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RANDOM INTERPOLATING SEQUENCES IN DIRICHLET SPACES

ANDREAS HARTMANN, KARIM KELLAY & BRETT D. WICK

ABSTRACT. We discuss random interpolating sequences in weighted Dirichlet spaces \mathcal{D}_α , $0 \leq \alpha \leq 1$. Our results in particular imply that almost sure interpolating sequences for \mathcal{D}_α are exactly the almost sure separated sequences when $0 \leq \alpha < 1/2$ (which covers the Hardy space $H^2 = \mathcal{D}_0$), and they are exactly the almost sure zero sequences for \mathcal{D}_α when $1/2 < \alpha < 1$. We also discuss the situation in the classical Dirichlet space $\mathcal{D} = \mathcal{D}_1$ where we get an almost optimal result. As a by-product we improve Rudowicz' sufficient condition for random Carleson measures in Hardy spaces.

1. INTRODUCTION

Understanding interpolating sequences is an important problem in complex analysis in one and several variables. The characterization of when a sequence of points is an interpolating sequences finds many applications to different problems in signal theory, control theory, operator theory, etc. In classical spaces like Hardy, Fock and Bergman spaces, interpolating sequences are now well understood objects, at least in one variable. In Dirichlet spaces, it turns however out that getting an exploitable description of such interpolating sequences is a notoriously difficult problem related to capacities. Crucial work has been undertaken in the 90s by Bishop and Marshall-Sundberg (see more precise indications below). However, while easier checkable sufficient conditions were given by Seip in the meantime, no real progress in the understanding of these sequences has been made since those works. In such a situation, a probabilistic approach can lead to a new vision of these interpolating sequences. Note that besides the Hardy and Bergman spaces, the Dirichlet space, and its weighted companions, are beyond the most prominent spaces of analytic functions on the unit disk. They appear naturally in problems on classical function theory, potential theory, as well as in operator theory when one investigates for instance weighted shifts.

Here we consider random sequences of the following kind. Let $\Lambda(\omega) = \{\lambda_n\}$ with $\lambda_n = \rho_n e^{i\theta_n(\omega)}$ where $\theta_n(\omega)$ is a sequence of independent random variables, all uniformly distributed on $[0, 2\pi]$ (Steinhaus sequence), and $\rho_n \in [0, 1)$ is a sequence of *a priori* fixed radii. Depending on distribution conditions on (ρ_n) as will be discussed below, we ask about the probability that $\Lambda(\omega)$ is interpolating for Dirichlet spaces \mathcal{D}_α , $0 \leq \alpha \leq 1$. Recall

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that the weighted Dirichlet space \mathcal{D}_α , $0 \leq \alpha \leq 1$, is the space of all analytic function f on the unit disc \mathbb{D} such that

$$\|f\|_\alpha^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^{1-\alpha} dA(z) < \infty,$$

where $dA(z) = dx dy / \pi$ stands for the normalized area measure on \mathbb{D} (we refer to [10] for Dirichlet spaces). If $\alpha = 0$, \mathcal{D}_0 is the Hardy space H^2 , and the classical Dirichlet space \mathcal{D} corresponds to $\alpha = 1$.

Recall that in a Hilbert space H of functions analytic in the unit disk \mathbb{D} equipped with a reproducing kernel k_λ , i.e. $f(\lambda) = \langle f, k_\lambda \rangle_H$ for every $\lambda \in \mathbb{D}$ and $f \in H$ (a so-called reproducing kernel Hilbert space), a sequence Λ of distinct points in \mathbb{D} is called (universal) interpolating if $\{(f(\lambda)/\|k_\lambda\|_H)_{\lambda \in \Lambda} : f \in H\} = \ell^2$ (for the difference between interpolating and universal interpolating sequences see below). Concerning the deterministic case of interpolation in the classical Dirichlet space \mathcal{D} , in unpublished work Bishop [4] and, independently, Marshall-Sundberg [11] characterized the interpolating sequences. The first published proof was given by Bøe [7] who provides a unifying scheme that applies to spaces that have a so-called complete Nevanlinna-Pick kernel (see [1, 18]) like Dirichlet spaces. For these spaces Λ is a (universal) interpolating sequence if and only if Λ is H -separated (i.e. $\inf_{\lambda \neq \lambda^* \in \Lambda} |\langle k_\lambda / \|k_\lambda\|_H, k_{\lambda^*} / \|k_{\lambda^*}\|_H \rangle| < \delta_\Lambda < 1$) and $\mu = \sum_{\lambda \in \Lambda} \delta_\lambda / \|k_\lambda\|_H^2$ is a Carleson measure for H (i.e. $\int_{\mathbb{D}} |f|^2 d\mu \leq C_\Lambda \|f\|_H^2$). There do exist complete characterizations of Carleson measures for \mathcal{D}_α ($0 < \alpha \leq 1$), but they are of a more complicated nature than the ones for $H^2 = \mathcal{D}_0$, and the corresponding conditions are not easily interpreted geometrically for the interpolating sequence, see for example [19, 3] as well as [2, 10] and references therein. However, in [18], Seip gave simple sufficient geometric conditions on a sequence to be (universal) interpolating for the Dirichlet spaces see Theorems 3.2 and 5.1, which will allow us to obtain sharp results in a certain sense for random interpolating sequences in \mathcal{D}_α for $\alpha \neq 1/2$ and $\alpha \neq 1$.

We also would like to observe that often, when the deterministic frame does not give a full answer to a problem, or if the deterministic conditions are not so easy to check, it is interesting to look at the random situation. In particular, it is interesting to ask for conditions ensuring that a sequence picked at random is interpolating almost surely or not (i.e., which are in a sense “generic situations”?). In this context, it is also worth mentioning the huge existing literature around gaussian analytic functions which investigates the zero distribution in classes of such functions [15].

The problems we would like to study in this paper are inspired by results by Cochran [9] and Rudowicz [16] who considered random interpolation in the Hardy space. Since interpolation in this space is again characterized by separation (in the pseudohyperbolic metric) and by the Carleson measure condition (note that the kernel is a complete Nevanlinna-Pick kernel), those authors were interested in a 0-1 law for separation, see [9], and a condition for being almost surely a Carleson measure [16], which led to a 0-1 law for interpolation. It is thus natural to discuss separation, Carleson measure type conditions and interpolation in Dirichlet spaces.

Concerning separation in Dirichlet spaces \mathcal{D}_α , $0 < \alpha < 1$, this turns out to be the same as in the Hardy spaces (see [18, p.22]), so that in that case Cochran's result perfectly characterizes the situation. The separation in the classical Dirichlet space, however, is much more delicate than in the Hardy space. We establish here a 0-1-type law for separation in \mathcal{D} . While our proof of this fact is inspired by Cochran's ideas, our proof requires a careful adaptation to the metric in that space.

Concerning Carleson measure type results in Dirichlet spaces, $0 < \alpha < 1$, we will first discuss the situation in the Hardy space and improve Rudowicz' result simplifying his proof. Our new proof, together with a 1-box condition by Seip (which requires itself separation), carries over to the Dirichlet situation and allows to discuss our results on interpolation in \mathcal{D}_α . As it turns out, we are able to exhibit a peculiar breakpoint in the behavior of such interpolating sequences depending on the weight α : for $0 \leq \alpha < 1/2$, almost sure separation implies almost sure interpolation, while for $1/2 < \alpha < 1$, almost sure zero sequences are almost surely interpolating. Partial results are given also for $\alpha = 1/2$.

Since zero sequences are of some importance as we have just seen, another central ingredient of our discussion is a rather immediate adaption of Bogdan's result on almost sure zero sequence in the Dirichlet space to the case of weighted Dirichlet spaces which we add for completeness in an annex.

In the classical Dirichlet space \mathcal{D} we are able to show that our sufficient condition for interpolation is almost optimal in a sense. Since in this case the condition for almost sure separation is much weaker than the Carleson-measure type condition, we cannot hope for optimality from the separation (as was the case in the Hardy space). It can already be seen from our improvement of the sufficient condition for Carleson measures in Hardy spaces, that the distribution conditions on (ρ_n) alone are not sufficient to obtain a 0-1 law.

Another result we would like to emphasize here is a new version of Seip's one box condition for (universal) interpolation in the Dirichlet space. As a matter of fact, his condition relating the points in a Carleson box — taking into account the natural logarithmic weight — to the side-length of the box is highly non-linear in the side-length. Our feeling is that usual Carleson windows are not necessarily adapted to spaces for which the underlying metric is not pseudohyperbolic. At least our probabilistic argument does not seem to work correctly for these windows. In order to circumvent that difficulty we give a version of Seip's one-box condition where the Carleson window is replaced by a flattened one. In that situation it is possible to arrange the parameters to obtain linear dependence on the side-length of the new window.

As usual, the definition of interpolating sequences is based on on the reproducing kernel of \mathcal{D}_α :

$$k_z(w) = \begin{cases} \frac{1}{\bar{z}w} \log \frac{1}{1 - \bar{z}w} & \text{if } \alpha = 1, \\ \frac{1}{(1 - \bar{z}w)^{1-\alpha}}, & \text{if } 0 \leq \alpha < 1. \end{cases} \quad (1.1)$$

Contrarily to the Hardy space situation, it turns out that in certain spaces (e.g. the Dirichlet space) there exist two notions of interpolation depending on whether the restriction operator $R_\Lambda : H \rightarrow \ell^2$, $R_\Lambda f = (f(\lambda)/\|k_\lambda\|_H)_{\lambda \in \Lambda}$ takes values in ℓ^2 or not.

Definition. Let $0 \leq \alpha \leq 1$. A sequence Λ of distinct points in \mathbb{D} said to be

- an interpolating sequence for \mathcal{D}_α if $R_\Lambda : f \rightarrow (f(\lambda)/\|k_\lambda\|_\alpha)_{\lambda \in \Lambda}$ is onto ℓ^2 , i.e. the interpolation problem $f(\lambda) = a_\lambda$ has a solution $f \in \mathcal{D}_\alpha$ whenever $(a_\lambda/\|k_\lambda\|_\alpha)_{\lambda \in \Lambda} \in \ell^2(\Lambda)$.
- a universal interpolating sequence for \mathcal{D}_α if it is interpolating and moreover R_Λ is well defined from \mathcal{D}_α into ℓ^2 .

Throughout this paper, even when not stated explicitly, when speaking about interpolation we mean universal interpolation. (Strictly speaking, in our theorems on interpolation, sufficient conditions always imply universal interpolation, while for necessary conditions it suffices to only impose standard interpolation.)

We will now discuss in details the results we have obtained in this paper.

1.1. Back to the Hardy space. As pointed out in the introduction, before considering the situation in the Dirichlet space, it seems appropriate to re-examine the situation in the Hardy space. Recall that Cochran established a 0-1 law for (pseudohyperbolic) separation (see Theorem 1.2.2 below) and Rudowicz showed that the same condition appearing for separation (see (1.5) below) implies almost surely the Carleson measure condition. This implies that interpolation is characterized by (1.5). As it turns out the situation in Dirichlet spaces is quite different. So, in order to better understand we start stating an improvement of Rudowicz' results on random Carleson measures in the Hardy space which will help to better understand the case of Dirichlet spaces.

Recall that the measure $dm_\Lambda = \sum_{\lambda \in \Lambda} (1 - |\lambda|^2)\delta_\lambda$ is called a Carleson measure if there is a constant C such that for every interval $I \subset \mathbb{T}$,

$$m_\Lambda(S_I) \leq C|I|,$$

where

$$S_I = \{z = re^{it} \in \mathbb{D} : e^{it} \in I, 1 - |I| \leq r < 1\}$$

is the usual Carleson window. We will prove that a weaker condition than Rudowicz' leads to Carleson measures almost surely in the Hardy space. We first need to introduce a notation:

$$N_n = \#\{\lambda \in \Lambda(\omega) : 1 - 2^{-n} \leq |\lambda| \leq 1 - 2^{-n-1}\}, \quad n = 0, 1, 2, \dots$$

Theorem 1.1. Let $\beta > 3/2$ and suppose

$$\sum_{n \geq 1} 2^{-n} N_n^\beta < +\infty.$$

Then the measure dm_Λ is a Carleson measure almost surely in the Hardy space.

