{[0, t] \times D^2} [\gamma(A + \eta u) + 1]^2 e^{2\gamma A + (2\gamma\eta + \gamma^2)u} Q_u(x, y) \mu(dx) \mu(dy),$$

which is uniformly bounded in t by item 4 of Assumption 4.1, and our choice of η (5.7). \square

5.3 Proof of Lemma 5.2

To prove concentration for Q , we consider the Doob martingale associated with Q (recall $\mathcal{F}_t := \sigma(X_s, s \in [0, t])$),

$$Q_t := \mathbb{E}[Q | \mathcal{F}_t],$$

(we need to prove first that $\mathbb{E}[Q] < +\infty$ but this is the easier part of the proof), and prove that $\text{ess sup}(Q)_\infty < \infty$ so that one can conclude using the following identity which is provided by the exponential martingale

$$\mathbb{E} \left[e^{\alpha Q - \frac{\alpha^2 \langle Q \rangle_\infty}{2}} \right] \leq \mathbb{E}[e^{\alpha \mathbb{E}[Q]}]. \quad (5.18)$$

We let Q^x denote the integrand in the definition of Q and Q_t^x to be the Doob martingale associated with it,

$$Q^x := \sum_{i=1}^{\infty} \mathbf{1}_{\{T_i^x < +\infty\}} (A + \eta T_i^x) e^{\gamma A + (\gamma + \frac{\eta}{2}) T_i^x}, \quad \text{and} \quad Q_t^x := \mathbb{E}[Q^x | \mathcal{F}_t]. \quad (5.19)$$

Our first task is to bound the expectation of Q^x , in a way which is uniform over x so that we have $\mathbb{E}[Q] < \infty$. This computation is also going to be used later to control the martingale bracket.

$$Q^x \leq \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} (A + \eta n) e^{(\gamma + \frac{\eta}{2}) n} \mathbf{1}_{\{T_i^x \in (n-1, n]\}} =: \sum_{n=1}^{\infty} Q^{x,n}. \quad (5.20)$$

We are going to prove

$$\mathbb{E}[\#\{i : T_i \in (n-1, n]\}] \leq \frac{4}{\sqrt{2\pi n}(\gamma + \eta)} e^{-\frac{(\gamma + \eta)^2 n}{2}}. \quad (5.21)$$

This implies that $\mathbb{E}[Q^{x,n}] \leq C\sqrt{n}e^{-\frac{n^2}{2}}$, and, by linearity, that $\mathbb{E}[Q]$ is bounded. Let us now prove (5.21). Assuming that A is chosen larger than $\gamma + \eta$ yields

$$\begin{aligned} \mathbb{P}(\exists i, T_i^x \in (n-1, n]) &\leq \mathbb{P}\left(\sup_{s \in [0, n]} X_s(x) \geq (\gamma + \eta)(n-1) + A\right) \\ &\leq \mathbb{P}\left(\sup_{s \in [0, n]} X_s(x) \geq (\gamma + \eta)n\right) \leq \frac{2}{\sqrt{2\pi n}(\gamma + \eta)} e^{-\frac{(\gamma + \eta)^2 n}{2}} \end{aligned} \quad (5.22)$$

where the last inequality is the standard Gaussian tail estimate. Using the Markov property for the Brownian Motion $(X_t(x))_{t \geq 0}$ at the k -th T_i^x in the interval $(n-1, n]$ the reader can check that that if $A \geq \gamma + 2$

$$\mathbb{P}[\#\{i : T_i \in (n-1, n]\} \geq k+1 \mid \#\{i : T_i \in (n-1, n]\} \geq k] \leq \mathbb{P}[\inf_{t \in [0, 1]} B_t \leq \gamma - A] \leq \frac{1}{2}. \quad (5.23)$$

Therefore we have from (5.22)

$$\begin{aligned} \mathbb{E}[\#\{i : T_i \in (n-1, n]\}] &= \sum_{k=1}^{\infty} \mathbb{P}[\#\{i : T_i \in (n-1, n]\} \geq k] \\ &\leq \mathbb{P}[\#\{i : T_i \in (n-1, n]\} \geq 1] \sum_{k=1}^{\infty} 2^{1-k} \leq \frac{4}{\sqrt{2\pi n}(\gamma + \eta)} e^{-\frac{(\gamma + \eta)^2 n}{2}}. \end{aligned} \quad (5.24)$$

Now we focus on controlling the martingale bracket. As (Q_t^x) is also martingale with respect to the Brownian Filtration associated with $(X_t(x))$, we can (from [Rev, Chap. V Th. 3.4]) write its variation in the form

$$dQ_t^x := A_t^x dX_t(x). \quad (5.25)$$

Then using the covariance structure of $X_t(x)$ one can compute the infinitesimal increment of the martingale bracket

$$d\langle Q^x, Q^y \rangle_t = A_t^x A_t^y Q_t(x, y) dt, \quad (5.26)$$

and thus obtain an expression for $\langle Q \rangle_\infty$

$$\langle Q \rangle_\infty = \int_{\mathbb{R}_+ \times D^2} A_u^x A_u^y Q_u(x, y) \mu(dx) \mu(dy) du. \quad (5.27)$$

Our remaining task is to find an expression for A_t^x and obtain a uniform bound in x for it. We must distinguish between two cases according to whether $t \in (T_k^x, R_k^x)$ or $t \in (R_k^x, T_{k+1}^x)$.

In the first case, using the strong Markov property for the Brownian motion $(X_t(x))_{t \geq 0}$ and denoting by \mathbf{E}_z the law of standard Brownian motion $(B_t)_{t \geq 0}$ starting at z we have

$$Q_t^x = \sum_{i=1}^k (A + \eta T_i^x) e^{(\gamma\eta + \frac{\gamma^2}{2})T_i^x} + \mathbf{E}_{X_t(x)} \left[\sum_{i=1}^{\infty} \mathbf{1}_{\{\widehat{T}_i^t < +\infty\}} (A + \eta(\widehat{T}_i^t + t)) e^{(\gamma\eta + \frac{\gamma^2}{2})(\widehat{T}_i^t + t)} \right] \quad (5.28)$$

where the sequence $\widehat{T}_k^t = \widehat{T}_k^t(B)$ (we drop the dependence in B to alleviate the notation) is recursively defined by $\widehat{T}_0^t := 0$ and for $k \geq 1$

$$\begin{aligned} \widehat{R}_k^t &:= \inf\{s \geq \widehat{T}_{k-1}^t : B_s \leq \gamma(t+s)\}, \\ \widehat{T}_k^t &:= \inf\{s \geq \widehat{R}_k^t : B_s = A + (\eta + \gamma)(t+s)\} \end{aligned}$$

From this we deduce that (5.25) holds for $t \in (T_k^x, R_k^x)$ with

$$A_t^x := \partial_z \left(\mathbf{E}_z \left[\sum_{i=1}^{\infty} \mathbf{1}_{\{\widehat{T}_i^t < +\infty\}} (A + \eta(\widehat{T}_i^t + t)) e^{(\gamma\eta + \frac{\gamma^2}{2})(\widehat{T}_i^t + t)} \right] \right) \Big|_{z=X_t(x)}. \quad (5.29)$$

In the second case, the same argument yields

$$Q_t^x = \sum_{i=1}^k (A + \eta T_i^x) e^{(\gamma\eta + \frac{\gamma^2}{2})T_i^x} + \mathbf{E}_{X_t(x)} \left[\sum_{i=1}^{\infty} \mathbf{1}_{\{T_i^t < +\infty\}} (A + \eta(T_i^t + t)) e^{(\gamma\eta + \frac{\gamma^2}{2})(T_i^t + t)} \right] \quad (5.30)$$

where the sequence T_k^t is recursively defined by $R_0^t := 0$ and for $k \geq 1$

$$\begin{aligned} T_k^t &:= \inf\{s \geq R_{k-1}^t : B_s = A + (\eta + \gamma)(t+s)\} \\ R_k^t &:= \inf\{s \geq T_k^t : B_s = \gamma(t+s)\}. \end{aligned}$$

Again, we deduce that (5.25) holds for $t \in [R_k^x, T_{k+1}^x)$ with

$$A_t^x := \partial_z \left(\mathbf{E}_z \left[\sum_{i=1}^{\infty} \mathbf{1}_{\{T_i^t < +\infty\}} (A + \eta(T_i^t + t)) e^{(\gamma\eta + \frac{\gamma^2}{2})(T_i^t + t)} \right] \right) \Big|_{z=X_t(x)}. \quad (5.31)$$

To conclude, we need to prove the following bounds

Lemma 5.3. *We have for every $t \geq 0$ and $z \geq \gamma t$*

$$\left| \partial_z \left(\mathbf{E}_z \left[\sum_{i=1}^{\infty} \mathbf{1}_{\{\widehat{T}_i^t < +\infty\}} (A + \eta(\widehat{T}_i^t + t)) e^{(\gamma\eta + \frac{\gamma^2}{2})(\widehat{T}_i^t + t)} \right] \right) \right| \leq C(A, \eta, \gamma)(t+1) e^{(\gamma\eta + \frac{\gamma^2}{2})t}. \quad (5.32)$$

