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# RANDOM INTERPOLATING SEQUENCES IN DIRICHLET SPACES

N. CHALMOUKIS, A. HARTMANN, K. KELLAY, AND B. D. WICK

ABSTRACT. We discuss random interpolating sequences in weighted Dirichlet spaces  $\mathcal{D}_\alpha$ ,  $0 \leq \alpha \leq 1$ . Our results in particular imply that almost sure interpolating sequences for  $\mathcal{D}_\alpha$  are exactly the almost sure separated sequences when  $0 \leq \alpha < 1/2$  (which covers the Hardy space  $H^2 = \mathcal{D}_0$ ), and they are exactly the almost sure zero sequences for  $\mathcal{D}_\alpha$  when  $1/2 < \alpha < 1$ . We show that this last result remains valid in the classical Dirichlet space  $\mathcal{D} = \mathcal{D}_1$  when one considers a weaker notion of interpolation, so-called simple interpolation. As a by-product we improve a sufficient condition by Rudowicz for random Carleson measures in Hardy spaces.

## 1. INTRODUCTION

Understanding interpolating sequences is an important problem in complex analysis in one and several variables. The characterization of when a sequence of points is an interpolating sequence finds many applications to different problems in signal theory, control theory, operator theory, etc. In classical spaces like Hardy, Fock and Bergman spaces, interpolating sequences are now well understood objects, at least in one variable [16, 25, 27]. In Dirichlet spaces, it turns however out that getting an exploitable description of such interpolating sequences is a notoriously difficult problem related to capacities. Crucial work has been undertaken in the 90s by Bishop and Marshall-Sundberg (see more precise indications below). However, while easier checkable sufficient conditions were given by Seip in the meantime, no real progress

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in the understanding of these sequences has been made since those works. In such a situation, a probabilistic approach can lead to a new vision of these interpolating sequences. Note that besides the Hardy and Bergman spaces, the Dirichlet space, and its weighted companions, are beyond the most prominent spaces of analytic functions on the unit disk. They appear naturally in problems on classical function theory, potential theory, as well as in operator theory when one investigates for instance weighted shifts.

Here we consider random sequences of the following kind. Let  $\Lambda(\omega) = \{\lambda_n\}$  with  $\lambda_n = \rho_n e^{i\theta_n(\omega)}$  where  $\theta_n(\omega)$  is a sequence of independent random variables, all uniformly distributed on  $[0, 2\pi]$  (Steinhaus sequence), and  $\rho_n \in [0, 1)$  is a sequence of *a priori* fixed radii. Depending on distribution conditions on  $(\rho_n)$  as will be discussed below, we ask about the probability that  $\Lambda(\omega)$  is interpolating for Dirichlet spaces  $\mathcal{D}_\alpha$ ,  $0 \leq \alpha \leq 1$ . Recall that the weighted Dirichlet space  $\mathcal{D}_\alpha$ ,  $0 \leq \alpha \leq 1$ , is the space of all analytic function  $f$  on the unit disc  $\mathbb{D}$  such that

$$\|f\|_\alpha^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^{1-\alpha} dA(z) < \infty,$$

where  $dA(z) = dx dy / \pi$  stands for the normalized area measure on  $\mathbb{D}$  (we refer to [15] for Dirichlet spaces). If  $\alpha = 0$ ,  $\mathcal{D}_0$  is the Hardy space  $H^2$ , and the classical Dirichlet space  $\mathcal{D}$  corresponds to  $\alpha = 1$ .

Recall that in a Hilbert space  $H$  of functions analytic in the unit disk  $\mathbb{D}$  equipped with a reproducing kernel  $k_\lambda$ , i.e.  $f(\lambda) = \langle f, k_\lambda \rangle_H$  for every  $\lambda \in \mathbb{D}$  and  $f \in H$  (a so-called reproducing kernel Hilbert space), a sequence  $\Lambda$  of distinct points in  $\mathbb{D}$  is called (universal) interpolating if  $\{(f(\lambda) / \|k_\lambda\|_H)_{\lambda \in \Lambda} : f \in H\} = \ell^2$  (for the difference between interpolating and universal interpolating sequences see below). Concerning the deterministic case of interpolation in the classical Dirichlet space  $\mathcal{D}$ , in unpublished work Bishop [7] and, independently, Marshall-Sundberg [17] characterized the interpolating sequences. The first published proof was given by Bøe [10] who provides a unifying scheme that applies to spaces that satisfy a certain property related to the so-called Pick property (see [2, 25]), and Dirichlet spaces fall in this category. For these spaces  $\Lambda$  is a (universal) interpolating sequence if and only if  $\Lambda$  is  $H$ -separated (i.e.  $\sup_{\lambda \neq \lambda^* \in \Lambda} |\langle k_\lambda / \|k_\lambda\|_H, k_{\lambda^*} / \|k_{\lambda^*}\|_H \rangle| < \delta_\Lambda < 1$ ) and  $\mu = \sum_{\lambda \in \Lambda} \delta_\lambda / \|k_\lambda\|_H^2$  is a Carleson measure for  $H$  (i.e.  $\int_{\mathbb{D}} |f|^2 d\mu \leq C_\Lambda \|f\|_H^2$ ). Recently, Aleman, Hartz, McCarthy and Richter [1] have shown that this characterization remains valid in arbitrary reproducing kernel Hilbert spaces satisfying the complete Pick property. Stegenga [26] characterized Carleson measures for Dirichlet spaces, but

this characterization is based on capacities which are notoriously difficult to estimate for arbitrary unions of intervals. There are other characterizations of Carleson measures in Dirichlet spaces, see [3, 4], as well as [6, 15] and references therein, but which are not easily interpreted geometrically for interpolating sequences. Finally, we mention related work by Cohn [14] based on multipliers.

In [25], Seip gave simple sufficient geometric conditions on a sequence to be (universal) interpolating for the Dirichlet spaces see Theorems 3.3 and 5.1, which, surprisingly, will allow us to obtain sharp results for random interpolating sequences in  $\mathcal{D}_\alpha$  for  $\alpha \in (0, 1)$ ,  $\alpha \neq 1/2$ . For  $\alpha = 1$ , the result by Bishop will give us the sharp result at least for simple interpolating sequences in  $\mathcal{D}$ .

We also would like to observe that more generally, when the deterministic frame does not give a full answer to a problem, or if the deterministic conditions are not so easy to check, it is interesting to look at the random situation. In particular, it is interesting to ask for conditions ensuring that a sequence picked at random is interpolating almost surely or not (i.e., which are in a sense “generic situations”?). In this context, it is also worth mentioning the huge existing literature around gaussian analytic functions which investigates the zero distribution in classes of such functions [22].

The problems we would like to study in this paper are inspired by results by Cochran [12] and Rudowicz [23] who considered random interpolation in the Hardy space. Since interpolation in this space is characterized by separation (in the pseudohyperbolic metric) and by the Carleson measure condition (note that the Hardy space was the pioneering space with a kernel satisfying the complete Pick property), those authors were interested in a 0-1 law for separation, see [12], and a condition for being almost surely a Carleson measure [23], which led to a 0-1 law for interpolation. It is thus natural to discuss separation, Carleson measure type conditions and interpolation in Dirichlet spaces.

Concerning separation in Dirichlet spaces  $\mathcal{D}_\alpha$ ,  $0 < \alpha < 1$ , this turns out to be the same as in the Hardy spaces (see [25, p.22]), so that in that case Cochran’s result perfectly characterizes the situation. The separation in the classical Dirichlet space, however, is much more delicate than in the Hardy space. We establish here a 0-1-type law for separation in  $\mathcal{D}$ . While our proof of this fact is inspired by Cochran’s ideas, our proof requires a careful adaptation to the metric in that space.

Concerning Carleson measure type results in Dirichlet spaces,  $0 < \alpha < 1$ , we will first discuss the situation in the Hardy space and improve

Rudowicz' result simplifying his proof. Our new proof carries over to the Dirichlet situation and allows, together with a 1-box condition by Seip (which requires itself separation), to discuss the results on interpolation in  $\mathcal{D}_1 = \mathcal{D}$ . For the spaces  $\mathcal{D}_\alpha$ ,  $0 < \alpha < 1$  we will present a different approach. As it turns out, we are able to exhibit a peculiar breakpoint in the behaviour of such interpolating sequences depending on the weight  $\alpha$ : for  $0 \leq \alpha < 1/2$ , almost sure separation corresponds to almost sure interpolation, while for  $1/2 < \alpha < 1$ , almost sure zero sequences correspond to almost sure interpolating sequences. Partial results are given also for  $\alpha = 1/2$ .