Note that Rudowicz [16] showed that the above condition with $\beta = 2$ is sufficient. There is still a gap remaining between the above condition and the Blaschke condition which corresponds to $\beta = 1$. However, the following result shows that for exponents $\beta > 1$ there exist non-trivial sequences (ρ_n) for which the Carleson measure condition is still satisfied almost surely.

Theorem 1.2. *For every $\alpha > 1$, there exists a sequence (N_n) such that $\sum 2^{-n} N_n^\alpha = +\infty$, for every $\beta \in [1, \alpha)$, $\sum 2^{-n} N_n^\beta < +\infty$, and the measure dm_Λ is a Carleson measure almost surely in the Hardy space.*

As a result, the Carleson measure condition alone is not sufficient to give 0-1 laws for interpolation in the Hardy space.

1.2. Interpolation in Dirichlet spaces \mathcal{D}_α , $0 < \alpha < 1$. For our paper, a key result for interpolation in Dirichlet spaces is [18, Theorem 4, p.38] which shows that pseudohyperbolic separation (see definitions below) and a certain 1-box condition are sufficient for (universal) interpolation in \mathcal{D}_α , $0 < \alpha < 1$. As already mentioned in the introduction above, separation in this case is as in the Hardy space, so that we essentially need to discuss the Carleson measure part of Seip's theorem.

1.2.1. Random zero sequences in Dirichlet spaces. A central role in our interpolation results will be played by random zero sequences. Indeed, for an interpolating sequence in the Dirichlet space it is necessary to be a zero sequence (interpolation implies that there are functions vanishing on the whole sequence except for one point λ , and multiplying this function by $(z - \lambda)$ yields a function in the Dirichlet space vanishing on the whole sequence). We recall some results on random zero set in Dirichlet spaces. Carleson proved in [8] that when

$$\sum_{\lambda \in \Lambda} \|k_\lambda\|_\alpha^{-2} < \infty \quad (1.2)$$

then the associated Blaschke product B to Λ belongs to \mathcal{D}_α , $0 < \alpha < 1$ (for $\alpha = 0$ this corresponds of the is Blaschke condition for the Hardy space). When $\alpha = 1$, Shapiro–Shields proved in [17] that the condition (1.2) is sufficient for $\{\lambda\}_{\lambda \in \Lambda}$ to be a zero set for the classical Dirichlet space \mathcal{D}_1 , see also [18, Theorem 1]. Note that if $0 \leq \alpha < 1$ then

$$\sum_{\lambda \in \Lambda} \|k_\lambda\|_\alpha^{-2} \asymp \sum_{\lambda \in \Lambda} (1 - |\lambda|)^{1-\alpha} \asymp \sum_n 2^{-(1-\alpha)k} N_n$$

and if $\alpha = 1$ then

$$\sum_{\lambda \in \Lambda} \|k_\lambda\|_1^{-2} \asymp \sum_{\lambda \in \Lambda} |\log(1 - |\lambda|)|^{-1} \asymp \sum_n n^{-1} N_n.$$

On the other hand, it was proved by Nagel–Shapiro–Shields in [14] that if $\{r_n\} \subset (0, 1)$ does not satisfy (1.2), then there is $\{\theta_n\}$ such that $\{r_n e^{i\theta_n}\}$ is not a zero set for \mathcal{D}_α . Bogdan

[7, Theorem 2] gives a condition on the radii $|\lambda_n|$ for the sequence $\Lambda(\omega)$ to be almost surely zeros sequence for \mathcal{D} :

$$P(\Lambda(\omega) \text{ is a zero set for } \mathcal{D}) = \begin{cases} 1 \\ 0 \end{cases} \text{ if and only if } \sum_n n^{-1} N_n \begin{cases} < \infty \\ = \infty. \end{cases} \quad (1.3)$$

Bogdan's arguments carry over to \mathcal{D}_α , $\alpha \in (0, 1)$. For the sake of completeness, we will prove in the annex, Section 8, the following result on almost sure zero sequences.

Theorem 1.3. *Let $0 \leq \alpha < 1$. Then*

$$P(\Lambda(\omega) \text{ is a zero set for } \mathcal{D}_\alpha) = \begin{cases} 1 \\ 0 \end{cases} \text{ if and only if } \sum_n 2^{-(1-\alpha)n} N_n \begin{cases} < \infty \\ = \infty. \end{cases} \quad (1.4)$$

1.2.2. Interpolation in Dirichlet spaces \mathcal{D}_α . As pointed out earlier, interpolation is intimately related with separation conditions and Carleson measure type conditions. Recall that a sequence Λ is called (pseudohyperbolically) separated if

$$\inf_{\substack{\lambda, \lambda^* \in \Lambda \\ \lambda \neq \lambda^*}} \rho(\lambda, \lambda^*) = \inf_{\substack{\lambda, \lambda^* \in \Lambda \\ \lambda \neq \lambda^*}} \frac{|\lambda - \lambda^*|}{|1 - \bar{\lambda}\lambda^*|} \geq \delta_\Lambda > 0.$$

Since in Dirichlet spaces \mathcal{D}_α , $0 \leq \alpha < 1$, the natural separation (\mathcal{D}_α -separated sequence) is indeed pseudohyperbolic separation [18, p.22], we recall Cochran's separation result on pseudohyperbolic separation.

Theorem (Cochran). *A sequence $\Lambda(\omega)$ is almost surely (pseudohyperbolically) separated if and only if*

$$\sum_n 2^{-n} N_n^2 < +\infty. \quad (1.5)$$

We should pause here to make a crucial observation. We have already mentioned that interpolating sequences are necessarily zero-sequences. Also separation is another necessary condition for interpolation. Now the condition for zero sequences (1.4) depends on α while the separation condition does not, and it follows that depending on α , it is one condition or the other which is dominating. From (1.4) and (1.5) it is not difficult to see that this breakpoint is exactly at $\alpha = 1/2$ (for $\alpha = 1/2$, (1.4) still implies (1.5)). This motivates already the necessary conditions of our next result. For the sufficiency we will need to appeal to Seip's one-box condition [18, Theorem 5, p.38] see also Theorem 3.2 which we will discuss later.

Theorem 1.4.

(i) *Let $0 < \alpha < 1/2$, then*

$$P(\Lambda(\omega) \text{ is interpolating for } \mathcal{D}_\alpha) = \begin{cases} 1 \\ 0 \end{cases} \text{ if and only if } \sum_n 2^{-n} N_n^2 \begin{cases} < \infty \\ = \infty. \end{cases}$$

(ii) Let $\alpha = 1/2$. If there exists $\beta > 2$ such that

$$\sum_k 2^{-n} N_n^\beta < \infty$$

then $P(\Lambda(\omega) \text{ is interpolating for } \mathcal{D}_{1/2}) = 1$.

Conversely, If $P(\Lambda(\omega) \text{ is interpolating for } \mathcal{D}_{1/2}) = 1$ then

$$\sum_n 2^{-n/2} N_n < \infty.$$

(iii) Let $1/2 < \alpha < 1$. Then

$$P(\Lambda(\omega) \text{ is interpolating for } \mathcal{D}_\alpha) = \begin{cases} 1 \\ 0 \end{cases} \text{ if and only if } \sum_n 2^{-(1-\alpha)n} N_n \begin{cases} < \infty \\ = \infty. \end{cases}$$

Our techniques, based on Seip's one-box condition, do not provide a complete answer in the case $\alpha = 1/2$ which needs further investigation.

An interesting reformulation of the above results connects random interpolation with random zero sequences and random separated sequences as stated in the following corollary.

Corollary 1.5. *The following statements hold:*

- (1) Let $0 \leq \alpha < 1/2$. The sequence $\Lambda(\omega)$ is almost surely separated if and only if it is interpolating almost surely.
- (2) Let $1/2 < \alpha < 1$. The sequence $\Lambda(\omega)$ is almost surely a zero sequence for \mathcal{D}_α if and only if it is almost surely interpolating in \mathcal{D}_α .

1.3. Interpolation in the classical Dirichlet space. For the classical Dirichlet space we will first establish a result on separation, and then use again a 1-box condition by Seip as stated in [18, Theorem 5, p.39].

1.3.1. Separation in the Dirichlet space. In the case $\alpha = 1$, the separation is given in a different way. Let

$$\rho_{\mathcal{D}}(z, w) = \sqrt{1 - \frac{|k_w(z)|^2}{k_z(z)k_w(w)}}, \quad z, w \in \mathbb{D}.$$

A sequence Λ is called \mathcal{D} -separated if

$$\inf_{\substack{\lambda, \lambda^* \in \Lambda \\ \lambda \neq \lambda^*}} \rho_{\mathcal{D}}(\lambda, \lambda^*) > \delta_\Lambda > 0$$

for some $\delta_\Lambda < 1$. This is equivalent to (see [18, p.23])

$$\frac{(1 - |\lambda|^2)(1 - |\lambda^*|^2)}{|1 - \bar{\lambda}\lambda^*|^2} \leq (1 - |\lambda|^2)^{\delta_\Lambda^2}, \quad \lambda, \lambda^* \in \Lambda. \quad (1.6)$$

For separation in the Dirichlet space \mathcal{D} we obtain the following 0-1 law.

Theorem 1.6.

$$P(\Lambda(\omega) \text{ is } \mathcal{D}\text{-separated for }) = \begin{cases} 1, & \text{if } \exists \gamma \in (1/2, 1) \text{ such that } \sum 2^{-\gamma n} N_n^2 < \infty, \\ 0, & \text{if } \forall \gamma \in (1/2, 1) \text{ such that } \sum_n 2^{-\gamma n} N_n^2 = \infty. \end{cases}$$

We observe that in both conditions we can replace the sum by a supremum (this amounts to replacing γ by a slightly bigger or smaller value). The lower bound $1/2$ for γ is not very important, since it is the behavior close to the value 1 which counts.

1.3.2. Interpolation in the Dirichlet space \mathcal{D} . In the Dirichlet space \mathcal{D} , the dyadic annuli (used in the Hardy space) are not adapted to the hyperbolic geometry. Instead we need to consider annuli with radii $2^{-\beta n}$, where $\beta > 1$ is some suitable constant. We will thus consider the following annuli:

$$A_{n,\beta} := \{z \in \mathbb{D} : 1 - 2^{-\beta n} \leq |z| < 1 - 2^{-\beta^{n+1}}\}.$$

Correspondingly we will set

$$N_{n,\beta} = \#(\Lambda \cap A_{n,\beta}).$$

We will now discuss random interpolating sequences in \mathcal{D} . Let us first mention that Bogdan's result reads as follows:

$$\sum_n \beta^{-n} N_{n,\beta} = \infty \implies P(\Lambda(\omega) \text{ is an interpolating sequence for } \mathcal{D}) = 0. \quad (1.7)$$

Indeed, it suffices to apply Lemma 6.1 to see that the above condition is equivalent to $\sum_n n^{-1} N_n = +\infty$ and to use (1.3). In order to show that this result is optimal in a sense we translate Theorems 1.1 and 1.2 to Dirichlet spaces. We actually get the same behavior in the power of $N_{k,\beta}$ as in the Hardy space.

Theorem 1.7. *If there exist $\beta > 1$ and $\kappa > 3/2$ such that*

$$\sum_n \beta^{-n} N_{n,\beta}^\kappa < \infty, \quad (1.8)$$

then $P(\Lambda(\omega) \text{ is an interpolating sequence for } \mathcal{D}) = 1$.

Observe that condition (1.8) implies the Bogdan condition so that the sequence is almost surely a zero sequence in \mathcal{D} and so, in view of Theorem 1.6, the sequence is also \mathcal{D} -separated.

Note that an annulus $A_{n,\beta}$ can always be covered by $\ln \beta / \ln \beta'$ annuli $A_{k,\beta'}$, when $1 < \beta' < \beta$ (thus independently on n). A natural question in this connection is whether the above condition depends on the choice of β . We refer to Lemma 6.1 to show that this is not the case.