Lemma 5.4. *We have for every $t \geq 0$ and $z \leq (\gamma + \eta t)$*

$$\left| \partial_z \left(\mathbf{E}_z \left[\sum_{i=1}^{\infty} \mathbf{1}_{\{T_i^t < +\infty\}} (A + \eta(T_i^t + t)) e^{(\gamma\eta + \frac{\gamma^2}{2})(T_i^t + t)} \right] \right) \right| \leq C(A, \eta, \gamma)(t + 1)e^{(\gamma\eta + \frac{\gamma^2}{2})t}. \quad (5.33)$$

These two results together with (5.26) and (5.29)-(5.31) yields the result (using the fact that $2\gamma\eta + \gamma^2 < 2$ and the fourth item in Assumption 4.1). \square

Proof of Lemma 5.3. We need to show that the functional $\mathbf{E}_z[\cdot]$ in the r.h.s. of (5.32) is Lipschitz in z for an adequate Lipschitz constant. We fix thus $x, y \geq \gamma t$ (we may consider only the case where $|x - y| < \varepsilon$ for an arbitrary $\varepsilon > 0$) and consider \mathbb{P} to be the coupling between \mathbf{E}_x and \mathbf{E}_y constructed with marginals that evolve independently until the first time they meet

$$\tau := \inf\{s > 0 : B_s^{(1)} = B_s^{(2)}\},$$

and jointly afterwards. We let $\widehat{T}_i^{(t,j)}, \widehat{R}_i^{(t,j)}$, $j = 1, 2$ denote the stopping time corresponding to each coordinate of the coupling (i.e. $\widehat{T}_i^{(t,j)} := \widehat{T}_i^t(B^{(j)})$). We have (again, here $\mathbf{E}_z[\cdot]$ corresponds to the expression in (5.32))

$$\begin{aligned} |\mathbf{E}_x[\cdot] - \mathbf{E}_y[\cdot]| &\leq \mathbb{P} \left[\tau > \min(\widehat{T}_1^{(t,1)}, \widehat{T}_1^{(t,2)}, 1) \right] \\ &\times \mathbb{E} \left[\sum_{j=1,2} \sum_{i=1}^{\infty} \mathbf{1}_{\{T_i^{(t,j)} < +\infty\}} \left[A + \eta(\widehat{T}_i^{(t,j)} + t) \right] e^{(\gamma\eta + \frac{\gamma^2}{2})(\widehat{T}_i^{(t,j)} + t)} \mid \tau > \min(\widehat{T}_1^{(t,1)}, \widehat{T}_1^{(t,2)}, 1) \right]. \end{aligned} \quad (5.34)$$

The first step is to bound the probability above. We are going to show that for some constant C (which does not depend on t) we have

$$\mathbb{P} \left[\tau > \min(\widehat{T}_1^{(t,1)}, \widehat{T}_1^{(t,2)}, 1) \right] \leq C|x - y|. \quad (5.35)$$

The bound $\mathbb{E}[\tau > 1] < (C/3)|x - y|$ is standard and thus by symmetry and union bound we only need to show that

$$\mathbb{E} \left[\tau > \widehat{T}_1^{(t,1)} \right] \leq \frac{C|x - y|}{3}. \quad (5.36)$$

Note that provided $A \geq 2$, for any choice of $x \geq \gamma t$ we have

$$\widehat{T}_1^{(t,1)} \geq \min\{s : |B_s^{(1)} - x| \geq 1\}, \quad (5.37)$$

and hence

$$\mathbb{E} \left[\tau > \widehat{T}_1^{(t,1)} \right] \leq \mathbf{P}_{(0, y-x)} [\mathcal{T}_\Delta > \mathcal{T}_{\{-1, 1\} \times \mathbb{R}}] \leq C|x - y|. \quad (5.38)$$

where $\Delta := \{(x, x) : x \in \mathbb{R}\}$ and \mathcal{T}_A denotes the hitting time of a set A by a two dimensional Brownian motion (here with initial condition $(0, y - x)$). The last inequality can be deduced by standard estimate for Brownian Motion (see Appendix B).

To complete the proof we need to show that the conditional expectation in (5.34) satisfies

$$\mathbb{E} \left[\sum_{j=1,2} \sum_{i=1}^{\infty} \dots \mid \tau > \min(\widehat{T}_1^{(t,1)}, \widehat{T}_1^{(t,2)}, 1) \right] \leq C(t + 1)e^{(\gamma\eta + \frac{\gamma^2}{2})t}. \quad (5.39)$$

To do so we use the Markov property for $(B_t^{(1)}, B_t^{(2)})_{t \geq 0}$ at time $\mathcal{T} = \min(\widehat{T}_1^{(t,1)}, \widehat{T}_1^{(t,2)}, 1)$ and consider the supremum over all possible realizations (r, z) of $(\mathcal{T}, B_{\mathcal{T}}^{(j)})$ and obtain

$$\begin{aligned}
& \mathbb{E} \left[\sum_{i=1}^{\infty} \mathbf{1}_{\{T_i^{(t,j)} < +\infty\}} \left[A + \eta(\widehat{T}_i^{(t,j)} + t) \right] e^{(\gamma\eta + \frac{\gamma^2}{2})(\widehat{T}_i^{(t,j)} + t)} \mid \tau > \min(\widehat{T}_1^{(t,1)}, \widehat{T}_1^{(t,2)}, 1) \right] \\
& \leq \sup_{r \in [t, t+1]} \max \left(\sup_{z > \gamma r} \mathbf{E}_z \left[\sum_{i=1}^{\infty} \mathbf{1}_{\{\widehat{T}_i^r < +\infty\}} \left[A + \eta(\widehat{T}_i^r + r) \right] e^{(\gamma\eta + \frac{\gamma^2}{2})(\widehat{T}_i^r + r)} \right], \right. \\
& \quad \left. \sup_{z \leq A + (\gamma + \eta)r} \mathbf{E}_z \left[\sum_{i=1}^{\infty} \mathbf{1}_{\{T_i^r < +\infty\}} \left[A + \eta(T_i^r + r) \right] e^{(\gamma\eta + \frac{\gamma^2}{2})(T_i^r + r)} \right] \right) = \mathcal{L}(t). \quad (5.40)
\end{aligned}$$

The two terms in the max lead us to make the distinction between the two cases $\widehat{R}_1^{(t,j)} < \mathcal{T}$ and $\widehat{R}_1^{(t,j)} > \mathcal{T}$. Using the Markov property at time $R_1^{(t,j)}$ in the first case, we obtain that for any $z > \gamma r$ we have

$$\begin{aligned}
& \mathbf{E}_z \left[\sum_{i=1}^{\infty} \mathbf{1}_{\{T_i^r < +\infty\}} \left[A + \eta(\widehat{T}_i^r + r) \right] e^{(\gamma\eta + \frac{\gamma^2}{2})(\widehat{T}_i^r + r)} \right] \\
& \leq \sup_{s \geq r} \mathbf{E}_{\gamma s} \left[\sum_{i=1}^{\infty} \mathbf{1}_{\{T_i^s < +\infty\}} \left[A + \eta(T_i^s + s) \right] e^{(\gamma\eta + \frac{\gamma^2}{2})(T_i^s + s)} \right]. \quad (5.41)
\end{aligned}$$

Now repeating the computation (5.20)-(5.22) we have

$$\begin{aligned}
& \mathbf{E}_{\gamma s} \left[\sum_{i=1}^{\infty} \mathbf{1}_{\{T_i^s < +\infty\}} \left[A + \eta(T_i^s + s) \right] e^{(\gamma\eta + \frac{\gamma^2}{2})(T_i^s + s)} \right] \leq C \sum_{n \geq 1} (n+s) e^{(\gamma\eta + \frac{\gamma^2}{2})(n+s)} \mathbf{E}_{\gamma s} [\#\{T_i^s \in [n-1, n]\}] \\
& \leq 2C \sum_{n \geq 1} (n+s) e^{(\gamma\eta + \frac{\gamma^2}{2})(n+s)} \mathbf{P}_{\gamma s} [\exists i, T_i^s \in [n-1, n]] \\
& \leq 2C \sum_{n \geq 1} \frac{(n+s)}{\sqrt{n}} e^{(\gamma\eta + \frac{\gamma^2}{2})(n+s) - \frac{(\eta s + \frac{\gamma + \eta)n}{2})^2}. \quad (5.42)
\end{aligned}$$

Using Laplace's method the last quantity can be bounded above by $e^{(\frac{\gamma^2}{2} - 2\eta^2)s}$, which is uniformly bounded in s (from (5.7) we have $\frac{\gamma^2}{2} - 2\eta^2 < 0$).