The natural endpoint  $\alpha = 1$  of the scale of Dirichlet spaces under consideration here follows more or less the same scheme as above, but requires to work with a weaker notion of interpolation if one asks for an if and only if statement. More precisely, in the classical Dirichlet space  $\mathcal{D}$  we are able to show that, almost surely, a sequence is *simple* interpolating (see definition below) if and only if it is almost surely a zero sequence for  $\mathcal{D}$ . For universal interpolation (see definition below) we still obtain an almost optimal result. We insist that the condition for almost sure separation in  $\mathcal{D}$  is much weaker than the Carleson-measure type condition so that we cannot hope for optimality from the separation (as was the case in the Hardy space). It can already be seen from our improvement of the sufficient condition for Carleson measures in Hardy spaces, that the Carleson measure condition is not sufficient to obtain a 0-1 law in  $H^2$ .

Since zero sequences are of some importance as we have just seen, another central ingredient of our discussion is a rather immediate adaptation of Bogdan's result on almost sure zero sequence in the Dirichlet space to the case of weighted Dirichlet spaces which we add for completeness in an annex.

As usual, the definition of interpolating sequences is based on the reproducing kernel of  $\mathcal{D}_\alpha$ :

$$(1.1) \quad k_z(w) = \begin{cases} \frac{1}{\bar{z}w} \log \frac{1}{1 - \bar{z}w} & \text{if } \alpha = 1, \\ \frac{1}{(1 - \bar{z}w)^{1-\alpha}}, & \text{if } 0 \leq \alpha < 1. \end{cases}$$

Contrarily to the Hardy space situation, it turns out that in certain spaces (e.g. the Dirichlet space) there exist two notions of interpolation depending on whether the restriction operator  $R_\Lambda : H \rightarrow \ell^2$ ,  $R_\Lambda f = (f(\lambda)/\|k_\lambda\|_H)_{\lambda \in \Lambda}$  takes values in  $\ell^2$  or not.

**Definition.** Let  $0 \leq \alpha \leq 1$ . A sequence  $\Lambda$  of distinct points in  $\mathbb{D}$  said to be

- a (simple) interpolating sequence for  $\mathcal{D}_\alpha$  if  $R_\Lambda : f \rightarrow (f(\lambda)/\|k_\lambda\|_\alpha)_{\lambda \in \Lambda}$  is onto  $\ell^2$ , i.e. the interpolation problem  $f(\lambda) = a_\lambda$  has a solution  $f \in \mathcal{D}_\alpha$  for every sequence  $(a_\lambda)$  with  $(a_\lambda/\|k_\lambda\|_\alpha)_{\lambda \in \Lambda} \in \ell^2(\Lambda)$ .
- a universal interpolating sequence for  $\mathcal{D}_\alpha$  if it is interpolating and moreover  $R_\Lambda$  is well defined from  $\mathcal{D}_\alpha$  into  $\ell^2$ .

Sequences which are interpolating for the Dirichlet space but not universally interpolating were discovered by Bishop in [7], and they were further analyzed in [5] and [13].

Throughout this paper, even when not stated explicitly, when speaking about interpolation we mean universal interpolation. In the only case we work with simple interpolating sequences this will be stated explicitly. In all other cases, in our theorems on interpolation, sufficient conditions always imply universal interpolation, while for necessary conditions it suffices to only impose standard interpolation.

We will now discuss in details the results we have obtained in this paper.

**1.1. Back to the Hardy space.** As pointed out in the introduction, before considering the situation in the Dirichlet space, it seems appropriate to re-examine the situation in the Hardy space. Recall that Cochran established a 0-1 law for (pseudohyperbolic) separation (see Theorem 1.3 below) and Rudowicz showed that Cochran's condition for separation implies almost surely the Carleson measure condition. This implies that interpolation is characterized by the condition ensuring almost sure separation. As it turns out the situation in Dirichlet spaces is quite different. So, in order to get a better understanding we start stating an improvement of Rudowicz' results on random Carleson measures in the Hardy space which will help to better understand the case of Dirichlet spaces.

Recall that the measure  $dm_\Lambda = \sum_{\lambda \in \Lambda} (1 - |\lambda|^2) \delta_\lambda$  is called a Carleson measure if there is a constant  $C$  such that for every interval  $I \subset \mathbb{T}$ ,

$$m_\Lambda(S_I) \leq C|I|,$$

where

$$S_I = \{z = re^{it} \in \mathbb{D} : e^{it} \in I, 1 - |I| \leq r < 1\}$$

is the usual Carleson window (see [16]). We will prove that a weaker condition than Rudowicz' leads to Carleson measures almost surely in

the Hardy space. We first need to introduce a notation:

$$N_n = \#\{\lambda \in \Lambda(\omega) : 1 - 2^{-n} \leq |\lambda| \leq 1 - 2^{-n-1}\}, \quad n = 0, 1, 2, \dots$$

**Theorem 1.1.** *Let  $\beta > 1$  and suppose*

$$\sum_{n \geq 1} 2^{-n} N_n^\beta < +\infty.$$

*Then the measure  $dm_\Lambda$  is a Carleson measure almost surely in the Hardy space.*

As a result, the Carleson measure condition alone is not sufficient to give a 0-1 law for interpolation in the Hardy space.

Note that Rudowicz [23] showed that the above condition with  $\beta = 2$  is sufficient. There is still a gap remaining between the above condition and the Blaschke condition which corresponds to  $\beta = 1$ . So *a priori* there might be sequences which are almost surely zero sequences without giving rise to almost sure Carleson measures.

**1.2. Interpolation in Dirichlet spaces  $\mathcal{D}_\alpha$ ,  $0 < \alpha < 1$ .** For our paper, a key result for interpolation in Dirichlet spaces is [25, Theorem 4, p.38] which shows that pseudohyperbolic separation (see definitions below) and a certain 1-box condition are sufficient for (universal) interpolation in  $\mathcal{D}_\alpha$ ,  $0 < \alpha < 1$ . As already mentioned in the introduction above, separation in this case is as in the Hardy space, so that we essentially need to discuss the Carleson measure part of Seip's theorem.

**1.2.1. Random zero sequences in Dirichlet spaces.** A central role in our interpolation results will be played by random zero sequences. Indeed, for an interpolating sequence in the Dirichlet space it is necessary to be a zero sequence (interpolation implies that there are functions vanishing on the whole sequence except for one point  $\lambda$ , and multiplying this function by  $(z - \lambda)$  yields a function in the Dirichlet space vanishing on the whole sequence). We recall some results on random zero set in Dirichlet spaces. Carleson proved in [11] that when

$$(1.2) \quad \sum_{\lambda \in \Lambda} \|k_\lambda\|_\alpha^{-2} < \infty$$

then the Blaschke product  $B$  associated to  $\Lambda$  belongs to  $\mathcal{D}_\alpha$ ,  $0 < \alpha < 1$  (for  $\alpha = 0$  this corresponds to the Blaschke condition for the Hardy space). When  $\alpha = 1$ , Shapiro–Shields proved in [24] that the condition (1.2) is sufficient for  $\{\lambda\}_{\lambda \in \Lambda}$  to be a zero set for the classical Dirichlet space  $\mathcal{D}_1$ , see also [25, Theorem 1]. Note that if  $0 \leq \alpha < 1$  then

$$\sum_{\lambda \in \Lambda} \|k_\lambda\|_\alpha^{-2} \asymp \sum_{\lambda \in \Lambda} (1 - |\lambda|)^{1-\alpha} \asymp \sum_n 2^{-(1-\alpha)n} N_n$$

and if  $\alpha = 1$  then

$$\sum_{\lambda \in \Lambda} \|k_\lambda\|_1^{-2} \asymp \sum_{\lambda \in \Lambda} |\log(1 - |\lambda|)|^{-1} \asymp \sum_n n^{-1} N_n.$$

On the other hand, it was proved by Nagel–Shapiro–Shields in [20] that if  $\{r_n\} \subset (0, 1)$  does not satisfy (1.2), then there is  $\{\theta_n\}$  such that  $\{r_n e^{i\theta_n}\}$  is not a zero set for  $\mathcal{D}_\alpha$ . Bogdan [10, Theorem 2] gives a condition on the radii  $|\lambda_n|$  for the sequence  $\Lambda(\omega)$  to be almost surely zero sequence for  $\mathcal{D}$ :

(1.3)

$$P(\Lambda(\omega) \text{ is a zero set for } \mathcal{D}) = \begin{cases} 1 & \text{if and only if } \sum_n n^{-1} N_n < \infty \\ 0 & = \infty. \end{cases}$$

Bogdan’s arguments carry over to  $\mathcal{D}_\alpha$ ,  $\alpha \in (0, 1)$ . For the sake of completeness, we will prove in the annex, Section 6, the following result on almost sure zero sequences.

