



HAL
open science

Twisted quadratic moments for Dirichlet L-functions at $s = 2$

Stéphane Louboutin

► **To cite this version:**

Stéphane Louboutin. Twisted quadratic moments for Dirichlet L-functions at $s = 2$. Publ. Math. Debrecen, In press. hal-02111813

HAL Id: hal-02111813

<https://hal.science/hal-02111813>

Submitted on 26 Apr 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Twisted quadratic moments for Dirichlet L -functions at $s = 2$

Stéphane R. LOUBOUTIN
Aix Marseille Université, CNRS, Centrale Marseille, I2M,
Marseille, FRANCE
stephane.louboutin@univ-amu.fr

April 26, 2019

To appear in **Publ. Math. Debrecen**

Abstract

Let c, n be given positive integers. Let $q > 2$ be coprime with c . Let X_q be the multiplicative group of order $\phi(q)$ of the Dirichlet characters modulo q . Set

$$M(q, c, n) := \frac{2}{\phi(q)} \sum_{\substack{\chi \in X_q \\ \chi(-1) = (-1)^n}} \chi(c) |L(n, \chi)|^2.$$

The goal of this paper is to explain how one can compute explicit formulas for $M(q, c, n)$ for a given small integers n and c . As an example, we give explicit formulas for $M(q, c, 2)$ for $c \in \{1, 2, 3, 4, 6\}$ and for $M(p, 5, 2)$ for p a prime integer. As a consequence, we show that a previously published formula for $M(p, 3, 2)$ is false.

⁰2010 Mathematics Subject Classification. Primary. 11M06. Key words and phrases. Dirichlet L -functions, mean values, quadratic moments.

1 Introduction

Let c, n be given positive integers. Let $q > 2$ be coprime with c . Let X_q be the multiplicative group of order $\phi(q)$ of the Dirichlet characters modulo q . Set

$$M(q, c, n) := \frac{2}{\phi(q)} \sum_{\substack{\chi \in X_q \\ \chi(-1) = (-1)^n}} \chi(c) |L(n, \chi)|^2.$$

In [Lou01] we developed a method for obtaining formulas for $M(q, 1, n)$. For example, by [Lou01, Theorem 2], we have

$$M(q, 1, 2) = \frac{\pi^2}{90} \times \left\{ \phi_4(q) + \frac{10}{q^2} \phi_2(q) \right\}, \quad (1)$$

where

$$\phi_k(q) = \prod_{p|q} \left(1 - \frac{1}{p^k} \right) \quad (k \in \mathbb{Z}_{\geq 1}).$$

In [Lou14], for a given $c > 1$ we developed a method for obtaining formulas for $M(q, c, 1)$. For example, in the cases that the multiplicative group $(\mathbb{Z}/c\mathbb{Z})^*$ is trivial of order 1, i.e. for $c \in \{1, 2\}$, by [Lou14, (1) and (3)] and [Lou15, Theorem 2], we have

$$M(q, c, 1) = \frac{\pi^2}{6c} \times \left\{ \phi_2(q) - \frac{3c\phi_1(q)}{q} \right\}. \quad (2)$$

In the cases that the multiplicative group $(\mathbb{Z}/c\mathbb{Z})^*$ is of order 2, i.e. for $c \in \{3, 4, 6\}$, by [Lou14, (4) and (5)] and [Lou15, Theorem 4], we have

$$M(q, c, 1) = \frac{\pi^2}{6c} \times \left\{ \phi_2(q) - \frac{3c\phi_1(q)}{q} - \frac{(c-1)(c-2)\chi_c(q)}{q} \prod_{p|q} \left(1 - \frac{\chi_c(p)}{p} \right) \right\},$$

where χ_c is the non-trivial character on the multiplicative group $(\mathbb{Z}/c\mathbb{Z})^*$. i.e. where $\chi_c(n) = 1$ if $n \equiv 1 \pmod{c}$ and $\chi_c(n) = -1$ if $n \equiv -1 \pmod{c}$. We point out that according to [Lou15, Section 4] as c gets bigger such explicit formula become very complicated, there is no known closed formula for $M(q, c, 2)$ and we gave formulas for $c \in \{1, 2, 3, 4, 5, 6, 8, 10\}$. Restricting himself to prime moduli, in [Liu, Corollary 1.1] H. Liu gives explicit formulas for $M(p, c, 2)$ for $c = 1, 2, 3, 4$, without explaining why he did not consider neither non-prime moduli nor values of $c > 4$. Here, focussing on the particular example of $M(q, c, 2)$, we explain how to extend Liu's results to non-prime moduli and values of $c > 4$. We will prove the following result and we will show that the formula for $M(p, 3, 2,)$ in [Liu, Corollary 1.1] is false (compare with [Lou15, Theorem 4]):

