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# NP-hardness of $\ell_0$ minimization problems: revision and extension to the non-negative setting

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**Abstract**—Sparse approximation arises in many applications and often leads to a constrained or penalized  $\ell_0$  minimization problem, which was proved to be NP-hard. This paper proposes a revision of existing analyses of NP-hardness of the penalized  $\ell_0$  problem and it introduces a new proof adapted from Natarajan’s construction (1995). Moreover, we prove that  $\ell_0$  minimization problems with non-negativity constraints are also NP-hard.

## I. INTRODUCTION

Sparse approximation appears in a wide range of applications, especially in signal processing, image processing and compressed sensing [1]. Given a signal data  $\mathbf{y} \in \mathbb{R}^m$  and a dictionary  $A$  of size  $m \times n$ , the aim is to find a signal  $\mathbf{x} \in \mathbb{R}^n$  that gives the best approximation  $\mathbf{y} \approx A\mathbf{x}$  and has the fewest non-zero coefficients (*i.e.*, sparsest solution). This task leads to solving one of the following constrained or penalized  $\ell_0$  minimization problems:

$$\min_{\|\mathbf{y}-A\mathbf{x}\|_2 \leq \epsilon} \|\mathbf{x}\|_0 \quad (\ell_0 C)$$

$$\min_{\|\mathbf{x}\|_0 \leq K} \|\mathbf{y}-A\mathbf{x}\|_2^2 \quad (\ell_0 C')$$

$$\min_{\mathbf{x}} \|\mathbf{y}-A\mathbf{x}\|_2^2 + \lambda \|\mathbf{x}\|_0 \quad (\ell_0 P)$$

in which  $\epsilon$ ,  $K$  and  $\lambda$  are positive quantities related to the noise standard deviation, the sparsity level and regularization strength, respectively. Letters  $C$  and  $P$  respectively indicate that the problem is constrained or penalized. Depending on application, the appropriate statement will be addressed. It is noteworthy that  $n$  and  $K$  often depend on  $m$  when one considers the size of problem.  $(\ell_0 C)$  and  $(\ell_0 C')$  are well known to be NP-hard [2, 3]. The NP-hardness of  $(\ell_0 P)$  was claimed to be a particular case of more general complexity analyses in [4, 5]. However, we point out that these complexity analyses do not rigorously apply to  $(\ell_0 P)$  as claimed. In this paper, we justify the complexity analyses in [4, 5] do not apply to problem  $(\ell_0 P)$ , and we provide a new proof for the NP-hardness of  $(\ell_0 P)$  adapted from Natarajan’s construction [2].

In several applications such as geoscience and remote sensing [6, 7], audio [8], chemometrics [9] and computed

tomography [10], the signal or image of interest is non-negative. In such contexts, one often addresses a minimization problem with both sparsity and non-negativity constraints [10–12]. Adding non-negativity constraints to  $\ell_0$  minimization problems yields the following problems:

$$\min_{\mathbf{x} \geq 0, \|\mathbf{y}-A\mathbf{x}\|_2 \leq \epsilon} \|\mathbf{x}\|_0 \quad (\ell_0 C+)$$

$$\min_{\mathbf{x} \geq 0, \|\mathbf{x}\|_0 \leq K} \|\mathbf{y}-A\mathbf{x}\|_2^2 \quad (\ell_0 C'+)$$

$$\min_{\mathbf{x} \geq 0} \|\mathbf{y}-A\mathbf{x}\|_2^2 + \lambda \|\mathbf{x}\|_0 \quad (\ell_0 P+)$$

Several papers address non-negative  $\ell_0$  minimization problems in the literature (see, *e.g.*, [13–16]). However, to the best of our knowledge, the complexity of these problems has not been addressed yet, the question of their NP-hardness being still open. Here we show that these problems are NP-hard and the proof can be derived from the NP-hardness of  $\ell_0$  minimization problems.

The rest of paper is organized as follows. In Section II, we discuss the issues related to NP-hardness of  $(\ell_0 P)$  in existing analyses and we present our proof. In Section III, we discuss about the NP-hardness of non-negative  $\ell_0$  minimization problems. We draw some conclusions in Section IV.