In the second case, repeating again (5.20)-(5.22) we have

$$\begin{aligned}
& \mathbf{E}_z \left[\sum_{i=1}^{\infty} \mathbf{1}_{\{T_i^r < +\infty\}} \left[A + \eta(T_i^r + r) \right] e^{(\gamma\eta + \frac{\gamma^2}{2})(T_i^r + r)} \right] \\
& \leq C \sum_{n \geq 1} (n+r) e^{(\gamma\eta + \frac{\gamma^2}{2})(n+r)} \mathbf{P}_z \left[\sup_{t \in [n-1, n]} B_t \geq (\gamma + \eta)(n-1+r) + A \right] \leq C' r e^{(\gamma\eta + \frac{\gamma^2}{2})r}, \quad (5.43)
\end{aligned}$$

which yields

$$\mathcal{L}(t) \leq C(A, \eta, \gamma)(t+1)e^{(\gamma\eta + \frac{\gamma^2}{2})t}. \quad (5.44)$$

□

Proof of Lemma 5.4. Like in the previous lemma, we prove (5.33) by providing Lipschitz bounds in the same way as (5.34). It is of course sufficient to consider the case when $|x - y| \leq 1$. The approach adopted in the proof of Lemma 5.3 does not work when both x and y are close to $A + (\gamma + \eta)t$, because in that case,

with an independent coupling the probability of the two motions merging before $\min(T_1^{(t,1)}, T_1^{(t,2)}, 1)$ can be made arbitrarily small.

Note however that this obstruction is not present if one decides to start the sum from $i = 2$. Setting

$$g_t(u) := (A + \eta(u + t))e^{(\gamma\eta + \frac{\gamma^2}{2})(u+t)} \quad (5.45)$$

and repeating the proof Lemma 5.3 one obtains

$$\left| \mathbf{E}_x \left[\sum_{i=2}^{\infty} \mathbf{1}_{\{T_i^t < +\infty\}} g_t(T_i^t) \right] - \mathbf{E}_y \left[\sum_{i=2}^{\infty} \mathbf{1}_{\{T_i^t < +\infty\}} g_t(T_i^t) \right] \right| \leq C(A, \eta, \gamma) |x - y| (t + 1) e^{(\gamma\eta + \frac{\gamma^2}{2})t}. \quad (5.46)$$

Then we must estimate the difference between the first terms and show that

$$\left| \mathbf{E}_x \left[\mathbf{1}_{\{T_1^t < +\infty\}} g_t(T_1^t) \right] - \mathbf{E}_y \left[\mathbf{1}_{\{T_1^t < +\infty\}} g_t(T_1^t) \right] \right| \leq C(A, \eta, \gamma) |x - y| (t + 1) e^{(\gamma\eta + \frac{\gamma^2}{2})t}. \quad (5.47)$$

To prove (5.47) we choose to couple the two Brownian motions in a “parallel” fashion

$$B_s^{(2)} = B_s^{(1)} + (y - x).$$

To alleviate the notation we let S_1 and S_2 designate the hitting time $T_1^t(B^{(1)})$ and $T_1^t(B^{(2)})$. Let us assume without loss of generality that $y < x$. Note that as $g_t(\cdot)$ is an increasing function and $S_2 > S_1$ we have

$$\mathbb{E} \left[\mathbf{1}_{\{S_2 < \infty\}} g_t(S_2) \right] \geq \mathbb{E} \left[\mathbf{1}_{\{S_1 < \infty\}} g_t(S_1) \right] \mathbb{P} [S_2 < \infty \mid S_1 < \infty]. \quad (5.48)$$

and thus the difference in (5.47) satisfies

$$\mathbf{E}_x [\mathbf{1}_{\{T_1^t < +\infty\}} g_t(T_1^t) \cdot] - \mathbf{E}_y [\mathbf{1}_{\{T_1^t < +\infty\}} g_t(T_1^t) \cdot] \leq \mathbb{P} [S_2 = \infty \mid S_1 < \infty] \mathbb{E} \left[\mathbf{1}_{\{S_1 < \infty\}} g_t(S_1) \right] \quad (5.49)$$

And we have already proved (cf. (5.43)) that

$$\mathbb{E} \left[\mathbf{1}_{\{S_1 < \infty\}} g_t(S_1) \right] \leq C(A, \eta, \gamma) (t + 1) e^{(\gamma\eta + \frac{\gamma^2}{2})t}, \quad (5.50)$$

Now using the strong Markov property for $B^{(2)}$ at S_1 we obtain that

$$\mathbb{P} [S_2 < \infty \mid S_1 < \infty] = \mathbf{P}_{y-x} [\exists s, B_s = (\gamma + \eta)s] = e^{-2(\gamma + \eta)(x-y)}, \quad (5.51)$$

where the last inequality is obtained by observing that $u(x) := \mathbf{P}_x[\cdot]$ is a solution of

$$u''(x) - 2(\gamma + \eta)u'(x) = 0.$$

Combining this with (5.49) and (5.50) we obtain that

$$\mathbf{E}_x [\mathbf{1}_{\{T_1^t < +\infty\}} g_t(T_1^t)] - \mathbf{E}_y [\mathbf{1}_{\{T_1^t < +\infty\}} g_t(T_1^t)] \leq C|x - y|(t + 1)e^{(\gamma\eta + \frac{\gamma^2}{2})t}. \quad (5.52)$$

For the other bound, let us define $\mathcal{T} := \min\{S_2, S'_2\}$ where

$$S'_2 := \inf\{s > S_1 : B_s^{(2)} = \gamma(t + s)\}.$$

To estimate $\mathbf{E}_y[\mathbf{1}_{\{T_1^t < +\infty\}} g_t(T_1^t)]$, we split it into two contribution, depending how S_2 compares to S'_2 . We show that

$$\begin{aligned} \mathbb{E} [g_t(S_2) \mathbf{1}_{\{S_2 < S'_2 < \infty\}}] &\leq \mathbb{E} [\mathbf{1}_{\{S_1 < \infty\}} g_t(S_1)], \\ \mathbb{E} [g_t(S_2) \mathbf{1}_{\{S'_2 < S_2 < \infty\}}] &\leq C|x - y|. \end{aligned} \quad (5.53)$$

and the sum of these two inequalities yields (note that $S'_2 < \infty$ with probability 1)

$$\mathbf{E}_y[\mathbf{1}_{\{T_1^t < +\infty\}} g_t(T_1^t)] - \mathbf{E}_x[\mathbf{1}_{\{T_1^t < +\infty\}} g_t(T_1^t)] \leq C'|x - y|. \quad (5.54)$$

Let us consider the Brownian Motion $\tilde{B}_s := B_{S_1+s}^{(2)} - B_{S_1}^{(2)}$. Note that $\mathcal{T} - S_1$ is a stopping time for \tilde{B} .

Note that, conditioned on the event $S_1 < \infty$, $K_s := \left[B_{S_1+s}^{(2)} - \gamma(t+s) \right] e^{\gamma(B_{S_1+s}^{(2)} - A) - \frac{\gamma^2(s+t)}{2}}$ is a martingale for the filtration $\tilde{\mathcal{F}}$ defined by

$$\tilde{\mathcal{F}}_u := \mathcal{F}_{S_1} \cup \sigma(\tilde{B}_s, s \leq u),$$

and thus $K_{s \wedge (\mathcal{T} - S_1)}$ is a positive martingale. Using Fatou's Lemma for the conditional expectation with respect to (for the first inequality)

$$\begin{aligned} \mathbb{E} \left[\mathbf{1}_{\{S_2 < S'_2 < \infty\}} [A + \eta(S_2 + t)] e^{(\gamma\eta + \frac{\gamma^2}{2})(S_2+t)} \right] &= \mathbb{E} \left[\mathbb{E} [K_{\mathcal{T} - S_1} | \mathcal{F}_{S_1}] \mathbf{1}_{\{S_1 < \infty\}} \right] \\ &\leq \mathbb{E} \left[\mathbf{1}_{\{S_1 < \infty\}} \left[B_{S_1}^{(2)} - \gamma(t+s) \right] e^{\gamma(B_{S_1}^{(2)} - A) - \frac{\gamma^2(s+t)}{2}} \right] \leq \mathbb{E} [\mathbf{1}_{\{S_1 < \infty\}} g_t(S_1)]. \end{aligned} \quad (5.55)$$