As in the Hardy space, we will see that when $\kappa > 1$, there exists sequences (ρ_n) satisfying condition (1.8) for this κ (but not for values bigger than κ) and such that the associated

sequences $\Lambda(\omega)$ are almost surely interpolating (see Theorem 7.1).

This paper is organized as follows. In the next section we present the improved version of the Rudowicz result concerning random Carleson measures in Hardy spaces which is the guideline for the corresponding result in the Dirichlet space. Indeed, this largely clarifies and simplifies not only the situation in the Hardy space, but also indicates the direction of investigation for the Dirichlet spaces (classical and weighted). In Section 3 we prove the sufficient condition for interpolation in \mathcal{D}_α , $0 < \alpha < 1$, as stated in Theorem 3.1. Here, we will also prove Corollary 1.4. In the following section we show the 0-1 law on separation in the classical Dirichlet space. This requires a subtle adaption of the Cochran discussion in the Hardy space to the much more intricate geometry in the Dirichlet space. Section 4 is devoted to the improvement of the random Carleson measure result of Rudowicz. In Section 5 we discuss a reformulation of Seip's one-box condition for interpolation in \mathcal{D} which will be needed in the proof of the interpolation result in the Dirichlet space. The short Section 6 contains some useful auxiliary results linking different distribution conditions. The proofs of the results on interpolating sequences in the classical Dirichlet space are contained in Section 7. The flattened one-box condition obtained in Section 6 allows to transfer the proofs from the Hardy and weighted Dirichlet space situation to the classical Dirichlet space. Actually, the core of the proof being probabilistic, we are able to get rid of analytic functions. In the final Section 8, we give some indications to the 0-1 law on zero-sequences in weighted Dirichlet spaces based on Bogdan's proof in the classical Dirichlet space.

A word on notation. We will write $A \lesssim B$ meaning that $A \leq cB$ for some positive constant c not depending on the parameters behind A and B . By $A \sim B$ (where A and B are strictly positive expressions) we mean that the quotient $A/B \rightarrow 1$ when passing to the suitable limit.

2. CARLESON CONDITION IN THE HARDY SPACE

Before considering Carleson measure conditions in the Dirichlet space, we will discuss the situation in the Hardy space, in particular we will prove here Theorems 1.1 and 1.2. We start with the following elementary lemma which is well known in probability theory and which will be very useful in the proof below of Theorem 1.1 as well as in Section 7 (it is essentially approximation of the binomial law by Poisson law). We refer for instance to [5] for the material on probability theory — essentially elementary — used in this paper.

Lemma 2.1. *If X is a binomial random variable with parameters p, N , then for $s = 0, 1, 2, \dots$,*

$$\lim_{\substack{N \rightarrow \infty \\ pN \rightarrow 0}} \frac{P(X = s)}{(pN)^s} = \lim_{\substack{N \rightarrow \infty \\ pN \rightarrow 0}} \frac{P(X \geq s)}{(pN)^s} = \frac{1}{s!}.$$

2.1. Proof of Theorem 1.1 and Theorem 1.2. Before proving the theorems let us make a first observation. It is known that the sequence (N_n) can be assumed increasing. Here is

an argument for that. Set $\tilde{N}_n = \sup_{1 \leq k \leq n} N_k$, then (\tilde{N}_n) is clearly increasing. Define $\varphi(k)$ such that $\tilde{N}_{\varphi(k)} > \tilde{N}_{\varphi(k)-1}$ which corresponds to the points where N_i is strictly larger than all the elements before. Note that for $\varphi(k) \leq n \leq \varphi(k+1) - 1$ we have $\tilde{N}_n = N_{\varphi(k)}$. Then

$$\sum_n 2^{-n} \tilde{N}_n = \sum_{k \geq 1} \sum_{n=\varphi(k)}^{\varphi(k+1)-1} 2^{-n} \tilde{N}_n = \sum_{k \geq 1} N_{\varphi(k)} \sum_{n=\varphi(k)}^{\varphi(k+1)-1} 2^{-n} \leq 2 \sum_{k \geq 1} N_{\varphi(k)} 2^{-\varphi(k)} < +\infty,$$

since the sequence $(\varphi(k))_k$ is strictly increasing.

We will also need to introduce some notation. Let

$$I_{n,k} = \{e^{it} : t \in [k2^{-n}, (k+1)2^{-n})\} \quad n \in \mathbb{N}, \quad k = 0, 1, \dots, 2^n$$

be dyadic intervals and $S_{n,k} = S_{I_{n,k}}$ the associated Carleson window. In order to check the Carleson measure condition for a positive Borel measure μ on \mathbb{D} it is clearly sufficient to check the Carleson measure condition for windows $S_{n,k}$:

$$\mu(S_{n,k}) \leq C|I_{n,k}| = C2^{-n},$$

for some fixed $C > 0$ and every $n \in \mathbb{N}$, $k = 0, \dots, 2^n$. Given n, k and $m \geq k$ let $X_{n,m,k}$ be the number of points of Λ contained in $S_{n,k} \cap A_m$ (we stratify the Carleson window $S_{n,k}$ into a disjoint union of layers $S_{n,k} \cap A_m$). Since A_m contains N_m points and the (normalized) length of $S_{n,k}$ is 2^{-n} , we have $X_{n,m,k} \sim B(2^{-n}, N_m)$ (binomial law). In order to show that dm_Λ is almost surely a Carleson measure we thus have to prove the existence of C such that

$$m_\Lambda(S_{n,k}) = \sum_{m \geq n} 2^{-m} X_{n,m,k} \leq C2^{-n}$$

almost surely, in other words we have to prove

$$\sup_{n,k} 2^n \sum_{m \geq n} 2^{-m} X_{n,m,k} \leq C$$

almost surely (in ω). The estimate above had already been investigated by Rudowicz [16]. Here we will proceed in a different way with respect to Rudowicz' argument to obtain an improved version of his result and which allows to better understand the Dirichlet space situation.

Proof of Theorem 1.1. In view of our preliminary remarks, we need to look at the random variable

$$Y_{n,k} = 2^n \sum_{m=n}^{+\infty} 2^{-m} X_{n,m,k},$$

where, as said above, $X_{n,m,k} \sim B(2^{-n}, N_m)$. Hence, saying that $Y_{n,k} \geq A$ means that there are Carleson windows for which the Carleson measure constant is at least A .

For $\gamma > 0$ to be fixed later, we will write

$$Y_{n,k} = \underbrace{2^n \sum_{m=n}^{n(1+\gamma)} 2^{-m} X_{n,m,k}}_{Y_{n,k}^0} + \underbrace{2^n \sum_{m=n(1+\gamma)+1}^{+\infty} 2^{-m} X_{n,m,k}}_{R_{n,k}}.$$

Let us also introduce

$$Z_{n,k} = \sum_{m=n}^{n(1+\gamma)} X_{n,m,k}.$$

Note that $Z_{n,k}$ is the sum of binomial variables having the same parameter $p = 2^{-n}$. The sum $Z_{n,k}$ is thus also binomially distributed with parameter $p = 2^{-n}$ and $N = N_n + \dots + N_{n(1+\gamma)}$. Now note that the event $(Y_{n,k} \geq A)$ is contained in the union $(Y_{n,k}^0 \geq A/2) \cup (R_{n,k} \geq A/2)$. Let us estimate the probability of the first event. Using the fact that $(Z_{n,k} \geq A/2)$ is easier to achieve than $(Y_{n,k}^0 \geq A/2)$ and by Lemma 2.1, we get assuming $A/2$ an integer,

$$P(Y_{n,k}^0 \geq A/2) \leq P(Z_{n,k} \geq A/2) \sim \frac{(pN)^{A/2}}{(A/2)!} = \frac{(2^{-n}(N_n + \dots + N_{n(1+\gamma)}))^{A/2}}{(A/2)!}.$$

Now remember that $\sum_n 2^{-n} N_n^\beta < \infty$ so that in particular $N_n \leq 2^{n/\beta}$ (at least for n sufficiently large). Hence

$$\frac{(2^{-n}(N_n + \dots + N_{n(1+\gamma)}))^{A/2}}{(A/2)!} \lesssim (2^{-n(1-(1+\gamma)/\beta)})^{A/2}.$$

We need to pick γ such that $(1 + \gamma)/\beta < 1$, equivalently

$$\gamma < \beta - 1. \quad (2.1)$$

In that case, for sufficiently large $A/2$,

$$P(Y_{n,k}^0 \geq A/2) \leq P(Z_{n,k} \geq A/2) \sim \frac{(pN)^{A/2}}{(A/2)!} \lesssim 2^{-4n}.$$

We estimate $R_{n,k}$ as Rudowicz did for $Y_{n,k}$ using Tchebychev's inequality:

$$\begin{aligned} P(R_{n,k} \geq A/2) &\lesssim \text{Var}(R_{n,k}) = 4^n \sum_{m \geq n(1+\gamma)} 4^{-m} \text{Var}(X_{n,m,k}) \\ &\lesssim 4^n \sum_{m \geq n(1+\gamma)} 4^{-m} \times 2^{-n} N_m = 2^n \sum_{m \geq n(1+\gamma)} 2^{-2m+m/\beta} \\ &\leq 2^{n(1-2(1+\gamma)+(1+\gamma)/\beta)} = 2^{-n(1+2\gamma-(1+\gamma)/\beta)} \end{aligned} \quad (2.2)$$

We will need $\eta := 1 + 2\gamma - (1 + \gamma)/\beta > 1$, equivalently $2\gamma > (1 + \gamma)/\beta = 1/\beta + \gamma/\beta$, i.e.

$$\gamma > \frac{1/\beta}{2 - 1/\beta} = \frac{1}{2\beta - 1}. \quad (2.3)$$

Note that the two conditions on γ , (2.1) and (2.3), require

$$\frac{1}{2\beta - 1} < \beta - 1 \iff 1 < (\beta - 1)(2\beta - 1) = 2\beta^2 - 3\beta + 1 \iff 0 < \beta(2\beta - 3),$$

which happens exactly for $\beta > 3/2$ (when β is positive as assumed). In that case, we can pick γ such that

$$\frac{1}{2\beta - 1} < \gamma < \beta - 1.$$

Hence, using that $(Y_{n,k} \geq A) \subset (Y_{n,k}^0 \geq A/2) \cup (R_{n,k} \geq A/2)$,

$$P(Y_{n,k} \geq A) \leq P(Y_{n,k}^0 \geq A/2) + P(R_{n,k} \geq A/2) \leq 2^{-4n} + 2^{-\eta n}.$$

In view of an application of the Borel-Cantelli Lemma we compute

$$\sum_{n \geq 0} \sum_{k=1}^{2^n} P(Y_{n,k} \geq A) \leq \sum_{n \geq 0} 2^n \times (2^{-4n} + 2^{-\eta n}) < \infty,$$

where we have used that $\eta > 1$. Hence, by the Borel-Cantelli Lemma, the event $Y_{n,k} \geq A$ can happen for at most a finite number of indices (n, k) . In particular the Carleson measure constant of dm_Λ is almost surely at most A except for a finite number of Carleson windows. \square

Theorem 1.2 shows that the gap $\beta \in (1, 3/2]$ can be removed if we relax the requirement that the Carleson measure condition should be satisfied almost surely for *every* sequence satisfying the distribution condition.