**Theorem 1.2.** *Let  $0 \leq \alpha < 1$ . Then*

(1.4)

$$P(\Lambda(\omega) \text{ is a zero set for } \mathcal{D}_\alpha) = \begin{cases} 1 & \text{if and only if } \sum_n 2^{-(1-\alpha)n} N_n < \infty \\ 0 & = \infty. \end{cases}$$

**1.2.2. Interpolation in Dirichlet spaces  $\mathcal{D}_\alpha$ .** As pointed out earlier, interpolation is intimately related with separation conditions and Carleson measure type conditions. Recall that a sequence  $\Lambda$  is called (pseudohyperbolically) separated if

$$\inf_{\substack{\lambda, \lambda^* \in \Lambda \\ \lambda \neq \lambda^*}} \rho(\lambda, \lambda^*) = \inf_{\substack{\lambda, \lambda^* \in \Lambda \\ \lambda \neq \lambda^*}} \frac{|\lambda - \lambda^*|}{|1 - \bar{\lambda}\lambda^*|} \geq \delta_\Lambda > 0.$$

Since in Dirichlet spaces  $\mathcal{D}_\alpha$ ,  $0 \leq \alpha < 1$ , the natural separation ( $\mathcal{D}_\alpha$ -separated sequence) is indeed pseudohyperbolic separation [25, p.22], we recall Cochran’s separation result on pseudohyperbolic separation.

**Theorem 1.3** (Cochran). *A sequence  $\Lambda(\omega)$  is almost surely (pseudohyperbolically) separated if and only if*

$$(1.5) \quad \sum_n 2^{-n} N_n^2 < +\infty.$$

We should pause here to make a crucial observation. We have already mentioned that interpolating sequences are necessarily zero-sequences. Also separation is another necessary condition for interpolation. Now the condition for zero sequences (1.4) depends on  $\alpha$  while the separation condition does not, and it follows that depending on  $\alpha$ , it is one condition or the other which is dominating. From (1.4) and (1.5) it



is not difficult to see that this breakpoint is exactly at  $\alpha = 1/2$  (for  $\alpha = 1/2$ , (1.4) still implies (1.5)). This motivates already the necessary conditions of our next result. For the sufficiency we will need to appeal to Seip's one-box condition [25, Theorem 5, p.38].

**Theorem 1.4.**

(i) *Let  $0 < \alpha < 1/2$ , then*

$$P(\Lambda(\omega) \text{ is interpolating for } \mathcal{D}_\alpha) = \begin{cases} 1 & \text{if and only if } \sum_n 2^{-n} N_n^2 \\ 0 & = \infty. \end{cases} < \infty$$

(ii) *Let  $\alpha = 1/2$ . If there exists  $\beta > 2$  such that*

$$\sum_k 2^{-k} N_k^\beta < \infty$$

*then  $P(\Lambda(\omega) \text{ is interpolating for } \mathcal{D}_{1/2}) = 1$ .*

*Conversely, If  $P(\Lambda(\omega) \text{ is interpolating for } \mathcal{D}_{1/2}) = 1$  then*

$$\sum_n 2^{-n/2} N_n < \infty.$$

(iii) *Let  $1/2 < \alpha < 1$ . Then*

$$P(\Lambda(\omega) \text{ is interpolating for } \mathcal{D}_\alpha) = \begin{cases} 1 & \text{if and only if } \sum_n 2^{-(1-\alpha)n} N_n \\ 0 & = \infty. \end{cases} < \infty$$

Our techniques, based on Seip's one-box condition, do not provide a complete answer in the case  $\alpha = 1/2$  which needs further investigation.

An interesting reformulation of the above results connects random interpolation with random zero sequences and random separated sequences as stated in the following corollary.

**Corollary 1.5.** *The following statements hold:*

- (1) *Let  $0 \leq \alpha < 1/2$ . The sequence  $\Lambda(\omega)$  is almost surely interpolating for  $\mathcal{D}_\alpha$  if and only if it is almost surely separated.*
- (2) *Let  $1/2 < \alpha < 1$ . The sequence  $\Lambda(\omega)$  is almost surely interpolating for  $\mathcal{D}_\alpha$  if and only if it is almost surely a zero sequence.*

**1.3. Interpolation in the classical Dirichlet space.** For the classical Dirichlet space we will first establish a result on separation, and then use again a 1-box condition by Seip as stated in [25, Theorem 5, p.39].

**1.3.1. Separation in the Dirichlet space.** In the case  $\alpha = 1$ , the separation is given in a different way. Let

$$\rho_{\mathcal{D}}(z, w) = \sqrt{1 - \frac{|k_w(z)|^2}{k_z(z)k_w(w)}}, \quad z, w \in \mathbb{D}.$$

A sequence  $\Lambda$  is called  $\mathcal{D}$ -separated if

$$\inf_{\substack{\lambda, \lambda^* \in \Lambda \\ \lambda \neq \lambda^*}} \rho_{\mathcal{D}}(\lambda, \lambda^*) > \delta_{\Lambda} > 0$$

for some  $\delta_{\Lambda} < 1$ . This is equivalent to (see [25, p.23])

$$(1.6) \quad \frac{(1 - |\lambda|^2)(1 - |\lambda^*|^2)}{|1 - \bar{\lambda}\lambda^*|^2} \leq (1 - |\lambda|^2)\delta_{\Lambda}^2, \quad \lambda, \lambda^* \in \Lambda.$$

For separation in the Dirichlet space  $\mathcal{D}$  we obtain the following 0-1 law.

**Theorem 1.6.**

$$P(\Lambda(\omega) \text{ is } \mathcal{D}\text{-separated for } ) = \begin{cases} 1, & \text{if } \exists \gamma \in (1/2, 1) \text{ such that } \sum 2^{-\gamma n} N_n^2 < \infty, \\ 0, & \text{if } \forall \gamma \in (1/2, 1) \text{ such that } \sum_n 2^{-\gamma n} N_n^2 = \infty. \end{cases}$$

We observe that in both conditions we can replace the sum by a supremum (this amounts to replacing  $\gamma$  by a slightly bigger or smaller value). The lower bound  $1/2$  for  $\gamma$  is not very important, since it is the behavior close to the value 1 which counts.

**1.3.2. Interpolation in the Dirichlet space  $\mathcal{D}$ .** Recall that Bogdan showed that  $\Lambda(\omega)$  is almost surely a zero sequence for  $\mathcal{D}$  if and only if  $\sum_n n^{-1} N_n < +\infty$ . This motivates already the necessary part of the following complete characterization of almost surely *simple* interpolating sequences for  $\mathcal{D}$ .

**Theorem 1.7.**

$$P(\Lambda(\omega) \text{ is simple interpolating for } \mathcal{D}) = \begin{cases} 1 \\ 0 \end{cases} \text{ if and only if } \sum_n n^{-1} N_n \begin{cases} < \infty \\ = \infty. \end{cases}$$

We can reformulate the above result in the same spirit as Corollary 1.5

**Corollary 1.8.** *A sequence is almost surely simple interpolating for  $\mathcal{D}$  if and only if it is almost surely a zero sequence for  $\mathcal{D}$ .*

Translating Theorem 1.1 to the Dirichlet space, we get the following result for universal interpolation which is optimal in a sense.