The second inequality is a consequence of the fact that $B_{S_1}^{(2)} = (\gamma + \eta)(S_1 + s) + A - (x - y)$. To prove the second inequality in (5.53), we use the Markov property at time S'_2 which yields

$$\mathbb{E} [g_t(S_2) \mathbf{1}_{\{S_2 < \infty\}} | S'_2] \mathbf{1}_{\{S'_2 < S_2; S'_2 < \infty\}} = \tilde{\mathbf{E}}_{B_{S'_2}} \left[g_{t+S'_2}(T_1^{t+S'_2}(\tilde{B})) \mathbf{1}_{\{T_1^{t+S'_2} < \infty\}} \right]. \quad (5.56)$$

where \tilde{B} denote a Brownian Motion independent of B . Setting $t + S'_2 = s$ and noting that $B_{S'_2} = \gamma s$, we obtain

$$\mathbb{E} [g_t(S_2) \mathbf{1}_{\{S'_2 < S_2 < \infty\}}] \leq \mathbb{P}[S'_2 < \infty] \max_{s \geq t} \mathbf{E}_{\gamma s} [g_s(T_1^s) \mathbf{1}_{\{T_1^s < \infty\}}]. \quad (5.57)$$

The second factor is uniformly bounded (cf. (5.42)). To control the first one we can control the conditional expectation since

$$\begin{aligned} \mathbb{P}[S'_2 < S_2 | S_1 < \infty] &= \mathbf{P}_{y-x} [\min\{s : B_s = \gamma s - A - \eta t\} < \min\{s : B_s = (\gamma + \eta)s\}] \\ &\leq \mathbf{P}_{y-x} [\min\{s : B_s = \gamma s - A - \eta t\} < \min\{s : B_s = \gamma s\}] = u(y - x) \end{aligned} \quad (5.58)$$

where u is the solution of the equation

$$u''(x) - 2\gamma u'(x) = 0,$$

with boundary condition $u(0) = 0$, $u(-A - \eta t) = 1$. A simple computation yields

$$u(y - x) = \frac{1 - e^{-2\gamma(x-y)}}{1 - e^{-2\gamma(A+\eta t)}} \leq C|x - y| \quad (5.59)$$

and completes our proof. \square

6 Small deviations of the GMC measure: Proof of Theorem 4.5

6.1 Sketch of proof

We assume the setup described in Section 5.1 and Assumption 4.1 (item 4 is not required).

We set $\tilde{X} = X - Z$ (recall that $Z = \frac{1}{\mu(D)} \int_D X(x) dx$) let \tilde{X}_t denote the corresponding martingale approximation,

$$\tilde{X}_t(z) = X_t(z) - \int_D X_t(x) \bar{h}(x) dx, \quad (6.1)$$

where $\bar{h} = h/\mu(D)$. In view of (5.1) the covariance structure of \tilde{X} is given by

$$\mathbb{E}[\tilde{X}_t(x)\tilde{X}_s(y)] = \int_0^{s \wedge t} \tilde{Q}_u(x, y) du. \quad (6.2)$$

where \tilde{Q} is defined in (4.10) and satisfies the same assumptions as Q .

Now using the fact that \mathcal{G}_∞ as the limit of \mathcal{G}_t (5.3) we can express $e^{-\gamma Z} \mathcal{G}_\infty$ as the following martingale limit

$$e^{-\gamma Z} \mathcal{G}_\infty = \lim_{t \rightarrow \infty} \int_D e^{\gamma \tilde{X}_t(x) - \frac{\gamma^2}{2} \mathbb{E}[\tilde{X}_t(x)]} e^{\frac{\gamma^2}{2} q(x)} h(x) dx. \quad (6.3)$$

It is obviously sufficient to prove the result for the limit above with $e^{\frac{\gamma^2}{2} q(x)} h(x)$ replaced by 1 (since the function is bounded from below) and for the sake of readability we assume that $D = [0, 1]^2$ and $\bar{h} = 1$. We set in that case $\tilde{M}_t := \int_D e^{\gamma \tilde{X}_t(x) - \frac{\gamma^2}{2} \mathbb{E}[\tilde{X}_t(x)]} dx$, and Theorem 4.5 reduces to showing

$$\mathbb{P}(\tilde{M}_\infty \leq s) \leq 2e^{-c|\ln s|^{-\beta} s^{-\frac{4}{\gamma^2}}} \quad (6.4)$$

Before stating the main idea of the proof, let us make a trivial observation: assuming that D is the square $[0, 1]^2$, we have for any smooth Gaussian field \tilde{X} (as opposed to the ln-correlated field which is only defined as a distribution), which satisfies $\int_D \tilde{X}(x) dx = 0$,

$$\int_D e^{\gamma \tilde{X}} dx \geq e^{\int_D \gamma \tilde{X} dx} = 1.$$

Things do not quite work as easily when considering the exponential of our ln-correlated field because in the exponential we are subtracting the variance which is infinite. However we can still control the probability of being small in two steps. Heuristically this goes as follows:

- (A) We apply Jensen to the field \tilde{X}_t , to show that $\tilde{M}_t \geq e^{-\frac{\gamma^2 t}{2}}$, for a value of t such that $e^{-\frac{\gamma^2 t}{2}} \geq 2s$.
- (B) We observe that the field $\tilde{X} - \tilde{X}_t$ which remains to be added to obtain \tilde{M} , has a small covariance when the distance is larger than e^{-t} so that conditioned on \tilde{M}_t , the r.v.

$$\tilde{M}_\infty = \int e^{\gamma(\tilde{X} - \tilde{X}_t) - \frac{\gamma^2}{2} \mathbb{E}[(\tilde{X} - \tilde{X}_t)^2]} d\tilde{M}_t$$

is the sum of a large number (of order e^{2t}) of almost independent positive contribution coming from regions of diameter e^{-t} , and thus should concentrate around its mean, which according to step (A) is larger than $2s$.

A quantitative implementation of this heuristic yields the desired exponent $4/\gamma^2$.

Turning this idea into a rigorous proof requires some care for the following reason: the independent variables appearing in the second step have inhomogeneous weight (this is the effect of \tilde{X}_t). Instead of using Jensen in step (A) we rely on the following observation (see Lemma 6.1): either \tilde{X}_t is larger or equal to -1 on most of D , or there exists a small region on which \tilde{X}_t takes high value. In the first case \tilde{X}_t can be replaced by -1 and we do not have to worry about inhomogeneities, and concentration in step (B) above can be proved with standard tools (see Proposition 6.2 below). To analyze the case where \tilde{X}_t takes high value in a small region, we show that the loss of concentration implied by the smallness of the region is more than compensated by the fact that \tilde{M}_t is very large, this is the more delicate part of the analysis.

6.2 Proof of Theorem 4.5

We assume throughout the proof that s is sufficiently small: this restriction only affects the value of the constant c in (4.7). We fix a parameter $\kappa > 1$ and given s one defines t_0 by the relation

$$e^{-\frac{\gamma^2}{2}t_0} = s|\ln s|^{2\kappa}. \quad (6.5)$$

Our first task is to show that \tilde{X}_{t_0} must either be larger than -1 on a large set, or assume a very large value on a region of small but still significant size. This is a simple consequence of the fact that $\int_D \tilde{X}_{t_0}(x) dx = 0$, but we register it as a lemma, with a convenient formulation to be used in our proof.

Lemma 6.1. *For all s sufficiently large, one of the following must hold*

- (i) $|\{x \in D : \tilde{X}_{t_0}(x) \geq -1\}| \geq |\ln s|^{-\kappa}$.
- (ii) $\exists n \geq n_0(s) := |\ln s|^\kappa/10$, $|\{x \in D : \tilde{X}_{t_0}(x) \geq n\}| \geq \frac{1}{n(\ln n)^2}$.

Proof. Let us suppose that (i) does not hold. Set $\mathcal{A} := \{x \in D : \tilde{X}_{t_0}(x) \geq -1\}$. Since \tilde{X}_{t_0} has zero spatial average, one has

$$\begin{aligned} \int_D (\tilde{X}_{t_0}(x))_+ dx &= \int_D (\tilde{X}_{t_0}(x))_- dx \\ &\geq \int_{D \cap \mathcal{A}} (\tilde{X}_{t_0}(x))_- dx \geq 1 - |\ln s|^{-\kappa}. \end{aligned}$$

One also gets the bound (by using that for all $u \geq -1$, $|\{x : \tilde{X}_{t_0}(x) \geq u\}| \leq |\mathcal{A}| \leq (\ln s)^{-\kappa}$)

$$\begin{aligned} \int_D (\tilde{X}_{t_0}(x))_+ dx &= \int_0^\infty |\{x : \tilde{X}_{t_0}(x) \geq u\}| du \\ &\leq \frac{1}{10} + \int_{(\ln s)^\kappa/10}^\infty |\{x : \tilde{X}_{t_0}(x) \geq u\}| du \leq \frac{1}{10} + \sum_{n=(\ln s)^\kappa/10}^\infty |\{x : \tilde{X}_{t_0}(x) \geq n\}|. \end{aligned} \quad (6.6)$$