Proof of Theorem 1.2. We set

$$N_n = \begin{cases} 2^{n/\alpha}, & \text{if there is } \ell \in \mathbb{N}, n = 2^\ell \\ 0 & \text{else.} \end{cases}$$

Then clearly $\sum_n 2^{-n} N_n^\alpha = +\infty$ and $\sum_n 2^{-n} N_n^\beta < +\infty$ for every $\beta < \alpha$. We will now show that this sequence is almost surely interpolating in the Hardy space. For this, we will use similar ideas and similar notation as in the previous proof. Write

$$Y_{n,k} = \underbrace{2^n \sum_{m=n}^{2n-1} 2^{-m} X_{n,m,k}}_{Y_{n,k}^0} + \underbrace{2^n \sum_{m=2n}^{+\infty} 2^{-m} X_{n,m,k}}_{R_{n,k}}.$$

As before, $P(R_{n,k} \geq A/2)$ can be estimated with Tchebychev's inequality (now $1 + \gamma = 2$). With the exact same computations as in (2.2) we obtain $P(R_{n,k} \geq A/2) \lesssim 2^{-n(3-2/\alpha)}$, which, since $\alpha > 1$, is again summable against 2^n . So we are mainly interested in $P(Y_{n,k}^0 \geq A/2)$. By construction there is a unique $L = 2^\ell \in [n, 2n - 1]$, for a suitable $\ell \in \mathbb{N}$, such that

$$Y_{n,k}^0 = 2^n \times 2^{-L} X_{n,L,k}.$$

Hence $P(Y_{n,k}^0 \geq A/2) = P(X_{n,L,k} \geq 2^{L-n}A/2)$. Since $X_{n,L,k} \sim B(2^{-n}, 2^{L/\alpha})$, using Lemma 2.1, we get

$$P(X_{n,L,k} \geq 2^{L-n}A/2) \simeq \frac{(2^{-n}2^{L/\alpha})^{2^{L-n}A/2}}{(2^{L-n}A/2)!}.$$

Recall the Stirling approximation:

$$K! \sim \sqrt{2\pi K} \left(\frac{K}{e}\right)^K \quad (2.4)$$

so that (assuming $K = 2^{L-n}A/2$ integer),

$$\begin{aligned} P(X_{n,L,k} \geq 2^{L-n}A/2) &\sim \frac{(2^{-n}2^{L/\alpha})^{2^{L-n}A/2}}{\sqrt{2\pi 2^{L-n}A/2}} \times \left(\frac{e}{2^{L-n}A/2}\right)^{2^{L-n}A/2} \\ &= \frac{1}{\sqrt{2\pi 2^{L-n}A/2}} \times \left(\frac{e \times 2^{-n}2^{L/\alpha}}{2^{L-n}A/2}\right)^{2^{L-n}A/2} \\ &= \frac{1}{\sqrt{\pi 2^{L-n}A}} \times \left(\frac{2e}{A} \times 2^{-(1-1/\alpha)L}\right)^{2^{L-n}A/2}. \end{aligned}$$

Since $\alpha > 1$ and choosing $A > 2e$ the expression between parenthesis is less than 1. In particular instead of taking the power $2^{L-n}A/2$ we raise this expression only to the power $A/2$ making it bigger. Moreover we can bound the first factor with the square-root in the denominator by one. As a result

$$P(X_{n,L,k} \geq 2^{L-n}A/2) \lesssim 2^{-(1-1/\alpha)L \times A/2}.$$

Now choose A such that $\eta = (1 - 1/\alpha)A/2 > 1$, so that $P(Y_{n,k}^0 \geq A/2) \leq 2^{-\eta m}$, and conclude as in the preceding proof. \square

3. PROOF OF THEOREM 1.4

The proof of Theorem 1.4 is based on the following seemingly more general result.

Theorem 3.1. *Let $0 < \alpha < 1$. Suppose $\Lambda(\omega)$ is almost surely separated and almost surely a zero sequence. If there exists $\beta > \frac{3-2\alpha}{2-2\alpha}$ such that*

$$\sum_n 2^{-n} N_n^\beta < \infty \quad (3.1)$$

then $P(\Lambda(\omega) \text{ is interpolating } \mathcal{D}_\alpha) = 1$.

As we have already mentioned in Subsection 1.2.2, imposing simultaneously Λ being a zero sequence and separated is rather artificial.

In order to prove Theorem 3.1 we adapt the proof of the preceding section to \mathcal{D}_α . Our discussions — here for \mathcal{D}_α , as well as as in Section 7 for \mathcal{D} — are based on one-box

conditions as introduced by Seip in [18]. We will in particular use the following result ([18, Theorem 4, p.38]).

Theorem 3.2 (Seip). *A separated sequence Λ in \mathbb{D} is a universal interpolating sequence for \mathcal{D}_α , $0 < \alpha < 1$, if there exist $0 < \kappa < 1 - \alpha$ and $C > 0$ such that for each arc $I \subset \mathbb{T}$*

$$\sum_{\lambda \in \Lambda \cap S_I} (1 - |\lambda|)^\kappa \leq C|I|^\kappa. \quad (3.2)$$

Observe in particular that the above condition implies

$$\sum_n 2^{-n\kappa} N_n < +\infty,$$

so that in particular $\sum_n 2^{-n(1-\alpha)} N_n < +\infty$, and thus that (3.2) implies that Λ is a zero sequence almost surely.

We have to show that the conditions of Theorem 3.1 imply (3.2) almost surely for some fixed $\kappa \in (0, 1)$. We use the same reasoning as in the preceding section. In particular with the same notation we have to prove

$$\sup_{n,k} 2^{\kappa n} \sum_{m \geq n} 2^{-\kappa m} X_{n,m,k} \leq C$$

almost surely. Again $X_{n,m,k} \sim B(2^{-n}, N_m)$, where we can now assume $N_m \leq 2^{m/\beta}$ for some $\beta > (3 - 2\alpha)/(2 - 2\alpha)$ (at least for sufficiently large m). For $\gamma > 0$ to be fixed later, we will write

$$Y_{n,k} = \underbrace{2^{\kappa n} \sum_{m=n}^{n(1+\gamma)} 2^{-\kappa m} X_{n,m,k}}_{Y_{n,k}^0} + \underbrace{2^{\kappa n} \sum_{m=n(1+\gamma)+1}^{+\infty} 2^{-\kappa m} X_{n,m,k}}_{R_{n,k}}.$$

We also use the notation $Z_{n,k}$ as in the preceding section, so that now we get

$$P(Y_{n,k}^0 \geq A/2) \leq P(Z_{n,k} \geq A/2) \sim \frac{(pN)^{A/2}}{(A/2)!} = \frac{(2^{-n} \sum_{m=n}^{n(1+\gamma)} N_m)^{A/2}}{(A/2)!}.$$

It can be observed that we now have less points N_m as in the Hardy space situation (where $N_m \leq 2^{m/\beta}$ with $\beta > 3/2$, while now we have $N_m \leq 2^{m/\beta}$ with $\beta > (3-2\alpha)/(2-2\alpha) > 3/2$). It is thus clear that for sufficiently large A we again conclude $P(Y_{n,k}^0 \geq A/2) \leq 2^{-4n}$. To be more precise, taking into account that $N_m \leq 2^{m/\beta}$, we see that $2^{-n} \sum_{m=n}^{n(1+\gamma)} N_m \lesssim 2^{-n(1-(1+\gamma)/\beta)}$ so that we get the same condition on γ as before: $(1 + \gamma)/\beta < 1$

$$\gamma < \beta - 1. \quad (3.3)$$

In that case, setting $\eta = 1 - (1 + \gamma)/\beta > 0$ we have $P(Y_{n,k}^0 \geq A/2) \leq 2^{-\eta \times A/2} \leq 2^{-4n}$ for sufficiently large A .

We estimate $R_{n,k}$ again using Tchebychev's inequality:

$$\begin{aligned} P(R_{n,k} \geq A/2) &\lesssim \text{Var}(R_{n,k}) = 4^{\kappa n} \sum_{m \geq n(1+\gamma)} 4^{-\kappa m} \text{Var}(X_{n,m,k}) \\ &\lesssim 4^{\kappa n} \sum_{m \geq n(1+\gamma)} 4^{-\kappa m} \times 2^{-n} N_m \leq 2^{(2\kappa-1)n} \sum_{m \geq n(1+\gamma)} 2^{-2\kappa m + m/\beta} \\ &\leq 2^{n(2\kappa-1-2\kappa(1+\gamma)+(1+\gamma)/\beta)} = 2^{-n(1+2\kappa\gamma-(1+\gamma)/\beta)}. \end{aligned}$$

We will need $\eta := 1 + 2\kappa\gamma - (1 + \gamma)/\beta > 1$, equivalently $2\kappa\gamma > (1 + \gamma)/\beta$, or $2\kappa\beta\gamma > 1 + \gamma$, i.e.

$$\gamma > \frac{1}{2\kappa\beta - 1}. \quad (3.4)$$

Now the two conditions on γ , (3.3) and (3.4), require

$$\begin{aligned} \frac{1}{2\kappa\beta - 1} < \beta - 1 &\iff 1 < 2\kappa\beta^2 - \beta - 2\kappa\beta + 1 \iff \beta(1 + 2\kappa) < 2\kappa\beta^2 \\ &\iff \beta > \frac{1 + 2\kappa}{2\kappa} = 1 + \frac{1}{2\kappa}. \end{aligned} \quad (3.5)$$

Observe that Seip's result requires $0 < \kappa < 1 - \alpha$ and that the above expression is decreasing in κ so that:

$$\frac{1 + 2\kappa}{2\kappa} > \frac{1 + 2(1 - \alpha)}{2(1 - \alpha)} = \frac{3 - 2\alpha}{2 - 2\alpha}.$$

By assumption, we have $\beta > (3 - 2\alpha)/(2 - 2\alpha)$ so that there exists $\kappa \in (0, 1 - \alpha)$ with (3.5). And for that κ we can find γ satisfying (3.3) and (3.4). From here, the proof follows the same lines as in the Hardy space discussed in the preceding section. This proves Theorem 3.1. \square

Let us give the proof of Corollary 1.4.

Proof of Corollary 1.4. (i) Let $0 < \alpha < 1/2$.

If Λ is interpolating almost surely, then it is separated almost surely, which implies $\sum_n 2^{-n} N_n^2 < +\infty$.

If $\sum_n 2^{-n} N_n^2 < +\infty$, then Λ is almost surely separated. Moreover, we can pick $\beta = 2$ in (3.1). Hence

$$\frac{3 - 2\alpha}{2 - 2\alpha} = 1 + \frac{1}{2(1 - \alpha)} < 2 = \beta.$$

Also, in this case $\sum_n 2^{-(1-\alpha)n} N_n < +\infty$, which gives (1.4) and hence, by Theorem 1.3 that Λ is a zero sequence almost surely. We conclude from Theorem 3.1 that Λ is almost surely interpolating.

(iii) Consider the case $1/2 < \alpha < 1$.

If Λ is interpolating almost surely, then it is a zero sequence almost surely, which implies $\sum_n 2^{-(1-\alpha)n} N_n < +\infty$ by Theorem 1.3.

Suppose $\sum_n 2^{-(1-\alpha)n} N_n < +\infty$. Then Λ is a zero sequence almost surely by Theorem 1.3. Also it is clear that the condition implies that $\sum_n 2^{-n} N_n^2 < +\infty$, which further implies that the sequence is almost surely separated. Pick

$$\beta = 1 + \frac{1}{2(1-\alpha)} + \epsilon > 1 + \frac{1}{2(1-\alpha)} = \frac{3-2\alpha}{2-2\alpha},$$

for some $\epsilon > 0$ to be fixed. The distribution condition implies $N_n \leq 2^{(1-\alpha)n}$ (at least for sufficiently large n). Hence

$$N_n^{\beta-1} \leq 2^{(1-\alpha) \times (\beta-1)n} = 2^{(1-\alpha) \times (\frac{1}{2(1-\alpha)} + \epsilon)n} = 2^{(1/2 + (1-\alpha)\epsilon)n}.$$

Then

$$2^{-\alpha n} \times N_n^{\beta-1} = 2^{(-\alpha + 1/2 + (1-\alpha)\epsilon)n}.$$

Now since $\alpha > 1/2$, there exists $\epsilon > 0$ such that $-\alpha + 1/2 + (1-\alpha)\epsilon < 0$. Hence

$$\sum_n 2^{-n} N_n^\beta = \sum_n 2^{-(1-\alpha)n} N_n \times 2^{-\alpha n} N_n^{\beta-1} < \infty.$$

We conclude from Theorem 3.1 that Λ is interpolating almost surely.

If $\sum_n 2^{-(1-\alpha)n} N_n = +\infty$. Then Λ is not a zero sequence almost surely by Theorem 1.3, and hence it is almost surely not interpolating.