**Theorem 1.9.** *If there exist  $\gamma \in (0, 1)$  such that*

$$(1.7) \quad \sum_n n^{-\gamma} N_n < \infty,$$

*then  $P(\Lambda(\omega) \text{ is an interpolating sequence for } \mathcal{D}) = 1$ .*

**1.4. Organization of the paper.** This paper is organized as follows. In the next section we present the improved version of the Rudowicz result concerning random Carleson measures in Hardy spaces which is the guideline for the corresponding result in the Dirichlet space. Indeed, this largely clarifies and simplifies not only the situation in the Hardy space, but also indicates the direction of investigation for the Dirichlet space. In Section 3 we prove the sufficient condition for interpolation in  $\mathcal{D}_\alpha$ ,  $0 < \alpha < 1$ . Here, we will also prove Corollary 1.4. In the following section we show the 0-1 law on separation in the classical Dirichlet space. This requires a subtle adaption of the Cochran discussion in the Hardy space to the much more intricate geometry in the Dirichlet space. Section 4 is devoted to the characterization of separated random interpolating sequences in the Dirichlet space. The proofs of the results on interpolating sequences in the classical Dirichlet space are contained in Section 5. Actually, as in the Hardy space, the core of the proof being probabilistic, we are able to get rid of analytic functions. In the final Section 6, we give some indications to the 0-1 law on zero-sequences in weighted Dirichlet spaces based on Bogdan's proof in the classical Dirichlet space.

A word on notation. Suppose  $A$  and  $B$  are strictly positive expressions. We will write  $A \lesssim B$  meaning that  $A \leq cB$  for some positive constant  $c$  not depending on the parameters behind  $A$  and  $B$ . By  $A \simeq B$  we mean  $A \lesssim B$  and  $B \lesssim A$ . We further use the notation  $A \sim B$  provided the quotient  $A/B \rightarrow 1$  when passing to the suitable limit.

## 2. CARLESON CONDITION IN THE HARDY SPACE

Before considering Carleson measure conditions in the Dirichlet space, we will discuss the situation in the Hardy space, in particular we will prove here Theorem 1.1.

**2.1. Proof of Theorem 1.1.** We start introducing some notation. Let

$$I_{n,k} = \{e^{2\pi it} : t \in [k2^{-n}, (k+1)2^{-n})\} \quad n \in \mathbb{N}, \quad k = 0, 1, \dots, 2^n$$

be dyadic intervals and  $S_{n,k} = S_{I_{n,k}}$  the associated Carleson window. In order to check the Carleson measure condition for a positive Borel measure  $\mu$  on  $\mathbb{D}$  it is clearly sufficient to check the Carleson measure condition for windows  $S_{n,k}$ :

$$\mu(S_{n,k}) \leq C|I_{n,k}| \simeq C2^{-n},$$

for some fixed  $C > 0$  and every  $n \in \mathbb{N}$ ,  $k = 0, \dots, 2^n$ . Given  $n$ ,  $k$  and  $m \geq k$  let  $X_{n,m,k}$  be the number of points of  $\Lambda$  contained in  $S_{n,k} \cap A_m$  (we stratify the Carleson window  $S_{n,k}$  into a disjoint union of layers  $S_{n,k} \cap A_m$ ). Since  $A_m$  contains  $N_m$  points and the (normalized) length of  $S_{n,k}$  is  $2^{-n}$ , we have  $X_{n,m,k} \sim B(2^{-n}, N_m)$  (binomial law). In order to show that  $dm_\Lambda$  is almost surely a Carleson measure we thus have to prove the existence of  $C$  such that

$$m_\Lambda(S_{n,k}) = \sum_{m \geq n} 2^{-m} X_{n,m,k} \leq C2^{-n}$$

almost surely, in other words we have to prove

$$\sup_{n,k} 2^n \sum_{m \geq n} 2^{-m} X_{n,m,k} \leq C$$

almost surely (in  $\omega$ ). The estimate above had already been investigated by Rudowicz [23]. Here we will proceed in a different way with respect to Rudowicz' argument to obtain an improved version of his result and which allows to better understand the Dirichlet space situation.

*Proof of Theorem 1.1.* In view of our preliminary remarks, we need to look at the random variable

$$Y_{n,k} = 2^n \sum_{m=n}^{+\infty} 2^{-m} X_{n,m,k},$$

where, as said above,  $X_{n,m,k} \sim B(2^{-n}, N_m)$ . Hence, saying that  $Y_{n,k} \geq A$  means that there are Carleson windows for which the Carleson measure constant is at least  $A$ . Also denote by  $G_{Y_{n,k}}$  the probability generating function of the random variable  $Y_{n,k}$ , i.e.  $G_{Y_{n,k}}(s) = \mathbb{E}(s^{Y_{n,k}})$ . It is well known that for a random variable  $X$  which follows binomial distribution with parameters  $p, N$  we have that  $G_X(s) = (1 - p + ps)^N$ .

By the hypothesis, for  $n$  sufficiently large,  $N_n \leq 2^{(1-\epsilon)n}$ ,  $\epsilon = 1 - 1/\beta$ . Introduce now two parameters  $A, s > 0$  to be specified later. By

Markov's inequality we have that

$$\begin{aligned}
\log P(Y_{n,k} \geq A) &= \log P(s^{Y_{n,k}} \geq s^A) \\
&\leq \log \left( \frac{1}{s^A} G_{Y_{n,k}}(s) \right) \\
&= \sum_{m \geq n} N_m \log(1 - 2^{-n} + 2^{-n} s^{2^{n-m}}) - A \log(s) \\
&\leq 2^{-n} \sum_{m \geq n} N_m (s^{2^{n-m}} - 1) - A \log(s) \\
&= \sum_{m \geq 0} N_{n+m} 2^{-(n+m)} 2^m (s^{2^{-m}} - 1) - A \log(s).
\end{aligned}$$

At this point notice that  $x(a^{1/x} - 1) \leq a$ , for all  $x \geq 1, a > 0$ , which together with the hypothesis on  $N_n$  gives

$$\log P(Y_{n,k} \geq A) \leq \sum_{m \geq 0} 2^{-\epsilon(n+m)} s - A \log(s) = \frac{2^\epsilon}{2^\epsilon - 1} s 2^{-\epsilon n} - A \log(s).$$

Now set  $s = 2^{\frac{\epsilon n}{2}}, A = \frac{4}{\epsilon}$  in the last inequality to get

$$\log P(Y_{n,k} \geq A) \leq \frac{2^\epsilon}{2^\epsilon - 1} 2^{-\frac{\epsilon n}{2}} - 2n \log(2).$$

Hence,  $P(Y_{n,k} \geq \frac{4}{\epsilon}) \leq C(\epsilon) 2^{-2n}$ .

In view of an application of the Borel-Cantelli Lemma we compute

$$\sum_{n \geq 0} \sum_{k=1}^{2^n} P(Y_{n,k} \geq A) \leq C(\epsilon) \sum_{n \geq 0} 2^n \times 2^{-2n} < \infty.$$

Hence, by the Borel-Cantelli Lemma, the event  $Y_{n,k} \geq A$  can happen for at most a finite number of indices  $(n, k)$ . In particular the Carleson measure constant of  $dm_\Lambda$  is almost surely at most  $A$  except for a finite number of Carleson windows.  $\square$

### 3. PROOF OF THEOREM 1.4

We start with the following elementary lemma which is well known in probability theory and which will be very useful in the proof below of Theorem 3.2. (it is essentially approximation of the binomial law by Poisson law). We refer for instance to [8] for the material on probability theory — essentially elementary — used in this paper.

**Lemma 3.1.** *If  $X$  is a binomial random variable with parameters  $p, N$ , then for every  $s = 0, 1, 2, \dots$ ,*

$$\lim_{\substack{N \rightarrow \infty \\ pN \rightarrow 0}} \frac{P(X = s)}{(pN)^s} = \lim_{\substack{N \rightarrow \infty \\ pN \rightarrow 0}} \frac{P(X \geq s)}{(pN)^s} = \frac{1}{s!}.$$

The proof of Theorem 1.4 is based on the following seemingly more general result.