## II. HARDNESS OF $\ell_0$ MINIMIZATION PROBLEMS

### A. Background on constrained $\ell_0$ minimization problems

Let us recall that an NP-complete problem is a problem in NP to which any other problem in NP can be reduced in polynomial time. Thus NP-complete problems are identified as the hardest problems in NP. An NP-complete problem is strongly NP-complete if it remains NP-complete when all of its numerical parameters are bounded by a polynomial in the length of the input. NP-hard problems are at least as hard as NP-complete problems. However, NP-hard problems do not need to be in NP and do not need to be decision problems. Formally, a problem is NP-hard (respectively, strongly NP-hard) if a NP-complete (respectively, strongly NP-complete) problem can be reduced in polynomial time to it. The reader is referred to [17, 18] for more information on this topic.

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In the literature, problem  $(\ell_0 C)$ , called SAS in [2], is well known to be NP-hard [2, Theorem 1]. The NP-hardness of  $(\ell_0 C)$  is a valuable extension of an earlier result: the problem of minimum weight solution to linear equations (equivalent to  $(\ell_0 C)$  with  $\epsilon = 0$ ) is NP-hard [17, p. 246]. Davis *et al.* proved that  $(\ell_0 C')$ , called  $M$ -optimal approximation in [3], is NP-hard for any  $K < m$  [3, Theorem 2.1]. Both analyses of Natarajan and Davis were made by a polynomial time reduction from the “exact cover by 3-sets” problem<sup>1</sup> which is known to be NP-complete [17, p. 221].

### B. Existing analyses on penalized $\ell_0$ minimization

In [4, 5], the NP-hardness of  $(\ell_0 P)$  is deduced as a particular case of more general complexity analyses. However, it turns out that the latter do not apply to  $(\ell_0 P)$ , as explained hereafter. Chen *et al.* [4] address the unconstrained  $\ell_q$ - $\ell_p$  minimization problem, defined by:

$$\min_{\mathbf{x}} \|\mathbf{y} - A\mathbf{x}\|_q^q + \lambda \|\mathbf{x}\|_p^p \quad (\ell_q\text{-}\ell_p)$$

where  $\lambda > 0$ ,  $q \geq 1$  and  $0 \leq p < 1$ . The authors showed that problem  $(\ell_q\text{-}\ell_p)$  is NP-hard with any  $\lambda > 0$ ,  $q \geq 1$  and  $0 \leq p < 1$  [4, Theorem 3]. Obviously,  $(\ell_0 P)$  is the case where  $q = 2$  and  $p = 0$ . The proof was done by i) introducing an invertible transformation which scales any instance of problem  $(\ell_q\text{-}\ell_p)$  to the problem  $(\ell_q\text{-}\ell_p)$  with  $\lambda = 1/2$ , and ii) establishing a polynomial time reduction from the partition problem which is known to be NP-complete [17] to the problem  $(\ell_q\text{-}\ell_p)$  with  $\lambda = 1/2$ . In other words, they showed that problem  $(\ell_q\text{-}\ell_p)$  with  $\lambda = 1/2$  is NP-hard and, because there exists an invertible transformation from any problem  $(\ell_q\text{-}\ell_p)$  to the one with  $\lambda = 1/2$ , every problem  $(\ell_q\text{-}\ell_p)$  is NP-hard. Similarly, they showed that  $(\ell_q\text{-}\ell_p)$  is strongly NP-hard [4, Theorem 5] by a reduction from the 3-partition problem which is known to be strongly NP-hard [17]. The invertible transform used in [4] is defined by:

$$\tilde{\mathbf{x}} = (2\lambda)^{1/p} \mathbf{x}, \quad \tilde{A} = (2\lambda)^{-1/p} A. \quad (1)$$

Unfortunately, (1) is not well-defined when  $p = 0$ . Therefore, [4, Theorems 3 and 5] do not apply to  $(\ell_0 P)$  when  $\lambda \neq 1/2$ .