If for all $n \geq |\ln s|^\kappa/10$, one has $|\{x : \tilde{X}_{t_0}(x) \geq n\}| \leq \frac{1}{n(\ln n)^2}$ then combining the above considerations leads to

$$1 - |\ln s|^{-\kappa} \leq \frac{1}{10} + \sum_{n=|\ln s|^\kappa/10}^\infty \frac{1}{n(\ln n)^2} \leq \frac{1}{10} + \frac{1}{\kappa \ln |\ln s|},$$

which is a contradiction. Therefore (ii) holds. □

Now define the events

$$\begin{aligned} A &:= \left\{ |\{x \in D : \tilde{X}_{t_0}(x) \geq -1\}| \geq (\ln s)^{-\kappa} \right\}, \\ B_n &:= \left\{ |\{x \in D : \tilde{X}_{t_0}(x) \geq n\}| \geq n^{-1}(\ln n)^{-2} \right\}, \\ \bar{B}_n &:= B_n \setminus B_{n-1}, \text{ for } n \geq n_0 + 1 \text{ and } \bar{B}_{n_0} = B_{n_0} \setminus A. \end{aligned} \quad (6.7)$$

According to Lemma 6.1 the events $(\bar{B}_n)_{n \geq n_0}$ and A partition the space. As a consequence we have for any choice of $n_1 > n_0$ (in the remainder of the proof we write \tilde{M} for \tilde{M}_∞)

$$\begin{aligned} \mathbb{P}[\tilde{M} \leq s] &= \mathbb{P}[\{\tilde{M} \leq s\} \cap A] + \sum_{n \geq n_0} \mathbb{P}[\{\tilde{M} \leq s\} \cap \bar{B}_n] \\ &\leq \max \left(\mathbb{P}[\tilde{M} \leq s | A], \max_{n \in \llbracket n_0, n_1 \rrbracket} \mathbb{P}[\tilde{M} \leq s | \bar{B}_n] \right) + \sum_{n > n_1} \mathbb{P}[\bar{B}_n]. \end{aligned} \quad (6.8)$$

Thus we need to control all the conditional probabilities above, with the possibility of discarding high values of n for which B_n is very unlikely. The adequate choice for n_1 turns out to be $n_1(s) := s^{-\frac{2}{\gamma^2}}$. Indeed there exists some constant $C > 0$ such that for all $x \in D$

$$\mathbb{E}[\tilde{X}_{t_0}(x)^2] \leq t_0 + C,$$

hence

$$\mathbb{E} \left[|\{x \in D : \tilde{X}_{t_0}(x) \geq n\}| \right] \leq e^{-\frac{n^2}{2(t_0+C)}}. \quad (6.9)$$

We get from the Markov inequality that

$$\mathbb{P}[B_n] \leq Cn(\ln n)^2 e^{-c\frac{n^2}{t_0}}, \quad (6.10)$$

$$\sum_{n \geq n_1} \mathbf{P}[\bar{B}_n] \leq e^{-cs^{-\frac{4}{\gamma^2}} |\ln s|^{-1}}. \quad (6.11)$$

Now to conclude we need to control the first term in the r.h.s. of (6.8), which amounts to control every conditional expectation. More precisely, we prove a bound for the conditional expectation with respect to \tilde{X}_{t_0} which is valid on the specified events. First we show that almost surely

$$\mathbf{P}[M \leq s | \tilde{X}_{t_0}] \mathbf{1}_A \leq e^{-|\ln s|^{-\beta} s^{-\frac{4}{\gamma^4}}}. \quad (6.12)$$

and then that

$$\mathbf{P}[M \leq s | \tilde{X}_{t_0}] \mathbf{1}_{B_n} \leq e^{-|\ln s|^{-\beta} s^{-\frac{4}{\gamma^4}}}. \quad (6.13)$$

To prove (6.12) we consider $\mathcal{A} := \{x : \tilde{X}_{t_0}(x) \geq -1\}$ and recall that $|\mathcal{A}| \geq |\ln s|^{-\kappa}$ when A holds. Hence replacing \tilde{X}_{t_0} by its lower bound on \mathcal{A} we have

$$\tilde{M} \geq c(\gamma) e^{-\frac{\gamma^2 t_0}{2}} \tilde{M}^{(t_0)}(\mathcal{A}), \quad (6.14)$$

where $\tilde{M}^{(t_0)}$ is chaos measure associated with $\tilde{X} - \tilde{X}_{t_0}$ meaning

$$\tilde{M}^{(t_0)}(dx) = \lim_{t \rightarrow \infty} e^{\gamma(\tilde{X}_t - \tilde{X}_{t_0}) - \frac{\gamma^2}{2} \mathbb{E}[(\tilde{X}_t - \tilde{X}_{t_0})^2]} dx, \quad (6.15)$$

and $c(\gamma)$ is to absorb various constant factors, including the one which is needed when replacing the variance of \tilde{X}_{t_0} by t_0 in the exponential cf. (4.2). Then (6.12) follows from the following general concentration result (proved in the Appendix C).

Proposition 6.2. *We have for every $t_0 \geq 0$, for every $\mathcal{A} \subset D$, with $|\mathcal{A}| \geq e^{-2t_0}$*

$$\mathbb{P} \left[\tilde{M}^{(t_0)}(\mathcal{A}) \leq \frac{|\mathcal{A}|}{2} \right] \leq 2e^{-ce^{2t_0} t_0^{-2\alpha} |\mathcal{A}|}, \quad (6.16)$$

Now, we must prove (6.13). In the case where B_n holds one sets

$$\mathcal{B}_n := \{x : \tilde{X}_{t_0}(x) \geq n\}$$

and we have $|\mathcal{B}_n| \geq \frac{1}{n(\ln n)^2}$. Again, we have the obvious lower estimate

$$\tilde{M} \geq c(\gamma) e^{\gamma n - \frac{\gamma^2 t_0}{2}} \tilde{M}^{(t_0)}(\mathcal{B}_n). \quad (6.17)$$

Now let us chose $t_1 = t_0 + \eta n$ for a small but fixed value of η and let \bar{X} denote the increment of \tilde{X} between t_0 and t_1

$$Y := \frac{1}{|\mathcal{B}_n|} \int_{\mathcal{B}_n} (\tilde{X}_{t_1}(x) - \tilde{X}_{t_0}(x)) dx$$

$$\bar{X}_{t_0, t_1} := \bar{X} := (\tilde{X}_{t_1} - \tilde{X}_{t_0}) - Y.$$

Now rewriting $\widetilde{M}^{(t_0)}(\mathcal{B}_n)$ using \bar{X} and Y we have

$$\widetilde{M} \geq c(\gamma) e^{\gamma(n+Y) - \frac{\gamma^2 t_1}{2}} \int_{\mathcal{B}_N} e^{\gamma \bar{X}(x)} \widetilde{M}^{(t_1)}(dx). \quad (6.18)$$

To conclude we need to show that the pre-factor is large and that the content of the exponential is in some sense concentrated. An explicit computation using (4.3) yields .

$$\mathbb{E}[Y^2] = \frac{1}{|\mathcal{B}_n|^2} \int_{t_0}^{t_1} \int_{\mathcal{B}_n \times \mathcal{B}_n} \tilde{Q}_u(z, z') dz dz' \leq Cn(\ln n)^2 e^{-2t}$$

Hence for fixed $\delta > 0$

$$\mathbb{P}[Y \leq -\delta n] \leq ce^{-cn(\ln n)^{-2} e^{2t}}, \quad (6.19)$$

from which we deduce

$$\mathbb{P} \left[ce^{\gamma n + \gamma Y - \frac{\gamma^2 t_1}{2}} \leq e^{\gamma n / 2} \right] \leq ce^{-cn(\ln n)^{-2} e^{2t}} \leq e^{-s - \frac{4}{\gamma^4}}. \quad (6.20)$$

We are left with showing that

$$\mathbb{P} \left[\int_{\mathcal{B}_N} e^{\gamma \bar{X}} \widetilde{M}^{(t_1)}(dx) \leq e^{-\gamma n / 2} \right]$$

is small. Repeating the reasoning of Lemma 6.1 we have either

- (i) $|\{x \in \mathcal{B}_n : \bar{X} \geq -1\}| \geq e^{-\sqrt{n}}$,
- (ii) There exists $m \geq m_0(n) := e^{\sqrt{n}/2}$ such that $|\{x \in \mathcal{B}_n : \bar{X} \geq m\}| \geq m^{-2}$.

In case (i) setting $\mathcal{A}_n := \{x \in \mathcal{B}_n : \bar{X} \geq -1\}$ we notice that conditioned to \tilde{X}_{t_1}

$$\int_{\mathcal{B}_n} e^{\gamma \bar{X}} \widetilde{M}^{(t_1)}(dx) \geq e^{-\gamma} \widetilde{M}^{(t_1)}(\mathcal{A}_n). \quad (6.21)$$

We can then use Proposition 6.2 to show that

$$\mathbb{P} \left[\widetilde{M}^{(t_1)}(\mathcal{A}_n) \leq \frac{|\mathcal{A}_n|}{2} \right] \leq e^{-e^{cn}}, \quad (6.22)$$

which is more than sufficient. Finally we need to show that the probability of being in case (ii) above is small. Note that \bar{X} has a variance of order n thus we have

$$\mathbb{E} [|\{x \in \mathcal{B}_n : \bar{X} \geq m\}|] \leq e^{-cm^2/n}. \quad (6.23)$$

And hence using Markov inequality (and eventually changing the value of c)

$$\mathbb{P} [\exists m \geq m_0(n), |\{x \in \mathcal{B}_n : \bar{X} \geq m\}| \geq m^{-2}] \leq \sum_{m \geq m_0} m^2 e^{-cm^2/n} \leq ce^{-ce^{\sqrt{n}}}. \quad (6.24)$$

Here as we have $n \geq n_0 = (-\ln s)^\kappa$, the right-hand side is smaller than $e^{-s - \frac{4}{\gamma^4}}$ provided $\kappa \geq 2$.