**Theorem 3.2.** *Let  $0 < \alpha < 1$ . Suppose  $\Lambda(\omega)$  is almost surely separated and almost surely a zero sequence. If there exists  $\beta > \frac{3 - 2\alpha}{2 - 2\alpha}$  such that*

$$(3.1) \quad \sum_n 2^{-n} N_n^\beta < \infty$$

*then  $P(\Lambda(\omega) \text{ is interpolating } \mathcal{D}_\alpha) = 1$ .*

As we have already mentioned in Subsection 1.2.2, imposing simultaneously  $\Lambda$  being a zero sequence and separated is rather artificial.

One could be tempted to repeat the proof used above in the Hardy space. However, it turns out that this does not give the best result, in particular when  $\alpha > 1/2$ . Our proof of Theorem 3.2 will be based on the following one-box condition introduced by Seip ([25, Theorem 4, p.38]), and on Tchebychevs inequality instead of Markovs. Note that Rudowicz already used Tchebychevs inequality, but his reasoning in the Hardy space worked only for  $\beta \geq 2$  in Theorem 1.1. We introduce a refinement of his argument which yields a more precise Carleson measure estimate, and which allows to obtain the sharp statements in Theorem 1.4.

**Theorem 3.3** (Seip). *A separated sequence  $\Lambda$  in  $\mathbb{D}$  is a universal interpolating sequence for  $\mathcal{D}_\alpha$ ,  $0 < \alpha < 1$ , if there exist  $0 < \kappa < 1 - \alpha$  and  $C > 0$  such that for each arc  $I \subset \mathbb{T}$*

$$(3.2) \quad \sum_{\lambda \in \Lambda \cap S_I} (1 - |\lambda|)^\kappa \leq C |I|^\kappa.$$

Observe in particular that the above condition implies

$$\sum_n 2^{-n\kappa} N_n < +\infty,$$

so that in particular  $\sum_n 2^{-n(1-\alpha)} N_n < +\infty$ , and thus that (3.2) implies that  $\Lambda$  is a zero sequence almost surely.

We have to show that the conditions of Theorem 3.2 imply (3.2) almost surely for some fixed  $\kappa \in (0, 1)$ . We use the same reasoning as in the preceding section. In particular with the same notation we have to prove

$$\sup_{n,k} 2^{\kappa n} \sum_{m \geq n} 2^{-\kappa m} X_{n,m,k} \leq C$$

almost surely. Again  $X_{n,m,k} \sim B(2^{-n}, N_m)$ , where we can now assume  $N_m \leq 2^{m/\beta}$  for some  $\beta > (3 - 2\alpha)/(2 - 2\alpha)$  (at least for sufficiently large  $m$ ).

The key idea here is now to split this sum into two pieces. The first piece can be estimated with the aid of binomial law, and for the second one we will use Tchebychevs inequality. For  $\gamma > 0$  to be fixed later, we will write

$$Y_{n,k} = \underbrace{2^{\kappa n} \sum_{m=n}^{n(1+\gamma)} 2^{-\kappa m} X_{n,m,k}}_{Y_{n,k}^0} + \underbrace{2^{\kappa n} \sum_{m=n(1+\gamma)+1}^{+\infty} 2^{-\kappa m} X_{n,m,k}}_{R_{n,k}}.$$

We also use the notation

$$Z_{n,k} := \sum_{m=n}^{n(1+\gamma)} X_{n,m,k},$$

so that now we get

$$P(Y_{n,k}^0 \geq A/2) \leq P(Z_{n,k} \geq A/2) \sim \frac{(pN)^{A/2}}{(A/2)!} = \frac{(2^{-n} \sum_{m=n}^{n(1+\gamma)} N_m)^{A/2}}{(A/2)!}.$$

It can be observed that we now have less points  $N_m$  as in the Hardy space situation (where  $N_m \leq 2^{m/\beta}$  with  $\beta > 3/2$ , while now we have  $N_m \leq 2^{m/\beta}$  with  $\beta > (3 - 2\alpha)/(2 - 2\alpha) > 3/2$ ). It is thus clear that for sufficiently large  $A$  we again conclude  $P(Y_{n,k}^0 \geq A/2) \leq 2^{-4n}$ . To be more precise, with  $N_m \leq 2^{m/\beta}$  in mind, we see that  $2^{-n} \sum_{m=n}^{n(1+\gamma)} N_m \lesssim 2^{-n(1-(1+\gamma)/\beta)}$  so that we get the same condition on  $\gamma$  as before:  $(1 + \gamma)/\beta < 1$

$$(3.3) \quad \gamma < \beta - 1.$$

In that case, setting  $\eta = 1 - (1 + \gamma)/\beta > 0$  we have  $P(Y_{n,k}^0 \geq A/2) \leq 2^{-\eta \times A/2} \leq 2^{-4n}$  for sufficiently large  $A$ .

We estimate  $R_{n,k}$  using Tchebychev's inequality:

$$\begin{aligned}
 P(R_{n,k} \geq A/2) &\lesssim \text{Var}(R_{n,k}) = 4^{\kappa n} \sum_{m \geq n(1+\gamma)} 4^{-\kappa m} \text{Var}(X_{n,m,k}) \\
 &\lesssim 4^{\kappa n} \sum_{m \geq n(1+\gamma)} 4^{-\kappa m} \times 2^{-n} N_m \leq 2^{(2\kappa-1)n} \sum_{m \geq n(1+\gamma)} 2^{-2\kappa m + m/\beta} \\
 &\leq 2^{n(2\kappa-1-2\kappa(1+\gamma)+(1+\gamma)/\beta)} = 2^{-n(1+2\kappa\gamma-(1+\gamma)/\beta)}.
 \end{aligned}$$

We will need  $\eta := 1+2\kappa\gamma-(1+\gamma)/\beta > 1$ , equivalently  $2\kappa\gamma > (1+\gamma)/\beta$ , or  $2\kappa\beta\gamma > 1 + \gamma$ , i.e.

$$(3.4) \quad \gamma > \frac{1}{2\kappa\beta - 1}.$$

Now the two conditions on  $\gamma$ , (3.3) and (3.4), require

$$\begin{aligned}
 \frac{1}{2\kappa\beta - 1} < \beta - 1 &\iff 1 < 2\kappa\beta^2 - \beta - 2\kappa\beta + 1 \iff \beta(1 + 2\kappa) < 2\kappa\beta^2 \\
 (3.5) \quad &\iff \beta > \frac{1 + 2\kappa}{2\kappa} = 1 + \frac{1}{2\kappa}.
 \end{aligned}$$

Observe that Seip's result requires  $0 < \kappa < 1 - \alpha$  and that the above expression is decreasing in  $\kappa$  so that:

$$\frac{1 + 2\kappa}{2\kappa} > \frac{1 + 2(1 - \alpha)}{2(1 - \alpha)} = \frac{3 - 2\alpha}{2 - 2\alpha}.$$

By assumption, we have  $\beta > (3 - 2\alpha)/(2 - 2\alpha)$  so that there exists  $\kappa \in (0, 1 - \alpha)$  with (3.5). And for that  $\kappa$  we can find  $\gamma$  satisfying (3.3) and (3.4). From here, the proof follows the same lines as in the Hardy space discussed in the preceding section. This proves Theorem 3.2.  $\square$

Let us give the proof of Theorem 1.4.

*Proof of Theorem 1.4.* (i) Let  $0 < \alpha < 1/2$ .

If  $\Lambda$  is interpolating almost surely, then it is separated almost surely, which implies  $\sum_n 2^{-n} N_n^2 < +\infty$ .

If  $\sum_n 2^{-n} N_n^2 < +\infty$ , then  $\Lambda$  is almost surely separated. Moreover, we can pick  $\beta = 2$  in (3.1). Hence

$$\frac{3 - 2\alpha}{2 - 2\alpha} = 1 + \frac{1}{2(1 - \alpha)} < 2 = \beta.$$

Also, in this case  $\sum_n 2^{-(1-\alpha)} N_n < +\infty$ , which gives (1.4) and hence, by Theorem 1.2 that  $\Lambda$  is a zero sequence almost surely. We conclude from Theorem 3.2 that  $\Lambda$  is almost surely interpolating.

(iii) Consider the case  $1/2 < \alpha < 1$ .



If  $\Lambda$  is interpolating almost surely, then it is a zero sequence almost surely, which implies  $\sum_n 2^{-(1-\alpha)n} N_n < +\infty$  by Theorem 1.2.