It remains to check that when Λ is almost surely not interpolating, then $\sum_n 2^{-(1-\alpha)n} N_n = +\infty$. By contraposition, if the sum is bounded, we have to show that the probability that Λ is interpolating is strictly positive. But we already know that in that case this probability is $1 > 0$.

(ii) $\alpha = 1/2$: In this case we only have a sufficient condition which follows by direct application of Theorems 1.3 and 3.1. \square

4. SEPARATED RANDOM SEQUENCES FOR THE DIRICHLET SPACE

We will now prove the separation result in \mathcal{D} .

Proof of Theorem 1.6. Separation with probability 0. Assume that for all $\gamma \in (1/2, 1)$ we have $\sup_k 2^{-\gamma k} N_k^2 = \infty$. As it turns out, under the condition of the Theorem, separation already fails in dyadic annuli (without taking into account radial Dirichlet separation).

Assume now that $\gamma_l \rightarrow 1$ as $l \rightarrow \infty$ and $\sup_k 2^{-\gamma_l k} N_k^2 = \infty$ for every l . For each $k = 1, 2, \dots$, let $I_k = [1 - 2^{-k+1}, 1 - 2^{-k}]$. Define

$$\Omega_k^{(l)} = \{\omega : \exists(i, j), i \neq j \text{ with } \rho_i, \rho_j \in I_k \text{ and } |\theta_i(\omega) - \theta_j(\omega)| \leq \pi 2^{-\gamma_l k}\}.$$

In view of (1.6), if $\omega \in \Omega_k^{(l)}$, this means that in the dyadic annulus A_k there are at least two points close in the Dirichlet metric. To be more precise, if $\omega \in \Omega_k^{(l)}$, then there is a pair of distinct point $\lambda_i(\omega)$ and $\lambda_j(\omega)$ such that $|\lambda_i|, |\lambda_j| \in I_k$ and $|\arg \lambda_i(\omega) - \arg \lambda_j(\omega)| \leq \pi 2^{-\gamma_l k}$. Hence

$$\frac{(1 - |\lambda_i|^2)(1 - |\lambda_j|^2)}{|1 - \overline{\lambda_i} \lambda_j|^2} \geq c \frac{2^{-2k}}{2^{-2k} + \pi 2^{-2\gamma_l k}},$$

where the constant c is an absolute constant. Hence

$$\frac{(1 - |\lambda_i|^2)(1 - |\lambda_j|^2)}{|1 - \overline{\lambda_i}\lambda_j|^2} \geq c \frac{1}{1 + \pi 2^{k(1-\gamma)}} \geq c' 2^{-k(1-\gamma)} \geq c'' (1 - |\lambda_i|)^{1-\gamma}.$$

Absorbing c'' into a suitable change of the power $\delta_l^2 := 1 - \gamma_l$ into $\delta'_l{}^2$ (which can be taken by choosing for instance $2\delta_l > \delta'_l > \delta_l$ provided k is large enough), then by (1.6)

$$\rho_{\mathcal{D}}(\lambda_i(\omega), \lambda_j(\omega)) \leq \delta'_l.$$

Our aim is thus to show that for every $l \in \mathbb{N}$, we can find almost surely $\lambda_i(\omega) \neq \lambda_j(\omega)$ such that $\rho_{\mathcal{D}}(\lambda_i(\omega), \lambda_j(\omega)) \leq \delta'_l$, i.e. $P(\Omega_k^{(l)}) = 1$.

Let us define a set $E := \{j : 2^{-\gamma j-1} N_j \leq 1\}$. Observe that when $k \notin E$, then at least two points are closer than $\pi 2^{-\gamma k}$ (this is completely deterministic), so that in that case $P(\Omega_k^{(l)}) = 1$. Hence if $E \subsetneq \mathbb{N}$, then we are done.

Consider now the case $E = \mathbb{N}$, and let $k \in E = \mathbb{N}$. We will use the Lemma on the probability of an uncrowded road [9, p. 740], which states

$$P(\Omega_k^{(l)}) = 1 - (1 - N_k 2^{-\gamma k-1})^{N_k-1}$$

(since $E = \mathbb{N}$ this is well defined).

We can assume that $N_k \geq 2$ (since obviously $\sum_{N_k < 2} 2^{-\gamma k} N_k < \infty$). In particular $N_k^2/2 \leq N_k(N_k - 1) \leq N_k^2$. Since $\log(1 - x) \leq -x$, we get

$$\begin{aligned} \sum_{k: N_k \geq 2} (N_k - 1) \log(1 - N_k 2^{-\gamma k-1}) &\leq - \sum_{k: N_k \geq 2} (N_k - 1) N_k 2^{-\gamma k-1} \\ &\leq -\frac{1}{2} \sum_{k: N_k \geq 2} N_k^2 2^{-\gamma k-1} \\ &= -\infty, \end{aligned}$$

by assumption. Hence, taking exponentials in the previous estimate,

$$\prod_{k \in E, N_k \geq 2} (1 - N_k 2^{-\gamma k-1})^{N_k-1} = 0,$$

which implies, by results on convergence on infinite products, that

$$\sum_k P(\Omega_k^{(l)}) = \infty.$$

Since the events $\Omega_k^{(l)}$ are independent, by the Borel–Cantelli Lemma,

$$P(\limsup \Omega_k^{(l)}) = 1,$$

where

$$\limsup \Omega_k^{(l)} = \bigcap_{n \geq 1} \bigcup_{k \geq n} \Omega_k^{(l)} = \{\omega : \omega \in \Omega_k^{(l)} \text{ for infinitely many } k\}.$$

In particular, since the probability of being in infinitely many $\Omega_k^{(l)}$ is one, there is at least one $\Omega_k^{(l)}$ which happens with probability one. So that again $P(\Omega_k^{(l)}) = 1$.

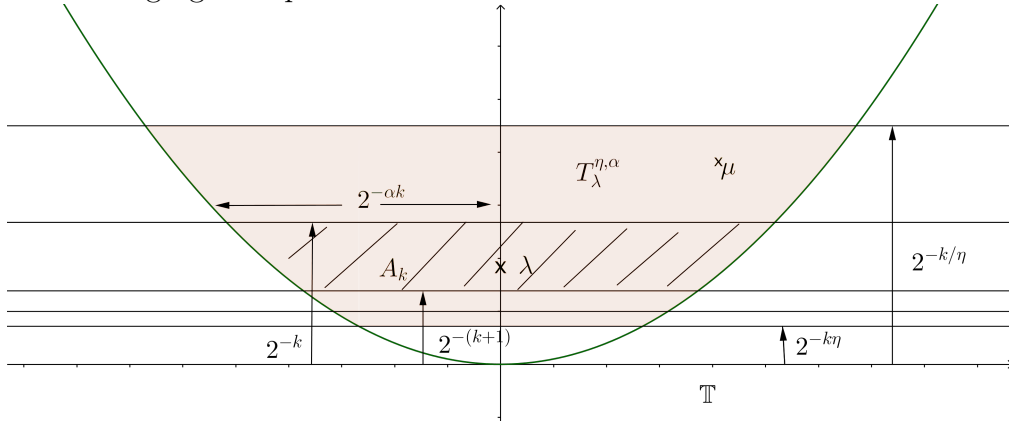
As a result, the probability that the sequence is δ_l' -separated in the Dirichlet metric is zero for every l . Since $\delta_l' \rightarrow 0$ when $l \rightarrow +\infty$, we deduce that

$$P(\omega : \{\lambda(\omega)\} \text{ is separated for } \mathcal{D}) = 0.$$

Separation with probability 1. Now we assume that $\sum_k 2^{-\gamma k} N_k^2 < +\infty$ for some $\gamma \in (1/2, 1)$. Let us begin defining a neighborhood in the Dirichlet metric. For that, fix $\eta > 1$ and $\alpha \in (0, 1)$. Given $\lambda \in \Lambda$, consider

$$T_\lambda^{\eta, \alpha} = \{z = re^{it} : (1 - |\lambda|)^\eta \leq 1 - r \leq (1 - |\lambda|)^\eta, |\theta - t| \leq (1 - r)^\alpha\}.$$

The following figure represents the situation:



Our aim is to prove that under the condition $\sum_k 2^{-\gamma k} N_k^2 < +\infty$, there exists $\eta > 1$ and $\alpha \in (0, 1)$ such that $T_\lambda^{\eta, \alpha}$ does not contain any other point of Λ except λ , and this is true for every $\lambda \in \Lambda$ with probability one. For this we need to estimate

$$P(T_\lambda^{\eta, \alpha} \cap \Lambda = \{\lambda\}).$$

Let us cover

$$T_\lambda^{\eta, \alpha} = \bigcup_{k/\eta}^{\eta k} (T_\lambda^{\eta, \alpha} \cap A_j),$$

and we need that for every $j \in [k/\eta, \eta k] \setminus \{k\}$, $(T_\lambda^{\eta, \alpha} \cap A_j) \cap \Lambda = \emptyset$ and $(T_\lambda^{\eta, \alpha} \cap A_k) \cap \Lambda = \{\lambda\}$. Note that $X_j = \#(T_\lambda^{\eta, \alpha} \cap A_j \cap \Lambda) \sim B(N_j, 2^{-\alpha j})$, $j \neq k$, and $X_k \sim B(N_k - 1, 2^{-\alpha k})$ (since we do not count λ in the latter case). Hence, since the arguments of the points are independent,

we have

$$\begin{aligned} P(T_\lambda^{\eta,\alpha} \cap \Lambda = \{\lambda\}) &= P\left(\left(\bigcap_{j=k/\eta, j \neq k}^{\eta k} (X_j = 0)\right) \cap (X_k = 1)\right) \\ &= \prod_{j=k/\eta, j \neq k}^{j=\eta k} \left(P(X_j = 0)\right) \times P(X_k = 1). \end{aligned}$$

From the binomial law we have $P(X_j = 0) = (1 - 2^{-\alpha j})^{N_j}$. Also, assuming $0 < \gamma < \alpha < 1$, we have $N_j 2^{-\alpha j} = o(j)$, so that $P(X_j = 0) = (1 - 2^{-\alpha j})^{N_j} \sim 1 - N_j 2^{-\alpha j}$. Moreover $P(X_k = 1) = N_k 2^{-\alpha k} (1 - 2^{-\alpha k})^{N_k - 1} \sim N_k 2^{-\alpha k}$. Hence we get

$$\begin{aligned} P(T_\lambda^{\eta,\alpha} \cap \Lambda = \{\lambda\}) &\sim \exp\left(\sum_{j=k/\eta, j \neq k}^{j=\eta k} \ln(P(X_j = 0))\right) \times N_k 2^{-\alpha k} \\ &\sim \left(1 - \sum_{j=k/\eta, j \neq k}^{j=\eta k} N_j 2^{-\alpha j}\right) \times N_k 2^{-\alpha k}. \end{aligned}$$

Again we use $\gamma < \alpha < 1$ to see now that the sum $\sum_{j=k/\eta, j \neq k}^{j=\eta k} N_j 2^{-\alpha j}$ is convergent and goes to zero when $k \rightarrow \infty$. This shows in particular that the fact of considering the event of having points in neighboring annuli of A_k containing λ can be neglected. Hence

$$P(T_\lambda^{\eta,\alpha} \cap \Lambda = \{\lambda\}) \sim N_k 2^{-\alpha k}.$$

We now sum over all $\lambda \in \Lambda$ by summing over all dyadic annuli A_k and the N_k points contained in each annuli:

$$\sum_{\lambda \in \Lambda} P(T_\lambda^{\eta,\alpha} \cap \Lambda = \{\lambda\}) \sim \sum_{k \in \mathbb{N}} N_k \times N_k 2^{-\alpha k} = \sum_{k \in \mathbb{N}} N_k^2 2^{-\alpha k}.$$

For $\alpha > \gamma$, this sum converges by assumption. Using the Borel-Cantelli lemma we deduce that $T_\lambda^{\eta,\alpha} \cap \Lambda = \{\lambda\}$ for all but finitely many λ with probability one. Obviously these finitely many neighborhoods $T_\lambda^{\eta,\alpha}$ contain finitely many points between which a lower Dirichlet distance exists. This achieves the proof of the separation. \square

It should be observed that the above proof only involves α but not η , so that it is the separation in the annuli which dominates the situation.