Using a different approach, Huo and Chen’s paper [5] addresses the penalized least-squares problem defined by:

$$\min_{\mathbf{x}} \|\mathbf{y} - A\mathbf{x}\|_2^2 + \lambda \sum_{i=1}^n \phi(|x_i|), \quad (\text{PLS})$$

where  $\phi$  is a penalty function mapping non-negative values to non-negative values. The authors showed that (PLS) is NP-hard if the penalty function  $\phi$  satisfies the following four conditions [5, Theorem 3.1]:

- C1.  $\phi(0) = 0$  and  $\forall 0 \leq \tau_1 < \tau_2$ ,  $\phi(\tau_1) \leq \phi(\tau_2)$ .
- C2. There exists  $\tau_0 > 0$  and a constant  $d > 0$  such that

$$\phi(\tau) \geq \phi(\tau_0) - d(\tau_0 - \tau)^2$$

<sup>1</sup>The latter problem, denoted by X3C in [2, 17], is stated as follows: Given a set  $S$  and a collection  $C$  of 3-element subsets of  $S$  (called triplets), is there a subcollection of disjoint triplets that exactly covers  $S$ ?

for every  $0 \leq \tau < \tau_0$ .

- C3. For the aforementioned  $\tau_0$ , if  $\tau_1, \tau_2 < \tau_0$  then

$$\phi(\tau_1) + \phi(\tau_2) \geq \phi(\tau_1 + \tau_2).$$

- C4. For every  $0 \leq \tau < \tau_0$ ,

$$\phi(\tau) + \phi(\tau_0 - \tau) > \phi(\tau_0). \quad (2)$$

The proof of [5, Theorem 3.1] is by a reduction from the NP-complete problem X3C to the decision version of (PLS); this leads to the NP-completeness of the decision version of (PLS) and so the NP-hardness of (PLS) [5, Appendix 1]. The authors claimed that the  $\ell_0$  penalty function satisfies conditions C1-C4 for  $\tau_0 = d = 1$ . Therefore, the (PLS) problem with the  $\ell_0$  penalty function is NP-hard [5, Corollary 3.2]. Unfortunately, it turns out that the  $\ell_0$  penalty does not fulfill condition C4 as claimed. Indeed, for  $\tau = 0$  the strict inequality (2) becomes  $\phi(0) > 0$ . Besides, in the proof [5, Appendix 1], the inputs of the decision problem are not guaranteed to have rational values. This might also violate the polynomiality of the reduction. Therefore, [5, Theorem 3.1] does not apply to  $(\ell_0 P)$ .

In [5], the authors also mention an alternate proof of NP-hardness of  $(\ell_0 P)$  from Huo and Ni’s earlier paper [19] as a special case of their results. In this proof [19, Appendix A.1], the relation between  $(\ell_0 P)$  and  $(\ell_0 C)$  is established using the principle of Lagrange multiplier. More precisely, the authors introduce an instance of  $(\ell_0 C)$  in which  $\epsilon$  is defined from the minimizer of  $(\ell_0 P)$  and argue that solving  $(\ell_0 P)$  is equivalent to solving the mentioned instance of  $(\ell_0 C)$ , which is known to be NP-hard [2]. There are a number of issues in the NP-hardness proof in [19]. For instance, the proposed transformation between  $(\ell_0 P)$  and  $(\ell_0 C)$  is not a polynomial time reduction. Besides, it is well known that  $(\ell_0 P)$  and  $(\ell_0 C)$  are not equivalent [20].

### C. New analysis on penalized $\ell_0$ minimization problems

To prove that a problem T is NP-hard, one must establish a polynomial time reduction (briefly called reduction hereafter) from some known NP-hard or NP-complete problem to T [18]. Roughly speaking, the reduction from a problem T1 to another problem T2 implies that T1 is not harder than T2. Therefore, if there exists a reduction from T1 to T2 and if T1 is NP-hard, T2 must be NP-hard too. The NP-hardness proofs in [2] and [3] use this principle. As an adaptation of Natarajan’s construction, we prove the NP-hardness of  $(\ell_0 P)$  using the same principle as follows.

**Theorem II.1.** *Problem  $(\ell_0 P)$  is NP-hard for  $0 < \lambda < 3$ .*

The proof is by a reduction from the known NP-complete problem X3C to  $(\ell_0 P)$ . The proof contains three steps: (1) Construct an instance of  $(\ell_0 P)$  from a given instance of X3C; (2) Construct a solution of  $(\ell_0 P)$  from a solution of X3C; (3) Construct a solution of X3C from a solution of  $(\ell_0 P)$ .