Suppose  $\sum_n 2^{-(1-\alpha)n} N_n < +\infty$ . Then  $\Lambda$  is a zero sequence almost surely by Theorem 1.2. Also it is clear that the condition implies that  $\sum_n 2^{-n} N_n^2 < +\infty$ , which further implies that the sequence is almost surely separated. Pick

$$\beta = 1 + \frac{1}{2(1-\alpha)} + \epsilon > 1 + \frac{1}{2(1-\alpha)} = \frac{3-2\alpha}{2-2\alpha},$$

for some  $\epsilon > 0$  to be fixed. The distribution condition implies  $N_n \leq 2^{(1-\alpha)n}$  (at least for sufficiently large  $n$ ). Hence

$$N_n^{\beta-1} \leq 2^{(1-\alpha)\times(\beta-1)n} = 2^{(1-\alpha)\times(\frac{1}{2(1-\alpha)}+\epsilon)n} = 2^{(1/2+(1-\alpha)\epsilon)n}.$$

Then

$$2^{-\alpha n} \times N_n^{\beta-1} = 2^{(-\alpha+1/2+(1-\alpha)\epsilon)n}.$$

Now since  $\alpha > 1/2$ , there exists  $\epsilon > 0$  such that  $-\alpha+1/2+(1-\alpha)\epsilon < 0$ . Hence

$$\sum_n 2^{-n} N_k^\beta = \sum_n 2^{-(1-\alpha)n} N_n \times 2^{-\alpha n} N_n^{\beta-1} < \infty.$$

We conclude from Theorem 3.2 that  $\Lambda$  is interpolating almost surely.

If  $\sum_n 2^{-(1-\alpha)n} N_n = +\infty$ . Then  $\Lambda$  is not a zero sequence almost surely by Theorem 1.2, and hence it is almost surely not interpolating.

Conversely, we need to check that if  $\Lambda$  is almost surely not interpolating, then we have  $\sum_n 2^{-(1-\alpha)n} N_n = +\infty$ . By contraposition, if the sum is bounded, we have to show that the probability that  $\Lambda$  is interpolating is strictly positive. But we already know that in that case this probability is  $1 > 0$ .

(ii)  $\alpha = 1/2$ : In this case we only have a sufficient condition which follows by direct application of Theorems 1.2 and 3.2.  $\square$

#### 4. SEPARATED RANDOM SEQUENCES FOR THE DIRICHLET SPACE

We will now prove the separation result in  $\mathcal{D}$ .

*Proof of Theorem 1.6. Separation with probability 0.* Assume that for all  $\gamma \in (1/2, 1)$  we have  $\sup_k 2^{-\gamma k} N_k^2 = \infty$ . As it turns out, under the condition of the Theorem, separation already fails in dyadic annuli (without taking into account radial Dirichlet separation).

Assume now that  $\gamma_l \rightarrow 1$  as  $l \rightarrow \infty$  and  $\sup_k 2^{-\gamma_l k} N_k^2 = \infty$  for every  $l$ . For each  $k = 1, 2, \dots$ , let  $I_k = [1 - 2^{-k+1}, 1 - 2^{-k})$ . Define

$$\Omega_k^{(l)} = \{\omega : \exists(i, j), i \neq j \text{ with } \rho_i, \rho_j \in I_k \text{ and } |\theta_i(\omega) - \theta_j(\omega)| \leq \pi 2^{-\gamma_l k}\}.$$

In view of (1.6), if  $\omega \in \Omega_k^{(l)}$ , this means that in the dyadic annulus  $A_k$  there are at least two points close in the Dirichlet metric. To be more precise, if  $\omega \in \Omega_k^{(l)}$ , then there is a pair of distinct point  $\lambda_i(\omega)$  and  $\lambda_j(\omega)$  such that  $|\lambda_i|, |\lambda_j| \in I_k$  and  $|\arg \lambda_i(\omega) - \arg \lambda_j(\omega)| \leq \pi 2^{-\gamma k}$ . Hence

$$\frac{(1 - |\lambda_i|^2)(1 - |\lambda_j|^2)}{|1 - \bar{\lambda}_i \lambda_j|^2} \geq c \frac{2^{-2k}}{2^{-2k} + \pi 2^{-2\gamma k}},$$

where the constant  $c$  is an absolute constant. Hence

$$\frac{(1 - |\lambda_i|^2)(1 - |\lambda_j|^2)}{|1 - \bar{\lambda}_i \lambda_j|^2} \geq c \frac{1}{1 + \pi 2^{k(1-\gamma)}} \geq c' 2^{-k(1-\gamma)} \geq c'' (1 - |\lambda_i|)^{1-\gamma}.$$

Absorbing  $c''$  into a suitable change of the power  $\delta_l^2 := 1 - \gamma_l$  into  $\delta_l'^2$  (which can be taken by choosing for instance  $2\delta_l > \delta_l' > \delta_l$  provided  $k$  is large enough), then by (1.6)

$$\rho_{\mathcal{D}}(\lambda_i(\omega), \lambda_j(\omega)) \leq \delta_l'.$$

Our aim is thus to show that for every  $l \in \mathbb{N}$ , we can find almost surely  $\lambda_i(\omega) \neq \lambda_j(\omega)$  such that  $\rho_{\mathcal{D}}(\lambda_i(\omega), \lambda_j(\omega)) \leq \delta_l'$ , i.e.  $P(\Omega_k^{(l)}) = 1$ . (Note that  $\delta_l' \rightarrow 0$  when  $l \rightarrow +\infty$ .)

Let us define a set  $E := \{j : 2^{-\gamma j-1} N_j \leq 1\}$ . Observe that when  $k \notin E$ , then at least two points are closer than  $\pi 2^{-\gamma k}$  (this is completely deterministic), so that in that case  $P(\Omega_k^{(l)}) = 1$ . Hence if  $E \subsetneq \mathbb{N}$ , then we are done.

Consider now the case  $E = \mathbb{N}$ , and let  $k \in E = \mathbb{N}$ . We will use the Lemma on the probability of an uncrowded road [12, p. 740], which states

$$P(\Omega_k^{(l)}) = 1 - (1 - N_k 2^{-\gamma k-1})^{N_k-1}$$

(since  $E = \mathbb{N}$  this is well defined).

We can assume that  $N_k \geq 2$  (since obviously  $\sum_{k: N_k < 2} 2^{-\gamma k} N_k < \infty$ ). In particular  $N_k^2/2 \leq N_k(N_k - 1) \leq N_k^2$ . Since  $\log(1 - x) \leq -x$ , we get

$$\begin{aligned} \sum_{k: N_k \geq 2} (N_k - 1) \log(1 - N_k 2^{-\gamma k-1}) &\leq - \sum_{k: N_k \geq 2} (N_k - 1) N_k 2^{-\gamma k-1} \\ &\leq -\frac{1}{2} \sum_{k: N_k \geq 2} N_k^2 2^{-\gamma k-1} \\ &= -\infty \end{aligned}$$

by assumption. Hence, taking exponentials in the previous estimate,

$$\prod_{k \in E, N_k \geq 2} (1 - N_k 2^{-\gamma k-1})^{N_k-1} = 0,$$

which implies, by results on convergence on infinite products, that

$$\sum_k P(\Omega_k^{(l)}) = \infty.$$

Since the events  $\Omega_k^{(l)}$  are independent, by the Borel–Cantelli Lemma,

$$P(\limsup \Omega_k^{(l)}) = 1,$$

where

$$\limsup \Omega_k^{(l)} = \bigcap_{n \geq 1} \bigcup_{k \geq n} \Omega_k^{(l)} = \{\omega : \omega \in \Omega_k^{(l)} \text{ for infinitely many } k\}.$$

In particular, since the probability of being in infinitely many  $\Omega_k^{(l)}$  is one, there is at least one  $\Omega_k^{(l)}$  which happens with probability one. So that again  $P(\Omega_k^{(l)}) = 1$ .