Theorem 1 Let $q > 1$ be coprime with $c \in \{1, 2, 3, 4, 6\}$. Define

c	u_c	v_c	c	u_c	v_c
1	10	0	3	210	80
2	70	0	4	490	360
			6	1830	2240

Let X_q^+ be the group of order $\phi(q)/2$ of the even Dirichlet characters modulo q . For $c \in \{3, 4, 6\}$, let χ_c be the only non-trivial character on the multiplicative group $(\mathbb{Z}/c\mathbb{Z})^* = \{\pm 1\}$. Then

$$\begin{aligned} M(q, c, 2) &:= \frac{2}{\phi(q)} \sum_{\chi \in X_q^+} \chi(c) |L(2, \chi)|^2 \\ &= \frac{\pi^4}{90c^2} \times \left\{ \phi_4(q) + \frac{u_c \phi_2(q)}{q^2} + \frac{v_c \chi_c(q)}{q^3} \prod_{p|q} \left(1 - \frac{\chi_c(p)}{p} \right) \right\}. \end{aligned}$$

As a Corollary we recover the formulas in [Liu, Corollary 1.1], correct his formula for $M(p, 3, 2)$ and give a new formula, the one for $M(p, 6, 2)$:

Corollary 2 Let $p > 3$ be a prime integer. We have

$$M(p, c, 2) = \frac{\pi^4}{90c^2} \times \frac{(p^2 - 1)(p^2 + u_c + 1)}{p^4} \quad (c \in \{1, 2\})$$

and

$$M(p, c, 2) = \frac{\pi^4}{90c^2} \times \frac{p^4 + u_c p^2 + v_c \chi_c(p)p - (u_c + v_c + 1)}{p^4} \quad (c \in \{3, 4, 6\}).$$

We also have the explicit formulas

$$M(p, 3, 2) = \frac{\pi^4}{810} \times \frac{p^4 + 210p^2 + 80p\chi_3(p) - 291}{p^4},$$

$$M(p, 4, 2) = \frac{\pi^4}{1440} \times \frac{p^4 + 490p^2 + 360\chi_4(p) - 851}{p^4}$$

and

$$M(p, 6, 2) = \frac{\pi^4}{3240} \times \frac{p^4 + 1830p^2 + 2240p\chi_6(p) - 4071}{p^4}.$$

Let us explain how we performed some numerical computation to check our formulas for $M(p, c, 2)$ and let us justify that the formula for $M(p, 3, 2)$ in [Liu, Corollary 1.1] is not correct. Let $p > 3$ be a prime integer. We refer to [Was, Chapter 4] for the justification of what follows. For χ_0 the trivial character modulo p we have $L(s, \chi_0) = (1 - p^{-s})\zeta(s)$ and $L(2, \chi_0) = \frac{p^2-1}{p^2} \frac{\pi^2}{6}$. For $\chi_0 \neq \chi \in X_p^+$, we have

$$L(2, \chi) = -W_\chi \frac{2\pi^2}{p^{3/2}} L(-1, \bar{\chi}) = W_\chi \frac{\pi^2}{p^{3/2}} B_{2, \bar{\chi}} = W_\chi \frac{\pi^2}{p^{5/2}} \sum_{a=1}^{p-1} \bar{\chi}(a) a^2,$$

where the root number W_χ is a complex number of absolute value equal to 1. Hence,

$$M(p, c, 2) = \frac{2\pi^4}{p-1} \times \left\{ \frac{(p^2-1)^2}{36p^4} + \sum_{\chi_0 \neq \chi \in X_p^+} \chi(c) \frac{\left| \sum_{a=1}^{p-1} \chi(a) a^2 \right|^2}{p^5} \right\}.$$

Taking $p = 5$ for which the only $\chi_0 \neq \chi \in X_5^+$ is the Legendre symbol $\left(\frac{\bullet}{5}\right)$ for which $\sum_{a=1}^4 \chi(a) a^2 = 1 - 2^2 - 3^2 + 4^2 = 4$, we obtain

$$M(5, c, 2) = \frac{8\pi^4}{5^5} \left(5 + \left(\frac{c}{5}\right) \right).$$

For $c \in \{1, 2, 3, 4\}$ this formula yields the same value than the ones in Corollary 2. In particular they give $M(5, 3, 2) = 32\pi^4/5^5$ whereas the formula in [Liu, Corollary 1.1] gives the wrong estimation $M(5, 3, 2) = 238\pi^4/(9 \cdot 5^5)$.