1) *Construction of an instance of  $(\ell_0 P)$  from a given instance of X3C:* Given an instance of X3C:  $S = \{s_1, s_2, \dots, s_m\}$  is a set of  $m$  elements.  $C$  is a collection of  $n$  triplets  $c_j$ ,  $1 \leq j \leq n$ . Without loss of generality we

can assume that  $m$  is a multiple of 3 since otherwise there is trivially no exact cover so no solution of X3C.

We now construct an instance of  $(\ell_0P)$ . Let  $\mathbf{y} = [1, 1, \dots, 1]^T \in \mathbb{R}^m$ . Let  $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$  where  $a_{ij} = 1$  if  $s_i \in c_j$  and  $a_{ij} = 0$  otherwise. Let  $\lambda \in \mathbb{Q}$ ,  $0 < \lambda < 3$ . Let

$$F(\mathbf{x}) := \|\mathbf{y} - A\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_0. \quad (3)$$

2) *Construction of a solution of  $(\ell_0P)$  from a solution of X3C:* Assume that there is a subcollection of disjoint triplets  $\hat{C}$  which exactly covers  $S$ . Let  $\mathbf{x}^* = [x_1^*, x_2^*, \dots, x_n^*]^T$  where  $x_j^* = 1$  if  $c_j \in \hat{C}$  and  $x_j^* = 0$  otherwise. We will prove that  $\mathbf{x}^*$  is a solution of  $(\ell_0P)$ .

Since  $\hat{C}$  exactly covers  $S$ ,  $|\hat{C}| = m/3$  and  $\mathbf{y} = A\mathbf{x}^*$ . Thus,  $\|\mathbf{x}^*\|_0 = m/3$  and

$$F(\mathbf{x}^*) = 0 + \lambda \frac{m}{3} = \lambda \frac{m}{3}.$$

Suppose that there exists  $\bar{\mathbf{x}}$  such that

$$F(\bar{\mathbf{x}}) < F(\mathbf{x}^*) = \lambda \frac{m}{3}. \quad (4)$$

Let us show that this leads to a contradiction.

Since  $F(\bar{\mathbf{x}}) \geq \lambda\|\bar{\mathbf{x}}\|_0$ , from (4) we have  $\|\bar{\mathbf{x}}\|_0 < m/3$ . Therefore, we can rewrite  $\|\bar{\mathbf{x}}\|_0 = m/3 - q$  for some  $q \in \mathbb{N}$ ,  $1 \leq q < m/3$ . Note that  $A\bar{\mathbf{x}}$  has  $m$  entries. Since the number of non-zero entries of  $A\bar{\mathbf{x}}$  identifies with the number of elements  $s_i$  recovered by the subcollection corresponding to  $\bar{\mathbf{x}}$ , this number cannot exceed  $3\|\bar{\mathbf{x}}\|_0 = m - 3q$ . As a result, the number of zero entries of  $A\bar{\mathbf{x}}$  must be between  $3q$  and  $m$ . Since  $\mathbf{y}$  is the all-one vector,  $\mathbf{y} - A\bar{\mathbf{x}}$  has at least  $3q$  entries valued 1, which implies

$$\|\mathbf{y} - A\bar{\mathbf{x}}\|_2^2 \geq 3q. \quad (5)$$

Hence,

$$F(\bar{\mathbf{x}}) \geq 3q + \lambda \left( \frac{m}{3} - q \right) = \lambda \frac{m}{3} + (3 - \lambda)q > \lambda \frac{m}{3}, \quad (6)$$

which contradicts (4). Therefore,  $\mathbf{x}^*$  is a solution of  $(\ell_0P)$ .