5. ONE-BOX CONDITION FOR CARLESON MEASURES AND INTERPOLATING SEQUENCES IN THE DIRICHLET SPACE \mathcal{D}

In this section we discuss a version of Seip's one-box conditions ([18, Theorem 5, p.39]) which we start recalling here.

Theorem 5.1 (Seip). *A \mathcal{D} -separated sequence Λ in \mathbb{D} is a universal interpolating sequence for \mathcal{D} if there exist $0 < \gamma < 1$ and $C > 0$ such that for each arc $I \subset \mathbb{T}$*

$$\sum_{\lambda \in \Lambda \cap S_I} \left(\log \frac{e}{1 - |\lambda|} \right)^{-\gamma} \leq C \left(\log \frac{e}{|I|} \right)^{-\gamma} \quad (5.1)$$

Again we observe that (5.1) implies in particular $\sum_n n^{-\gamma} N_n < \infty$ and hence $\sum_n n^{-1} N_n < \infty$ which by Bogdan's result implies that Λ is a zero sequence almost surely.

We will introduce a version of the Carleson windows adapted to the Dirichlet spaces and taking into account the logarithmic condition. Let $\eta > 0$ be a fixed parameter. Given an arbitrary interval $I \subset \mathbb{T}$, we define

$$h_I = 2^{-|I|^{-1/\eta}}$$

and set

$$\tilde{S}_\eta(I) = \{z = re^{it} : e^{it} \in I, 1 - h_I \leq |z| < 1\}, \quad (5.2)$$

which is a flat Carleson window (height/length goes to zero). We are now in a position to prove a version of Seip's theorem which allows one to remove the highly non-linear right-hand side in (5.1).

Theorem 5.2. *Given $\eta > 0$ and $\gamma \in (0, 1)$. A \mathcal{D}_1 -separated sequence $\Lambda \subset \mathbb{D}$ satisfying for every arc $I \subset \mathbb{T}$*

$$\sum_{\lambda \in \tilde{S}_\eta(I)} \left(\log \frac{e}{1 - |\lambda|} \right)^{-\gamma} \lesssim |I|^{\gamma/\eta} \quad (5.3)$$

is interpolating for \mathcal{D}_1 .

Corollary 5.3. *With the same notation as the theorem suppose that Λ is \mathcal{D}_1 -separated and $\eta \in (0, 1)$. If*

$$\sum_{\lambda \in \tilde{S}_\eta(I)} \left(\log \frac{e}{1 - |\lambda|} \right)^{-\eta} \lesssim |I| \quad (5.4)$$

then Λ is interpolating.

Proof of Corollary 5.3. Since $\eta \in (0, 1)$ there exists $\gamma \in (0, 1)$ such that $\eta/\gamma < 1$ (pick $\gamma \in (\eta, 1)$). Hence

$$\left(\sum_{\lambda \in \tilde{S}_\eta(I)} \left(\log \frac{e}{1 - |\lambda|} \right)^{-\gamma} \right)^{\eta/\gamma} \leq \sum_{\lambda \in \tilde{S}_\eta(I)} \left(\log \frac{e}{1 - |\lambda|} \right)^{-\eta}$$

and it is enough to apply the theorem. □

Remark 5.4. *The above two results are true under a seemingly weaker testing condition. It suffices actually to test for **dyadic intervals**:*

$$I = I_{n,k} = \{e^{it} \in \mathbb{T} : \frac{k}{2^n}\pi \leq t < \frac{k+1}{2^n}\pi\}, \quad n \in \mathbb{N}, k = 0, \dots, 2^n.$$

Indeed, suppose we have (5.3) for such $I_{n,k}$, where $\eta > 0$ and $\gamma \in (0, 1)$. Let I be an arbitrary interval in \mathbb{T} . Let $n \in \mathbb{N}$ be such that $2^{-(n+1)}\pi \leq |I| < 2^{-n}\pi$. Also there exists k such that $I \subset I_{n,k} \cup I_{n,k+1}$ (indices are modulo 2^n). Then

$$\tilde{S}_\eta(I) \subset \tilde{S}_\eta(I_{n,k}) \cup \tilde{S}_\eta(I_{n,k+1}).$$

(Observe that $\tilde{S}_\eta(I_{n,k})$ is higher than $\tilde{S}_\eta(I)$ but I will in general meet (at most) two dyadic intervals). Now using (5.4) for dyadic intervals we get

$$\begin{aligned} \sum_{\lambda \in \tilde{S}_\eta(I)} \left(\log \frac{e}{1-|\lambda|} \right)^{-\gamma} &\leq \sum_{\lambda \in \tilde{S}_\eta(I_{n,k}) \cup \tilde{S}_\eta(I_{n,k+1})} \left(\log \frac{e}{1-|\lambda|} \right)^{-\gamma} \\ &\lesssim |I_{n,k}|^{\gamma/\eta} + |I_{n,k+1}|^{\gamma/\eta} \\ &\simeq 2^{-n\gamma/\eta} \\ &\lesssim |I|^{\gamma/\eta} \end{aligned}$$

as desired. The argument is the same for Corollary 5.3.

Proof. The proof follows essentially the same scheme as Seip's [18, p 39]. We need some definitions. Let $\eta > 0$ be fixed, and introduce a non-dyadic paving of the dyadic annuli. To this end, let

$$\tilde{I}_{n,k} = \{(k-1)n^{-\eta}\pi \leq \theta \leq kn^{-\eta}\pi\}.$$

Then, keeping in mind that $h_I = 2^{-|I|^{-1/\eta}}$ and $|\tilde{I}_{n,k}| = n^{-\eta}$ we get $h_{\tilde{I}_{n,k}} = 2^{-n}$. Then $\tilde{S}_\eta(\tilde{I}_{n,k})$ is a ‘‘flat’’ Carleson window for which we will also consider the upper top half

$$\tilde{T}_{n,k} = \left\{ z = re^{i\theta} : 2^{-n-1} < 1 - |z| \leq 2^{-n} ; e^{i\theta} \in \tilde{I}_{n,k} \right\}.$$

Given $\lambda \in \Lambda$, let $E_\lambda = \{(n, k) : \tilde{T}_{n,k} \cap [0, \lambda] \neq \emptyset\}$. It is clear that for every $(n, k) \in E_\lambda$ we have

$$n \leq \frac{\ln \frac{1}{1-|\lambda|}}{\ln 2},$$

and the biggest n with $(n, k) \in E_\lambda$ is comparable to $\ln \frac{1}{1-|\lambda|} / \ln 2$. Then

$$f(\lambda) = \int_0^\lambda f'(z) dz = \sum_{(n,k) \in E_\lambda} \int_{[0,\lambda] \cap \tilde{T}_{n,k}} f'(z) dz.$$

We introduce $\zeta_{n,k} \in \tilde{T}_{n,k}$ with $|f'(\zeta_{n,k})|(1 - |\zeta_{n,k}|) \geq \frac{1}{2} \sup_{\zeta \in \tilde{T}_{n,k}} |f'(\zeta)|(1 - |\zeta|)$. This, together with $|[0, \lambda] \cap \tilde{T}_{n,k}| \leq 2^{-n-1} \leq 1 - |\zeta_{n,k}|$, yields

$$|f(\lambda)| \leq 2 \sum_{(n,k) \in E_\lambda} |f'(\zeta_{n,k})|(1 - |\zeta_{n,k}|).$$

The condition $(n, k) \in E_\lambda$ can be rewritten as $\lambda \in \tilde{S}_{n,k} := \tilde{S}_\eta(\tilde{T}_{n,k})$. As did Seip, we now introduce a weight factor and apply the Cauchy-Schwarz inequality

$$\begin{aligned} |f(\lambda)| &\leq 2 \sum_{(n,k) \in E_\lambda} |f'(\zeta_{n,k})|(1 - |\zeta_{n,k}|) \left(\ln \frac{e}{1 - |\zeta_{n,k}|} \right)^{\frac{\gamma-\gamma}{2}} \\ &\leq \left(\sum_{(n,k) \in E_\lambda} |f'(\zeta_{n,k})|^2 (1 - |\zeta_{n,k}|)^2 \left(\ln \frac{e}{1 - |\zeta_{n,k}|} \right)^\gamma \right)^{1/2} \left(\sum_{(n,k) \in E_\lambda} \left(\ln \frac{e}{1 - |\zeta_{n,k}|} \right)^{-\gamma} \right)^{1/2}. \end{aligned}$$

Remember that $1 - |\zeta_{n,k}| \simeq 2^{-n}$ and so, since $\gamma \in (0, 1)$,

$$\sum_{(n,k) \in E_\lambda} \left(\ln \frac{e}{1 - |\zeta_{n,k}|} \right)^{-\gamma} \simeq \sum_{n \leq \ln(1-|\lambda|)^{-1}/\ln 2} n^{-\gamma} \simeq \left(\ln \frac{e}{1 - |\lambda|} \right)^{1-\gamma}.$$

Hence

$$|f(\lambda)|^2 \lesssim \left(\ln \frac{e}{1 - |\lambda|} \right)^{1-\gamma} \sum_{(n,k) \in E_\lambda} |f'(\zeta_{n,k})|^2 (1 - |\zeta_{n,k}|)^2 \left(\ln \frac{e}{1 - |\zeta_{n,k}|} \right)^\gamma.$$

We now compute

$$\sum_{\lambda \in \Lambda} |f(\lambda)|^2 \ln \frac{e}{1 - |\lambda|} \leq \sum_{\lambda \in \Lambda} \left(\ln \frac{e}{1 - |\lambda|} \right)^{-\gamma} \sum_{(n,k) \in E_\lambda} |f'(\zeta_{n,k})|^2 (1 - |\zeta_{n,k}|)^2 \left(\ln \frac{e}{1 - |\zeta_{n,k}|} \right)^\gamma.$$

We will reorder all the (n, k) appearing in some E_λ by ℓ . Correspondingly we will use the notation $\tilde{S}_{(\ell)} = \tilde{S}_{n,k}$, and $\zeta_{(\ell)} = \zeta_{n,k}$. In view of an application of Fubini's theorem, this means that when we first sum over ℓ then we have to sum over all λ which are contained in $\tilde{S}_{(\ell)}$. More precisely

$$\begin{aligned} \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \ln \frac{1}{1 - |\lambda|} &\leq \sum_{\lambda \in \Lambda} \left(\ln \frac{e}{1 - |\lambda|} \right)^{-\gamma} \sum_{\ell: \lambda \in \tilde{S}_{(\ell)}} |f'(\zeta_{(\ell)})|^2 (1 - |\zeta_{(\ell)}|)^2 \left(\ln \frac{e}{1 - |\zeta_{(\ell)}|} \right)^\gamma \\ &= \sum_{\ell} |f'(\zeta_{(\ell)})|^2 (1 - |\zeta_{(\ell)}|)^2 \left(\ln \frac{e}{1 - |\zeta_{(\ell)}|} \right)^\gamma \sum_{\lambda \in \tilde{S}_{(\ell)}} \left(\ln \frac{e}{1 - |\lambda|} \right)^{-\gamma}. \end{aligned}$$

Now since $\zeta_{(\ell)} = \zeta_{n,k}$ we have $(\ln(1 - |\zeta_{(\ell)}|))^\gamma \simeq n^\gamma \simeq |I|^{\gamma/\eta}$. Applying (5.3) we are led to an estimate of $\sum_{\ell} |f'(\zeta_{(\ell)})|^2 (1 - |\zeta_{(\ell)}|)^2$. We conclude observing that the sequence $(\zeta_{(\ell)})_\ell$ is a (union of at most 4) separated sequences in the pseudohyperbolic metric (the ‘‘top halves’’ $\tilde{T}_{n,k}$ are in dyadic annuli and are much larger than usual dyadic boxes and

hence pseudohyperbolically large in the tangential direction), and thus it defines a Carleson measure for the Bergman space. Since f' is in the Bergman space when f is in the Dirichlet space, we obtain the Carleson measure condition. This achieves the proof. \square

6. DIFFERENT DISTRIBUTION CONDITIONS

In this section we will state some auxiliary results on the connection between the distribution conditions given above and conditions more adapted to the Dirichlet space. Let $\beta > 1$ and

$$A_{n,\beta} := \{z \in \mathbb{D} : 1 - 2^{-\beta n} \leq |z| < 1 - 2^{-\beta^{n+1}}\}.$$

Note that an annulus $A_{n,\beta}$ can always be covered by $\ln \beta / \ln \beta'$ annuli $A_{k,\beta'}$, when $1 < \beta' < \beta$ (thus independently on n). We will set

$$N_{n,\beta} = \#(\Lambda \cap A_{n,\beta}).$$

We need to connect the conditions on N_k and $N_{n,\beta}$. This will be given by the following result.

