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► **To cite this version:**

Raymond Mortini, Rudolf Rupp. The Riemann-Carathéodory result implies Borsuk's retract theorem. *Topology and its Applications*, 2017, 232 (1), pp.29-33. <10.1016/j.topol.2017.09.037>. <hal-02114242>

HAL Id: hal-02114242

<https://hal.science/hal-02114242v1>

Submitted on 29 Apr 2019

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THE RIEMANN-CARATHÉODORY RESULT IMPLIES BORSUK'S RETRACT THEOREM

RAYMOND MORTINI AND RUDOLF RUPP

ABSTRACT. We show that the Riemann-Carathéodory theorem on continuous extensions of conformal maps easily implies Borsuk's theorem characterizing those compacta K in the plane that are retracts.

25.9.2017

1. INTRODUCTION

One of the first major theorems one encounters in an introductory course in topology is Tietze's extension theorem: let X be a T_4 space (that is a topological space such that each pair of disjoint closed sets can be separated by open sets) and let A be a closed subset of X . Then every continuous function $f : A \rightarrow [-1, 1]$ admits a continuous extension to X with the same target space.

In case of complex-valued functions, one easily sees that one can replace $[-1, 1]$ by the closed unit disk $\mathbf{D} = \{z \in \mathbb{C} : |z| \leq 1\}$: just consider the function

$$\frac{\operatorname{Re} f + i \operatorname{Im} f}{\max\{|f|, 1\}}.$$

Here we are interested in a simple proof of Borsuk's Theorem [1] giving a large class of compact sets $K \subseteq \mathbb{C}$ that allow for every T_4 -space these continuous extensions with target space K . Using the notion of a *retract* (see below), one can actually restrict to the space $X = \mathbb{C}$:

Definition 1.1. A non-void compact set $K \subseteq \mathbb{R}^n$ is called a *retract* (of \mathbb{R}^n), if the identity function on K has a continuous extension I_K to \mathbb{R}^n with $I_K(\mathbb{R}^n) = K$. I_K is called a *retraction map* of \mathbb{R}^n onto K (or a retraction map associated with K).

Proposition 1.2 (well-known). *Let X be a T_4 -space, $A \subseteq X$ closed, $K \subseteq \mathbb{R}^n$ a retract, and $f : A \rightarrow K$ continuous. Then f admits a continuous extension F to X with $F(X) \subseteq K$.*

Proof. Choose $r > 0$ such that $K \subseteq [-r, r]^n$. By the vector-valued Tietze theorem, f admits a continuous extension to some map $f^* : X \rightarrow [-r, r]^n$. Since K is assumed to be a retract, there is a retraction map $I_K : \mathbb{R}^n \rightarrow K$ of \mathbb{R}^n onto K . Hence, $F = I_K \circ f^*$ is the desired extension of f . \square

2010 *Mathematics Subject Classification.* Primary 30C99 Secondary 54D05; 54C15; 54C55.

Key words and phrases. conformal maps; continuous extensions; planar retracts; Tietze's extension theorem; Borsuk's extension theorem.

The proof of Borsuk's result in [1] is rather difficult. We present a simple function theoretic proof based on Riemann's mapping theorem for conformal maps ([6, p. 283]) and Carathéodory's refinement:

Theorem 1.3. *Let $f : \mathbb{D} \rightarrow G$ be a conformal map of the unit disk \mathbb{D} onto the bounded domain $G \subseteq \mathbb{C}$. Then the following assertions are equivalent:*

- (1) f has a continuous extension to $\overline{\mathbb{D}}$;
- (2) ∂G is a curve; that is $\partial G = \{\phi(\xi) : \xi \in \partial\mathbb{D}\}$ with continuous ϕ ;
- (3) ∂G is locally connected;
- (4) $\mathbb{C} \setminus G$ is locally connected,

where, as usual, a topological space X is *locally connected* if for every $x \in X$ and every neighborhood $N(x) \subseteq X$ of x there exists a connected open set C with $x \in C \subseteq N(x)$. A nice, easy readable proof of Theorem 1.3 can be found in Pommerenke's book [5]. Here is now Borsuk's extension theorem:

Theorem 1.4 (Borsuk). *Let X be a T_4 -space and let $A \subseteq X$ be a closed subset. Suppose that $K \subseteq \mathbb{C}$ is a compact set satisfying the following three properties:*

- (1) K is connected;
- (2) K is locally connected;
- (3) $\mathbb{C} \setminus K$ is connected.

Then every continuous function $f : A \rightarrow K$ admits a continuous extension $F : X \rightarrow K$.