A Dirichlet GFF and Markov Property

Dirichlet GFF

We refer to [Dub, Section 4.2] for references concerning this subsection. Consider a bounded simply connected domain $D \subset \mathbb{C}$ equipped with a smooth (up to the boundary of D) conformal metric $g := e^{\omega(z)}|dz|^2$. Define the Sobolev space $H_0^1(D)$ as the closure of smooth functions with compact support in D , i.e. $C_0^\infty(D)$, with respect to the Dirichlet energy $\int |df|_g^2 dv_g$. This space does not depend on the particular choice of g by conformal invariance of the Dirichlet energy. The Dirichlet Green function G is the integral kernel for the mapping $f \in L^2(D, dv_g) \mapsto u \in H_0^1(D)$ defined by

$$-\Delta_g u = f, \quad u \in H_0^1(D),$$

that is

$$u(x) = \int_D G(x, y) f(y) v_g(dy).$$

Again the Dirichlet Green function does not depend on the choice of the metric g .

Denote by $H^{-1}(D)$ the dual space of $H_0^1(D)$ and by $\langle \cdot, \cdot \rangle_g$ the duality bracket obtained by extending the mapping $(f, f') \in C_0^\infty(D)^2 \mapsto \langle f, f' \rangle_g := \int_D f f' dv_g$ to $H^{-1}(D) \times H_0^1(D)$.

The Dirichlet Gaussian Free Field (Dirichlet GFF) X on D is a Gaussian random variable taking values in $H^{-1}(D)$ characterized by its mean and covariance kernel for test functions $f, f' \in H_0^1(D)$

$$\mathbb{E}[\langle X, f \rangle_g] = 0 \quad \text{and} \quad \mathbb{E}[\langle X, f \rangle_g \langle X, f' \rangle_g] = 2\pi \iint_{M^2} f(x) G(x, y) f'(y) v_g(dx) v_g(dy).$$

With these definitions the Dirichlet GFF does not depend on the metric g in the sense that

$$(\langle X, f \rangle_g)_{f \in H_0^1(D)} \stackrel{law}{=} (\langle X, f \rangle_{g_0})_{f \in H_0^1(D)},$$

where the index 0 stands for quantities evaluated in the Euclidean background metric $g_0 := |dz|^2$, i.e. with $\omega = 0$, and where (with a slight abuse of notations) we have identified in the above expression the metric g with the function e^ω . With another harmless abuse of notation, we use $\int X(z) f(z) d^2z$ for $\langle X, f \rangle_0$.

Notice that this definition of the Dirichlet GFF extends to simply connected domains of Riemannian manifolds using local charts and conformal invariance of the Dirichlet Green function.

Domain Markov property

Let M be a compact Riemannian manifold without boundary equipped with a metric g . Let D be some strict domain of M with a smooth Jordan curve \mathcal{C} as boundary and let μ be a probability measure of the form $\mu(dz) := e^{\theta(z)} \mathcal{H}(dz)$ for some continuous function θ and $\mathcal{H}(dz)$ is the one-dimensional Hausdorff measure of the volume form v_g restricted to \mathcal{C} . Let X_g be the GFF on M in the metric g . Then the field $\tilde{X}_g := X_g - \mu(X_g)$ is a centered Gaussian distribution with covariance kernel given by the Green function G_μ on M of the Laplacian Δ_g with zero average over \mathcal{C} in the μ measure. In particular it satisfies

$$-\Delta_g \int_M G_\mu(\cdot, y) f(y) v_g(dy) = f(\cdot) - \int_{\mathcal{C}} f(y) \mu(dy) \tag{A.1}$$

together with $\int_{\mathcal{C}} G_\mu(\cdot, y) \mu(dy) = 0$.

Consider the harmonic extension operator P defined by

$$\Delta_g P f = 0 \text{ in } D, \quad P f|_{\mathcal{C}} = f$$

and denote by p its integral kernel, i.e. $P(f)(x) = \int_{\mathcal{C}} p(x, z) f(z) \mu(dz)$. Denote by $P(\tilde{X}_g)$ the harmonic extension inside D of the boundary values of \tilde{X}_g restricted to \mathcal{C} . We claim

Proposition A.1. *The field $\tilde{X}_g - P(\tilde{X}_g)$ is a Dirichlet GFF inside D independent of $P(\tilde{X}_g)$.*

Proof. Indeed this is a simple consequence of the fact that the covariance kernel of the field $\tilde{X}_g - P(\tilde{X}_g)$ is given by

$$\tilde{G}(x, y) := G_\mu(x, y) - \int_{\mathcal{C}} p(x, z)G_\mu(z, y)\mu(dz) - \int_{\mathcal{C}} p(y, z)G_\mu(z, x)\mu(dz) + \iint_{\mathcal{C}^2} p(x, z)p(y, z')G_\mu(z, z')\mu(dz),$$

which is symmetric and satisfies

$$-\Delta_g \tilde{G}(\cdot, y) = \delta_y(\cdot) \quad \text{in } D, \quad \tilde{G}(\cdot, y)|_{\mathcal{C}} = 0. \quad (\text{A.2})$$

Hence \tilde{G} is the Dirichlet Green function inside D . Furthermore $G_\mu - \tilde{G}$ is harmonic in both variables inside D and coincides with G_μ when one of the variables x, y is sent to the boundary \mathcal{C} . Hence $G_\mu - \tilde{G}$ is equal to $\iint_{\mathcal{C}^2} p(x, z)p(y, z')G_\mu(z, z')\mu(dz)$, which is nothing but the covariance kernel of $P(\tilde{X}_g)$. \square

The important point to get the above decomposition of the field \tilde{X}_g is that the spatial average requirement is localized on the boundary \mathcal{C} , hence the boundary average in the right-hand side of (A.1) does not show up in the right-hand side of (A.2), hence allowing us to identify the Dirichlet Green function.

B Hitting time for Brownian motion

Consider a $2d$ Brownian motion B starting from $(0, y)$ and denote \mathcal{T}_A the hitting time of a set A by B . Set $\Delta := \{(x, x) : x \in \mathbb{R}\}$. We claim that for some $C > 0$ and all $|y| \leq 1/2$

$$\mathbf{P}_{(0, y)} [\mathcal{T}_\Delta > \mathcal{T}_{\{-1, 1\} \times \mathbb{R}}] \leq C|x - y|. \quad (\text{B.1})$$

We can assume that $y > 0$ without loss of generality. Consider the square of side-length $1/2$ with one side being a segment of Δ centered at the orthogonal projection of $(y, 0)$ on Δ : $(y/\sqrt{2}, y/\sqrt{2})$ and which contains $(0, y)$. For y sufficiently small this square does not touch $\{-1, 1\} \times \mathbb{R}$ and thus the probability is then bounded above by the probability that B exits the square by not using the side included in Δ . Now using the fact that the projections of B along the diagonal and in the orthogonal direction are independent this latter probability can be computed using standard estimates for the one dimensional Brownian Motion.