As a result, the probability that the sequence is  $\delta'_l$ -separated in the Dirichlet metric is zero for every  $l$ . Since  $\delta'_l \rightarrow 0$  when  $l \rightarrow +\infty$ , we deduce that

$$P(\omega : \{\lambda(\omega)\} \text{ is separated for } \mathcal{D}) = 0.$$

*Separation with probability 1.* Now we assume that  $\sum_k 2^{-\gamma k} N_k^2 < +\infty$  for some  $\gamma \in (1/2, 1)$ . Let us begin defining a neighborhood in the Dirichlet metric. For that, fix  $\eta > 1$  and  $\alpha \in (0, 1)$ . Given  $\lambda \in \Lambda$ , so that for some  $k$ ,  $\lambda \in A_k$ . Consider

$$T_\lambda^{\eta, \alpha} = \{z = re^{it} : (1 - |\lambda|)^\eta \leq 1 - r \leq (1 - |\lambda|)^\eta, |\theta - t| \leq (1 - r)^\alpha\}.$$

Figure 1 represents the situation.

Our aim is to prove that under the condition  $\sum_k 2^{-\gamma k} N_k^2 < +\infty$ , there exists  $\eta > 1$  and  $\alpha \in (0, 1)$  such that  $T_\lambda^{\eta, \alpha}$  does not contain any other point of  $\Lambda$  except  $\lambda$ , and this is true for every  $\lambda \in \Lambda$  with probability one. For this we need to estimate

$$P(T_\lambda^{\eta, \alpha} \cap \Lambda = \{\lambda\}).$$

Let us cover

$$T_\lambda^{\eta, \alpha} = \bigcup_{j=k/\eta}^{\eta k} (T_\lambda^{\eta, \alpha} \cap A_j),$$

and we need that for every  $j \in [k/\eta, \eta k] \setminus \{k\}$ ,  $(T_\lambda^{\eta, \alpha} \cap A_j) \cap \Lambda = \emptyset$  and  $(T_\lambda^{\eta, \alpha} \cap A_k) \cap \Lambda = \{\lambda\}$ . Note that  $X_j = \#(T_\lambda^{\eta, \alpha} \cap A_j \cap \Lambda) \sim B(N_j, 2^{-\alpha j})$ ,  $j \neq k$ , and  $X_k \sim B(N_k - 1, 2^{-\alpha k})$  (since we do not count  $\lambda$  in the latter

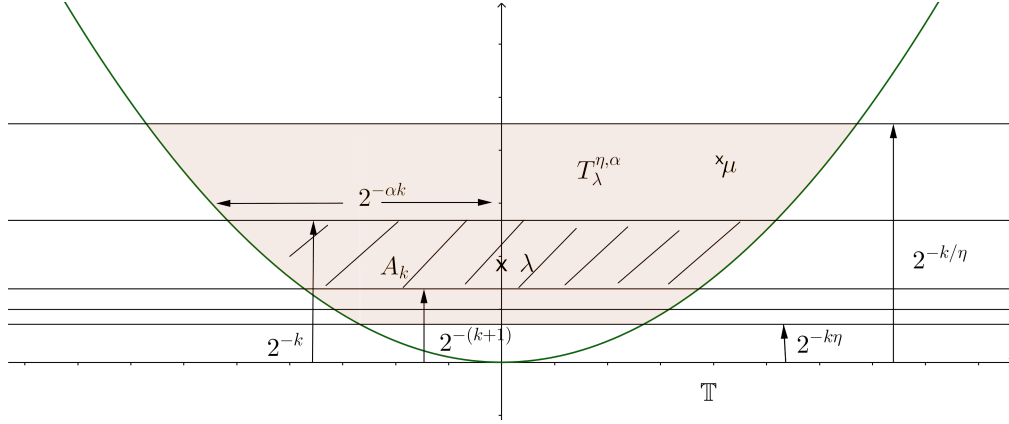


FIGURE 1. Dirichlet neighborhood.

case). Hence, since the arguments of the points are independent, we have

$$\begin{aligned} P(T_\lambda^{\eta, \alpha} \cap \Lambda = \{\lambda\}) &= P\left(\left(\bigcap_{j=k/\eta, j \neq k}^{\eta k} (X_j = 0)\right) \cap (X_k = 1)\right) \\ &= \prod_{j=k/\eta, j \neq k}^{j=\eta k} \left(P(X_j = 0)\right) \times P(X_k = 1). \end{aligned}$$

From the binomial law we have  $P(X_j = 0) = (1 - 2^{-\alpha j})^{N_j}$ , for  $j \in \{k/\eta, \eta k\} \setminus \{k\}$ . Also, assuming  $0 < \gamma < \alpha < 1$ , we have  $N_j 2^{-\alpha j} = o(j)$ , so that

$$P(X_j = 0) = (1 - 2^{-\alpha j})^{N_j} \sim 1 - N_j 2^{-\alpha j}.$$

Moreover

$$P(X_k = 1) = N_k 2^{-\alpha k} (1 - 2^{-\alpha k})^{N_k - 1} \sim N_k 2^{-\alpha k}.$$

Hence we get

$$\begin{aligned} P(T_\lambda^{\eta, \alpha} \cap \Lambda = \{\lambda\}) &\sim \exp\left(\sum_{j=k/\eta, j \neq k}^{j=\eta k} \ln(P(X_j = 0))\right) \times N_k 2^{-\alpha k} \\ &\sim \left(1 - \sum_{j=k/\eta, j \neq k}^{j=\eta k} N_j 2^{-\alpha j}\right) \times N_k 2^{-\alpha k}. \end{aligned}$$

Again we use  $\gamma < \alpha < 1$  to see now that the sum  $\sum_{j=k/\eta, j \neq k}^{\eta k} N_j 2^{-\alpha j}$  is convergent and goes to zero when  $k \rightarrow \infty$ . This shows in particular

that the fact of considering the event of having points in neighboring annuli of  $A_k$  containing  $\lambda$  can be neglected. Hence

$$P(T_\lambda^{\eta,\alpha} \cap \Lambda = \{\lambda\}) \sim N_k 2^{-\alpha k}.$$

We now sum over all  $\lambda \in \Lambda$  by summing over all dyadic annuli  $A_k$  and the  $N_k$  points contained in each annuli:

$$\sum_{\lambda \in \Lambda} P(T_\lambda^{\eta,\alpha} \cap \Lambda = \{\lambda\}) \sim \sum_{k \in \mathbb{N}} N_k \times N_k 2^{-\alpha k} = \sum_{k \in \mathbb{N}} N_k^2 2^{-\alpha k}.$$

For  $\alpha > \gamma$ , this sum converges by assumption. Using the Borel-Cantelli lemma we deduce that  $T_\lambda^{\eta,\alpha} \cap \Lambda = \{\lambda\}$  for all but finitely many  $\lambda$  with probability one. Obviously these finitely many neighborhoods  $T_\lambda^{\eta,\alpha}$  contain finitely many points between which a lower Dirichlet distance exists. This achieves the proof of the separation.  $\square$

It should be observed that the above proof only involves  $\alpha$  but not  $\eta$ , so that it is the separation in the annuli which dominates the situation.

## 5. PROOF OF THEOREM 1.9 AND THEOREM 1.7

In order to prove Theorems 1.9 and 1.7 we will use Seip's one box condition [25, Theorem 5, p.39] as well as the corollary below which follows from a theorem of Bishop ([7, Theorem 1.3]).

**Theorem 5.1** (Seip). *A  $\mathcal{D}$ -separated sequence  $\Lambda$  in  $\mathbb{D}$  is a universal interpolating sequence for  $\mathcal{D}$  if there exist  $0 < \gamma < 1$  and  $C > 0$  such that for each arc  $I \subset \mathbb{T}$*

$$(5.1) \quad \sum_{\lambda \in \Lambda \cap S_I} \left( \log \frac{e}{1-|\lambda|} \right)^{-\gamma} \leq C \left( \log \frac{e}{|I|} \right)^{-\gamma}.$$

**Theorem 5.2** (Bishop). *A  $\mathcal{D}$ -separated sequence  $\Lambda$  in  $\mathbb{D}$  is a (simple) interpolating sequence for  $\mathcal{D}$  if*

- (1)  $\sum_{\lambda \in \Lambda} \left( \log \frac{e}{1-|\lambda|} \right)^{-1} < +\infty$  (zero sequence), and
- (2)  $\exists \eta \in (0, 1)$  such that for all  $\nu \in \Lambda$ , we have

$$\sum_{\lambda \in \Lambda \cap S(I_\nu^\eta)} \left( \log \frac{e}{1-|\lambda|} \right)^{-1} \leq \left( \log \frac{e}{1-|\nu|} \right)^{-1}$$

where  $I_\nu^\eta$  is the interval centered at  $\nu/|\nu|$  of length  $(1-|\nu|)^\eta$ .