Lemma 6.1. *Given $\beta > 1$ and $\gamma > 0$. We have $\sum_k k^{-\gamma} N_k < +\infty$ if and only if $\sum_k \beta^{-\gamma k} N_{k,\beta} < +\infty$.*

Proof. Using the fact that $\lfloor \beta^{k+1} \rfloor \leq \beta^{k+1}$ we have

$$\sum_k \beta^{-\gamma k} N_{k,\beta} \leq \sum_k \beta^{-\gamma k} \sum_{\ell=\lfloor \beta^k \rfloor}^{\lfloor \beta^{k+1} \rfloor} N_\ell = \sum_k \beta^\gamma \sum_{\ell=\lfloor \beta^k \rfloor}^{\lfloor \beta^{k+1} \rfloor} \beta^{-\gamma(k+1)} N_\ell \leq \beta^\gamma \sum_k \sum_{\ell=\lfloor \beta^k \rfloor}^{\lfloor \beta^{k+1} \rfloor} \ell^{-\gamma} N_\ell < \infty.$$

Conversely, since $\lfloor \beta^k \rfloor \geq \beta^{k-1}$ for sufficiently large k and with $\ell \geq \lfloor \beta^k \rfloor$ we get

$$\sum_k \sum_{\ell=\lfloor \beta^k \rfloor}^{\lfloor \beta^{k+1} \rfloor - 1} \ell^{-\gamma} N_\ell \leq \sum_k \sum_{\ell=\lfloor \beta^k \rfloor}^{\lfloor \beta^{k+1} \rfloor - 1} \beta^{-\gamma(k-1)} N_\ell = \beta^\gamma \sum_k \beta^{-\gamma k} \sum_{\ell=\lfloor \beta^k \rfloor}^{\lfloor \beta^{k+1} \rfloor - 1} N_\ell \lesssim \sum_k \beta^{-\gamma k} N_{k,\beta} < \infty$$

(note that in the next to last inequality we need to cover in particular $N_{\lfloor \beta^k \rfloor}$ by $N_{k-1,\beta}$ and $N_{k,\beta}$). \square

In particular we have the following consequences.

Corollary 6.2. *Given $\beta > 1$.*

- (1) *For every $\eta > 0$, if $\sum_k \beta^{-\eta k} N_{k,\beta}^2 = +\infty$ then $\sum_k k^{-\eta/2} N_k = +\infty$.*
- (2) *If for some $\eta > 0$, we have $\sum_k \beta^{-\eta k} N_{k,\beta}^2 < +\infty$ then $\sum_k k^{-\eta'/2} N_k < +\infty$ for every $\eta' > \eta$.*
- (3) *If for some $\eta \in (0, 2)$, we have $\sum_k \beta^{-\eta k} N_{k,\beta}^2 < +\infty$ then $\sum_k k^{-1} N_k < +\infty$*

Proof. (1) If $\sum_k \beta^{-\eta k} N_{k,\beta}^2 = +\infty$ then $\sum_k \beta^{-\eta k/2} N_{k,\beta} = +\infty$. The Lemma 6.1 gives the result.

(2) The condition implies that $\sum_k \beta^{-\eta'/2k} N_{k,\beta} < +\infty$, and again Lemma 6.1 gives the result.

(3) This result is an immediate consequence of (2) observing that we can pick $\eta' \in (\eta, 2) \subset (0, 2)$. \square

7. PROOF OF THEOREM 1.7 AND AN ASSOCIATED RESULT

7.1. **Proof of Theorem 1.7.** Let $\eta \in (0, 1)$ be the parameter in the Seip type result Corollary 5.3 and which we will fix later. Let $\beta > 1$ and $\kappa > 3/2$ and suppose now

$$\sum_{m \geq 1} \beta^m N_{m,\beta}^\kappa < +\infty.$$

For each n we decompose \mathbb{T} into intervals of length $|I_{n,k}| = \beta^{-\eta n}$: $\mathbb{T} = \bigcup_{k=1}^{\beta^{n\eta}} I_{n,k}$. The height corresponding to the flattened windows $\tilde{S}_{n,k}$ on the intervals $I_{n,k}$, and defined in (5.2), is given by

$$h_{I_{n,k}} = 2^{-|I_{n,k}|^{-1/\eta}} = 2^{-\beta^{-\eta n/\eta}} = 2^{-\beta^n}.$$

We now use a ‘‘superdyadic’’ paving of the disk: for this, given n, k , let

$$\tilde{S}_{n,k} = \tilde{S}_\eta(I_{n,k}) = \left\{ z = re^{i\theta} : 1 - 2^{-\beta^n} \leq |z| < 1 ; e^{i\theta} \in I_{n,k} \right\}$$

and

$$\hat{T}_{n,k} = \left\{ z = re^{i\theta} : 1 - 2^{-\beta^n} < |z| \leq 2^{-\beta^{n+1}} ; e^{i\theta} \in I_{n,k} \right\},$$

which corresponds to the upper ‘‘big half’’ of $\tilde{S}_{n,k}$.

We now cover $\tilde{S}_{n,k}$ by super-dyadic slices:

$$\tilde{S}_{n,k} = \bigcup_{m \geq n} \tilde{S}_{n,k} \cap A_{m,\beta}.$$

As before, to each slice of $\tilde{S}_{n,k}$ we associate a random variable $X_{n,m,k}^{(\beta)}$ counting the points of $A_{m,\beta}$ in $\tilde{S}_{n,k}$. In particular $X_{n,m,k}^{(\beta)} \sim B(\beta^{-\eta m}, N_{m,\beta})$. Corollary 5.3 requires

$$\sum_{\lambda \in \tilde{S}_{n,k}} \left(\ln \frac{e}{1 - |\lambda|} \right)^{-\eta} \lesssim |I_{n,k}| = \beta^{-n\eta},$$

which translates to

$$\sup_{n,k} \beta^{n\eta} \sum_{m \geq n} \beta^{-m\eta} X_{n,m,k}^{(\beta)} \leq C$$

almost surely (in ω).

In order to repeat the ideas of the situation in the Hardy space, we introduce

$$Y_{n,k}^{(\beta)} = \beta^{n\eta} \sum_{m=n}^{+\infty} \beta^{-m\eta} X_{n,m,k}^{(\beta)}.$$

For $\gamma > 0$ to be fixed later, we will write

$$Y_{n,k}^{(\beta)} = \underbrace{\beta^{n\eta} \sum_{m=n}^{n(1+\gamma)} \beta^{-m\eta} X_{n,m,k}^{(\beta)}}_{Y_{n,k}^{(\beta),0}} + \underbrace{\beta^{n\eta} \sum_{m=n(1+\gamma)+1}^{+\infty} \beta^{-m\eta} X_{n,m,k}^{(\beta)}}_{R_{n,k}^{(\beta)}}.$$

As before

$$Z_{n,k}^{(\beta)} = \sum_{m=n}^{n(1+\gamma)} X_{n,m,k}^{(\beta)}.$$

Note that $Z_{n,k}^{(\beta)}$ is the sum of binomial variables having all the same parameter $p = \beta^{-n\eta}$. The sum $Z_{n,k}^{(\beta)}$ is thus also binomially distributed with parameter $p = \beta^{-n\eta}$ and $N = N_{n,\beta} + \cdots + N_{n(1+\gamma),\beta}$. Now note that the event $(Y_{n,k}^{(\beta)} \geq A)$ is contained in the union $(Y_{n,k}^{(\beta),0} \geq A/2) \cup (R_{n,k}^{(\beta)} \geq A/2)$. Let us estimate the probability of the first event. Using the fact that $(Z_{n,k}^{(\beta)} \geq A/2)$ is easier to achieve than $(Y_{n,k}^{(\beta),0} \geq A/2)$ and Lemma 2.1, we get assuming $A/2$ entire,

$$P(Y_{n,k}^{(\beta),0} \geq A/2) \leq P(Z_{n,k}^{(\beta)} \geq A/2) \sim \frac{(pN)^{A/2}}{(A/2)!} = \frac{(\beta^{-n\eta}(N_{n,\beta} + \cdots + N_{n(1+\gamma),\beta}))^{A/2}}{(A/2)!}.$$

Now remember that $\sum_n \beta^{-n} N_{n,\beta}^\kappa < \infty$ so that in particular $N_{n,\beta} \leq \beta^{n/\kappa}$ (at least for n sufficiently large). Hence

$$\frac{(\beta^{-n\eta}(N_{n,\beta} + \cdots + N_{n(1+\gamma),\beta}))^{A/2}}{(A/2)!} \lesssim (\beta^{-n(\eta-(1+\gamma)/\kappa)})^{A/2}.$$

In order to get the power of β negative, we need to pick γ such that $(1 + \gamma)/\kappa < \eta$, equivalently

$$\gamma < \eta\kappa - 1. \quad (7.1)$$

In that case, for sufficiently large $A/2$,

$$P(Y_{n,k}^{(\beta),0} \geq A/2) \leq P(Z_{n,k}^{(\beta)} \geq A/2) \sim \frac{(pN)^{A/2}}{(A/2)!} \lesssim \beta^{-4n}.$$

Again we estimate $R_{n,k}^{(\beta)}$ using Tchebychev's inequality:

$$\begin{aligned} P(R_{n,k}^{(\beta)} \geq A/2) &\lesssim \text{Var}(R_{n,k}^{(\beta)}) = \beta^{2n\eta} \sum_{m \geq n(1+\gamma)} \beta^{-2m\eta} \text{Var}(X_{n,m,k}^{(\beta)}) \\ &\lesssim \beta^{2n\eta} \sum_{m \geq n(1+\gamma)} \beta^{-2m\eta} \times \beta^{-n\eta} N_{m,\beta} = \beta^{n\eta} \sum_{m \geq n(1+\gamma)} \beta^{-2m\eta+m/\kappa} \\ &\leq \beta^{n\eta(1-2(1+\gamma)+(1+\gamma)/(\eta\kappa))} = \beta^{-n\eta(1+2\gamma-(1+\gamma)/(\eta\kappa))}. \end{aligned}$$

We will need $\delta := 1 + 2\gamma - (1 + \gamma)/(\eta\kappa) > 1$, equivalently $2\gamma > (1 + \gamma)/(\eta\kappa) = 1/(\eta\kappa) + \gamma/(\eta\kappa)$, i.e.

$$\gamma > \frac{1/(\eta\kappa)}{2 - 1/(\eta\kappa)} = \frac{1}{2\eta\kappa - 1}. \quad (7.2)$$

Note that the two conditions on γ , (7.1) and (7.2), require

$$\begin{aligned} \frac{1}{2\eta\kappa - 1} < \eta\kappa - 1 &\iff 1 < (\eta\kappa - 1)(2\eta\kappa - 1) = 2\eta^2\kappa^2 - 2\eta\kappa + 1 \\ &\iff \kappa > \frac{3}{2\eta}. \end{aligned}$$

By assumption $\kappa > 3/2$ so that for $1 > \eta > 3/(2\kappa)$ we have $\kappa > 3/(2\eta)$. We will apply Seip's theorem with such $\eta \in (3/(2\kappa), 1)$. In that case, we can thus pick γ such that

$$\frac{1}{2\eta\kappa - 1} < \gamma < \eta\kappa - 1.$$

Now, as already said, for $(Y_{n,k}^{(\beta)} \geq A)$ it is necessary that $(Y_{n,k}^{(\beta),0} \geq A/2)$ or $(R_{n,k}^{(\beta)} \geq A/2)$. Hence

$$P(Y_{n,k}^{(\beta)} \geq A) \leq P(Y_{n,k}^{(\beta),0} \geq A/2) + P(R_{n,k}^{(\beta)} \geq A/2) \leq \beta^{-4n} + \beta^{-\delta\eta n}.$$

In view of an application of the Borel-Cantelli Lemma we compute

$$\sum_{n \geq 0} \sum_{k=1}^{\beta^{n\eta}} P(Y_{n,k}^{(\beta)} \geq A) \leq \sum_{n \geq 0} \beta^{n\eta} \times (\beta^{-4n} + \beta^{-\delta\eta n}) < \infty,$$

where we have used that $\delta > 1$ (and $\eta < 1 < 4$). Hence, by the Borel-Cantelli Lemma, the event $Y_{n,k} \geq A$ can happen for at most a finite number of indices (n, k) . In particular the Carleson measure constant of dm_Λ is almost surely at most A except for a finite number of Carleson windows. \square

As in the Hardy space, Theorem 3.1 leaves a gap $1 < \kappa \leq 3/2$ for which we now discuss the corresponding version in the classical Dirichlet space.