C Proof of Proposition 6.2

Let us reformulate the result. The concentration estimate can be deduced from a control of the Laplace transform of $\tilde{M}^t(\mathcal{A})$. We first prove a result for $M^{0, (t)}$ which is obtained by replacing $\tilde{X} - \tilde{X}_t$ by $X^0 - X_t^0$, where X^0 is the field associated with the special covariance function (recall (4.1)) $Q_u^0(x, y) = \rho(e^u(x - y))$ for some fixed positive continuous function ρ with $\rho(0) = 1$ and with support included in the ball of radius 1. Or more precisely we prove that for $p \in (1, 4/\gamma^2)$ all $r \leq e^{2t}$

$$\mathbb{E} \left[\exp \left(-r \left(M^{0, (t)}(\mathcal{A}) - |\mathcal{A}| \right) \right) \right] \leq \exp \left(c_p r^p e^{2t(1-p)} |\mathcal{A}| \right). \quad (\text{C.1})$$

From this result we can obtain some information on the Laplace transform of $\tilde{M}^t(\mathcal{A})$ using Kahane's convexity inequality. Setting $t' := t - \alpha \ln t + c$ (with c a constant and $\alpha > 1/v$, v being the exponent appearing in (4.3)) and $\varepsilon(t) = e^{-2t}$, we have

$$\int_t^\infty Q_u(x, y) du \leq \int_{t'}^\infty Q_u^0(x, y) du + \varepsilon(t), \quad (\text{C.2})$$

(the case where $|x - y| \leq e^{-t\alpha}$ can be handled using (4.2) which has to be valid on both sides and the other case is handled by (4.3)). Hence if Z is a standard Gaussian independent of X^0 we have

$$\begin{aligned} \mathbb{E} \left[\exp \left(-r \widetilde{M}^{(t)}(\mathcal{A}) \right) \right] &\leq \mathbb{E} \left[\exp \left(-r e^{\gamma \varepsilon(t)^{1/2} Z - \frac{\gamma^2 \varepsilon(t)}{2}} M^{0,(t')}(\mathcal{A}) \right) \right] \\ &\leq \mathbb{E} \left[\exp \left(-\frac{9r}{10} M^{0,(t')}(\mathcal{A}) \right) \right] + \mathbb{P} \left[Z \geq c\varepsilon(t)^{-1/2} \right]. \end{aligned} \quad (\text{C.3})$$

The first term can be estimated using (C.1) and the second one is of order $\exp(-ce^{-2t})$. Thus we obtain for $r \leq e^{2t'}$

$$\mathbb{E} \left[\exp \left(-r \widetilde{M}^{(t)}(\mathcal{A}) - |\mathcal{A}| \right) \right] \leq \exp \left(\frac{r}{10} |\mathcal{A}| + c_p r^p e^{2t(1-p)} |\mathcal{A}| \right) + \exp(r|\mathcal{A}| - ce^{-2t}). \quad (\text{C.4})$$

We deduce (using $|\mathcal{A}| \leq 1$)

$$\begin{aligned} \mathbb{P}(\widetilde{M}^{(t_0)}(\mathcal{A}) \leq |\mathcal{A}|/2) &\leq e^{-\frac{r}{2} |\mathcal{A}|} \mathbb{E} \left[\exp \left(-r \widetilde{M}^{(t_0)}(\mathcal{A}) - |\mathcal{A}| \right) \right] \\ &\leq \exp \left(-\frac{2r}{5} |\mathcal{A}| + c_p r^p e^{2t(1-p)} |\mathcal{A}| \right) + \exp(r|\mathcal{A}|/2 - ce^{-2t_0} |\mathcal{A}|), \end{aligned}$$

which applied to $r = \delta e^{2t'_0}$ for δ sufficiently small yields the result.

Now to prove (C.1) we can split \mathcal{A} according to the intersection with squares of side-length e^{-t} forming a partition of D . Let $(D_i^t)_{i \in \llbracket 1, 4q \rrbracket}$ with $q_t = e^{2t}/4$ be such squares (say we fix the value of t so that e^t is an even integer). Now we have

$$M^{0,(t)}(\mathcal{A}) - |\mathcal{A}| = \sum_{i=1}^{4q} M^{0,(t)}(\mathcal{A} \cap D_i^t) - |\mathcal{A} \cap D_i^t| =: \sum_{i=1}^{4q} Z_i. \quad (\text{C.5})$$

Now using independence of the field at distance t we can split the sum above in four groups of independent random variables. Reordering the indices we can assume that these four groups are $(1, \dots, q)$, $(q+1, \dots, 2q)$... We have from Jensen's inequality

$$\mathbb{E} \left[\exp \left(-r \left(M^{0,(t)}(\mathcal{A}) - |\mathcal{A}| \right) \right) \right] \leq \left(\prod_{j=1}^4 \mathbb{E} \left[\exp \left(-4r \sum_{i=1}^q Z_{qj+i} \right) \right] \right)^{1/4}. \quad (\text{C.6})$$

To conclude it is thus sufficient to control the exponential moment of $\sum_{i=1}^q Z_{qj+i}$ which is a sum of independent random variables. The important input is to be able to control the p moment of Z_i .

Lemma C.1. *Denote by M the GMC associated to the field X^0 . If \mathcal{B} is a subset of D then we have for every $1 < p < 4/\gamma^2$*

$$\mathbb{E} [(M(\mathcal{B}) - |\mathcal{B}|)^p] \leq C_p |\mathcal{B}|. \quad (\text{C.7})$$

Proof. It is sufficient to prove the bound for $\mathbb{E}[|M(\mathcal{B})|^p]$ and then use that $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$.

Now we have by Cameron-Martin

$$\mathbb{E} [M(\mathcal{B})^p] \leq \mathbb{E} [M(\mathcal{B})M(D)^{p-1}] = \int_{\mathcal{B}} \mathbb{E}_x [M(D)^{p-1}] dx, \quad (\text{C.8})$$

where \mathbb{P}_x denote tilted measure where $\gamma Q(x, \cdot)$ is added to X . One concludes by showing that $\mathbb{E}_x [M(D)^{p-1}]$ is uniformly bounded in x , which is done in [Da-Ku-Rh-Va, Section 3.4]. \square

Setting $\mathcal{A}_i := \mathcal{A} \cap D_i^{(t)}$, scaling (the field $X^0 - X_t^0$ has same law as $X^0(e^t \cdot)$) yields immediately

$$\mathbb{E}[|Z_i|^p] = e^{-2t(p-1)} \mathbb{E}[M(\mathcal{A}_i)^p] \leq C_p e^{-2t(p-1)} |\mathcal{A}_i|. \quad (\text{C.9})$$

Now we use the relation

$$e^x - x - 1 \leq c_p |x|^p$$

valid for all $x \in (-\infty, 4]$. Then for $r \leq e^{2t}$ (using the fact that $Z_i \geq -|\mathcal{A}_i| \geq -e^{-2t}$), we have

$$\mathbb{E} \left[\exp \left(-4r \sum_{i=1}^q Z_{qj+i} \right) \right] \leq \exp \left(c_p r^p \sum_{i=1}^q \mathbb{E}[|Z_i|^p] \right), \quad (\text{C.10})$$

Using (C.9) this yields (C.1)

$$\mathbb{E} \left[\exp \left(-r(M^{0,(t)}(\mathcal{A}) - |\mathcal{A}|) \right) \right] \leq \exp \left(c_p r^p e^{-2t(p-1)} |\mathcal{A}| \right). \quad (\text{C.11})$$

D Motivations: $2d$ quantum gravity off conformal invariance?

We stress from the very beginning that this section is entirely speculative or conjectural from a mathematical perspective. Our motivations for constructing a path integral for the Mabuchi K-energy finds its roots in a deeper understanding of $2d$ Euclidean quantum gravity. We adopt a point of view advocated by A. Bilal, F. Ferrari, S. Klevtsov and S. Zelditch in a series of works [Bi-Fe-Kl, Fe-Kl-Ze, Fe-Kl-Ze2], which we recast in terms of the scaling limit of Random Planar Maps (RPM) in order to be more adapted to a probabilistic readership.

Recall that RPM have been introduced as a way of discretizing $2d$ -quantum gravity. For that one usually considers planar lattices that can be embedded onto a compact Riemann surface M (without boundary), which we require to be of genus $\mathbf{h} \geq 2$. To fix the ideas, we consider finite triangulations of M as our lattices. So let \mathcal{T}_N be the (finite) set of triangulations of M with N faces (up to orientation preserving homeomorphisms). Now we need to embed conformally such lattices onto M . There is a subtlety here: there are infinitely many (non diffeomorphic) conformal structures on M as its genus is higher than 2. We wish to get rid of this degree of freedom. Let us recall how it goes.

Let $\text{Met}(M)$ be the space of Riemannian metrics on M . Two metrics $g, g' \in \text{Met}(M)$ are said equivalent if

$$g' = \psi^*(e^\omega g)$$

where $\omega \in C^\infty(M)$ and ψ is a diffeomorphism (ψ^* is the pushforward). Let \mathcal{M} be the set of equivalence classes, called the moduli space. Uniformization theorem tells us that each equivalence class contains a metric g with uniformized scalar curvature $K_g = -2$ (called hyperbolic metric). So we consider a fixed family $(g_\tau)_{\tau \in \mathcal{M}}$ of hyperbolic metrics on M parameterized by moduli $\tau \in \mathcal{M}$.

The procedure to embed triangulations conformally onto M is now the following. Each $T \in \mathcal{T}_N$ can be equipped with a metric structure h_T , where each triangle is given volume $1/N$. The metric structure consists in gluing flat equilateral triangles: the exact definition of the metric structure is given in the lecture notes [Rh-Va1] in the case of the sphere and the case we consider here does not present additional difficulties. Then the uniformization theorem tells us that for each T there exists a unique $\tau_T \in \mathcal{M}$ (τ_T is called the modulus of T) along with an orientation preserving diffeomorphism $\psi_T : T \rightarrow M$ and a conformal factor $\omega_T \in C^\infty(M)$ such that ¹²

$$h_T = \psi_T^*(e^{\omega_T} g_{\tau_T}). \quad (\text{D.1})$$

In what follows, we wish to work with triangulations with fixed modulus so we introduce $\mathcal{T}_{N,\tau}$ the set of triangulations of M with N faces and modulus τ , namely those T such that $\tau_T = \tau$.