**Corollary 5.3.** *A  $\mathcal{D}$ -separated sequence  $\Lambda$  in  $\mathbb{D}$  is a (simple) interpolating sequence for  $\mathcal{D}$  if there exist  $C > 0$  such that for each arc*

$I \subset \mathbb{T}$

$$(5.2) \quad \sum_{\lambda \in \Lambda \cap S_I} \left( \log \frac{e}{1 - |\lambda|} \right)^{-1} \leq C \left( \log \frac{e}{|I|} \right)^{-1}.$$

In order to deduce the corollary from Bishop's theorem observe that the first hypothesis of the Theorem follows immediately by choosing  $I = \mathbb{T}$ , and the second one from the observation that  $\log(e/|I_\nu^\eta|) \simeq \log(e/(1 - |\nu|))$ .

Observe that both conditions (5.1) and (5.2) imply in particular  $\sum_n n^{-1} N_n < \infty$  which by Bogdan's result implies that  $\Lambda$  is a zero sequence almost surely.

*Proof of Theorem 1.9 and Theorem 1.7.* We proceed in a similar manner as in the proof of Theorem 1.1. Using the usual dyadic discretization the conditions in Theorem 5.1 and Theorem 5.3 translate to

$$\sum_{m \geq n} \frac{1}{m^\gamma} X_{n,m,k} \leq C \frac{1}{n^\gamma} \text{ almost surely,}$$

where  $0 < \gamma < 1$  corresponds to Seip's condition and  $\gamma = 1$  to Bishop's. As usual we have to estimate the tail of the random variables

$$Y_{n,k} = \sum_{m \geq n} \left( \frac{n}{m} \right)^\gamma X_{n,m,k}.$$

To do that, introduce again two positive parameters  $s, A$ . Using the formula for the generating function of a binomially distributed random variable and Markov's inequality we can estimate as follows

$$\begin{aligned} \log P(Y_{n,k} \geq A) &= \log P(s^{Y_{n,k}} > s^A) \\ &\leq \sum_{m \geq n} N_m \log(1 - 2^{-n} + 2^{-n} s^{(\frac{n}{m})^\gamma}) - A \log(s) \\ &\leq 2^{-n} \sum_{m \geq n} \frac{N_m}{m^\gamma} \left( \frac{m}{n} \right)^\gamma (s^{(\frac{n}{m})^\gamma} - 1) n^\gamma - A \log(s) \\ &\leq n^\gamma 2^{-n} s \sum_{m \geq n} \frac{N_m}{m^\gamma} - A \log(s). \end{aligned}$$

Setting  $s = 2^{n/2}$  and  $A = 4$  the above calculation gives

$$P(Y_{n,k} > 4) \lesssim C 2^{-2n}.$$

Again, an application of the Borel-Cantelli Lemma concludes the proof.  $\square$

## 6. ANNEX : PROOF OF THEOREM 1.2

Carleson proved in [11, Theorem 2.2] that, for  $0 < \alpha < 1$ , if

$$\sum_{\lambda \in \Lambda} (1 - |\lambda|)^{1-\alpha} < \infty$$

then the Blaschke product  $B$  associated to  $\Lambda$  belongs to  $\mathcal{D}_\alpha$ . So the sufficiency part of Theorem 1.2 follows immediately from this result (and is moreover deterministically true).

For the proof of the converse we will need the following two lemmas. The first one is a version of the Borel-Cantelli Lemma [8, Theorem 6.3].

**Lemma 6.1.** *If  $\{A_n\}$  is a sequence of measurable subsets in a probability space  $(X, P)$  such that  $\sum P(A_n) = \infty$  and*

$$(6.1) \quad \liminf_{n \rightarrow \infty} \frac{\sum_{j,k \leq n} P(A_j \cap A_k)}{[\sum_{k \leq n} P(A_k)]^2} \leq 1,$$

then  $P(\limsup_{n \rightarrow \infty} A_n) = 1$ .

The second Lemma is due to Nagel, Rudin and Shapiro [20, 21] who discussed tangential approach regions of functions in  $\mathcal{D}_\alpha$ .

**Lemma 6.2.** *Let  $f \in \mathcal{D}_\alpha$ ,  $0 < \alpha < 1$ . Then, for a.e.  $\zeta \in \mathbb{T}$ , we have  $f(z) \rightarrow f^*(\zeta)$  as  $z \rightarrow \zeta$  in each region*

$$|z - \zeta| < \kappa(1 - |z|)^{1-\alpha}, \quad (\kappa > 1).$$

*Proof of Theorem 1.2.* In view of our preliminary observations, we are essentially interested in the converse implication. So suppose  $\sum_n 2^{-(1-\alpha)n} N_n = +\infty$  or equivalently

$$(6.2) \quad \sum_n (1 - \rho_n)^{1-\alpha} = +\infty.$$

We have to show that  $\Lambda$  is not a zero sequence almost surely. For this, introduce the intervals  $I_\ell = (e^{-i(1-\rho_\ell)^{1-\alpha}}, e^{i(1-\rho_\ell)^{1-\alpha}})$  and let  $F_\ell = e^{i\theta_\ell} I_\ell$ . Denoting by  $m$  normalized Lebesgue measure on  $\mathbb{T}$ , observe that

$$m(F_\ell) = m(I_\ell) = (1 - \rho_\ell)^{1-\alpha}.$$

We have for every  $\varphi \in F_\ell$ ,  $\lambda_\ell \in \Omega_{\kappa, \varphi} = \{z \in \mathbb{D} : |z - e^{i\varphi}| < \kappa(1 - |z|)^{1-\alpha}\}$ . By Lemma 6.2 it suffices to prove that  $|\limsup_\ell F_\ell| > 0$

a.s. (the latter condition means that there is a set of strictly positive measure on  $\mathbb{T}$  to which  $\Lambda$  accumulates in Dirichlet tangential approach regions according to Lemma 6.2, which is of course not possible for a zero sequence). Let  $E$  denote the expectation with respect to the Steinhaus sequence  $(\theta_n)$ . By Fubini's theorem we have  $E[m(F_j \cap F_k)] = m(I_j)m(I_k)$ ,  $j \neq k$ , (the expected size of intersection of two intervals only depends on the product of the length of both intervals). By Fatou's Lemma and (6.2)

$$\begin{aligned} E \left[ \liminf_{n \rightarrow \infty} \frac{\sum_{j,k \leq n} m(F_j \cap F_k)}{[\sum_{k \leq n} m(F_k)]^2} \right] &\leq \liminf_{n \rightarrow \infty} E \left[ \frac{\sum_{j,k \leq n} m(F_j \cap F_k)}{[\sum_{k \leq n} m(F_k)]^2} \right] \\ &= \liminf_{n \rightarrow \infty} \frac{\sum_{j,k \leq n} E[m(F_j \cap F_k)]}{[\sum_{k \leq n} m(F_k)]^2} \\ &= \liminf_{n \rightarrow \infty} \frac{\sum_{j,k \leq n, j \neq k} m(I_j)m(I_k) + \sum_{k \leq n} m(I_k)}{[\sum_{k \leq n} m(I_k)]^2} \\ &= \liminf_{n \rightarrow \infty} \left( 1 + \frac{\sum_{k \leq n} m(I_k)(1 - m(I_k))}{[\sum_{k \leq n} m(I_k)]^2} \right). \end{aligned}$$

Now, since  $1 - m(I_k) \rightarrow 1$ , and by (6.2), keeping in mind that  $m(I_k) = (1 - \rho_k)^{1-\alpha}$ , we have

$$\lim_{n \rightarrow \infty} \frac{\sum_{k \leq n} m(I_k)(1 - m(I_k))}{[\sum_{k \leq n} m(I_k)]^2} = 0.$$

This implies that (6.1) holds on a set  $B$  of positive probability and hence, by the zero-one law, on a set of probability one. From Lemma 6.1 we conclude  $P(\limsup_{n \rightarrow \infty} F_n) = 1$  a.s., which is what we had to show.  $\square$

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