Theorem 1.4 shows in particular that any planar compact set K satisfying conditions (1)-(3) is a retract (just put $A = K$, $X = \mathbb{C}$ and f the identity function on K). On the other hand, it is easy to see that any retract K in \mathbb{R}^n satisfies properties (1)-(3), where property (3) is a consequence of Brouwer's fix-point theorem (see [2] or [3] for the planar case). Thus, we have a complete description of the planar retracts:

Theorem 1.5 (Borsuk). *A compact subset K of the plane is a retract if and only if K is connected, locally connected and polynomially convex.*

It is worth pointing out that these three conditions are far from being sufficient for K to be a retract in \mathbb{R}^n whenever $n \geq 3$. Indeed we have the following example.

Example 1.6. The set $K := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z = 0\}$ is not a retract in \mathbb{R}^3 , although it satisfies i)-iii) above.

Proof. Suppose that K is a retract for \mathbb{R}^3 . Let $R = (R_1, R_2, R_3) : \mathbb{R}^3 \rightarrow K$ be a retraction map. Then $R_3 \equiv 0$ on \mathbb{R}^3 . Now, if $(x_0, y_0, z_0) \in K$, then $z_0 = 0$, and so $r(x, y) := (R_1(x, y, 0), R_2(x, y, 0))$ is a retraction map of \mathbb{R}^2 onto the unit circle C in \mathbb{R}^2 . But $\mathbb{R}^2 \setminus C$ is not connected. A contradiction to the remarks above. \square

For our proof of Theorem 1.4, we need the following Lemma, which is also well-known. The proof is only presented for completeness.

Lemma 1.7. *Let K_1, K_2 be two compact subsets of \mathbb{R}^n such that*

- (1) $K_1 \cap K_2$ is a retract;
- (2) $K_1 \cup K_2$ is a retract.

Then the K_j are both retracts.

Proof. We show that K_1 is a retract. For $x \in K_1$, let $f(x) = x$. Since $K_1 \cap K_2$ is a retract, there is a continuous map $I_{K_1 \cap K_2}$ on \mathbb{R}^n extending $f|_{K_1 \cap K_2}$ and with range space $K_1 \cap K_2$. Let $f_2 : K_1 \cup K_2 \rightarrow K_1$ be defined by $f_2(x) = I_{K_1 \cap K_2}(x)$ if $x \in K_2$ and $f_2(x) = x$ if $x \in K_1$. Then f_2 is well-defined and hence continuous. Since $K_1 \cup K_2$ is a retract, by Proposition 1.2, there exists on \mathbb{R}^n a continuous extension f_3 of f_2 with $f_3(\mathbb{R}^n) \subseteq K_1 \cup K_2$. Then $F = f_2 \circ f_3$ is a continuous map on \mathbb{R}^n that extends the identity on K_1 and has values in K_1 . \square

2. THE PROOF OF THEOREM 1.4

Proof. Let $\widehat{\mathbb{C}}$ be the one-point compactification of \mathbb{C} . Due to the fact that $\widehat{\mathbb{C}} \setminus K$ is connected, the outer boundary of K (this is, via definition, the set of all those boundary points of K that belong to the boundary of the unbounded component of $\mathbb{C} \setminus K$) coincides with ∂K .

Case 1 $K^\circ \neq \emptyset$.

Let $\Omega := \widehat{\mathbb{C}} \setminus K$. We first show that every Riemann map

$$R : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \rightarrow \widehat{\mathbb{C}} \setminus K$$

with $R(\infty) = \infty$ admits a continuous extension to the boundary \mathbb{T} of \mathbb{D} .

To see this, let $a \in K^\circ$ and consider for $w \in \Omega$ the map $f_1(w) = \frac{1}{w-a}$. Since $a \in K^\circ$, we see that $\delta := \inf_{w \in \Omega} |w - a| > 0$. Hence the image G of Ω under f_1 is a bounded, simply connected domain containing the origin. Moreover, since f_1 is a homeomorphism of $\widehat{\mathbb{C}}$, the assumptions on K imply that $f_1(K) \cap \mathbb{C}$ is a locally connected closed subspace of \mathbb{C} and coincides with $\mathbb{C} \setminus G$. Now let $\psi : \mathbb{D} \rightarrow \mathbb{C}$ be defined by $\psi(z) := (f_1 \circ R)(1/z)$ if $0 < |z| < 1$ and $\psi(0) = 0$. Then ψ is a conformal map of \mathbb{D} onto $G = f_1(\Omega)$. By Theorem 1.3, ψ admits a continuous extension to \mathbb{T} , also denoted by ψ . Thus we conclude that for $z \in \widehat{\mathbb{C}} \setminus \mathbb{D}$

$$R(z) = (f_1^{-1} \circ \psi)(1/z)$$

is continuous on $\{|z| \geq 1\}$. Moreover $R(\mathbb{T}) = \partial\Omega = \partial K$. Now we define the retraction of \mathbb{C} onto K by

$$f(w) = \begin{cases} R\left(\frac{R^{-1}(w)}{|R^{-1}(w)|}\right) & \text{if } w \in \mathbb{C} \setminus K \\ w & \text{if } w \in K. \end{cases}$$