2 Proof of Theorem 1

Lemma 3 *Assume that $\gcd(c, d) = 1$. Set*

$$S_{\cot^k}(d) = \sum_{a=1}^{d-1} \cot^k \left(\frac{\pi a}{d} \right),$$

with the convention that $S_{\cot^k}(1) = 0$, and

$$S_2(c, d) := \sum_{a=1}^{d-1} \cot^2 \left(\frac{\pi a}{d} \right) \cot^2 \left(\frac{\pi ac}{d} \right), \quad (3)$$

with the convention that $S_2(c, 1) = 0$ for $c \geq 1$. Then

$$M(q, c, 2) = \frac{\pi^4}{2q^4} \sum_{d|q} \mu(q/d) \left((d-1) + 2S_{\cot^2}(d) + S_2(c, d) \right). \quad (4)$$

In particular,

$$M(p, c, 2) = \frac{\pi^4}{2q^4} \left((p-1) + 2S_{\cot^2}(p) + S_2(c, p) \right). \quad (5)$$

Proof. From [Lou01, Proposition 3] we have

$$L(2, \chi) = -\frac{\pi^2}{2q^2} \sum_{a=1}^{q-1} \chi(a) \cot'(\pi a/q) \quad (\chi \in X_q^+)$$

and as in the proof of [Lou01, Proposition 3] we obtain

$$M(q, c, 2) = \frac{\pi^4}{2q^4} \sum_{\substack{a=1 \\ \gcd(a,q)=1}}^{q-1} \cot'(\pi a/q) \cot'(\pi ac/q),$$

with $\cot' = -1 - \cot^2$ the derivative of \cot . Using $\sum_{d|a \text{ and } q} \mu(d) = 0$ or 1 according as $\gcd(a, q) > 1$ or $\gcd(a, q) = 1$, we obtain

$$M(q, c, 2) = \frac{\pi^4}{2q^4} \sum_{\substack{d>1 \\ d|q}} \mu(q/d) \sum_{a=1}^{d-1} \left(1 + \cot^2\left(\frac{\pi a}{d}\right)\right) \left(1 + \cot^2\left(\frac{\pi ac}{d}\right)\right) \quad (6)$$

and the desired result, thanks to the conventions $S_{\cot^2}(1) = S_2(c, 1) = 0$. •

Lemma 4 *Let $c > 1$ be an integer. It holds that*

$$\begin{aligned} (\cot^2 x)(\cot^2(cx)) &= \frac{1}{c^2} \cot^4 x - \frac{2(c^2 - 1)}{3c^2} \cot^2 x + \frac{c^4 - 1}{15c^2} \\ &\quad - \frac{2}{c^2} \sum_{k=1}^{c-1} \frac{2 \cot^3(k\pi/c) + \cot(k\pi/c)}{\sin^2(k\pi/c)} + \frac{1}{c^2} \sum_{k=1}^{c-1} \frac{\cot^2(k\pi/c)}{\sin^4(k\pi/c)} (\cot(k\pi/c) - \cot x)^2. \end{aligned}$$

Proof. Adapt the proof of [Lou14, Lemma 4]. •

Lemma 5 *Let $d > 1$ be an integer and $\theta \in (0, \pi) \setminus \{\pi a/d; 1 \leq a \leq d-1\}$. Set*

$$T_m(\theta, d) := \frac{1}{d} \sum_{a=1}^{d-1} \frac{1}{(\cot \theta - \cot(\pi a/d))^m}.$$