Theorem 7.1. *For every $\kappa > 1$, for every $\beta > 1$ there exists $(N_{n,\beta})$ such that*

$$\sum_n \beta^{-n} N_{n,\beta}^\kappa < \infty, \quad \text{and} \quad \sum_n \beta^{-n} N_{n,\beta}^{\kappa'} = \infty, \quad \kappa' > \kappa \quad (7.3)$$

and $P(\Lambda(\omega) \text{ is an interpolating sequence for } \mathcal{D}) = 1$.

Proof. Observe that condition (7.3) implies the Bogdan condition so that the sequence is almost surely a zero sequence in \mathcal{D} .

The proof follows the lines of the situation in the Hardy space. Let again $\eta \in (0, 1)$ be the parameter appearing in Corollary 5.3 and which will be fixed later. Given $\kappa > 1$, let $\kappa' > \kappa$, and set

$$N_{n,\beta} = \begin{cases} \beta^{n/\kappa'}, & \text{if there is } \ell \in \mathbb{N}, n = 2^\ell \\ 0 & \text{else.} \end{cases}$$

We will use a similar notation as in the previous section:

$$Y_{n,k}^{(\beta)} = \underbrace{\beta^{n\eta} \sum_{m=n}^{2n-1} \beta^{-m\eta} X_{n,m,k}^{(\beta)}}_{Y_{n,k}^{(\beta),0}} + \underbrace{\beta^{n\eta} \sum_{m=2n}^{+\infty} \beta^{-m\eta} X_{n,m,k}^{(\beta)}}_{R_{n,k}^{(\beta)}}$$

where again $X_{n,m,k}^{(\beta)} \sim B(\beta^{-n\eta}, N_{m,\beta})$ (we look at flattened Carleson windows based on intervals of length $\beta^{-n\eta}$). As before, $P(R_{n,k}^{(\beta)} \geq A/2)$ can be estimated with Tchebychev's inequality, more precisely

$$\begin{aligned} P(R_{n,k}^{(\beta)} \geq A/2) &\lesssim \text{Var}(R_{n,k}^{(\beta)}) = \beta^{2n\eta} \sum_{m \geq 2n} \beta^{-2m\eta} \text{Var}(X_{n,m,k}^{(\beta)}) \\ &\lesssim \beta^{2n\eta} \sum_{m \geq 2n} \beta^{-2m\eta} \times \beta^{-n\eta} N_{m,\beta} = \beta^{n\eta} \sum_{m \geq 2n} \beta^{-2m\eta + m/\kappa'} \\ &\leq \beta^{n\eta(1-4+2/(\eta\kappa'))} = \beta^{-n\eta(3-2/(\eta\kappa'))}. \end{aligned}$$

Since $\kappa' > 1$ there exists $\eta \in (0, 1)$ such that $\eta\kappa' > 1$ and hence $3 - 2/(\eta\kappa') > 1$ so that $P(R_{n,k}^{(\beta)} \geq A/2)$ is summable against $\beta^{n\eta}$.

So we are mainly interested in $P(Y_{n,k}^{(\beta),0} \geq A/2)$. By construction there is a unique $L \in [n, 2n - 1]$ such that

$$Y_{n,k}^{(\beta),0} = \beta^{n\eta} \times \beta^{-L\eta} X_{n,L,k}^{(\beta)}.$$

Hence $P(Y_{n,k}^{(\beta),0} \geq A/2) = P(X_{n,L,k}^{(\beta)} \geq \beta^{(L-n)\eta} A/2)$. Since $X_{n,L,k}^{(\beta)} \sim B(\beta^{-\eta n}, \beta^{L/\kappa'})$, using Lemma 2.1 we get

$$P(X_{n,L,k}^{(\beta)} \geq \beta^{(L-n)\eta} A/2) \sim \frac{(\beta^{-\eta n} \beta^{L/\kappa'})^{\beta^{(L-n)\eta} A/2}}{(\beta^{(L-n)\eta} A/2)!}.$$

Using again Stirling's approximation (2.4) (and assuming $K = \beta^{(L-n)\eta} A/2$ to be integer), we get

$$\begin{aligned} P(X_{n,L,k}^{(\beta)} \geq \beta^{(L-n)\eta} A/2) &\sim \frac{(\beta^{-\eta n} \beta^{L/\kappa'})^{\beta^{(L-n)\eta} A/2}}{\sqrt{2\pi} \beta^{(L-n)\eta} A/2} \times \left(\frac{e}{\beta^{(L-n)\eta} A/2} \right)^{\beta^{(L-n)\eta} A/2} \\ &= \frac{1}{\sqrt{2\pi} \beta^{(L-n)\eta} A/2} \times \left(\frac{e \times \beta^{-\eta n} \beta^{L/\kappa'}}{\beta^{(L-n)\eta} A/2} \right)^{\beta^{(L-n)\eta} A/2} \\ &= \frac{1}{\sqrt{\pi} \beta^{(L-n)\eta} A} \times \left(\frac{2e}{A} \times \beta^{-(\eta-1/\kappa')L} \right)^{\beta^{(L-n)\eta} A/2}. \end{aligned}$$

Since $\kappa' > \kappa > 1$ there exists $\eta \in (0, 1)$ such that $\eta - 1/\kappa' > 0$. Also, choosing $A > 2e$ the expression between parenthesis is less than 1. In particular instead of taking the power

$\beta^{(L-n)\eta}A/2$ we raise this expression only to the power $A/2$ making it bigger. Moreover we can bound the first factor with the square-root in the denominator by one. As a result

$$P(X_{n,L,k}^{(\beta)} \geq \beta^{(L-n)\eta}A/2) \lesssim \beta^{-(\eta-1/\kappa')LA/2}.$$

Now, since $L \geq n$, we reach

$$P(X_{n,L,k}^{(\beta)} \geq \beta^{(L-n)\eta}A/2) \lesssim \beta^{-(\eta-1/\kappa')nA/2}.$$

Now choose A such that $(\eta - 1/\kappa')A/2 > \eta$, and we can conclude as in the preceding proof. \square

8. ANNEX : PROOF OF THEOREM 1.3

Carleson proved in [8, Theorem 2.2] that, for $0 < \alpha < 1$, if

$$\sum_{\lambda \in \Lambda} (1 - |\lambda|)^\alpha < \infty$$

then the Blaschke product B associated to Λ belongs to \mathcal{D}_α . So the sufficiency part of Theorem 1.3 follows immediately from this result (and is moreover deterministically true).

For the proof of the converse we will need the following two lemmas. The first one is a version of the Borel-Cantelli Lemma [5, Theorem 6.3].

Lemma 8.1. *If $\{A_n\}$ is a sequence of measurable subsets in a probability space (X, P) such that $\sum P(A_n) = \infty$ and*

$$\liminf_{n \rightarrow \infty} \frac{\sum_{j,k \leq n} P(A_j \cap A_k)}{[\sum_{k \leq n} P(A_k)]^2} \leq 1, \quad (8.1)$$

then $P(\limsup_{n \rightarrow \infty} A_n) = 1$.

The second Lemma is due to Nagel, Rudin and Shapiro [14] who discussed tangential approach regions of functions in \mathcal{D}_α .

Lemma 8.2. *Let $f \in \mathcal{D}_\alpha$, $0 < \alpha < 1$. Then, for a.e. $\zeta \in \mathbb{T}$, we have $f(z) \rightarrow f^*(\zeta)$ as $z \rightarrow \zeta$ in each region*

$$|z - \zeta| < \kappa(1 - |z|)^\alpha, \quad (\kappa > 1).$$

Proof of Theorem 1.3. In view of our preliminary observations, we are essentially interested in the converse implication. So suppose $\sum_n 2^{-(1-\alpha)n} N_n = +\infty$ or equivalently

$$\sum_n (1 - \rho_n)^{1-\alpha} = +\infty. \quad (8.2)$$

We have to show that Λ is not a zero sequence almost surely. For this, introduce the intervals $I_\ell = (e^{-i(1-\rho_\ell)^\alpha}, e^{i(1-\rho_\ell)^\alpha})$ and let $F_\ell = e^{i\theta_\ell} I_\ell$. Denoting by m normalized Lebesgue measure on \mathbb{T} , observe that

$$m(F_\ell) = m(I_\ell) = (1 - \rho_\ell)^{1-\alpha}.$$

We have for every $\varphi \in F_\ell$, $\lambda_\ell \in \Omega_{\kappa, \varphi} = \{z \in \mathbb{D} : |z - e^{i\varphi}| < \kappa(1 - |z|)^\alpha\}$. By Lemma 8.2 it suffices to prove that $|\limsup_\ell F_\ell| > 0$ a.s. (the latter condition means that there is a set of strictly positive measure on \mathbb{T} to which Λ accumulates in Dirichlet tangential approach regions according to Lemma 8.2, which is of course not possible for a zero sequence). Let E denote the expectation with respect to the Steinhaus sequence (θ_n) . By Fubini's theorem we have $E[m(F_j \cap F_k)] = m(I_j)m(I_k)$, $j \neq k$, (the expected size of intersection of two intervals only depends on the product of the length of both intervals). By Fatou's Lemma and (8.2)

$$\begin{aligned} E \left[\liminf_{n \rightarrow \infty} \frac{\sum_{j,k \leq n} m(F_j \cap F_k)}{[\sum_{k \leq n} m(F_k)]^2} \right] &\leq \liminf_{n \rightarrow \infty} E \left[\frac{\sum_{j,k \leq n} m(F_j \cap F_k)}{[\sum_{k \leq n} m(F_k)]^2} \right] \\ &= \liminf_{n \rightarrow \infty} \frac{\sum_{j,k \leq n} E[m(F_j \cap F_k)]}{[\sum_{k \leq n} m(F_k)]^2} \\ &= \liminf_{n \rightarrow \infty} \frac{\sum_{j,k \leq n, j \neq k} m(I_j)m(I_k) + \sum_{k \leq n} m(I_k)}{[\sum_{k \leq n} m(I_k)]^2} \\ &= \liminf_{n \rightarrow \infty} \left(1 + \frac{\sum_{k \leq n} m(I_k)(1 - m(I_k))}{[\sum_{k \leq n} m(I_k)]^2} \right). \end{aligned}$$

Now, since $1 - m(I_k) \rightarrow 1$, and by (8.2), keeping in mind that $m(I_k) = (1 - \rho_k)^{1-\alpha}$, we have

$$\lim_{n \rightarrow \infty} \frac{\sum_{k \leq n} m(I_k)(1 - m(I_k))}{[\sum_{k \leq n} m(I_k)]^2} = 0.$$

This implies that (8.1) holds on a set B of positive probability and hence, by the zero-one law, on a set of probability one. From Lemma 8.1 we conclude $P(\limsup_{n \rightarrow \infty} F_n) = 1$ a.s., which is what we had to show.

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