3) *Construction of a solution of X3C from a solution of  $(\ell_0P)$ :* Assume that  $\mathbf{x}^*$  is a solution of  $(\ell_0P)$ . We will consider four cases as follows.

a) *Case  $\|\mathbf{x}^*\|_0 > m/3$ :* We deduce that X3C has no solution. Indeed, assume that  $\hat{C}$  is an exact cover for  $S$ . Define  $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$  where  $x_j = 1$  if  $c_j \in \hat{C}$  and  $x_i = 0$  otherwise. Then we have

$$F(\mathbf{x}) = \lambda \frac{m}{3} < \lambda\|\mathbf{x}^*\|_0 \leq F(\mathbf{x}^*)$$

which contradicts the fact that  $\mathbf{x}^*$  is a solution of  $(\ell_0P)$ .

b) *Case  $\|\mathbf{x}^*\|_0 < m/3$ :* We deduce that X3C has no solution. Indeed, assume that  $\hat{C}$  is an exact cover for  $S$ . Let  $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$  where  $x_j = 1$  if  $c_j \in \hat{C}$  and  $x_i = 0$  otherwise. Then we have  $F(\mathbf{x}) = \lambda \frac{m}{3}$ . Since  $\|\mathbf{x}^*\|_0 < m/3$ , we can write  $\|\mathbf{x}^*\|_0 = m/3 - q$  for some  $q \in \mathbb{N}$  and  $1 \leq q < m/3$ . Similar to (6), we have  $F(\mathbf{x}^*) > \lambda \frac{m}{3}$ . Since  $F(\mathbf{x}) = \lambda \frac{m}{3}$ , we obtain  $F(\mathbf{x}^*) > F(\mathbf{x})$  which contradicts the fact that  $\mathbf{x}^*$  is a solution of  $(\ell_0P)$ .

c) *Case where  $\|\mathbf{x}^*\|_0 = m/3$  and  $\mathbf{y} \neq A\mathbf{x}^*$ :* We deduce that X3C has no solution. Indeed, assume that  $\hat{C}$  is an exact cover for  $S$ . Define  $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$  where  $x_j = 1$  if  $c_j \in \hat{C}$  and  $x_i = 0$  otherwise. Then we have

$$F(\mathbf{x}) = \lambda \frac{m}{3} < \|\mathbf{y} - A\mathbf{x}^*\|_2^2 + \lambda\|\mathbf{x}^*\|_0 = F(\mathbf{x}^*)$$

which contradicts the fact that  $\mathbf{x}^*$  is a solution of  $(\ell_0P)$ .

d) *Case where  $\|\mathbf{x}^*\|_0 = m/3$  and  $\mathbf{y} = A\mathbf{x}^*$ :* Let  $\hat{C}$  be the collection of triplets  $c_j$  such that the  $j^{\text{th}}$  entry of  $\mathbf{x}^*$  is non-zero. Obviously,  $\hat{C}$  is an exact cover for  $S$  so a solution of X3C.

Thus Theorem II.1 is proved.

It is notable that the proof above is also valid when  $F(\mathbf{x}) := \|\mathbf{y} - A\mathbf{x}\|_p^p + \lambda\|\mathbf{x}\|_0$  for any  $p \geq 1$ . Indeed, one only needs to check whether (5) still holds when the  $\ell_2$  norm is replaced by the  $\ell_p$  norm with  $p \geq 1$ . This is the case since  $\mathbf{y} - A\bar{\mathbf{x}}$  has at least  $3q$  entries equal to 1. Therefore, we have the following generalization of Theorem II.1.

**Theorem II.2.** *Problem  $\min_{\mathbf{x}} \|\mathbf{y} - A\mathbf{x}\|_p^p + \lambda\|\mathbf{x}\|_0$  is NP-hard for  $p \geq 1$  and  $0 < \lambda < 3$ .*

### III. HARDNESS OF NON-NEGATIVE $\ell_0$ MINIMIZATION PROBLEMS

The NP-hardness of non-negative  $\ell_0$  minimization problems is a consequence of NP-hard proofs of  $(\ell_0C)$  [2],  $(\ell_0C')$  [3] and  $(\ell_0P)$  (Theorem II.1). Indeed, all these proofs consist in a reduction from X3C and the solution that established equivalence is binary. Therefore, the additional non-negativity constraints do not change the validity of these proofs. In other words, one can repeat the same proofs as in [2, 3] and that of Theorem II.1 for the corresponding non-negative  $\ell_0$  minimization problems  $(\ell_0C+)$ ,  $(\ell_0C'+)$  and  $(\ell_0P+)$ . Another way to prove the NP-hardness of non-negative  $\ell_0$  minimization problems is by a reduction from the corresponding  $\ell_0$  minimization problems which are known to be NP-hard. In this reduction, the instance of non-negative problems is defined by