Then

$$T_1(\theta, d) = (\sin^2 \theta) (\cot \theta - \cot(d\theta)),$$

$$T_2(\theta, d) = (\sin^4 \theta) (d \cot^2(d\theta) - 2(\cot \theta)(\cot(d\theta)) + \cot^2 \theta + (d-1))$$

and

$$\begin{aligned} -2(2 \cot^3 \theta + \cot \theta) \frac{T_1(\theta, d)}{\sin^2 \theta} + (\cot^2 \theta) \frac{T_2(\theta, d)}{\sin^4 \theta} \\ = (d-3) \cot^2 \theta - 3 \cot^4 \theta + 2(\cot^3 \theta + \cot \theta) \cot(d\theta) + d(\cot^2 \theta)(\cot^2(d\theta)). \end{aligned}$$

Proof. For $T_1(\theta, d)$, see [Lou14, Lemma 5] and take $\alpha = \cot \theta$. For $T_2(\theta, d)$, notice that $T_2(\theta, d) = (\sin^2 \theta) \frac{dT_1(\theta, d)}{d\theta}$. •

Now, noticing that $\sum_{d|q} \mu(q/d)d^k = q^k \phi_k(q)$ and $\phi_0(q) = 0$ for $q > 1$, Theorem 1 follows from (4) and the following Proposition:

Proposition 6 *Assume that $\gcd(c, d) = 1$. Set*

$$F_c(d) := \sum_{k=1}^{c-1} (\cot^3(k\pi/c) + \cot(k\pi/c)) \cot(kd\pi/c),$$

with the convention that $F_1(d) = 0$ for $d \geq 1$. Then

$$(d-1) + 2S_{\cot^2}(d) + S_2(c, d) = \frac{Q_c(d)}{45c^2} + \frac{d^2}{c^2} S_2(d, c) + \frac{2d}{c^2} F_c(d),$$

where $Q_c(d) := d^4 + 5(7c^2 - 9c + 4)d^2 - (3c^4 + 5c^2 + 3)$. Notice that $S_2(d, c)$ and $F_c(d)$ depend only on d modulo c . In particular, we have

c	$S_2(d, c)$	$F_c(d)$	$45c^2((d-1) + 2S_{\cot^2}(d) + S_2(c, d))$
1	0	0	$d^4 + 10d^2 - 11$
2	0	0	$d^4 + 70d^2 - 71$
3	$\frac{2}{9}$	$\frac{8}{9}\chi_3(d)$	$d^4 + 210d^2 + 80d\chi_3(d) - 291$
4	2	$4\chi_4(d)$	$d^4 + 490d^2 + 360d\chi_4(d) - 851$
6	$\frac{164}{9}$	$\frac{224}{9}\chi_6(d)$	$d^4 + 1830d^2 + 2240d\chi_6(d) - 4071$

Proof. Using (3) and Lemma 4, we obtain

$$\begin{aligned} c^2 S_2(c, d) &= S_{\cot^4}(d) - \frac{2(c^2 - 1)}{3} S_{\cot^2}(d) + \frac{c^4 - 1}{15} (d - 1) \\ &\quad - 2d \sum_{k=1}^{c-1} (2 \cot^3(k\pi/c) + \cot(k\pi/c)) \frac{T_1(k\pi/c, d)}{\sin^2(k\pi/c)} \\ &\quad + d \sum_{k=1}^{c-1} (\cot^2(k\pi/c)) \frac{T_2(k\pi/c, d)}{\sin^4(k\pi/c)}. \end{aligned}$$

Using the last assertion in Lemma 5 for $\theta = k\pi/c$, we obtain

$$\begin{aligned} c^2 S_2(c, d) &= S_{\cot^4}(d) - \frac{2(c^2 - 1)}{3} S_{\cot^2}(d) + \frac{c^4 - 1}{15} (d - 1) \\ &\quad + d \left((d - 3) S_{\cot^2}(c) - 3 S_{\cot^4}(c) \right) \\ &\quad + 2d \sum_{k=1}^{c-1} (\cot^3(k\pi/c) + \cot(k\pi/c)) \cot(kd\pi/c) \\ &\quad + d^2 \sum_{k=1}^{c-1} (\cot^2(k\pi/c)) (\cot^2(kd\pi/c)). \end{aligned}$$

Now, for $n > 1$ the $\cot(\pi k/n)$, $1 \leq k \leq n-1$, are the roots of

$$\frac{(X+i)^n - (X-i)^n}{2in} = X^{n-1} - \frac{(n-1)(n-2)}{6}X^{n-3} + \dots \in \mathbb{Q}[X].$$