¹²Recall that in the decomposition (D.1), the functions ω_T and ψ_T are unique except if the metric g_{τ_T} possesses non trivial isometries. This situation is rather unlikely to happen as the set of such moduli has measure 0 with respect to a natural measure on \mathcal{M} called the Weil-Petersson volume form. We could pursue our discussion while including these special moduli but, for simplicity, we exclude this situation by restricting to those τ such that g_τ has trivial isometry group.

Now we explain how to couple quantum gravity to matter fields which stand for models of statistical physics, whose partition function denoted $\mathcal{Z}(T)$ can be defined on each triangulation $T \in \mathcal{T}_N$, hence every $\mathcal{T}_{N,\tau}$. Call $\mathcal{Z}_{N,\tau}$ the partition function of the matter field on triangulations of size N

$$\mathcal{Z}_{N,\tau} = \sum_{T \in \mathcal{T}_{N,\tau}} \mathcal{Z}(T). \quad (\text{D.2})$$

The main point is to determine the scaling limit of the random geometry on M induced by this model as $N \rightarrow \infty$. Random geometry is understood in the following sense (further details can be found in [Da-Ku-Rh-Va]): given N , we can pick a $T \in \mathcal{T}_{N,\tau}$ at random by defining a probability law

$$\mathbb{P}_{N,\tau}(T) := \frac{\mathcal{Z}(T)}{\mathcal{Z}_{N,\tau}}. \quad (\text{D.3})$$

It induces via (D.1) a random function ω_T on M and the question is to find what is the limiting law for such a random function sampled according to $\mathbb{P}_{N,\tau}(T)$ as $N \rightarrow \infty$. The answer is in most cases unknown (even heuristically) as it depends crucially on the choice of the matter field.

Yet, starting from the eighties with the seminal work [Pol], physicists have designed a toolbox to guess what the limiting law should be. It is based on the way the matter field reacts to local changes of geometries. We base the following discussion on the situation when the partition function $\mathcal{Z}(T) := \mathcal{Z}(e^{\omega_T} g_{\tau_T})$ for matter fields is expressed in terms of regularized determinant of Laplacian and Gaussian integrals. For $g \in \text{Met}(M)$, consider the formal Gaussian integral

$$\mathcal{Z}_{q,m}(g) := \int_{\Sigma} e^{-\frac{1}{4\pi} \int_M (|dX|_g^2 + iqK_g X + m^2 X^2)} \text{d}v_g DX \quad (\text{D.4})$$

where $q \in \mathbb{R}$ is a parameter called background charge m is a mass parameter, Σ is a functional space of maps $X : M \rightarrow \mathbb{R}$ and DX stands for the formal Lebesgue measure on Σ . In the case $q = m = 0$, this integral is ill-defined because of the zero-mode divergence (the null eigenvalue of Laplacian) but can be given sense by ζ -regularization [Ray-Sin]: it results that

$$\mathcal{Z}_{q=0,m=0}(g) := \left(\frac{\det'(-\Delta_g)}{V_g} \right)^{-\mathbf{c}_{\text{mat}}/2} \quad (\text{D.5})$$

where $\det'(-\Delta_g)$ has been defined in subsection 2.1 and $\mathbf{c}_{\text{mat}} = 1$. This is the usual interpretation of the partition function of Gaussian Free Field. More generally, when $\mathbf{c}_{\text{mat}} \leq 1$, the right-hand side of (D.5) can be considered as a relevant partition function for matter fields (for $\mathbf{c}_{\text{mat}} = -0$, it corresponds to pure gravity and for $\mathbf{c}_{\text{mat}} = -2$ to uniform spanning trees ...). Such an expression has an interesting metric dependence called *gravitational anomaly*: if \hat{g} is another metric conformal to g then (2.11) entails

$$\ln \frac{\mathcal{Z}_{\text{mat}}(\hat{g})}{\mathcal{Z}_{\text{mat}}(g)} = \frac{\mathbf{c}_{\text{mat}}}{96\pi} S_{\text{L}}^{\text{cl},0}(\hat{g}, g). \quad (\text{D.6})$$

For symmetry reasons¹³, the limiting law of the random function ω_T must "balance"¹⁴ this gravitational anomaly. Liouville CFT (1.4) is then expected to describe the limiting law of ω_T as it is the only "reasonable QFT" able to counterbalance Liouville type gravitational anomalies, the parameter γ in (1.5) being tuned in terms of \mathbf{c}_{mat} through the famous central charge balance¹⁵

$$1 + 6Q^2 - 26 + \mathbf{c}_{\text{mat}} = 0.$$

¹³Called background independence.

¹⁴Here we are voluntarily vague to keep the discussion reasonably short. Further explanations can be found for instance in [Bi-Fe-Kl].

¹⁵Recall that $1 + 6Q^2$ is the central charge of Liouville CFT. The ghost contribution is -26 , see [Gui-Rh-Va] for a brief mathematical description of the ghost system and other references.

This ansatz has been successfully applied to the coupling of quantum gravity with matter fields satisfying (D.7), hence CFTs (e.g. Ising model, lattice Gaussian Free-Field, the $O(N)$ dilute and dense loop models with $0 \leq N < 2$, etc... we refer to [Kos] for a review and references).

Very little is known about models of 2d quantum gravity beyond this CFT framework. Yet, the study of gravitational anomalies provides serious hints about the type of path integrals ruling the limiting law of the random function ω_T . The simplest non conformal QFT is probably the massive Gaussian Free Field (D.4). Computing the gravitational anomaly for this model is unclear at fixed mass $m > 0$ but it can be understood perturbatively in the limit $m \rightarrow 0$ [Fe-Kl-Ze]. The reader should keep in mind that there is a serious discontinuity when $m \rightarrow 0$ in the path integral (D.4). We propose here to focus on a simplified model arising as a suitable $m \rightarrow 0$ limit of the path integral (D.4)

$$\mathcal{Z}_{q,m=0^+}(g) := \int_{\Sigma} e^{-\frac{1}{4\pi} \int_M (|dX|_g^2 + iq(K_g - \bar{K}_g)X)} dv_g DX$$

where \bar{K}_g stands for mean curvature $\bar{K}_g := \frac{1}{V_g} \int_M K_g dv_g$. This expression can be rewritten in terms of GFF expectation

$$\mathcal{Z}_{q,m=0^+}(g) := \left(\frac{\det'(-\Delta_g)}{V_g} \right)^{-1/2} \mathbb{E}[e^{-\frac{iq}{4\pi} \int_M K_g X_g dv_g}].$$

By using a Girsanov transform, this quantity can then be expressed in terms of the Polyakov action

$$\mathcal{P}(g) := \iint_{M^2} K_g(x) G_g(x, y) K_g(y) v_g(dx) v_g(dy).$$

Indeed we have

$$\mathcal{Z}_{q,m=0^+}(g) = \left(\frac{\det'(-\Delta_g)}{V_g} \right)^{-1/2} \exp\left(-\frac{6q^2}{96\pi} \mathcal{P}(g)\right).$$

Thanks to this expression, one can prove that the gravitational anomaly exhibits a Mabuchi K-energy term¹⁶: if \hat{g} is another metric conformal to g then

$$\ln \frac{\mathcal{Z}_{q,m=0^+}(\hat{g})}{\mathcal{Z}_{q,m=0^+}(g)} = \frac{1 - 6q^2}{96\pi} S_L^{\text{cl},0}(\hat{g}, g) + \frac{q^2(1 - \mathbf{h})}{4\pi} S_M^{\text{cl}}(\hat{g}, g). \quad (\text{D.7})$$

This suggests the following conjecture: sample a triangulation of size N according to the probability law (D.3) with $\mathcal{Z}(T) := \mathcal{Z}_{q,m=0^+}(e^{\omega_T} g_{T\tau})$. In the scaling limit as $N \rightarrow \infty$ the law of the random function ω_T is described by the path integral (1.11) in the background metric g_τ , conditioned on having volume 1 (see subsection 3.1), with parameters

$$\gamma = (4 + q^2)^{1/2} - q \quad \text{and} \quad \beta = \frac{q^2(\mathbf{h} - 1)}{4\pi}.$$

In conclusion, (1.11) thus appears as a candidate for modeling fluctuating metrics in quantum gravity coupled to matter field slightly non conformally invariant. Let us also mention that quantizing Mabuchi K-energy also appears as a collective field theory for Dyson gas [Wieg] or Laughlin states in Quantum Hall Effect [La-Ca-Wi] but a precise relation with our path integral is still unclear for us.

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¹⁶This computation is a particular case of the results in [Fe-Kl-Ze] but a direct check can also be carried out.

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