$$\tilde{\mathbf{y}} = \mathbf{y}, \quad \tilde{A} = [A, -A], \quad \tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{bmatrix}$$

where  $\mathbf{x}^+ = \max\{\mathbf{x}, \mathbf{0}\}$ ,  $\mathbf{x}^- = \max\{-\mathbf{x}, \mathbf{0}\}$ . Naturally, by this construction, one gets  $\tilde{\mathbf{x}} \geq \mathbf{0}$ ,  $\|\tilde{\mathbf{x}}\|_0 = \|\mathbf{x}\|_0$  and  $\tilde{A}\tilde{\mathbf{x}} = A\mathbf{x}$ . The proofs (skipped for brevity) contain three steps similar to that of Theorem II.1.

Therefore, we can state the following theorem without proof.

**Theorem III.1.**  *$(\ell_0C+)$ ,  $(\ell_0C'+)$  are NP-hard. The same for  $(\ell_0P+)$  with  $0 < \lambda < 3$ .*

In the same spirit and using the same argument as at the end of Section II-C one can directly extend Theorem II.2 to the non-negative setting.

**Theorem III.2.** *Problem  $\min_{\mathbf{x} \geq \mathbf{0}} \|\mathbf{y} - A\mathbf{x}\|_p^p + \lambda\|\mathbf{x}\|_0$  is NP-hard for  $p \geq 1$  and  $0 < \lambda < 3$ .*

#### IV. CONCLUSION

NP-hardness of penalized  $\ell_0$  minimization problems cannot be deduced from previous complexity analyses, as stated in [4, 5]. Here, we introduced a new proof of NP-hardness of penalized  $\ell_0$  minimization problems when the regularization parameter  $\lambda$  is smaller than 3, by an adaptation of Natarajan's construction [2], while the case  $\lambda \geq 3$  is still open. Besides, we showed that the  $\ell_0$  minimization problems with non-negative constraints are also NP-hard.

This work can be extended in several directions. For instance, researchers interested in what makes NP-hard problems even harder might be interested in the strong NP-hardness of the aforementioned optimization problems. As it is widely believed that X3C is strongly NP-complete, one might easily deduce the strong NP-hardness of  $(\ell_0 C)$ ,  $(\ell_0 C')$  and other problems which are reduced from X3C. However, to the best of our knowledge, X3C is only proved to be NP-complete [17, pp. 53, 221] and the strong NP-completeness has not been rigorously shown yet. Therefore, we believe that the question of strong NP-hardness of (non-negative)  $\ell_0$  minimization problems is not trivial and needs more work in future.

Besides, as (non-negative)  $\ell_0$  minimization problems are NP-hard, it would be interesting to know if the associated decision problems are in NP (so being NP-complete). Let us consider the decision problem associated with  $(\ell_0 C)$ : given  $\mathbf{y} \in \mathbb{Q}^m$ ,  $A \in \mathbb{Q}^{m \times n}$ , a positive rational number  $\epsilon$  and a positive integer  $K$ , does there exist  $\mathbf{x} \in \mathbb{R}^n$  such that  $\|\mathbf{y} - A\mathbf{x}\|_2 \leq \epsilon$  and  $\|\mathbf{x}\|_0 \leq K$ ? This decision problem should be in NP since if one can guess a rational solution  $\mathbf{x}$ , it can be verified in polynomial time if  $\|\mathbf{y} - A\mathbf{x}\|_2 \leq \epsilon$  and  $\|\mathbf{x}\|_0 \leq K$ . Similarly, we conjecture that the decision version of other optimization problems mentioned in the paper are in NP as well.

Another perspective is the approximability of aforementioned NP-hard problems. The hardness of approximating  $(\ell_0 C)$  was discussed in [21, 22]. It was shown that approximating  $(\ell_0 C)$  to within a factor of  $(1-\alpha) \ln(n)$ ,  $0 < \alpha < 1$  is NP-hard [22]. Examining whether similar results can be obtained on other NP-hard problems presented in the paper would require more involved theoretical analysis, which is left for future work.

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