It remains to show that f is continuous on ∂K (the continuity outside ∂K is obvious). Let $\eta_0 \in \partial K$. Choose a sequence (w_n) outside K converging to η_0 . Under R , let z_n be the unique preimage of w_n in $\mathbb{C} \setminus \overline{\mathbb{D}}$. Note that the sequence (z_n) does not converge in general, since R is not necessarily injective on the boundary. However, it is clear that $|z_n| \rightarrow 1$. Now

$$(2.1) \quad f(w_n) = R\left(\frac{R^{-1}(w_n)}{|R^{-1}(w_n)|}\right)$$

$$(2.2) \quad = R\left(\frac{z_n}{|z_n|}\right).$$

Hence, if (z_{n_j}) converges to some $\alpha \in \mathbb{T}$, we have $z_{n_j}/|z_{n_j}| \rightarrow \alpha$, too, and so $R(\frac{z_{n_j}}{|z_{n_j}|}) \rightarrow R(\alpha)$. On the other hand, $R(z_{n_j}) = w_{n_j} \rightarrow \eta_0 = f(\eta_0)$. Thus, for each cluster point α of (z_n) we obtain the same limit $R(\alpha) = f(\eta_0)$ and so $f(w_n) \rightarrow f(\eta_0)$. Thus f is continuous at η_0 by the gluing Lemma.

To sum up, we have shown that K is a retract if $K^\circ \neq \emptyset$.

Case 2 $K^\circ = \emptyset$.

Let $z_0 \in K$ be a point in K with maximal real part. Now we add to K a disk D such that $K \cap D = \{z_0\}$; for example

$$D = \{w \in \mathbb{C} : |w - (z_0 + 1)| \leq 1\}.$$

Then $K' = K \cup D$ is connected, locally connected and has connected complement. Since the interior of K' is not empty, K' is a retract by the first case. Now $D \cap K = \{z_0\}$ is a retract. Hence, by Lemma 1.7, K is a retract, too. Now it remains to apply Proposition 1.2 to conclude. \square

We note that there exist many topological T_4 -spaces, X , that have the extension property $(A, K) \rightarrow (X, K)$ above, but for which the target space K , $K \subseteq \mathbb{C}$, is not necessarily a retract. For example, let X be discrete (that is every singleton is an open set). Then every function $f : A \rightarrow K$, $A \subseteq X$, is continuous for every compact set K and admits a continuous extension F to X with $F(X) \subseteq K$. Here is a more natural example, which has been communicated to us by Edward Tymchatyn: let $X = [0, 1]$ be endowed with the usual Euclidean topology and $K = \mathbb{T} = S^1$ the unit circle. Since the topological (=covering) dimension of X is one, every continuous map $f : A \subseteq X \rightarrow S^1$, A closed, admits an extension to a map $F \in C(X, S^1)$ (see [4]). The unit circle S^1 , though, is not a retract.

Question 2.1. Let $K \subseteq \mathbb{C}$ be compact and let X be a T_4 space. Suppose that for every closed subset A of X and for every continuous map $f : A \rightarrow K$ there is a continuous extension $F : X \rightarrow K$ of f with the same target space K . Under which conditions on X , the planar compactum K necessarily is a retract (that is, in view of Theorem 1.5, a connected, locally connected, polynomially convex set)?

Let \mathcal{C} be that class of spaces X for which K necessarily is a retract when extension is possible. Then, for example, $\mathbb{C} \in \mathcal{C}$, but neither $[0, 1]$ nor any discrete space X .

Acknowledgements We thank Edward Tymchatyn for valuable discussions on Question 2.1.

REFERENCES

- [1] K. Borsuk. *Theory of retracts*, Monografie Matematyczne. 44. Warszawa: PWN - Polish Scientific Publishers, 1967. 1, 2
- [2] R. Engelking, K. Sieklucki. *Topology, a geometric approach*, Heldermann Verlag, Berlin, 1992. 2
- [3] R. Mortini, Einige Anmerkungen zum Fortsetzungssatz von Tietze, Elem. Math. 60 (2005), 150–153. 2
- [4] A.R. Pears, *Dimension theory of general spaces*, Cambridge Univ. Press London, 1975. 4
- [5] C. Pommerenke. *Boundary behaviour of conformal maps*, Springer, Berlin, 1992. 2
- [6] W. Rudin, *Real and Complex Analysis*, third edition, McGraw-Hill, New York, 1986 2

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