Hence, $S_{\cot^2}(n) = \frac{(n-1)(n-2)}{3}$ and $S_{\cot^4}(n) = \frac{(n-1)(n-2)(n^2+3n-13)}{45}$. The desired result follows. •

Corollary 7 For $p \neq 2, 5$ a prime integer, we have

$$M(q, 5, 2) = \frac{\pi^4}{2250p^4} \times \begin{cases} p^4 + 994p^2 + 1008p - 2003 & \text{if } p \equiv 1 \pmod{5}, \\ p^4 + 706p^2 + 144p - 2003 & \text{if } p \equiv 2 \pmod{5}, \\ p^4 + 706p^2 - 144p - 2003 & \text{if } p \equiv 3 \pmod{5}, \\ p^4 + 994p^2 - 1008p - 2003 & \text{if } p \equiv 4 \pmod{5}. \end{cases}$$

Proof. Follows from (5), Proposition 6 and the computation of $S_2(r, 5)$ and $F_5(r)$ for the value r of d modulo 5 ranging in $\{1, 2, 3, 4\}$. •

Remarks 8 As in [Zag], for q a positive integer set

$$d(q; a_1, \dots, a_n) = (-1)^{n/2} \sum_{k=1}^{q-1} \cot\left(\frac{ka_1}{q}\right) \cdots \cot\left(\frac{ka_n}{q}\right).$$

In [Zag, (47)], Zagier gave a reciprocity law for these generalized Dedekind sums under the assumption that q and the a_k 's be pairwise coprime. For $S_2(c, q) = d(q; 1, 1, c, c)$ this assumption is not fulfilled. Hence, Proposition 6, where $F_c(x) := d(c; 1, 1, 1, x) + d(c; 1, x)$, which can be viewed as a reciprocity law for the sums $S_2(c, d)$ does not follow from Zagier's reciprocity law.

3 Conclusion

[Lou14, Lemma 4] which deals with $(\cot x)(\cot(cx))$ and the present Corollary 4 which deals with $(\cot^2 x)(\cot^2(cx))$ could easily be generalized to evaluate $(\cot^m x)(\cot^n(cx))$ for small values of m and n . Lemma 5 can be very easily generalized to evaluate $T_m(\theta, d)$ for small values of m . Lemma 3 can be generalized to express $M(q, c, n)$ in terms of the

$$S_{k,l}(c, d) := \sum_{a=1}^{d-1} \cot^k\left(\frac{\pi a}{d}\right) \cot^l\left(\frac{\pi ac}{d}\right),$$

see [Lou01]. Hence, following the method developed here one could obtain explicit formulas for $M(q, c, n)$ and $M(p, c, n)$ for other small values of n and

c . As explained here and in [Lou15], the formulas for $M(q, c, n)$ would get more complicated as $\phi(c)$ increases. We would get $\phi(c)$ twisted quadratic moments formulas, one for each value of d modulo c (notice that $\phi(c) = 1$ if and only if $c = 1, 2$ and $\phi(c) = 2$ if and only if $c = 3, 4, 6$). Only asymptotic estimates as in [Bet] and [LL] could yield explicit formulas for all c 's.

References

- [Bet] S. Bettin. On the reciprocity law for the twisted second moment of Dirichlet L -functions. *Trans. Amer. Math. Soc.* **368** (2016), 6887–6914.
- [Liu] H. Liu. On the mean values of Dirichlet L -functions. *J. Number Theory* **147** (2015), 172–183.
- [LL] S. H. Lee and S. Lee. On the twisted quadratic moment for Dirichlet L -functions. *J. Number Theory* **174** (2017), 427–435.
- [Lou01] S. Louboutin. The mean value of $|L(k, \chi)|^2$ at positive rational integers $k \geq 1$. *Colloq. Math.* **90** (2001), 69–76.
- [Lou14] S. Louboutin. A twisted quadratic moment for Dirichlet L -functions. *Proc. Amer. Math. Soc.* **142** (2014), 1539–154
- [Lou15] S. Louboutin. Twisted quadratic moments for Dirichlet L -functions. *Bull. Korean Math. Soc.* **52** (2015), 2095–2105.
- [Was] L. C. Washington. *Introduction to Cyclotomic Fields*. Second Edition. Graduate Texts in Mathematics, **83**. Springer-Verlag, New York, 1997.
- [Zag] D. Zagier. Higher dimensional Dedekind sums. *Math. Ann.* **202** (1973), 149–172.